Quantum phase transitions of metals

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Outline

1. Quantum phase transitions of a semi-metal

   Graphene, Dirac fermions and the Gross-Neveu model

2. Spin density wave order on the square lattice

   From a large Fermi surface to Fermi pockets

3. Competing orders I

   Unconventional superconductivity and the phase diagrams of the cuprates and the pnictides

4. Competing orders II

   Bond order, geometric phases, and gauge theories

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The Hubbard Model

$$H = -\sum_{i,j} t_{ij} c^{\dagger}_{i\alpha} c_{j\alpha} + U \sum_i \left(n_{i\uparrow} - \frac{1}{2} \right) \left(n_{i\downarrow} - \frac{1}{2} \right) - \mu \sum_i c^{\dagger}_{i\alpha} c_{i\alpha}$$

In the limit of large U, and at a density of one particle per site, this maps onto the Heisenberg antiferromagnet

$$H_{AF} = \sum_{i<j} J_{ij} S_i^a S_j^a$$

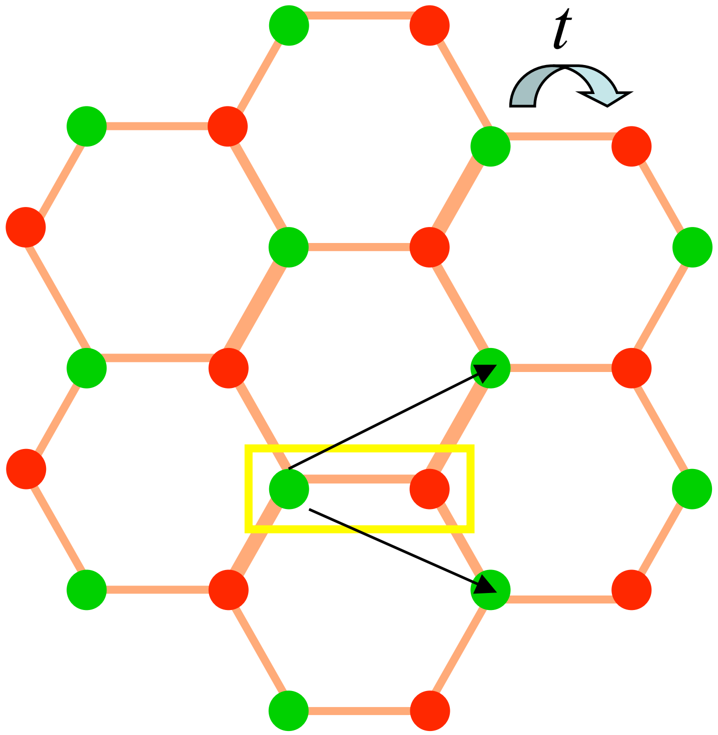
where $a = x, y, z,$

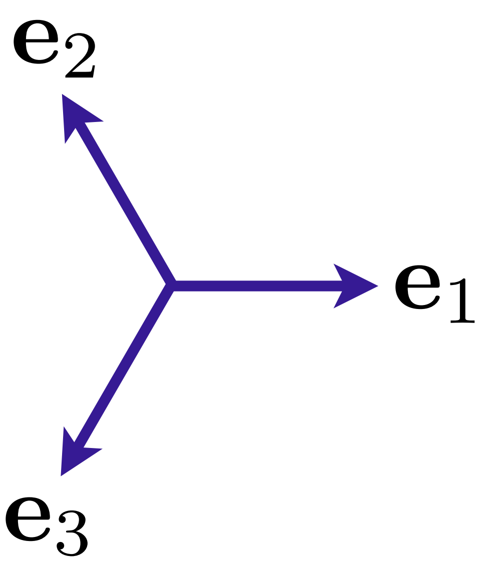
$$S_i^a = \frac{1}{2} c^{a\dagger}_{i\alpha} \sigma^a_{\alpha\beta} c_{i\beta},$$

with σ^a the Pauli matrices and

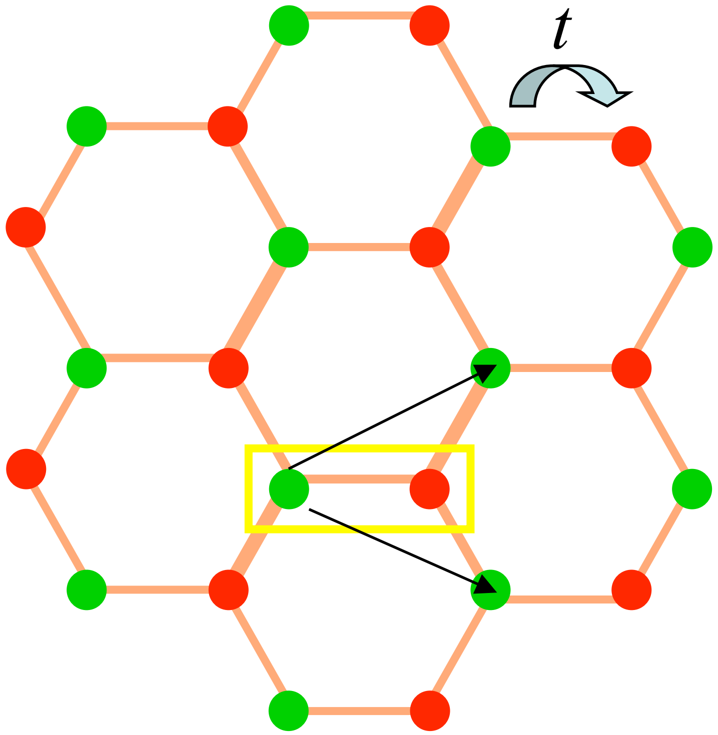
$$J_{ij} = \frac{4t_{ij}^2}{U}$$

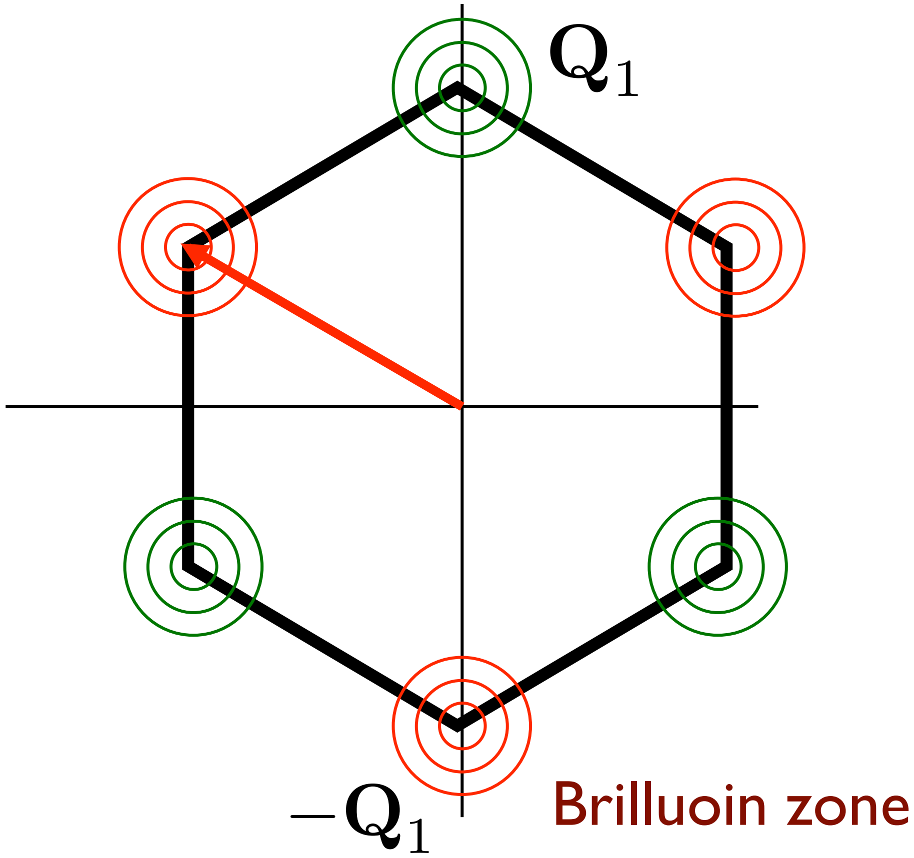
Graphene

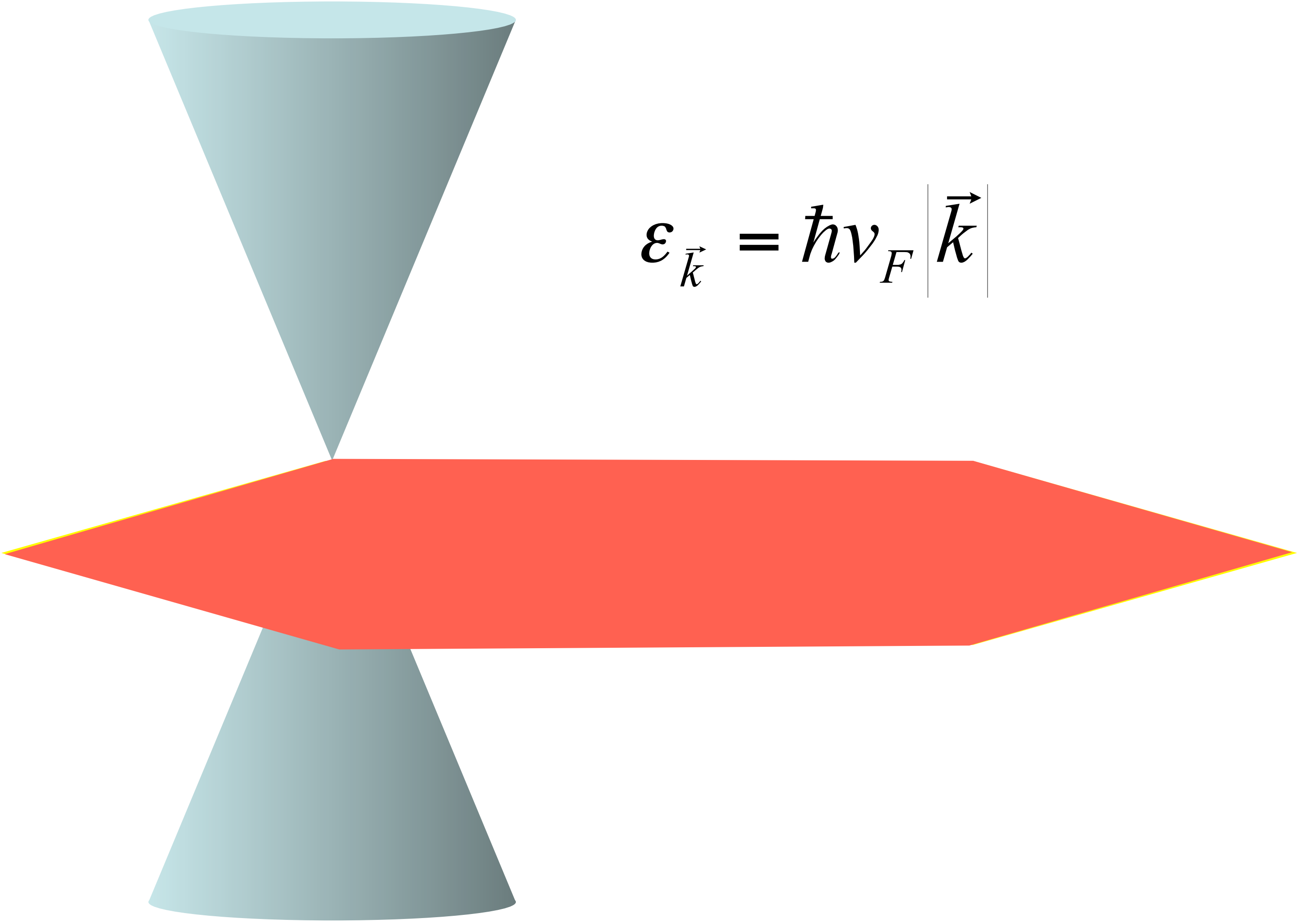




Graphene



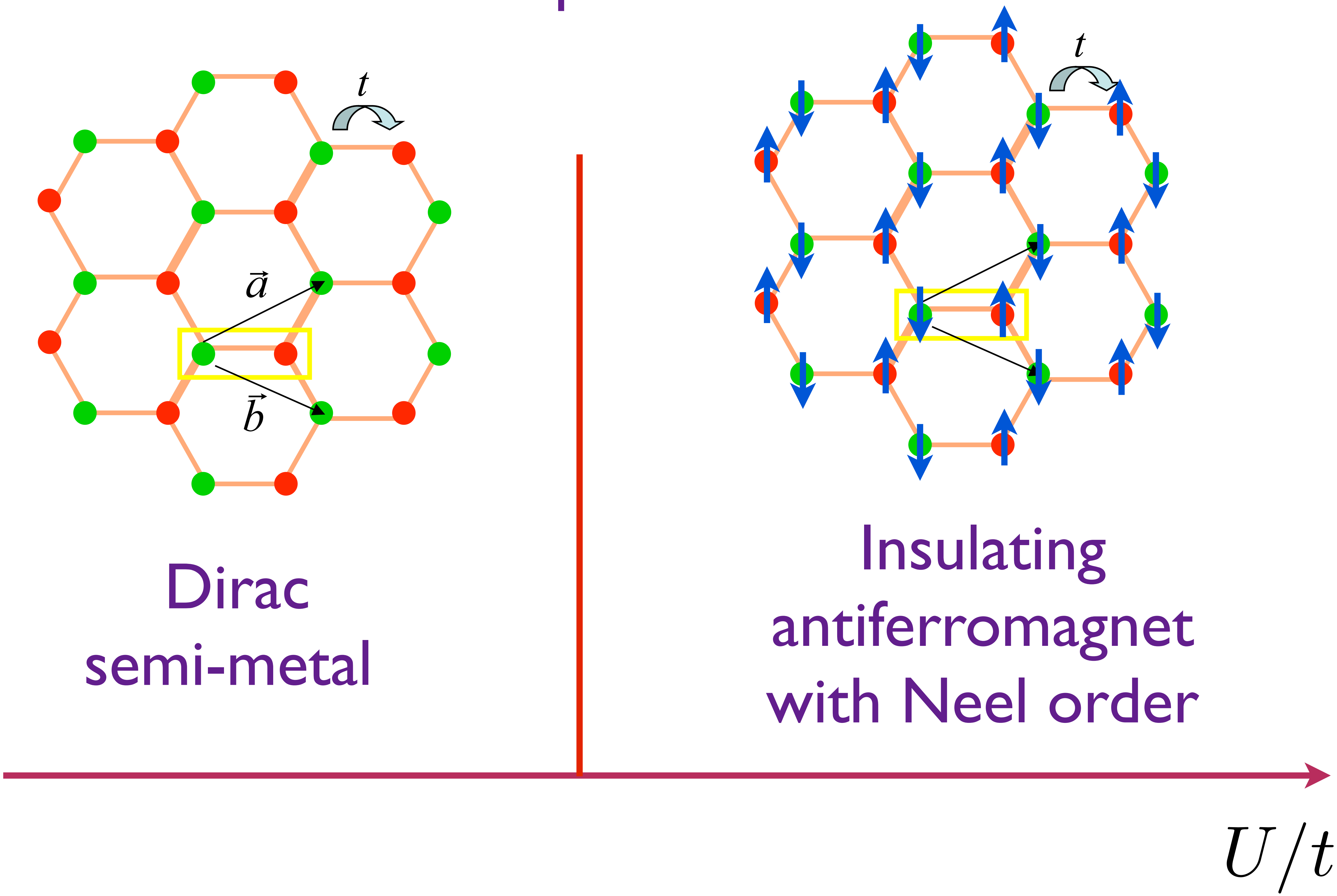




$$\varepsilon_{\vec{k}} = \hbar v_F \left|\vec{k}\right|$$

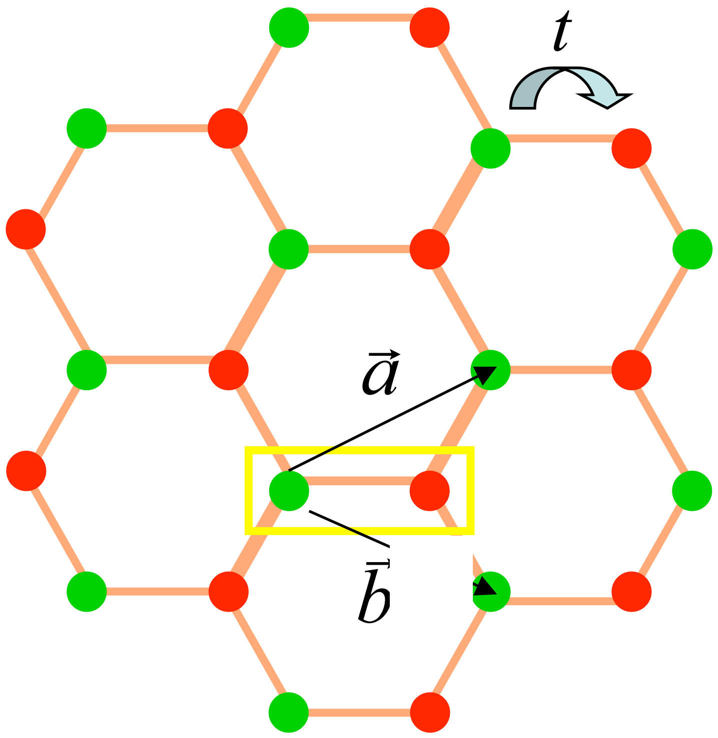
Semi-metal with massless Dirac fermions

Graphene

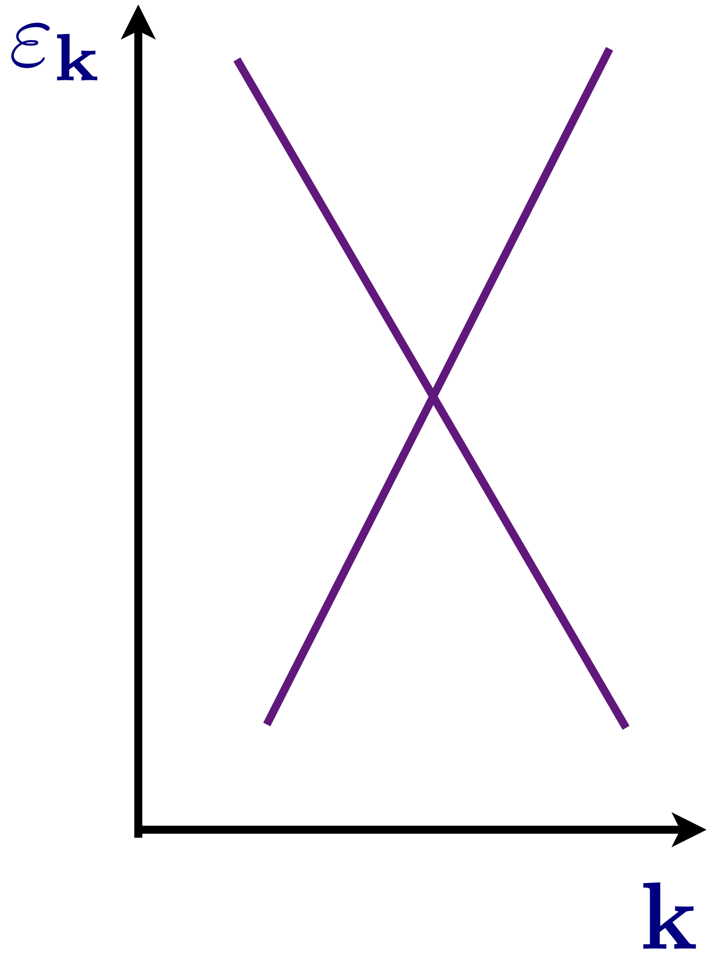


Dirac
semi-metal

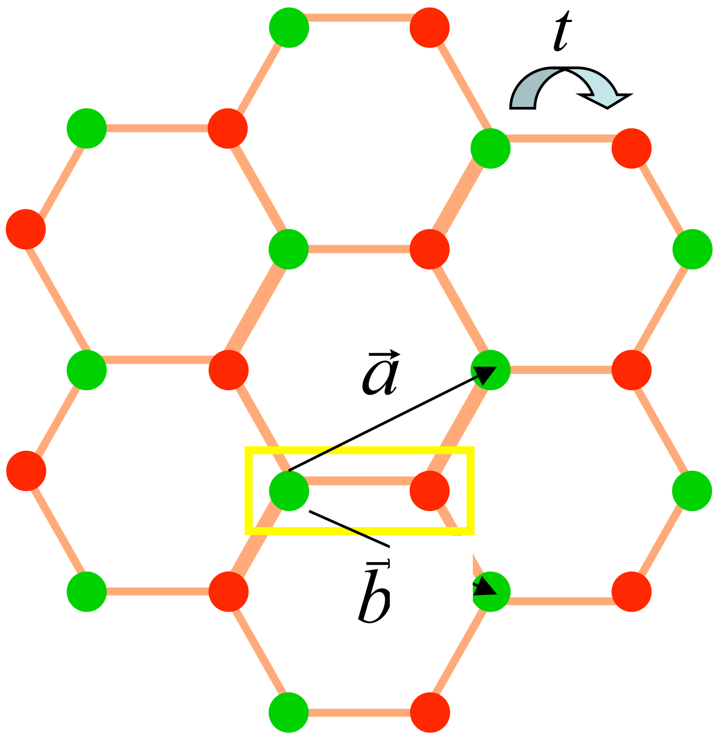
Insulating
antiferromagnet
with Neel order

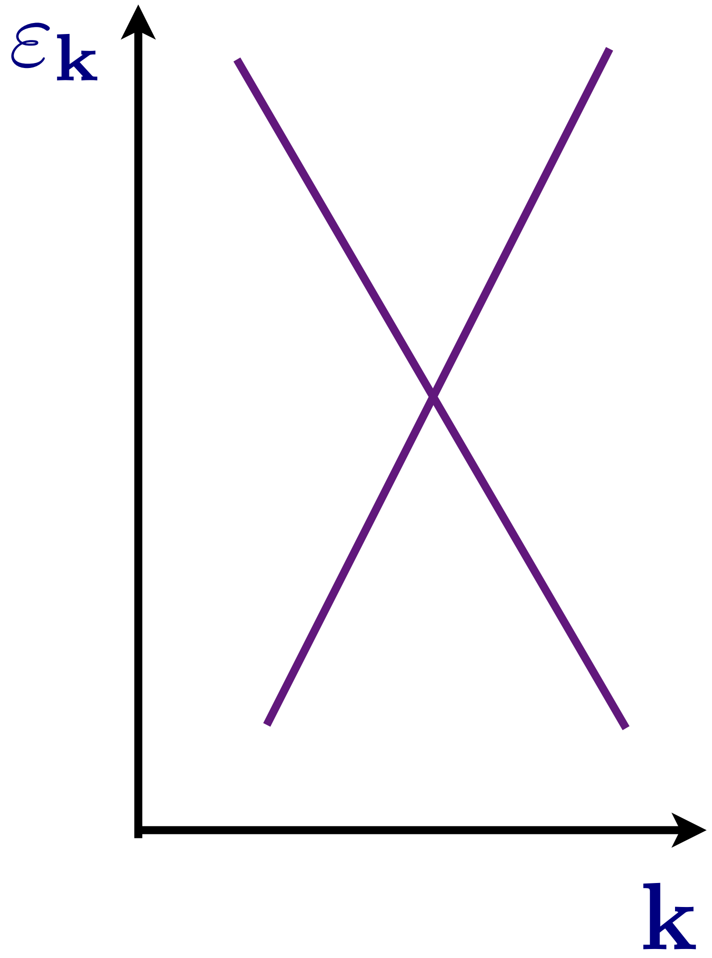


$$H_0 = -\sum_{i<j} t_{ij} c^{\dagger}_{i\alpha} c_{i\alpha}$$



- Begin with free electrons.

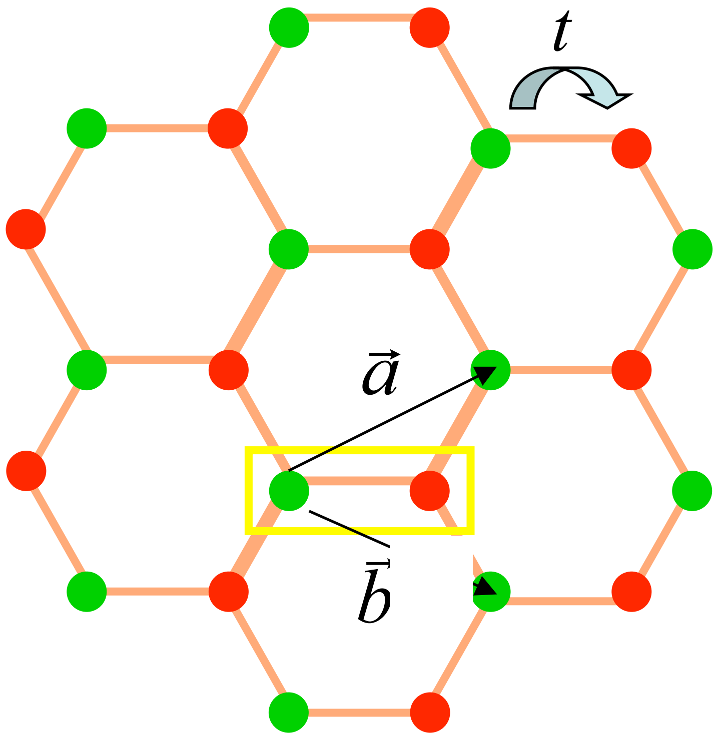


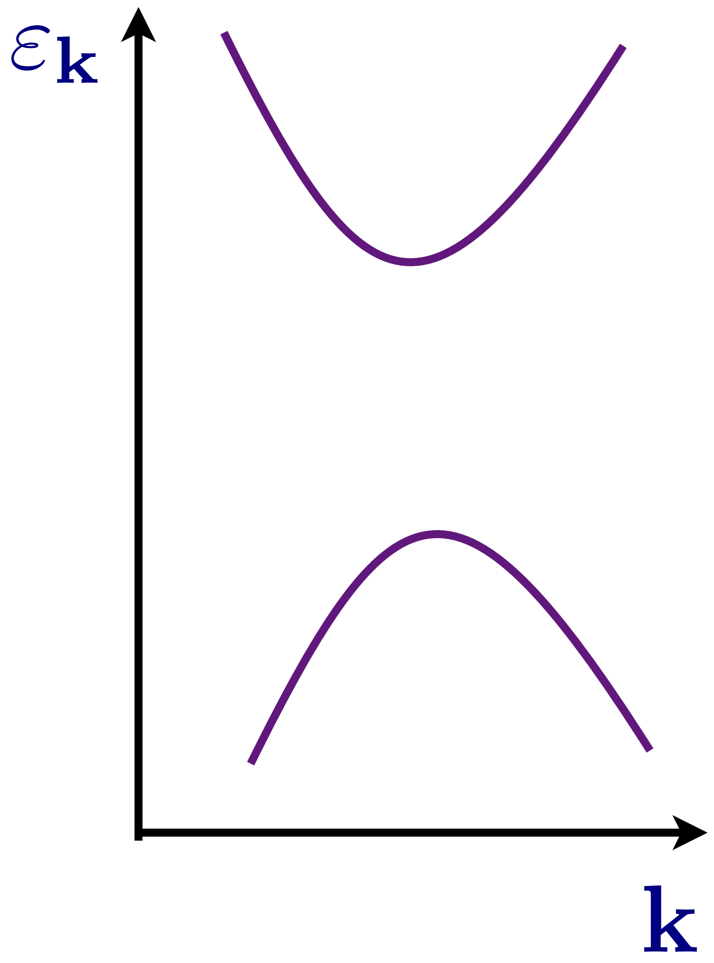


$$H_0 = -\sum_{i<j} t_{ij} c^\dagger_{i\alpha} c_{i\alpha}$$

- Begin with free electrons.
- Add local antiferromagnetism with order parameter $\vec{\varphi}$

$$H_{sdw} = -\sum_i \vec{\varphi}(\mathbf{r}_i)(-1)^i c^\dagger_{i\alpha} \vec{\sigma}_{\alpha\beta} c_{i\beta}$$





$$H_0 = -\sum_{i<j} t_{ij} c_{i\alpha}^{\dagger} c_{i\alpha}$$

- Begin with free electrons.

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$$H_{sdw} = -\sum_{i} \vec{\varphi}(\mathbf{r}_i)(-1)^i c_{i\alpha}^{\dagger} \vec{\sigma}_{\alpha\beta} c_{i\beta}$$

- The phase with $\langle\vec{\varphi}\rangle \neq 0$ is an insulator with a gap between conduction and valence bands.

Honeycomb lattice at half filling.

We define the unit length vectors

$$\mathbf{e}_1 = (1,0) \quad , \quad \mathbf{e}_2 = (-1/2, \sqrt{3}/2) \quad , \quad \mathbf{e}_3 = (-1/2, -\sqrt{3}/2). \tag{1}$$

Note that $\mathbf{e}_i \cdot \mathbf{e}_j = -1/2$ for $i \neq j$, and $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 = 0$.
We take the origin of co-ordinates of the honeycomb lattice at the center of an *empty hexagon*. The A sublattice sites closest to the origin are at $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, while the B sublattice sites closest to the origin are at $-\mathbf{e}_1$, $-\mathbf{e}_2$, and $-\mathbf{e}_3$.
The reciprocal lattice is generated by the wavevectors

$$\mathbf{G}_1 = \frac{4\pi}{3}\mathbf{e}_1 \quad , \quad \mathbf{G}_2 = \frac{4\pi}{3}\mathbf{e}_2 \quad , \quad \mathbf{G}_3 = \frac{4\pi}{3}\mathbf{e}_3 \tag{2}$$

The first Brillouin zone is a hexagon whose vertices are given by

$$\mathbf{Q}_1 = \frac{1}{3}(\mathbf{G}_2 - \mathbf{G}_3) \quad , \quad \mathbf{Q}_2 = \frac{1}{3}(\mathbf{G}_3 - \mathbf{G}_1) \quad , \quad \mathbf{Q}_3 = \frac{1}{3}(\mathbf{G}_1 - \mathbf{G}_2), \tag{3}$$

and $-\mathbf{Q}_1$, $-\mathbf{Q}_2$, and $-\mathbf{Q}_3$.

We define the Fourier transform of the fermions by

$$c_A(\mathbf{k}) = \sum_{\mathbf{r}} c_A(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} \tag{4}$$

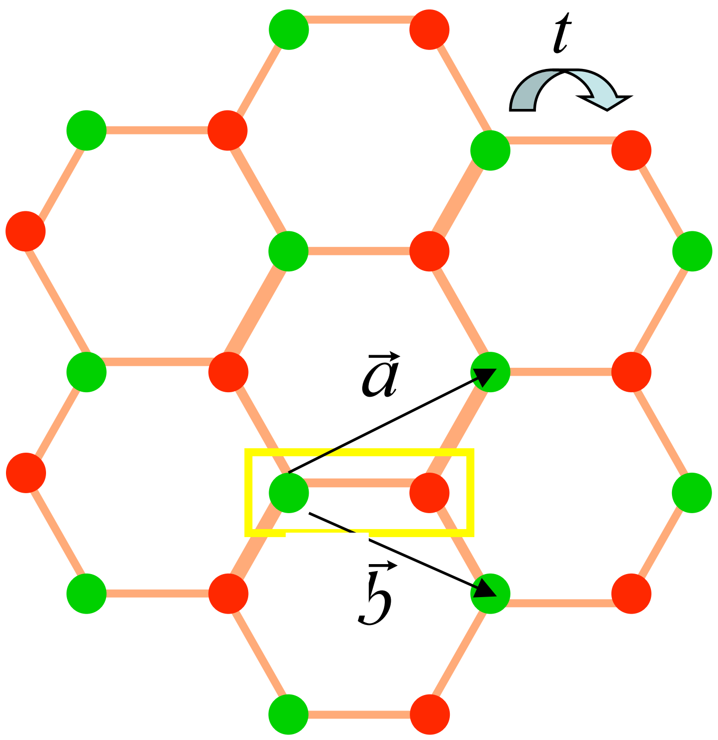
and similarly for c_B.
The hopping Hamiltonian is

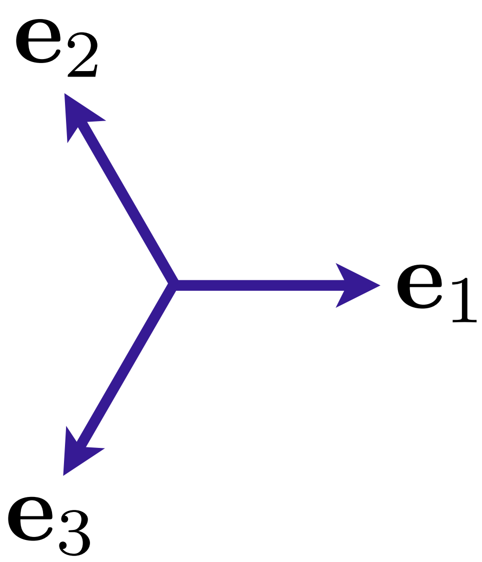
$$H_0 = -t \sum_{\langle ij \rangle} \left(c^\dagger_{Ai\alpha} c_{Bj\alpha} + c^\dagger_{Bj\alpha} c_{Ai\alpha} \right) \tag{5}$$

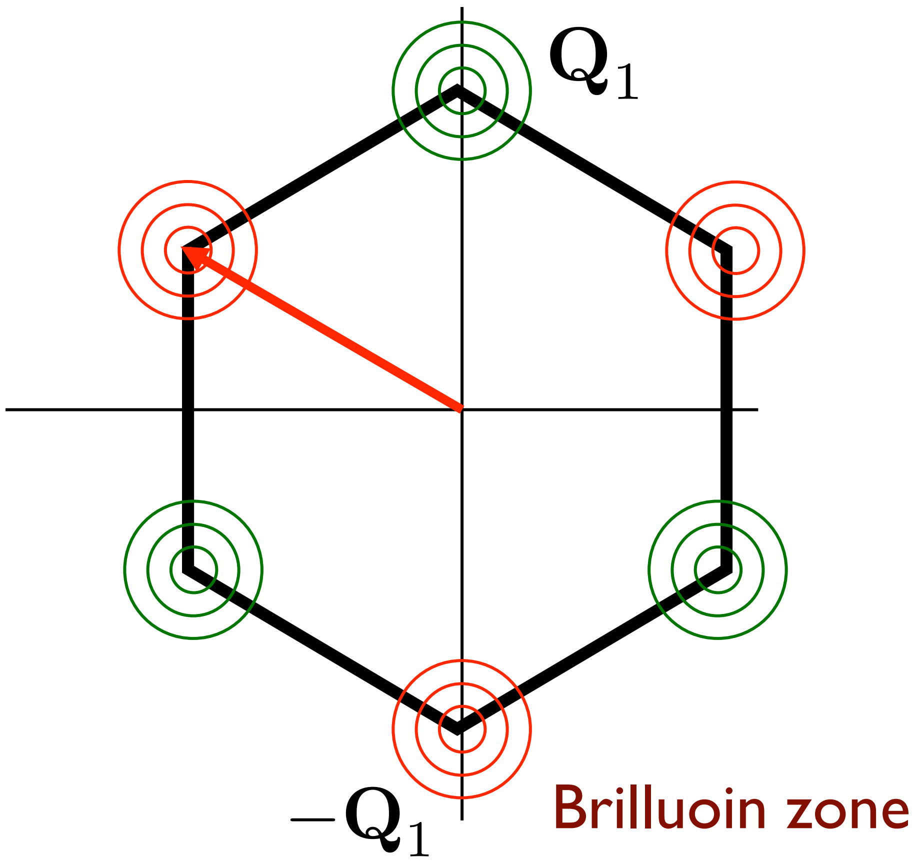
where α is a spin index. If we introduce Pauli matrices τ^a in sublattice space ($a = x, y, z$), this Hamiltonian can be written as

$$\begin{aligned} H_0 &= \int \frac{d^2k}{4\pi^2} c^\dagger(\mathbf{k}) \Big[-t \Big(\cos(\mathbf{k}\cdot\mathbf{e}_1) + \cos(\mathbf{k}\cdot\mathbf{e}_2) + \cos(\mathbf{k}\cdot\mathbf{e}_3) \Big) \tau^x \\ &\quad + t \Big(\sin(\mathbf{k}\cdot\mathbf{e}_1) + \sin(\mathbf{k}\cdot\mathbf{e}_2) + \sin(\mathbf{k}\cdot\mathbf{e}_3) \Big) \tau^y \Big] c(\mathbf{k}) \end{aligned} \tag{6}$$

The low energy excitations of this Hamiltonian are near $\mathbf{k} \approx \pm \mathbf{Q}_1$.


t
$\vec{a}$
$\vec{b}$


$\mathbf{e}_2$
$\mathbf{e}_1$
$\mathbf{e}_3$


$\mathbf{Q}_1$
$-\mathbf{Q}_1$
Brilluoin zone

In terms of the fields near $\mathbf{Q}_1$ and $-\mathbf{Q}_1$, we define

$$\begin{aligned} \Psi_{A1\alpha}(\mathbf{k}) &= c_{A\alpha}(\mathbf{Q}+\mathbf{k}) \\ \Psi_{A2\alpha}(\mathbf{k}) &= c_{A\alpha}(-\mathbf{Q}+\mathbf{k}) \\ \Psi_{B1\alpha}(\mathbf{k}) &= c_{B\alpha}(\mathbf{Q}+\mathbf{k}) \\ \Psi_{B2\alpha}(\mathbf{k}) &= c_{B\alpha}(-\mathbf{Q}+\mathbf{k}) \end{aligned} \tag{7}$$

We consider Ψ to be a 8 component vector, and introduce Pauli matrices ρ^a which act in the $1, 2$ valley space. Then the Hamiltonian is

$$H_0 = \int \frac{d^2k}{4\pi^2} \Psi^\dagger(\mathbf{k}) \Big(v\tau^y k_x + v\tau^x \rho^z k_y \Big) \Psi(\mathbf{k}), \tag{8}$$

where $v = 3t/2$; below we set $v = 1$. Now define $\overline{\Psi} = \Psi^\dagger \rho^z \tau^z$. Then we can write the imaginary time Lagrangian as

$$\mathcal{L}_0 = -i\overline{\Psi}\left(\omega\gamma_0 + k_x\gamma_1 + k_y\gamma_2\right)\Psi \tag{9}$$

where

$$\gamma_0 = -\rho^z\tau^z \quad \gamma_1 = \rho^z\tau^x \quad \gamma_2 = -\tau^y \tag{10}$$

Exercise: Observe that $\mathcal{L}_0$ is invariant under the scaling transformation $x' = xe^{-\ell}$ and $\tau' = \tau e^{-\ell}$. Write the Hubbard interaction U in terms of the Dirac fermions, and show that it has the tree-level scaling transformation $U' = Ue^{-\ell}$. So argue that all short-range interactions are *irrelevant* in the Dirac semi-metal phase.

Antiferromagnetism

We use the operator equation (valid on each site i):

$$U\left(n_\uparrow - \frac{1}{2}\right)\left(n_\downarrow - \frac{1}{2}\right) = -\frac{2U}{3} S^{a2} + \frac{U}{4} \tag{11}$$

Then we decouple the interaction via

$$\exp\left(\frac{2U}{3}\sum_i \int d\tau S_i^{a2}\right) = \int \mathcal{D}J_i^a(\tau) \exp\left(-\sum_i \int d\tau \left[\frac{3}{8U} J_i^{a2} - J_i^a S_i^a\right]\right. \tag{12}$$

We now integrate out the fermions, and look for the saddle point of the resulting effective action for J_i^a. At the saddle-point we find that the lowest energy is achieved when the vector has opposite orientations on the A and B sublattices. Anticipating this, we look for a continuum limit in terms of a field φ^a where

$$J_A^a = \varphi^a \quad , \quad J_B^a = -\varphi^a \tag{13}$$

The coupling between the field φ^a and the Ψ fermions is given by

$$\begin{aligned} \sum_i J_i^a c_{i\alpha}^\dagger \sigma_{\alpha\beta}^a c_{i\beta} &= \varphi^a \left(c_{A\alpha}^\dagger \sigma_{\alpha\beta}^a c_{A\beta} - c_{B\alpha}^\dagger \sigma_{\alpha\beta}^a c_{B\beta} \right) \\ &= \varphi^a \Psi^\dagger \tau^z \sigma^a \Psi = -\varphi^a \overline{\Psi} \rho^z \sigma^a \Psi \end{aligned} \tag{14}$$

From this we motivate the low energy theory

$$\mathcal{L} = \overline{\Psi} \gamma_\mu \partial_\mu \Psi + \frac{1}{2} \left[(\partial_\mu \varphi^a)^2 + s \varphi^{a2} \right] + \frac{u}{24} \left(\varphi^{a2} \right)^2 - \lambda \varphi^a \overline{\Psi} \rho^z \sigma^a \Psi \tag{15}$$

Note that the matrix $\rho^z \sigma^a$ commutes with all the γ_μ; hence $\rho^z \sigma^a$ is a matrix in “flavor” space. This is the Gross-Neveu model, and it describes the quantum phase transition from the Dirac semi-metal to an insulating Néel state In mean-field theory, the

Dirac semi-metal is obtained for $s > 0$ with $\langle \varphi^a \rangle = 0$. The Néel state obtains for $s < 0$, and we have $\varphi^a = N_0 \delta_{az}$ (say), and so the dispersion of the electrons is

$$\omega_k = \pm \sqrt{k^2 + \lambda^2 N_0^2} \tag{16}$$

near the points $\pm \mathbf{Q}_1$. These form the conduction and valence bands of the insulator.

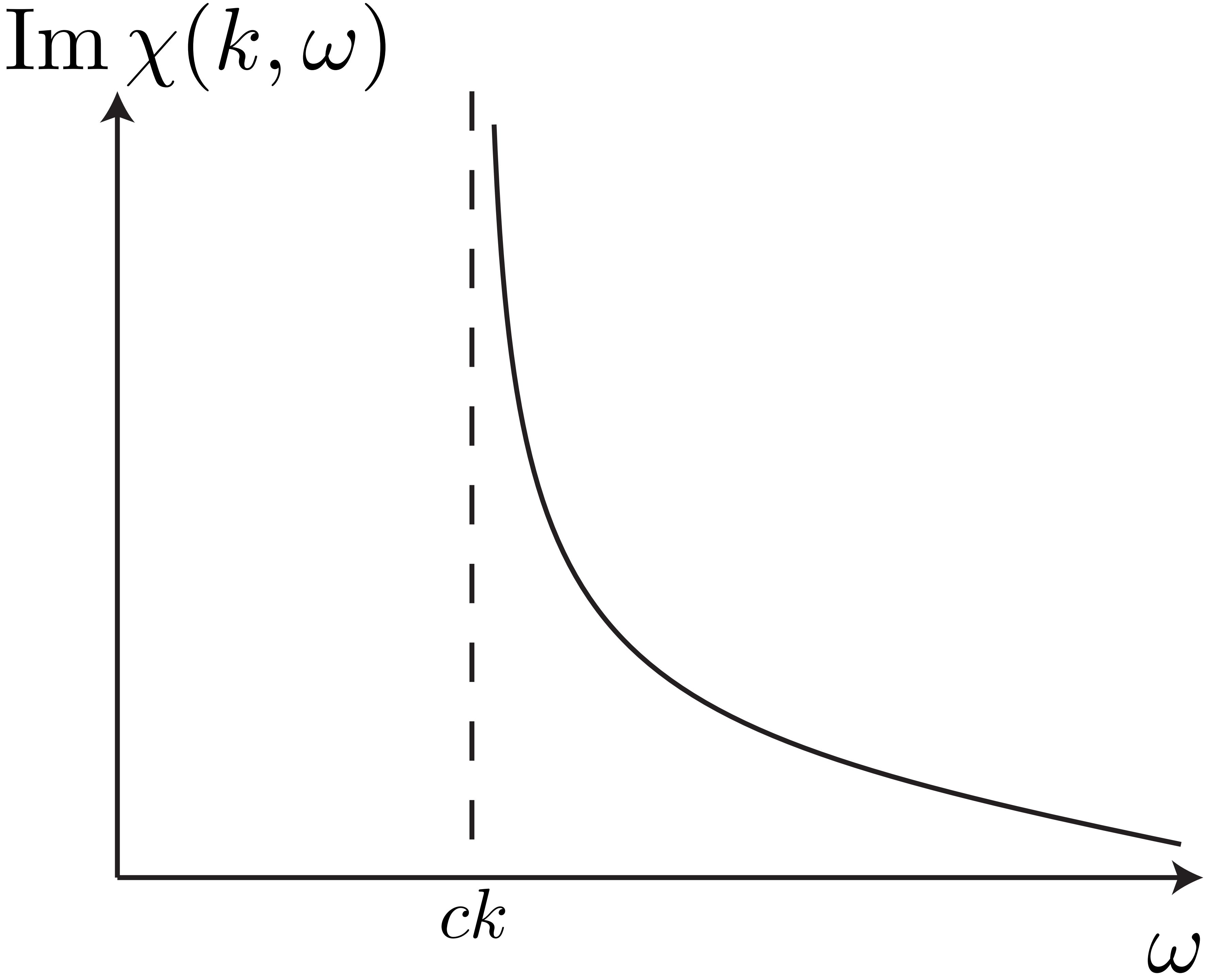
Exercise: Perform a tree-level RG transformation on $\mathcal{L}$. The quadratic gradient terms are invariant under $\Psi' = \Psi e^{\ell}$ and $\varphi' = \varphi e^{\ell/2}$. Show that this leads to $s' = s e^{2\ell}$. Thus s is a relevant perturbation which drives the system into either the semi-metal or antiferromagnetic insulator. The quantum critical point is reached by tuning s to its critical value ($= 0$ at tree level). Show that the couplings u and λ are both relevant perturbations at this critical point. Thus, while interactions are irrelevant in the Dirac semi-metal (and in the insulator), they are strongly relevant at the quantum-critical point.

An analysis of this quantum critical point requires a RG analysis which goes beyond tree-level. Such an analysis can be controlled in an expansion in $1/N$ (where N is the number of fermion flavors) or $(3-d)$ (where d is the spatial dimensionality. For reviews see cond-mat/0109419 or Chapter 17, *Quantum Phase Transitions*, by S. Sachdev, Second Edition (forthcoming).

An important result of such an analysis is the following structure in the electron Green's function:

$$\left\langle \Psi(k,\omega); \Psi^{\dagger}(k,\omega) \right\rangle \sim \frac{i\omega + vk_x\tau^y + vk_y\tau^x\rho^z}{(\omega^2 + v^2k_x^2 + v^2k_y^2)^{1-\eta/2}} \qquad (17)$$

where $\eta > 0$ is the *anomalous dimension* of the fermion. Note that this leads to a fermion spectral density which has no quasiparticle pole: thus the quantum critical point has no well-defined quasiparticle excitations.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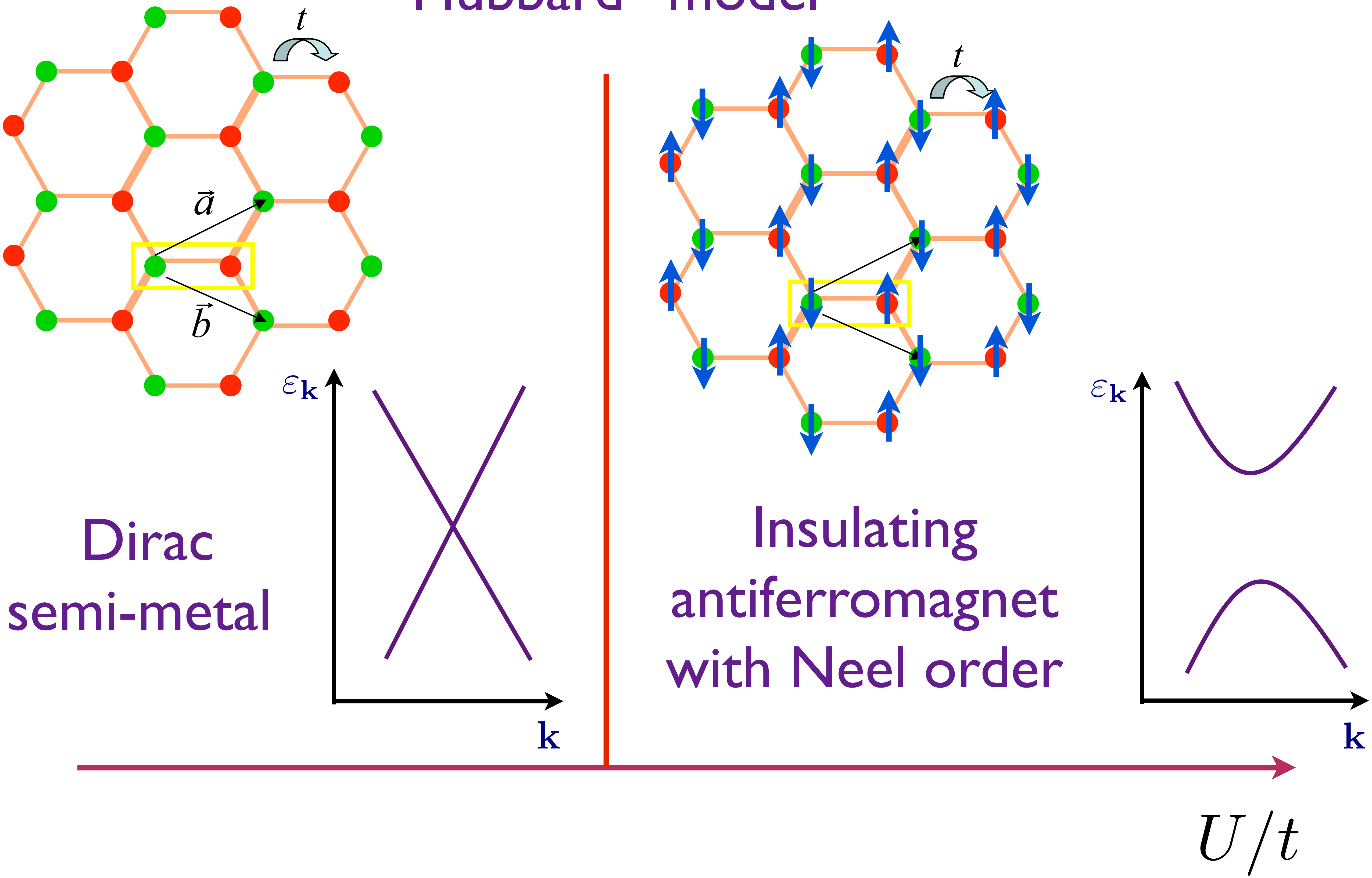
“Hubbard” model



Dirac
semi-metal

Insulating
antiferromagnet
with Neel order

Quantum phase transition described by a strongly-coupled conformal field theory without well-defined quasiparticles

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$t_{ij} \rightarrow$ "hopping". $U \rightarrow$ local repulsion, $\mu \rightarrow$ chemical potential

Spin index $\alpha = \uparrow, \downarrow$

$$n_{i\alpha} = c^\dagger_{i\alpha} c_{i\alpha}$$

$$c^\dagger_{i\alpha} c_{j\beta} + c_{j\beta} c^\dagger_{i\alpha} = \delta_{ij} \delta_{\alpha\beta}$$
$$c_{i\alpha} c_{j\beta} + c_{j\beta} c_{i\alpha} = 0$$

Will study on the honeycomb and square lattices

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$$H = -\sum_{i<j} t_{ij} c^\dagger_{i\alpha} c_{j\alpha} + U \sum_i \left(n_{i\uparrow} - \frac{1}{2} \right) \left(n_{i\downarrow} - \frac{1}{2} \right) - \mu \sum_i c^\dagger_{i\alpha} c_{i\alpha}$$

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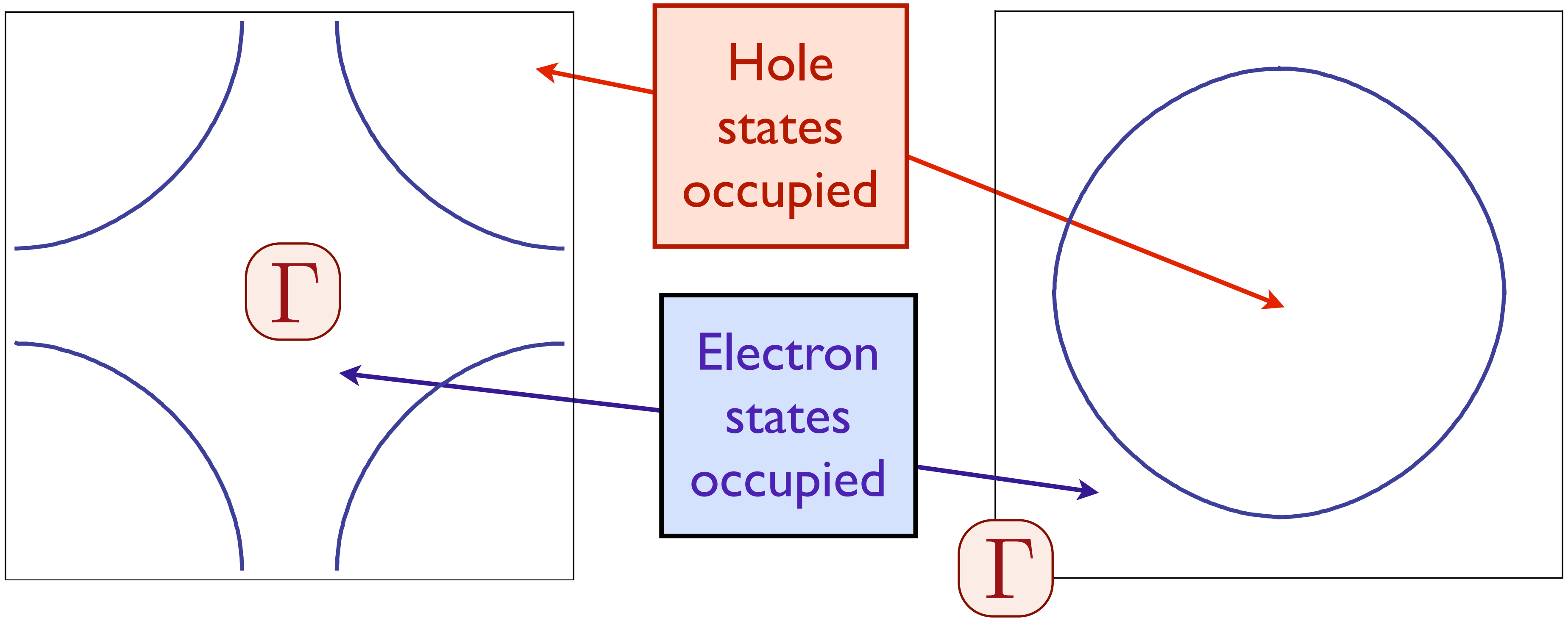
Will study on the honeycomb and ***square*** lattices

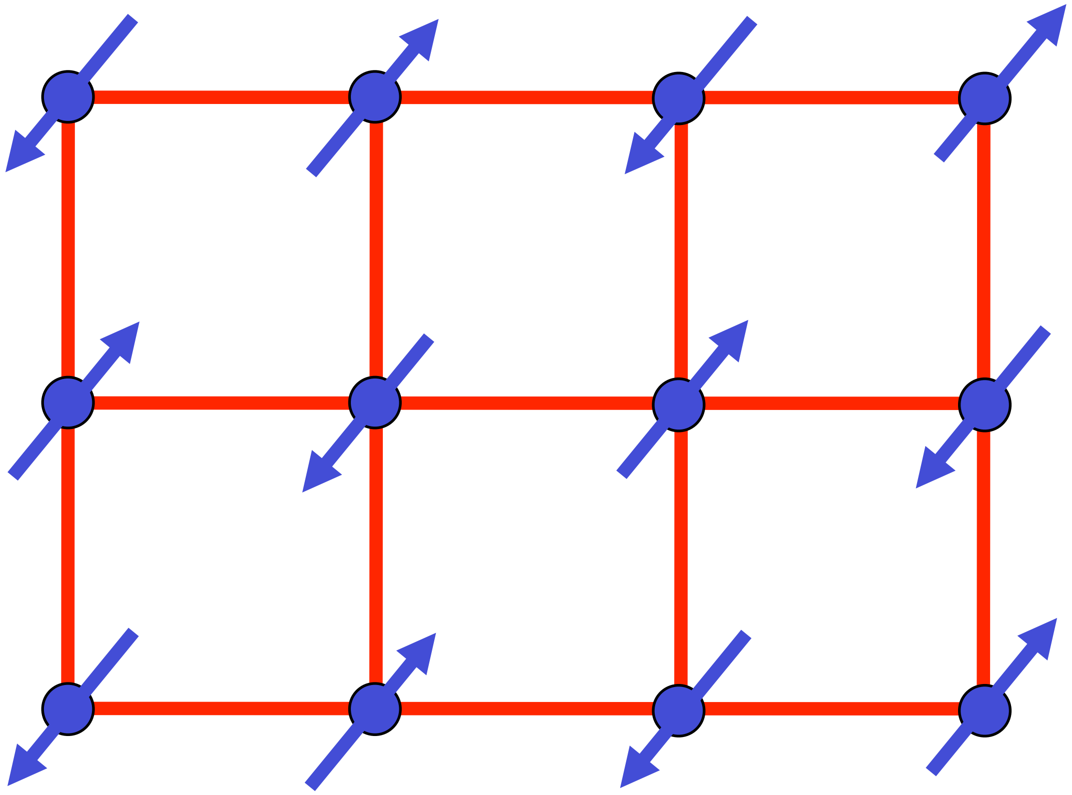
Fermi surface+antiferromagnetism



Hole states occupied

Electron states occupied

+

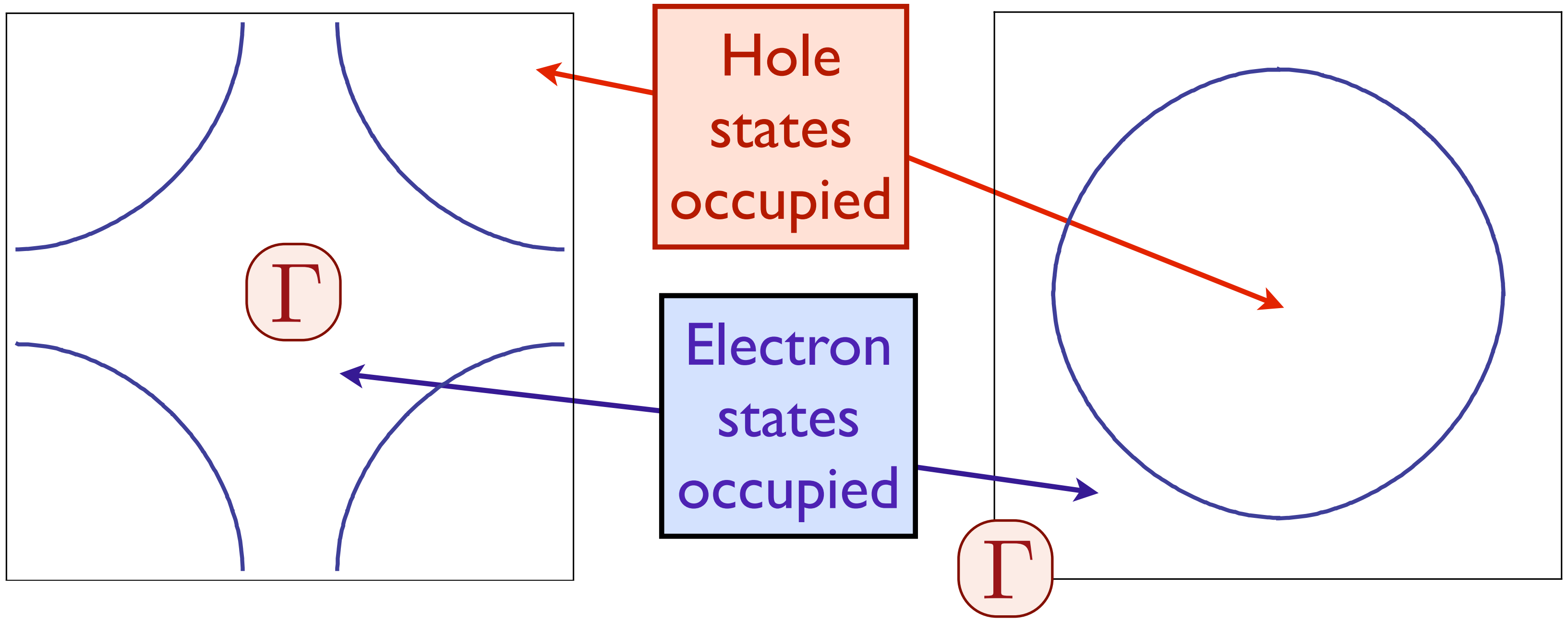


The electron spin polarization obeys

$$\left\langle \vec{S}(\mathbf{r},\tau) \right\rangle = \vec{\varphi}(\mathbf{r},\tau)e^{i\mathbf{K}\cdot\mathbf{r}}$$

where **K** is the ordering wavevector.

Fermi surfaces in electron- and hole-doped cuprates



Hole states occupied

Electron states occupied

Γ

Γ

Effective Hamiltonian for quasiparticles:

$$H_0 = -\sum_{i<j} t_{ij} c_{i\alpha}^{\dagger} c_{j\alpha} \equiv \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} c_{\mathbf{k}\alpha}^{\dagger} c_{\mathbf{k}\alpha}$$

with t_{ij} non-zero for first, second and third neighbor, leads to satisfactory agreement with experiments. The area of the occupied electron states, $\mathcal{A}_e$, from Luttinger's theory is

$$\mathcal{A}_e = \begin{cases} 2\pi^2(1-x) & \text{for hole-doping } x \\ 2\pi^2(1+p) & \text{for electron-doping } p \end{cases}$$

The area of the occupied hole states, $\mathcal{A}_h$, which form a closed Fermi surface and so appear in quantum oscillation experiments is $\mathcal{A}_h = 4\pi^2 - \mathcal{A}_e$.

Spin density wave theory

Just as we did for the honeycomb lattice, decouple the Hubbard interaction via a field $\vec{\varphi}$ which couples to the local spin. Also, as on the honeycomb lattice, we assume the tendency to two-sublattice antiferromagnetic spin ordering; on the square lattice this is at wavevector $\mathbf{K} = (\pi, \pi)$. So the coupling between $\vec{\varphi}$ and the electrons takes the form

$$H_{\rm sdw} = \sum_{\mathbf{k},\mathbf{q},\alpha,\beta} \vec{\varphi}_{\mathbf{q}} \cdot c^{\dagger}_{\mathbf{k}+\mathbf{q},\alpha} \vec{\sigma}_{\alpha\beta} c_{\mathbf{k}+\mathbf{K},\beta}$$

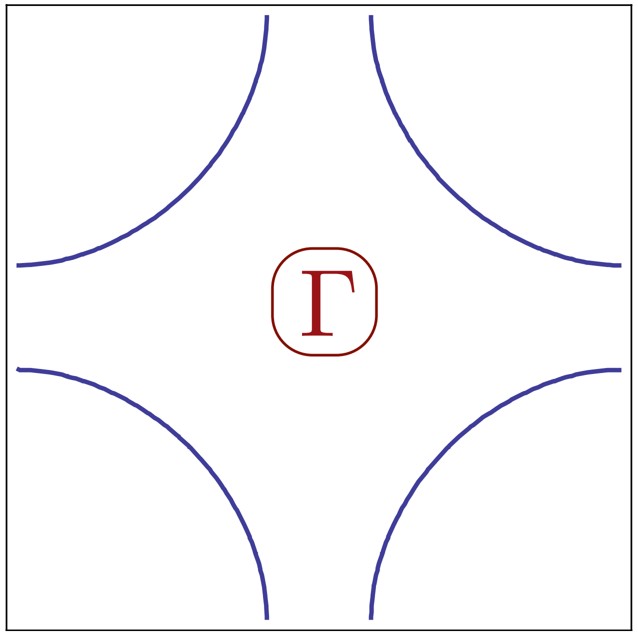
where $\vec{\sigma}$ are the Pauli matrices, the boson momentum $\mathbf{q}$ is small, while the fermion momenum $\mathbf{k}$ extends over the entire Brillouin zone. In the antiferromagnetically ordered state, we may take $\vec{\varphi} \propto (0,0,1)$, and the electron dispersions obtained by diagonalizing $H_0 + H_{\rm sdw}$ are

$$E_{\mathbf{k}\pm} = \frac{\varepsilon_{\mathbf{k}} + \varepsilon_{\mathbf{k}+\mathbf{K}}}{2} \pm \sqrt{\left(\frac{\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}+\mathbf{K}}}{2}\right) + \varphi^2}$$

This leads to the Fermi surfaces shown in the following slides as a function of increasing $|\vec{\varphi}|$.

Hole-doped cuprates

Increasing SDW order



S. Sachdev, A. V. Chubukov, and A. Sokol, *Phys. Rev. B* **51**, 14874 (1995).
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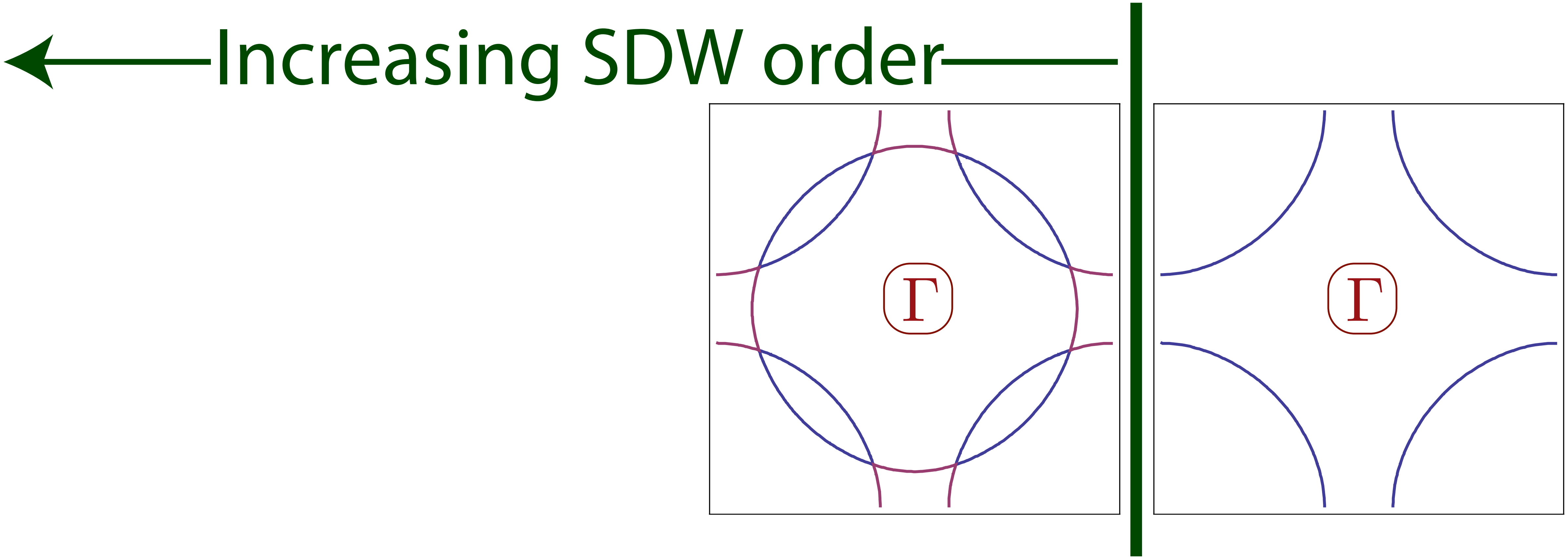
Hole-doped cuprates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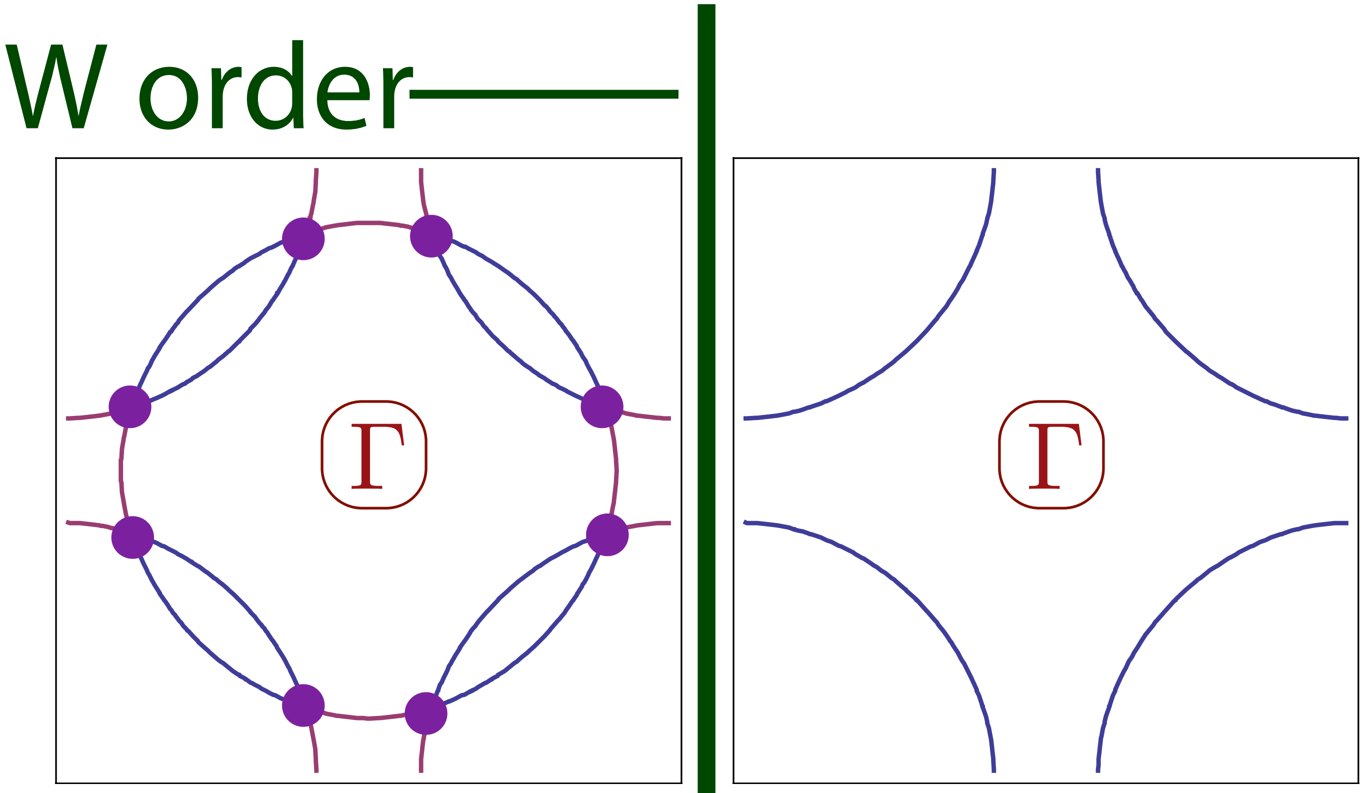
Hole-doped cuprates

← Increasing SDW order



Hot spots

where $\varepsilon_{\mathbf{k}} = \varepsilon_{\mathbf{k}+\mathbf{K}}$

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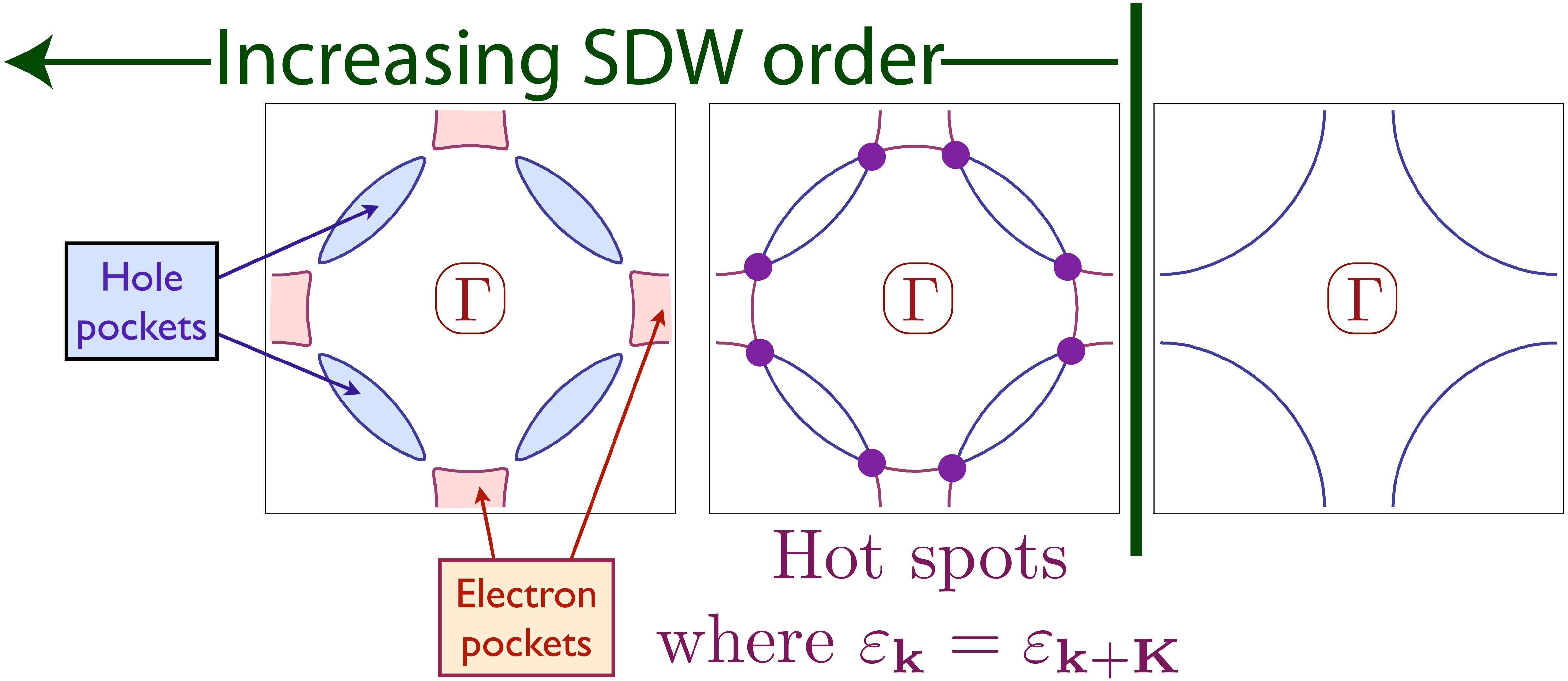

Increasing SDW order

Hole pockets

Electron pockets

Hot spots

where $\varepsilon_{\mathbf{k}} = \varepsilon_{\mathbf{k}+\mathbf{K}}$

Fermi surface breaks up at hot spots into electron and hole “pockets”

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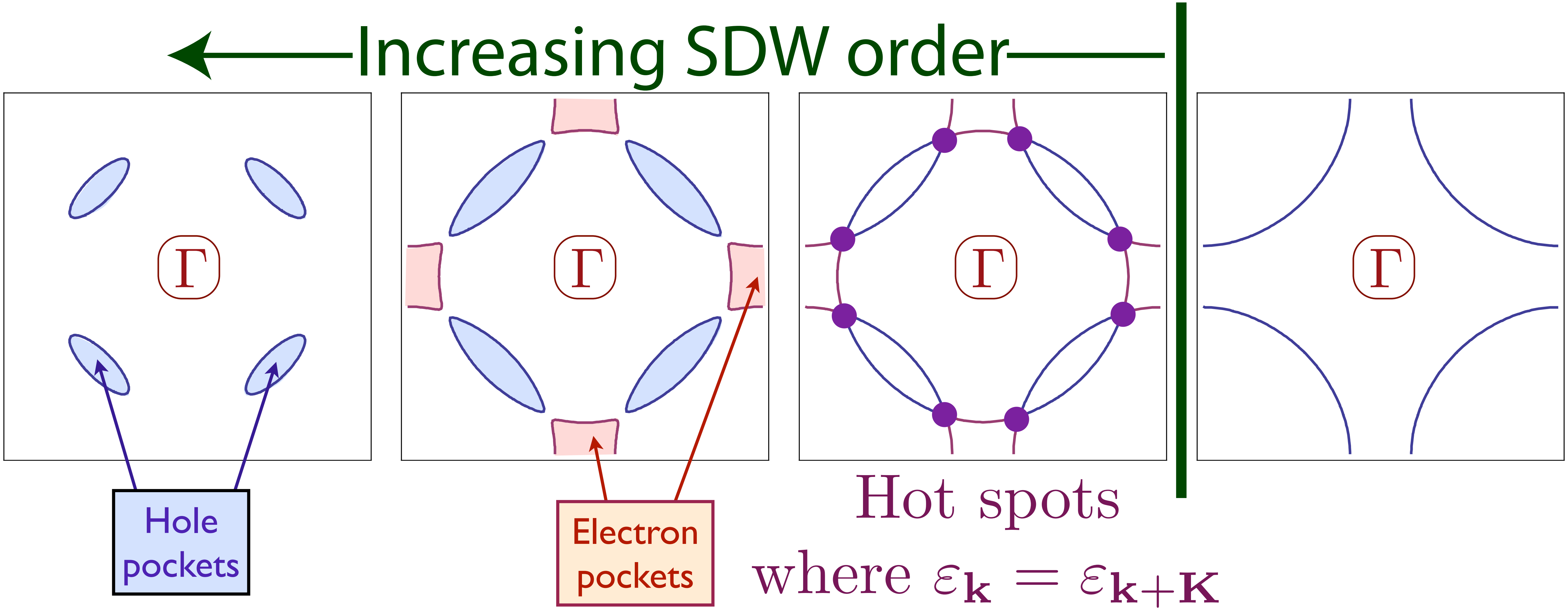

Increasing SDW order

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Fermi surface breaks up at hot spots into electron and hole "pockets"

S. Sachdev, A. V. Chubukov, and A. Sokol, *Phys. Rev. B* **51**, 14874 (1995).
A. V. Chubukov and D. K. Morr, *Physics Reports* **288**, 355 (1997).

Evidence for small Fermi pockets

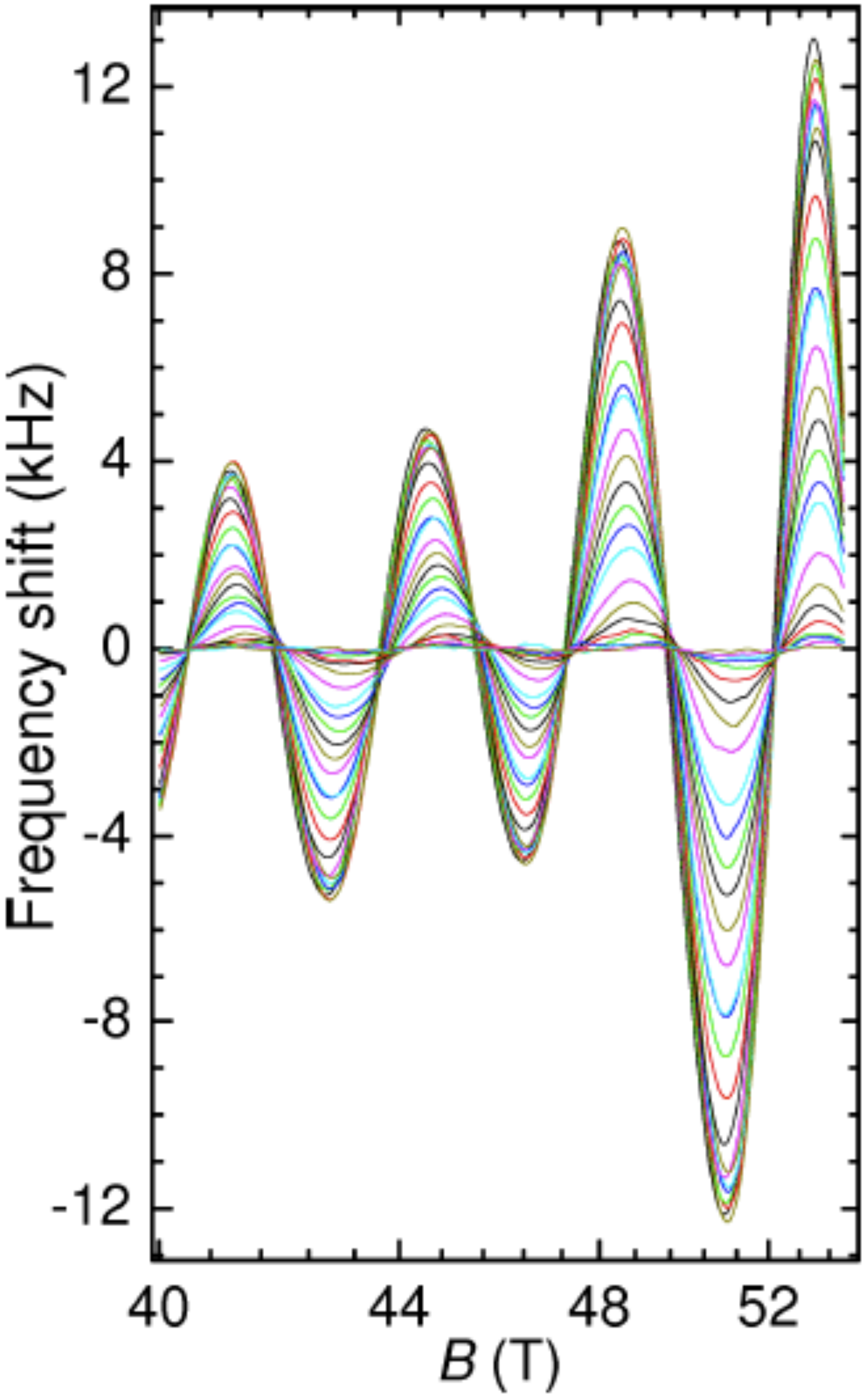


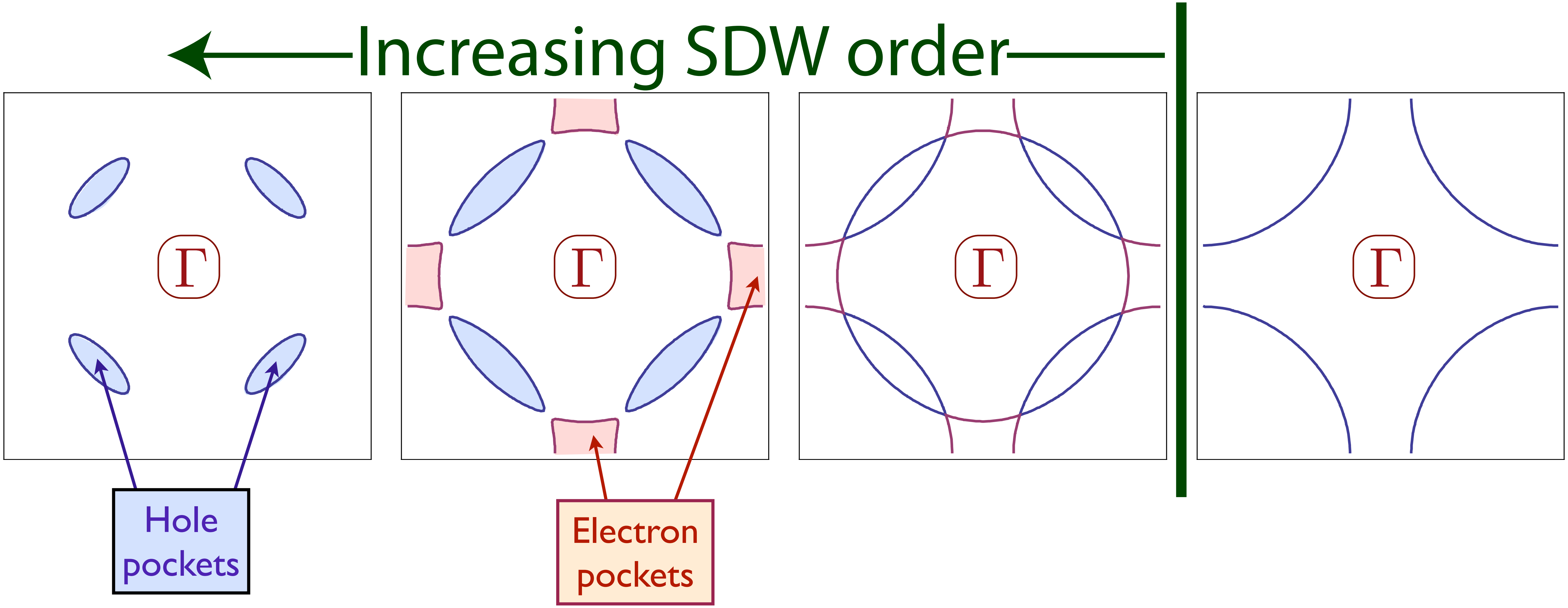
FIG. 2: Magnetic quantum oscillations measured in $YBa_2Cu_3O_{6+x}$ with $x \approx 0.56$ (after background polynomial subtraction). This restricted interval in $B = |\mathbf{B}|$ furnishes a dynamic range of $\sim$ 50 dB between T = 1 and 18 K. The actual T values are provided in Fig. 3.

Fermi liquid behaviour in an underdoped high Tc superconductor

Suchitra E. Sebastian, N. Harrison,
M. M. Altarawneh, Ruixing Liang, D. A. Bonn,
W. N. Hardy, and G. G. Lonzarich

arXiv:0912.3022

Hole-doped cuprates



Large Fermi surface breaks up into electron and hole pockets

S. Sachdev, A. V. Chubukov, and A. Sokol, *Phys. Rev. B* **51**, 14874 (1995).
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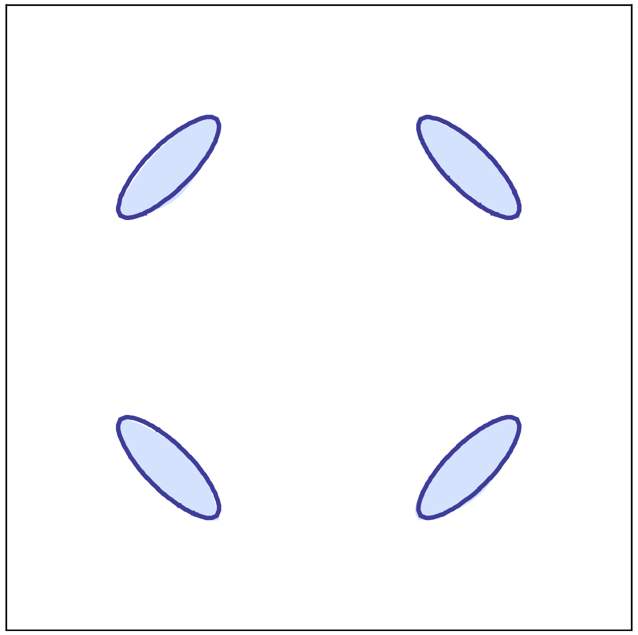
Square lattice Hubbard model with hole doping

$\langle \vec{\varphi} \rangle \neq 0$

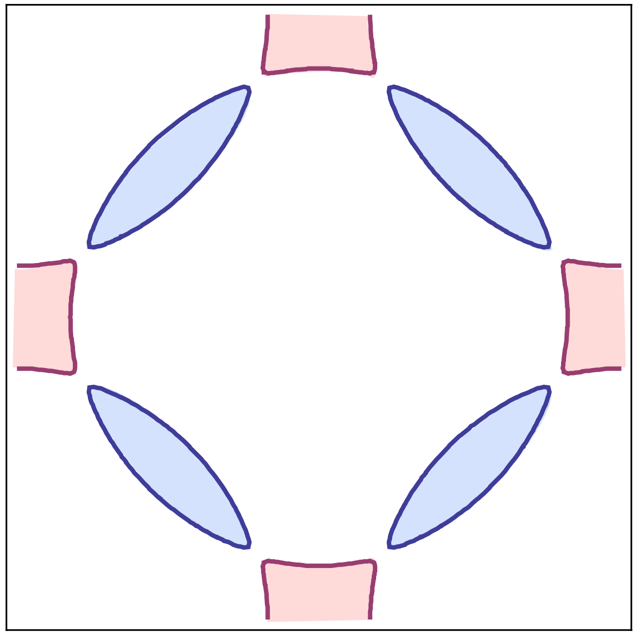
and large



Metal with
hole pockets

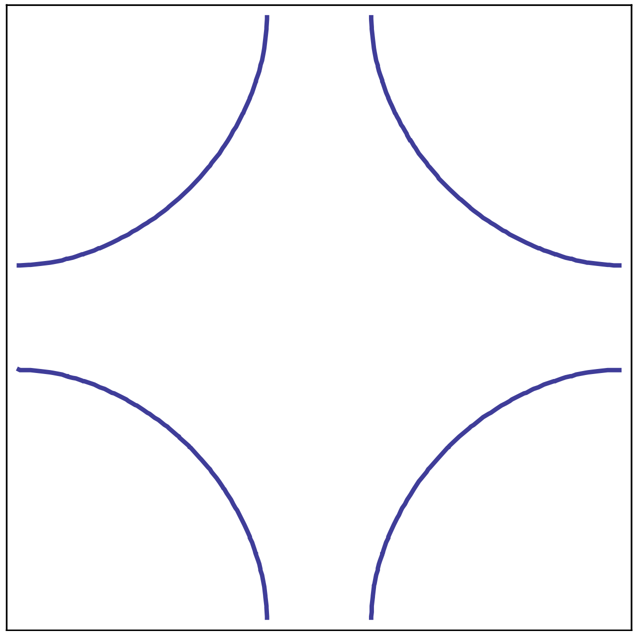
$\langle \vec{\varphi} \rangle \neq 0$

and small



Metal with
electron and
hole pockets

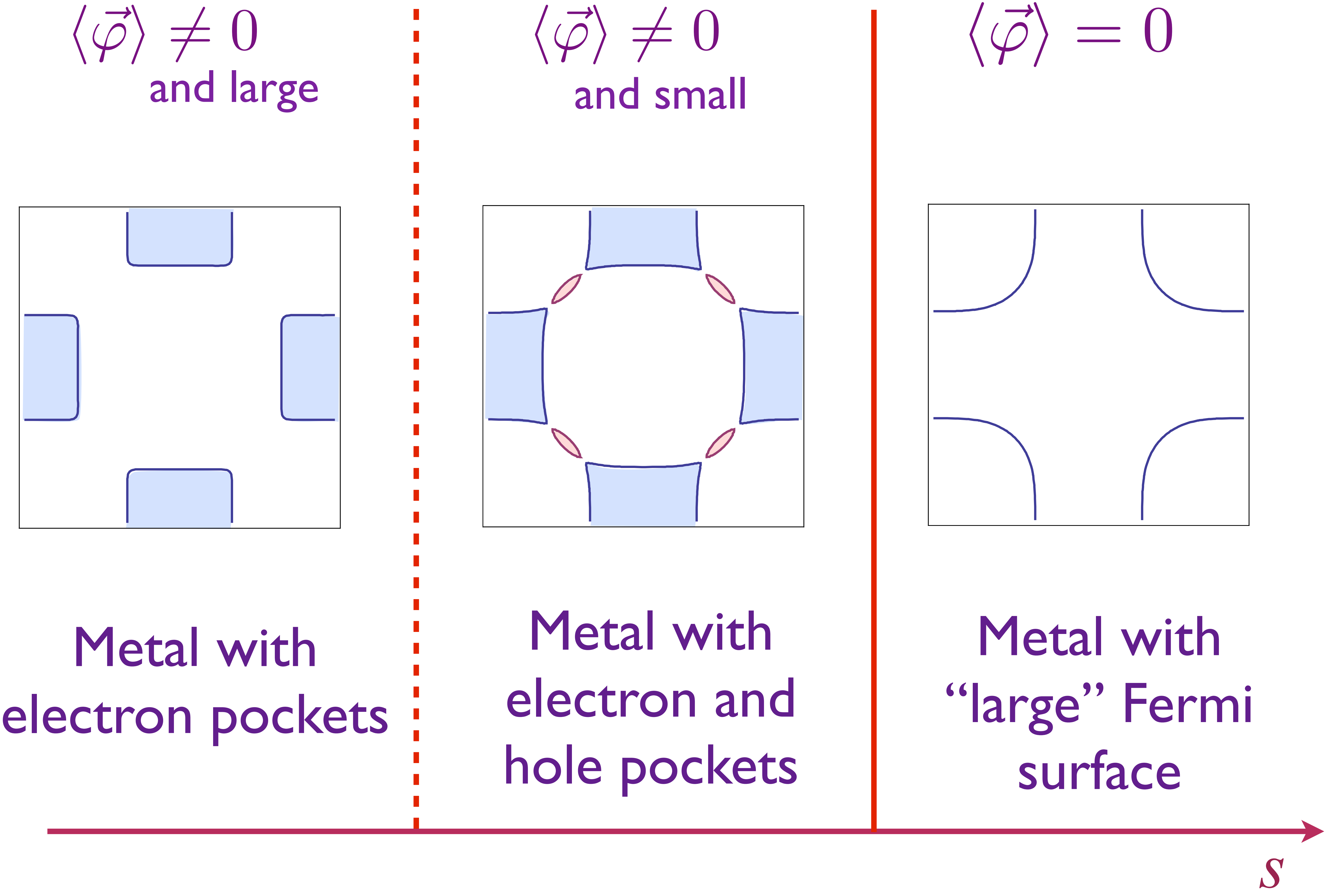
$\langle \vec{\varphi} \rangle = 0$



Metal with
“large” Fermi
surface

s

Square lattice Hubbard model with electron doping



$\langle \vec{\varphi} \rangle \neq 0$
and large

Metal with electron pockets

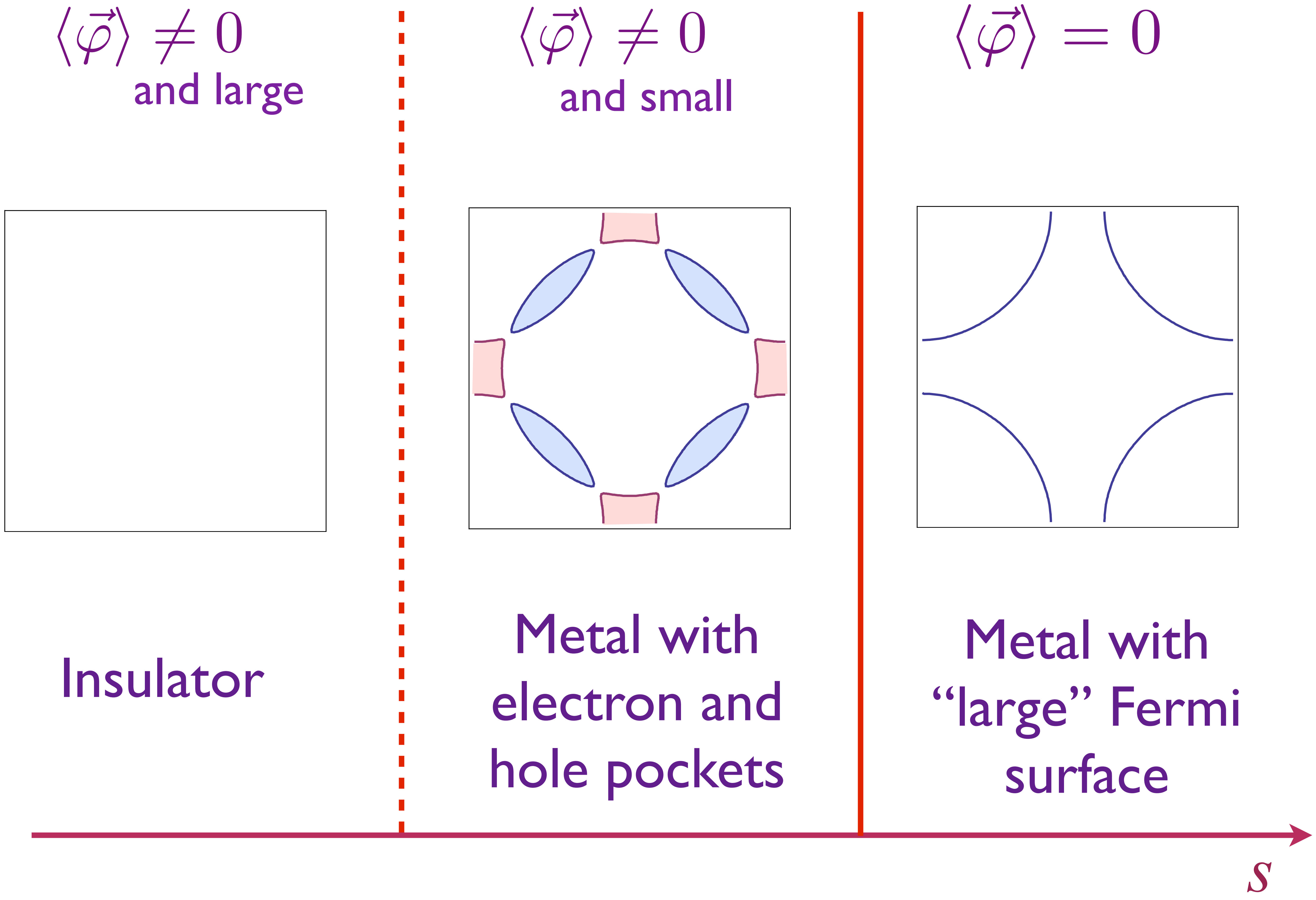
$\langle \vec{\varphi} \rangle \neq 0$
and small

Metal with electron and hole pockets

$\langle \vec{\varphi} \rangle = 0$

Metal with "large" Fermi surface

Square lattice Hubbard model with no doping



$\langle \vec{\varphi} \rangle \neq 0$

and large

Insulator

$\langle \vec{\varphi} \rangle \neq 0$

and small

Metal with electron and hole pockets

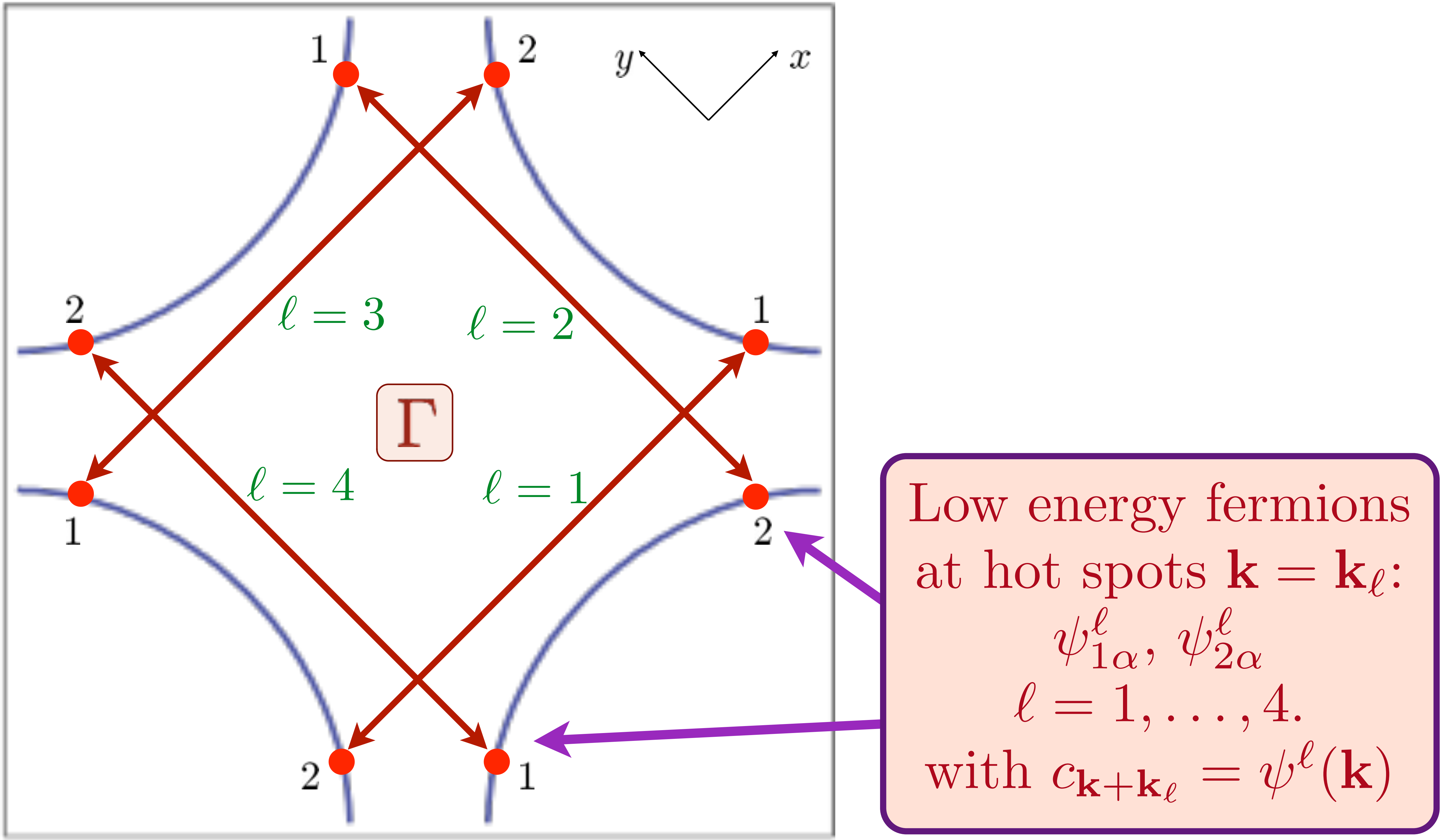
$\langle \vec{\varphi} \rangle = 0$

Metal with "large" Fermi surface

Start from the “spin-fermion” model

$$\mathcal{Z} = \int \mathcal{D}c_\alpha \mathcal{D}\vec{\varphi} \exp\left(-\mathcal{S}\right)$$

$$\begin{aligned}\mathcal{S} = \int d\tau \sum_{\mathbf{k}} c^\dagger_{\mathbf{k}\alpha} \left(\frac{\partial}{\partial \tau} - \varepsilon_{\mathbf{k}} \right) c_{\mathbf{k}\alpha} \\ - \lambda \int d\tau \sum_i c^\dagger_{i\alpha} \vec{\varphi}_i \cdot \vec{\sigma}_{\alpha\beta} c_{i\beta} e^{i\mathbf{K}\cdot\mathbf{r}_i} \\ + \int d\tau d^2 r \left[\frac{1}{2} \left(\boldsymbol{\nabla}_r \vec{\varphi} \right)^2 + \frac{\widetilde{\zeta}}{2} \left(\partial_\tau \vec{\varphi} \right)^2 + \frac{s}{2} \vec{\varphi}^2 + \frac{u}{4} \vec{\varphi}^4 \right]\end{aligned}$$



Low energy fermions at hot spots $\mathbf{k} = \mathbf{k}_\ell$:

$\psi^\ell_{1\alpha}$, $\psi^\ell_{2\alpha}$

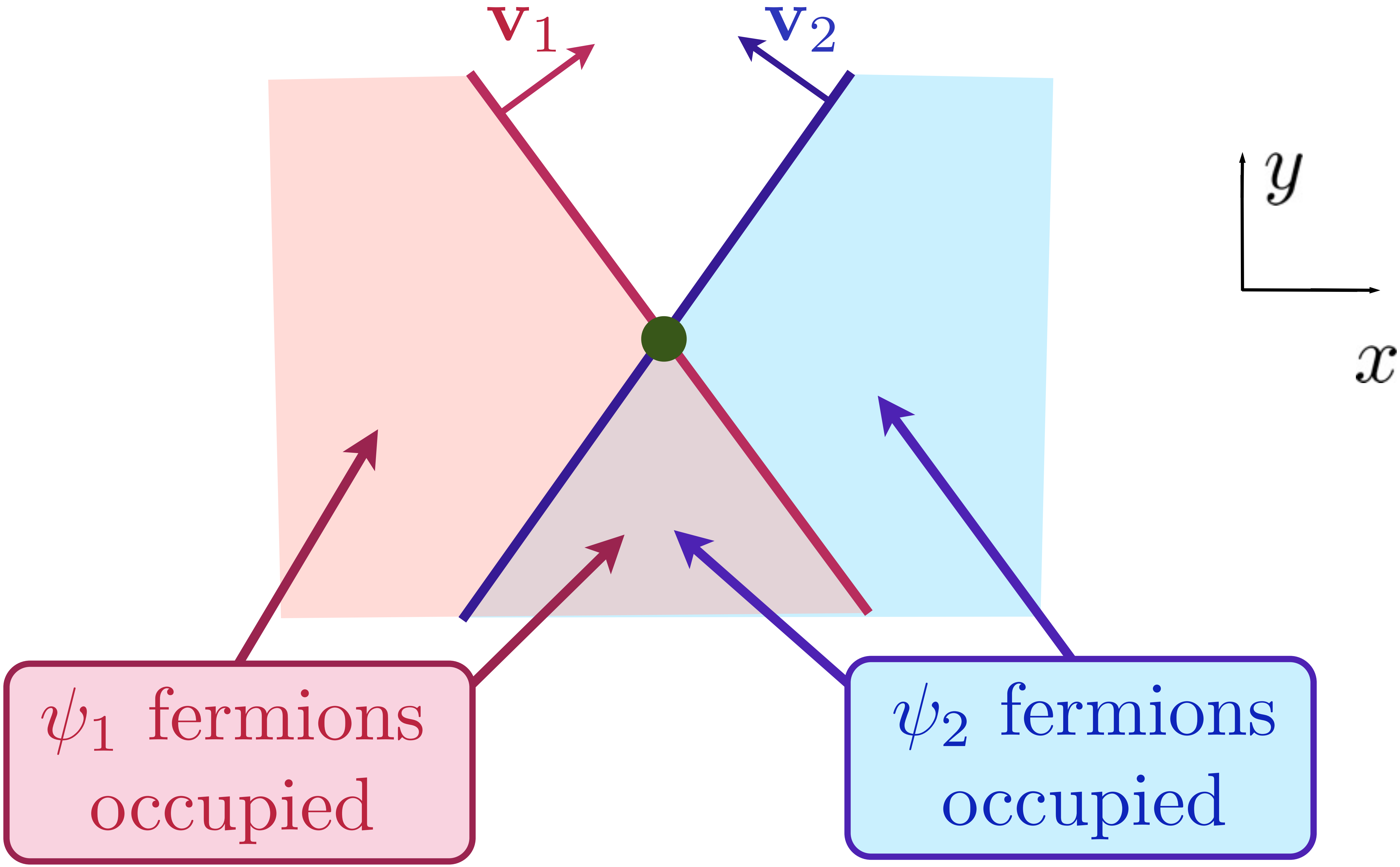
$\ell = 1, \ldots, 4.$

with $c_{\mathbf{k}+\mathbf{k}_\ell} = \psi^\ell(\mathbf{k})$

$$\mathcal{L}_f = \psi^{\ell\dagger}_{1\alpha}\left(\zeta\partial_\tau - i\mathbf{v}^\ell_1 \cdot \boldsymbol{\nabla}_r\right)\psi^\ell_{1\alpha} + \psi^{\ell\dagger}_{2\alpha}\left(\zeta\partial_\tau - i\mathbf{v}^\ell_2 \cdot \boldsymbol{\nabla}_r\right)\psi^\ell_{2\alpha}$$

$$\mathbf{v}^{\ell=1}_1 = (v_x, v_y),\ \mathbf{v}^{\ell=1}_2 = (-v_x, v_y)$$

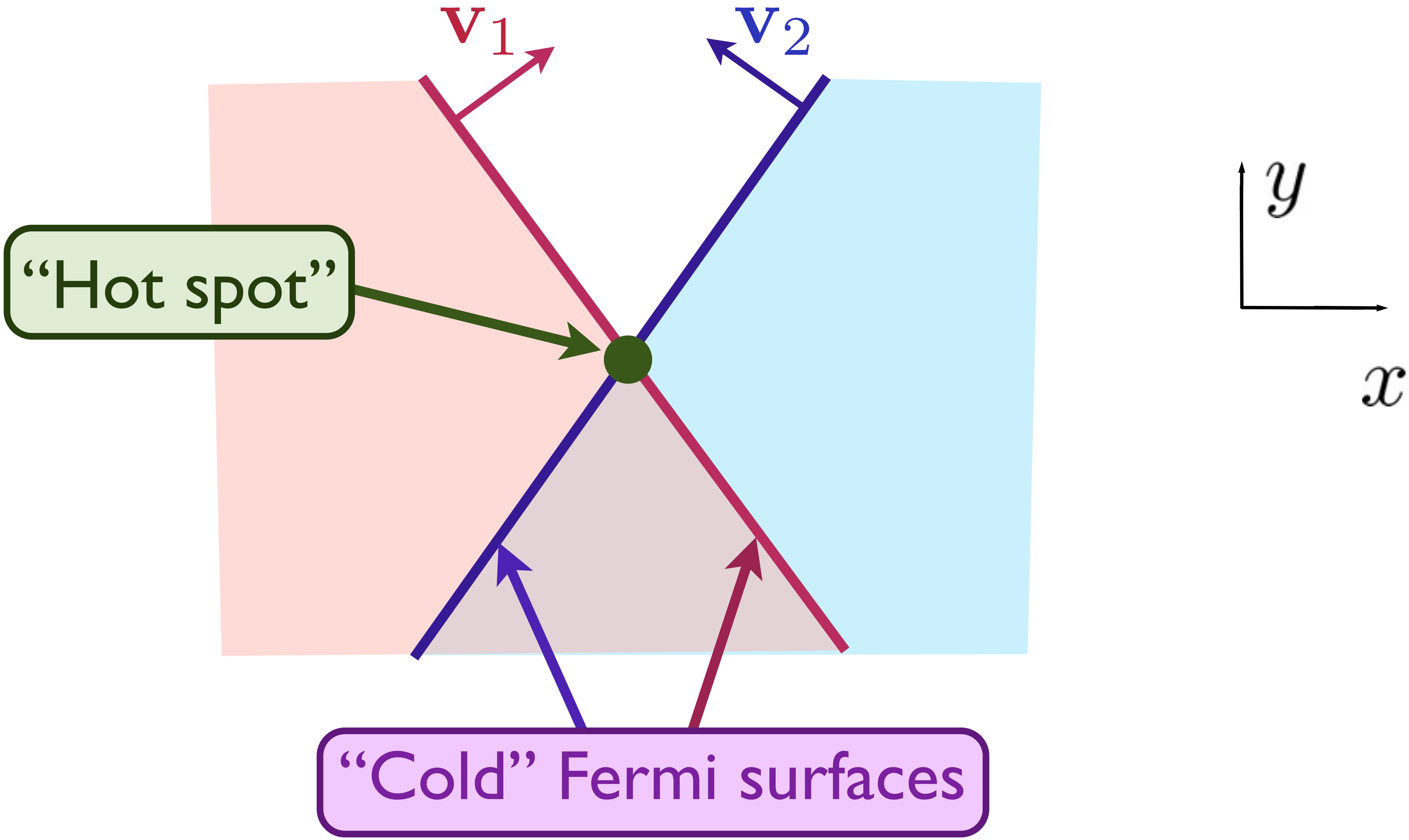
$$\mathcal{L}_f = \psi^{\ell\dagger}_{1\alpha}\left(\zeta\partial_\tau - i\mathbf{v}^\ell_1\cdot\boldsymbol{\nabla}_r\right)\psi^\ell_{1\alpha} + \psi^{\ell\dagger}_{2\alpha}\left(\zeta\partial_\tau - i\mathbf{v}^\ell_2\cdot\boldsymbol{\nabla}_r\right)\psi^\ell_{2\alpha}$$



ψ_1 fermions occupied

ψ_2 fermions occupied

$$\mathcal{L}_f = \psi^{\ell\dagger}_{1\alpha}\left(\zeta\partial_\tau - i\mathbf{v}^\ell_1\cdot\boldsymbol{\nabla}_r\right)\psi^\ell_{1\alpha} + \psi^{\ell\dagger}_{2\alpha}\left(\zeta\partial_\tau - i\mathbf{v}^\ell_2\cdot\boldsymbol{\nabla}_r\right)\psi^\ell_{2\alpha}$$



$$\mathcal{L}_f = \psi_{1\alpha}^{\ell\dagger} \left(\zeta\partial_\tau - i\mathbf{v}_1^\ell \cdot \boldsymbol{\nabla}_r\right) \psi_{1\alpha}^\ell + \psi_{2\alpha}^{\ell\dagger} \left(\zeta\partial_\tau - i\mathbf{v}_2^\ell \cdot \boldsymbol{\nabla}_r\right) \psi_{2\alpha}^\ell$$

Order parameter: $\mathcal{L}_\varphi = \frac{1}{2}\left(\boldsymbol{\nabla}_r \vec{\varphi}\right)^2 + \frac{\widetilde{\zeta}}{2}\left(\partial_\tau \vec{\varphi}\right)^2 + \frac{s}{2}\vec{\varphi}^2 + \frac{u}{4}\vec{\varphi}^4$

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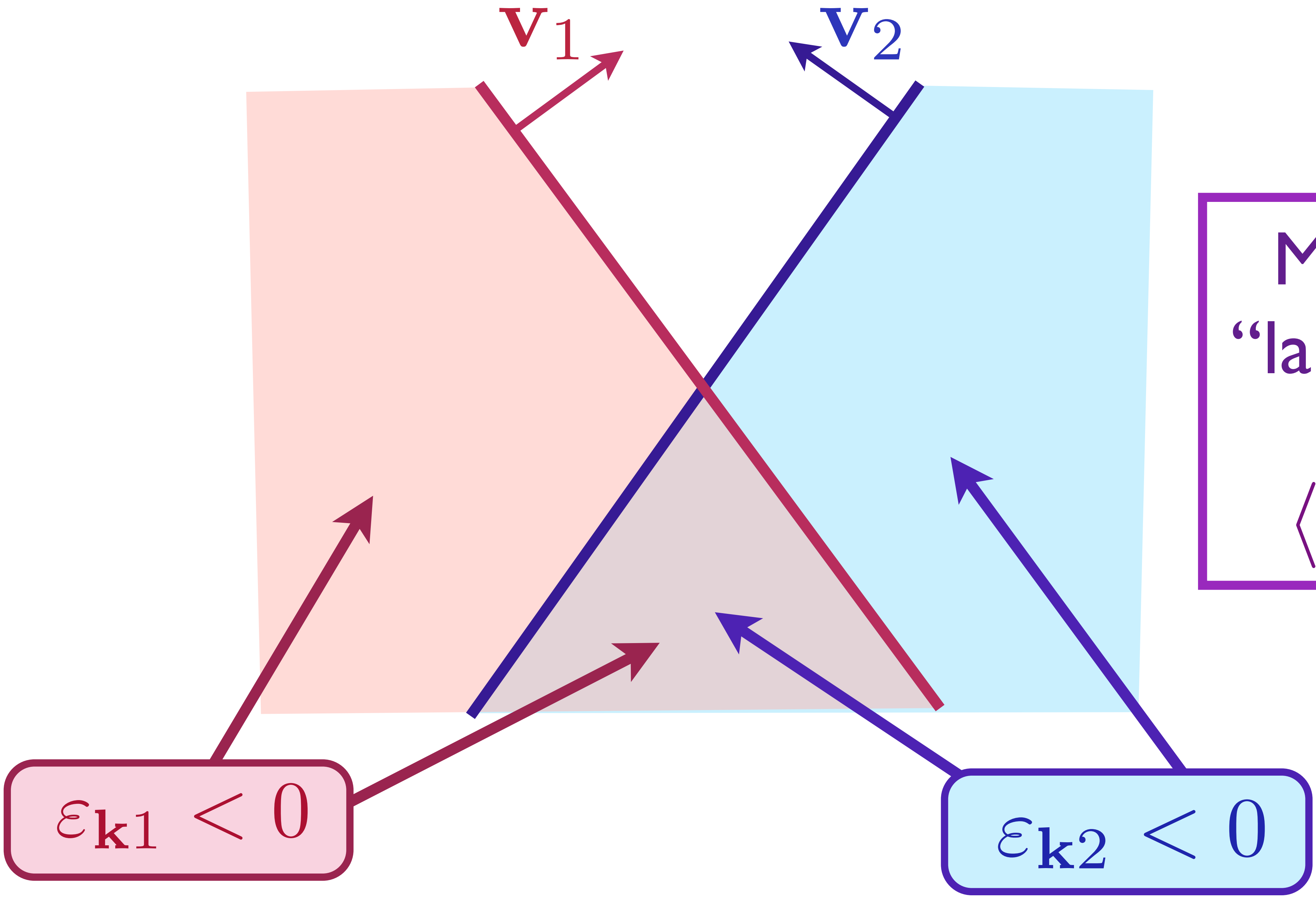
"Yukawa" coupling: $\mathcal{L}_c = -\lambda\vec{\varphi}\cdot\left(\psi^{\ell\dagger}_{1\alpha}\vec{\sigma}_{\alpha\beta}\psi^\ell_{2\beta} + \psi^{\ell\dagger}_{2\alpha}\vec{\sigma}_{\alpha\beta}\psi^\ell_{1\beta}\right)$

$$\mathcal{L}_f = \psi_{1\alpha}^{\ell\dagger}\left(\zeta\partial_\tau - i\mathbf{v}_1^\ell \cdot \boldsymbol{\nabla}_r\right)\psi_{1\alpha}^\ell + \psi_{2\alpha}^{\ell\dagger}\left(\zeta\partial_\tau - i\mathbf{v}_2^\ell \cdot \boldsymbol{\nabla}_r\right)\psi_{2\alpha}^\ell$$

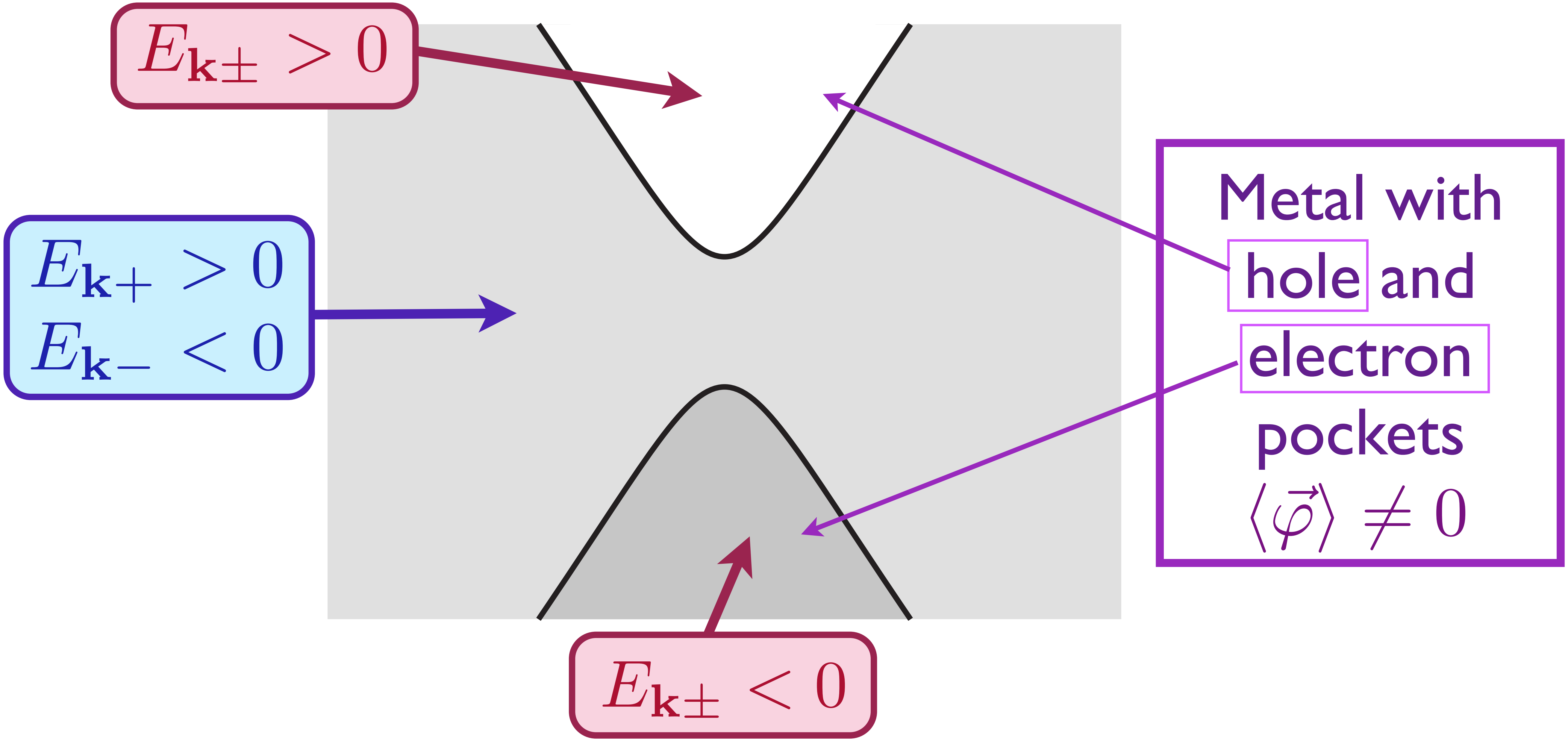


Metal with
"large" Fermi
surface
$\langle\vec{\varphi}\rangle = 0$

Fermion dispersions: $\varepsilon_{\mathbf{k}1} = \mathbf{v}_1 \cdot \mathbf{k}$ and $\varepsilon_{\mathbf{k}2} = \mathbf{v}_2 \cdot \mathbf{k}$

$$\mathcal{L}_f = \psi^{\ell\dagger}_{1\alpha}\left(\zeta\partial_\tau - i\mathbf{v}^\ell_1\cdot\boldsymbol{\nabla}_r\right)\psi^\ell_{1\alpha} + \psi^{\ell\dagger}_{2\alpha}\left(\zeta\partial_\tau - i\mathbf{v}^\ell_2\cdot\boldsymbol{\nabla}_r\right)\psi^\ell_{2\alpha}$$

$$-\lambda\vec{\varphi}\cdot\left(\psi^{\ell\dagger}_{1\alpha}\vec{\sigma}_{\alpha\beta}\psi^\ell_{2\beta} + \psi^{\ell\dagger}_{2\alpha}\vec{\sigma}_{\alpha\beta}\psi^\ell_{1\beta}\right)$$



$E_{\mathbf{k}\pm} > 0$

$E_{\mathbf{k}+} > 0$
$E_{\mathbf{k}-} < 0$

Metal with hole and electron pockets
$\langle\vec{\varphi}\rangle \neq 0$

$E_{\mathbf{k}\pm} < 0$

Fermion dispersions:

$$E_{\mathbf{k}\pm} = \frac{\varepsilon_{\mathbf{k}1}+\varepsilon_{\mathbf{k}2}}{2} \pm \sqrt{\left(\frac{\varepsilon_{\mathbf{k}1}-\varepsilon_{\mathbf{k}2}}{2}\right)^2 + \lambda^2|\vec{\varphi}|^2}$$

$$\mathcal{L}_f = \psi^{\ell\dagger}_{1\alpha}\left(\zeta\partial_\tau - i\mathbf{v}^{\ell}_1\cdot\boldsymbol{\nabla}_r\right)\psi^{\ell}_{1\alpha} + \psi^{\ell\dagger}_{2\alpha}\left(\zeta\partial_\tau - i\mathbf{v}^{\ell}_2\cdot\boldsymbol{\nabla}_r\right)\psi^{\ell}_{2\alpha}$$

Order parameter: $\mathcal{L}_\varphi = \frac{1}{2}\left(\boldsymbol{\nabla}_r\vec{\varphi}\right)^2 + \frac{\widetilde{\zeta}}{2}\left(\partial_\tau\vec{\varphi}\right)^2 + \frac{s}{2}\vec{\varphi}^2 + \frac{u}{4}\vec{\varphi}^4$

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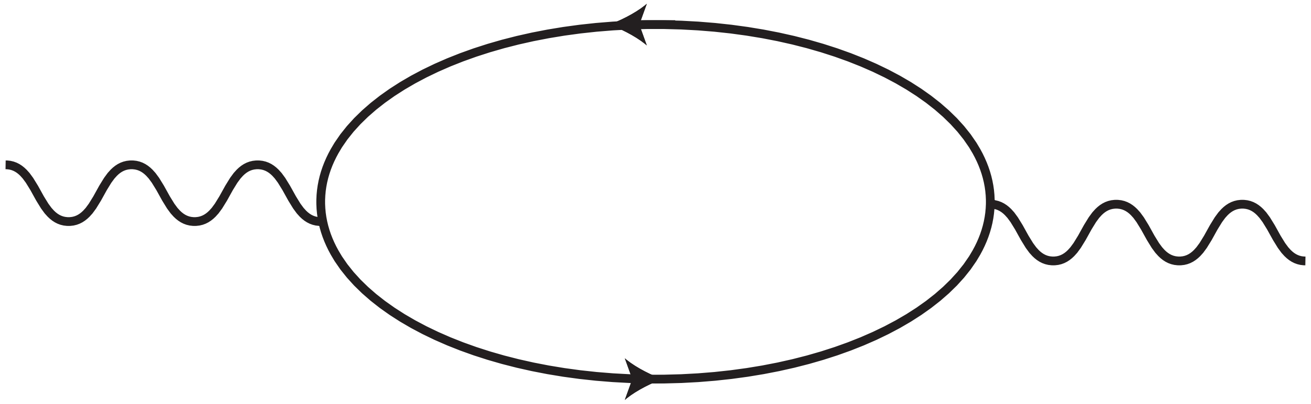
Hertz theory

Integrate out fermions and obtain an effective action for the boson field $\vec{\varphi}$ alone. Because the fermions are gapless, this is potentially dangerous, and will lead to non-local terms in the $\vec{\varphi}$ effective action. Hertz focused on only the simplest such non-local term. However, there are an infinite number of non-local terms at higher order, and these lead to a breakdown of the Hertz theory in $d = 2$.

Ar. Abanov and A.V. Chubukov, *Phys. Rev. Lett.* **93**, 255702 (2004)

Hertz action.

Upon integrating the fermions out, the leading term in the $\vec{\varphi}$ effective action is $-\Pi(q,\omega_n)|\vec{\varphi}(q,\omega_n)|^2$, where $\Pi(q,\omega_n)$ is the fermion polarizability. This is given by a simple fermion loop diagram



$$\Pi(q,\omega_n) = \int \frac{d^d k}{(2\pi)^d} \int \frac{d\epsilon_n}{2\pi} \frac{1}{[-i\zeta(\epsilon_n+\omega_n)+\mathbf{v}_1\cdot(\mathbf{k}+\mathbf{q})][-i\zeta\epsilon_n+\mathbf{v}_2\cdot\mathbf{k}]} \tag{1}$$

We define oblique co-ordinates $p_1 = \mathbf{v}_1 \cdot \mathbf{k}$ and $p_2 = \mathbf{v}_2 \cdot \mathbf{k}$. It is then clear that the integrand in (1) is independent of the $(d-2)$ transverse momenta, whose integral yields an overall factor Λ^{d-2} (in $d=2$ this factor is precisely 1). Also, by shifting the integral over k_1 we note that the integral is independent of q. So we have

$$\Pi(q,\omega_n) = \frac{\Lambda^{d-2}}{|\mathbf{v}_1 \times \mathbf{v}_2|} \int \frac{dp_1 dp_2 d\epsilon_n}{8\pi^3} \frac{1}{[-i\zeta(\epsilon_n + \omega_n) + p_1][-i\zeta\epsilon_n + p_2]}. \tag{2}$$

Next, we evaluate the frequency integral to obtain

$$\begin{aligned} \Pi(q,\omega_n) &= \frac{\Lambda^{d-2}}{\zeta|\mathbf{v}_1 \times \mathbf{v}_2|} \int \frac{dp_1 dp_2}{4\pi^2} \frac{[\mathrm{sgn}(p_2) - \mathrm{sgn}(p_1)]}{-i\zeta\omega_n + p_1 - p_2} \\ &= -\frac{|\omega_n|\Lambda^{d-2}}{4\pi|\mathbf{v}_1 \times \mathbf{v}_2|}. \end{aligned} \tag{3}$$

In the last step, we have dropped a frequency-independent, cutoff-dependent constant which can absorbed into a redefinition of r. Notice also that the factor of ζ has cancelled.

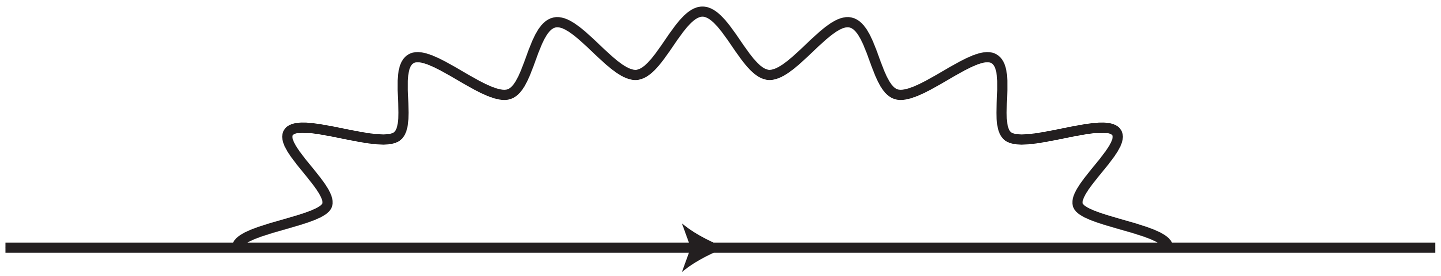
Inserting this fermion polarizability in the effective action for $\vec{\varphi}$, we obtain the Hertz action for the SDW transition:

$$\begin{aligned} \mathcal{S}_H = \int \frac{d^d k}{(2\pi)^d} T \sum_{\omega_n} \frac{1}{2} \left[k^2 + \gamma|\omega_n| + s\right] |\vec{\varphi}(k,\omega_n)|^2 \\ + \frac{u}{4} \int d^d x d\tau \left(\vec{\varphi}^2(x,\tau)\right)^2. \end{aligned} \tag{4}$$

Exercise: Perform a tree-level RG rescaling on $\mathcal{S}_H$. Now we rescale co-ordinates as $x' = xe^{-\ell}$ and $\tau' = \tau e^{-z\ell}$. Here z is the dynamic critical exponent. Show that the gradient and non-local terms become invariant for $z = 2$ (previous theories considered here had $z = 1$). Then show that the transformation of the quartic term is $u' = ue^{(2-d)\ell}$. This led Hertz to conclude that the SDW quantum critical point was described by a Gaussian theory for the SDW order parameter in $d \geq 2$.

Fate of the fermions.

Let us, for now, assume the validity of the Hertz Gaussian action, and compute the leading correction to the electronic Green's function. This is given by the following Feynman graph for the electron self energy, Σ. At zero momentum for the ψ_1 fermion we have



$$\Sigma_1(0,\omega_n) = \lambda^2 \int \frac{d^d q}{(2\pi)^d} \int \frac{d\epsilon_n}{2\pi} \frac{1}{\left[q^2 + \gamma|\epsilon_n|\right]\left[-i\zeta(\epsilon_n+\omega_n) + \mathbf{v}_2\cdot\mathbf{q}\right]}. \tag{5}$$

We first perform the integral over the $\mathbf{q}$ direction parallel to $\mathbf{v}_2$, while ignoring the subdominant dependence on this momentum in the boson propagator. The dependence on ζ immediately disappears, and we have

$$\begin{aligned}\Sigma_1(0,\omega_n) &= i\,\frac{\lambda^2}{|v_2|}\int\frac{d^{d-1}q}{(2\pi)^{d-1}}\int\frac{d\epsilon_n}{2\pi}\frac{\mathrm{sgn}(\epsilon_n+\omega_n)}{|q|^2+\gamma|\epsilon_n|}\\ &= i\,\frac{\lambda^2}{\pi|v_2|\gamma}\mathrm{sgn}(\omega_n)\int\frac{d^{d-1}q}{(2\pi)^{d-1}}\ln\left(\frac{|q|^2+\gamma|\omega_n|}{|q|^2}\right). \qquad (6)\end{aligned}$$

Evaluation of the q integral shows that

$$\Sigma_1(0,\omega_n) \sim |\omega_n|^{(d-1)/2} \qquad (7)$$

The most important case is $d = 2$, where we have

$$\Sigma_1(0,\omega_n) = i\,\frac{\lambda^2}{\pi|v_2|\sqrt{\gamma}}\mathrm{sgn}(\omega_n)\sqrt{|\omega_n|} \quad , \quad d = 2. \qquad (8)$$

Strong coupling physics in $d = 2$

The theory so far has the boson propagator

$$\sim \frac{1}{q^2 + \gamma|\omega|}$$

which scales with dynamic exponent $z_b = 2$, and now a fermion propagator

$$\sim \frac{1}{-i\zeta\omega + c_1|\omega|^{(d-1)/2} + \mathbf{v} \cdot \mathbf{q}}.$$

First note that for $d < 3$, the bare $-i\zeta\omega$ term is less important than the contribution from the self energy at low frequencies. This indicates that ζ is *irrelevant* in the critical theory, and we can set $\zeta \to 0$. Fortunately, all the loop diagrams evaluated so far are independent of ζ.

Setting $\zeta = 0$, we see that the fermion propagator scales with dynamic exponent $z_f = 2/(d-1)$. For $d > 2$, $z_f < z_b$, and so at small momenta the boson fluctuations have lower energy than the fermion fluctuations. Thus it seems reasonable to assume that the fermion fluctuations are not as singular, and we can focus on an effective theory of the SDW order parameter $\vec{\varphi}$ alone. In other words, the Hertz assumptions appear valid for $d > 2$.

However, in $d = 2$, we have $z_f = z_b = 2$. Thus fermionic and bosonic fluctuations are equally important, and it is not appropriate to integrate the fermions out at an initial stage. We have to return to the original theory of coupled bosons and fermions. This turns out to be strongly coupled, and exhibits complex critical behavior. For more details, see

M. A. Metlitski and S. Sachdev, arXiv:1005.1288 (Physical Review B **82**, 075127 (2010)).

$$\mathcal{L}_f = \psi_{1\alpha}^{\ell\dagger}\left(\zeta\partial_\tau - i\mathbf{v}_1^\ell\cdot\boldsymbol{\nabla}_r\right)\psi_{1\alpha}^\ell + \psi_{2\alpha}^{\ell\dagger}\left(\zeta\partial_\tau - i\mathbf{v}_2^\ell\cdot\boldsymbol{\nabla}_r\right)\psi_{2\alpha}^\ell$$

Order parameter: $$\mathcal{L}_\varphi = \frac{1}{2}\left(\boldsymbol{\nabla}_r\vec{\varphi}\right)^2 + \frac{\widetilde{\zeta}}{2}\left(\partial_\tau\vec{\varphi}\right)^2 + \frac{s}{2}\vec{\varphi}^2 + \frac{u}{4}\vec{\varphi}^4$$

"Yukawa" coupling: $$\mathcal{L}_c = -\lambda\vec{\varphi}\cdot\left(\psi_{1\alpha}^{\ell\dagger}\vec{\sigma}_{\alpha\beta}\psi_{2\beta}^\ell + \psi_{2\alpha}^{\ell\dagger}\vec{\sigma}_{\alpha\beta}\psi_{1\beta}^\ell\right)$$

Perform RG on both fermions and $\vec{\varphi}$, using a *local* field theory.

$$\mathcal{L}_f = \psi_{1\alpha}^{\ell\dagger}\left(\zeta\partial_\tau - i\mathbf{v}_1^\ell \cdot \boldsymbol{\nabla}_r\right)\psi_{1\alpha}^\ell + \psi_{2\alpha}^{\ell\dagger}\left(\zeta\partial_\tau - i\mathbf{v}_2^\ell \cdot \boldsymbol{\nabla}_r\right)\psi_{2\alpha}^\ell$$

Order parameter: $\mathcal{L}_\varphi = \frac{1}{2}\left(\boldsymbol{\nabla}_r\vec{\varphi}\right)^2 + \frac{\widetilde{\zeta}}{2}\left(\partial_\tau\vec{\varphi}\right)^2 + \frac{s}{2}\vec{\varphi}^2 + \frac{u}{4}\vec{\varphi}^4$

"Yukawa" coupling: $\mathcal{L}_c = -\lambda\vec{\varphi}\cdot\left(\psi_{1\alpha}^{\ell\dagger}\vec{\sigma}_{\alpha\beta}\psi_{2\beta}^\ell + \psi_{2\alpha}^{\ell\dagger}\vec{\sigma}_{\alpha\beta}\psi_{1\beta}^\ell\right)$

Under the rescaling $x' = xe^{-\ell}$, $\tau' = \tau e^{-z\ell}$, the spatial gradients are fixed if the fields transform as

$$\vec{\varphi}' = e^{(d+z-2)\ell/2}\vec{\varphi} \quad ;\text{“} \quad \psi' = e^{(d+z-1)\ell/2}\psi.$$

Then the Yukawa coupling transforms as

$$\lambda' = e^{(4-d-z)\ell/2}\lambda$$

For $d = 2$, with $z = 2$ the bare time-derivative terms ζ, $\widetilde{\zeta}$ are irrelevant, but the Yukawa coupling is invariant. Thus we have to work at fixed $\lambda = 1$, and cannot expand in powers of λ: critical theory is *strongly coupled.*

Outline

1. Quantum phase transitions of a semi-metal

   Graphene, Dirac fermions and the Gross-Neveu model

2. Spin density wave order on the square lattice

   From a large Fermi surface to Fermi pockets

3. Competing orders I

   Unconventional superconductivity and the phase diagrams of the cuprates and the pnictides

4. Competing orders II

   Bond order, geometric phases, and gauge theories

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1. Unconventional pairing near a SDW transition in a metal

2. Phenomenological theory of competition between SDW order and superconductivity (SC)

3. Fermi surface theory of interplay between SDW and SC

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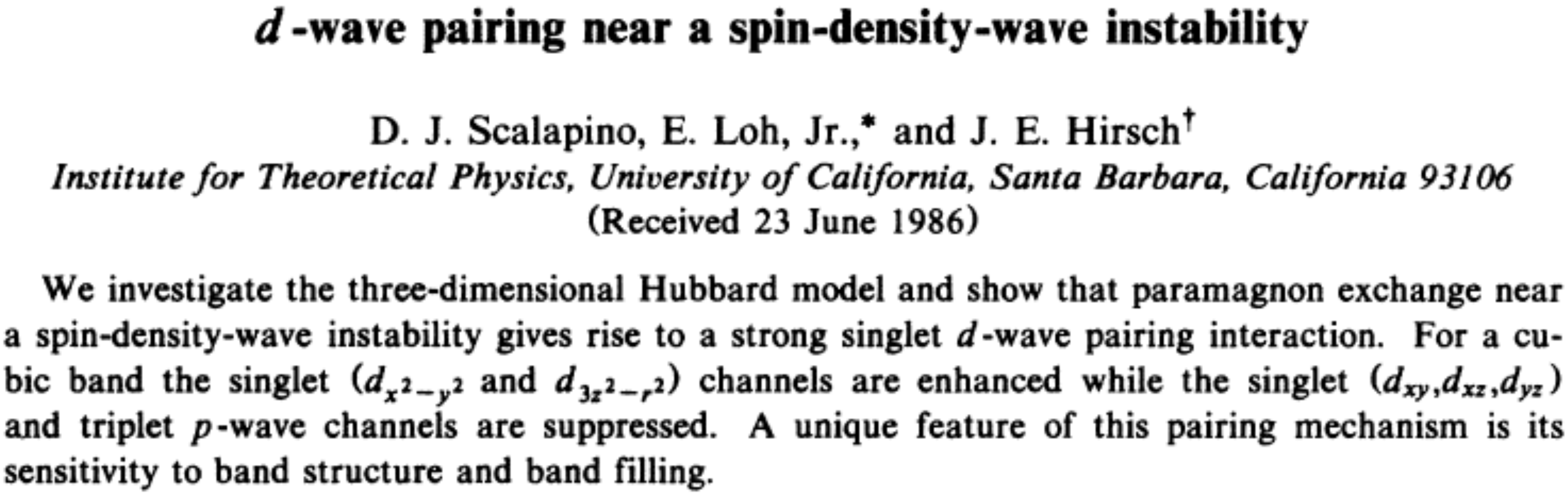


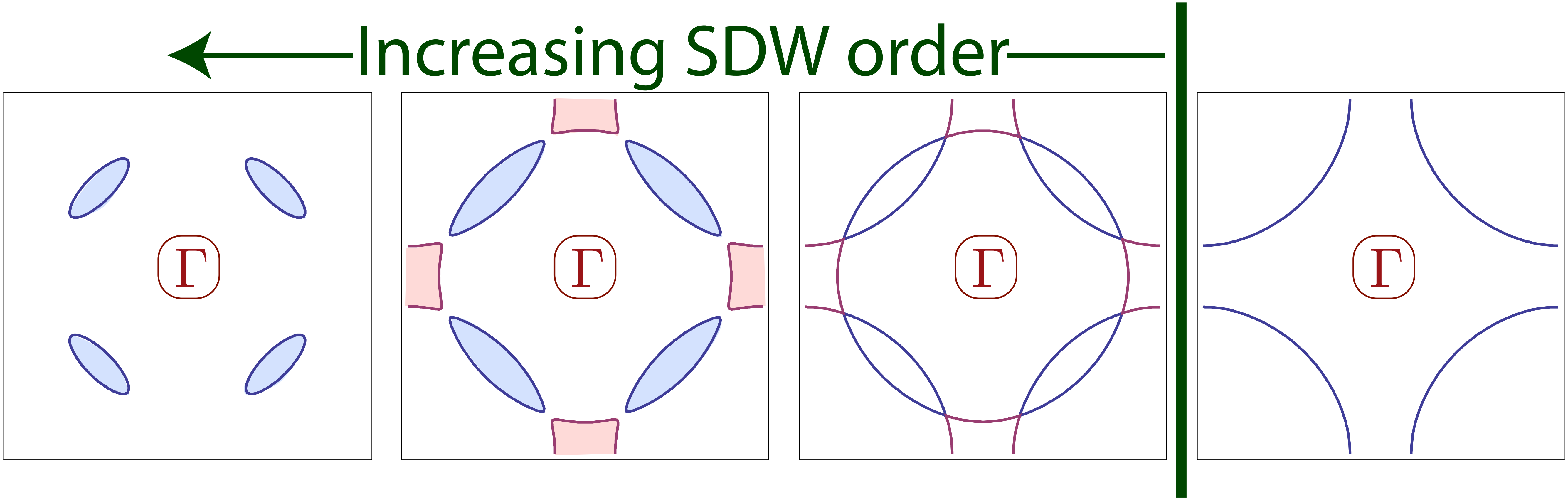
d-wave pairing near a spin-density-wave instability

D. J. Scalapino, E. Loh, Jr.,* and J. E. Hirsch†
Institute for Theoretical Physics, University of California, Santa Barbara, California 93106
(Received 23 June 1986)

We investigate the three-dimensional Hubbard model and show that paramagnon exchange near a spin-density-wave instability gives rise to a strong singlet *d*-wave pairing interaction. For a cubic band the singlet ($d_{x^2-y^2}$ and $d_{3z^2-r^2}$) channels are enhanced while the singlet (d_{xy},d_{xz},d_{yz}) and triplet *p*-wave channels are suppressed. A unique feature of this pairing mechanism is its sensitivity to band structure and band filling.

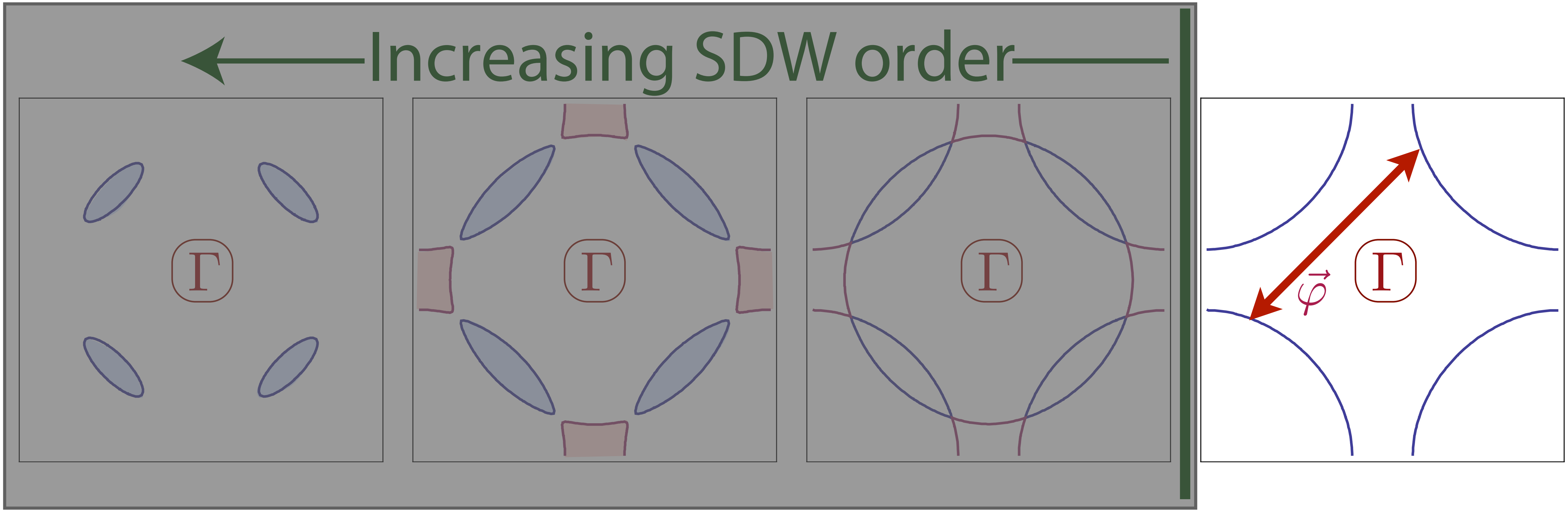
Physical Review B **34**, 8190 (1986)

Spin density wave theory in hole-doped cuprates



Increasing SDW order

Spin-fluctuation exchange theory of d-wave superconductivity in the cuprates



Fermions at the *large* Fermi surface exchange fluctuations of the SDW order parameter $\vec{\varphi}$.

Pairing by SDW fluctuation exchange

We now allow the SDW field $\vec{\varphi}$ to be dynamical, coupling to electrons as

$$H_{\rm sdw} = - \sum_{\mathbf{k},\mathbf{q},\alpha,\beta} \vec{\varphi}_{\mathbf{q}} \cdot c^{\dagger}_{\mathbf{k},\alpha} \vec{\sigma}_{\alpha\beta} c_{\mathbf{k}+\mathbf{K}+\mathbf{q},\beta}.$$

Exchange of a $\vec{\varphi}$ quantum leads to the effective interaction

$$H_{ee} = -\frac{1}{2} \sum_{\mathbf{q}} \sum_{\mathbf{p},\gamma,\delta} \sum_{\mathbf{k},\alpha,\beta} V_{\alpha\beta,\gamma\delta}(\mathbf{q}) c^{\dagger}_{\mathbf{k},\alpha} c_{\mathbf{k}+\mathbf{q},\beta} c^{\dagger}_{\mathbf{p},\gamma} c_{\mathbf{p}-\mathbf{q},\delta},$$

where the pairing interaction is

$$V_{\alpha\beta,\gamma\delta}(\mathbf{q}) = \vec{\sigma}_{\alpha\beta} \cdot \vec{\sigma}_{\gamma\delta} \frac{\chi_0}{\xi^{-2} + (\mathbf{q} - \mathbf{K})^2},$$

with $\chi_0 \xi^2$ the SDW susceptibility and ξ the SDW correlation length.

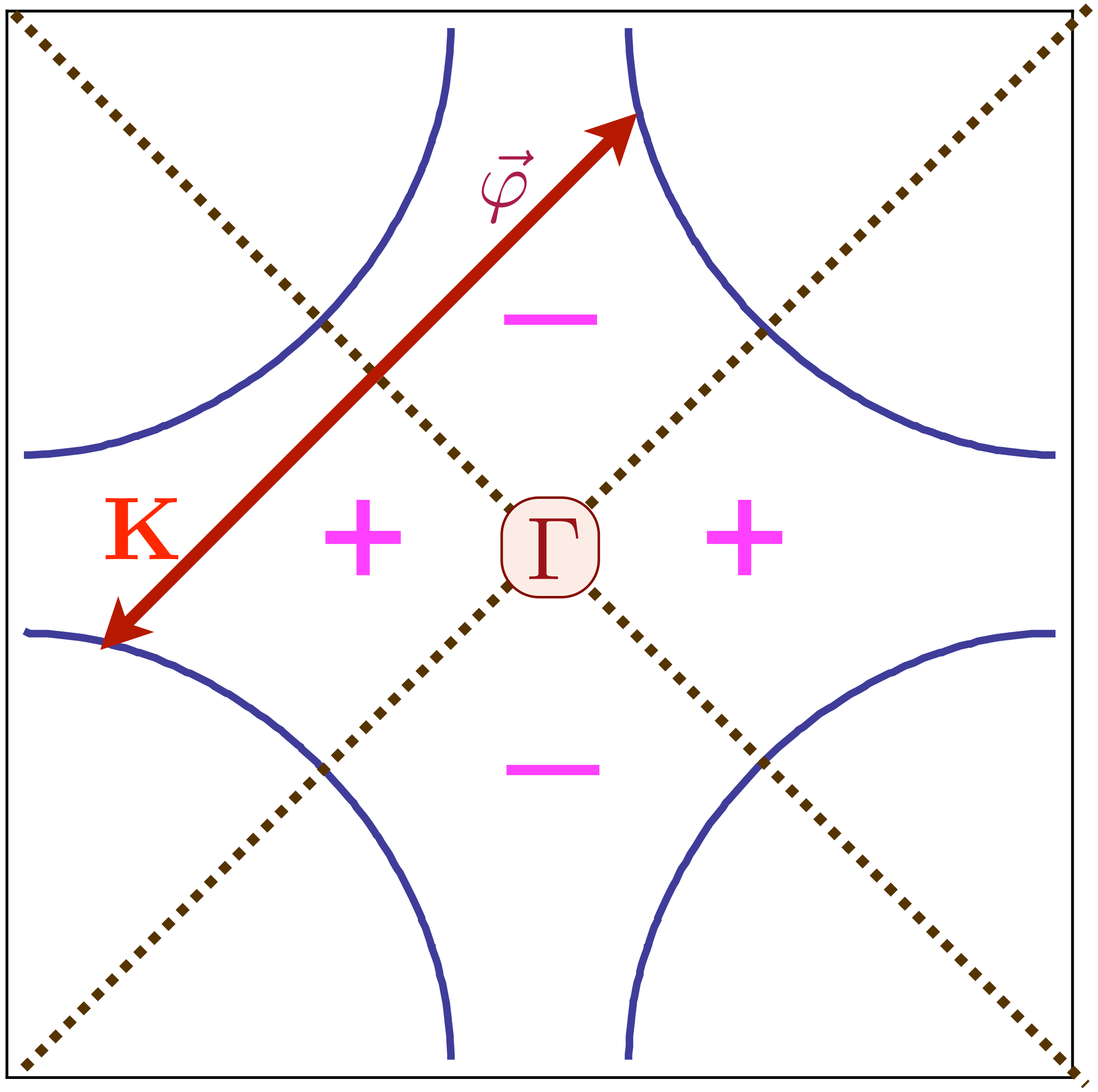
BCS Gap equation

In BCS theory, this interaction leads to the 'gap equation' for the pairing gap $\Delta_{\mathbf{k}} \propto \langle c_{\mathbf{k}\uparrow} c_{-\mathbf{k}\downarrow} \rangle$.

$$\Delta_{\mathbf{k}} = - \sum_{\mathbf{p}} \left(\frac{3\chi_0}{\xi^{-2} + (\mathbf{p} - \mathbf{k} - \mathbf{K})^2} \right) \frac{\Delta_{\mathbf{p}}}{2\sqrt{\varepsilon_{\mathbf{p}}^2 + \Delta_{\mathbf{p}}^2}}$$

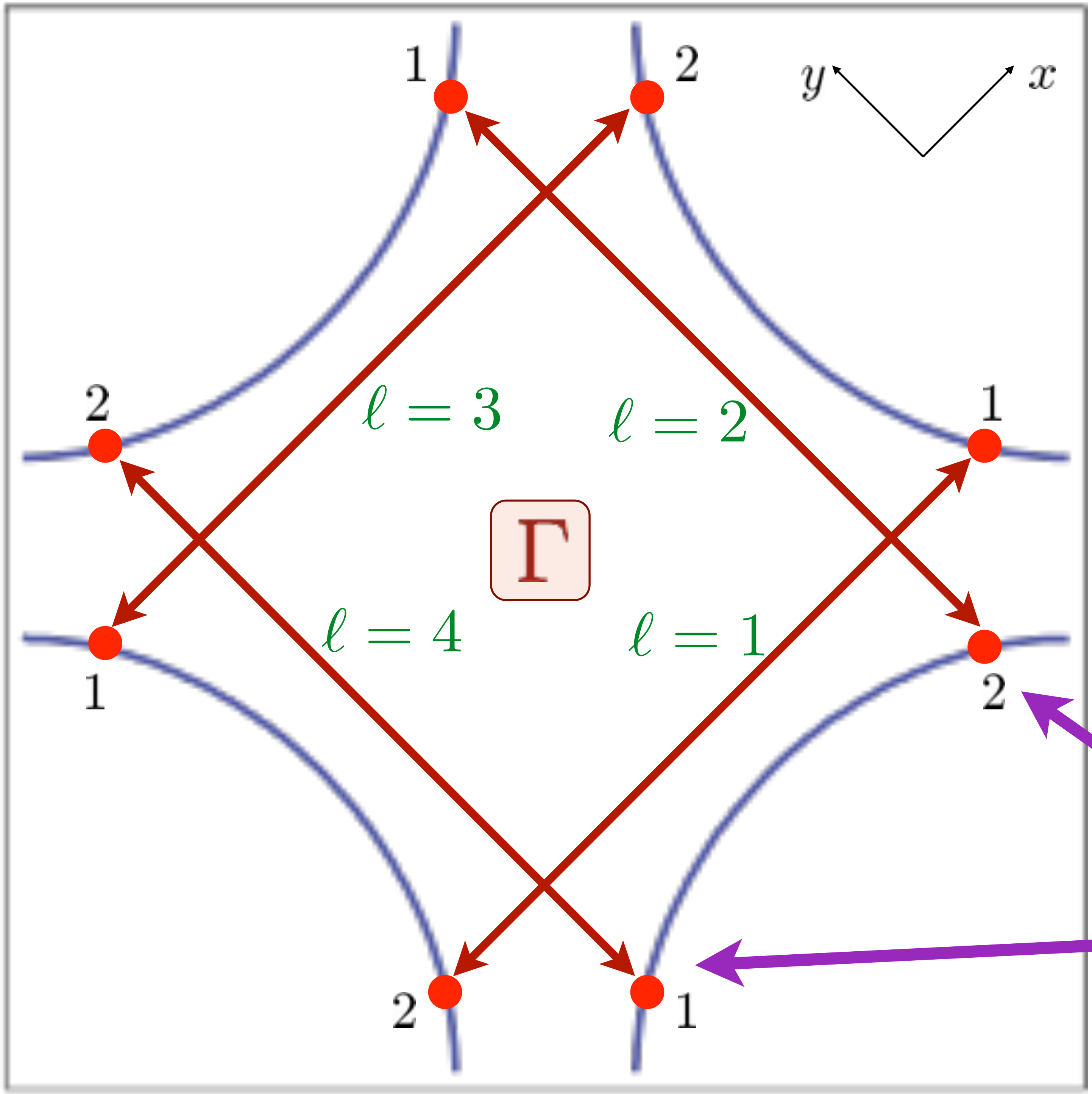
Non-zero solutions of this equation require that $\Delta_{\mathbf{k}}$ and $\Delta_{\mathbf{p}}$ have opposite signs when $\mathbf{p} - \mathbf{k} \approx \mathbf{K}$.

d-wave pairing of the large Fermi surface



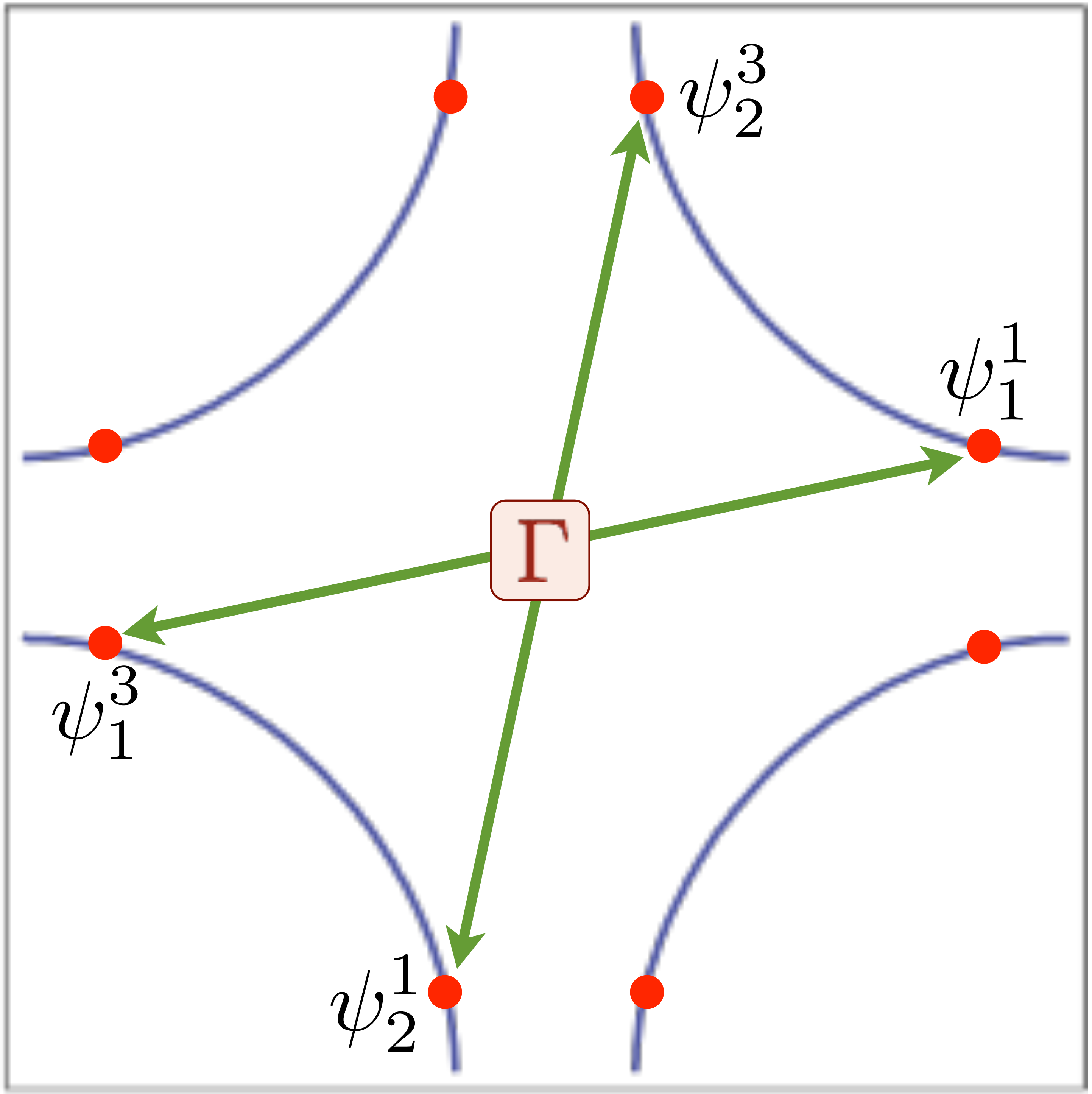
$$\langle c_{\mathbf{k}\uparrow} c_{-\mathbf{k}\downarrow} \rangle \propto \Delta_{\mathbf{k}} = \Delta_0 (\cos(k_x) - \cos(k_y))$$

d-wave pairing in the theory of hotspots



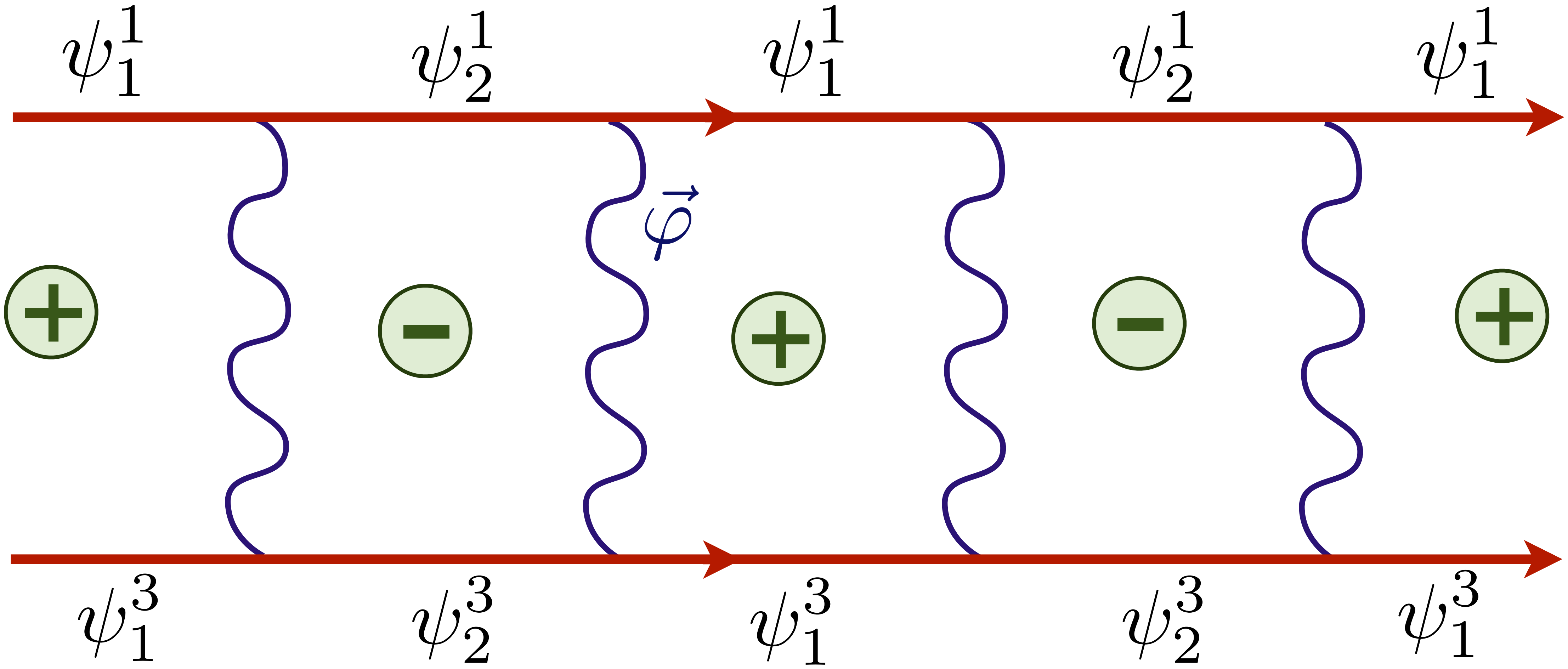
Low energy fermions
$\psi^{\ell}_{1\alpha}$, $\psi^{\ell}_{2\alpha}$
$\ell = 1, \ldots, 4$

d-wave pairing in the theory of hotspots

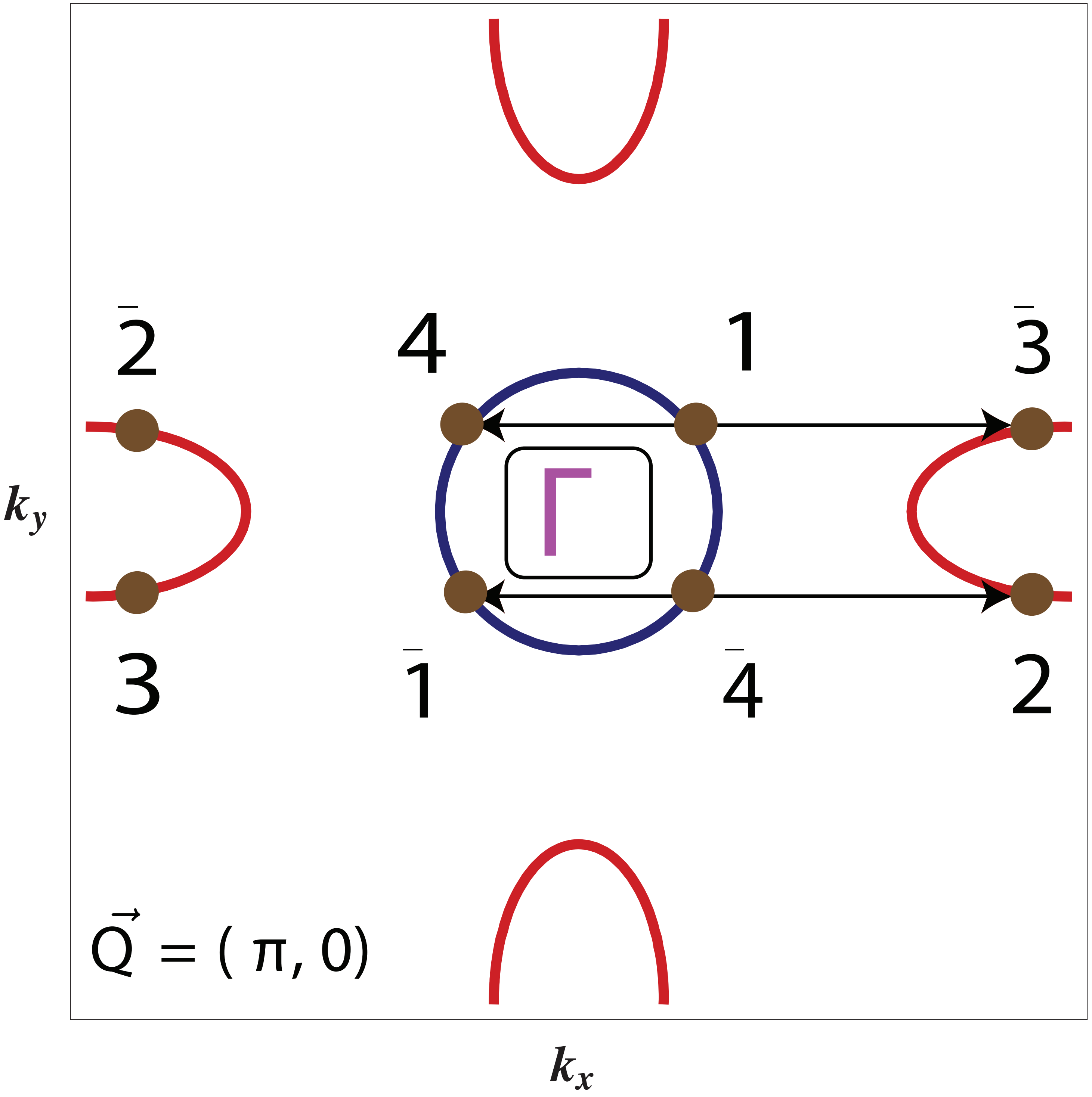


Hot spots have strong instability to d-wave pairing near SDW critical point. This instability is stronger than the BCS instability of a Fermi liquid.

Pairing order parameter: $\varepsilon^{\alpha\beta}\left(\psi^3_{1\alpha}\psi^1_{1\beta} - \psi^3_{2\alpha}\psi^1_{2\beta}\right)$



d-wave Cooper pairing instability in particle-particle channel

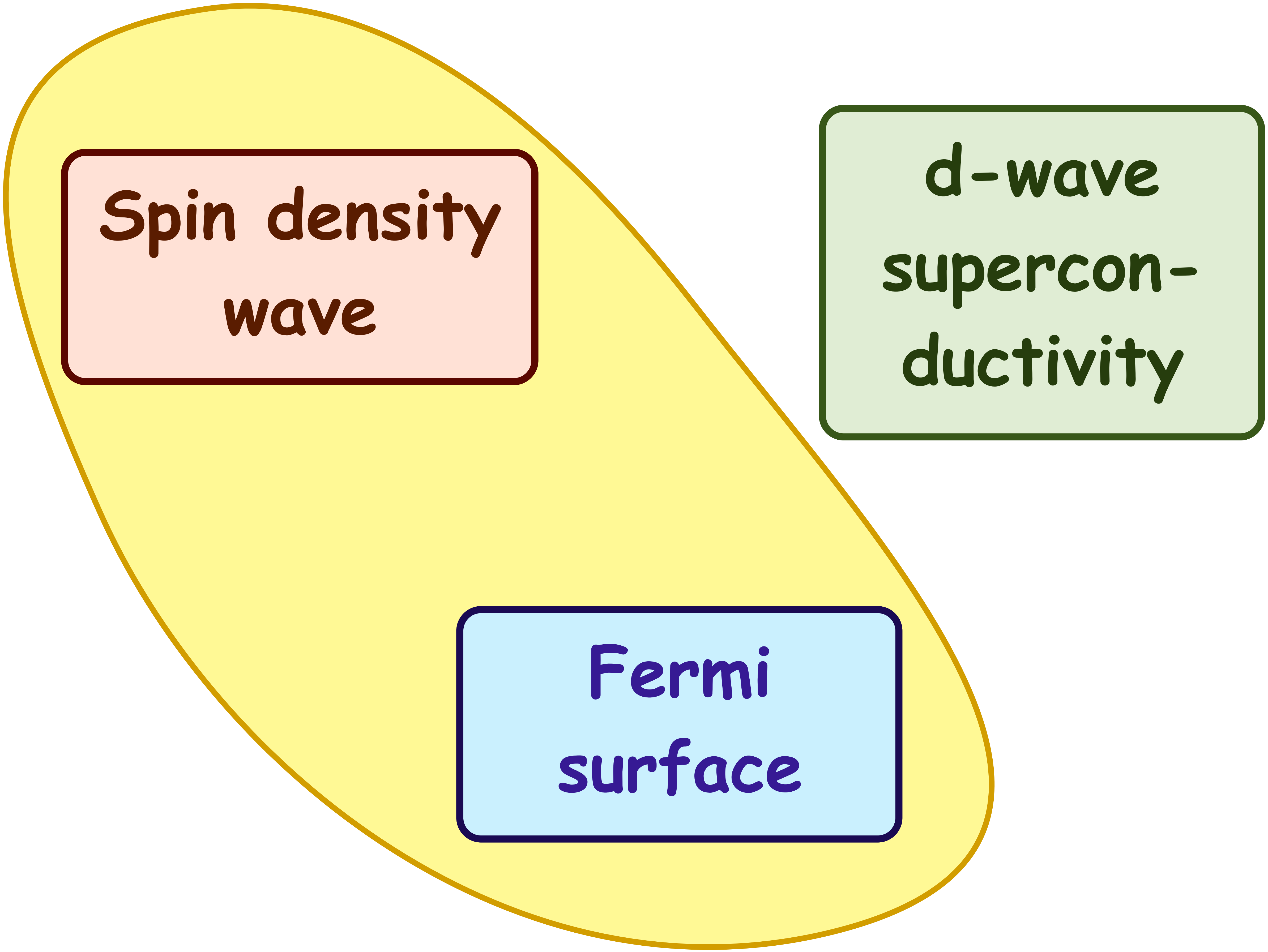


Similar theory applies to the pnictides, and leads to $s_\pm$ pairing.

Spin density wave

d-wave superconductivity

Fermi surface

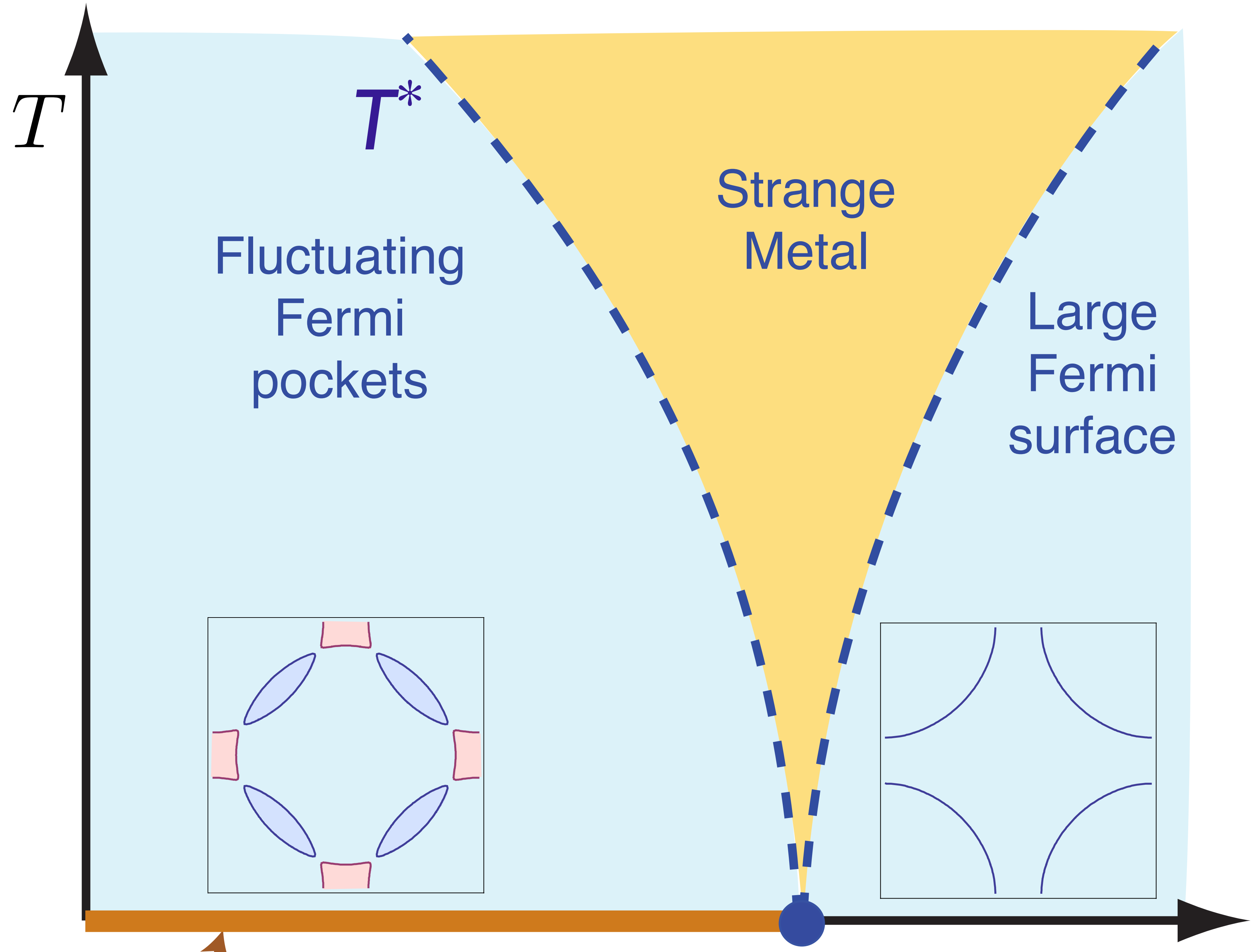


Spin density wave

d-wave superconductivity

Fermi surface

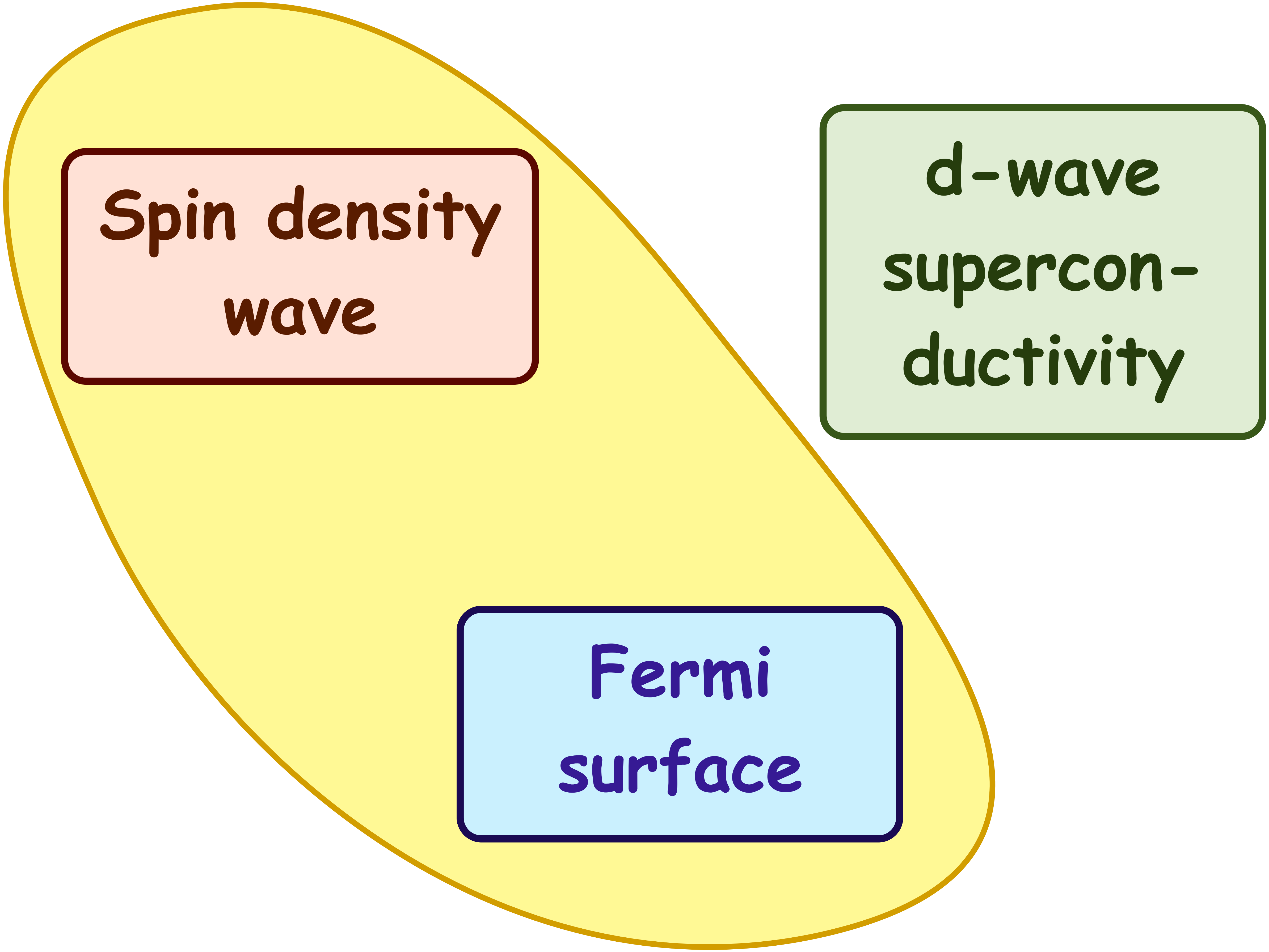
Theory of quantum criticality in the cuprates



x_m

Spin density wave (SDW)

Underlying SDW ordering quantum critical point in **metal** at $x = x_m$



Spin density wave

d-wave superconductivity

Fermi surface

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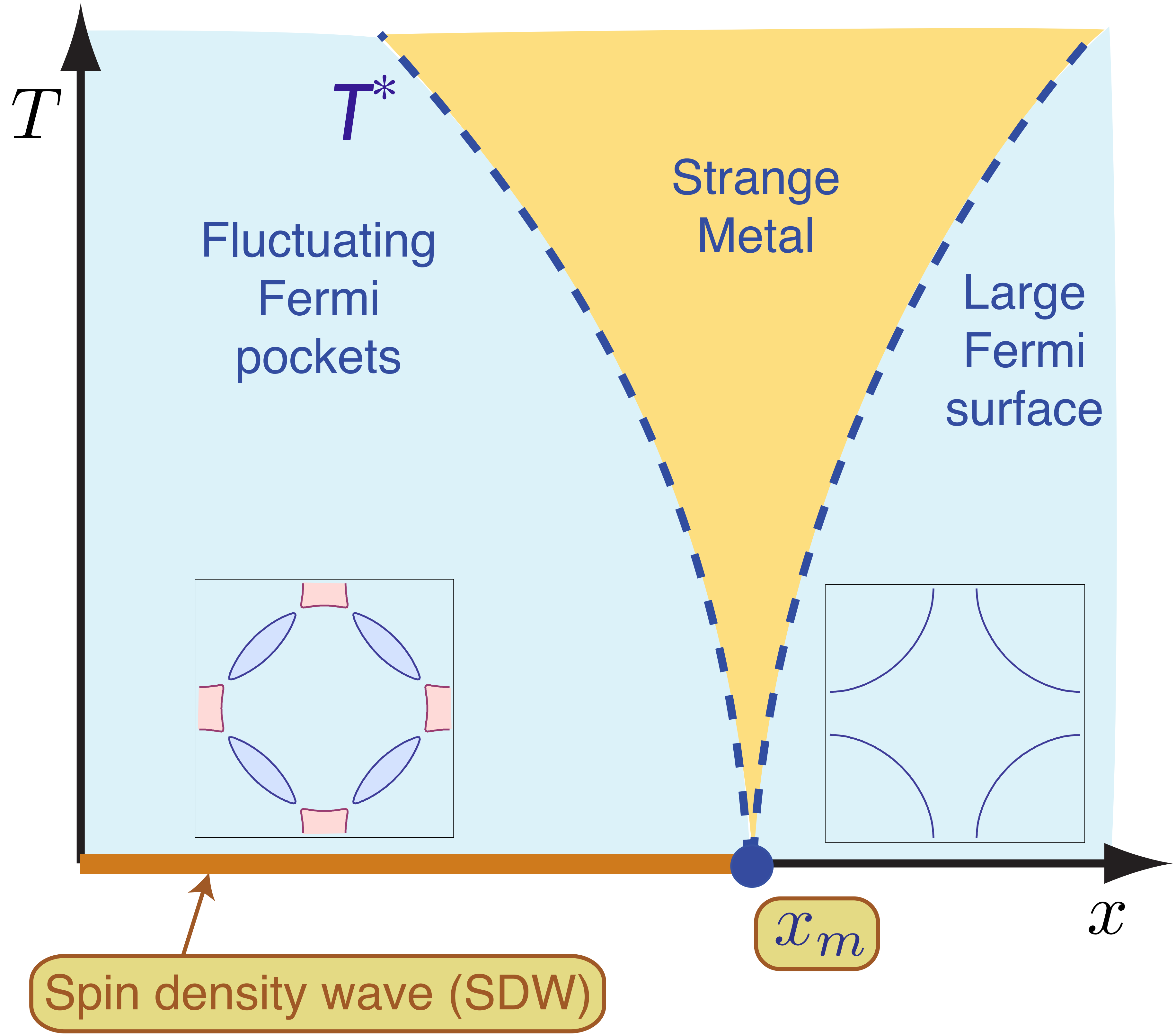

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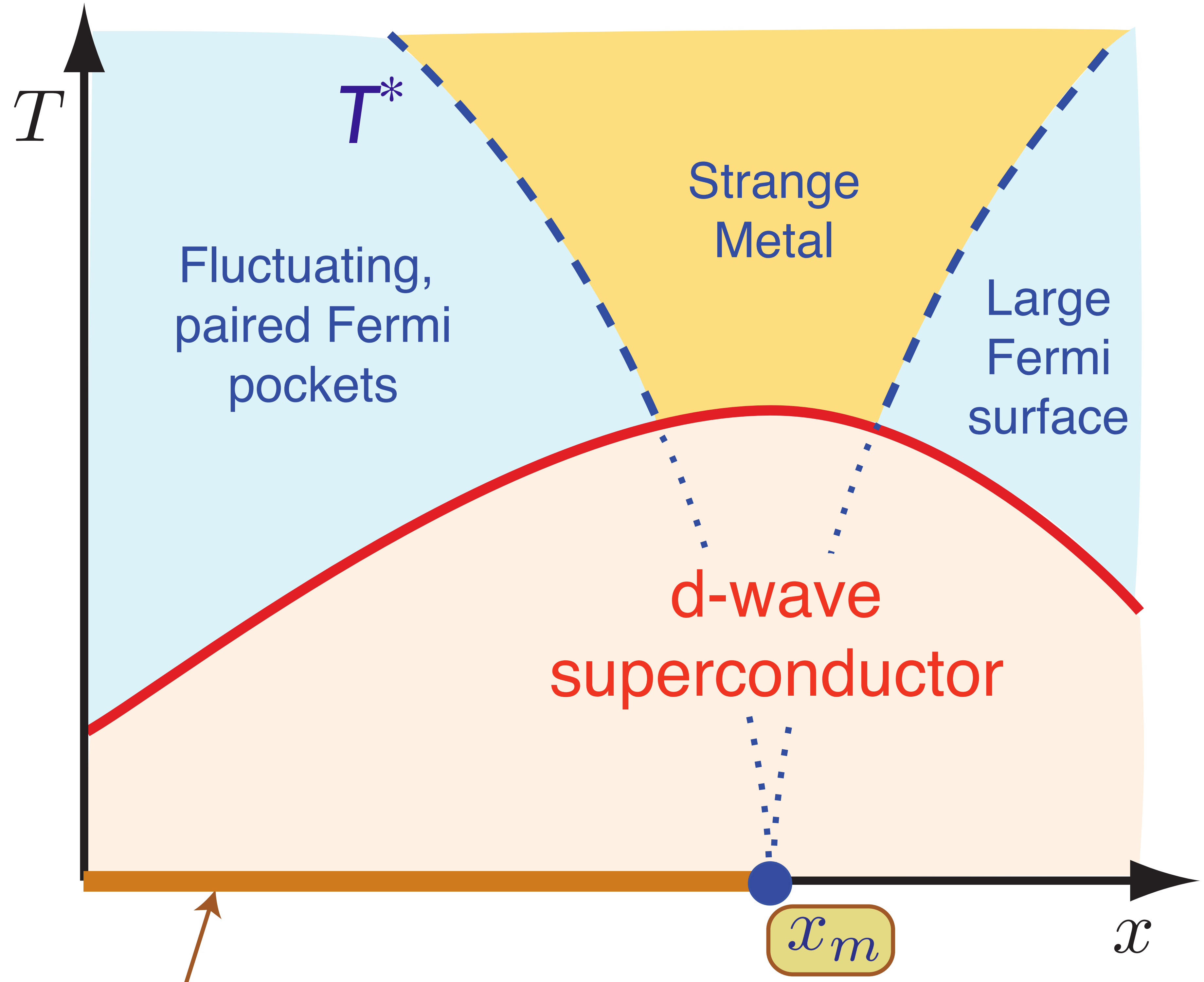
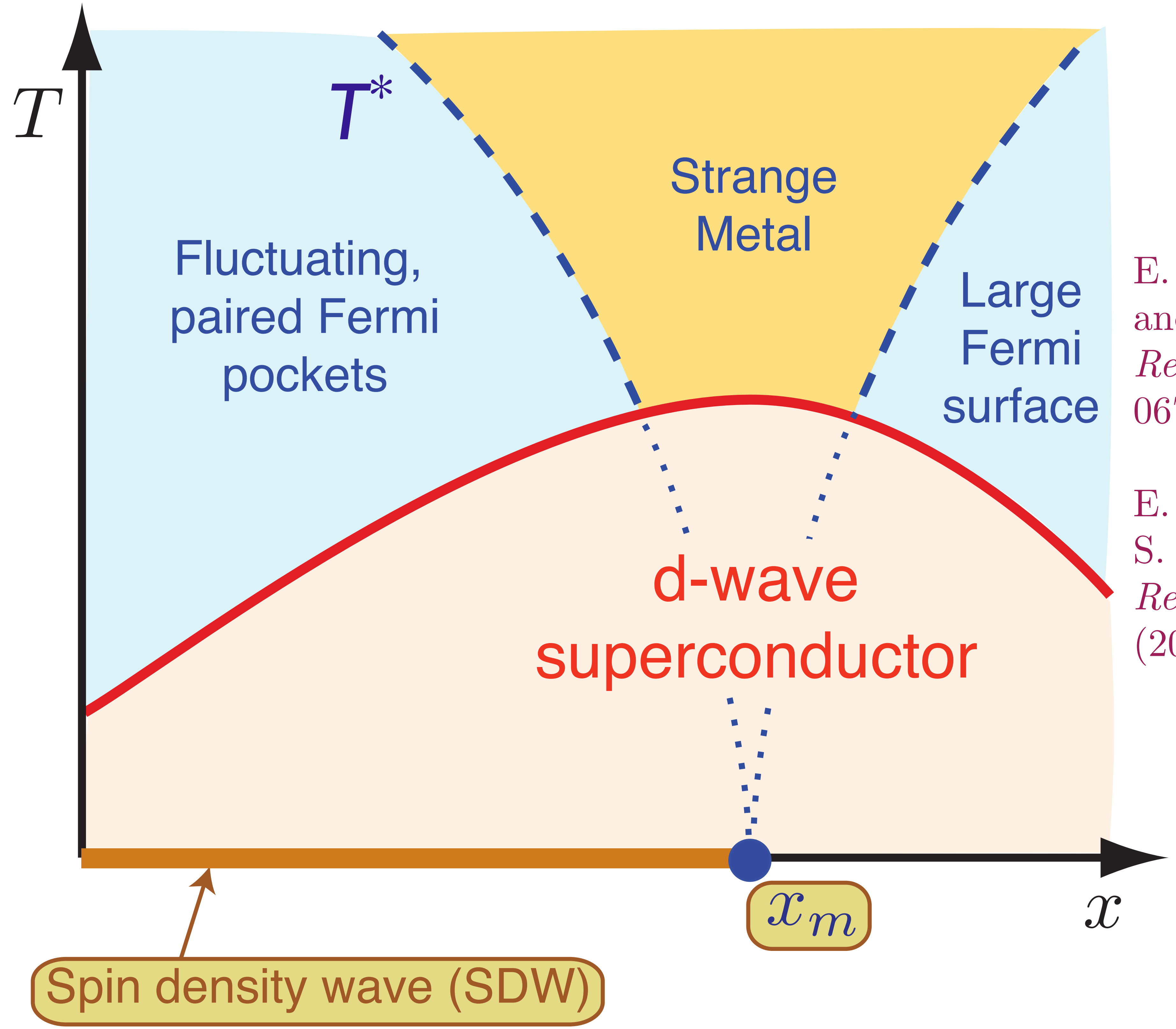


Spin density wave (SDW)

Onset of d-wave superconductivity hides the critical point $x = x_m$

Theory of quantum criticality in the cuprates



E. Demler, S. Sachdev and Y. Zhang, *Phys. Rev. Lett.* **87**, 067202 (2001).

E. G. Moon and S. Sachdev, *Phy. Rev. B* **80**, 035117 (2009)

Competition between SDW order and superconductivity moves the actual quantum critical point to $x = x_s < x_m$.

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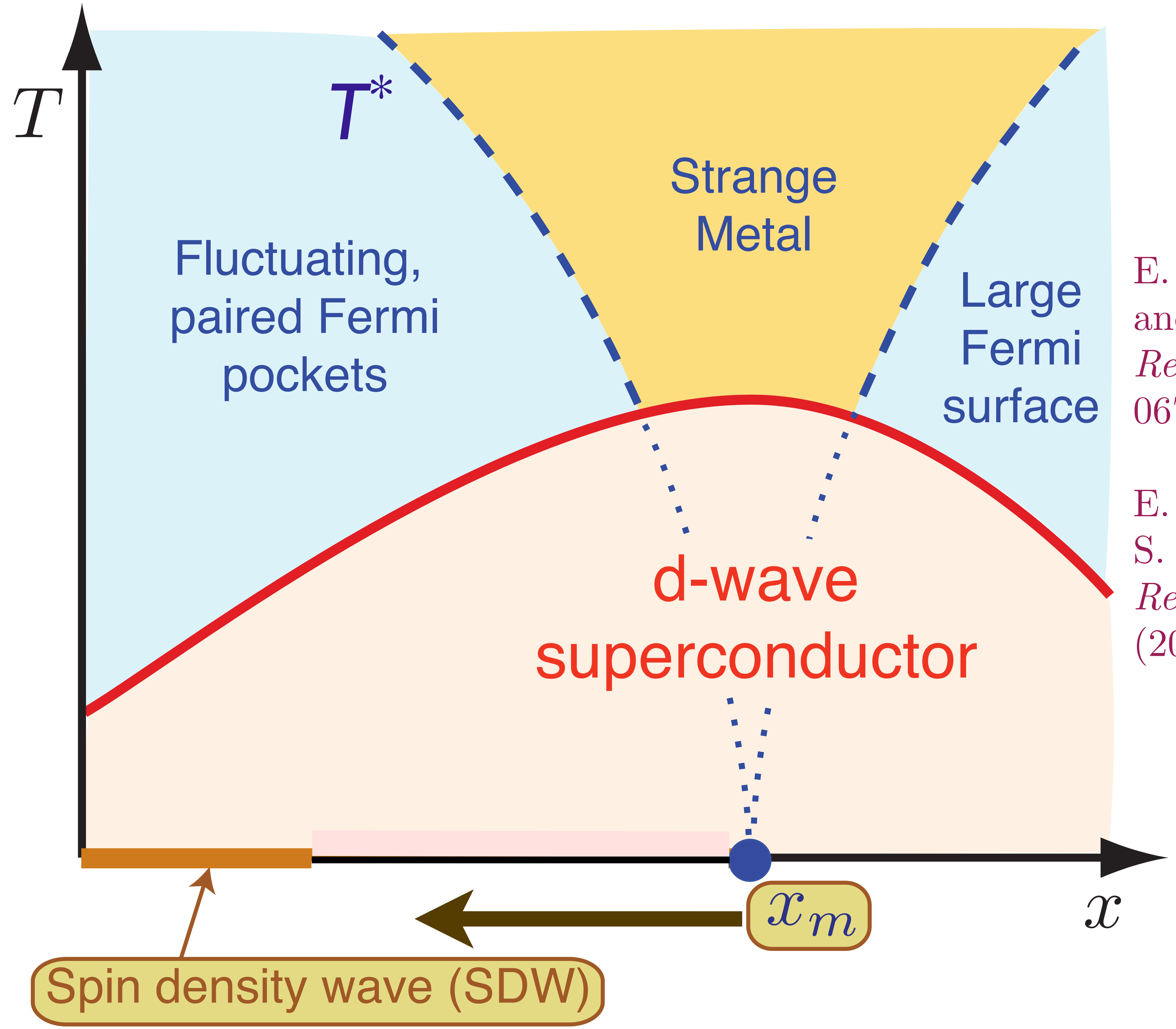

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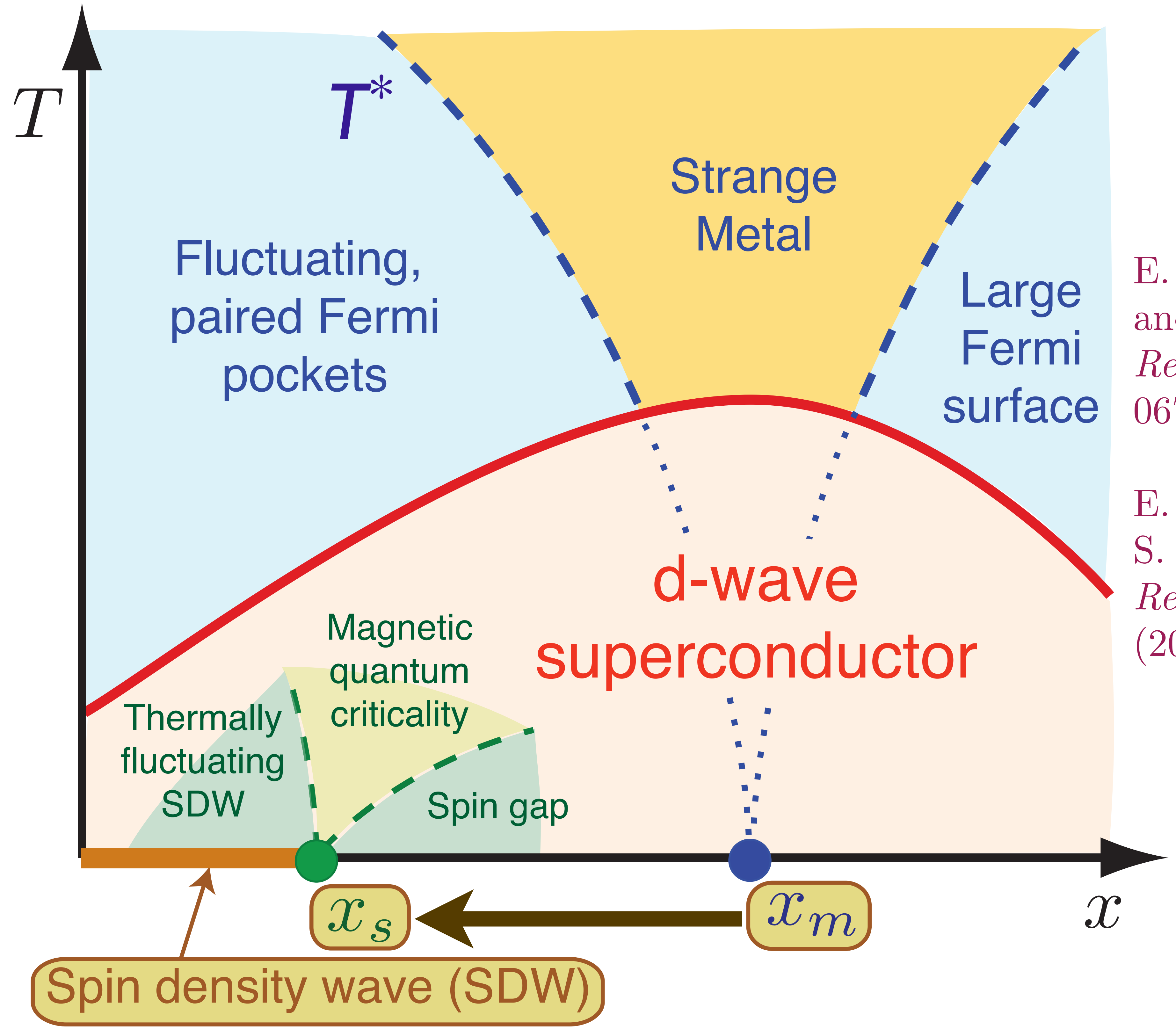

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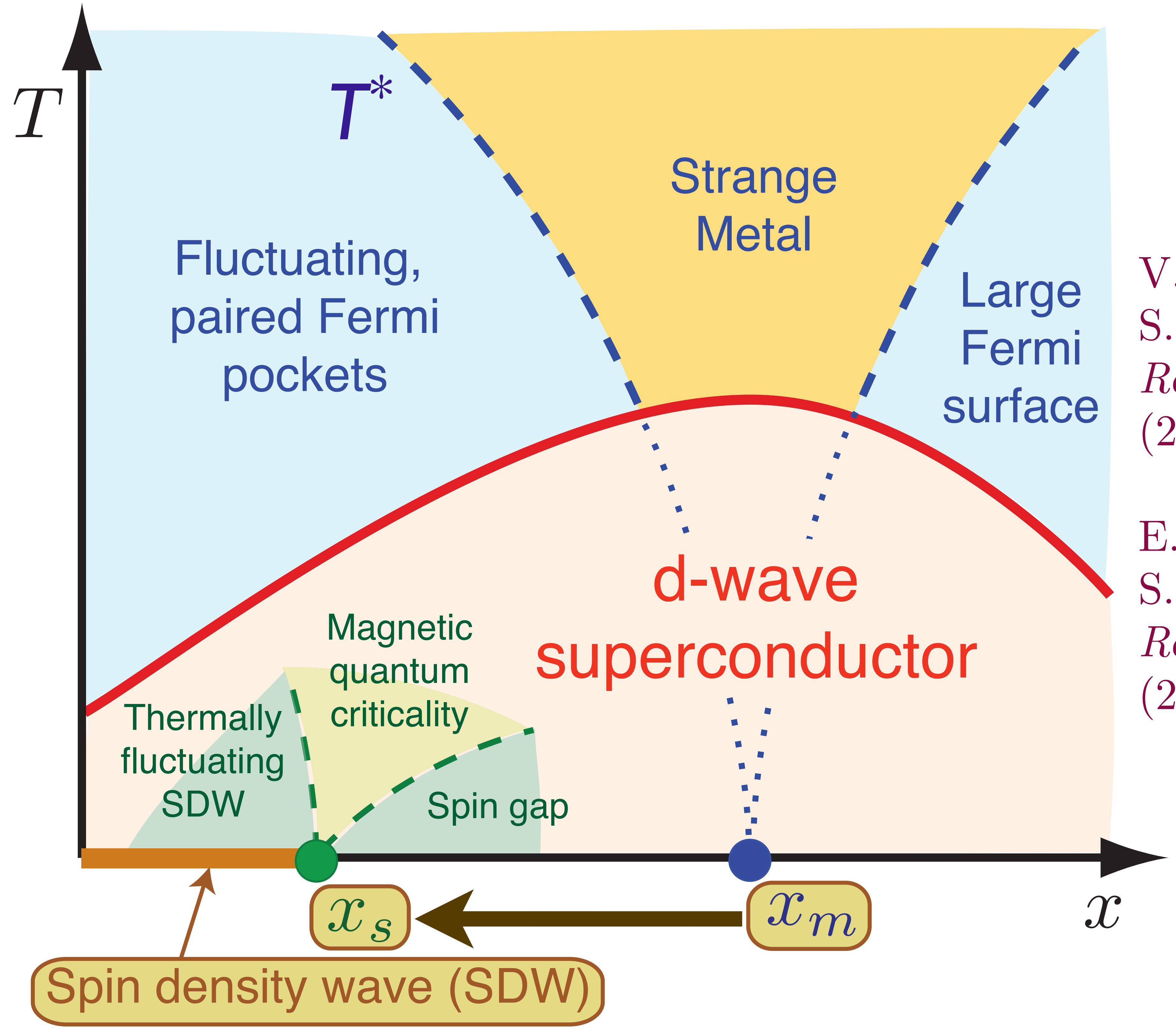

V. Galitski and S. Sachdev, *Phys. Rev.* B **79**, 134512 (2009).

E. G. Moon and S. Sachdev, *Phys. Rev. B* **80**, 035117 (2009)

Physics of competition: *d*-wave SC and SDW "eat up" same pieces of the large Fermi surface.

Lecture 3

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Phenomenological quantum theory of competing orders

Competition between superconductivity (SC) and spin-density wave (SDW) order

Write down a Landau-Ginzburg action for the quantum fluctuations of the SDW order ($\vec{\varphi}$) and superconductivity (Δ):

$$
\begin{aligned}
\mathcal{S} \quad = \quad & \int d^2r d\tau \left[\frac{1}{2} (\partial_\tau \vec{\varphi})^2 + \frac{c^2}{2} (\nabla_x \vec{\varphi})^2 + \frac{s}{2} \vec{\varphi}^2 + \frac{u}{4} \left(\vec{\varphi}^2 \right)^2 \right. \\
& \qquad \qquad \qquad \qquad \left. + \kappa \, \vec{\varphi}^2 \left| \Delta \right|^2 \right] \\
& + \int d^2 r \left[\left| (\nabla_x - i(2e/\hbar c) \mathcal{A}) \Delta \right|^2 - |\Delta|^2 + \frac{|\Delta|^4}{2} \right]
\end{aligned}
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where $\kappa > 0$ is the repulsion between the two order parameters, and $\nabla \times \mathcal{A} = H$ is the applied magnetic field.

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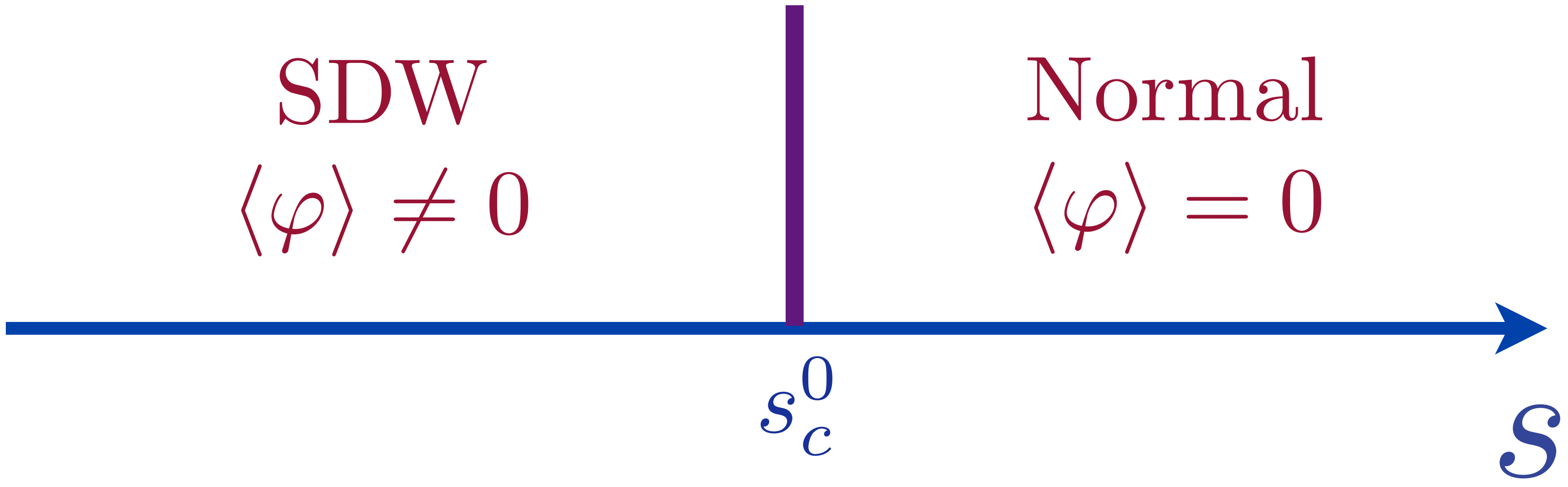

SDW

$\langle \varphi \rangle \neq 0$

Normal

$\langle \varphi \rangle = 0$

s_c^0

s

No SC

$\Delta = 0$

Phenomenological quantum theory of competing orders

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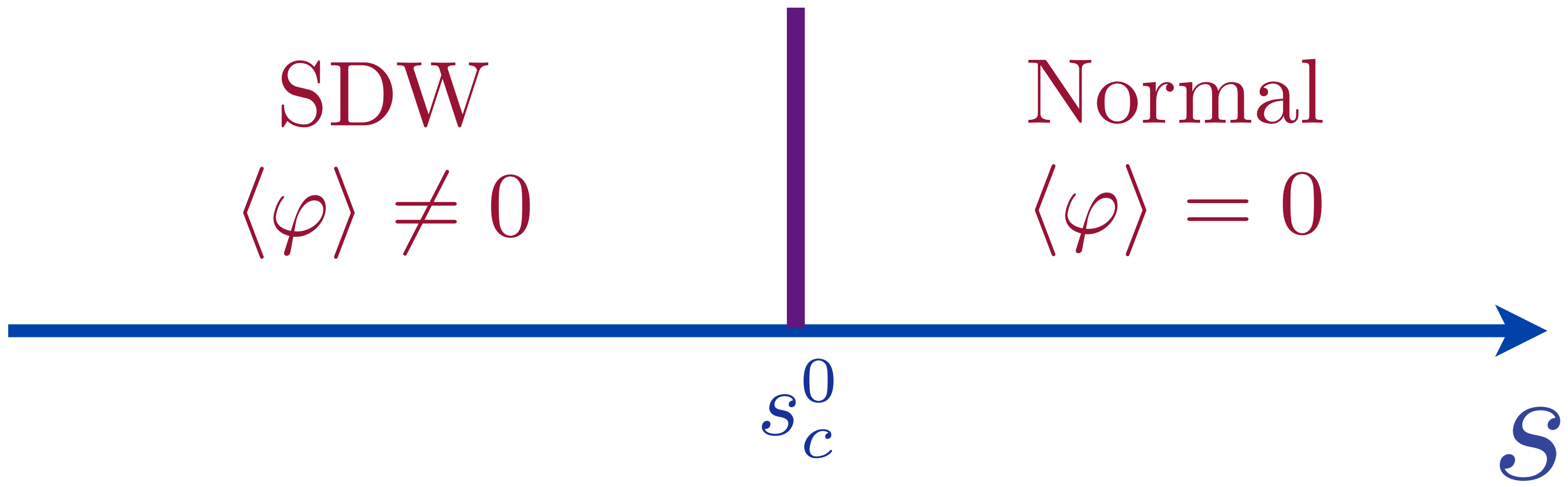

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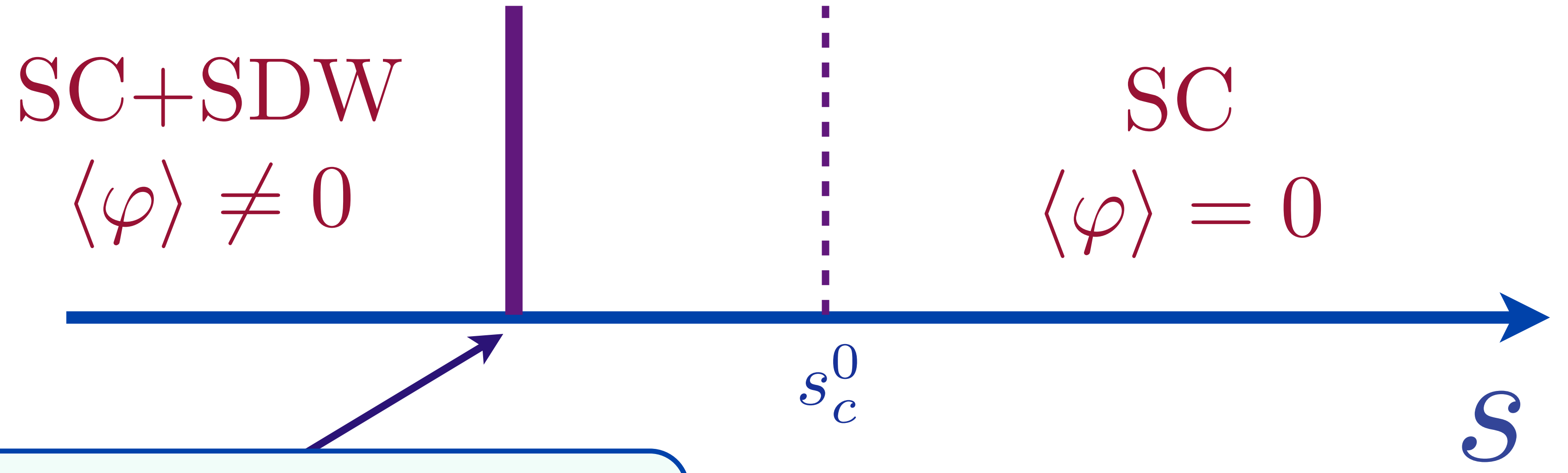

SC+SDW

$\langle \varphi \rangle \neq 0$

SC

$\langle \varphi \rangle = 0$

s_c^0

s

SC

$\Delta \neq 0$

$$s_c = s_c^0 - \kappa |\Delta|^2$$

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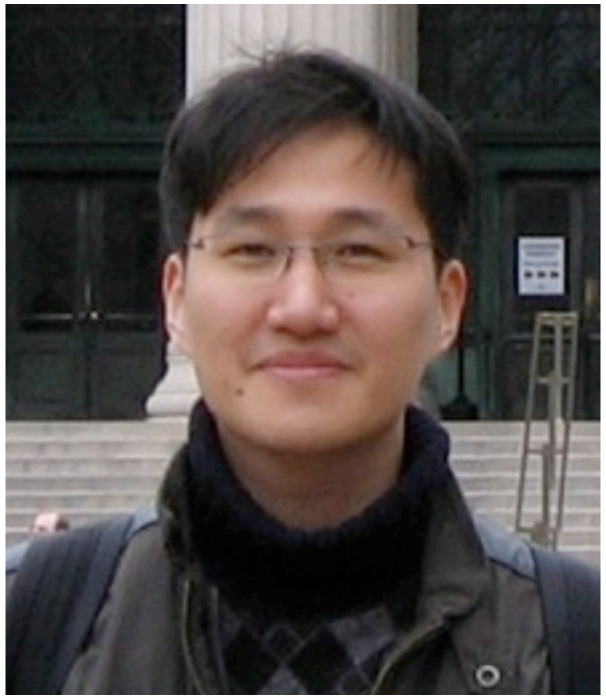
$$\mathcal{L}_f = \psi_{1\alpha}^{\ell\dagger}\left(\zeta\partial_\tau - i\mathbf{v}_1^\ell \cdot \boldsymbol{\nabla}_r\right)\psi_{1\alpha}^\ell + \psi_{2\alpha}^{\ell\dagger}\left(\zeta\partial_\tau - i\mathbf{v}_2^\ell \cdot \boldsymbol{\nabla}_r\right)\psi_{2\alpha}^\ell$$

Order parameter: $\mathcal{L}_\varphi = \frac{1}{2}\left(\boldsymbol{\nabla}_r\vec{\varphi}\right)^2 + \frac{\widetilde{\zeta}}{2}\left(\partial_\tau\vec{\varphi}\right)^2 + \frac{s}{2}\vec{\varphi}^2 + \frac{u}{4}\vec{\varphi}^4$

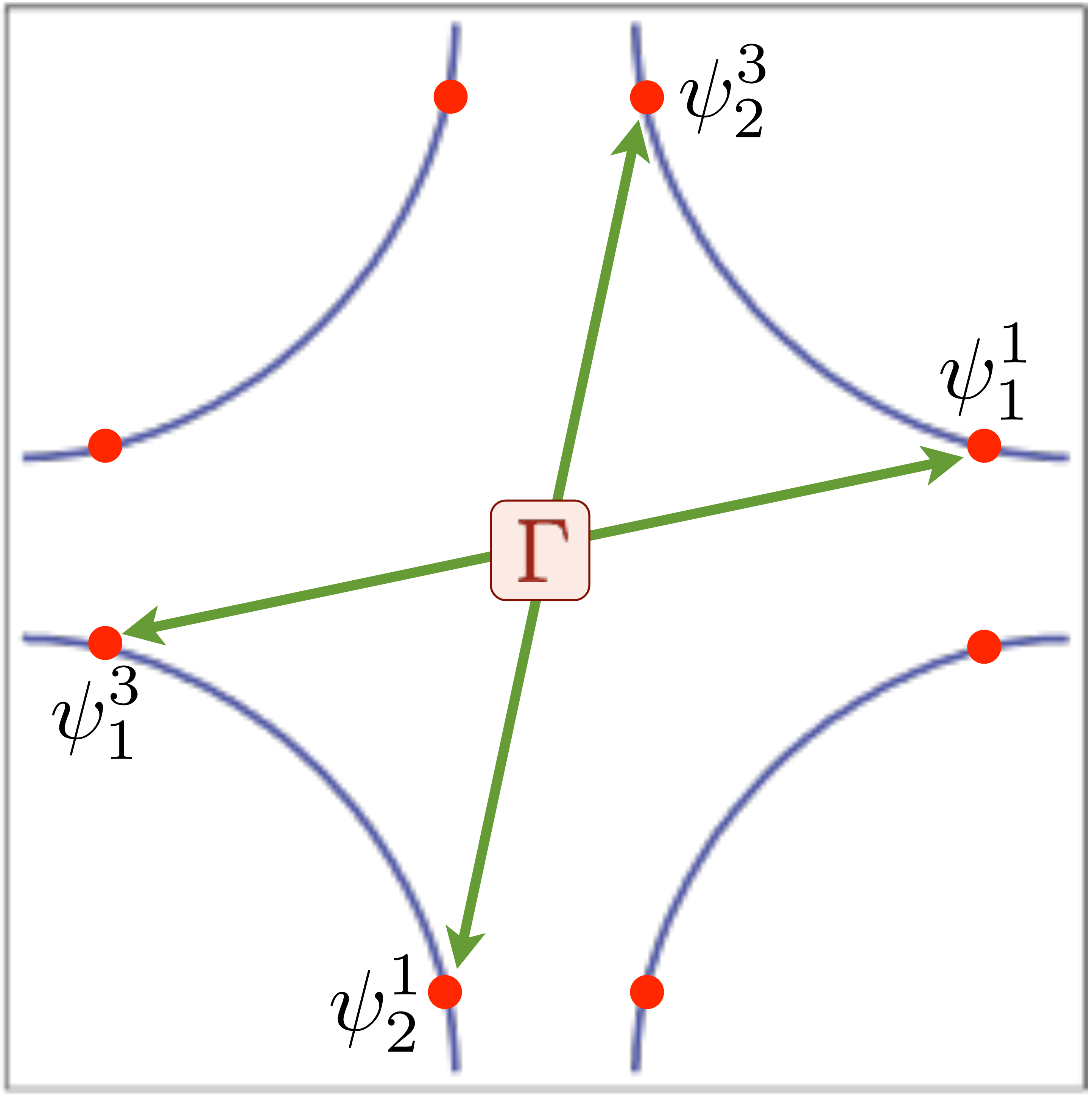
"Yukawa" coupling: $\mathcal{L}_c = -\lambda\vec{\varphi}\cdot\left(\psi_{1\alpha}^{\ell\dagger}\vec{\sigma}_{\alpha\beta}\psi_{2\beta}^\ell + \psi_{2\alpha}^{\ell\dagger}\vec{\sigma}_{\alpha\beta}\psi_{1\beta}^\ell\right)$

Return to theory of SDW ordering in a metal

Eun Gook Moon
and S. Sachdev,
arXiv:1005.3312



d-wave pairing in the theory of hotspots



Hot spots have strong instability to d-wave pairing near SDW critical point. This instability is stronger than the BCS instability of a Fermi liquid.

Pairing order parameter: $\varepsilon^{\alpha\beta}\left(\psi^3_{1\alpha}\psi^1_{1\beta} - \psi^3_{2\alpha}\psi^1_{2\beta}\right)$

$$\mathcal{L}_f = \psi^{\ell\dagger}_{1\alpha}\left(\zeta\partial_\tau - i\mathbf{v}^\ell_1\cdot\boldsymbol{\nabla}_r\right)\psi^\ell_{1\alpha} + \psi^{\ell\dagger}_{2\alpha}\left(\zeta\partial_\tau - i\mathbf{v}^\ell_2\cdot\boldsymbol{\nabla}_r\right)\psi^\ell_{2\alpha}$$

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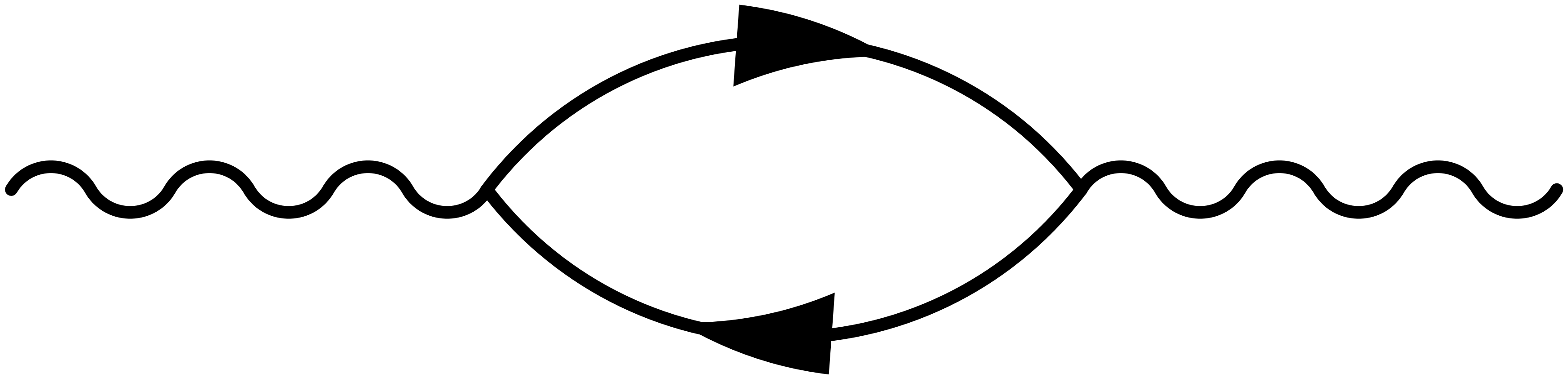
Pairing: $\mathcal{L}_\Delta = \Delta\varepsilon^{\alpha\beta}\left(\psi^3_{1\alpha}\psi^1_{1\beta} - \psi^3_{2\alpha}\psi^1_{2\beta}\right.$
$\left. -\psi^4_{1\alpha}\psi^2_{1\beta} - \psi^4_{2\alpha}\psi^2_{2\beta}\right) + \text{H.c.}$

Include the possibility of pairing in the metal.
And then compute the shift in the critical value
of the SDW transition, $s_c - s_c^0$ due to a non-zero Δ.

Shift in SDW order due to superconductivity in weak-coupling.

We integrate out the ψ fermions, and determine the correction to the coupling s. To leading order, this is given by the fermion contribution to the SDW susceptibility, χ, given by the following graph:



So we have for N_f hotspots $s_c - s_c^0 = N_f(\chi(\Delta) - \chi(0))$, where

$$\chi(\Delta) = 2\lambda^2 \int_{k,\omega} \frac{(\omega^2 - \varepsilon_1(k)\varepsilon_2(k) + \Delta^2)}{(\omega^2 + \varepsilon_1^2(k) + \Delta^2)(\omega^2 + \varepsilon_2^2(k) + \Delta^2)}, \qquad (1)$$

where $\varepsilon_i(k) = \vec{v}_i \cdot k$.

From this we find

$$s_c - s_c^0 \;=\; -N_f\ \lambda^2 \frac{C}{|\sin(\theta_1-\theta_2)|}\ \frac{|\Delta|}{v_{f1}v_{f2}}, \tag{2}$$

where the Fermi velocities are defined as

$$\vec{v_1} = v_{f1}(\cos(\theta_1),\sin(\theta_1))\ ,\ \ \vec{v_2} = v_{f2}(\cos(\theta_2),\sin(\theta_2))$$

and

$$\begin{aligned} C = \frac{1}{4\pi^2}\int dq_x dq_y \Bigg[\\ \frac{1}{|q_x|+|q_y|} - \frac{1}{\sqrt{q_x^2+1}+\sqrt{q_y^2+1}} \\ \times\left(1+\frac{1}{\sqrt{q_x^2+1}\sqrt{q_y^2+1}}\right)\Bigg]. \end{aligned} \tag{3}$$

Examination of this integral shows that the sign of C is not immediately evident. The small q region near the hot spots contributes a positive integrand (*i.e.* competition between SDW and SC), while the large q region away from the hot spots contributes a negative integrand (attraction between SDW and SC). Evaluation of the integral shows that

$$C = 0! \tag{4}$$

So there is exact cancellation in the present weak-coupling computation.

Extension of this computation to incommensurate order shows that $C > 0$, implying competition between orders.

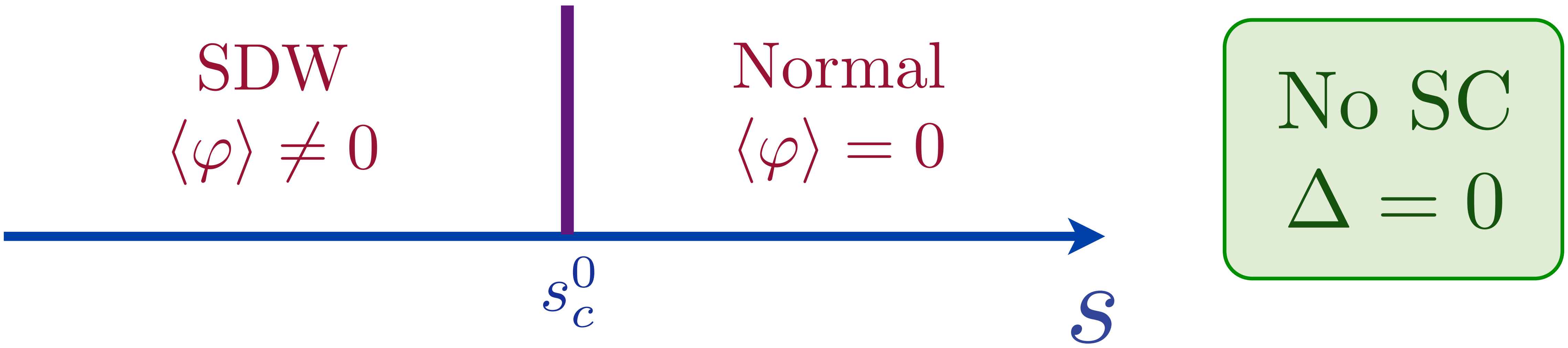
In general, the sign of $s_c - s_c^0$ will be determined by non-universal details. So weak-coupling theory allows both attraction and competition between SDW and SC.

In strong coupling, there is only competition between SDW and SC.

E. G. Moon and S. Sachdev, *Phy. Rev. B* **80**, 035117 (2009)

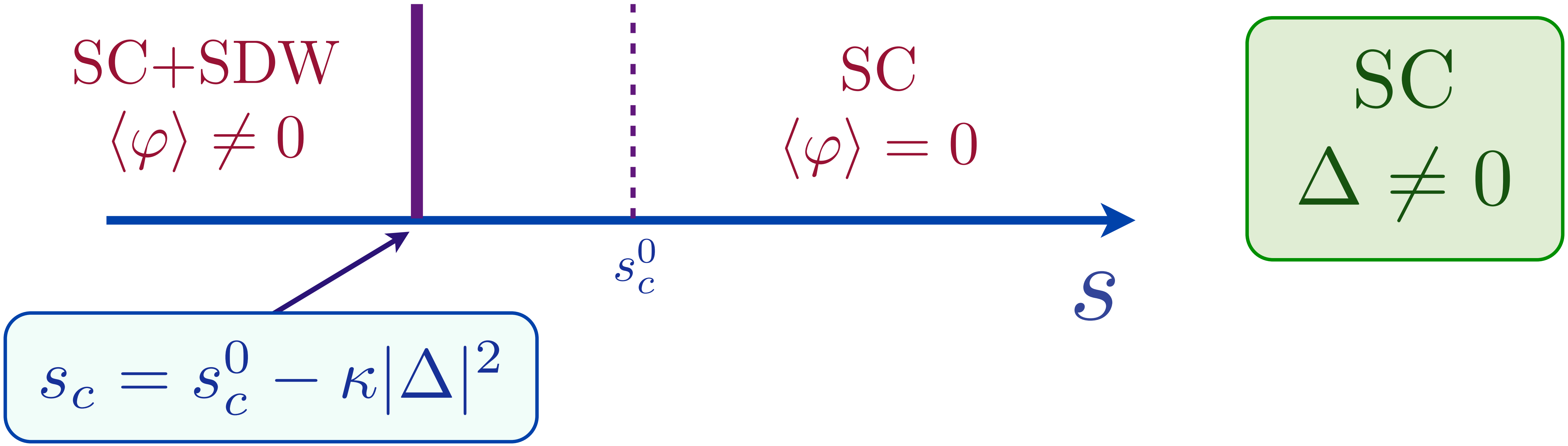
Phenomenological quantum theory of competing orders

Competition between superconductivity (SC) and spin-density wave (SDW) order



No SC

$\Delta = 0$

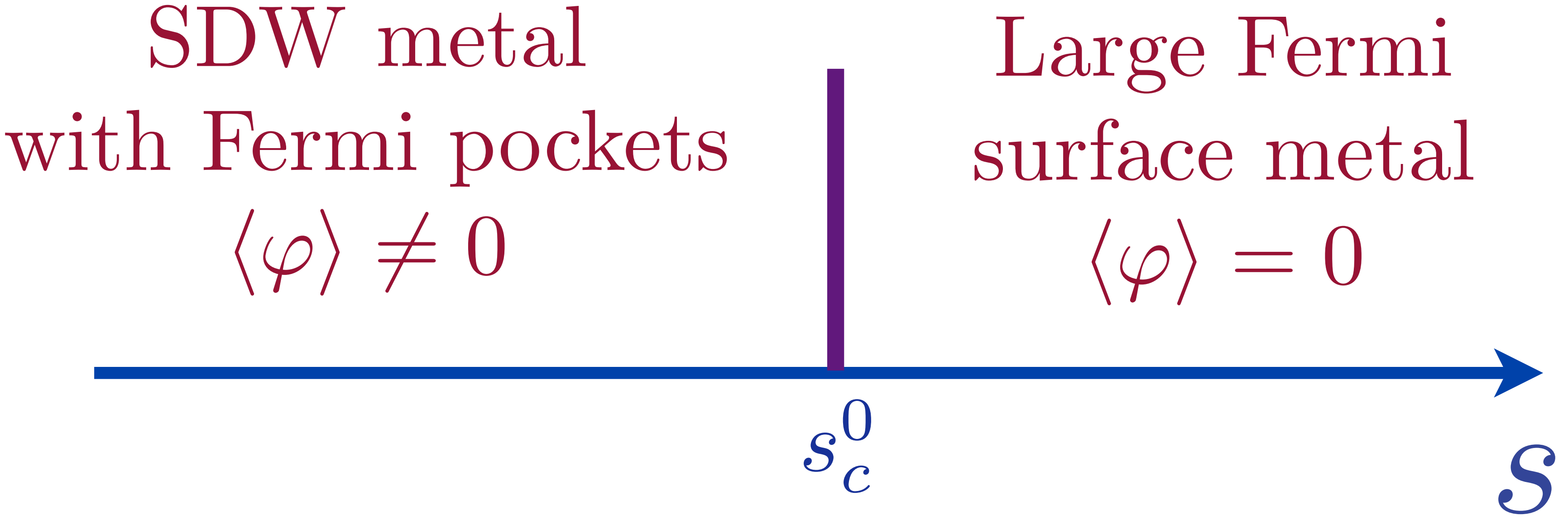


SC

$\Delta \neq 0$

$$s_c = s_c^0 - \kappa |\Delta|^2$$

Fermi surface theory of competing orders



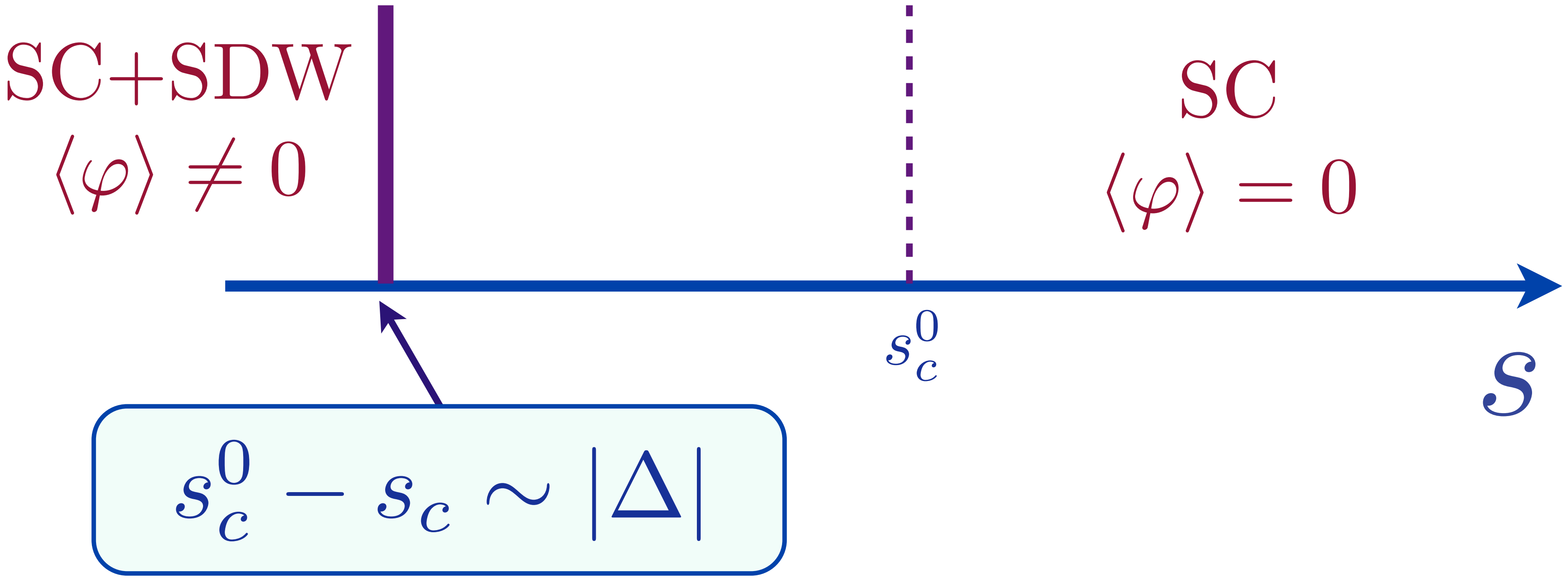
SDW metal
with Fermi pockets
$\langle\varphi\rangle \neq 0$

Large Fermi
surface metal
$\langle\varphi\rangle = 0$

s_c^0

s

No SC
$\Delta = 0$



SC+SDW
$\langle\varphi\rangle \neq 0$

SC
$\langle\varphi\rangle = 0$

s_c^0

s

$s_c^0 - s_c \sim |\Delta|$

SC
$\Delta \neq 0$

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Decouple quartic term by a Hubbard-Stratanovich field $\lambda(x,\tau)$, and integrate out the SDW quantum fluctuations represented by $\vec{\varphi}(x,\tau)$

$$\begin{aligned}
\mathcal{S}_{\rm eff}\left[\Delta(x), \lambda(x,\tau)\right] &= \int d^2 r d\tau \frac{\lambda^2}{u} \\
&+ \frac{3}{2}\mathrm{Tr}\ln\left[-\frac{1}{2}\partial_\tau^2 - \frac{c^2}{2}\nabla_x^2 + \frac{s}{2} + \kappa|\Delta|^2 + i\lambda\right] \\
&+ \int d^2 r\left[|(\nabla_x - i(2e/\hbar c)\mathcal{A})\Delta|^2 - |\Delta|^2 + \frac{|\Delta|^4}{2}\right]
\end{aligned}$$

Solve the saddle point equations

$$\frac{\delta \mathcal{S}_{\rm eff}}{\delta \Delta} = 0 \quad , \quad \frac{\delta \mathcal{S}_{\rm eff}}{\delta \lambda} = 0$$

to determine the optimal values of $\Delta(x)$ and $i\lambda(x)$. These will have the same symmetry as that of a Abrikosov vortex lattice. The solutions yield the phase diagram as a function of s and $\nabla \times \mathcal{A} = H$.

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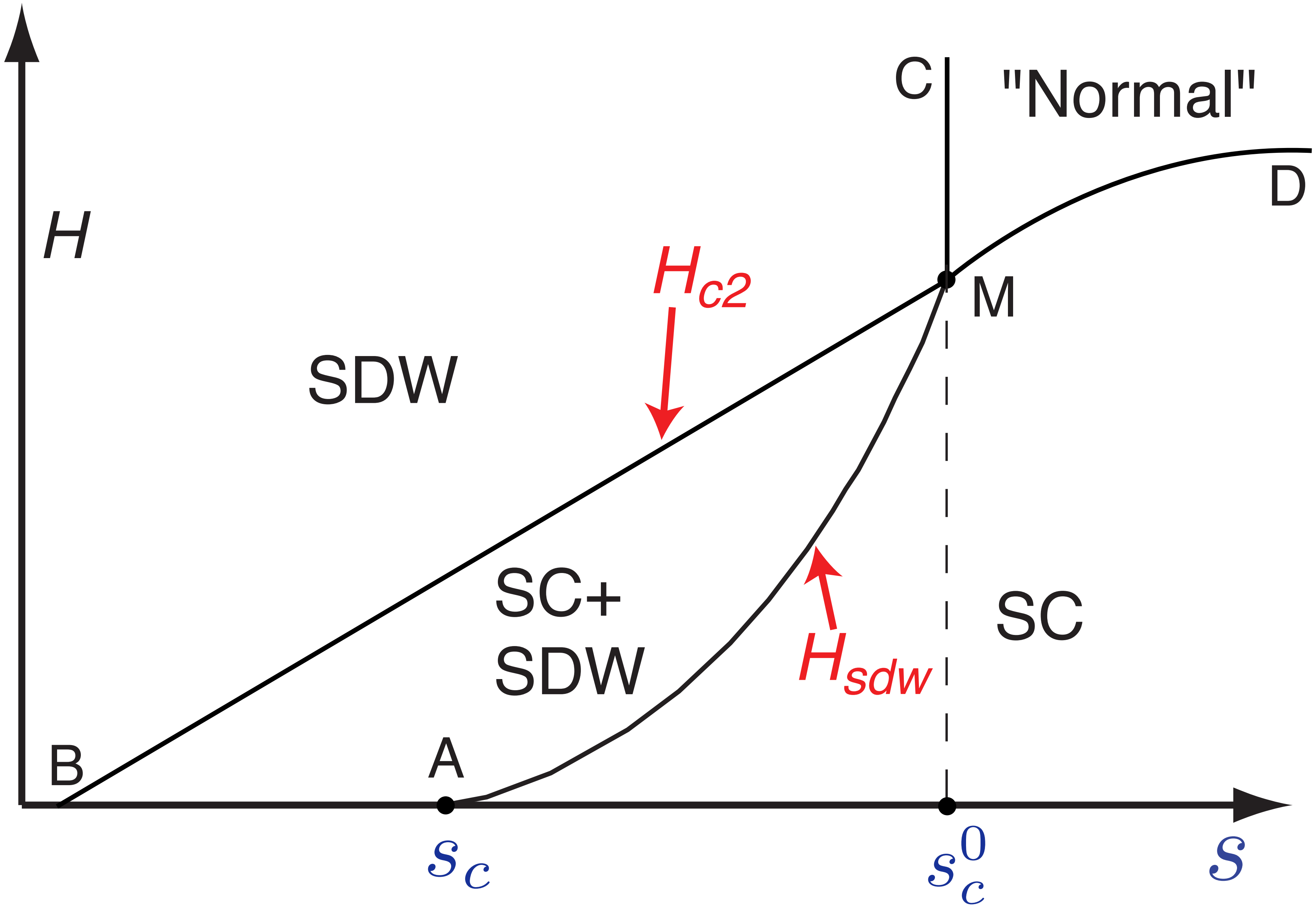

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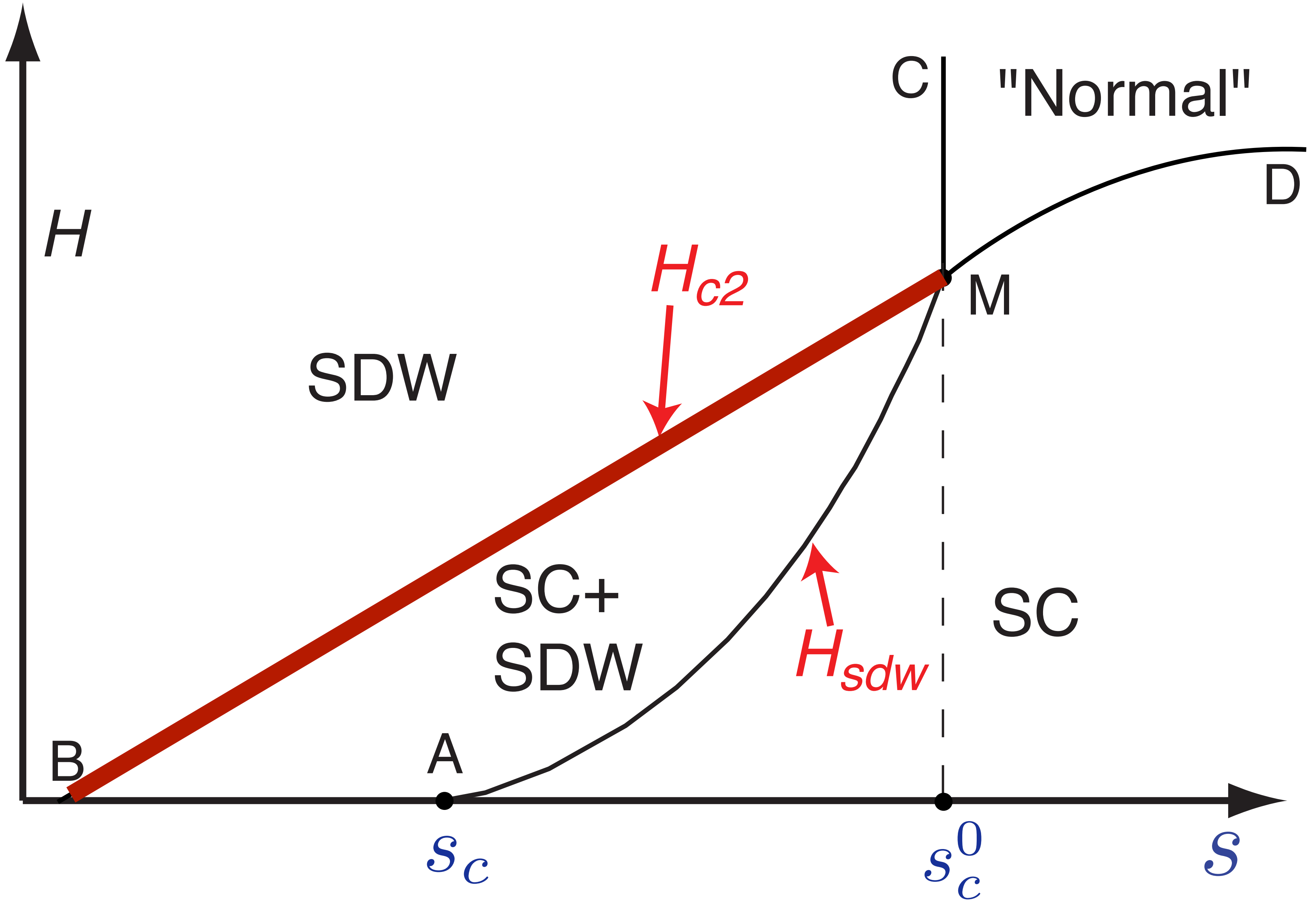
Phenomenological quantum theory of competing orders

Competition between superconductivity (SC) and spin-density wave (SDW) order



- Upper-critical field, H_{c2}, decreases as SDW is enhanced with decreasing doping (s)

E. Demler, S. Sachdev and Y. Zhang, *Phys. Rev. Lett.* **87**, 067202 (2001).

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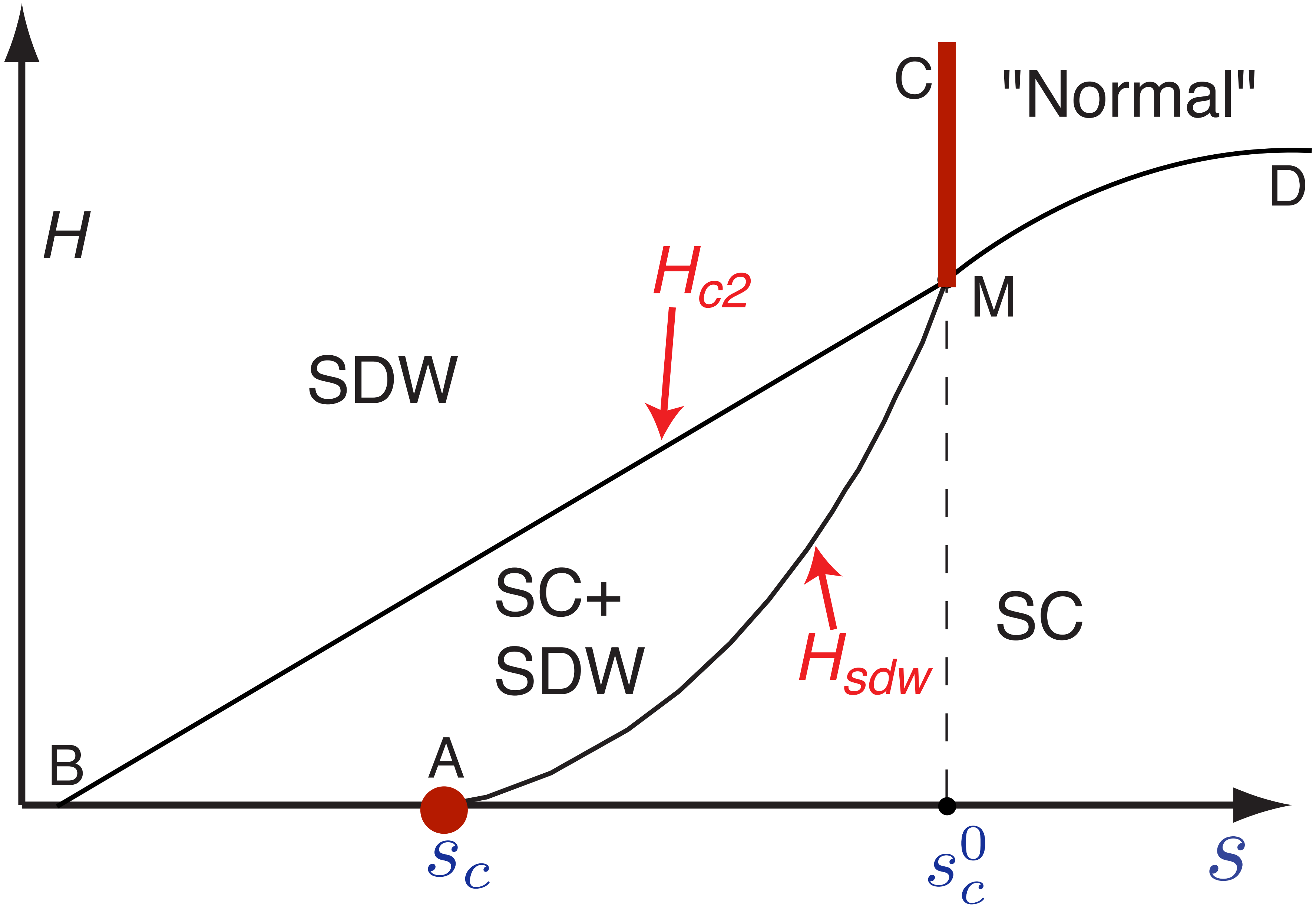

- SDW order is more stable in the metal than in the superconductor: $s_c < s_c^0$.

E. Demler, S. Sachdev and Y. Zhang, *Phys. Rev. Lett.* **87**, 067202 (2001).

Phenomenological quantum theory of competing orders

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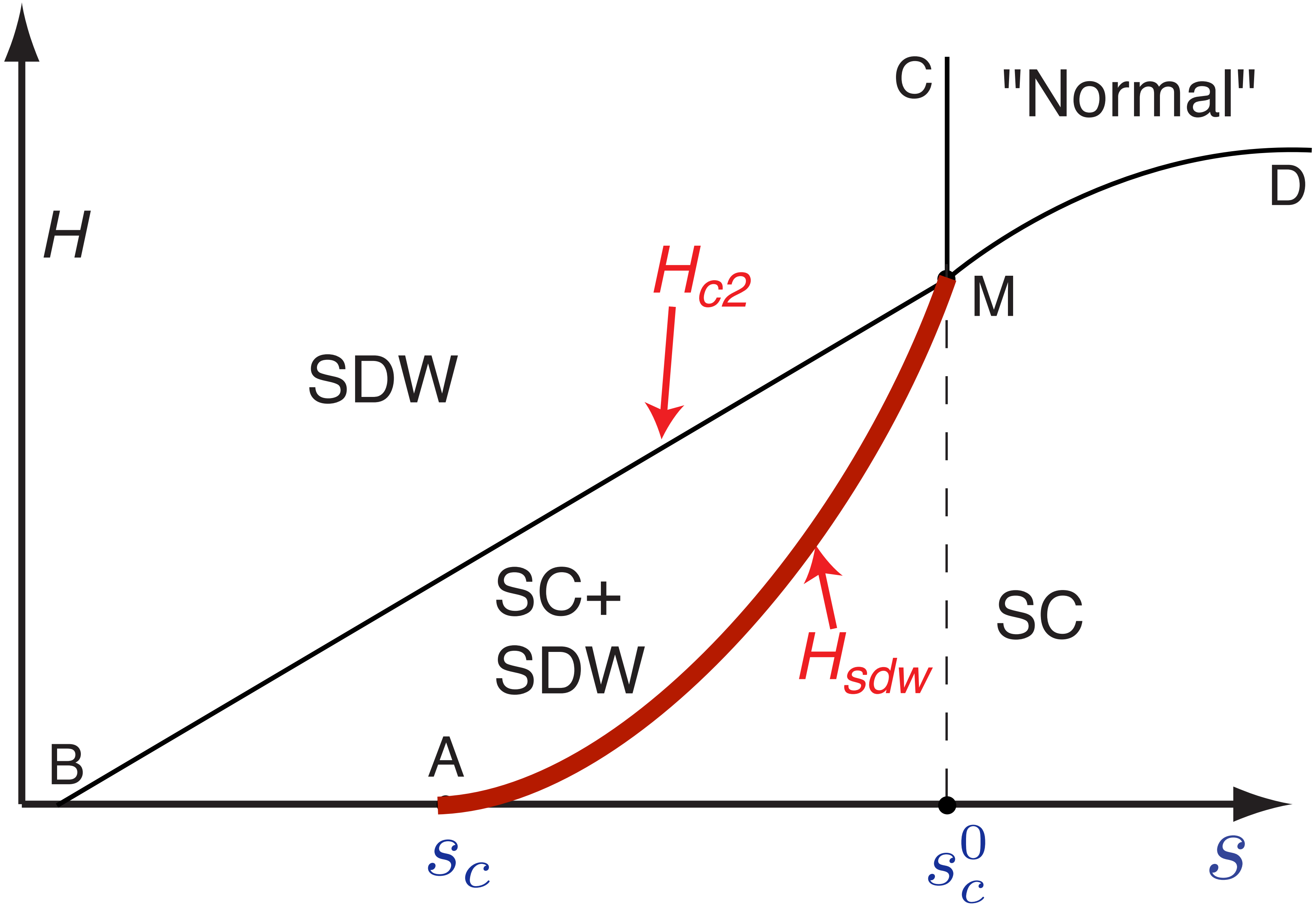
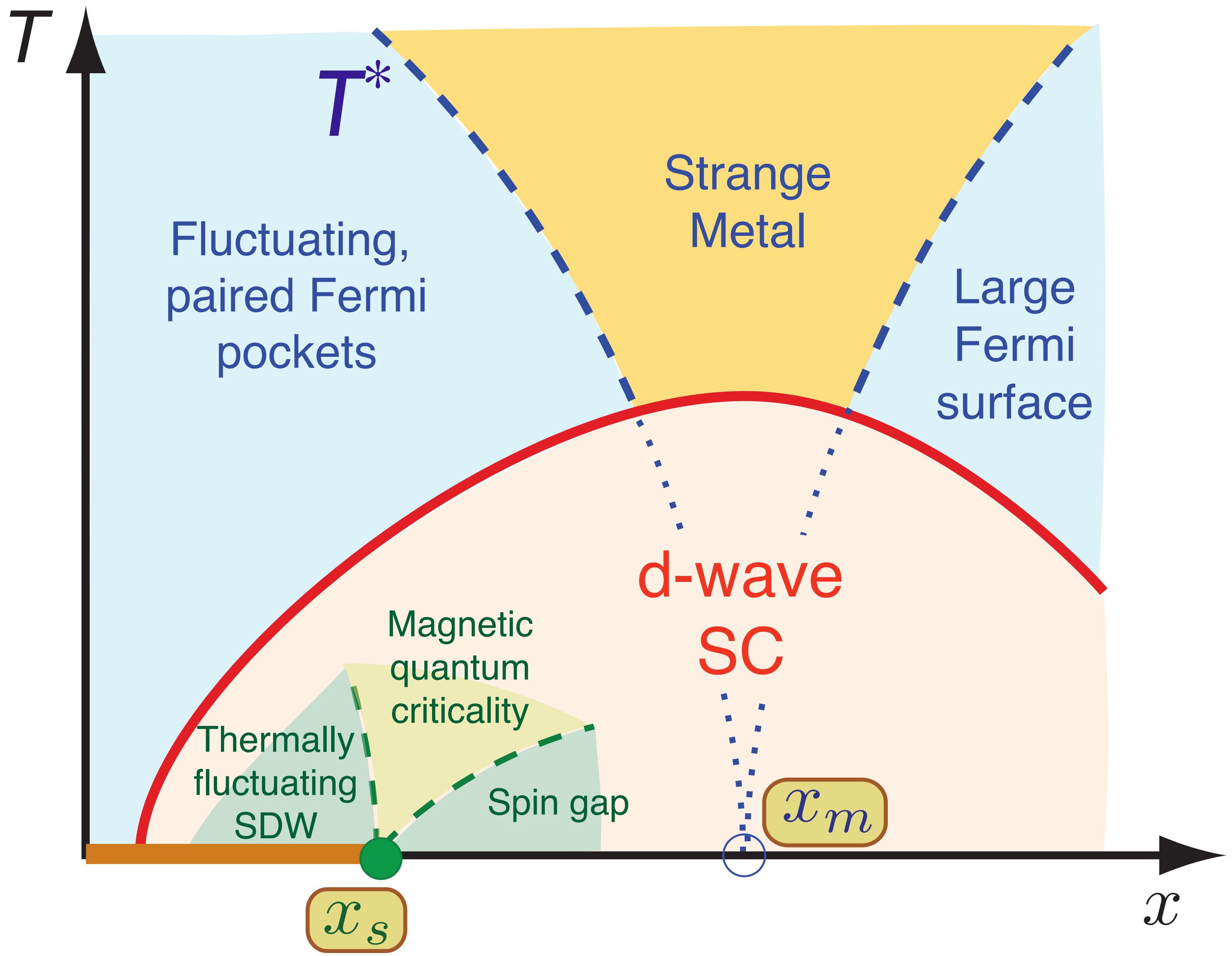
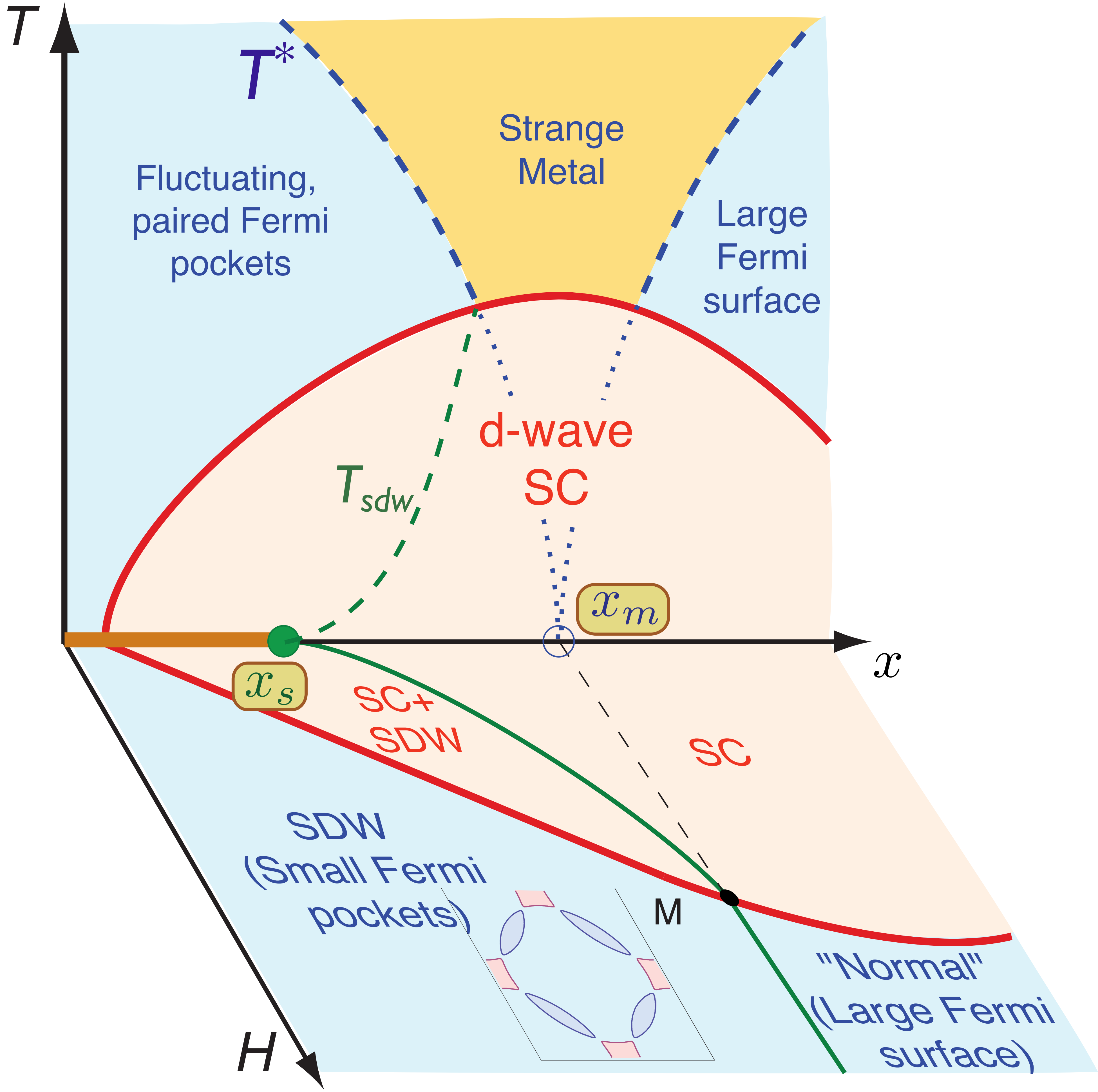
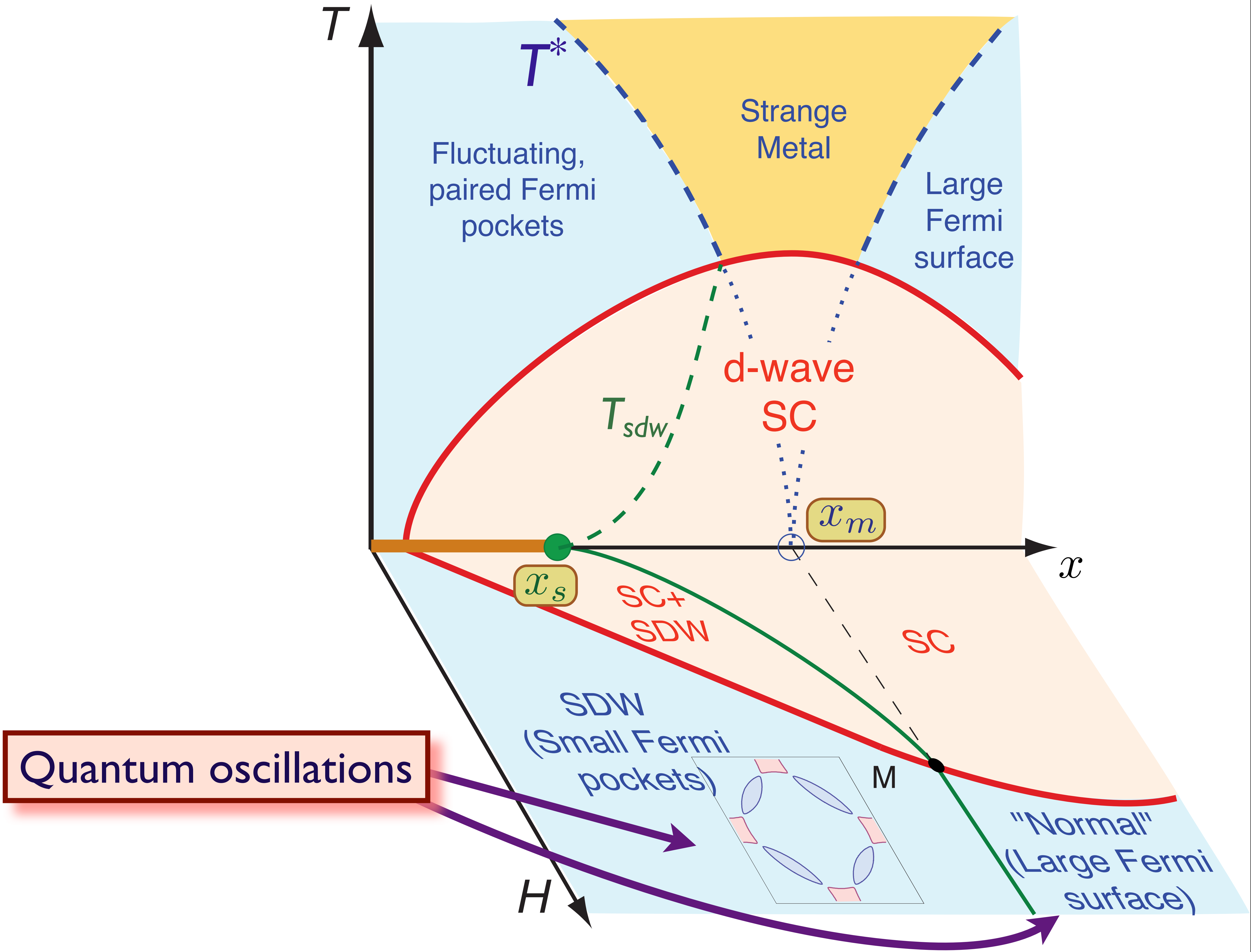


- For doping with $s_c < s < s_c^0$, SDW order appears at a quantum phase transition at $H = H_{sdw} > 0$.

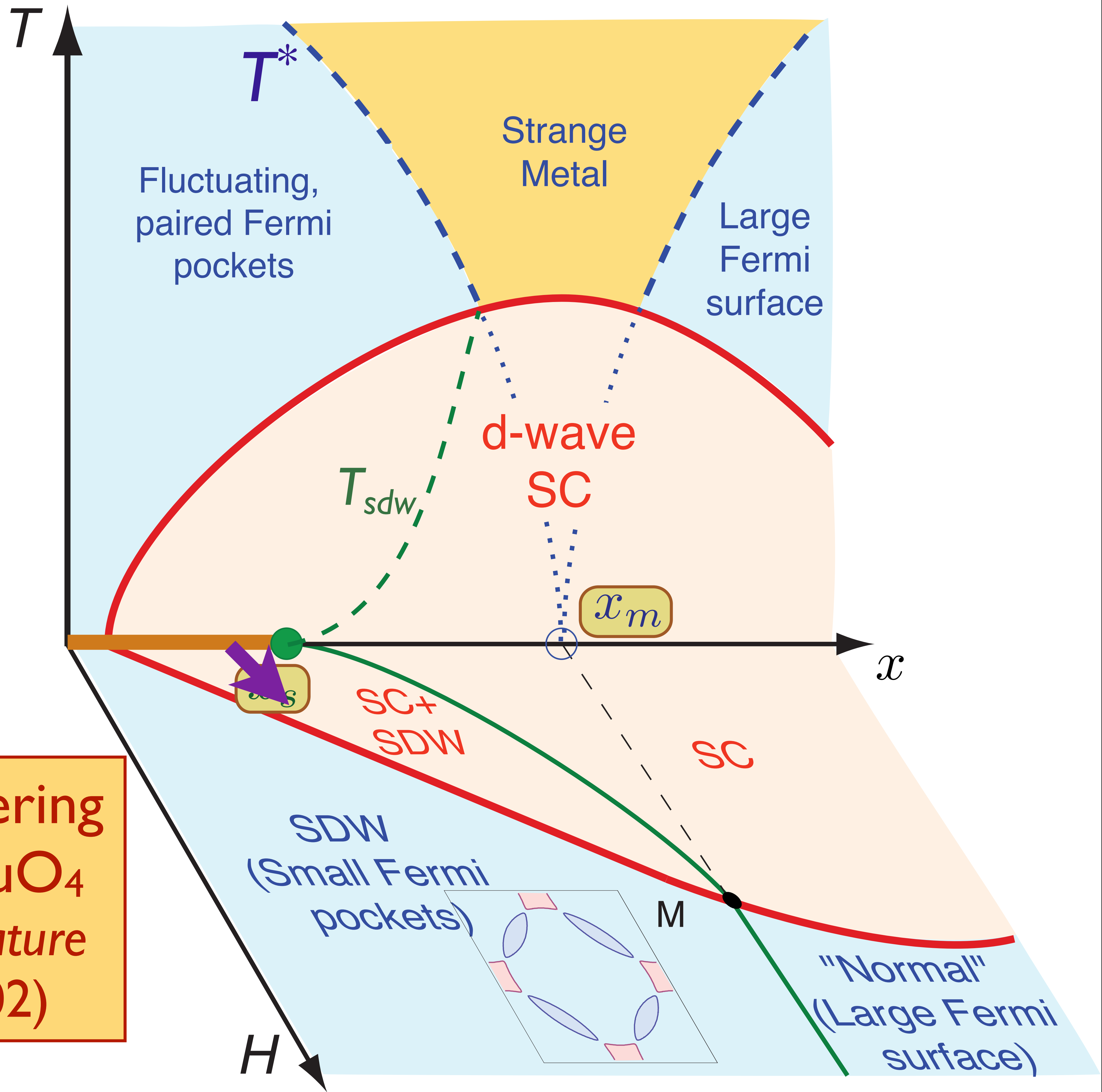
E. Demler, S. Sachdev and Y. Zhang, *Phys. Rev. Lett.* **87**, 067202 (2001).


T
T^*
Fluctuating, paired Fermi pockets
Strange Metal
Large Fermi surface
d-wave SC
Magnetic quantum criticality
Thermally fluctuating SDW
Spin gap
x_m
x_s
x

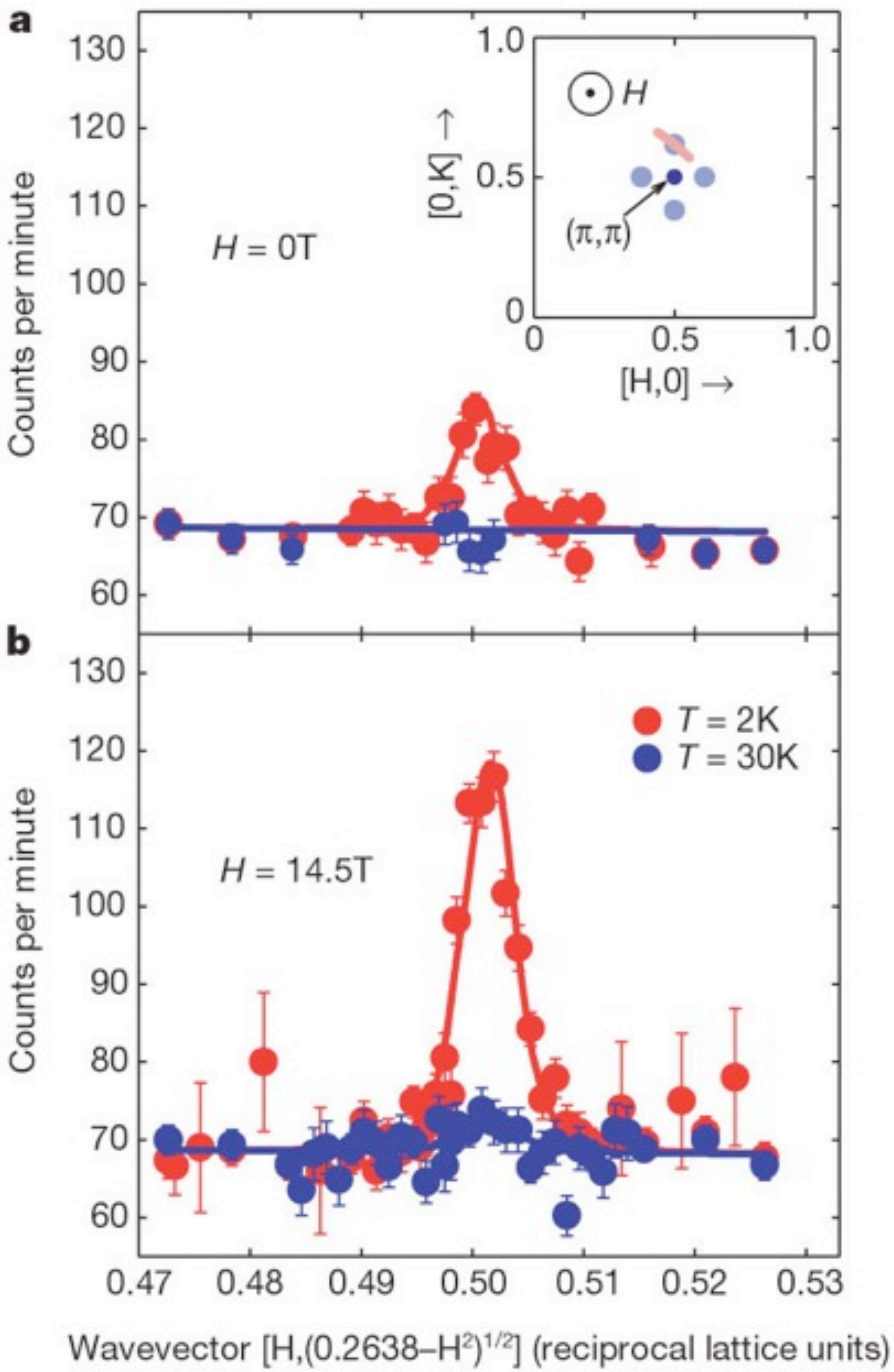

T
T^*
Fluctuating, paired Fermi pockets
Strange Metal
Large Fermi surface
d-wave SC
T_{sdw}
x_m
x
x_s
SC+ SDW
SC
SDW (Small Fermi pockets)
M
"Normal" (Large Fermi surface)
H

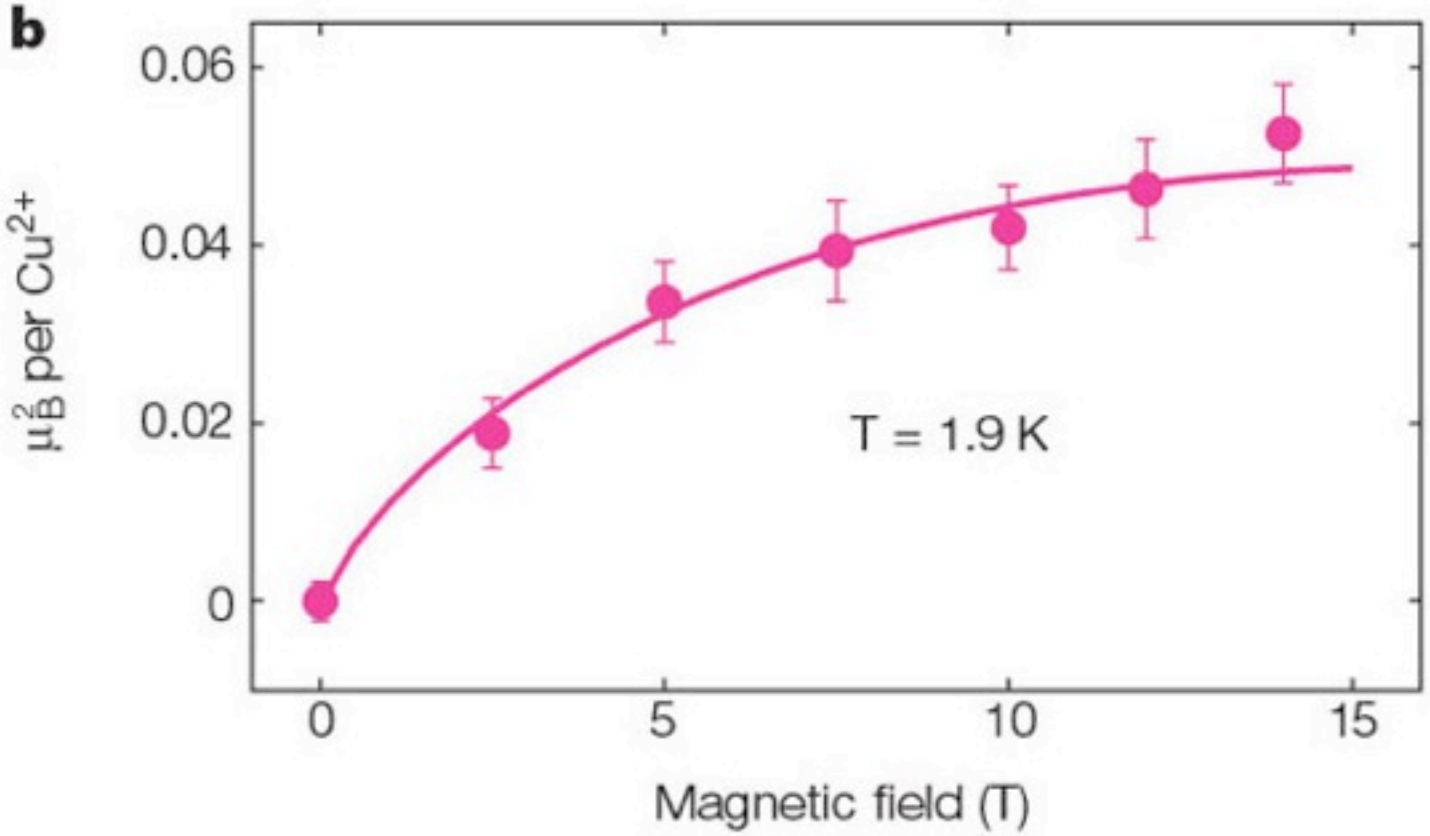


Quantum oscillations



Neutron scattering on $La_{1.9}Sr_{0.1}CuO_4$
B. Lake *et al.*, *Nature*
415, 299 (2002)





B. Lake, H. M. Rønnow, N. B. Christensen,
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N. Mangkorntong, T. Sasagawa, M. Nohara,
H. Takagi, and T. E. Mason,
Nature **415**, 299 (2002)

B. Lake, G. Aeppli, K. N. Clausen, D. F. McMorrow,
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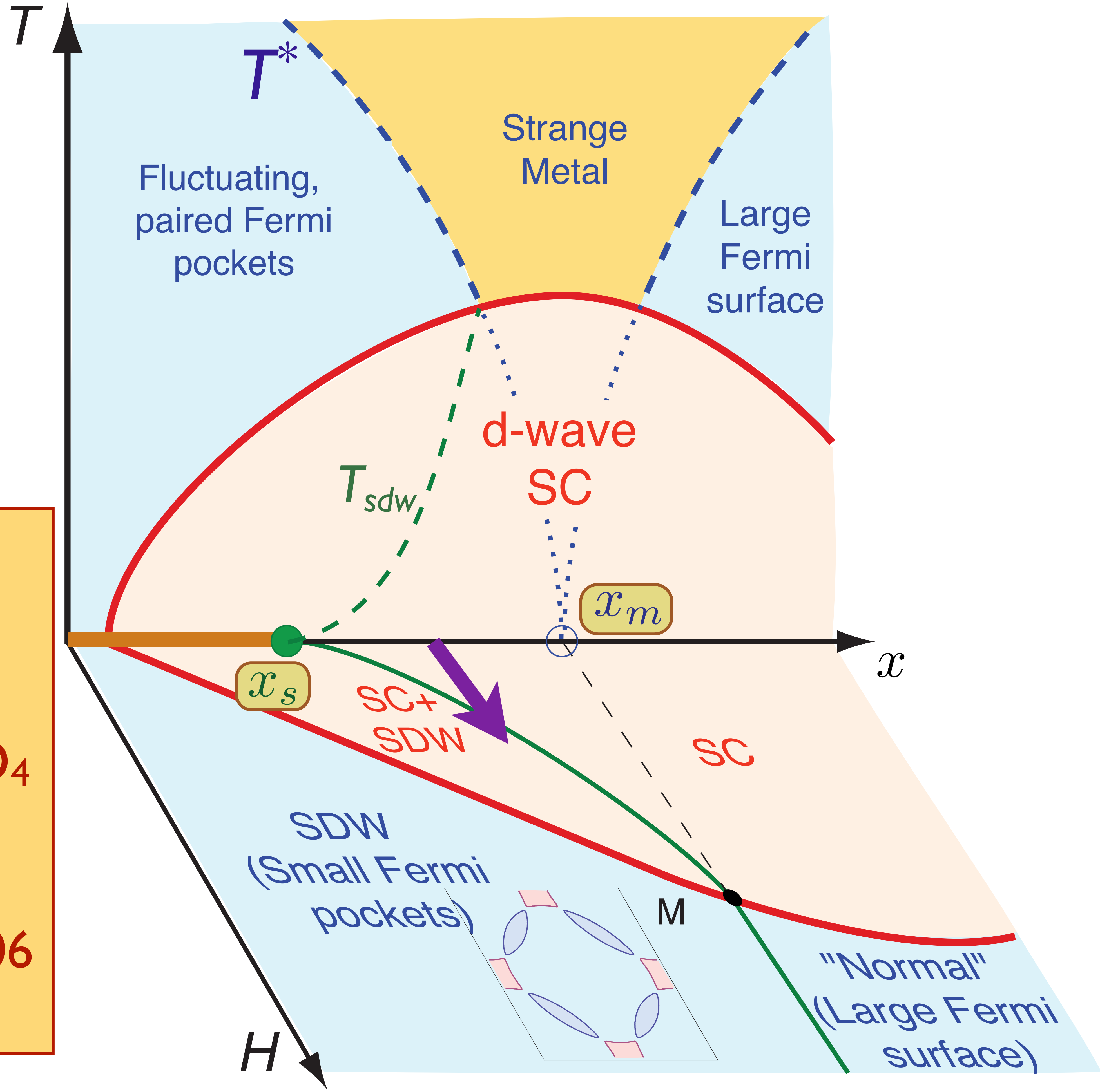
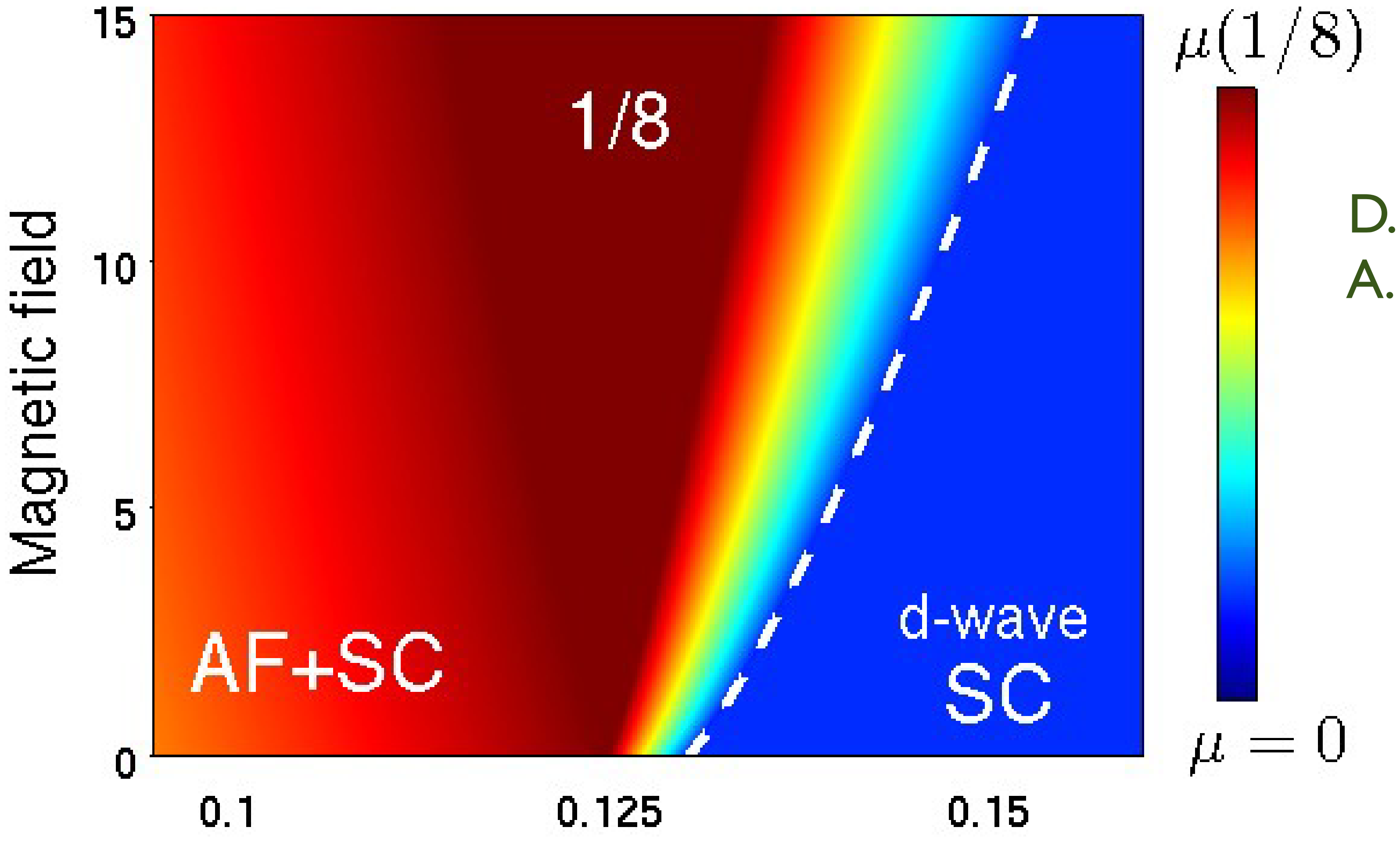
Science **291**, 1759 (2001).

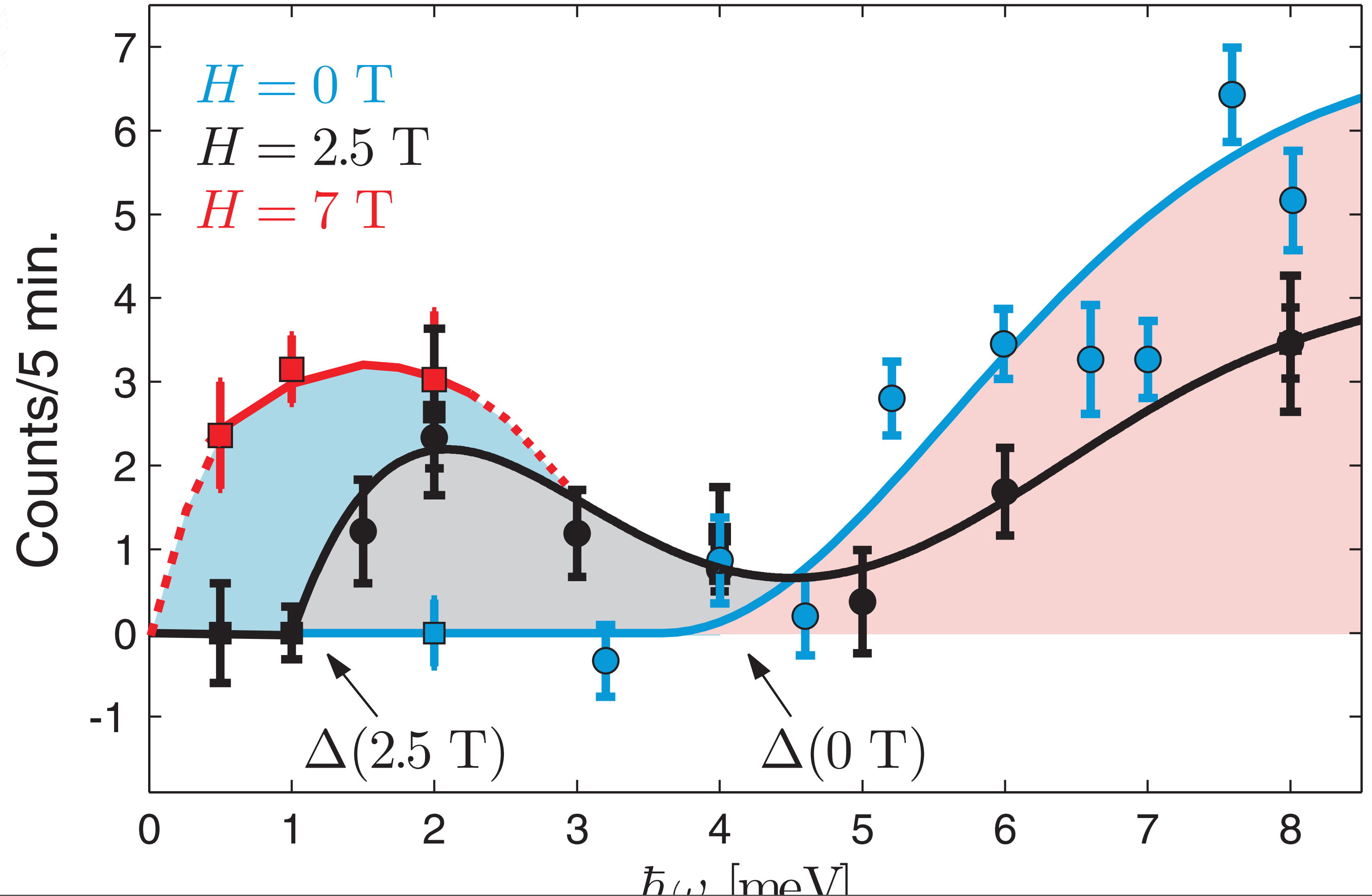


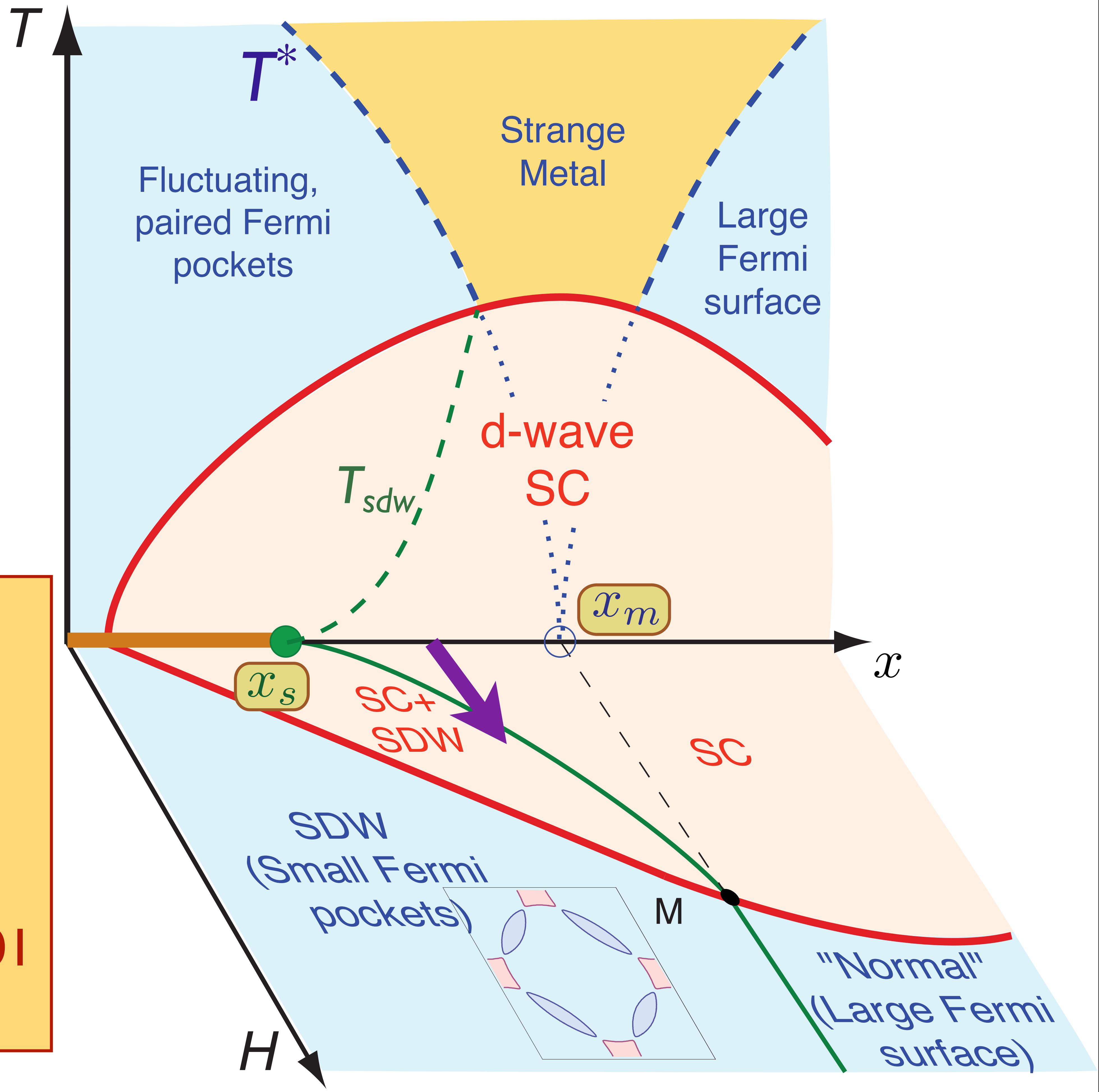
Neutron scattering and μSR on $La_{1.855}Sr_{0.145}CuO_4$
J. Chang *et al.*, *Physical Review Letters* **102**, 177006 (2009)



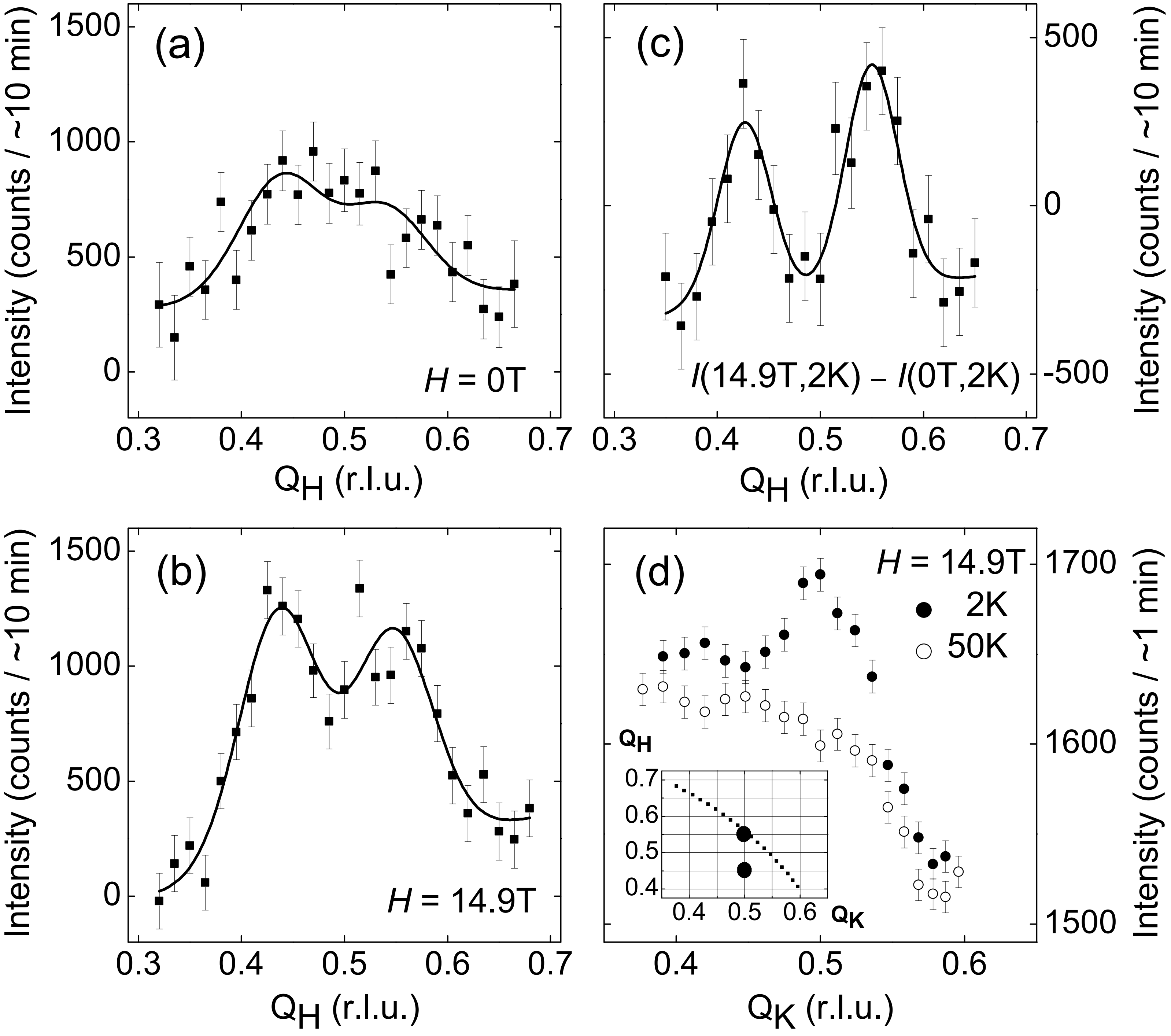
J. Chang, Ch. Niedermayer, R. Gilardi, N.B. Christensen, H.M. Ronnow, D.F. McMorrow, M. Ay, J. Stahn, O. Sobolev, A. Hiess, S. Pailhes, C. Baines, N. Momono, M. Oda, M. Ido, and J. Mesot, *Physical Review B* **78**, 104525 (2008).

J. Chang, N. B. Christensen, Ch. Niedermayer, K. Lefmann, H. M. Roennow, D. F. McMorrow, A. Schneidewind, P. Link, A. Hiess, M. Boehm, R. Mottl, S. Pailhes, N. Momono, M. Oda, M. Ido, and J. Mesot, *Phys. Rev. Lett.* **102**, 177006 (2009).

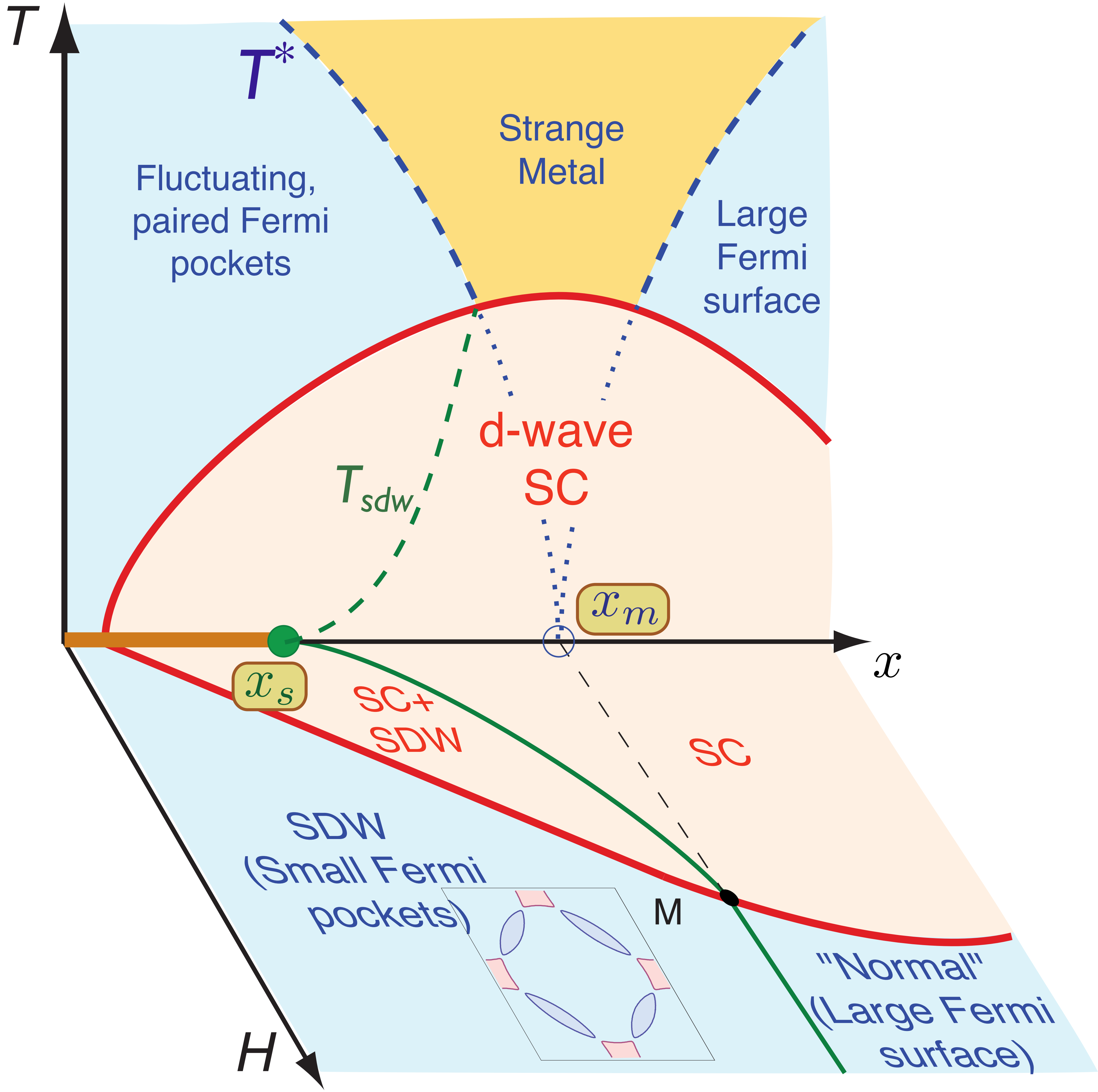




Neutron scattering on $YBa_2Cu_3O_{6.45}$
D. Haug *et al.*, *Physical Review Letters* **103**, 017001 (2009)



D. Haug, V. Hinkov, A. Suchaneck, D. S. Inosov, N. B. Christensen, Ch. Niedermayer, P. Bourges,
Y. Sidis, J. T. Park, A. Ivanov, C. T. Lin, J. Mesot, and B. Keimer,
Physical Review Letters **103**, 017001 (2009)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d-wave SC
T_{sdw}
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x
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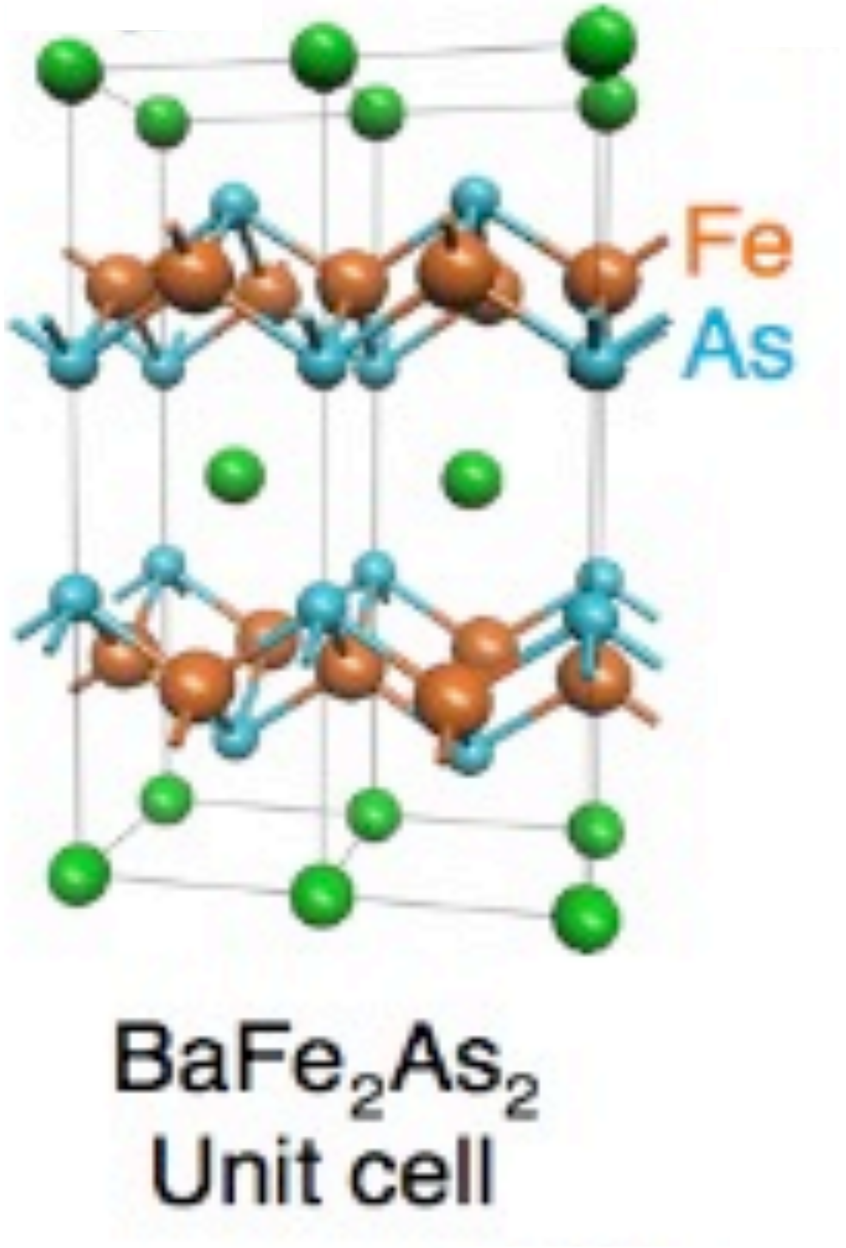
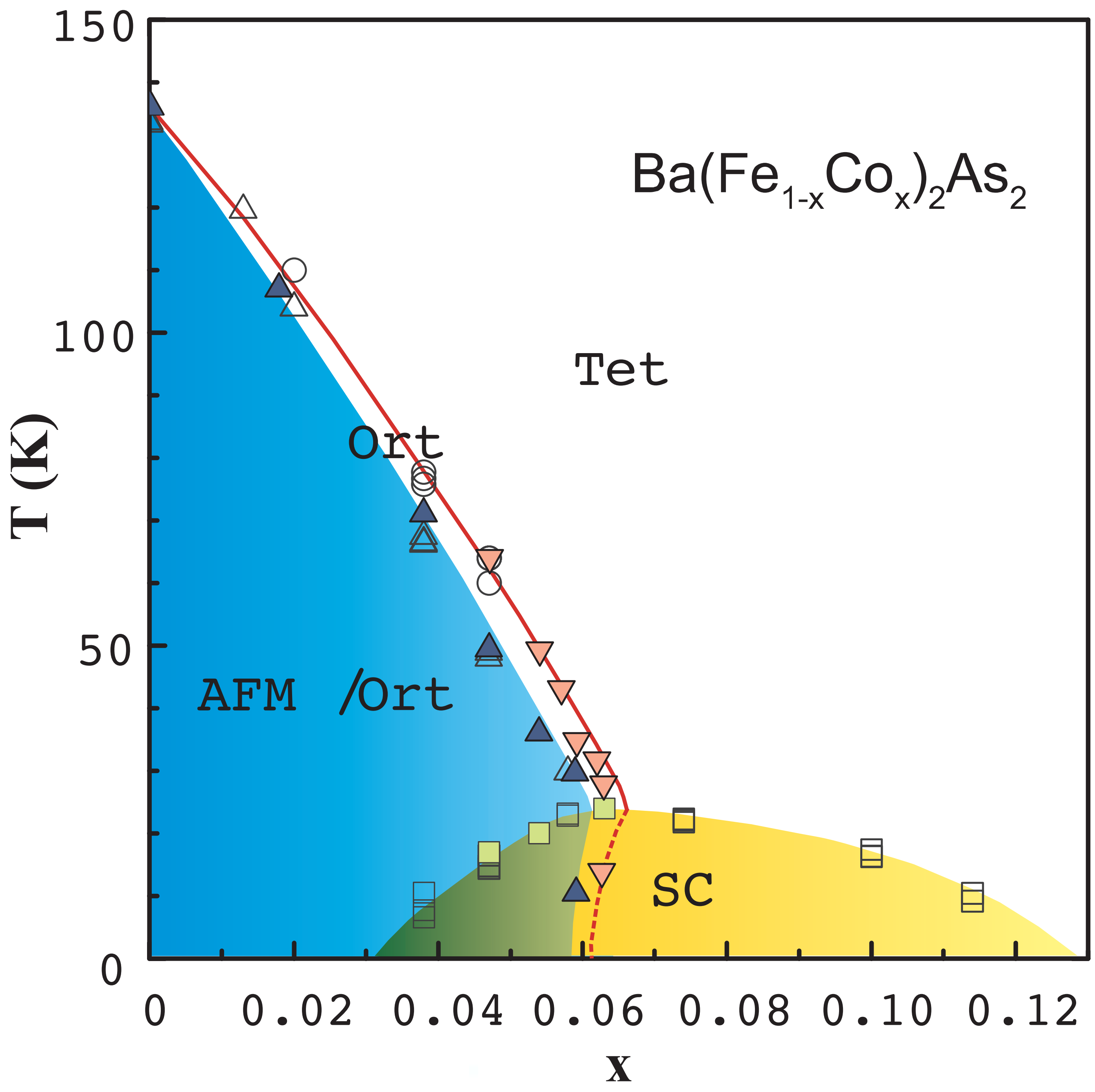
SC+ SDW
SC
SDW (Small Fermi pockets)
M
"Normal" (Large Fermi surface)
H

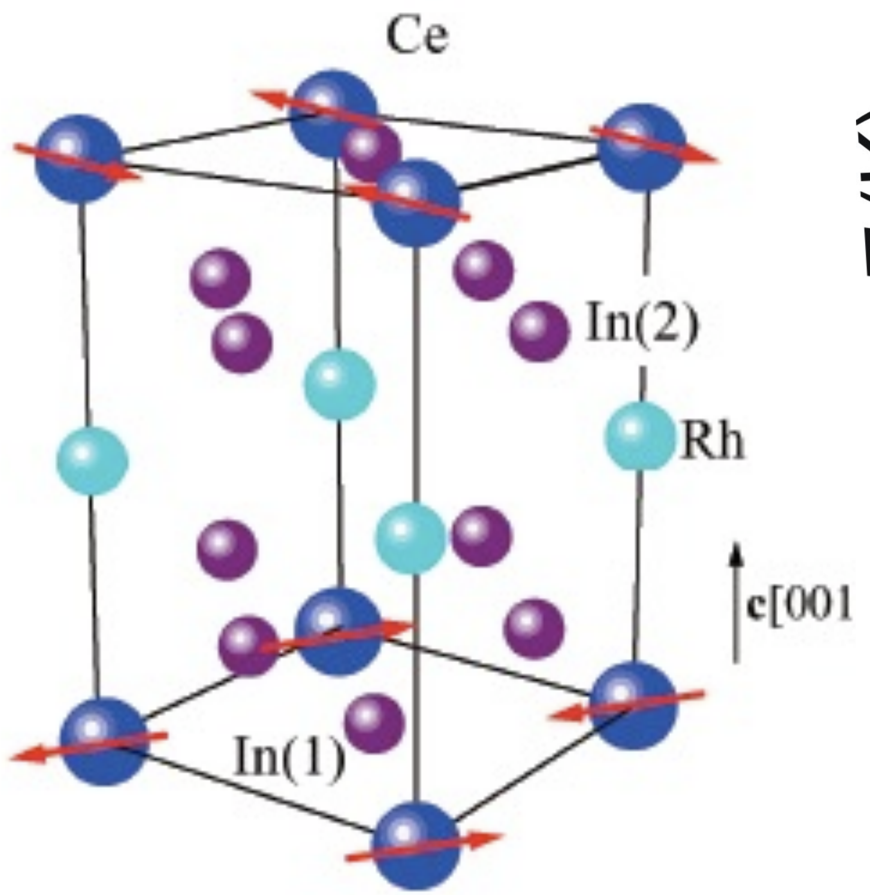
Similar phase diagram for the pnictides

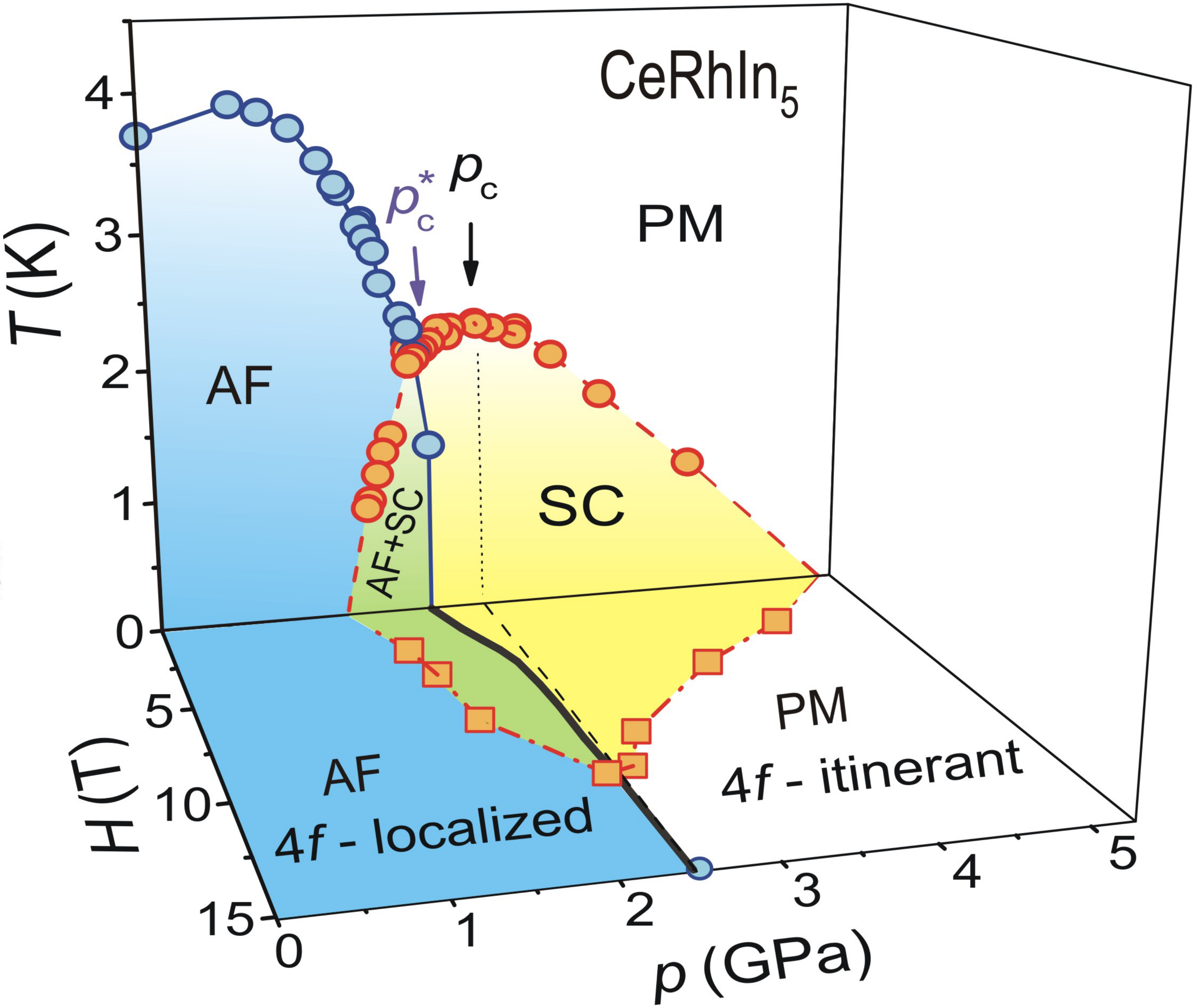




S. Nandi, M. G. Kim, A. Kreyssig, R. M. Fernandes, D. K. Pratt, A. Thaler, N. Ni, S. L. Bud'ko, P. C. Canfield, J. Schmalian, R. J. McQueeney, and A. I. Goldman, *Physical Review Letters* **104**, 057006 (2010)

Similar phase diagram for $CeRhIn_5$





G. Knebel, D. Aoki, and J. Flouquet, arXiv:0911.5223

Outline

1. Quantum phase transitions of a semi-metal

   Graphene, Dirac fermions and the Gross-Neveu model

2. Spin density wave order on the square lattice

   From a large Fermi surface to Fermi pockets

3. Competing orders I

   Unconventional superconductivity and the phase diagrams of the cuprates and the pnictides

4. Competing orders II

   Bond order, geometric phases, and gauge theories

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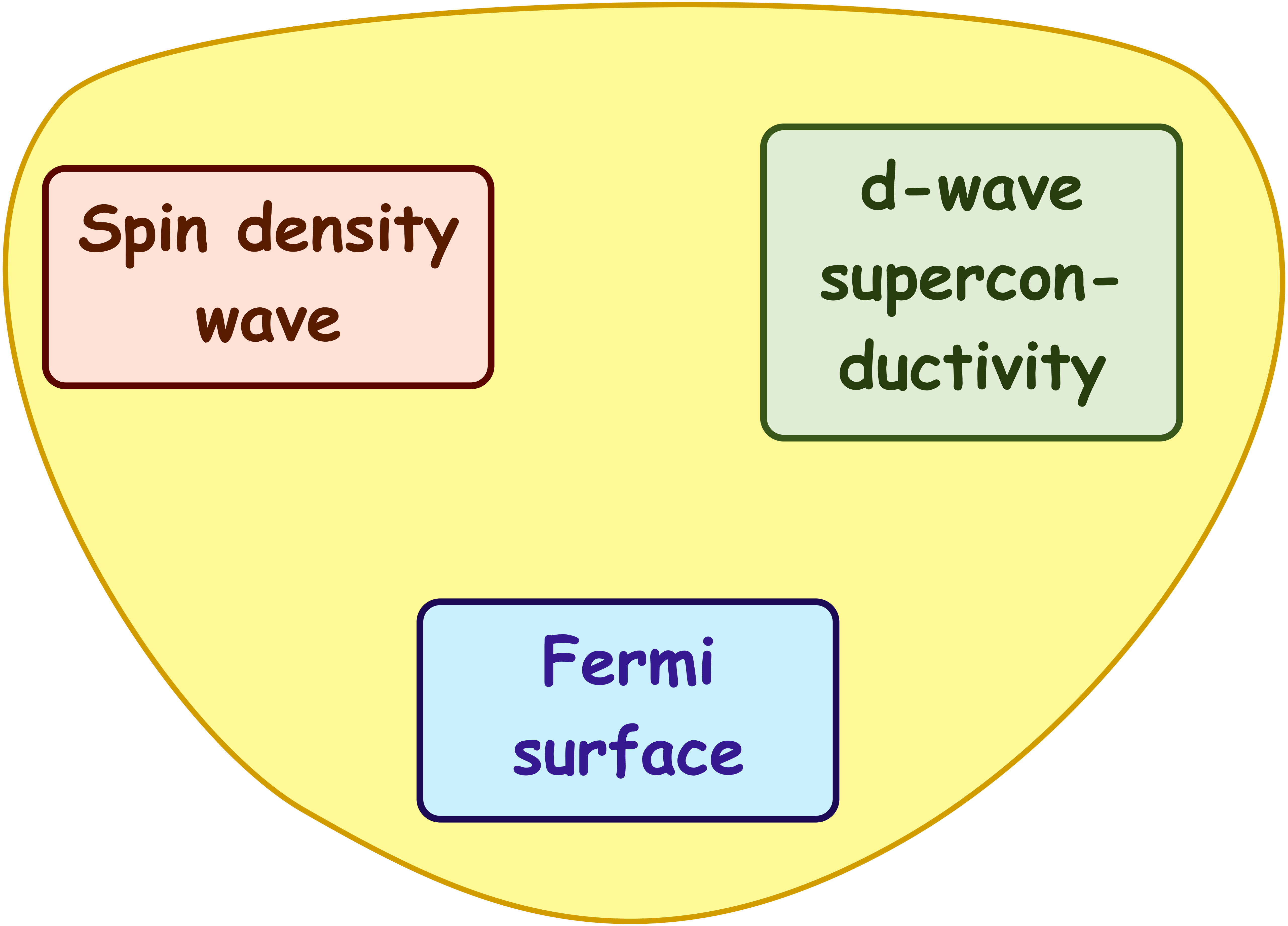

Spin density wave

d-wave supercon-ductivity

Fermi surface

Spin density wave

d-wave superconductivity

Fermi surface

Valence-bond order

Lecture 4

1. SDW transition in a metal: instability to a modulated bond order, which is locally "Ising-nematic".

2. Fluctuations in *orientation* of local antiferromagnetic order: Berry phases associated with skyrmion density.

3. Gauge theory of fluctuating skyrmion density ("hedgehogs"): valence bond solid order.

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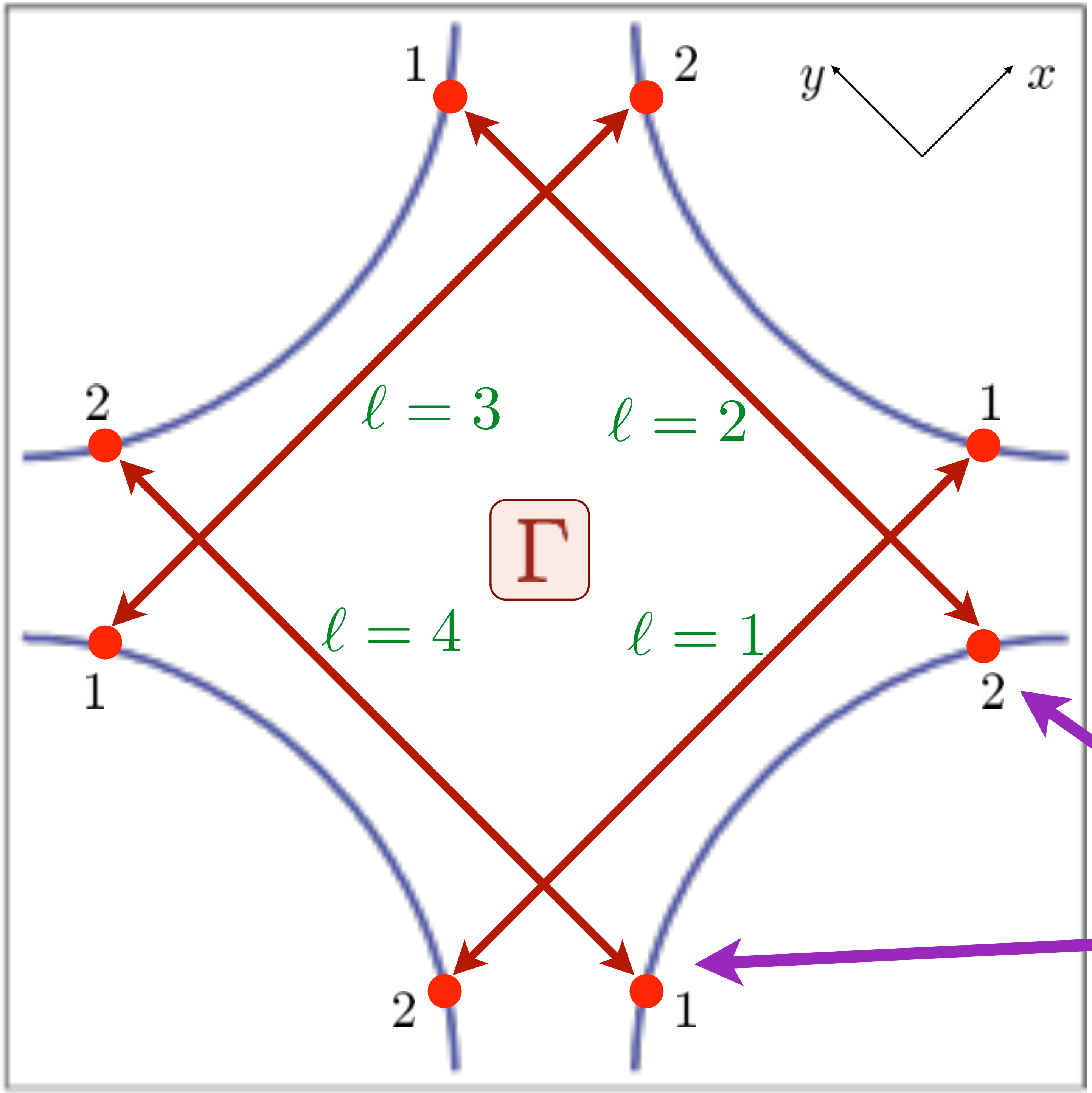
d-wave pairing in the theory of hotspots



Low energy fermions
$\psi^{\ell}_{1\alpha}$, $\psi^{\ell}_{2\alpha}$
$\ell = 1, \ldots, 4$

$$\mathcal{L}_f = \psi_{1\alpha}^{\ell\dagger}\left(\zeta\partial_\tau - i\mathbf{v}_1^\ell\cdot\boldsymbol{\nabla}_r\right)\psi_{1\alpha}^\ell + \psi_{2\alpha}^{\ell\dagger}\left(\zeta\partial_\tau - i\mathbf{v}_2^\ell\cdot\boldsymbol{\nabla}_r\right)\psi_{2\alpha}^\ell$$

Order parameter: $\mathcal{L}_\varphi = \frac{1}{2}\left(\boldsymbol{\nabla}_r\vec{\varphi}\right)^2 + \frac{\widetilde{\zeta}}{2}\left(\partial_\tau\vec{\varphi}\right)^2 + \frac{s}{2}\vec{\varphi}^2 + \frac{u}{4}\vec{\varphi}^4$

"Yukawa" coupling: $\mathcal{L}_c = -\lambda\vec{\varphi}\cdot\left(\psi_{1\alpha}^{\ell\dagger}\vec{\sigma}_{\alpha\beta}\psi_{2\beta}^\ell + \psi_{2\alpha}^{\ell\dagger}\vec{\sigma}_{\alpha\beta}\psi_{1\beta}^\ell\right)$

Return to theory of SDW ordering in a metal

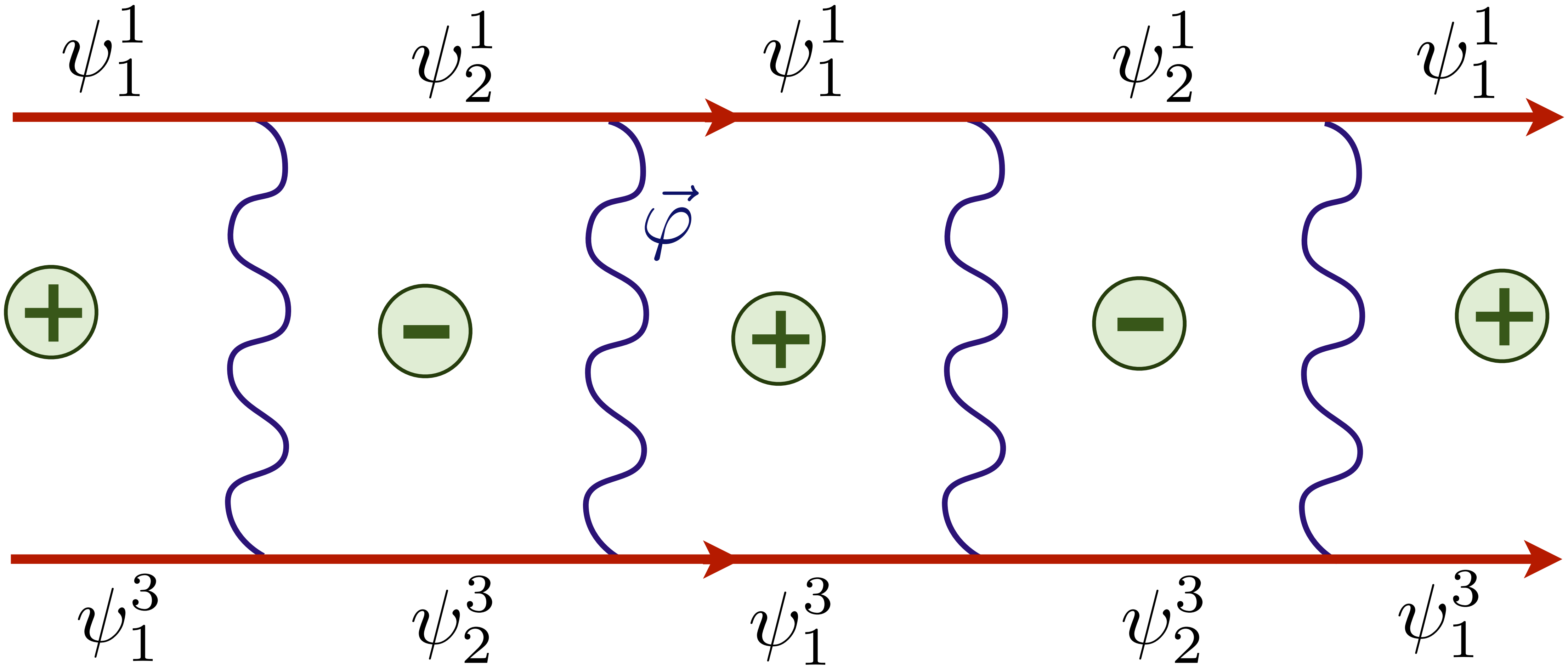
M.A. Metlitski and
S. Sachdev,
arXiv:1005.1288

Emergent Pseudospin symmetry

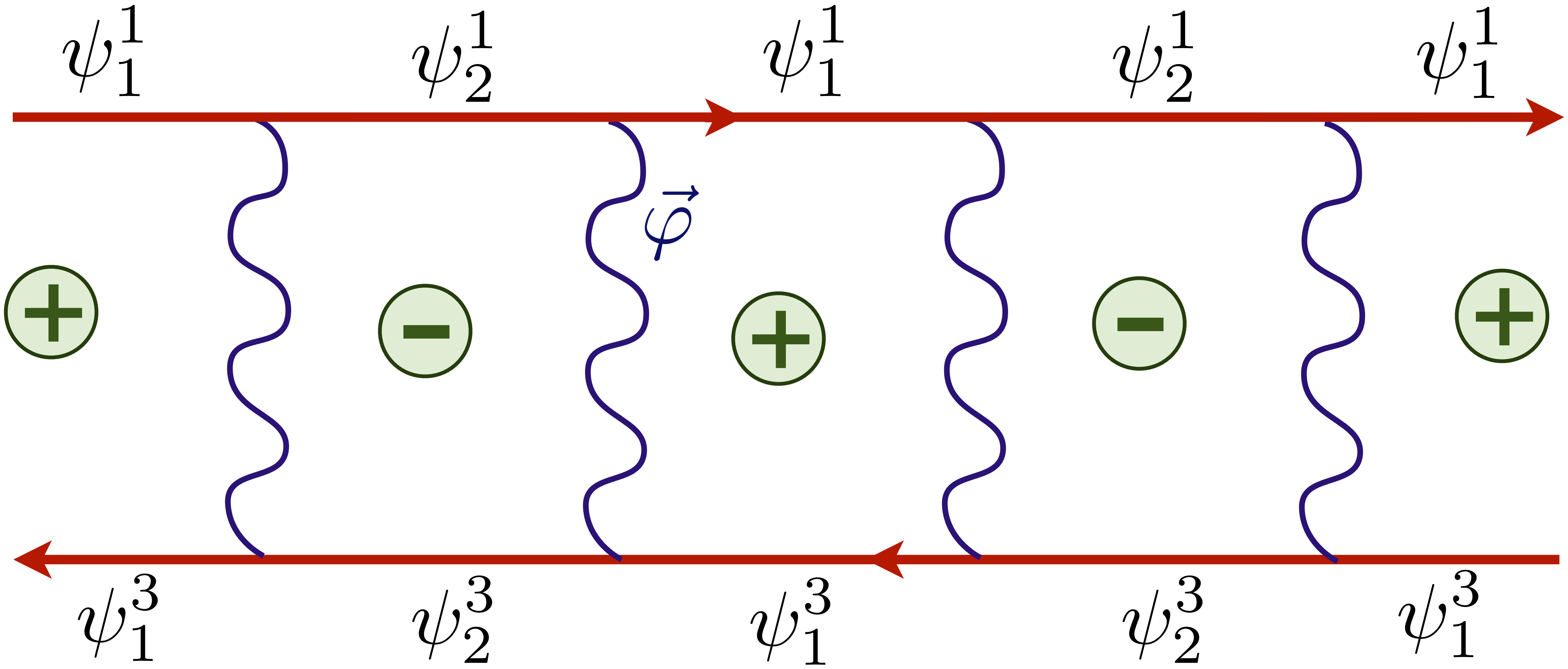
Continuum theory of hotspots in invariant under:

$$\left(\begin{array}{c} \psi^\ell_\uparrow \\ \psi^{\ell\dagger}_\downarrow \end{array} \right) \rightarrow U^\ell \left(\begin{array}{c} \psi^\ell_\uparrow \\ \psi^{\ell\dagger}_\downarrow \end{array} \right)$$

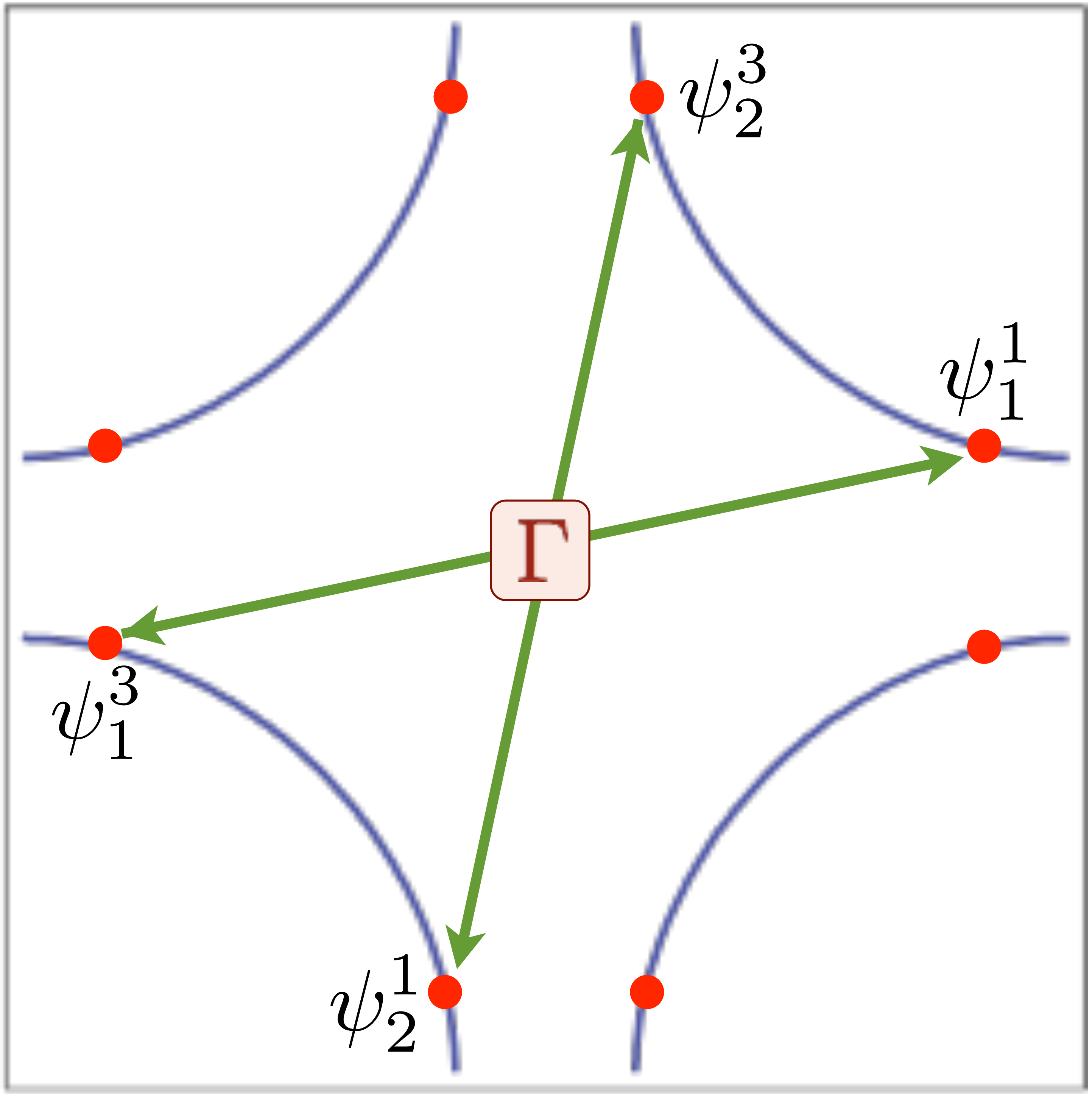
where U^ℓ are arbitrary SU(2) matrices which can be *different* on different hotspots ℓ.



d-wave Cooper pairing instability in particle-particle channel



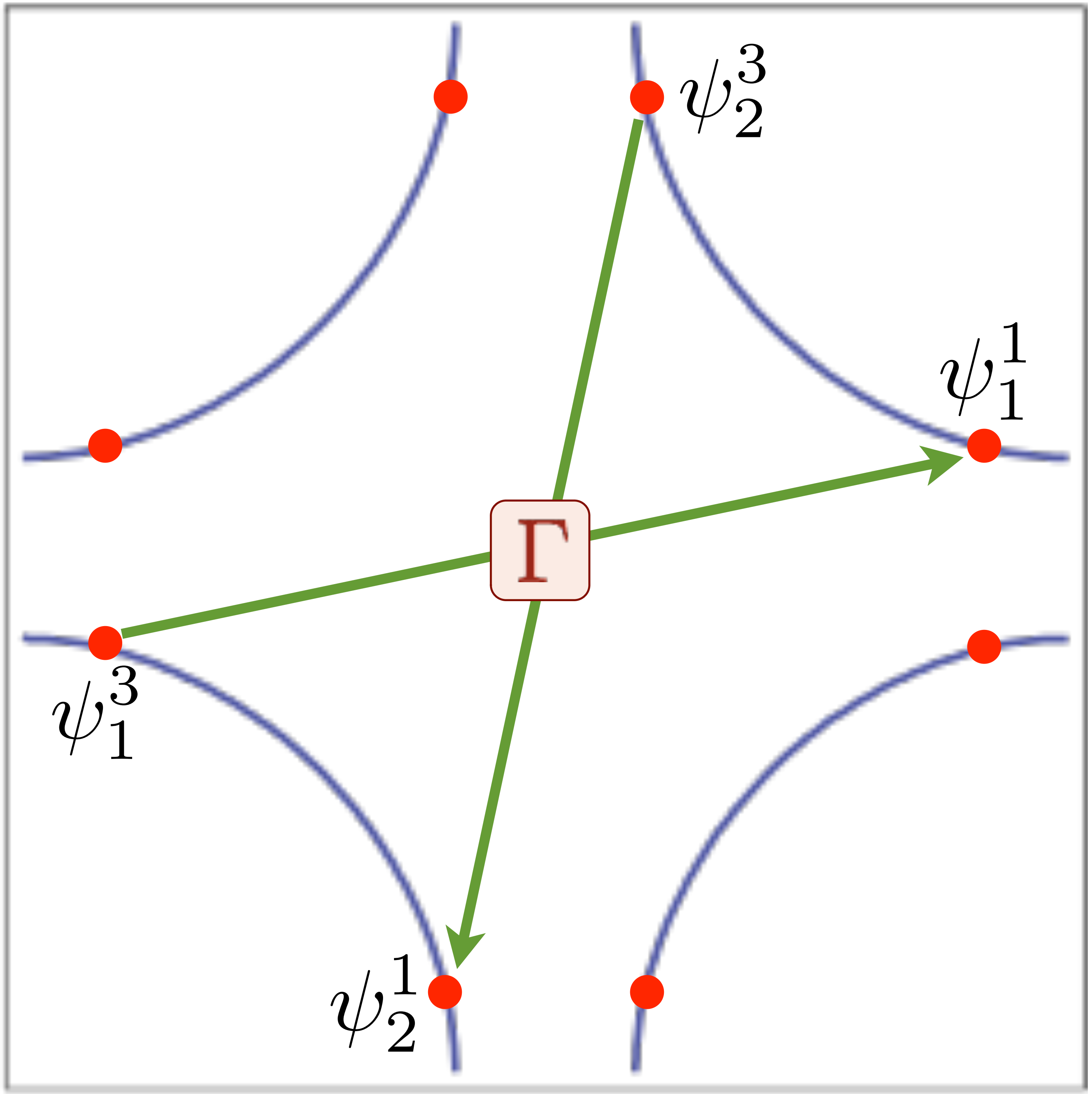
Bond density wave (with local Ising-nematic order) instability in particle-hole channel



Recall *d*-wave pairing instability in the particle-particle channel

Pairing order parameter:

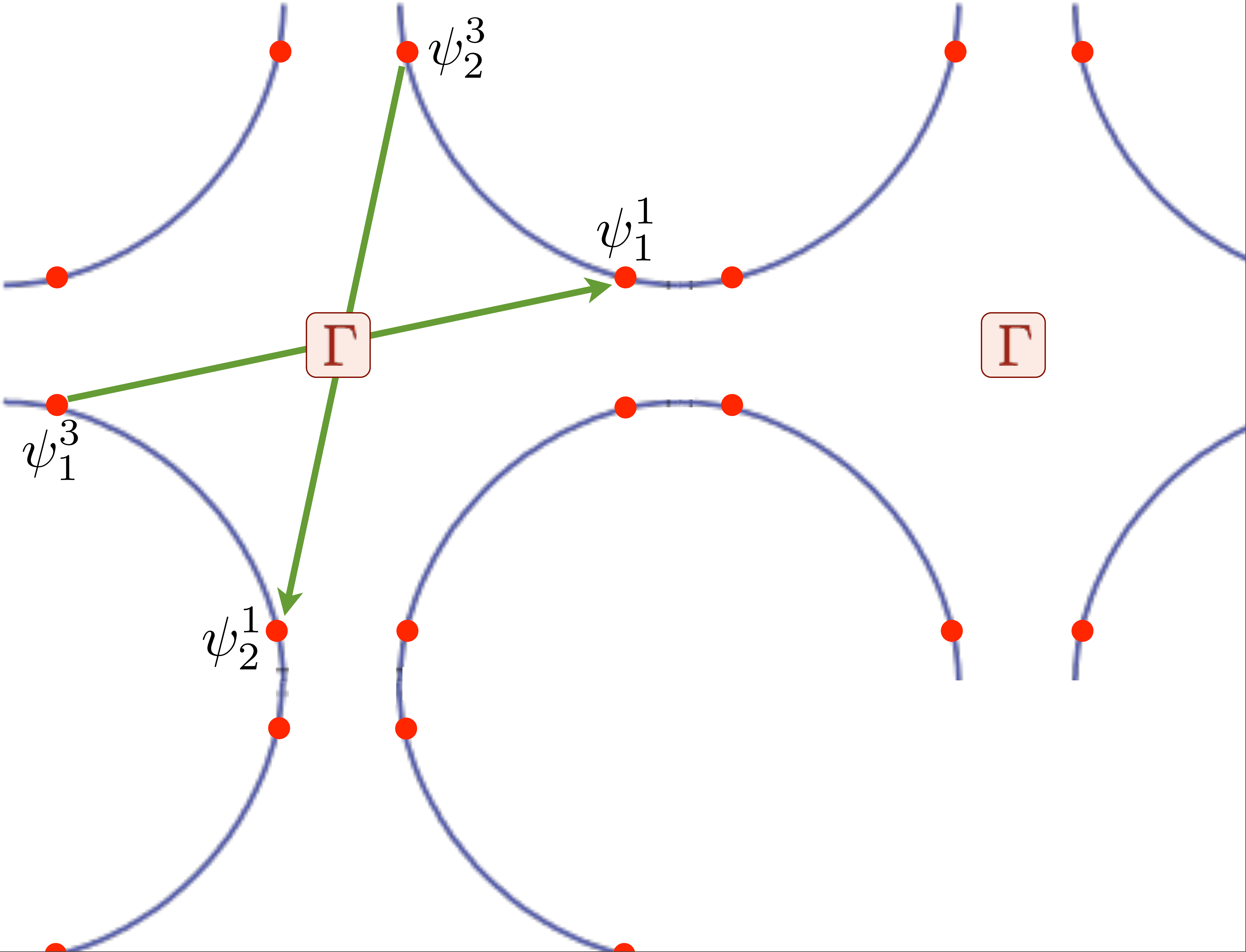
$$\varepsilon^{\alpha\beta}\left(\psi^3_{1\alpha}\psi^1_{1\beta} - \psi^3_{2\alpha}\psi^1_{2\beta}\right)$$

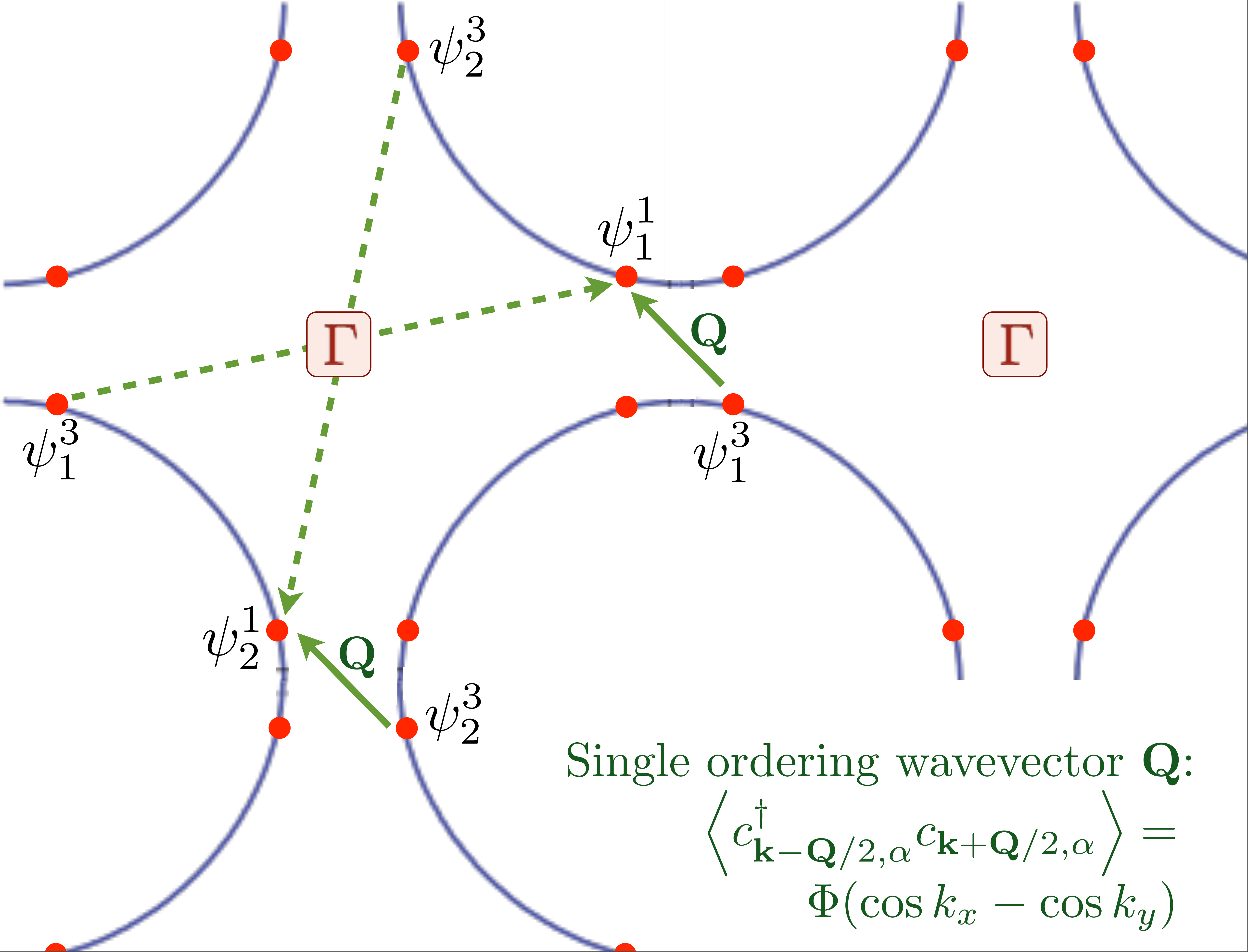


d-wave pairing has a partner instability in the particle-hole channel

Density-wave order parameter:

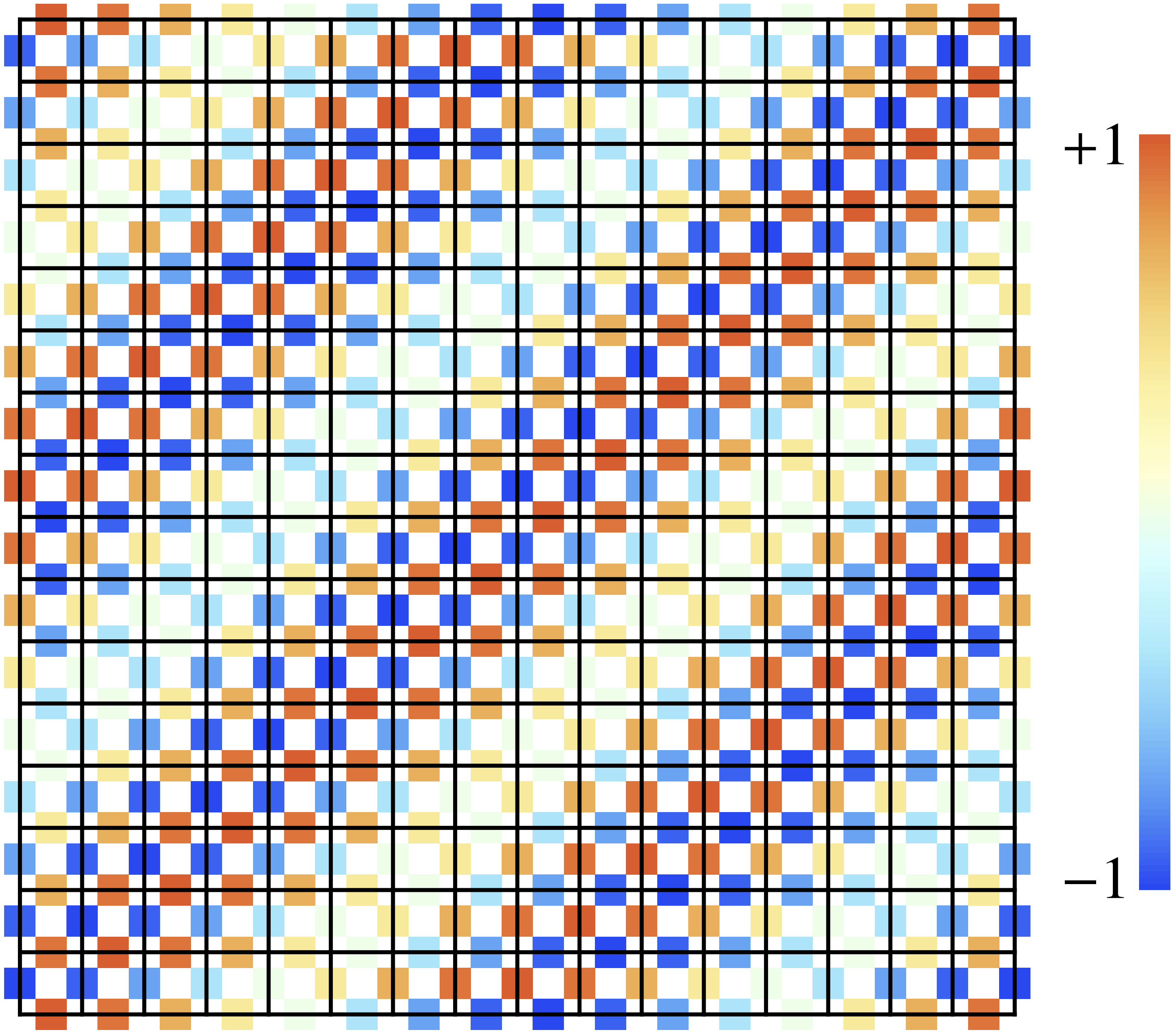
$$\left(\psi^{3\dagger}_{1\alpha}\psi^{1}_{1\alpha} - \psi^{3\dagger}_{2\alpha}\psi^{1}_{2\alpha}\right)$$


ψ_2^3
ψ_1^1
Γ
Γ
ψ_1^3
ψ_2^1



Single ordering wavevector $\mathbf{Q}$:

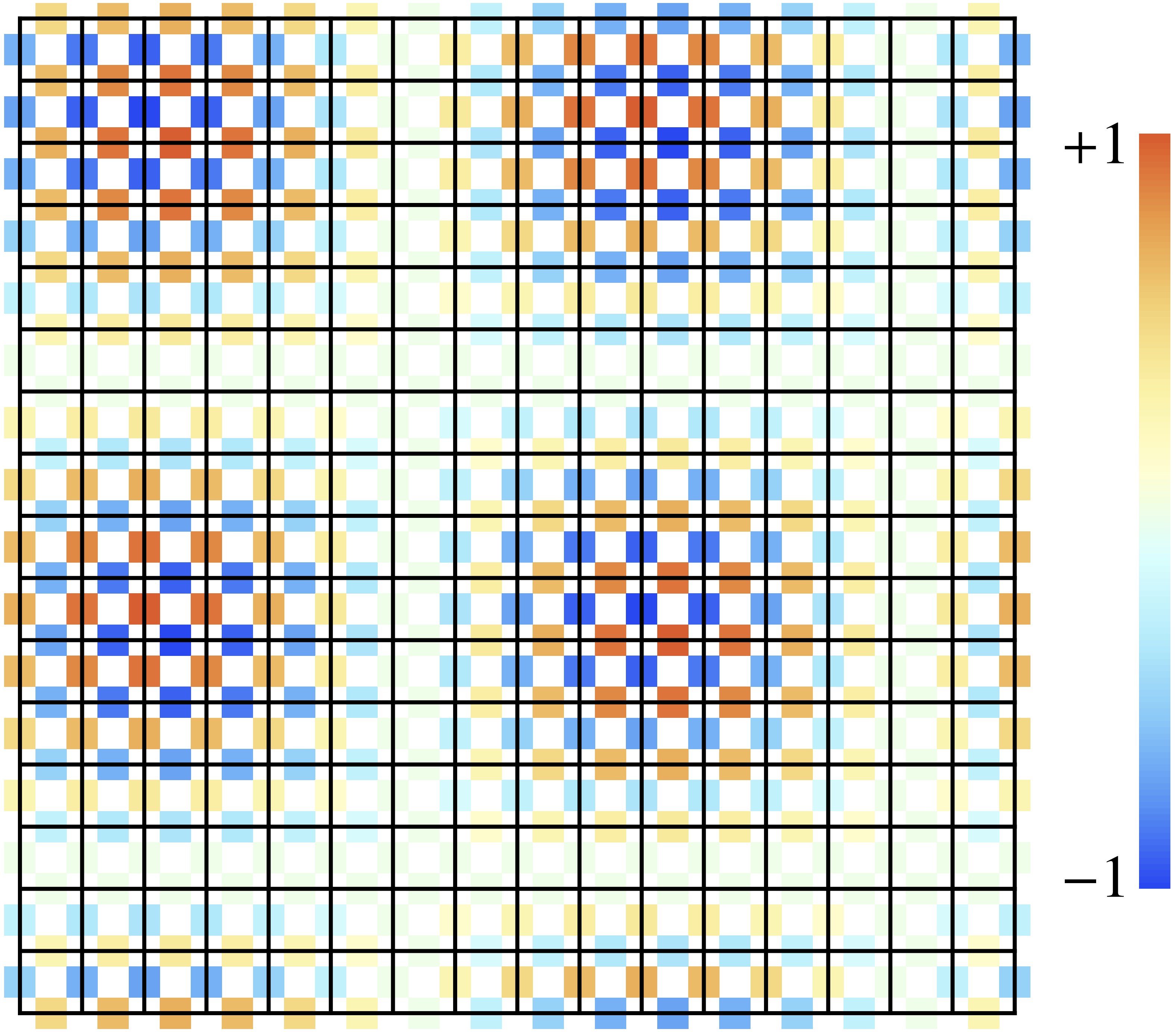
$$\left\langle c^\dagger_{\mathbf{k}-\mathbf{Q}/2,\alpha} c_{\mathbf{k}+\mathbf{Q}/2,\alpha} \right\rangle = \Phi(\cos k_x - \cos k_y)$$



"Bond density" measures amplitude for electrons to be in spin-singlet valence bond: VBS order

No modulations on sites. Modulated bond-density wave with local Ising-nematic ordering:

$$\left\langle c^\dagger_{\mathbf{k}-\mathbf{Q}/2,\alpha} c_{\mathbf{k}+\mathbf{Q}/2,\alpha} \right\rangle = \Phi(\cos k_x - \cos k_y)$$



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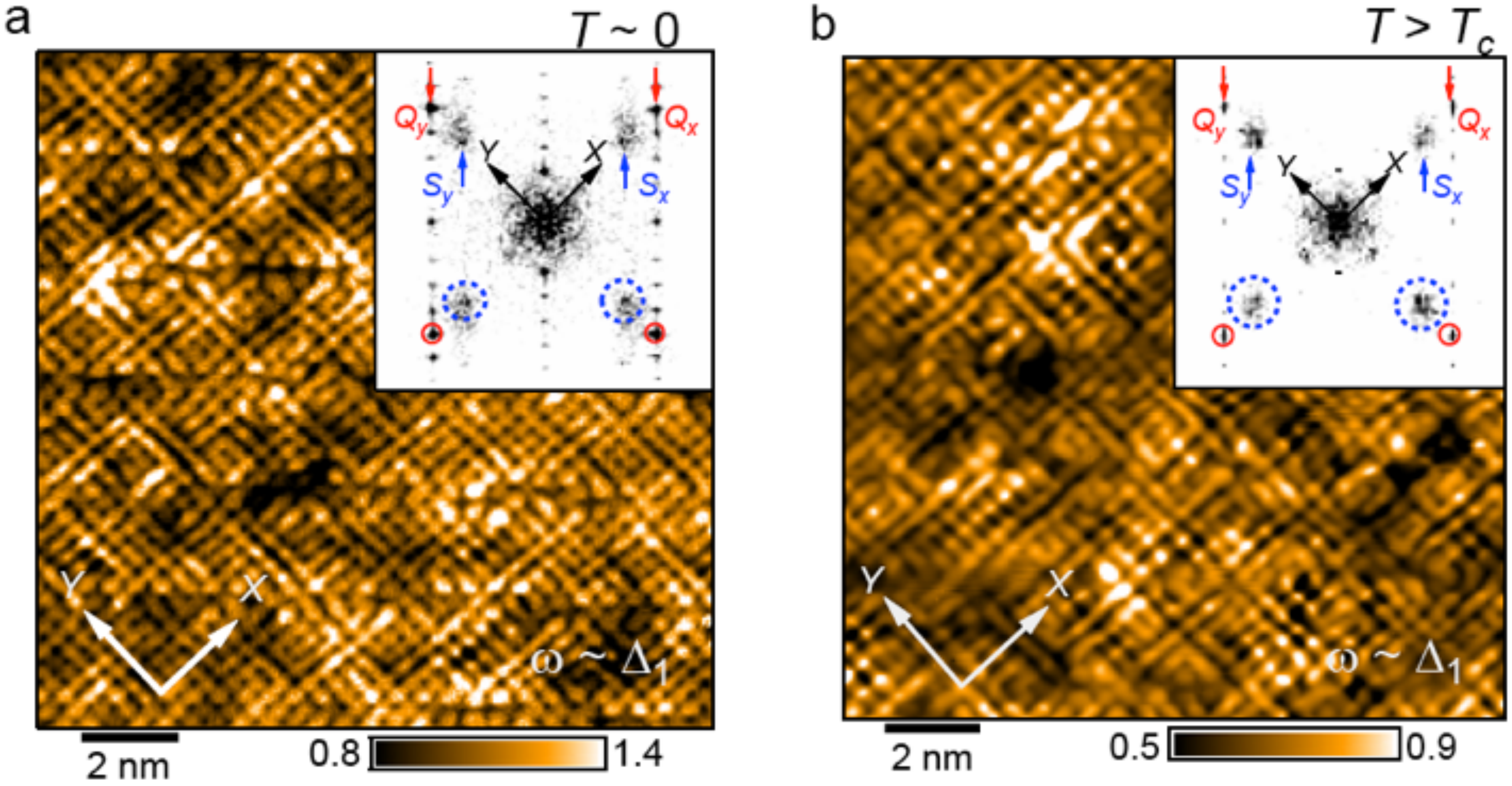
STM measurements of $Z(r)$, the energy asymmetry in density of states in $Bi_2Sr_2CaCu_2O_{8+\delta}$.



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Y. Kohsaka, Chung Koo
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Nature **466**, 347 (2010).

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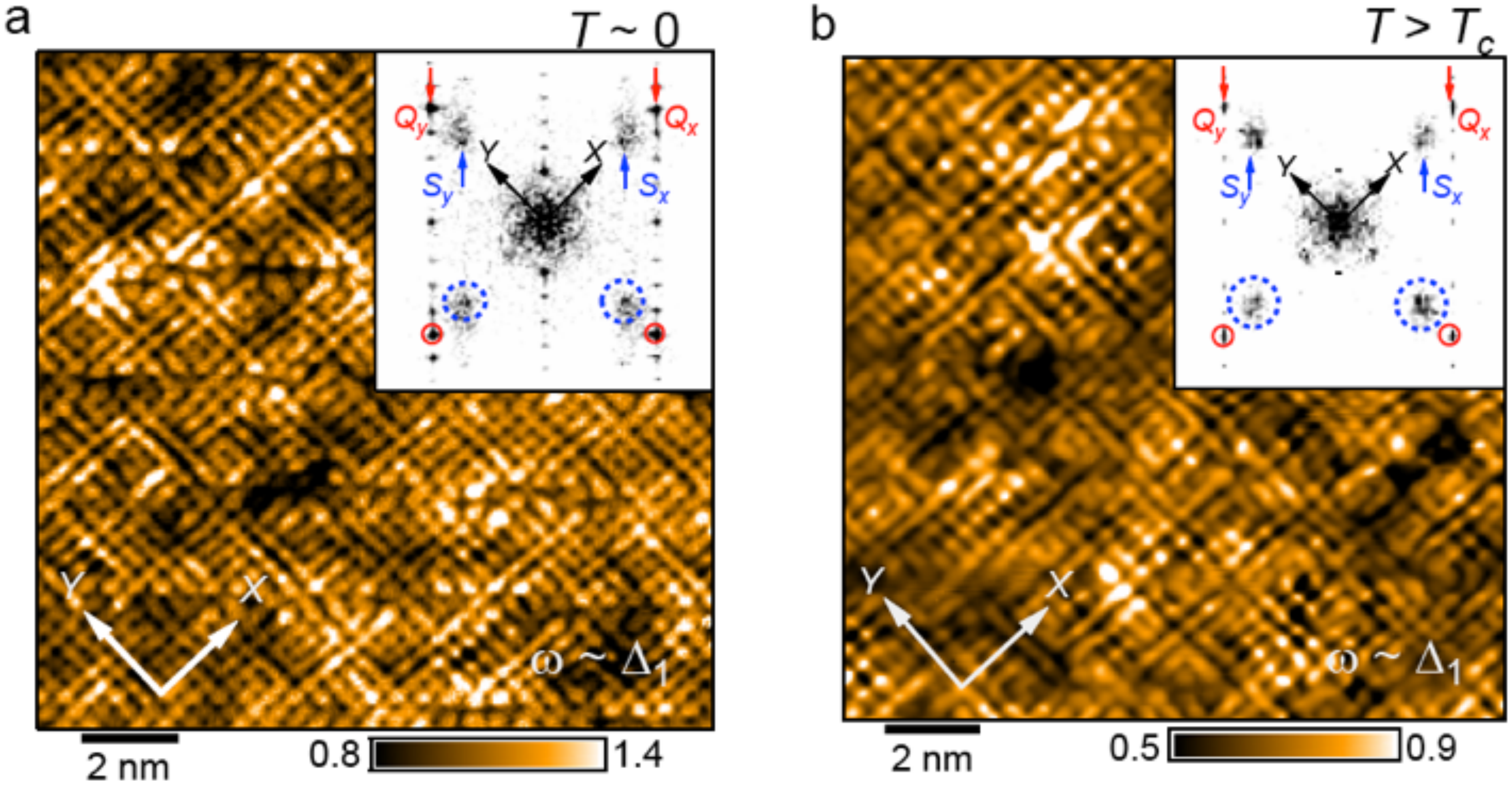

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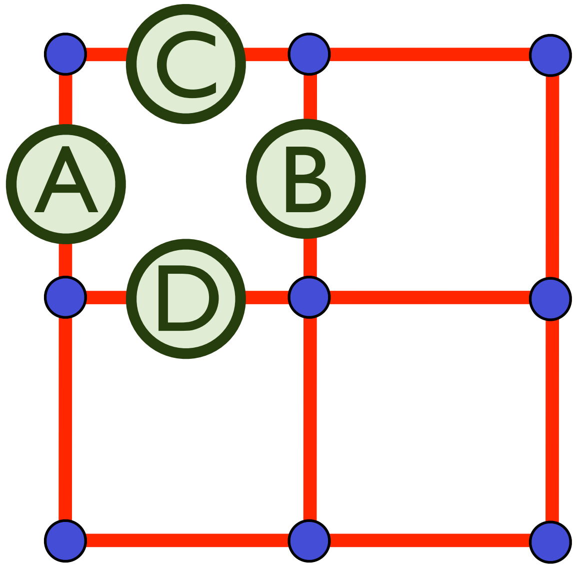

$$O_N = Z_A + Z_B - Z_C - Z_D$$

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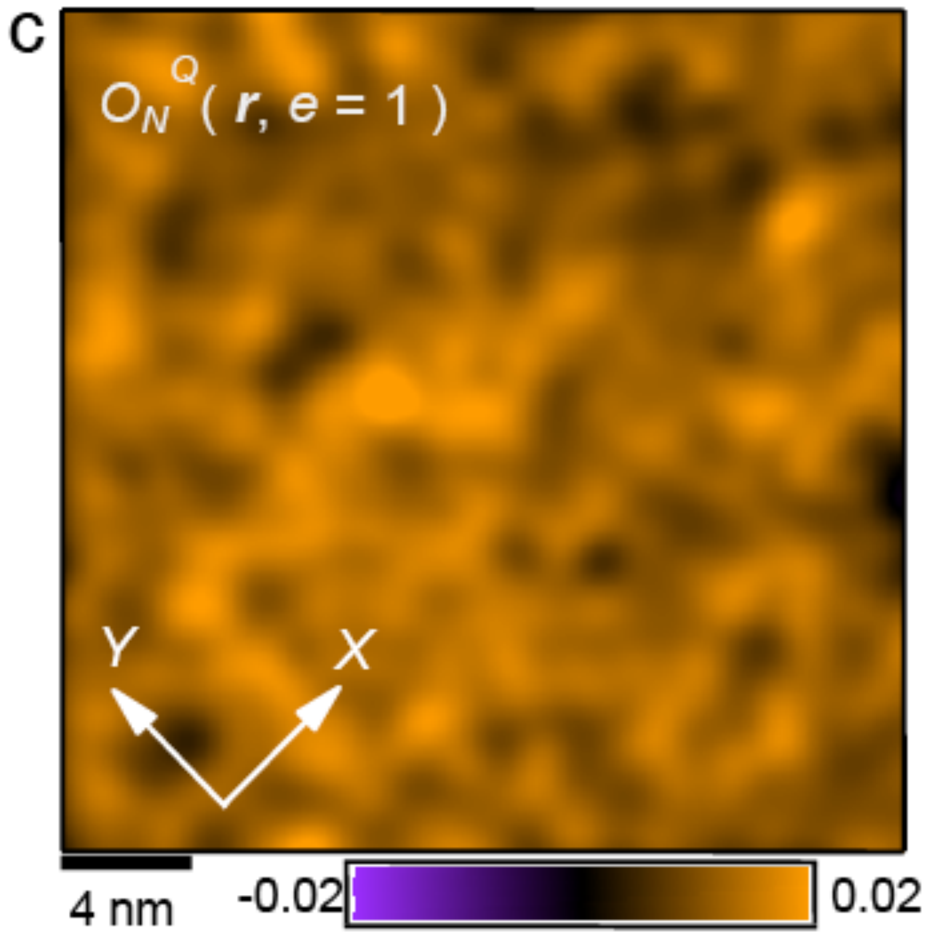

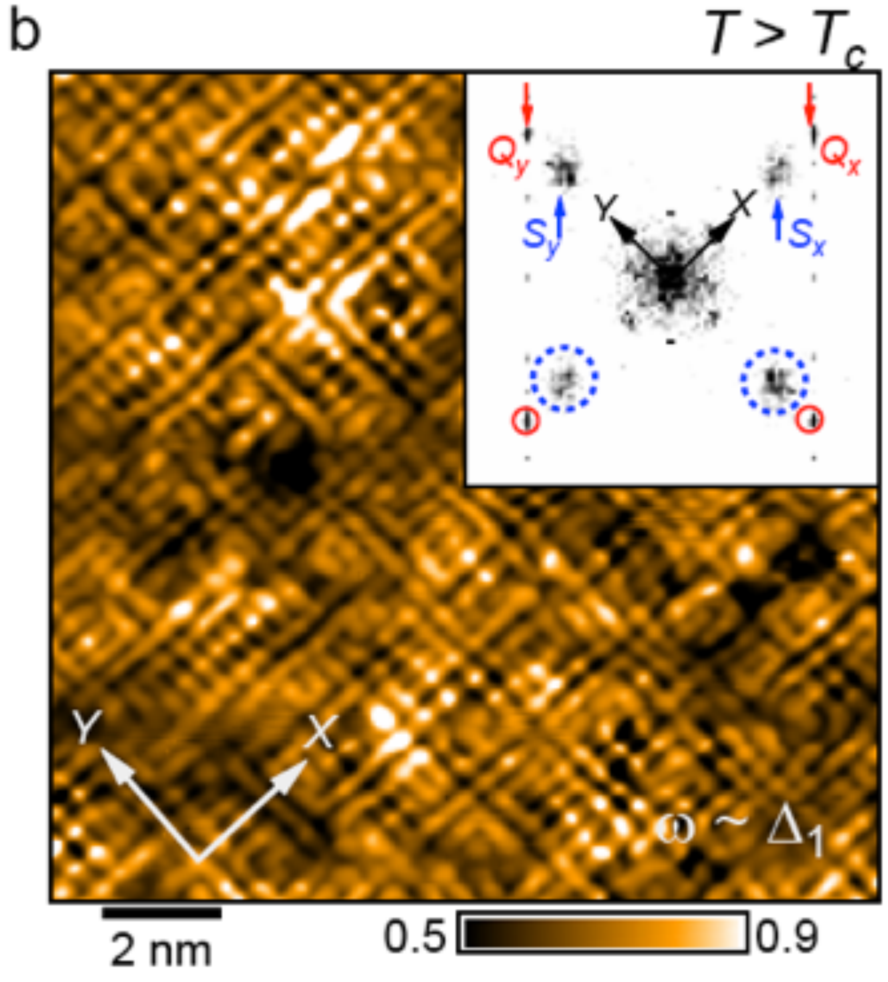

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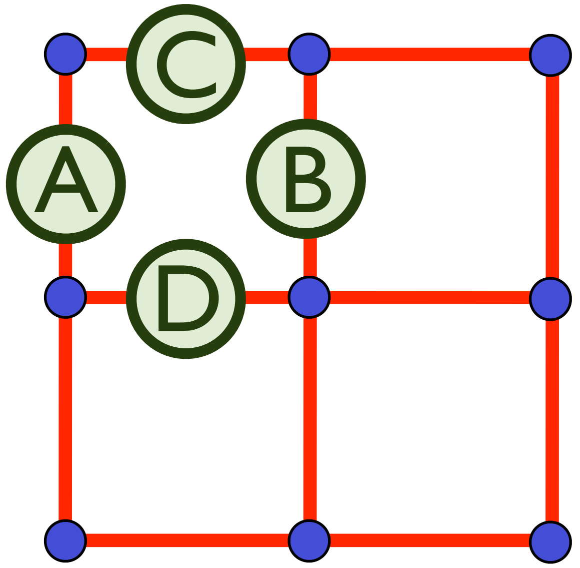

$$O_N = Z_A + Z_B - Z_C - Z_D$$

Strong anisotropy of electronic states between *x* and *y* directions: Electronic "Ising-nematic" order

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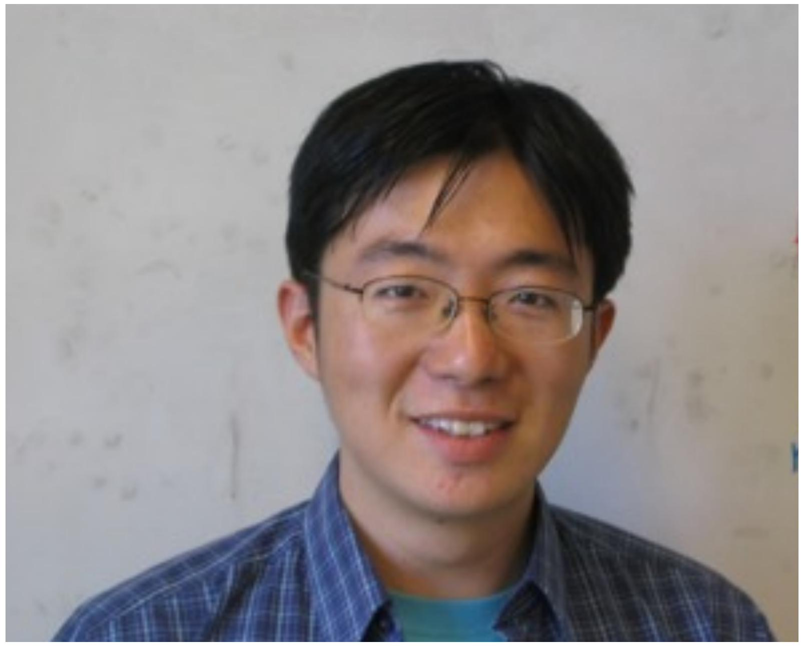


Cenke Xu

Harvard → UCSB

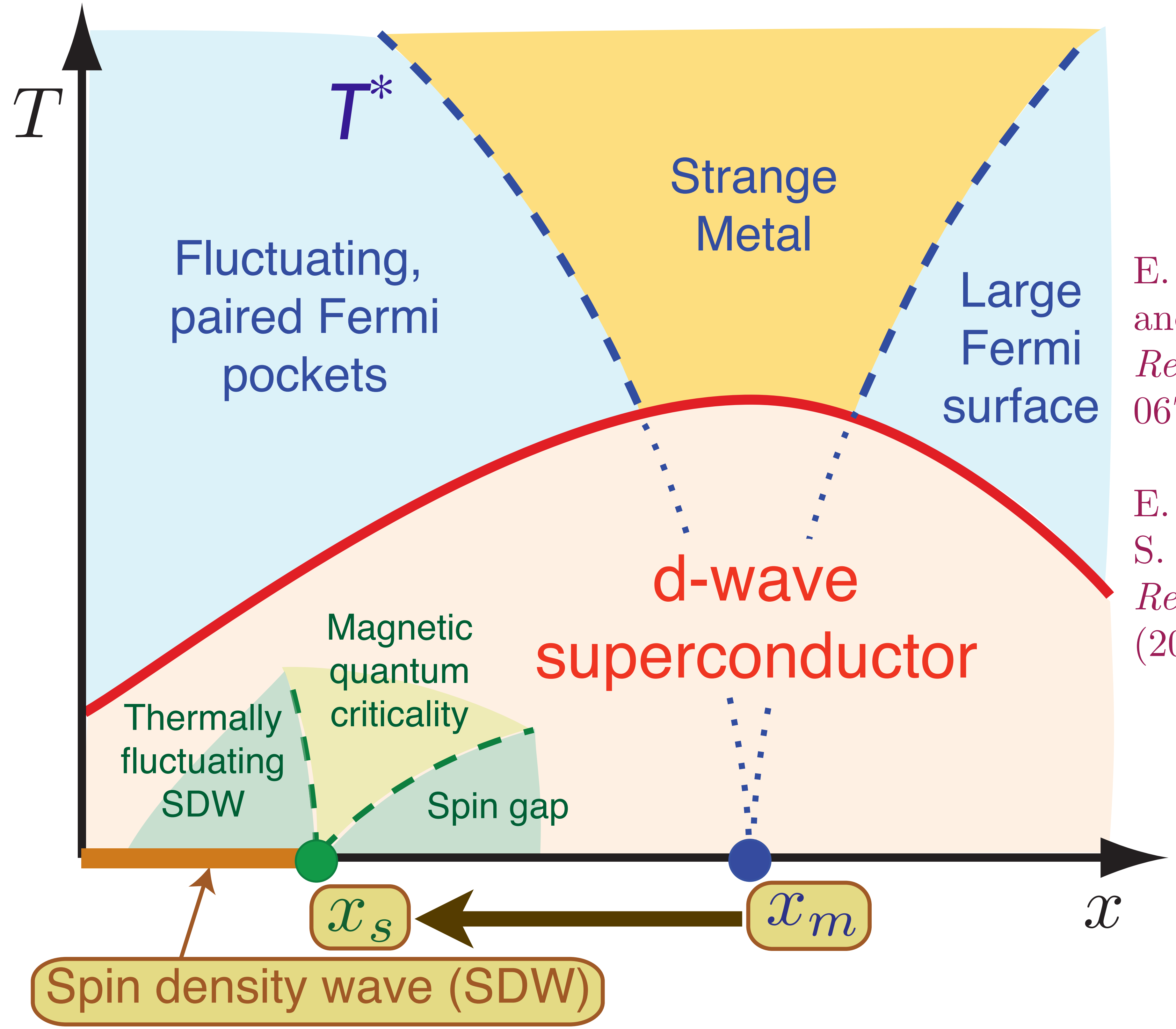
Liang Fu

Harvard

Physical Review Letters **105**, 057201 (2010)

and to appear

Theory of quantum criticality in the cuprates



E. Demler, S. Sachdev and Y. Zhang, *Phys. Rev. Lett.* **87**, 067202 (2001).

E. G. Moon and S. Sachdev, *Phy. Rev. B* **80**, 035117 (2009)

Competition between SDW order and superconductivity moves the actual quantum critical point to $x = x_s < x_m$.

Start from the "spin-fermion" model

$$
\begin{aligned}
\mathcal{Z} &= \int \mathcal{D}c_\alpha \mathcal{D}\vec{\varphi} \exp\left(-\mathcal{S}\right) \\
\mathcal{S} &= \int d\tau \sum_{\mathbf{k}} c^\dagger_{\mathbf{k}\alpha} \left(\frac{\partial}{\partial\tau} - \varepsilon_{\mathbf{k}} \right) c_{\mathbf{k}\alpha} \\
&\quad - \lambda \int d\tau \sum_i c^\dagger_{i\alpha} \vec{\varphi}_i \cdot \vec{\sigma}_{\alpha\beta} c_{i\beta} e^{i\mathbf{K}\cdot\mathbf{r}_i} \\
&+ \int d\tau d^2 r \left[\frac{1}{2} \left(\boldsymbol{\nabla}_r \vec{\varphi} \right)^2 + \frac{\widetilde{\zeta}}{2} \left(\partial_\tau \vec{\varphi} \right)^2 + \frac{s}{2} \vec{\varphi}^2 + \frac{u}{4} \vec{\varphi}^4 \right]
\end{aligned}
$$

Modify SDW order parameter $\vec{\varphi}$ to a fixed length field $\vec{n}$, with $\vec{n}^2 = 1$

$$\mathcal{Z} = \int \mathcal{D}c_\alpha \mathcal{D}\vec{n}\delta\left(\vec{n}^2 - 1\right)\exp\left(-\mathcal{S}\right)$$

$$\mathcal{S} = \int d\tau \sum_{\mathbf{k}} c^\dagger_{\mathbf{k}\alpha}\left(\frac{\partial}{\partial\tau} - \varepsilon_{\mathbf{k}}\right)c_{\mathbf{k}\alpha}$$

$$- \lambda \int d\tau \sum_i c^\dagger_{i\alpha}\vec{n}_i \cdot \vec{\sigma}_{\alpha\beta}c_{i\beta}e^{i\mathbf{K}\cdot\mathbf{r}_i}$$

$$+ \int d\tau d^2 r \frac{1}{2g}\left[\left(\boldsymbol{\nabla}_r\vec{n}\right)^2 + \frac{1}{c^2}\left(\partial_\tau\vec{n}\right)^2\right]$$

This is proposed as a theory for $s_c < s < s_c^0$ (or doping $x_s < x < x_m$).

Fluctuating Néel states

Begin with the electronic Hamiltonian on the square lattice

$$H_0 = -\sum_{i,j} t(\mathbf{r}_i - \mathbf{r}_j) c_\alpha^\dagger(\mathbf{r}_i) c_\alpha(\mathbf{r}_j) \tag{1}$$

Now we allow for a spatially varying Néel order $n^a(\mathbf{r})$ $(a = x, y, z)$. Then the Hamiltonian is

$$H_0 = -\sum_{i,j} t(\mathbf{r}_i - \mathbf{r}_j) c_\alpha^\dagger(\mathbf{r}_i) c_\alpha(\mathbf{r}_j) + N_0 \sum_i \eta_i n^a(\mathbf{r}_i) c_\alpha^\dagger(\mathbf{r}_i) \sigma^a_{\alpha\beta} c_\beta(\mathbf{r}_i) \tag{2}$$

where $\eta_i = \pm 1$ on the two sublattices.
We begin with the Hamiltonian in Eq. (2), and assume $n^a(\mathbf{r})$ is a slowly varying unit vector. In any local region, without loss of generality, we can choose co-ordinates so that $n^a(\mathbf{r})$ is close to the North pole $(0, 0, 1)$. In this co-ordinate system we parameterize the variations in the Néel order in terms of the complex field φ via

$$n^a = \left(\frac{\varphi + \varphi^*}{2}, \frac{\varphi - \varphi^*}{2i}, \sqrt{1 - |\varphi|^2} \right). \tag{3}$$

We assume $|\varphi| \ll 1$ and slowly varying. Inserting Eq. (3) into Eq. (2) we obtain the Hamiltonian $H = H_0 + H_1$ with

$$H_0 = \sum_{\mathbf{k}} \left(\varepsilon_{\mathbf{k}} c^\dagger(\mathbf{k}) c(\mathbf{k}) + N_0 c^\dagger(\mathbf{k}+\mathbf{Q}) \sigma^z c(\mathbf{k}) \right), \qquad (4)$$

where $\mathbf{Q} = (\pi, \pi)$ and

$$\varepsilon_{\mathbf{k}} = -\sum_{\mathbf{s}} t(\mathbf{s}) \cos(\mathbf{k} \cdot \mathbf{s}), \qquad (5)$$

with $t(-\mathbf{s}) = t(\mathbf{s})$. Throughout the summation over momenta extends over the entire square lattice Brillouin zone. Also, we will drop the α spin indices of the c_α, all Pauli matrices in this present section will be assumed to act on the α space, and the α indices will be traced over. The coupling to the spatial variations in the Néel order parameterized by φ are given to the needed order in φ

by

$$
\begin{aligned}
H_1 \quad = \quad & N_0 \sum_{\mathbf{k}_1, \mathbf{k}_2} \left[\varphi^*(\mathbf{k}_1) c^\dagger(\mathbf{k}_2 + \mathbf{Q}) \sigma^+ \text{H.c.} \right] \qquad (6) \\
& - \frac{N_0}{2} \sum_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3} \varphi^*(\mathbf{k}_1) \varphi(\mathbf{k}_1 + \mathbf{k}_2) c^\dagger(\mathbf{k}_3 + \mathbf{Q}) \sigma^z c(\mathbf{k}_3 - \mathbf{k}_1)
\end{aligned}
$$

We are now interesting in computing the response of the observable properties of H to a slow variation in the Néel order $n^a(\mathbf{r})$. A key choice we have to make here is that of a suitable observable. We are interested in the nature of the phase where Néel order is ‘disordered’ and so it is natural that the observable should be spin rotation invariant. Also, because we will use the observable to characterize a ‘competing order’, it should preferably vanish in the spatially uniform Néel state, and be induced only when there are spatial variations in the Néel order. Finally, for convenience, the observable should be a fermion bilinear. With these constraints, it turns out that a unique choice is forced upon us: it is the observable

$$\mathcal{O}(\mathbf{k},\mathbf{r}) = \int_q \left\langle c^\dagger(\mathbf{k}+\mathbf{Q}+\mathbf{q}/2)c(\mathbf{k}-\mathbf{q}/2)\right\rangle e^{-i\mathbf{q}\cdot\mathbf{r}} \qquad (7)$$

Here the integral over $\mathbf{q}$ is over *small* momenta, characteristic of those carried by the bosonic fields; thus the variation of $\mathcal{O}(\mathbf{k},\mathbf{r})$ with $\mathbf{r}$ is slow. In the simplest case, the right-hand-side has support only at $\mathbf{q}=0$, so that $\mathcal{O}(\mathbf{k},\mathbf{r})$ takes the $\mathbf{r}$-independent value

$$\mathcal{O}(\mathbf{k}) = \left\langle c^\dagger(\mathbf{k}+\mathbf{Q})c(\mathbf{k})\right\rangle . \qquad (8)$$

On the other hand, $\mathbf{k}$ is an arbitrary momentum in the Brillouin zone, and we will find very useful information in the $\mathbf{k}$ dependence of $\mathcal{O}(\mathbf{k})$. It is easy to check from H_0 that $\mathcal{O}(\mathbf{k})=0$ in the uniform Néel state, as we required; only $\left\langle c^\dagger(\mathbf{k}+\mathbf{Q})\sigma^z c(\mathbf{k})\right\rangle \neq 0$ in the uniform Néel state.

We now proceed to a computation of $\mathcal{O}(\mathbf{k},\mathbf{r})$ in powers of φ using the Hamiltonian H_0+H_1. We will need to work to second order in φ, and also to second order in spatial gradients of φ; as stated earlier, all fermion momenta are allowed to be arbitrary at all stages.

First, let us collect the propagators of H_0. The single fermion Green's function of H_0 is written in terms of its 'normal' and 'anomalous' parts as

$$\begin{aligned}\langle c(\mathbf{k})\,;\,c^\dagger(\mathbf{p})\rangle &= \delta_{\mathbf{k},\mathbf{p}}G(\mathbf{k}) + \delta_{\mathbf{k}+\mathbf{Q},\mathbf{p}}\sigma^z F(\mathbf{k}) \\ G(\mathbf{k}) &\equiv \frac{u_{\mathbf{k}}^2}{-i\omega + E_{1\mathbf{k}}} + \frac{v_{\mathbf{k}}^2}{-i\omega + E_{2\mathbf{k}}} \\ F(\mathbf{k}) &\equiv u_{\mathbf{k}} v_{\mathbf{k}} \left(\frac{1}{-i\omega + E_{1\mathbf{k}}} - \frac{1}{-i\omega + E_{2\mathbf{k}}}\right),\end{aligned} \tag{9}$$

The eigenenergies in Eq. (9) are

$$E_{1,2\mathbf{k}} = \frac{\varepsilon_{\mathbf{k}} + \varepsilon_{\mathbf{k}+\mathbf{Q}}}{2} \pm \sqrt{\left(\frac{\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}+\mathbf{Q}}}{2}\right)^2 + N_0^2}, \tag{10}$$

and the parameters are

$$u_{\mathbf{k}} = \cos(\theta_{\mathbf{k}}/2) \quad , \quad v_{\mathbf{k}} = \sin(\theta_{\mathbf{k}}/2) \tag{11}$$

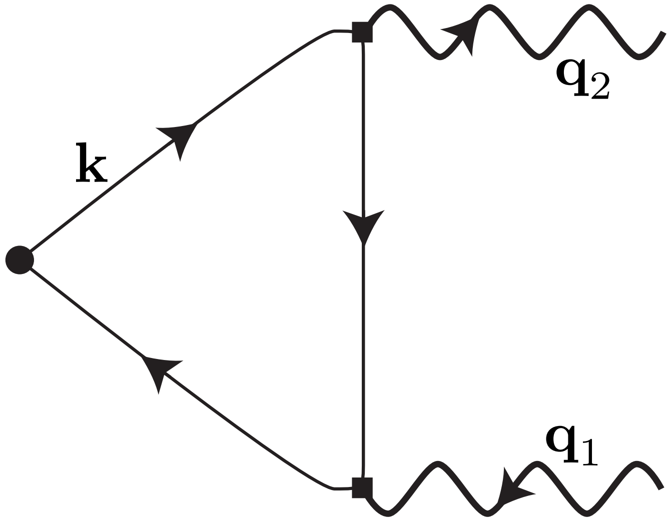
with

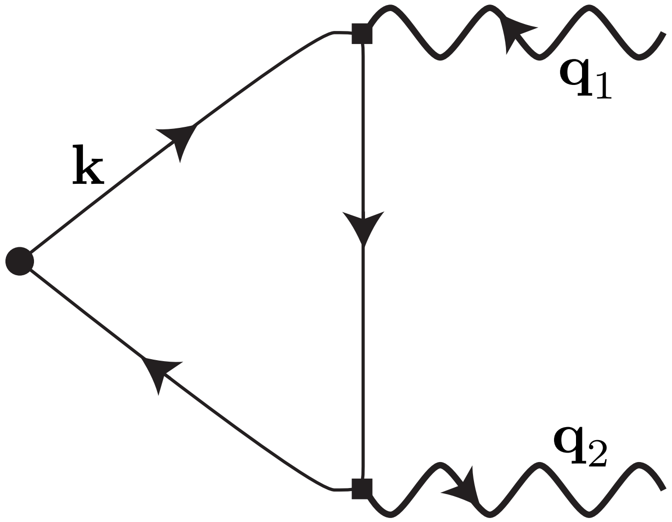
$$\tan\theta_{\mathbf{k}} = \frac{N_0}{(\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}+\mathbf{Q}})/2} \quad , \quad 0 < \theta_{\mathbf{k}} < \pi \tag{12}$$

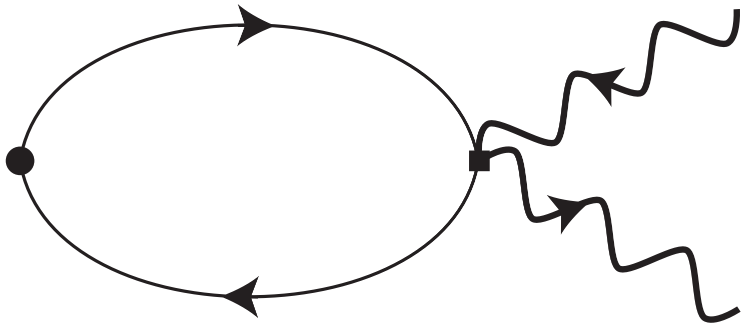
Note that these relations imply

$$u_{\mathbf{k}+\mathbf{Q}} = v_{\mathbf{k}} \quad , \quad v_{\mathbf{k}+\mathbf{Q}} = u_{\mathbf{k}} \quad , \quad E_{1,\mathbf{k}+\mathbf{Q}} = E_{1\mathbf{k}} \quad , \quad E_{2,\mathbf{k}+\mathbf{Q}} = E_{2\mathbf{k}}. \tag{13}$$

The contributions to $\langle c^\dagger(\mathbf{k}+\mathbf{Q}+\mathbf{q}/2)c(\mathbf{k}-\mathbf{q}/2)\rangle$ to second order in φ are shown below.







The last diagram vanishes identically, while the first two evaluate to

$$\left\langle c^{\dagger}(\mathbf{k}+\mathbf{Q})c(\mathbf{k}+\mathbf{q}_1-\mathbf{q}_2)\right\rangle = \sum_{\mathbf{q}_1,\mathbf{q}_2} J(\mathbf{k},\mathbf{q}_1,\mathbf{q}_2)\varphi^*(\mathbf{q}_2)\varphi(\mathbf{q}_1) \quad (14)$$

where

$$\begin{aligned} J(\mathbf{k},\mathbf{q}_1,\mathbf{q}_2) \;=\;& N_0^2\sum_{\omega}\Big[F(\mathbf{k})G(\mathbf{k}+\mathbf{Q}-\mathbf{q}_2)G(\mathbf{k}+\mathbf{q}_1-\mathbf{q}_2)\\ -\;& G(\mathbf{k}+\mathbf{Q})F(\mathbf{k}-\mathbf{q}_2)G(\mathbf{k}+\mathbf{q}_1-\mathbf{q}_2)\\ +\;& G(\mathbf{k}+\mathbf{Q})G(\mathbf{k}-\mathbf{q}_2)F(\mathbf{k}+\mathbf{q}_1-\mathbf{q}_2)\\ -\;& F(\mathbf{k})F(\mathbf{k}-\mathbf{q}_2)F(\mathbf{k}+\mathbf{q}_1-\mathbf{q}_2)\Big] - (\mathbf{q}_1 \leftrightarrow -\mathbf{q}_2) \end{aligned}$$

We now expand this to second order in $\mathbf{q}_1$ and $\mathbf{q}_2$. This leads to very lengthy expressions, which we simplified using Mathematica. In the end, a simple final result was obtained:

$$J(\mathbf{k},\mathbf{q}_1,\mathbf{q}_2) = (\mathbf{q}_1\times\mathbf{q}_2)\left(\frac{\partial\varepsilon_{\mathbf{k}+\mathbf{Q}}}{\partial\mathbf{k}}\times\frac{\partial\varepsilon_{\mathbf{k}}}{\partial\mathbf{k}}\right)\sum_{\omega}\frac{N_0^3}{(-i\omega+E_{1\mathbf{k}})^3(-i\omega+E_{2\mathbf{k}})^3} \quad (15)$$

Now we combine Eqs. (14) and (15). The Fourier transform of $(\mathbf{q}_1 \times \mathbf{q}_2)\varphi^*(\mathbf{q}_2)\varphi(\mathbf{q}_1)$ is $\partial_x\varphi\partial_y\varphi^* - \partial_y\varphi\partial_x\varphi^*$ and to second order in φ this equals $-2i(\partial_x n^x \partial_y n^y - \partial_x n^y \partial_y n^x)$. In a spin rotationally invariant form, this expression is proportional to the skyrmion density, and so we have one of our main results:

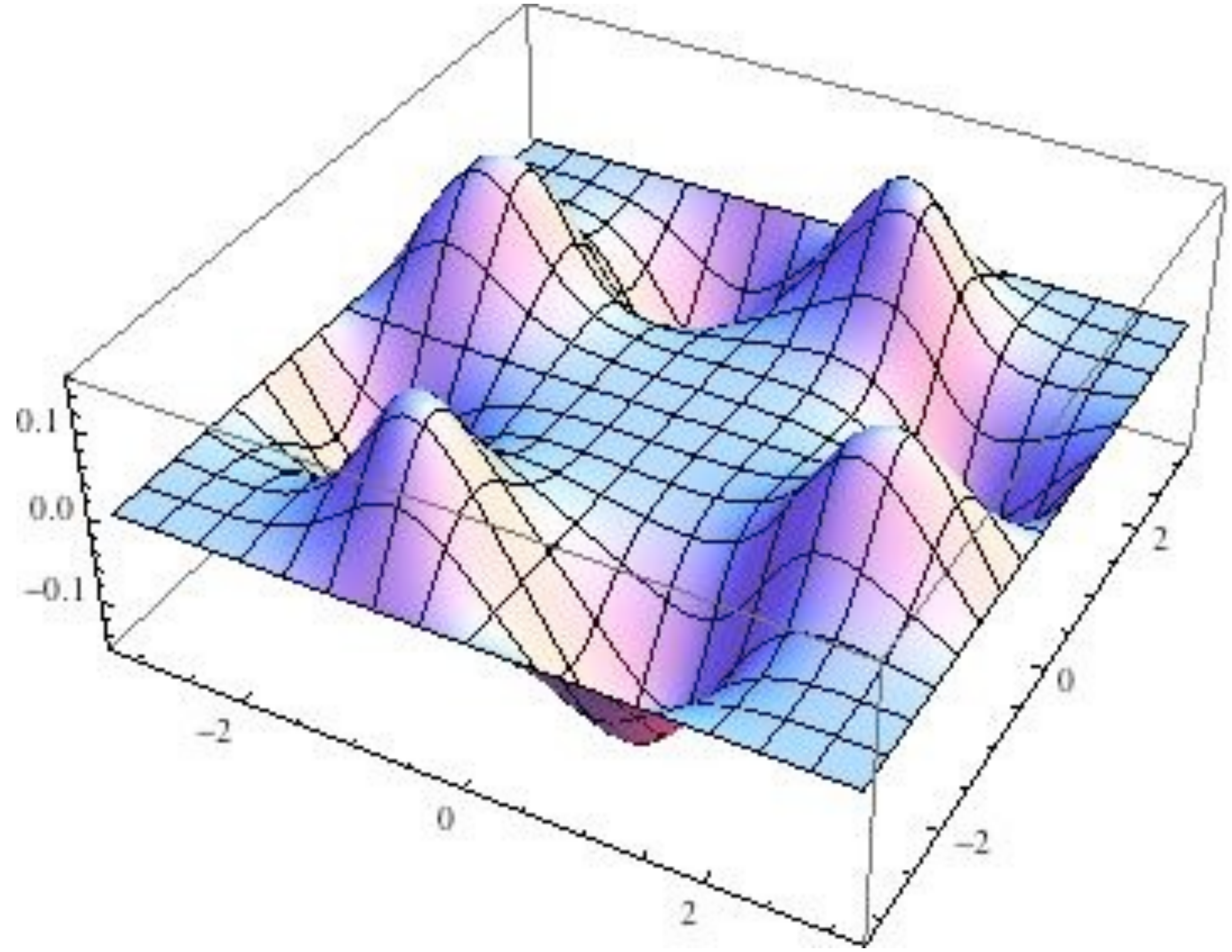
$$\boxed{\left\langle c^\dagger(\mathbf{k}+\mathbf{Q})c(\mathbf{k})\right\rangle(\mathbf{r}) = -i\mathcal{F}(\mathbf{k})\epsilon_{abc}n^a(\mathbf{r})\partial_x n^b(\mathbf{r})\partial_y n^c(\mathbf{r})} \tag{16}$$

where

$$\begin{aligned} \mathcal{F}(\mathbf{k}) &= \left(\frac{\partial\varepsilon_{\mathbf{k}+\mathbf{Q}}}{\partial\mathbf{k}} \times \frac{\partial\varepsilon_{\mathbf{k}}}{\partial\mathbf{k}}\right)\sum_\omega \frac{2N_0^3}{(-i\omega + E_{1\mathbf{k}})^3(-i\omega + E_{2\mathbf{k}})^3} \\ &= 6N_0^3\left(\frac{\partial\varepsilon_{\mathbf{k}+\mathbf{Q}}}{\partial\mathbf{k}} \times \frac{\partial\varepsilon_{\mathbf{k}}}{\partial\mathbf{k}}\right)\frac{(\mathrm{sgn}(E_{1\mathbf{k}}) - \mathrm{sgn}(E_{2\mathbf{k}}))}{(E_{1\mathbf{k}} - E_{2\mathbf{k}})^5}. \end{aligned} \tag{17}$$

In the last step, we have evaluated frequency summation at zero temperature. In the remaining analysis we will assume we are dealing with a fully gapped insulator with $E_{1\mathbf{k}} > 0$ and $E_{2\mathbf{k}} < 0$ over the entire Brillouin zone. A plot of $\mathcal{F}(\mathbf{k})$ for the insulating case is shown below for

$\varepsilon_{\mathbf{k}} = \cos k_x - \cos k_y + 0.4\cos(k_x + k_y) + 0.4\cos(k_x - k_y)$ and $N_0 = 1$.



The integral of $\mathcal{F}(\mathbf{k})$ is zero over the Brillouin zone. However, note that it has the same symmetry as the function $(\cos k_x - \cos k_y)\sin k_x \sin k_y$; so the integral of $\mathcal{F}(\mathbf{k})(\cos k_x - \cos k_y)\sin k_x \sin k_y$ is non-zero. This suggest we define the charge $\mathcal{Q}_t$ by

$$\boxed{\mathcal{Q}_t = -i \sum_{\mathbf{k}} c^{\dagger}(\mathbf{k}) c(\mathbf{k}+\mathbf{Q}) (\cos k_x - \cos k_y) \sin k_x \sin k_y .} \tag{18}$$

Note $\mathcal{Q}_t^{\dagger} = \mathcal{Q}_t$.

Eq. (16) implies that any quantum fluctuation which leads to a non-zero value of the skyrmion density $\epsilon_{abc} n^a \partial_x n^b \partial_y n^c$ will induce a change in $\mathcal{O}$. A change in $\mathcal{O}$ implies corresponding change in $\mathcal{Q}_t$ because the two observables have identical symmetries. A hedgehog tunneling event is one in which the spatial integral of $\epsilon_{abc} n^a \partial_x n^b \partial_y n^c$ (the skyrmion number) changes by 4π. Thus, before the hedgehog event $\langle \mathcal{Q}_t \rangle = 0$, while after the hedgehog tunneling event, we have $\langle \mathcal{Q}_t \rangle \neq 0$. We can normalize $\mathcal{Q}_t$ so that $\langle \mathcal{Q}_t \rangle = 1$ for each hedgehog, and the normalization constant will depend upon Eq. (17) and the details on the band structure. Then we have

$$\boxed{\mathcal{Q}_t \cong \text{skyrmion number.}} \tag{19}$$

This is the key result.

Competing orders

We now discuss the implications of the main result in Eq. (16) in the 'quantum disordered' phase where Néel order has been lost. Such a phase will have a proliferation of hedgehog/monpole tunnelling events, and so Eq. (16) implies that there will be correspondingly large fluctuations in the charge $\mathcal{Q}_t$. We can therefore expect that fluctuations in variables conjugate to $\mathcal{Q}_t$ will be suppressed, and will therefore have long-range order: this is the competing order induced by the geometric phase in Eq. (16). Thus any quantum variable conjugate to $\mathcal{Q}_t$ is a bona-fide competing order. There are many possibilities, but for now let us verify that the traditional VBS order does satisfy the requirements. Specifically, the VBS order is $V = V_x + iV_y$ defined by

$$\begin{aligned} V_x &= i \sum_{\mathbf{k}} c^\dagger(\mathbf{k}) c(\mathbf{k} + \mathbf{Q}_x) \sin k_x \\ V_y &= i \sum_{\mathbf{k}} c^\dagger(\mathbf{k}) c(\mathbf{k} + \mathbf{Q}_y) \sin k_y \end{aligned} \qquad (20)$$

where $\mathbf{Q}_x = (\pi, 0)$ and $\mathbf{Q}_y = (0, \pi)$. Note that these are Fourier transforms of

$$V_x = (-1)^{x_i} c^\dagger(\mathbf{r}_i) c(\mathbf{r}_i + \hat{x}) + \text{c.c.} \tag{21}$$

and similarly for V_y. Now we can commute the commutators

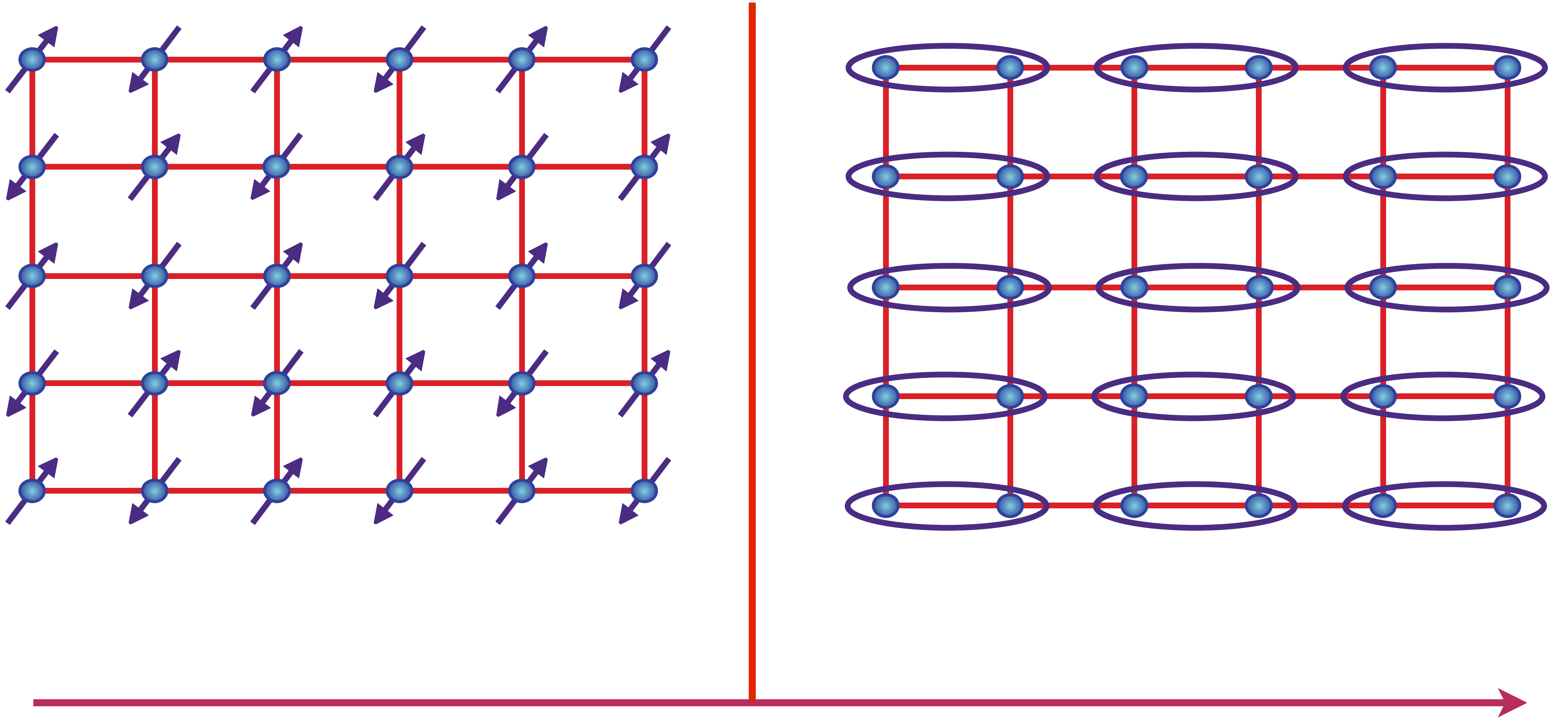
$$\begin{aligned}
[\mathcal{Q}_t, V_x] &= -\sum_{\mathbf{k}} c^\dagger(\mathbf{k}) c(\mathbf{k} + \mathbf{Q}_y) \sin k_y \frac{(\cos(k_x) - \cos(3k_x))}{2} \simeq iV_y \\
[\mathcal{Q}_t, V_y] &= \sum_{\mathbf{k}} c^\dagger(\mathbf{k}) c(\mathbf{k} + \mathbf{Q}_x) \sin k_x \frac{(\cos(k_y) - \cos(3k_y))}{2} \simeq -iV_x
\end{aligned}$$

Here the $\simeq$ means that the two operators have the same symmetry under the square lattice space group.
Thus we have the key result

$$[\mathcal{Q}_t, V] \simeq V. \tag{22}$$

This means that V is a raising order for $\mathcal{Q}_t$. But this is precisely the effect of the monopole tunneling event: in other words, V has the same quantum numbers as a monopole operator. Then we conclude that V is a competing order which becomes long-range in the quantum-disordered Néel phase.

S=1/2 square lattice antiferromagnetic insulator



“Frustration”

Lecture 4

1. SDW transition in a metal: instability to a modulated bond order, which is locally "Ising-nematic".

2. Fluctuations in *orientation* of local antiferromagnetic order: Berry phases associated with skyrmion density.

3. Gauge theory of fluctuating skyrmion density ("hedgehogs"): valence bond solid order.

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Gauge theory of quantum-disordered Néel state

By a suitable Hubbard-Stratonovich transformation of some interaction term, we can think of $\mathcal{Q}_t$ as a dynamic fluctuating field. The quantum-disordered phase has a low energy photon field A_μ (μ is a spacetime index), whose flux measures the skyrmion density.

$$\partial_x A_y - \partial_y A_x = \frac{1}{2}\epsilon_{abc} n^a \partial_x n^b \partial_y n^c. \tag{1}$$

Thus we can write an effective Lagrangian for this photon phase in terms of the dynamic fields $\mathcal{Q}_\mu$ and A_μ:

$$\mathcal{L}_{\rm eff} = \frac{\mathcal{Q}_\mu^2}{2K} + \frac{1}{2e^2}\left(\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda\right)^2 + \frac{i}{2\pi K}\mathcal{Q}_\mu \epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda \tag{2}$$

Here the last term represents the linear response computed to the skyrmion density computed earlier, after appropriate renormalization. The spatial components of $\mathcal{Q}_\mu$ are similarly defined by an appropriate response to the electric components of the photon. The term proportional to $1/e^2$ is the usual Maxwell action for A_μ, obtained after integrating out the z_α and ψ_p. Now we can perform the standard duality transformation of 2+1 dimensional electrodynamics on $\mathcal{L}_{\mathrm{eff}}$. Here these correspond to decoupling the Maxwell term by a Hubbard-Stratonovich field and then integrating out the photon; this yields

$$\mathcal{L}_{\mathrm{eff}} = \frac{\mathcal{Q}_\mu^2}{2K} + \frac{e^2}{8\pi^2}\left(\partial_\mu\phi - \frac{\mathcal{Q}_\mu}{K}\right)^2 \tag{3}$$

where $e^{i\phi}$ is the monopole operator. This allows to conclude that the correlations of $\partial_\mu\phi$ are the same as those of $\mathcal{Q}_\mu$, or in other words, we have the operator correspondence $\partial_\mu\phi \simeq \mathcal{Q}_\mu$.
Now let us look for an order parameter $V \sim e^{i\phi}$ so that $\partial_t\phi \simeq \mathcal{Q}_t$. Or more precisely, we need a V so that $-i(V^\dagger\partial_t V - V\partial_t V^\dagger) \simeq \mathcal{Q}_t$. It is easy to check that the VBS order parameter V does indeed satisfy the needed requirements.

Spin density wave

d-wave superconductivity

Fermi surface

Valence-bond order