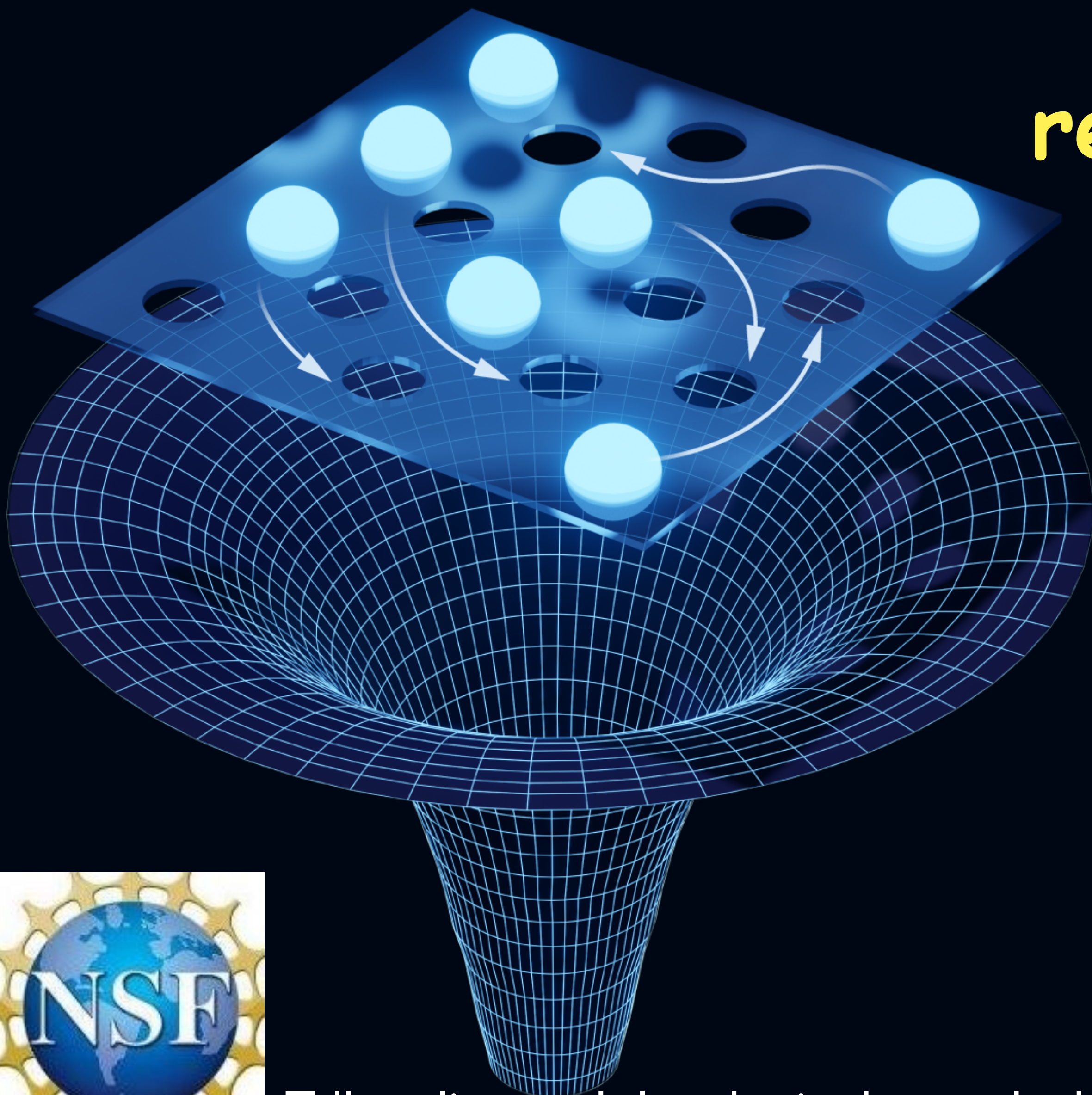


Quantum glasses, reparameterization invariance, Sachdev-Ye-Kitaev models



29th Solvay Conference on Physics:

“The Structure and Dynamics of Disordered Systems”

October 19-21, 2023

Subir Sachdev



Talk online: sachdev.physics.harvard.edu



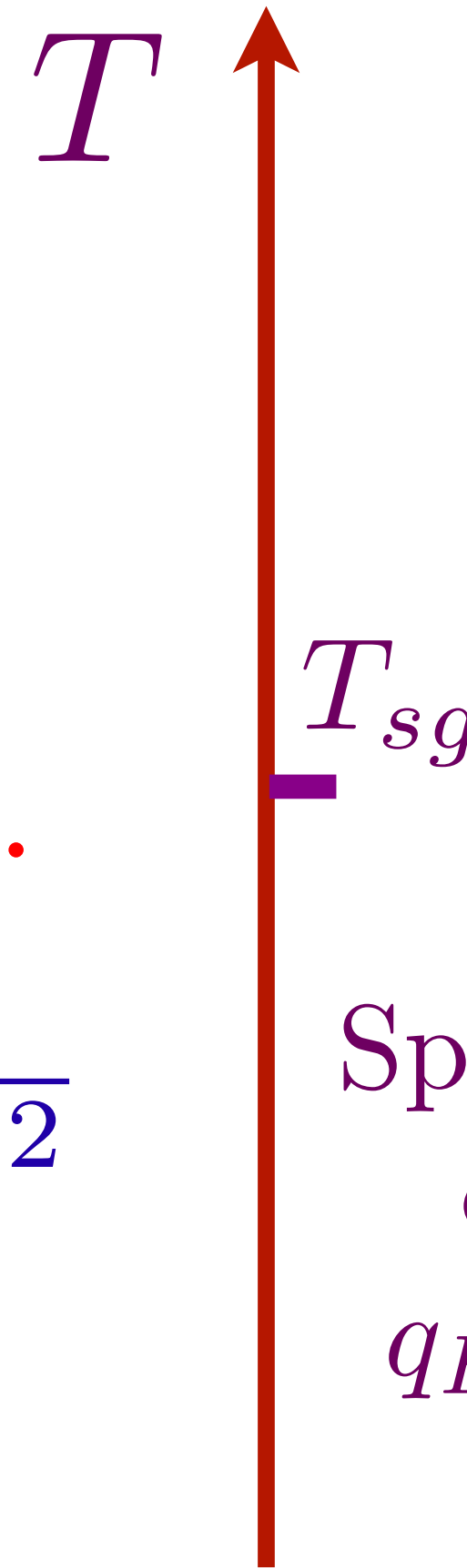
Sherrington-Kirkpatrick model

$$H = \frac{1}{2\sqrt{N}} \sum_{i,j=1}^N J_{ij} Z_i Z_j$$

$$\mathcal{Z} = \sum_{Z_i = \pm 1} e^{-H/T}$$

$$\overline{J_{ij}} = 0, \quad \overline{J_{ij}^2} = J^2, \quad \text{Different } J_{ij} \text{ uncorrelated.}$$

$$\text{Edwards-Anderson order parameter } q_{EA} = \overline{\langle Z_i \rangle^2}$$



Quantum Ising model

$$H = \frac{1}{2\sqrt{N}} \sum_{i,j=1}^N J_{ij} Z_i Z_j - g \sum_{i=1}^N X_i$$

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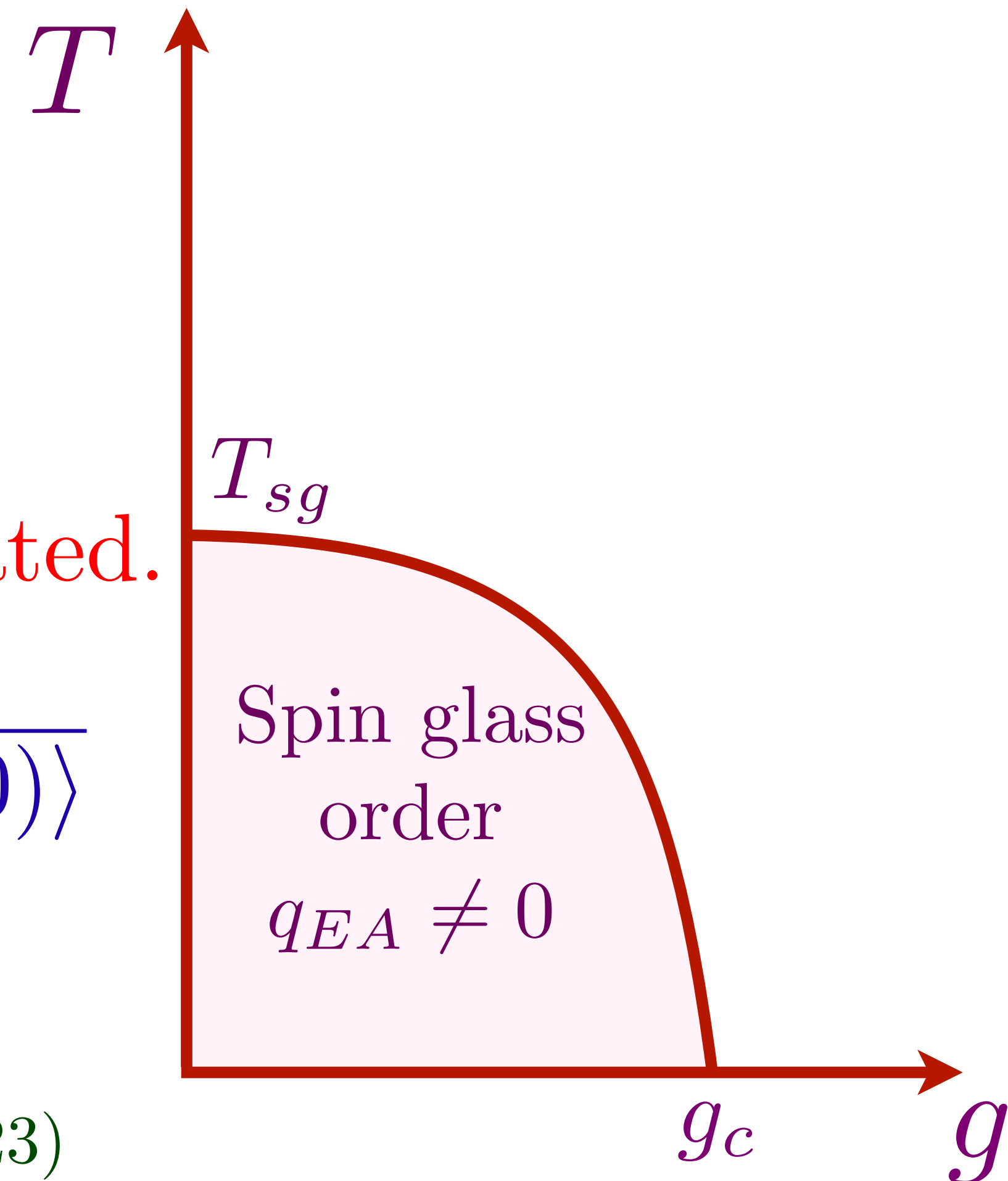
$$\text{EA order parameter } q_{EA} = \lim_{\tau \rightarrow \infty} \overline{\langle Z_i(\tau) Z_i(0) \rangle}$$

Related models describe numerous recent quantum simulators:

Ebadi *et al.*, Rydberg atoms in optical tweezers, *Science* (2022)

King *et al.*, Superconducting Qubits, D-Wave Systems, *Nature* (2023)

Maciejewski *et al.*, Superconducting Qubits, Rigetti (2023)



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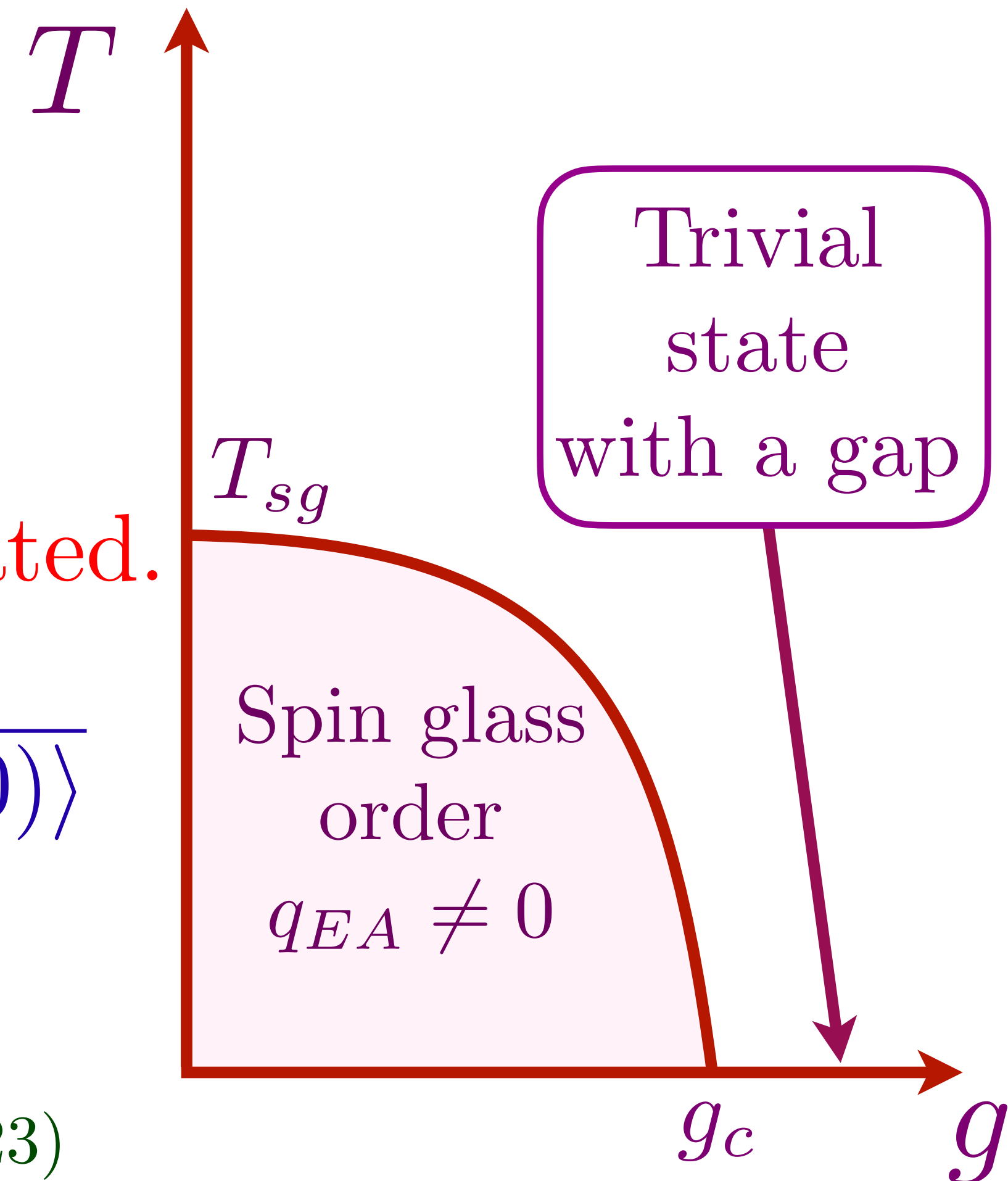
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T

Quantum-critical
“Planckian” dynamics
(with log violations):
Relaxation time $\sim \hbar/k_B T$

T_{sg}

Spin glass
order
 $q_{EA} \neq 0$

g_c

g

Quantum Ising model

$$H = \frac{1}{2\sqrt{N}} \sum_{i,j=1}^N J_{ij} Z_i Z_j - g \sum_{i=1}^N X_i$$

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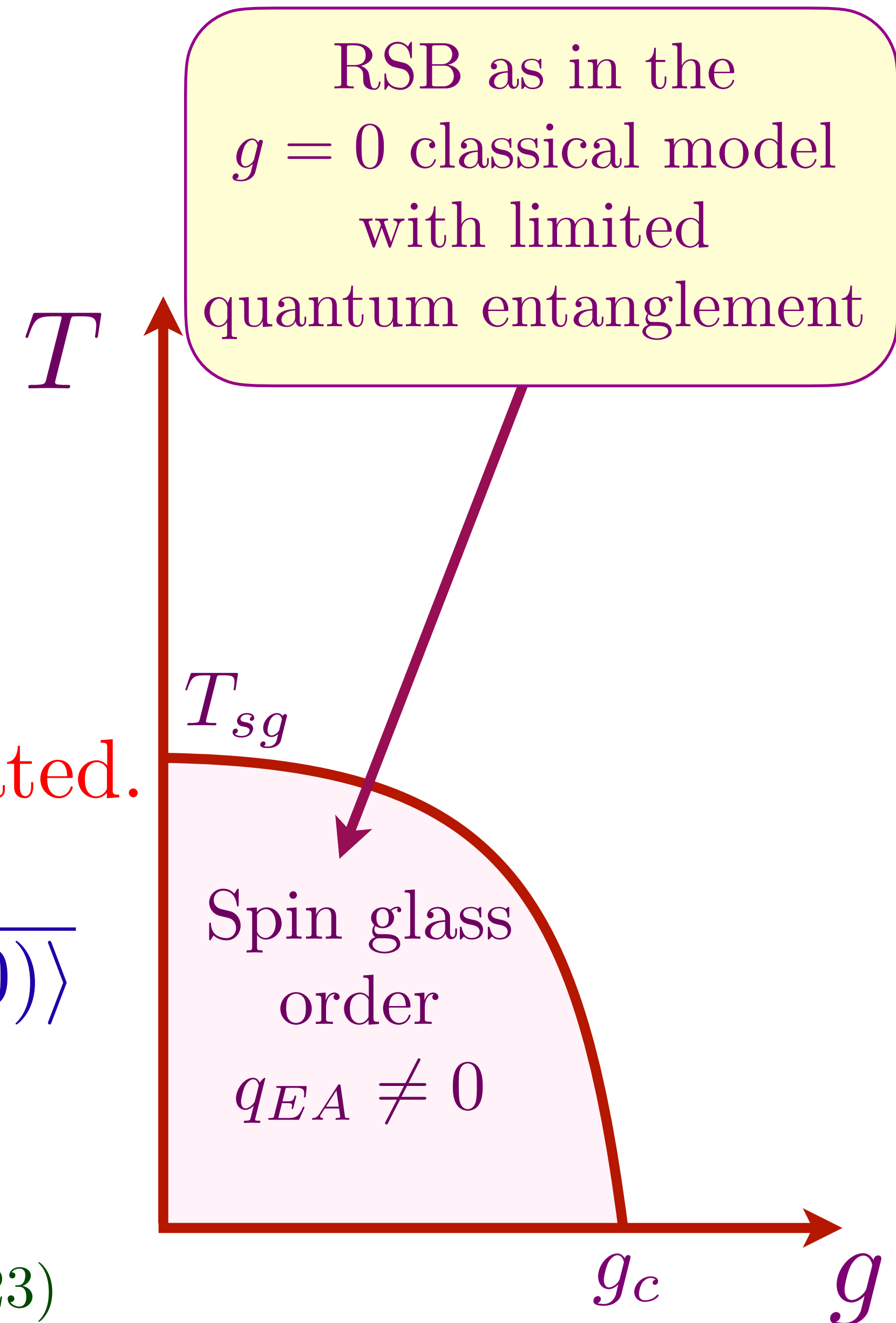
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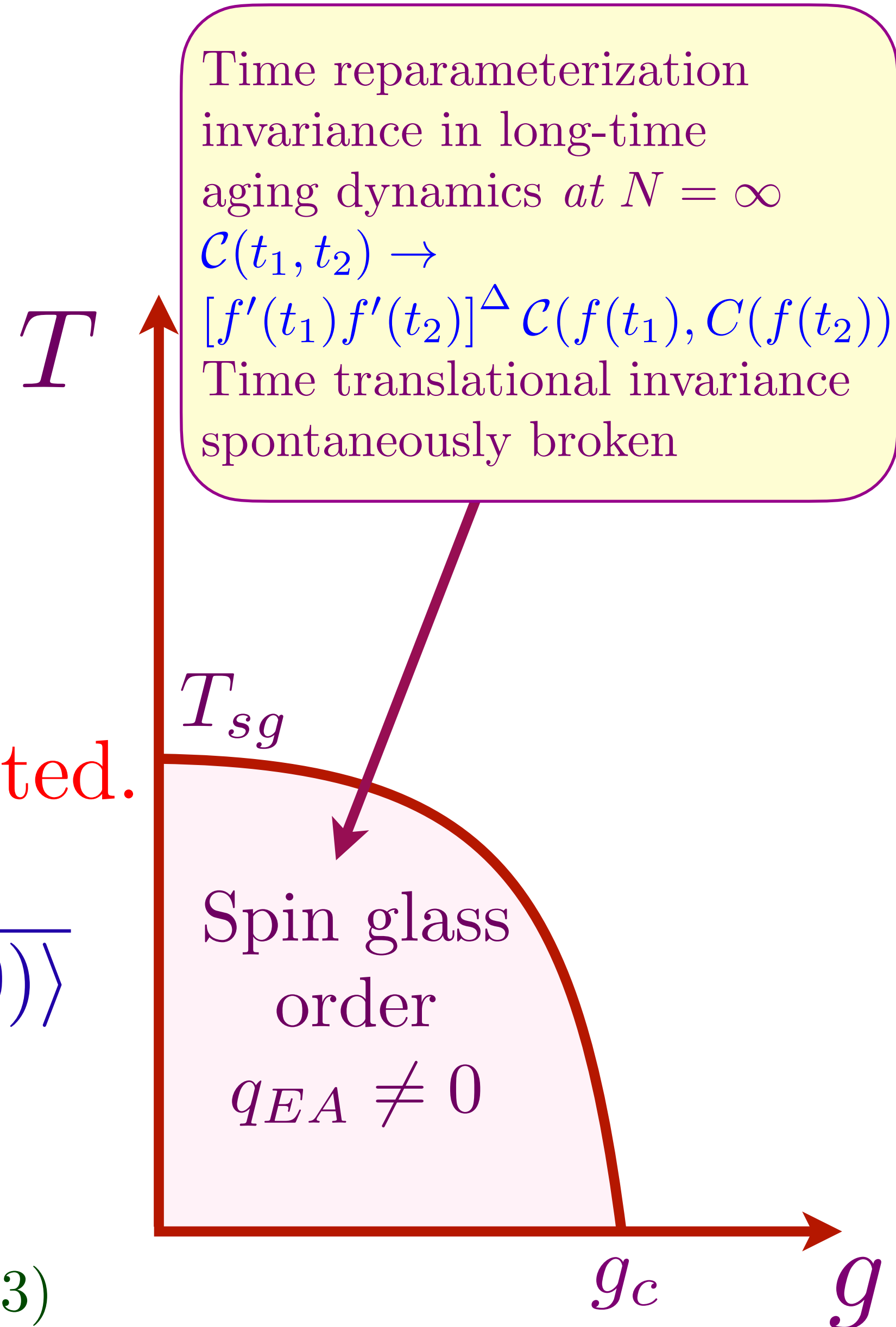
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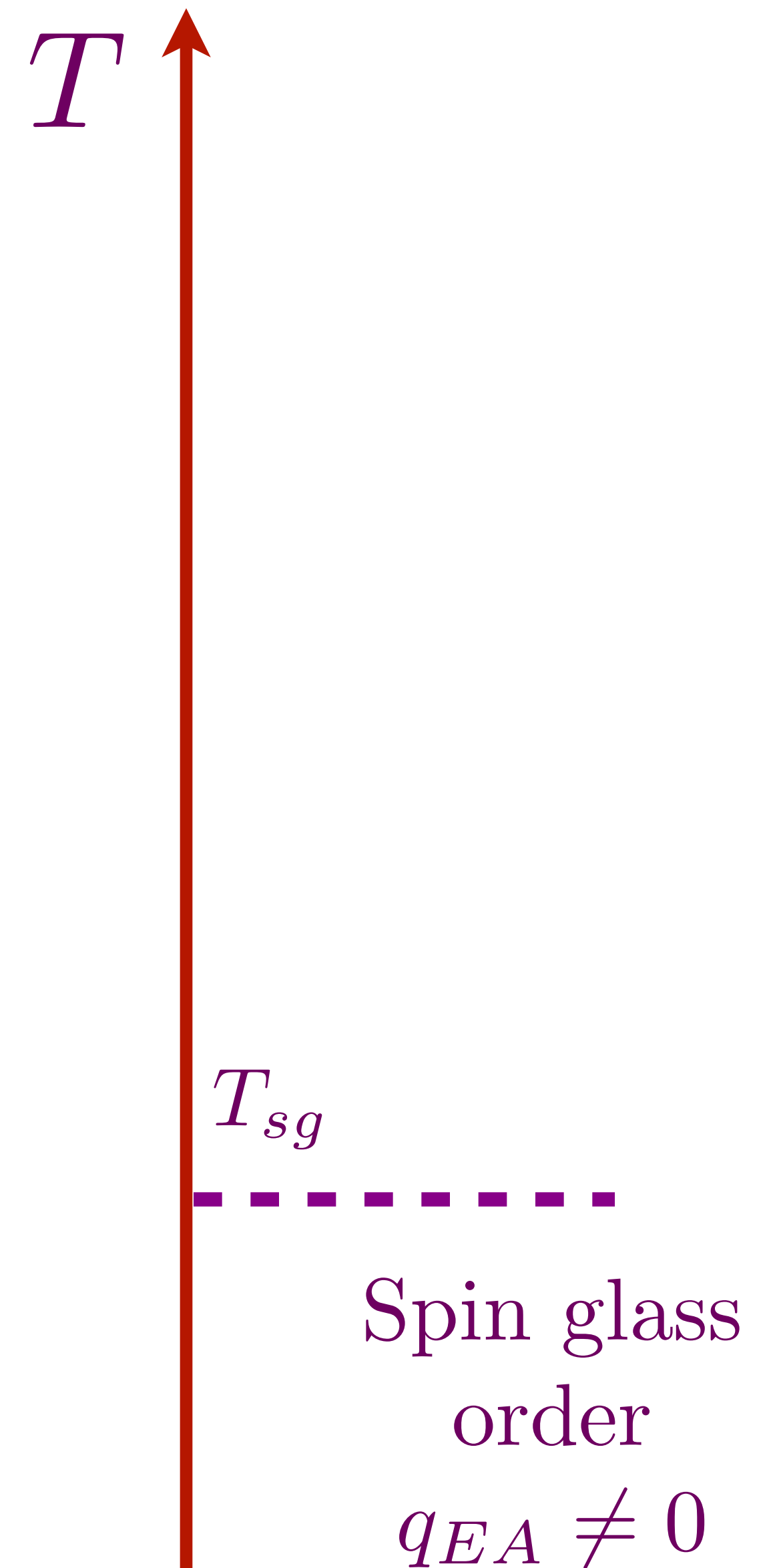


Quantum Heisenberg model

$$H = \frac{1}{2\sqrt{N}} \sum_{i,j=1}^N J_{ij} (X_i X_j + Y_i Y_j + Z_i Z_j)$$

$$\overline{J_{ij}} = 0, \quad \overline{J_{ij}^2} = J^2, \quad \text{Different } J_{ij} \text{ uncorrelated.}$$

- No trivial ground state—2-fold degenerate state on each site (LSM theorem, anomalies...)
- Can be generalized to $SU(M)$ symmetry, with $M^2 - 1$ ‘Pauli’ operators.



Related models describe underdoped cuprates,
atoms in optical cavities (Schleier-Smith, Esslinger, Lev)

Quantum Heisenberg model

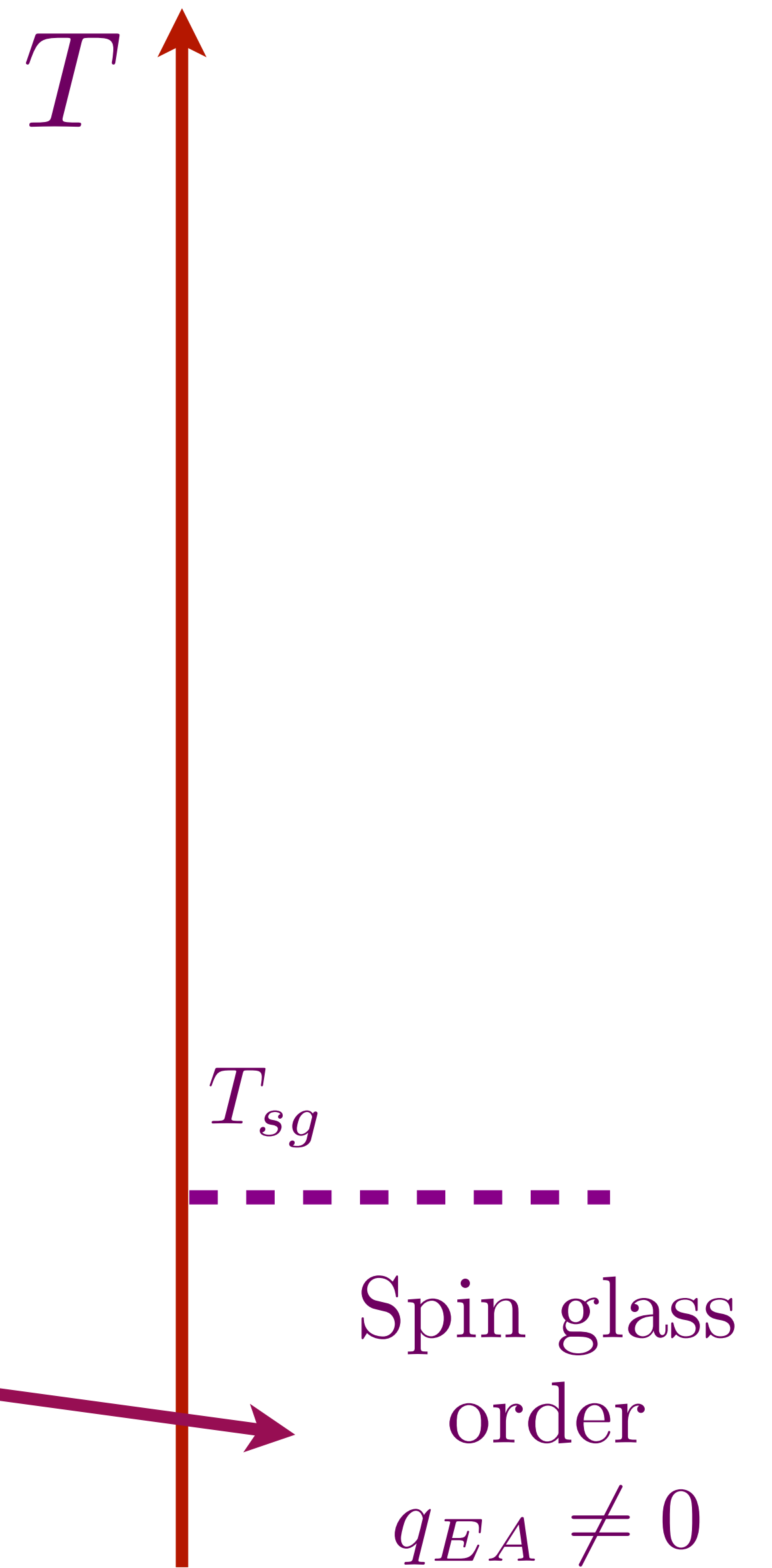
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Time reparameterization invariance
in long-time aging dynamics *at* $N = \infty$.
Time translational invariance
spontaneously broken.
Limited quantum entanglement.

Related models describe underdoped cuprates,
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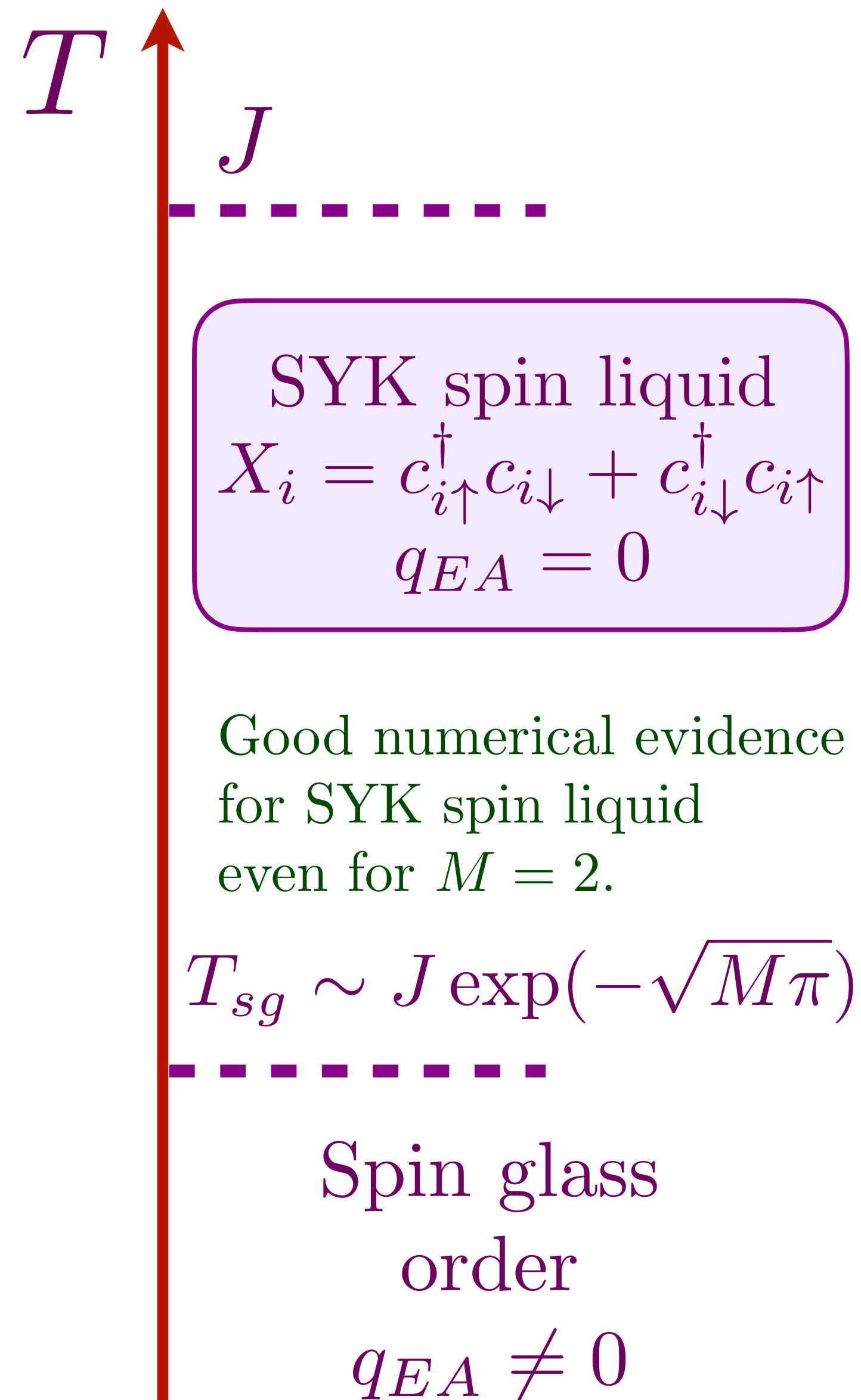
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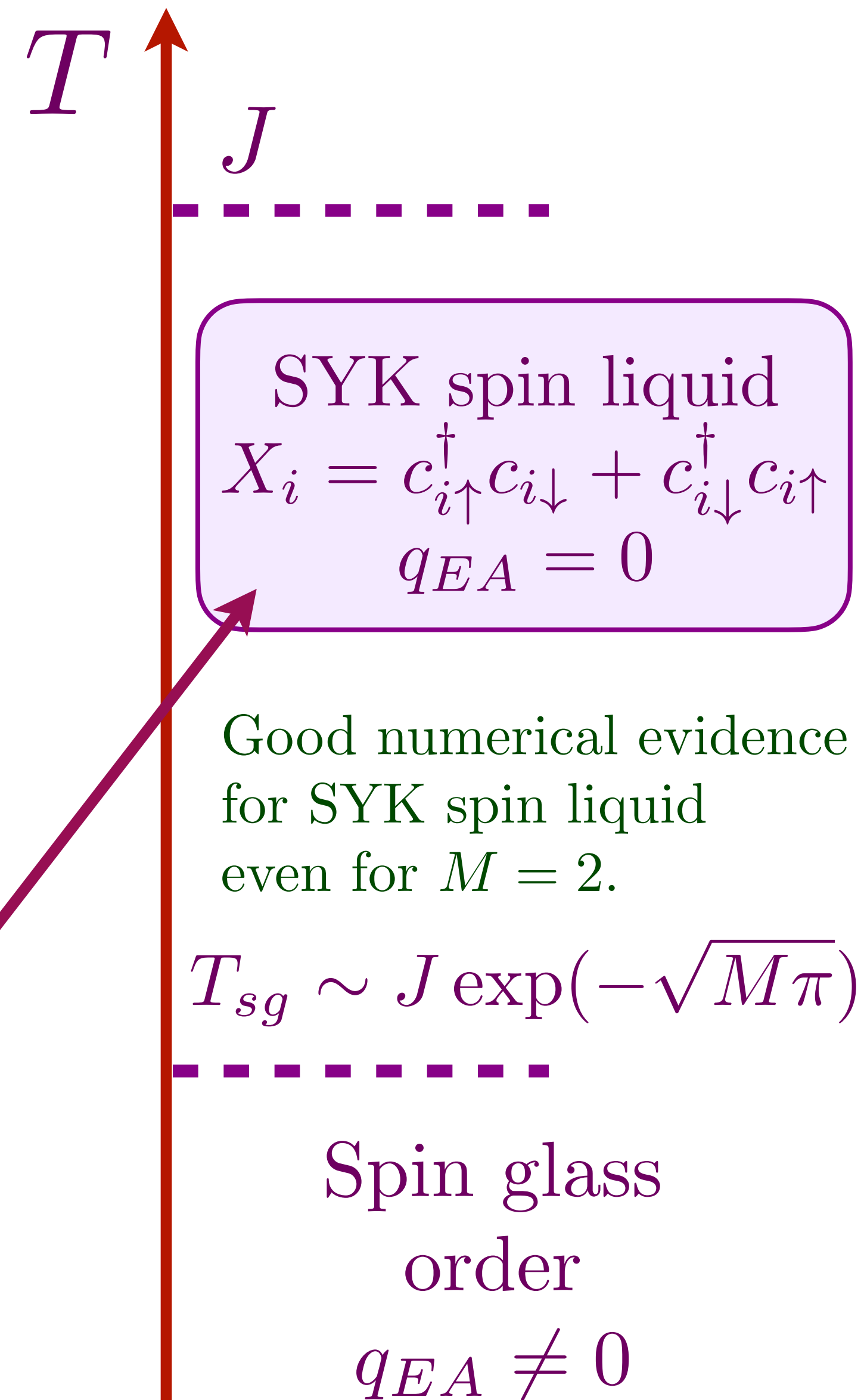
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- Can be generalized to $SU(M)$ symmetry, with $M^2 - 1$ ‘Pauli’ operators.

Complex quantum entanglement
 Maximal quantum chaos
 Planckian dynamics
 Relaxation time $\sim \hbar/(k_B T)$

Related models describe underdoped cuprates,
 atoms in optical cavities (Schleier-Smith, Esslinger, Lev)



Quantum Heisenberg model

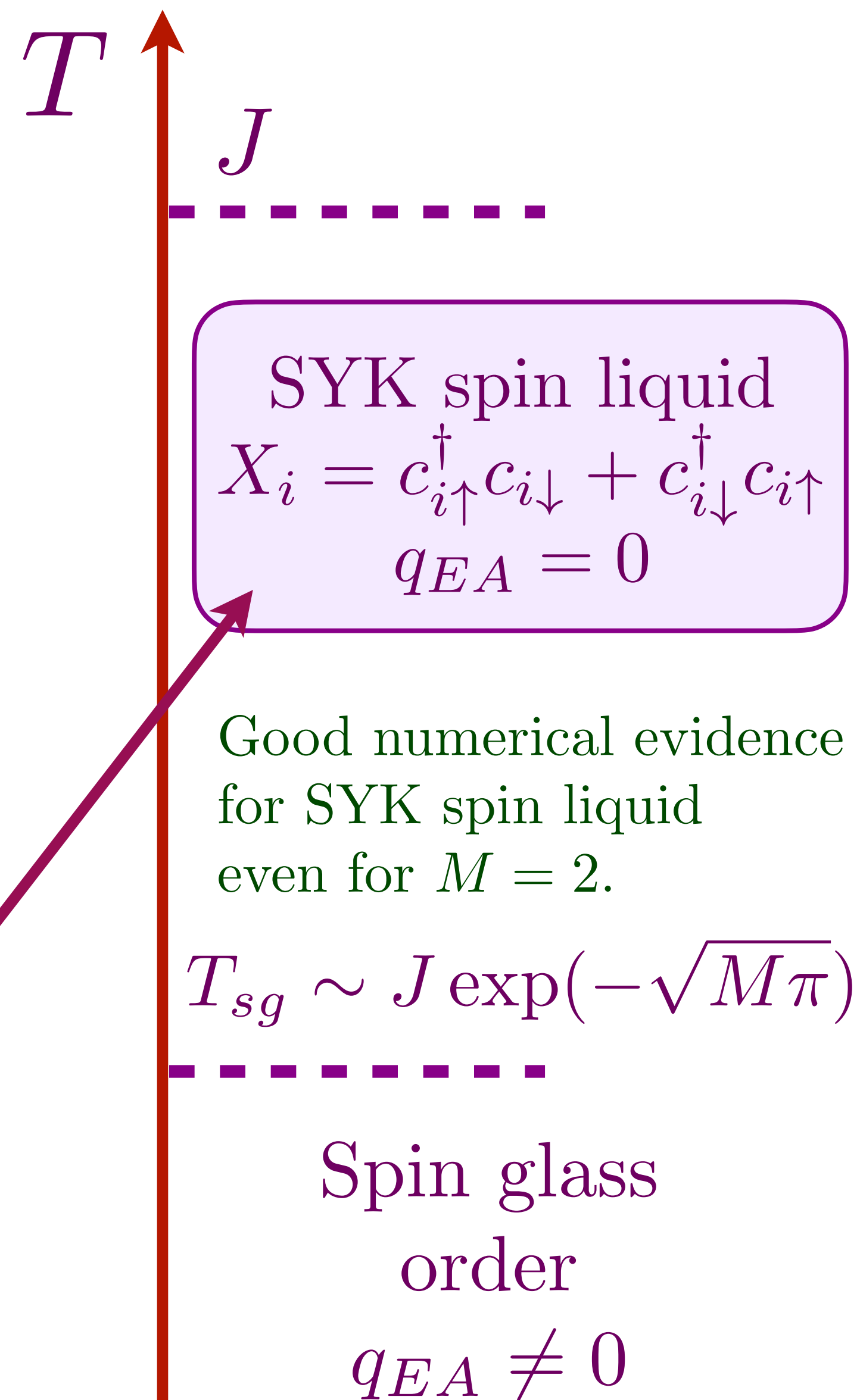
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- No trivial ground state—2-fold degenerate state on each site (LSM theorem, anomalies...)
- Can be generalized to $SU(M)$ symmetry, with $M^2 - 1$ ‘Pauli’ operators.

Time reparameterization invariance in low-frequency dynamics: plays no role at $N = \infty$, but is crucial for finite N quantum gravity theory.
Time translational invariance preserved.

Related models describe underdoped cuprates, atoms in optical cavities (Schleier-Smith, Esslinger, Lev)



Fermions with random and all-to-all hopping

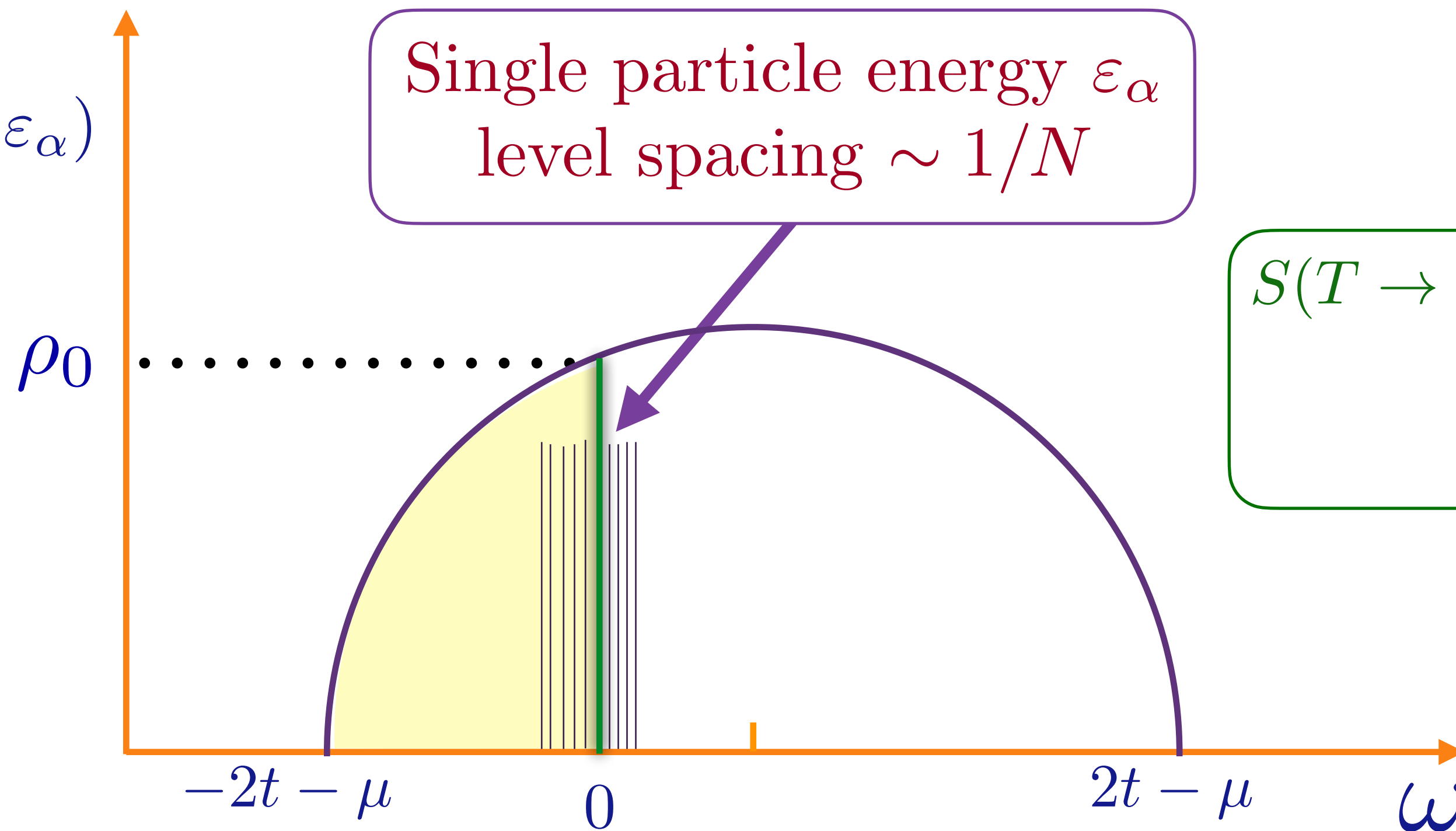
$$H = \frac{1}{(N)^{1/2}} \sum_{i,j=1}^N t_{ij} c_i^\dagger c_j - \mu \sum_i c_i^\dagger c_i$$

$$c_i c_j + c_j c_i = 0 \quad , \quad c_i c_j^\dagger + c_j^\dagger c_i = \delta_{ij}$$

$$\frac{1}{N} \sum_i c_i^\dagger c_i = Q$$

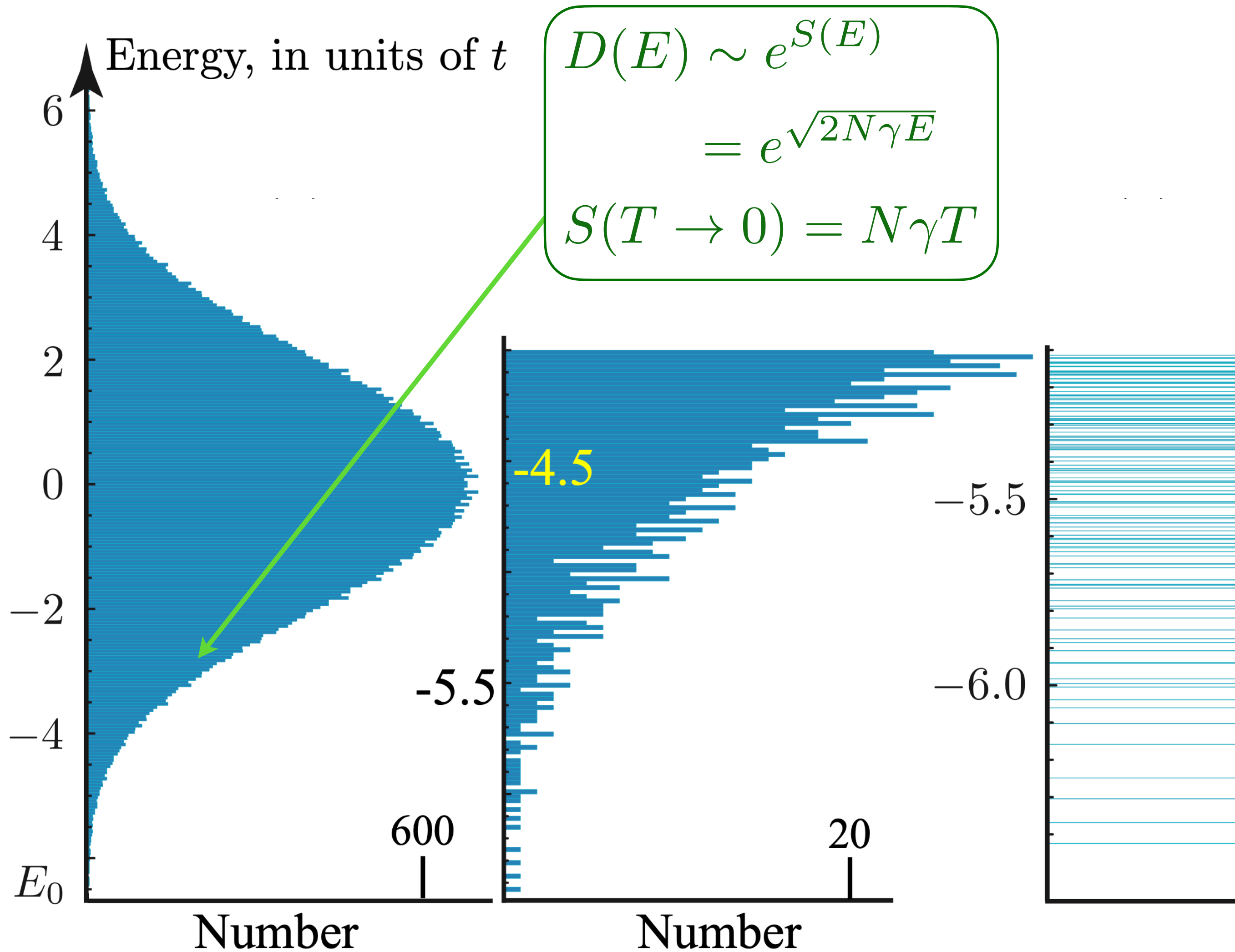
t_{ij} are independent random variables with $\overline{t_{ij}} = 0$ and $\overline{|t_{ij}|^2} = t^2$

$$\rho(\omega) = \frac{1}{N} \sum_\alpha \delta(\omega - \varepsilon_\alpha)$$



Fermions with random and all-to-all hopping

$$D(E) = \sum_i \delta(E - E_i); \quad E_0 + E_i \Rightarrow \text{Many body eigenvalue}$$



$$D(E) \sim e^{S(E)}$$

$$= e^{\sqrt{2N\gamma E}}$$

$$S(T \rightarrow 0) = N\gamma T$$

For random matrix model:
 $E_0 + E_i = \sum_{\alpha} n_{\alpha} \varepsilon_{\alpha}$
 $n_{\alpha} = 0, 1,$
 occupation number

$$D(E) \sim N$$

The SYK model: fermions with random and all-to-all interactions

(See also: the “2-Body Random Ensemble” in nuclear physics; did not obtain the large N limit;
Brody et al. *Rev. Mod. Phys.* **53**, 385 (1981))

$$\mathcal{H} = \frac{1}{(2N)^{3/2}} \sum_{\alpha, \beta, \gamma, \delta=1}^N U_{\alpha\beta;\gamma\delta} c_{\alpha}^{\dagger} c_{\beta}^{\dagger} c_{\gamma} c_{\delta} - \mu \sum_{\alpha} c_{\alpha}^{\dagger} c_{\alpha}$$

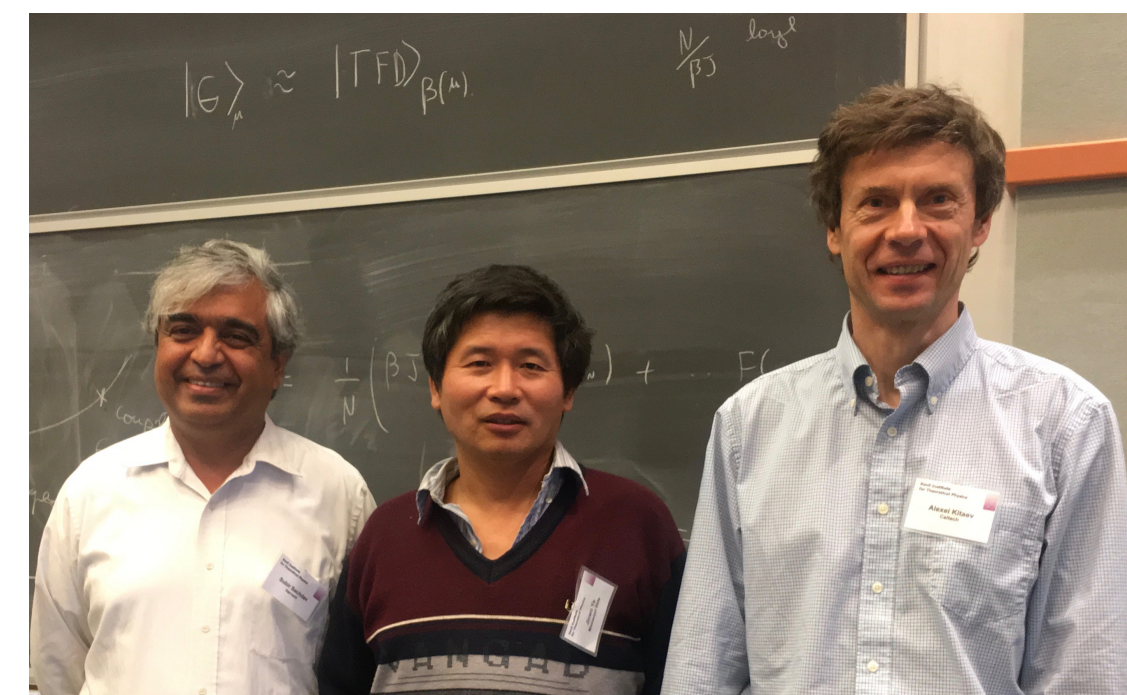
$$c_{\alpha} c_{\beta} + c_{\beta} c_{\alpha} = 0 \quad , \quad c_{\alpha} c_{\beta}^{\dagger} + c_{\beta}^{\dagger} c_{\alpha} = \delta_{\alpha\beta}$$

$$\mathcal{Q} = \frac{1}{N} \sum_{\alpha} c_{\alpha}^{\dagger} c_{\alpha}; \quad [\mathcal{H}, \mathcal{Q}] = 0; \quad 0 \leq \mathcal{Q} \leq 1$$

$U_{\alpha\beta;\gamma\delta}$ are independent random variables with $\overline{U_{\alpha\beta;\gamma\delta}} = 0$ and $\overline{|U_{\alpha\beta;\gamma\delta}|^2} = U^2$
 $N \rightarrow \infty$ yields critical strange metal.

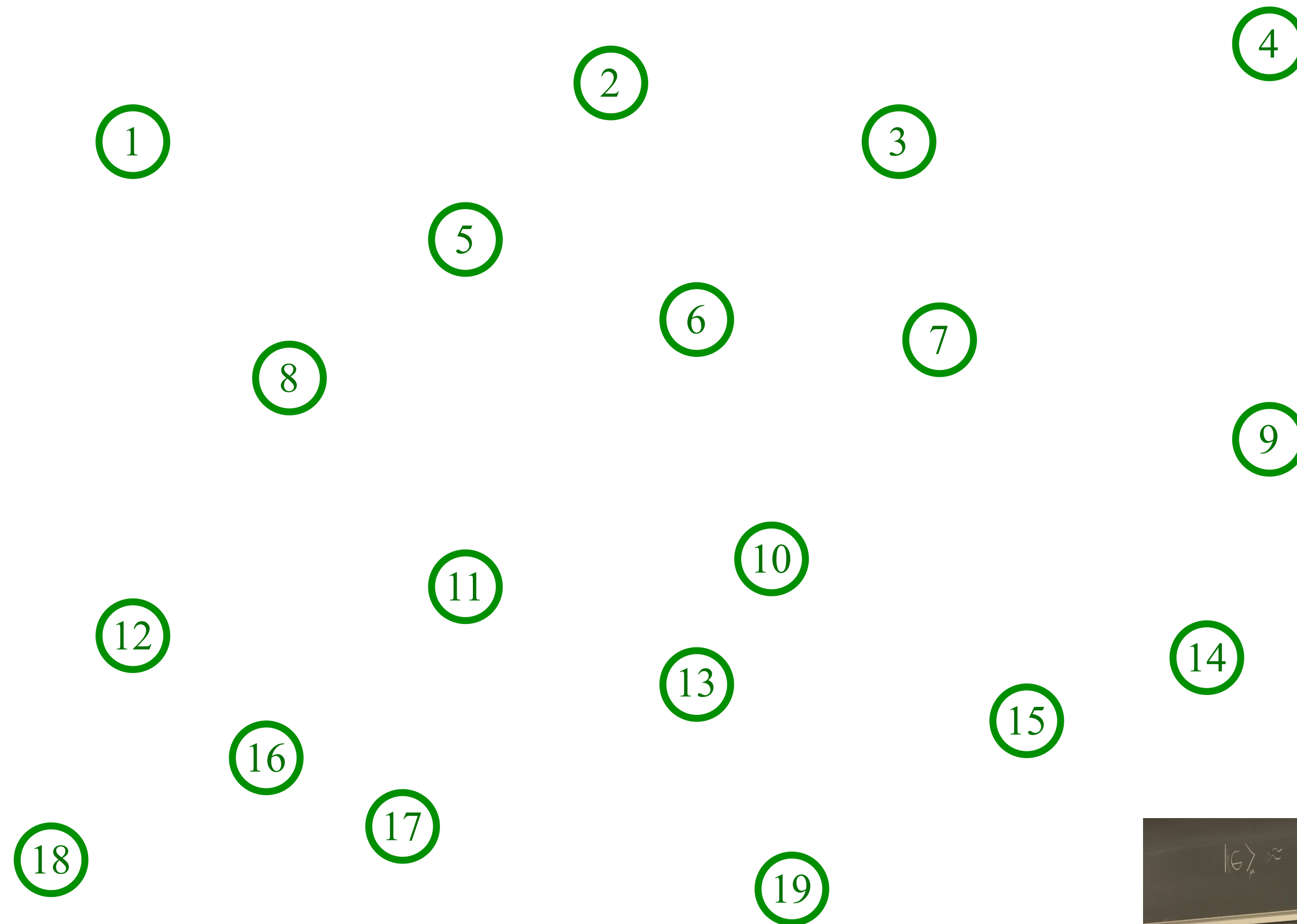
Has no spin glass phase!

Proposed realizations involving Majorana zero modes (Chew 2017, Pikulin 2017), quantum processors (Garcia-Alvarez 2016, Babbush 2018), ultracold gases (Dan-shita 2017, Tigran 2021, Uhrich 2023) and disordered graphene flakes (Chen 2018, Brzezinska 2022). Simulations of the SYK model have been achieved on quantum processors (Jafferis 2022) and controllable nuclear-spin-chain simulators (Luo 2019), and thermoelectric transport in graphene flakes is consistent with SYK behavior (Anderson 2023).

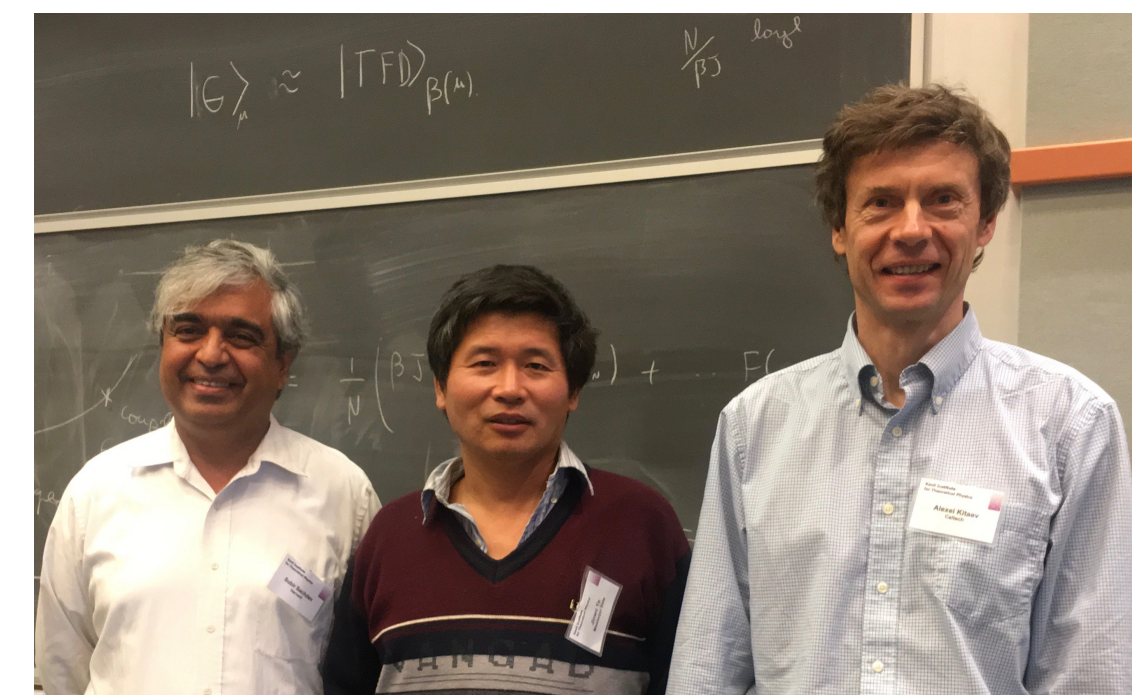


The SYK model

Sachdev, Ye (1993); Kitaev (2015)

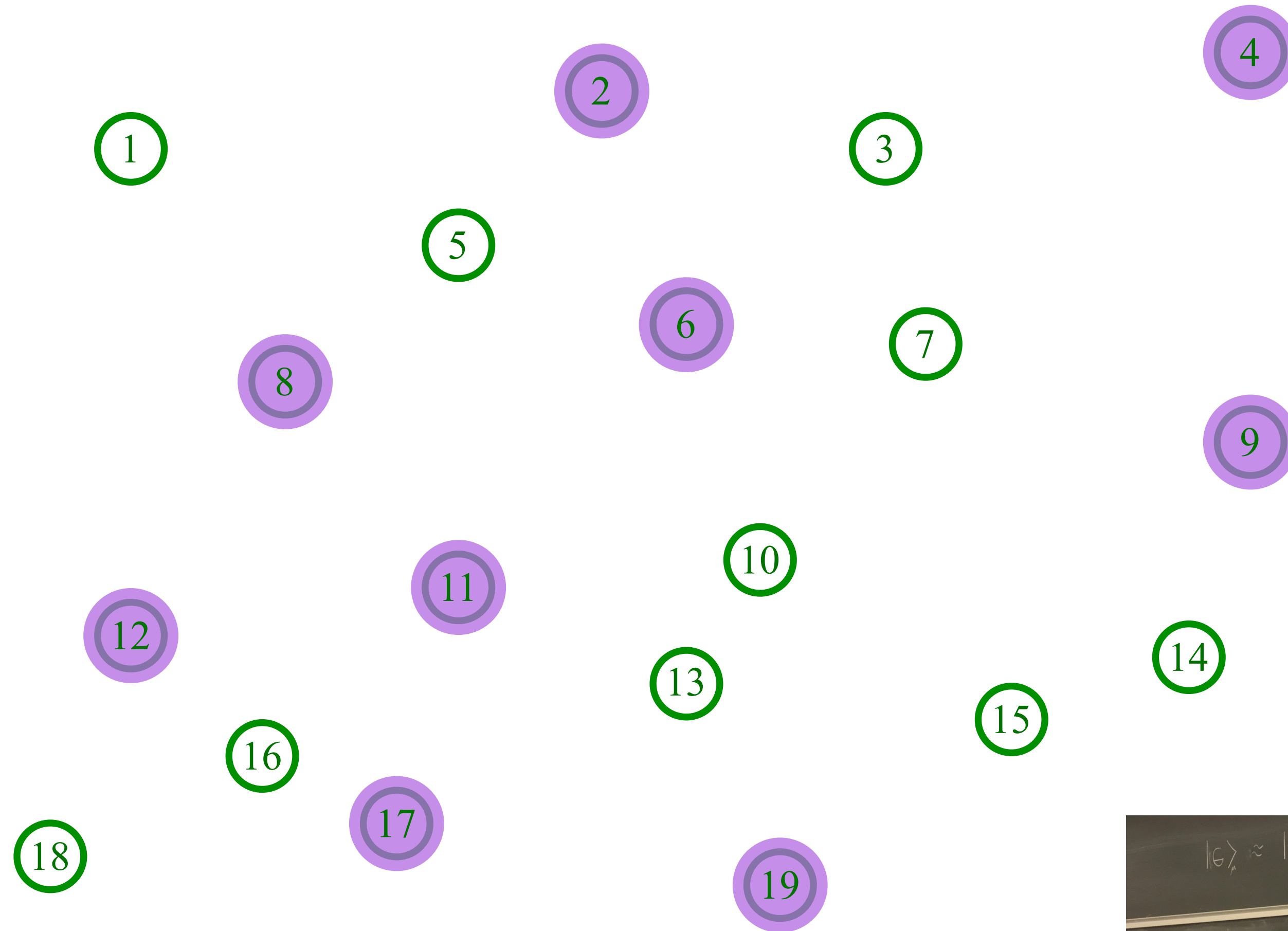


Pick a set of random positions

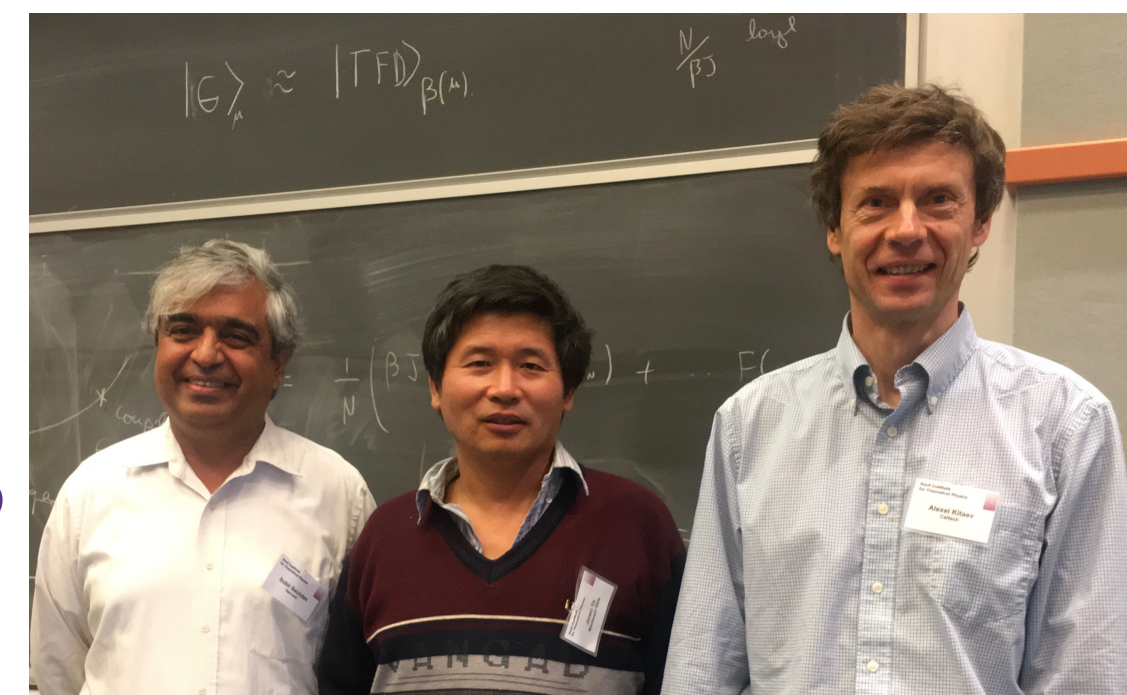


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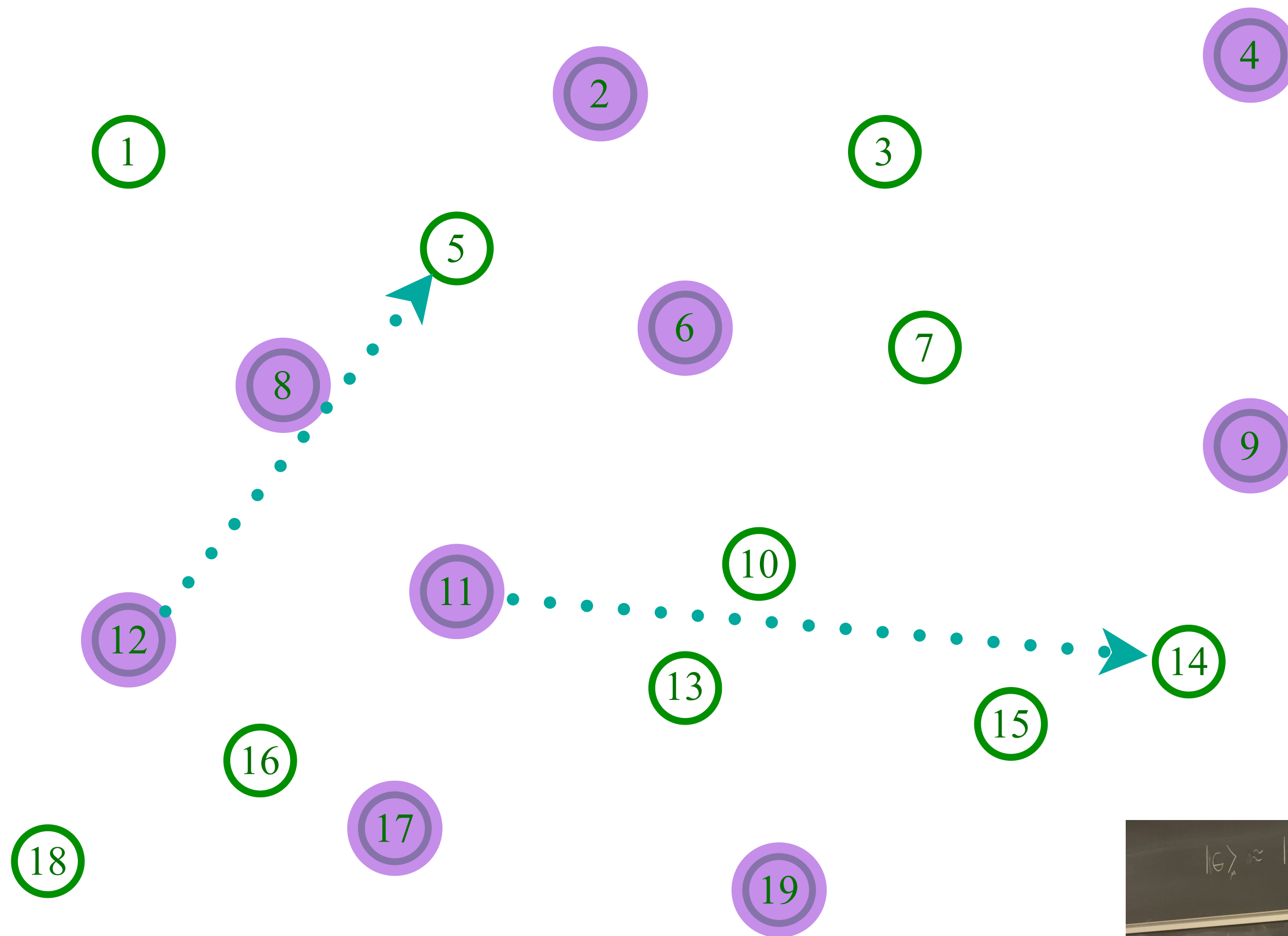
Place electrons randomly on some sites



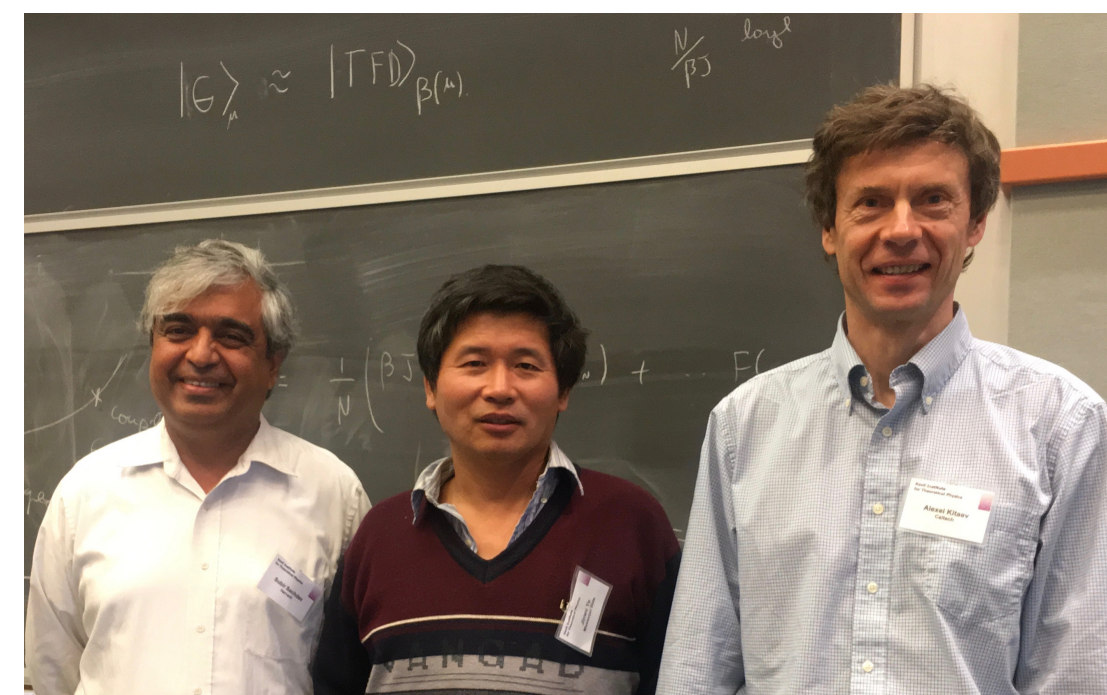
The SYK model

Sachdev, Ye (1993); Kitaev (2015)

$$U_{11,12;5,14}$$



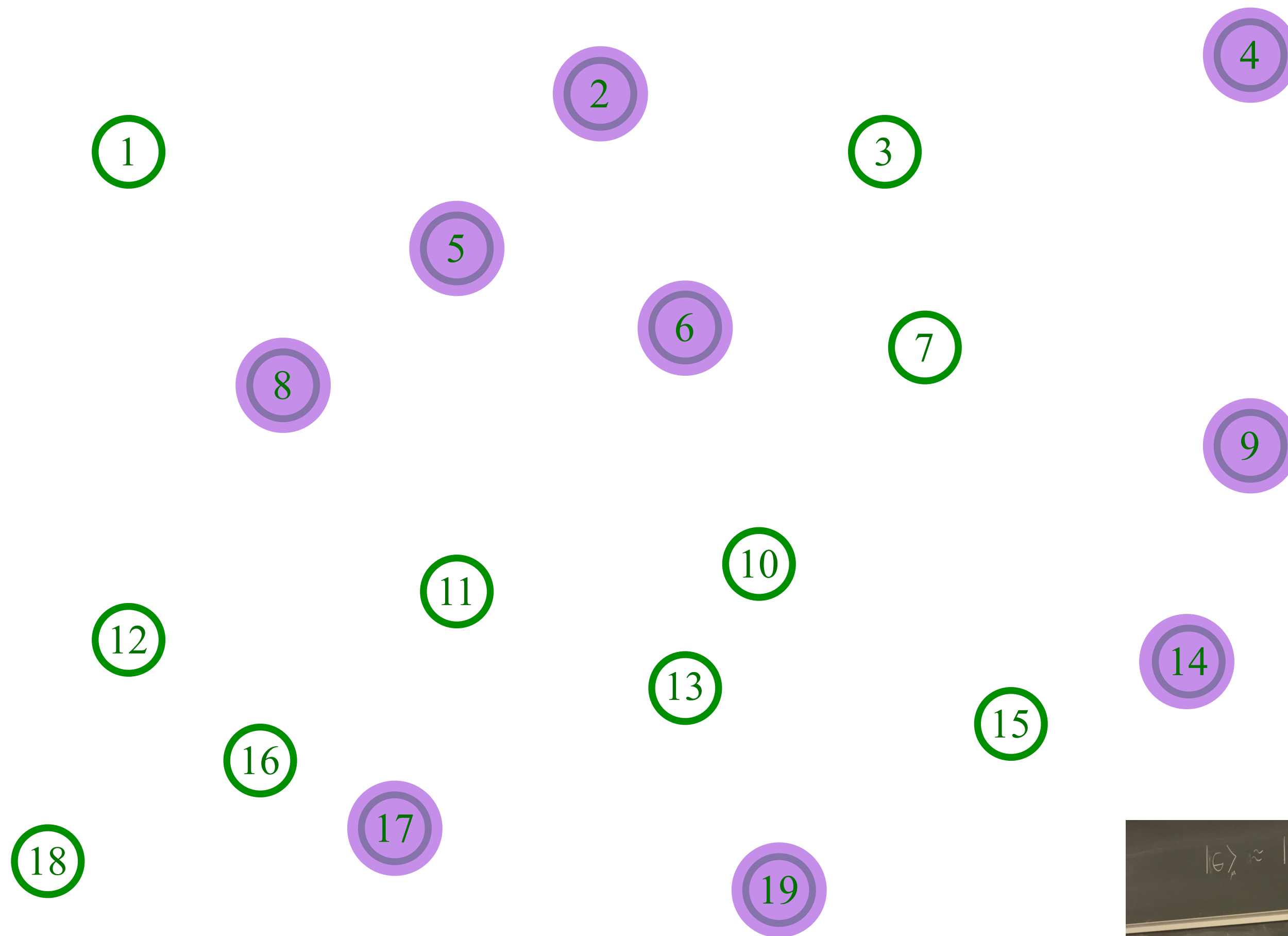
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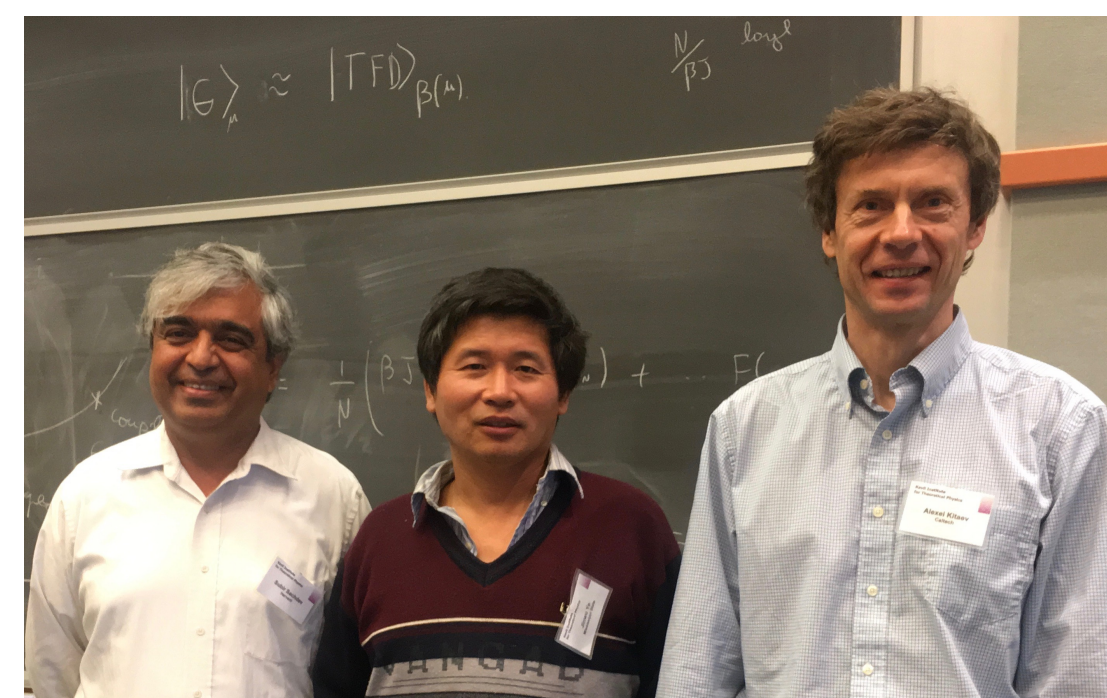
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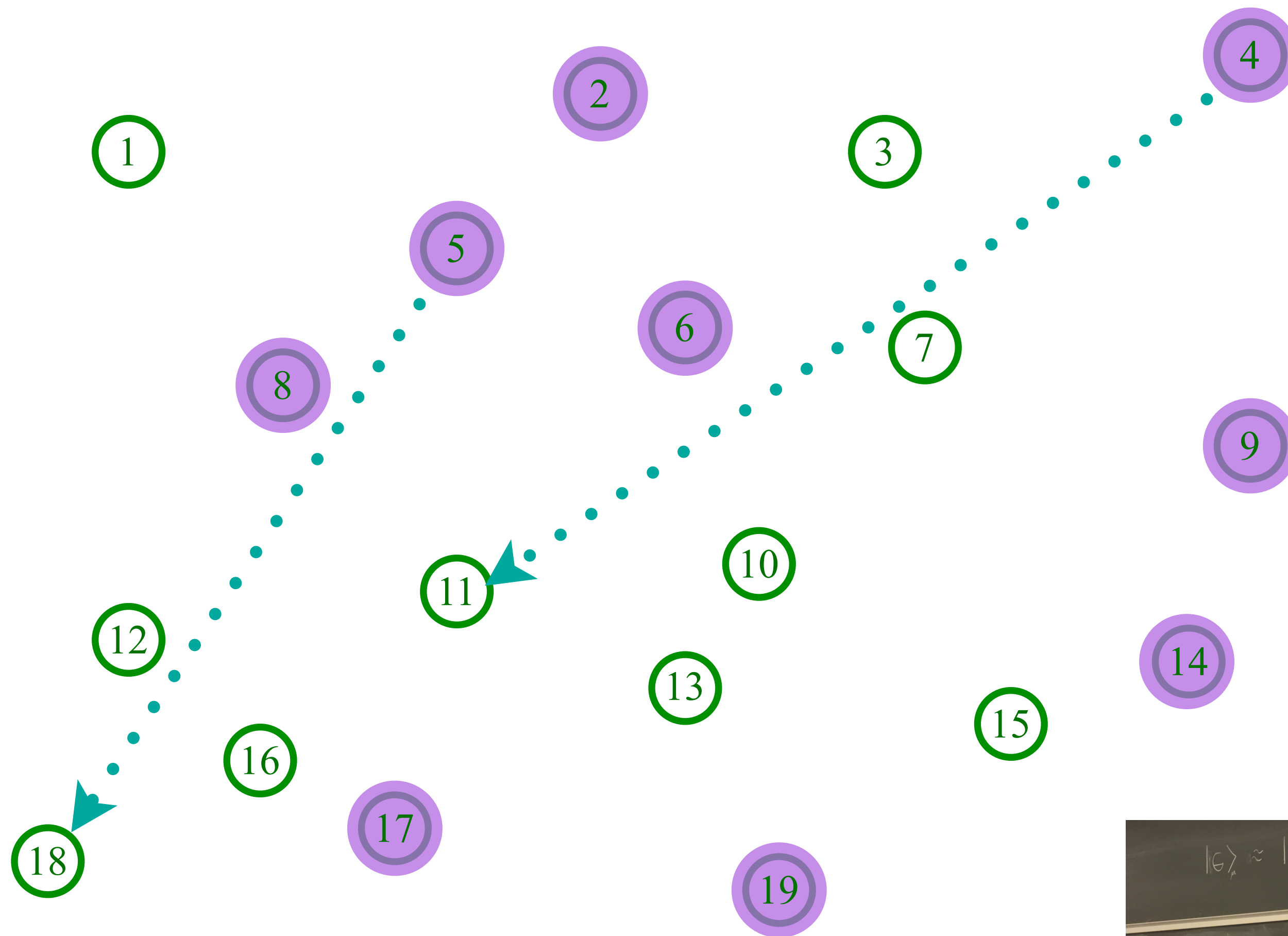
Entangle electrons pairwise randomly



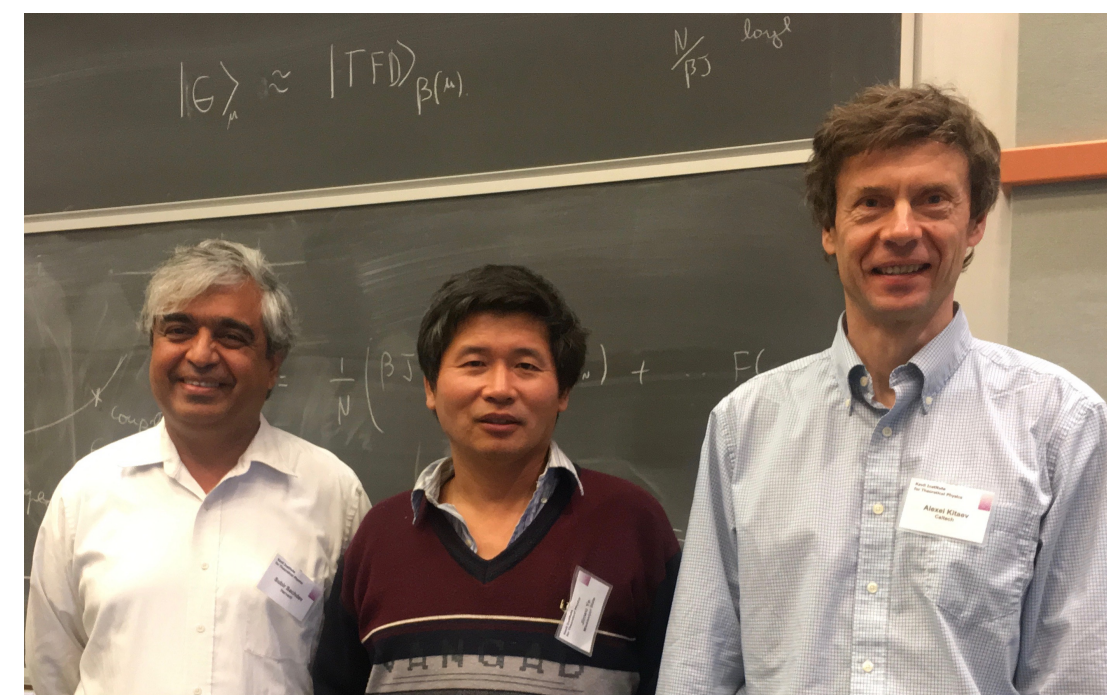
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$$U_{4,5;11,18}$$



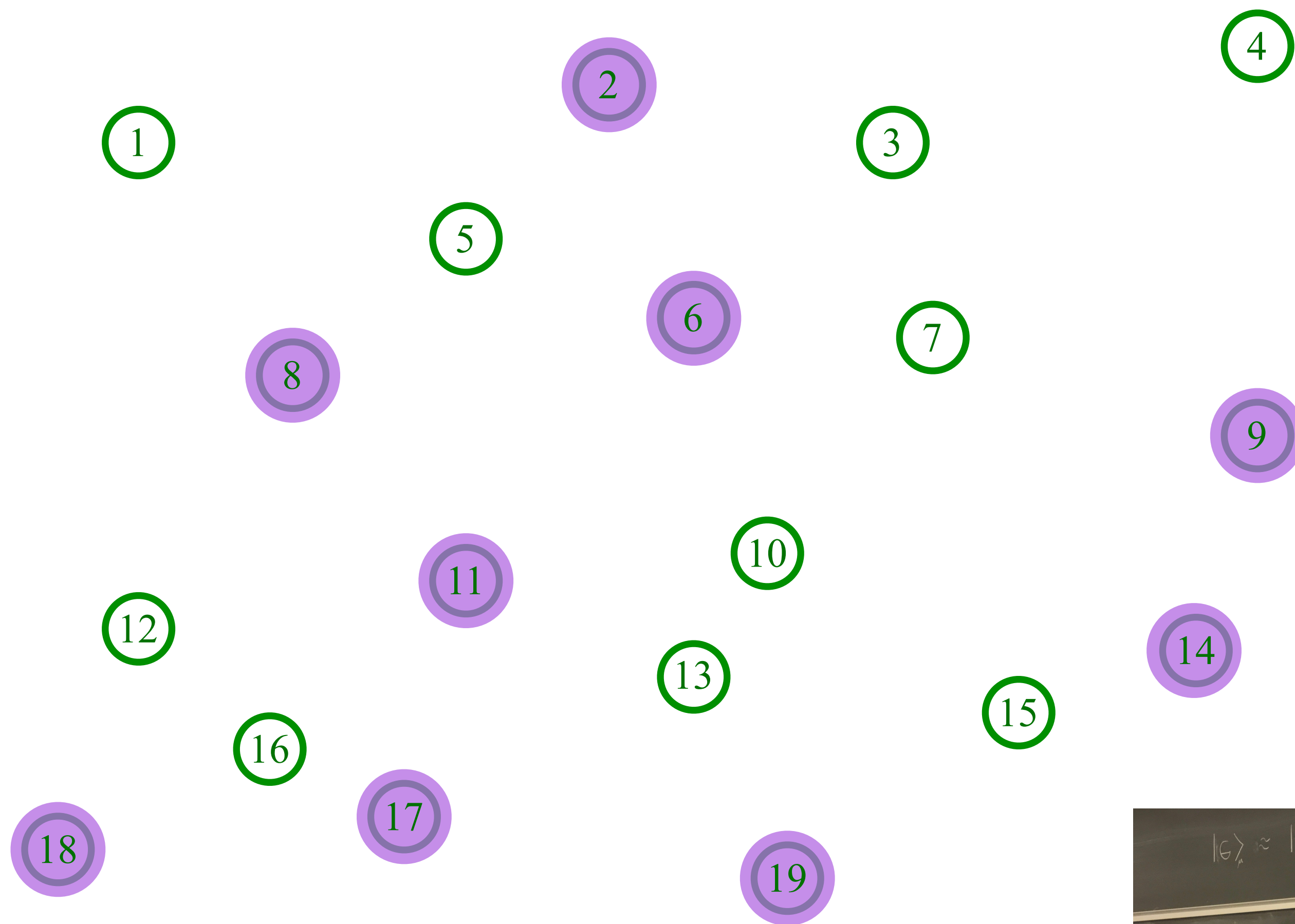
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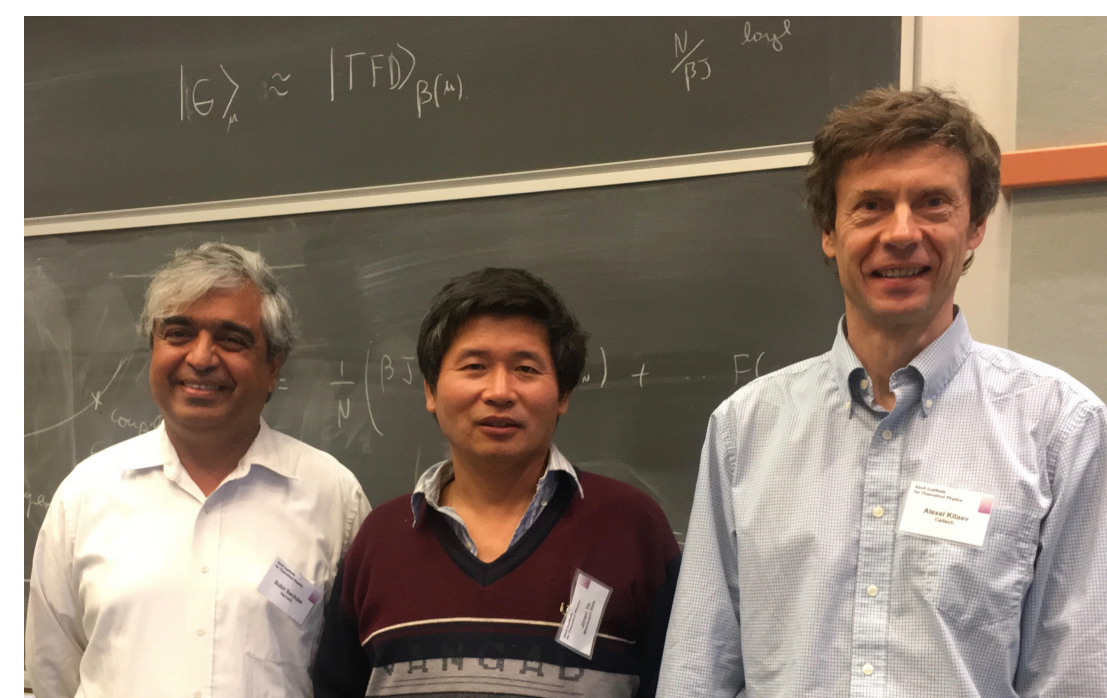
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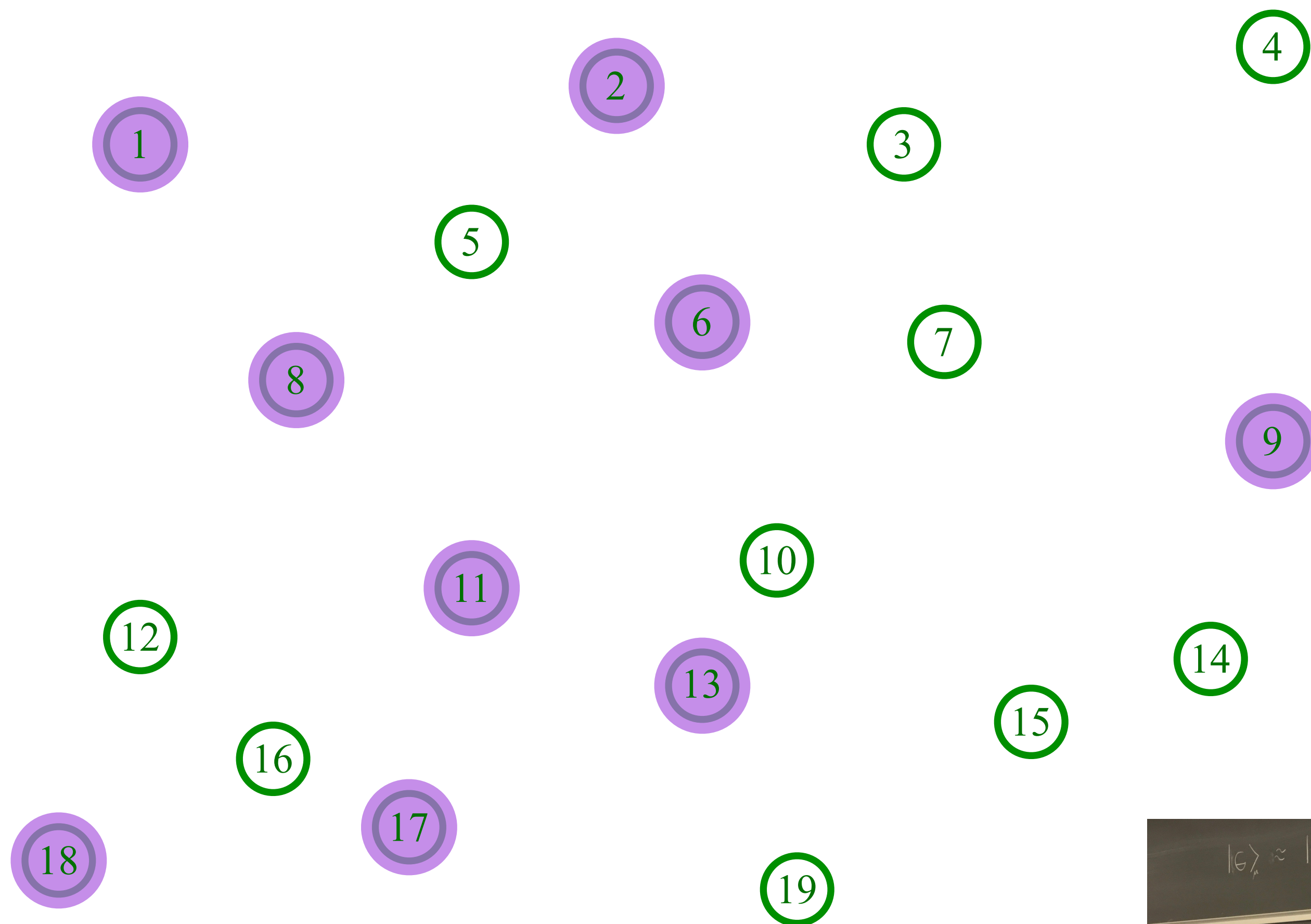
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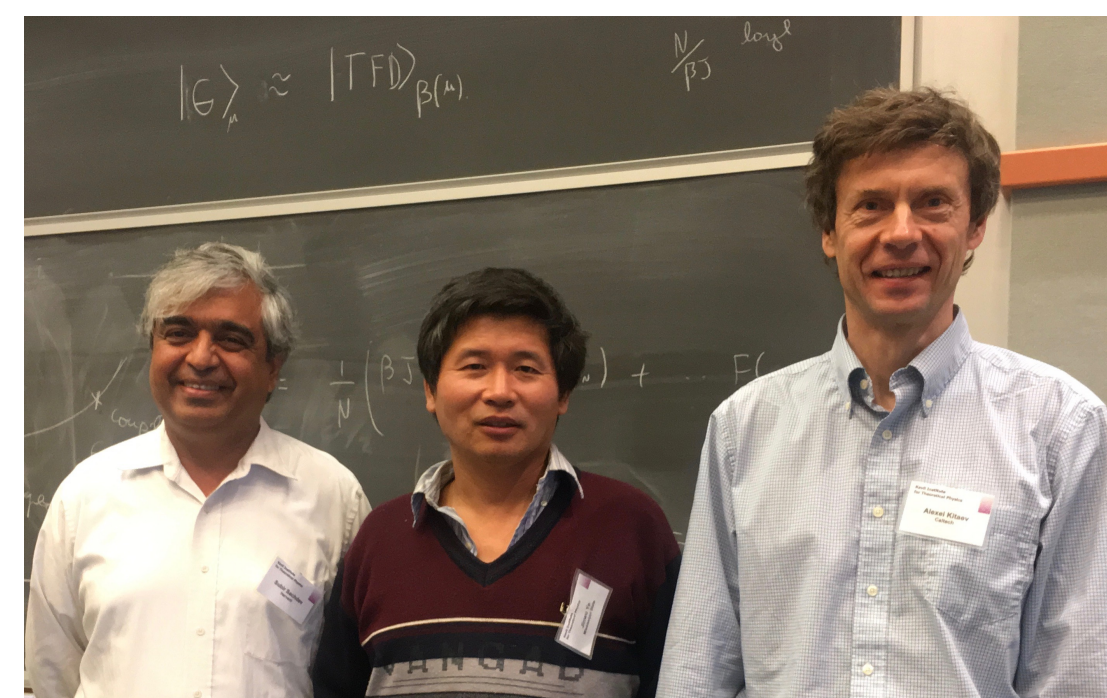
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Sachdev, Ye (1993); Kitaev (2015)

$$U_{14,19;1,13}$$



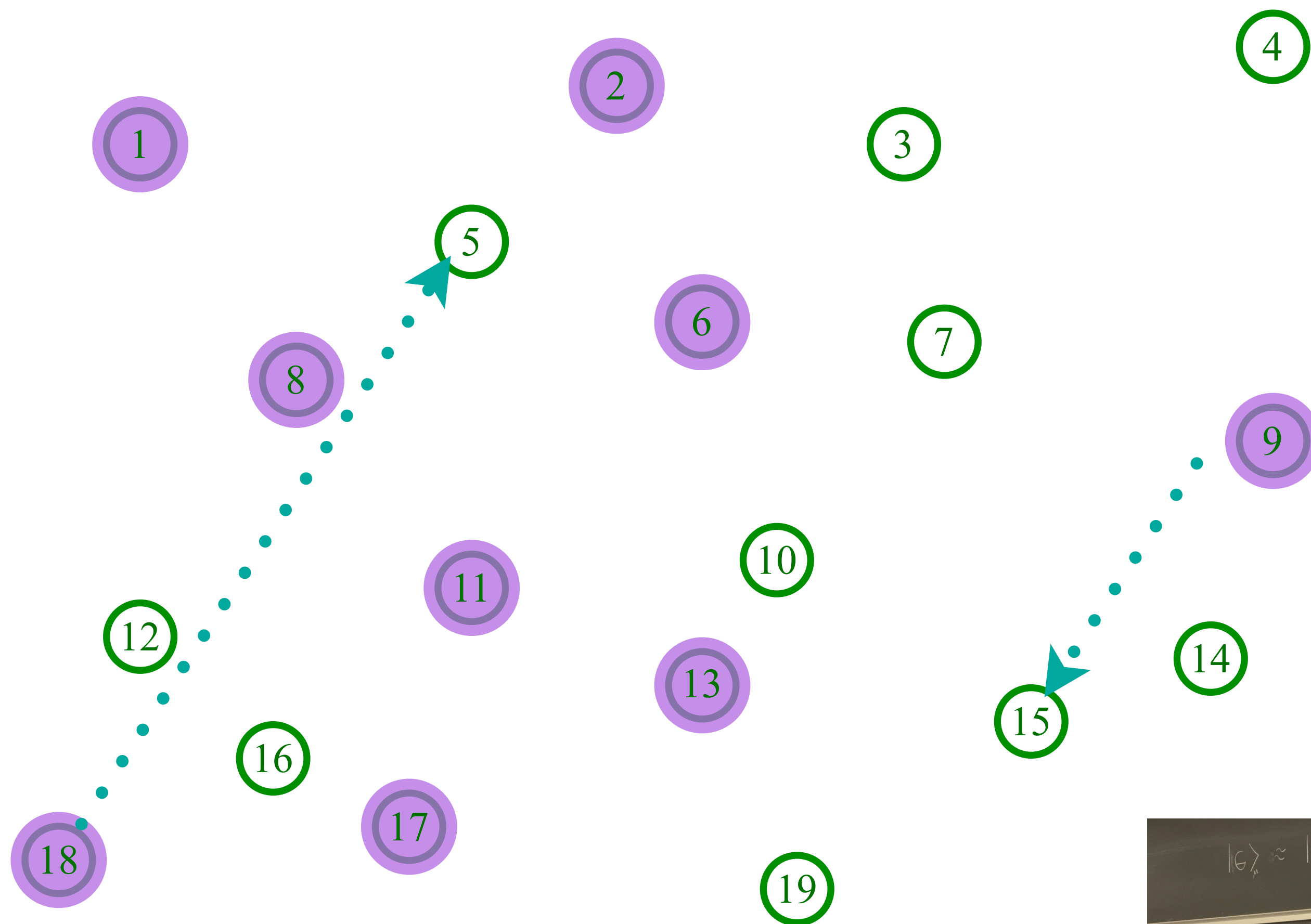
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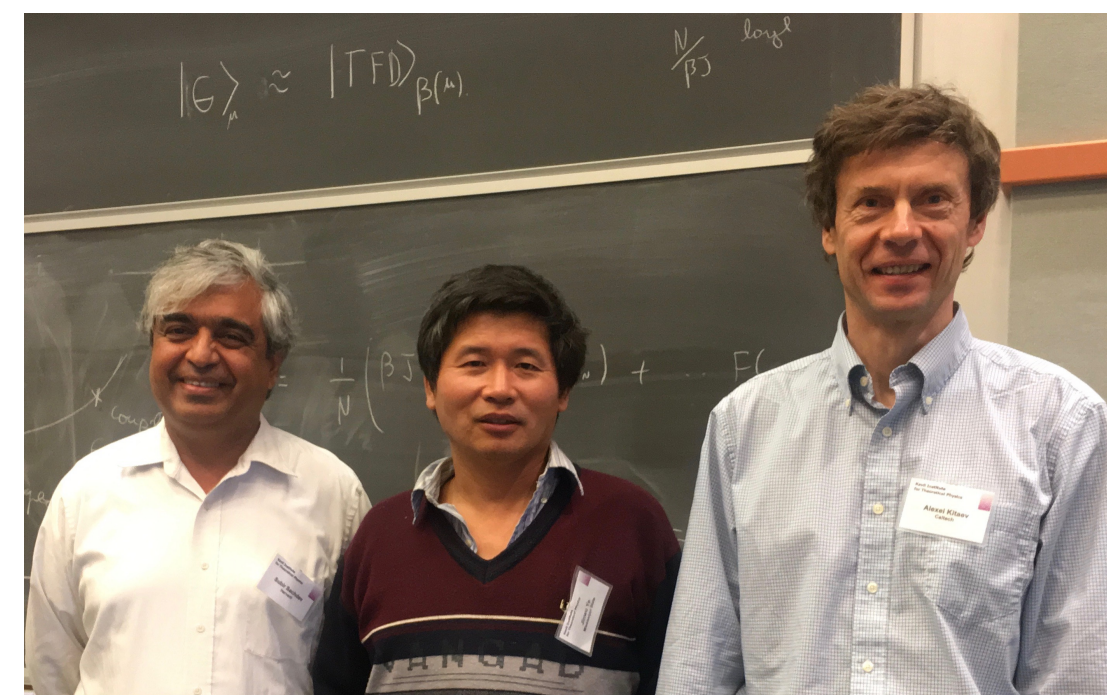
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Sachdev, Ye (1993); Kitaev (2015)

$$U_{9,18;5,15}$$



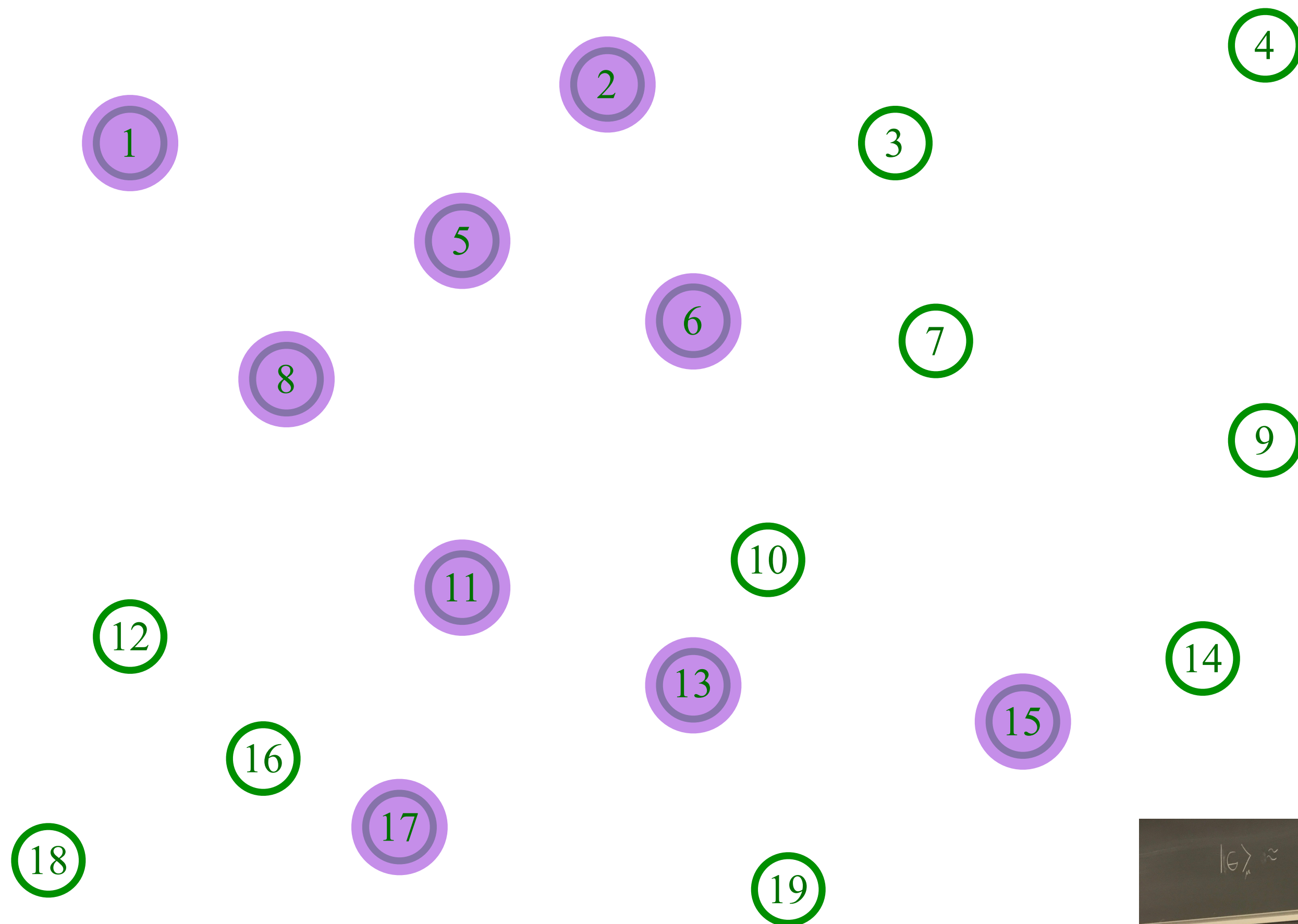
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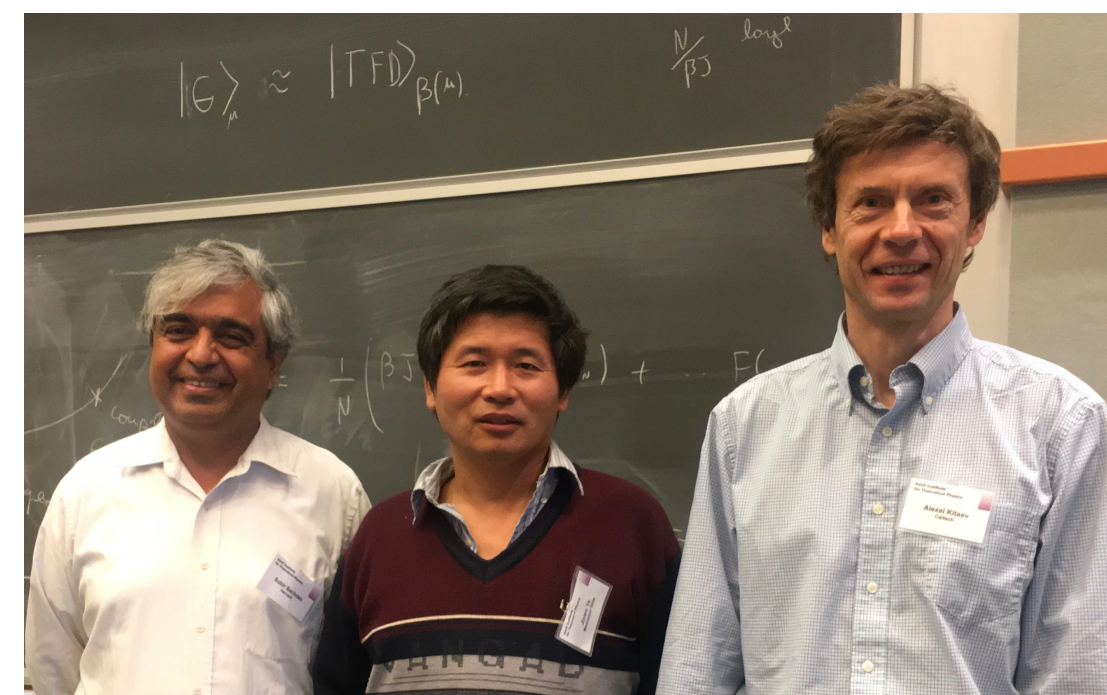
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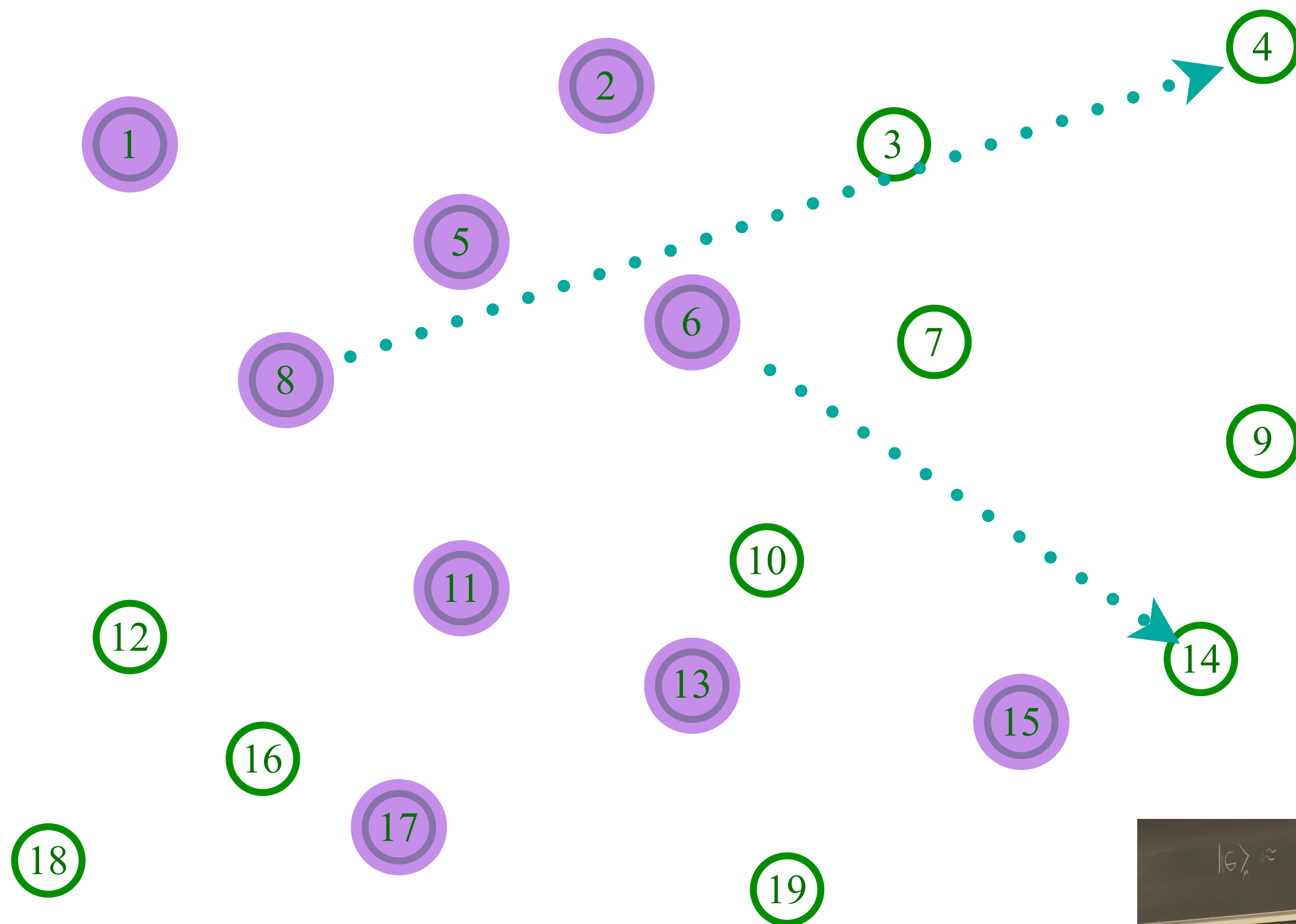
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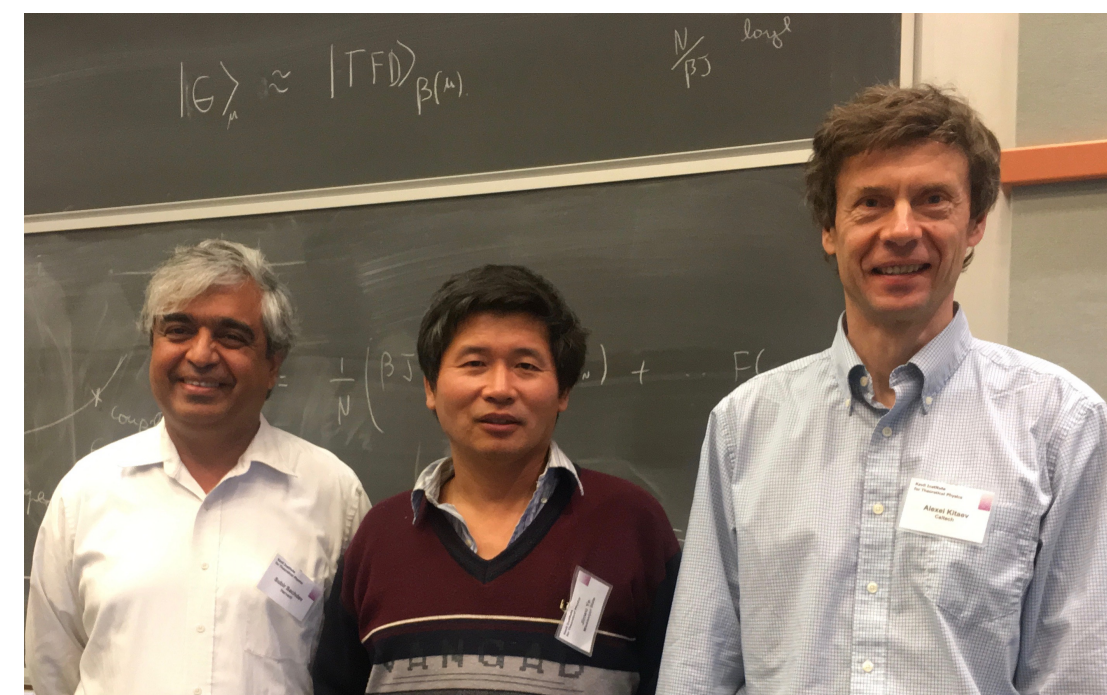
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$$U_{6,8;4,14}$$



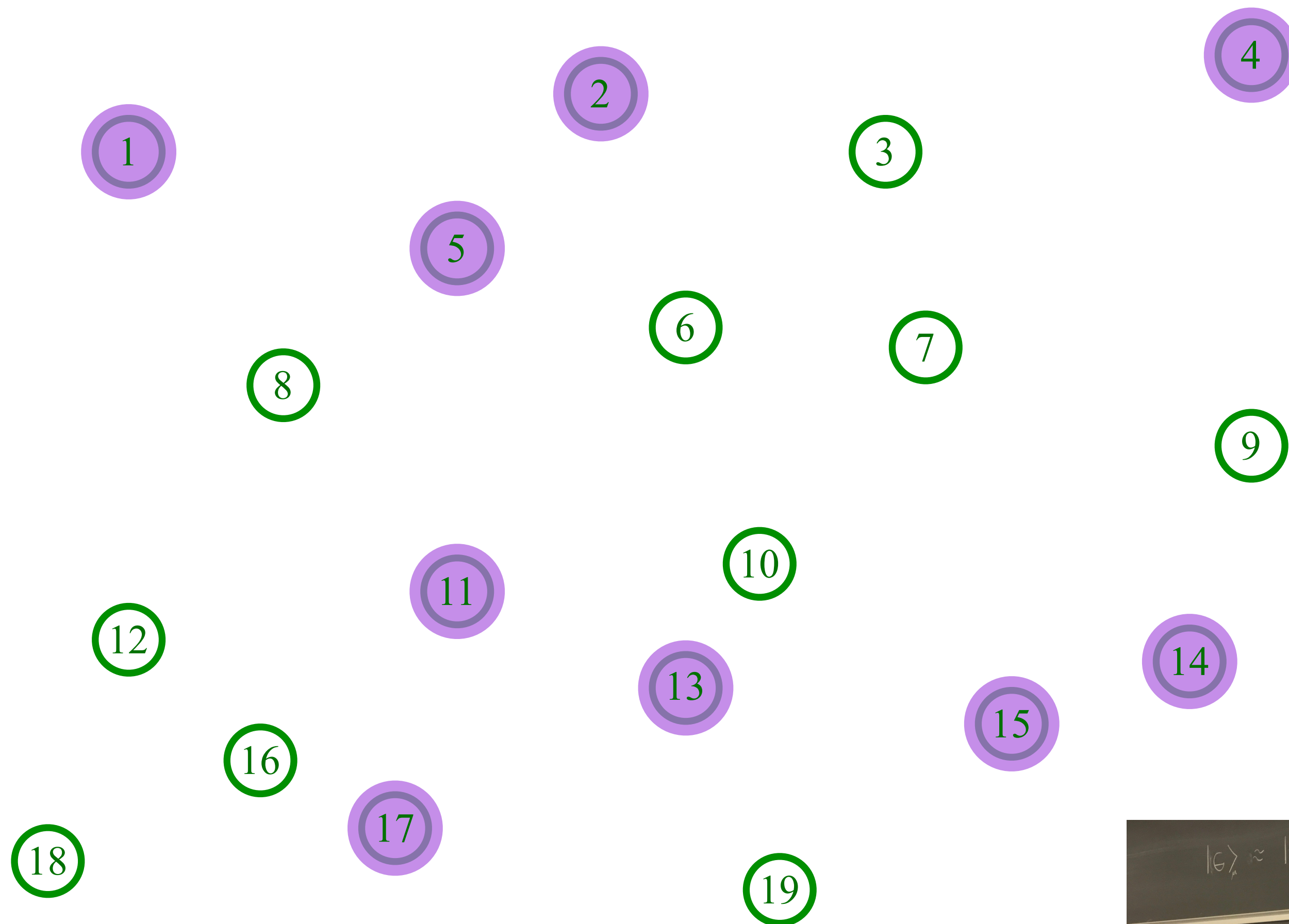
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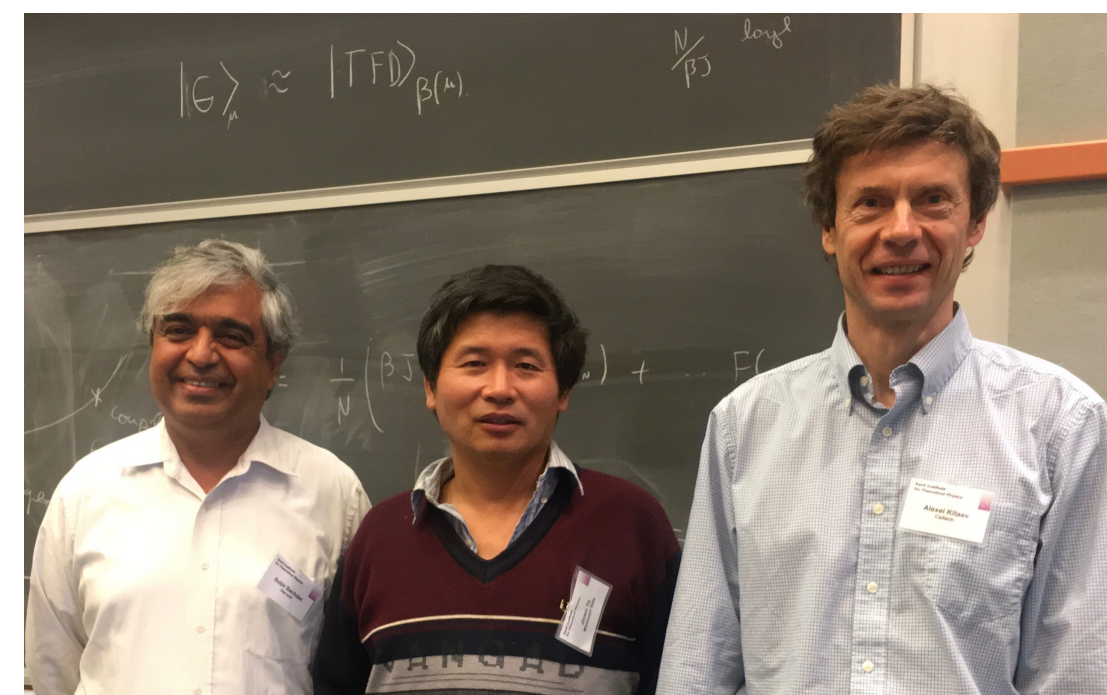
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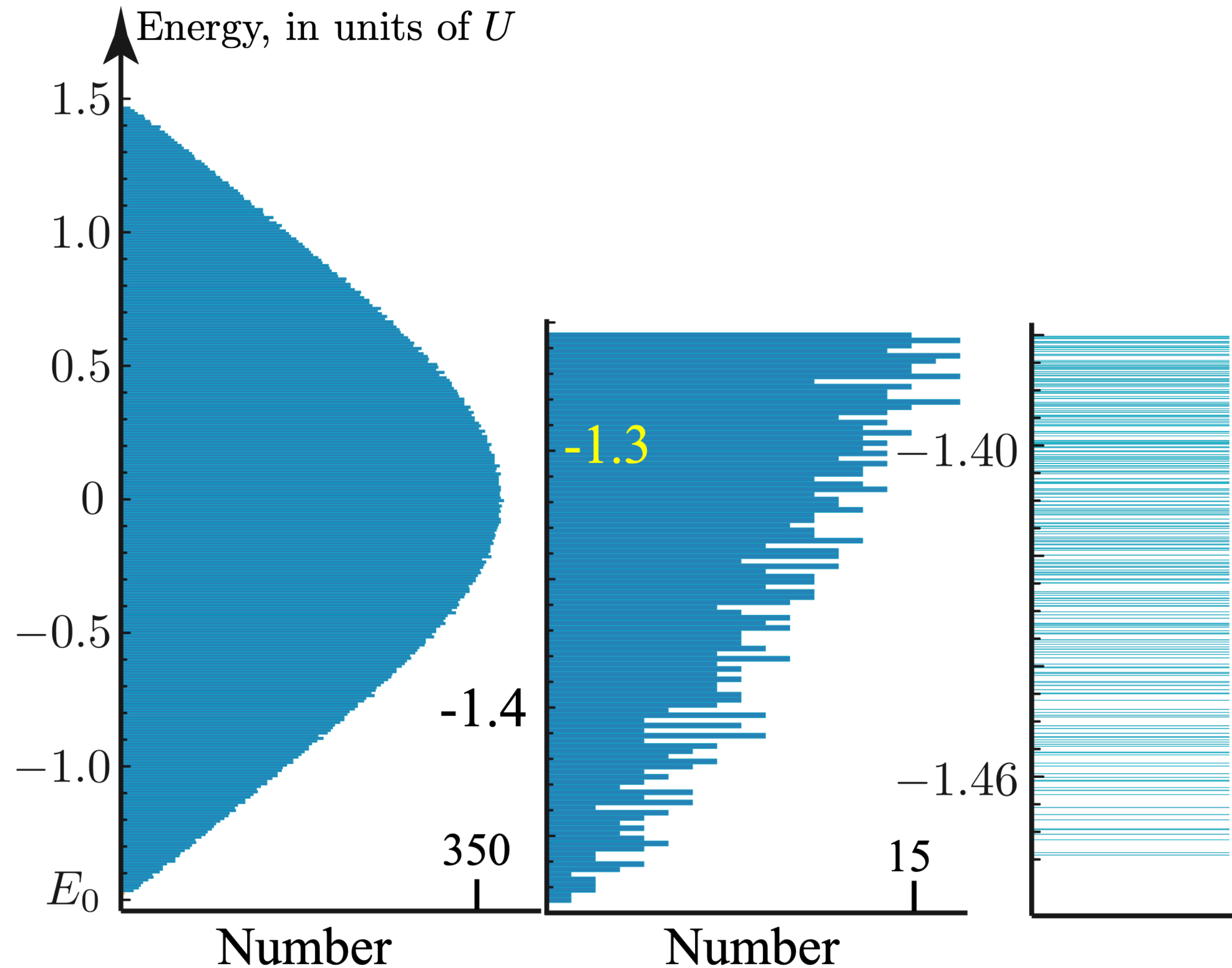


Entangle electrons pairwise randomly



The SYK model: fermions with random and all-to-all interactions

$$D(E) = \sum_i \delta(E - E_i); \quad E_0 + E_i \Rightarrow \text{Many body eigenvalue}$$



Complex SYK model

The SYK model: fermions with random and all-to-all interactions

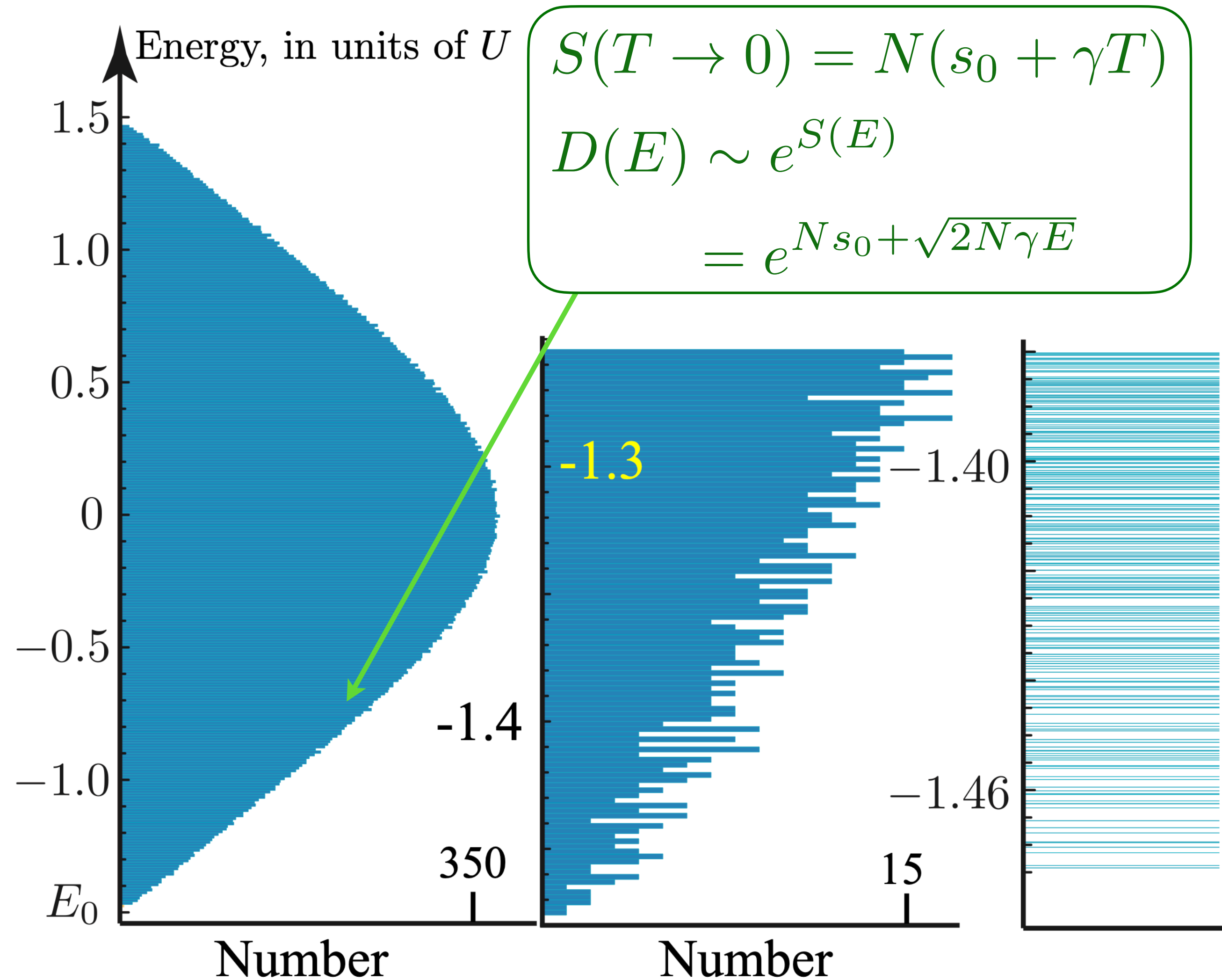
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At $Q = 1/2$

$$s_0 = \frac{\text{Catalan}}{\pi} + \frac{\ln 2}{4}$$

$$= 0.46484769917\dots$$

Georges, Parcollet and Sachdev
Phys. Rev. B **63**, 134406 (2001)



Complex SYK model

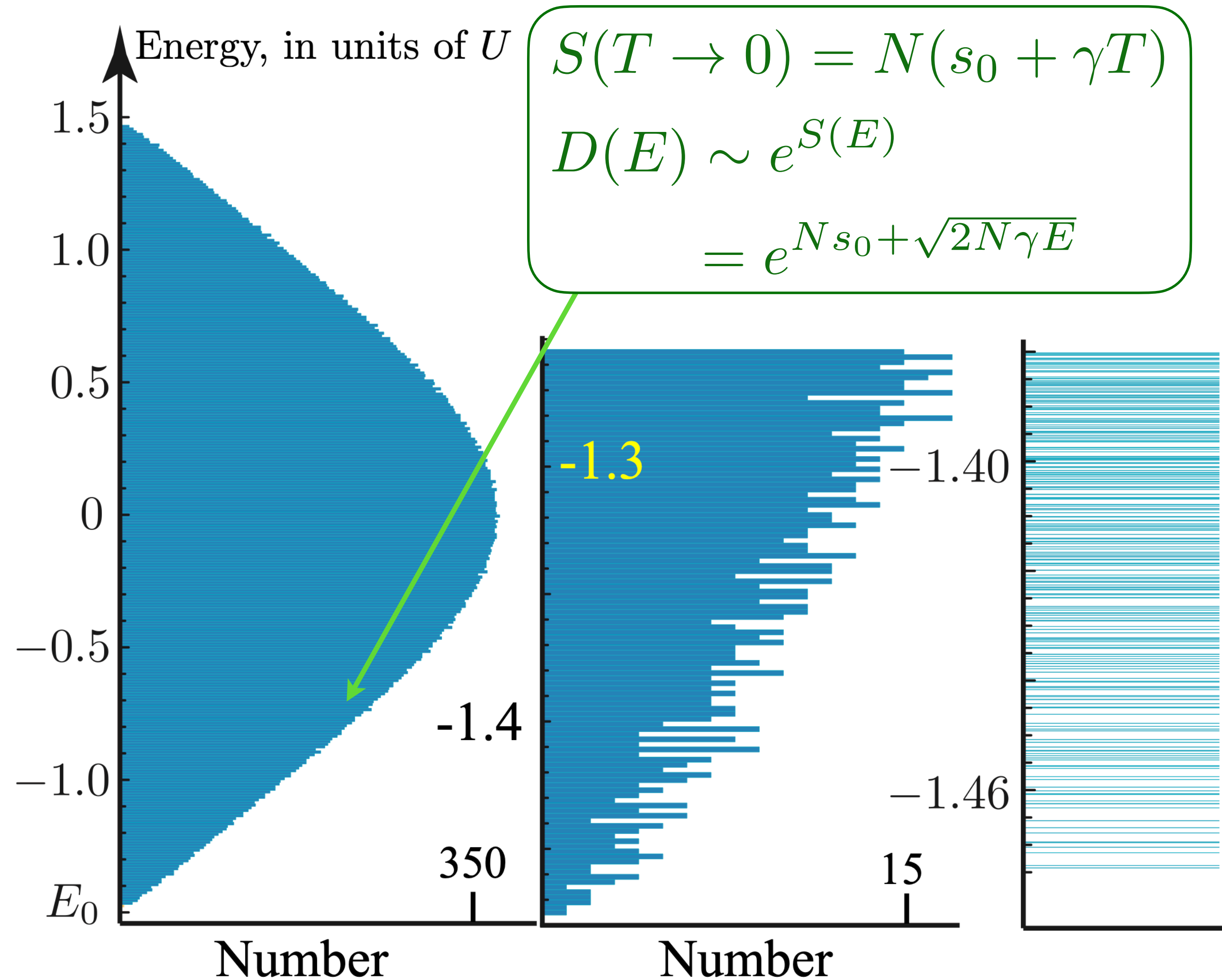
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Energy level spacing $\sim e^{-N s_0}$!

No quasiparticle decomposition: wavefunctions change chaotically from one state to the next.

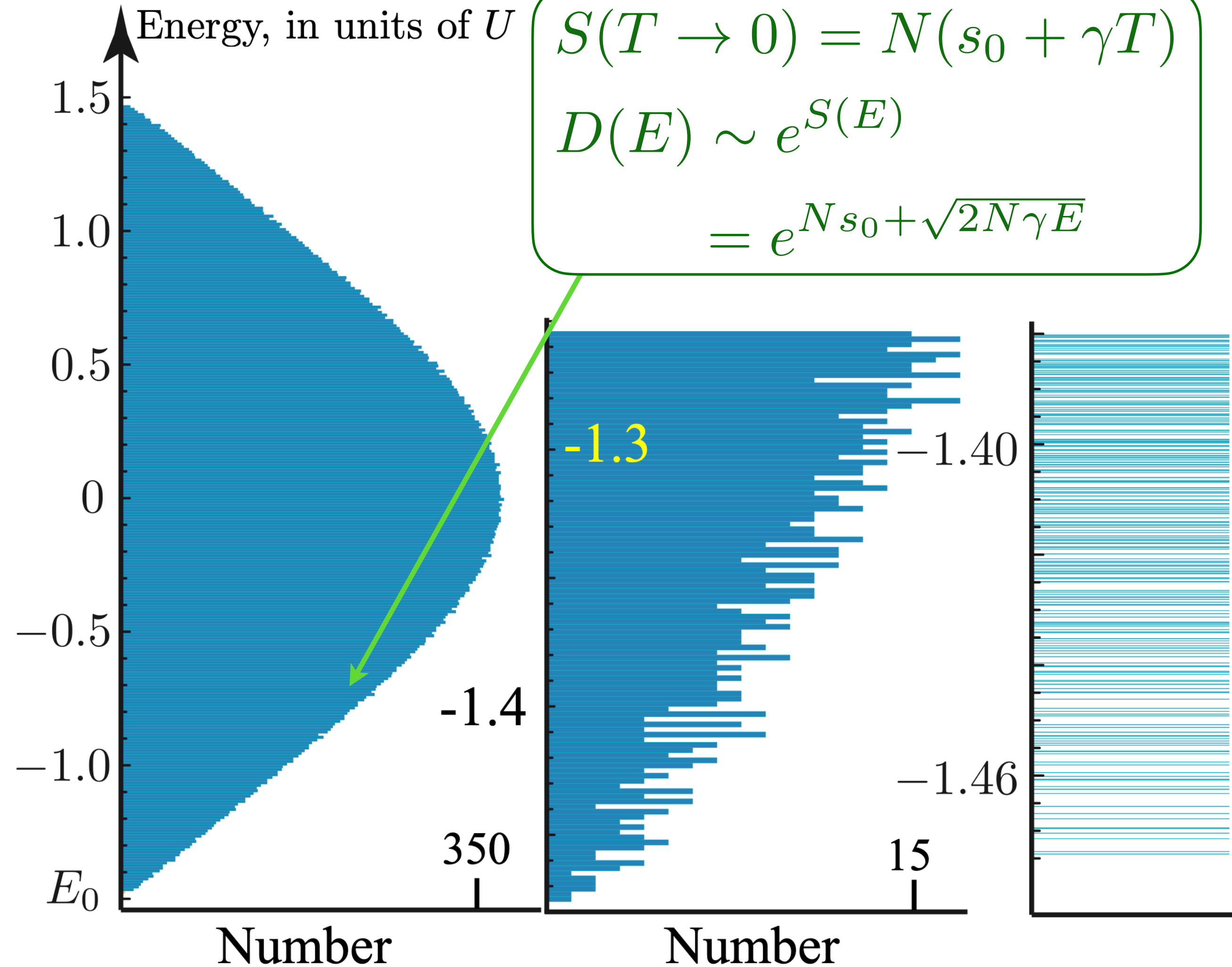
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$$s_0 = \frac{\text{Catalan}}{\pi} + \frac{\ln 2}{4} = 0.46484769917 \dots$$



$$S(T \rightarrow 0) = N(s_0 + \gamma T)$$

$$D(E) \sim e^{S(E)}$$

$$= e^{N s_0 + \sqrt{2N\gamma E}}$$

$$D(E) \sim N^{-1} \exp(N s_0) \sinh(\sqrt{2N\gamma E})$$

Georges, Parcollet and Sachdev
Phys. Rev. B **63**, 134406 (2001)

Gu, Kitaev, Sachdev and Tarnopolsky
J. High Energ. Phys. **157** (2020)

Cotler et al. J. High Energ. Phys. **118** (2017)

Complex SYK model

The SYK model: fermions with random and all-to-all interactions

The (averaged) partition function can be written as path integral over the bilocal fermion Green's function $G(\tau_1, \tau_2) \sim \frac{1}{N} \sum_{\alpha} c_{\alpha}(\tau_1) c_{\alpha}^{\dagger}(\tau_2)$

$$\overline{\mathcal{Z}} = \int \mathcal{D}G(\tau_1, \tau_2) \exp(-N S_{\text{eff}}[G])$$

The large N saddle point equation $\delta S_{\text{eff}}/\delta G = 0$ for $G(\tau_1, \tau_2) = G_s(\tau_1 - \tau_2)$ is

$$G_s(i\omega) = \frac{1}{i\omega + \mu - \Sigma_s(i\omega)} \quad , \quad \Sigma_s(\tau) = -U^2 G_s^2(\tau) G_s(-\tau)$$

Time reparameterization symmetry:

At frequencies $\ll U$, the path integral for is invariant under time reparameterization $f(\sigma)$

$$\tau = f(\sigma)$$

$$G(\tau_1, \tau_2) = [f'(\sigma_1) f'(\sigma_2)]^{-1/4} \tilde{G}(\sigma_1, \sigma_2)$$

There is also an emergent U(1) gauge symmetry. Hints that the low energy theory is quantum gravity+electromagnetism!

Yukawa-SYK model

Esterlis and Schmalian
Phys. Rev. B **100**, 115132 (2019)
Wang *Phys. Rev. Lett.* **124**,
017002 (2020)

$$H = \sum_{ij} t_{ij} \psi_i^\dagger \psi_j + \sum_{\ell} \frac{1}{2} (\pi_{\ell}^2 + \omega_{\ell}^2 \phi_{\ell}^2) + \sum_{ij\ell} g_{ij\ell} \psi_i^\dagger \psi_j \phi_{\ell}$$

Yukawa couplings $g_{ij\ell}$ are independent random numbers
coupling fermions ψ_i to oscillators ϕ_{ℓ} .

- Yields a critical state, similar to that of the SYK model

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- Model in $d = 2$ with dispersing bosons and fermions and spatially random Yukawa interactions yields a universal theory of strange metals, consistent with numerous observations.
- Strong disorder (non-self-averaging) effects at low T similar to random transverse-field Ising model.

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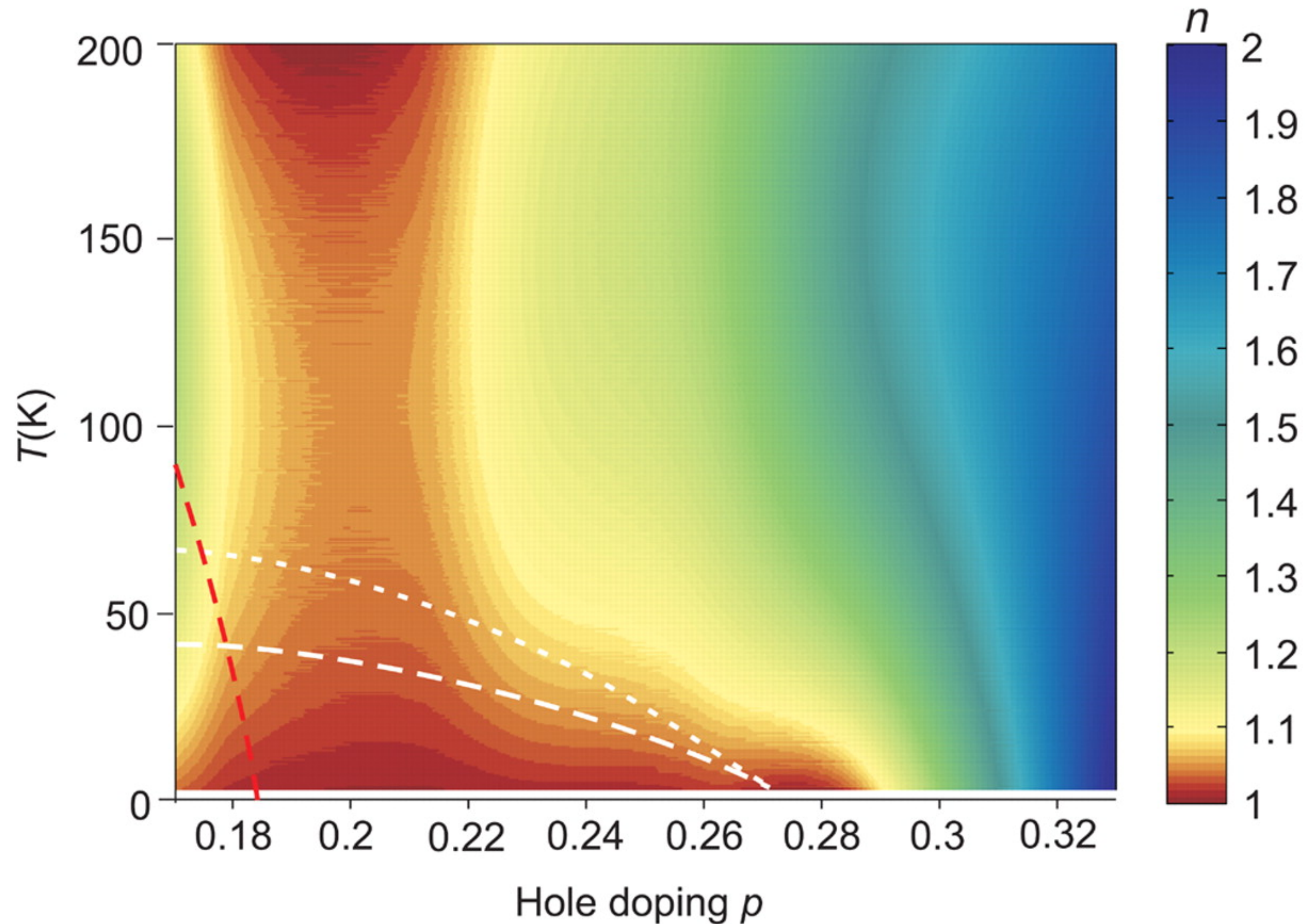
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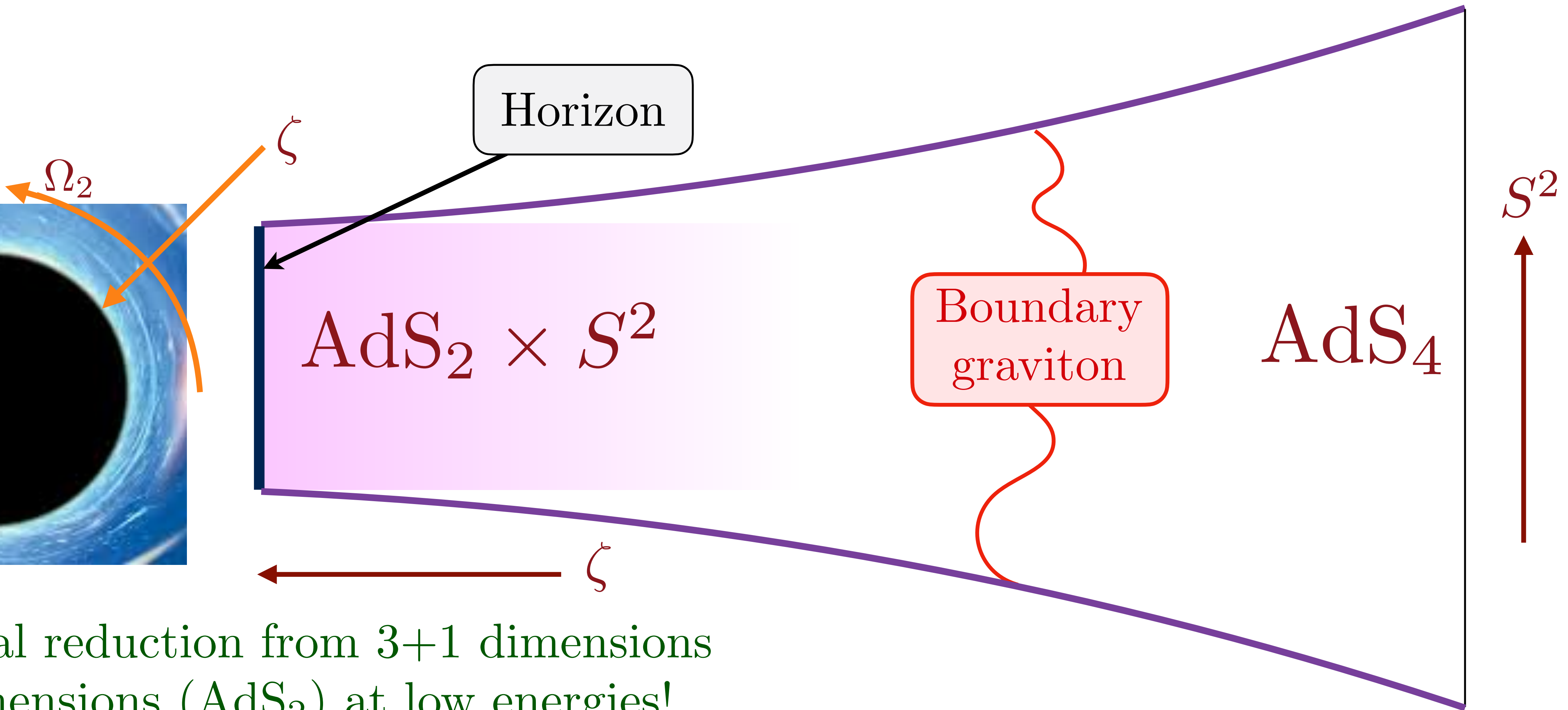
- Yields a critical state, similar to that of the SYK model
- Model in $d = 2$ with dispersing bosons and fermions and spatially random Yukawa interactions yields a universal theory of strange metals, consistent with numerous observations.
- Strong disorder (non-self-averaging) effects at low T similar to random transverse-field Ising model.
- Superconductivity is possible at low T .
- The scalar field ϕ can represent either a symmetry-breaking order parameter, or a fractionalized particle.

Anomalous Criticality in the Electrical Resistivity of $\text{La}_{2-x}\text{Sr}_x\text{CuO}_4$

R. A. Cooper, Y. Wang, B. Vignolle, O. J. Lipscombe, S. M. Hayden, Y. Tanabe, T. Adachi, Y. Koike, M. Nohara, H. Takagi, Cyril Proust, N. E. Hussey, *Science* **323**, 603 (2009)



Reissner-Nordstrom black hole of Einstein-Maxwell theory



Dimensional reduction from 3+1 dimensions to 1+1 dimensions (AdS_2) at low energies!

The isometry group of AdS_2 is the 0+1 dimensional conformal group $SL(2, \mathbb{R})$.

Thermodynamics of quantum black holes with charge Q
 from Euclidean path integral outside horizon

$$\begin{aligned} \mathcal{Z}(Q, T) &= \int \mathcal{D}g_{\mu\nu} \mathcal{D}A_\mu \exp \left(-\frac{1}{\hbar} I_{\text{Einstein gravity+Maxwell EM}}^{(3+1)}[g_{\mu\nu}, A_\mu] \right) \\ &\approx \exp \left(\frac{A_0 c^3}{4\hbar G} \right) \int \mathcal{D}g_{\mu\nu} \mathcal{D}A_\mu \exp \left(-\frac{1}{\hbar} I_{\text{JT gravity of AdS}_2+\text{boundary graviton}}^{(1+1)}[g_{\mu\nu}, A_\mu] \right) \\ &= \int \mathcal{D}f(\tau) \mathcal{D}\phi(\tau) \exp \left(-\frac{1}{\hbar} I_{\text{SYK}}[\text{time reparameterizations } f(\tau), \text{ phase rotations } \phi(\tau)] \right) \end{aligned}$$

Saddle-point:

$$S_{BH}(T \rightarrow 0, Q) = \frac{A(T)c^3}{4G\hbar} = \frac{A_0 c^3}{4G\hbar} \left(1 + \frac{2(\pi A_0)^{1/2} T}{\hbar c} + \dots \right)$$

$A_0 = 2GQ^2/c^4$ is the area of the
 charged black hole horizon at $T = 0$.

Quantum simulation of charged black holes by the SYK model

- For generic charged black holes in 3+1 dimensions, the SYK model yields, in terms of $A_0 = 2GQ^2/c^4$ the horizon area at $T = 0$:

Iliesiu, Murthy, Turiaci
arXiv:2209.13608 (2022)

Bekenstein-Hawking

Developments from the SYK model

$$D(E) \sim \left(\frac{A_0 c^3}{\hbar G} \right)^{-347/90} \exp \left(\frac{A_0 c^3}{4\hbar G} \right) \sinh \left(\left[\frac{\sqrt{\pi} A_0^{3/2} c^2}{\hbar^2 G} E \right]^{1/2} \right)$$

There is no degeneracy, but an exponentially small level spacing down to the ground state.

$D(E)$

