SYK tutorial

2023 Jacques Solvay International Chair in Physics
June 22, 2023

Subir Sachdev

Review articles: arXiv:2304.13744, 2305.01001

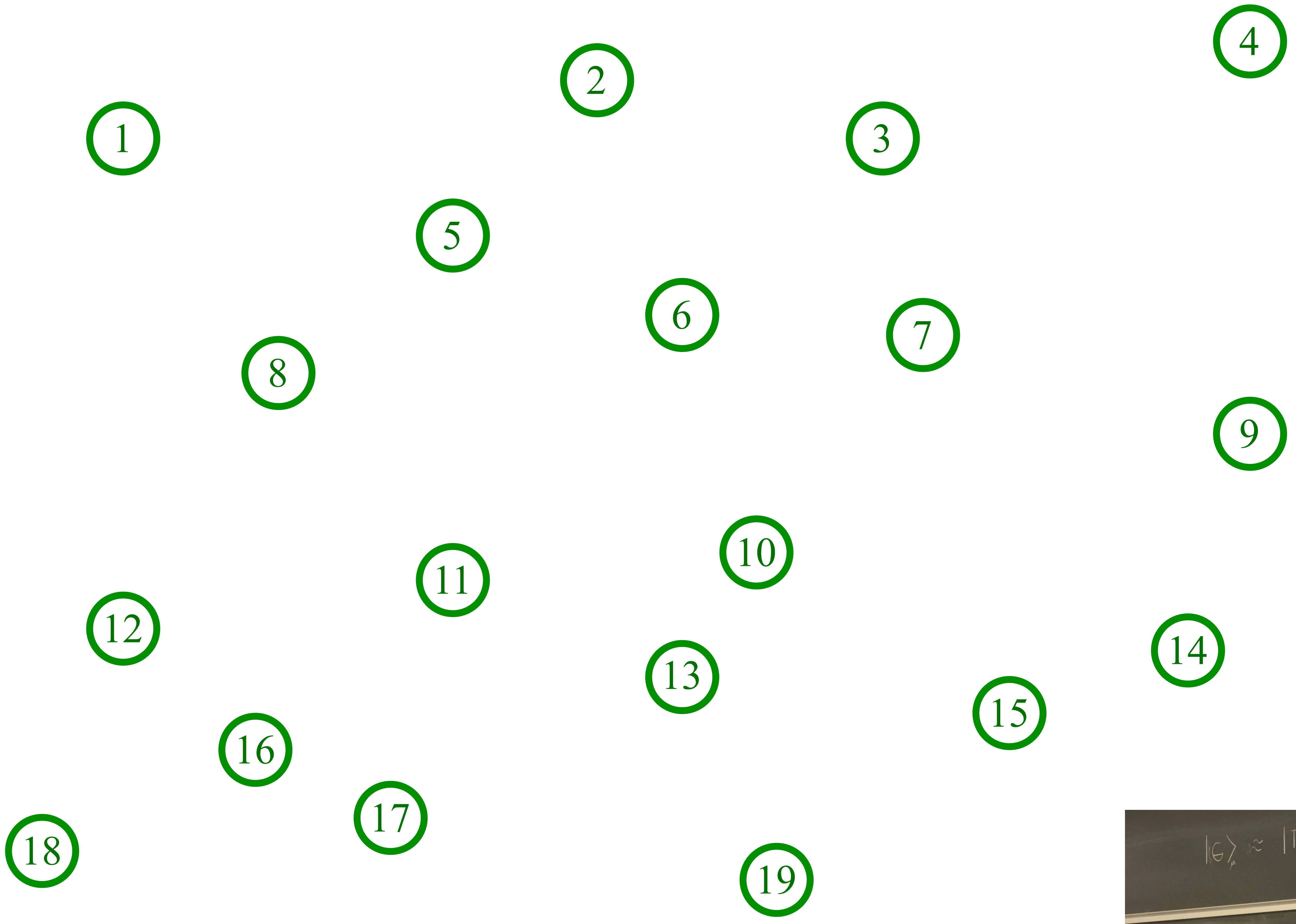
PHYSICS
HARVARD

Talk online: sachdev.physics.harvard.edu

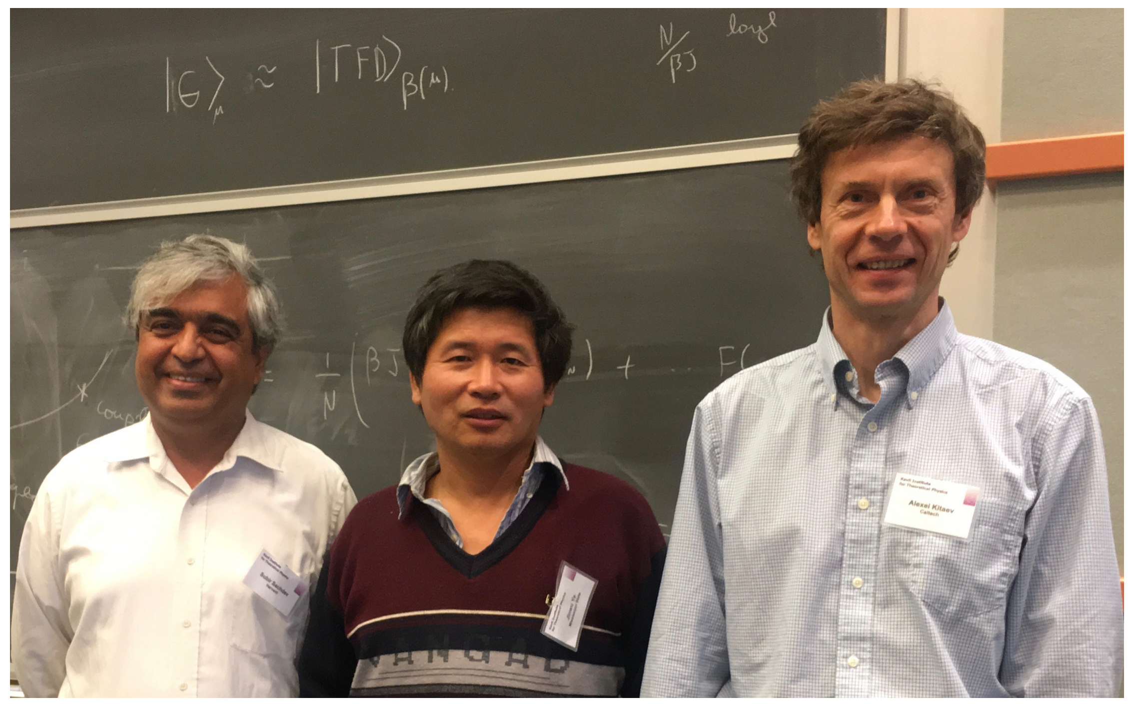
1. Large-N theory of the SYK model
2. Finite-N theory of the SYK model
3. Quantum Einstein-Maxwell gravity theory of charged black holes
4. Universal theory of strange metals

The SYK model

Sachdev, Ye (1993); Kitaev (2015)

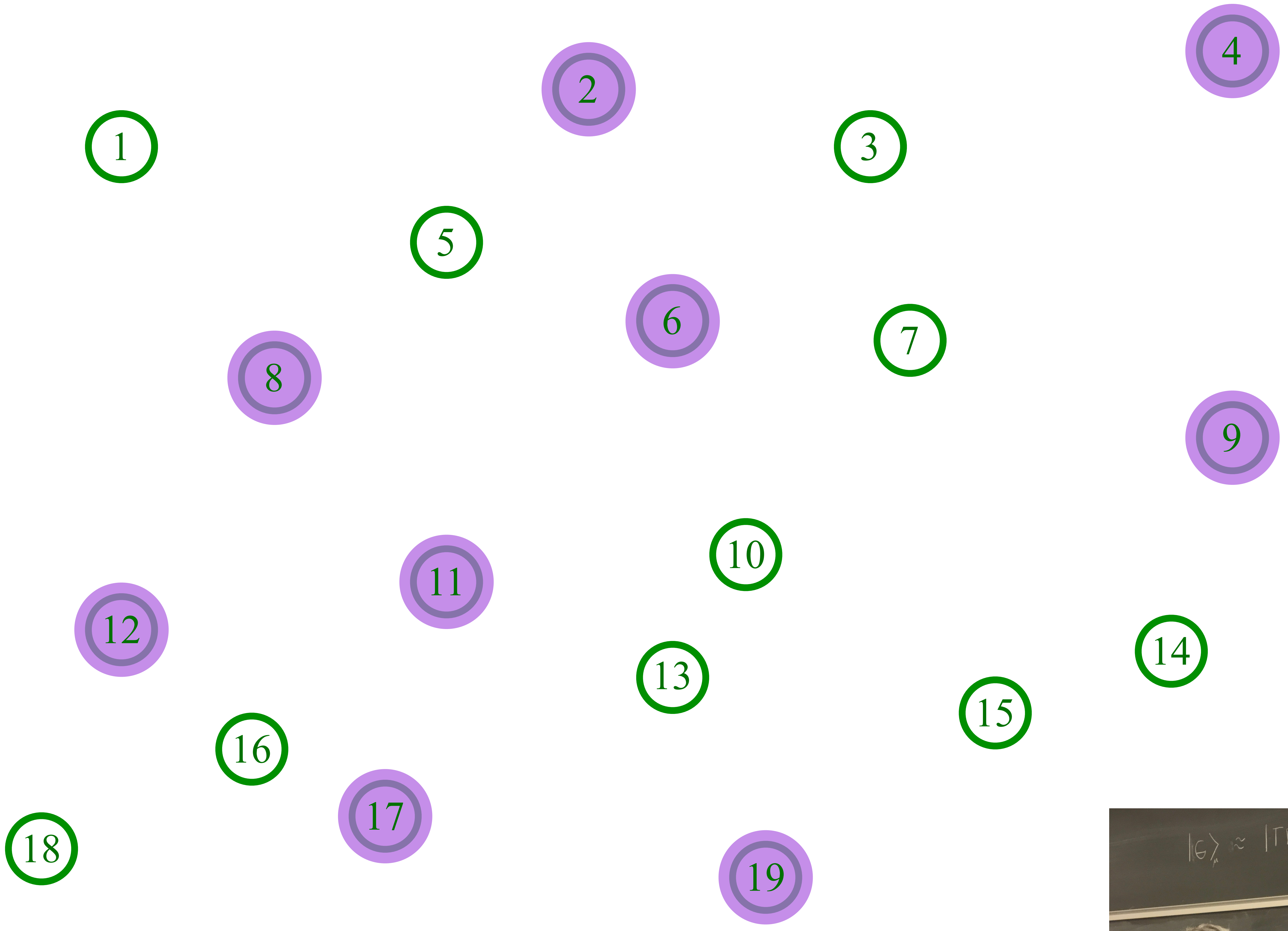


Pick a set of random positions

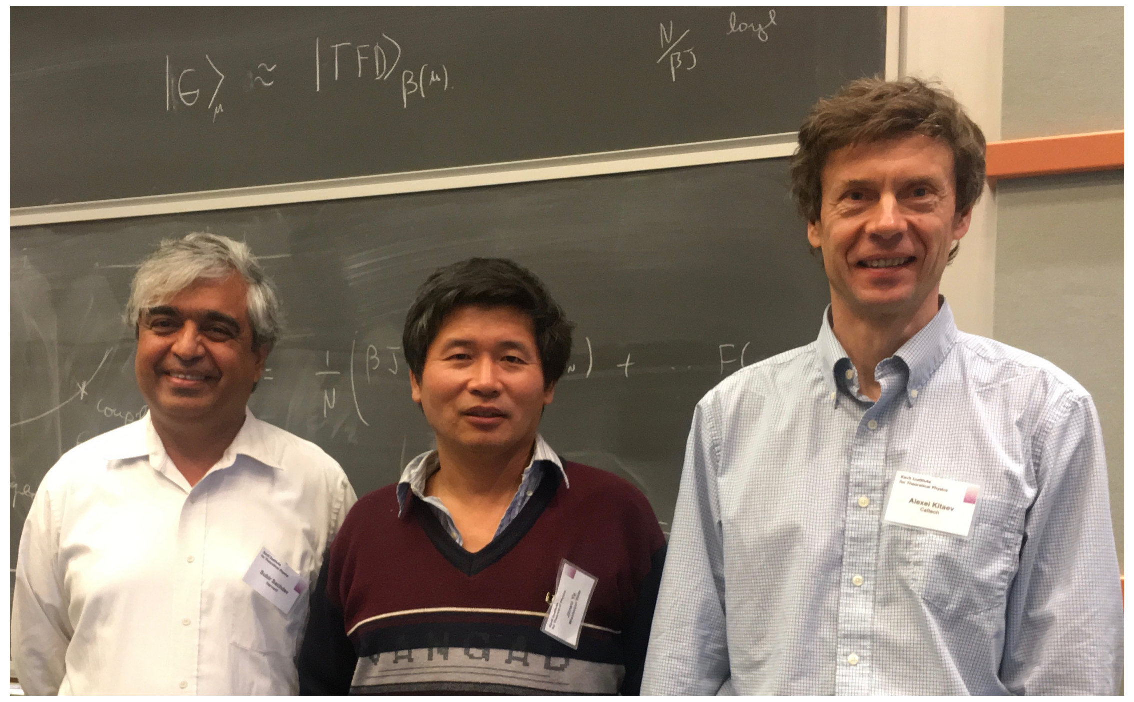


The SYK model

Sachdev, Ye (1993); Kitaev (2015)



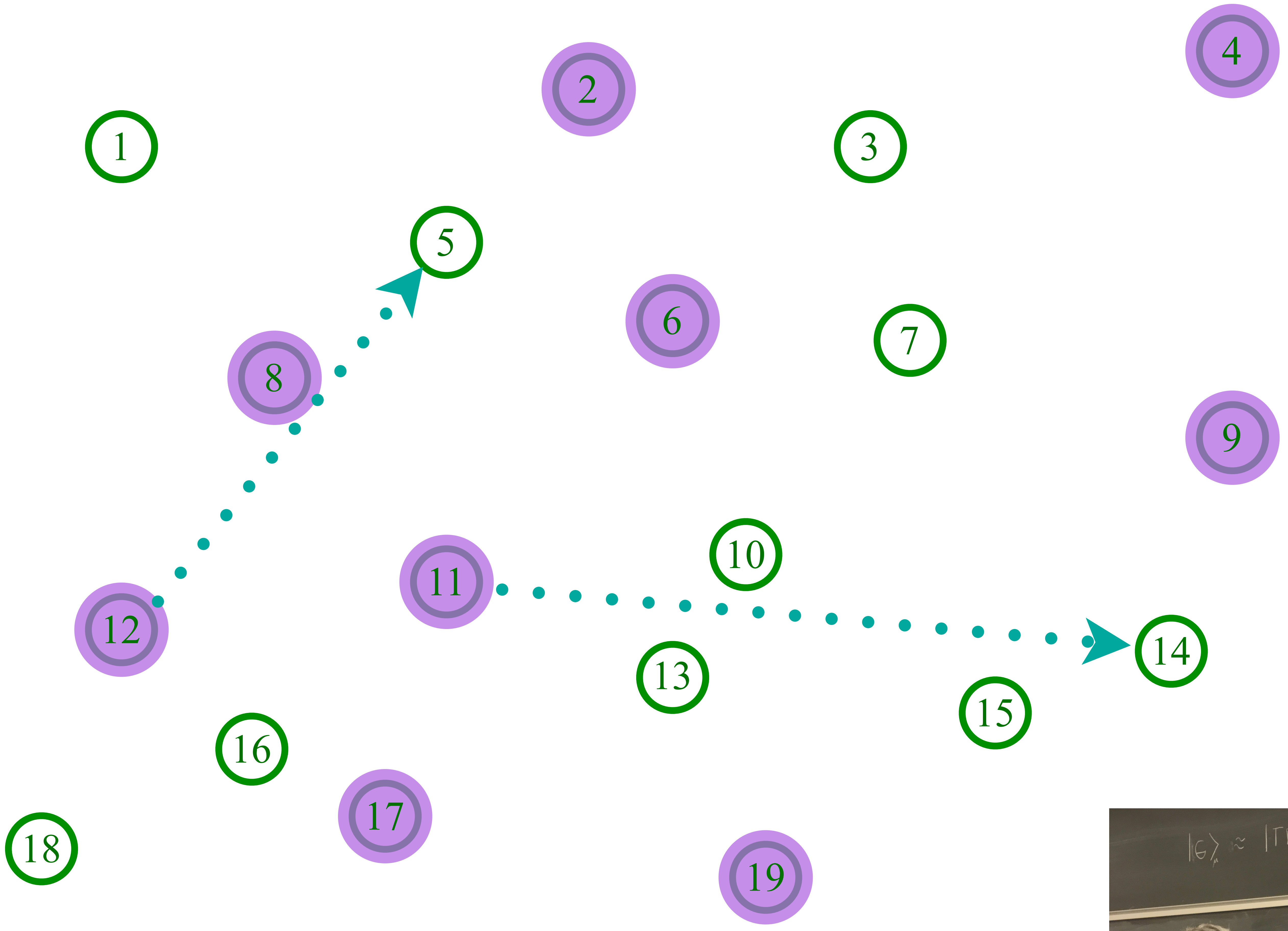
Place electrons randomly on some sites



The SYK model

Sachdev, Ye (1993); Kitaev (2015)

$$U_{11,12;5,14}$$



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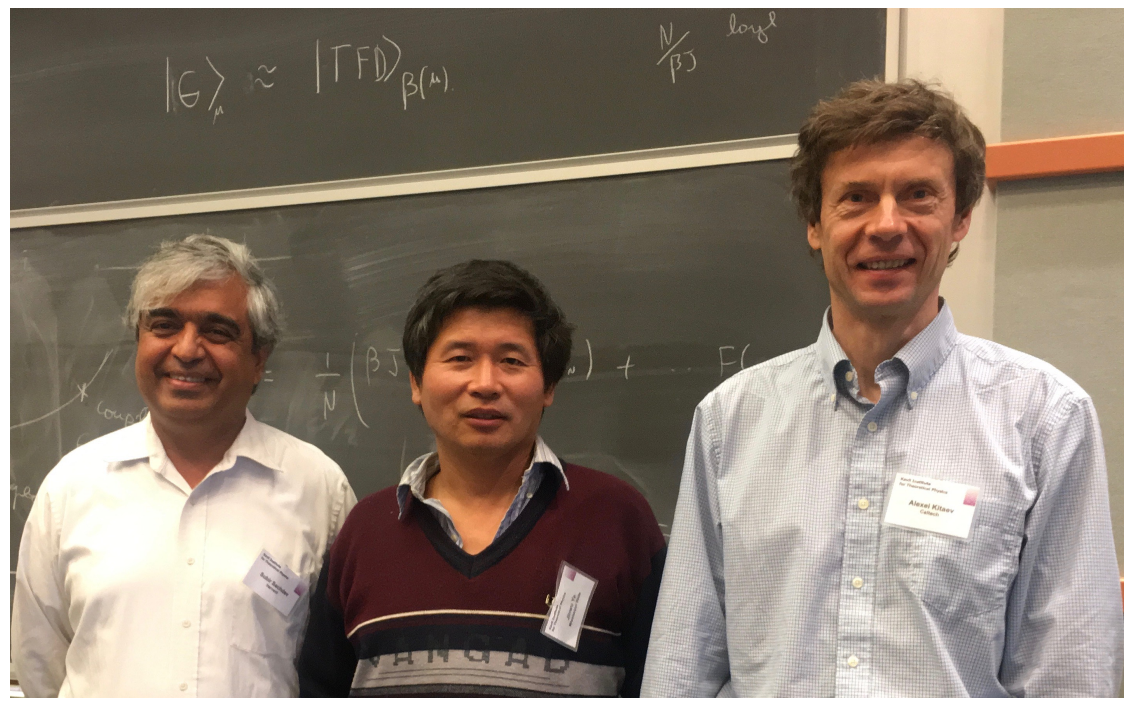

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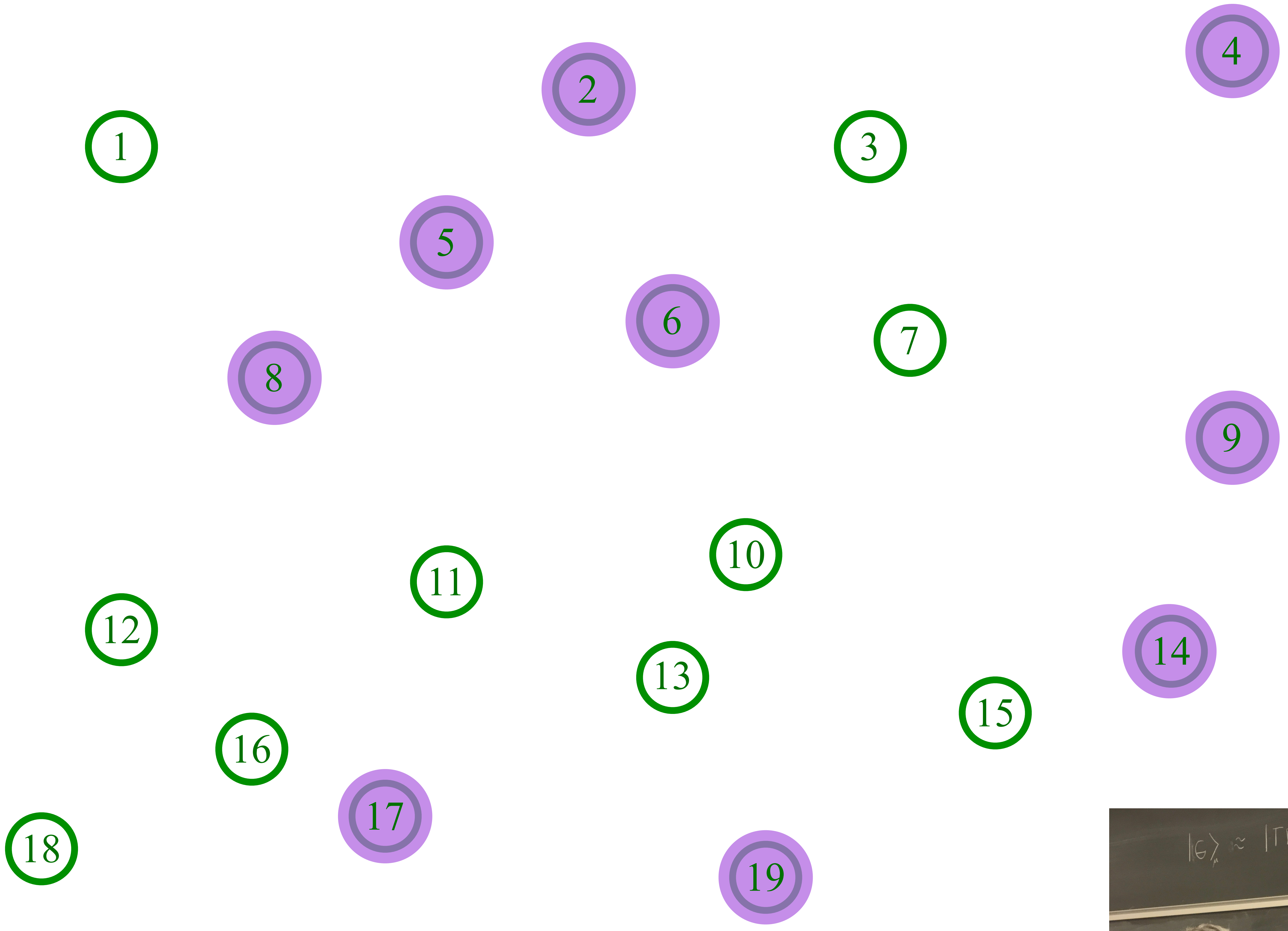
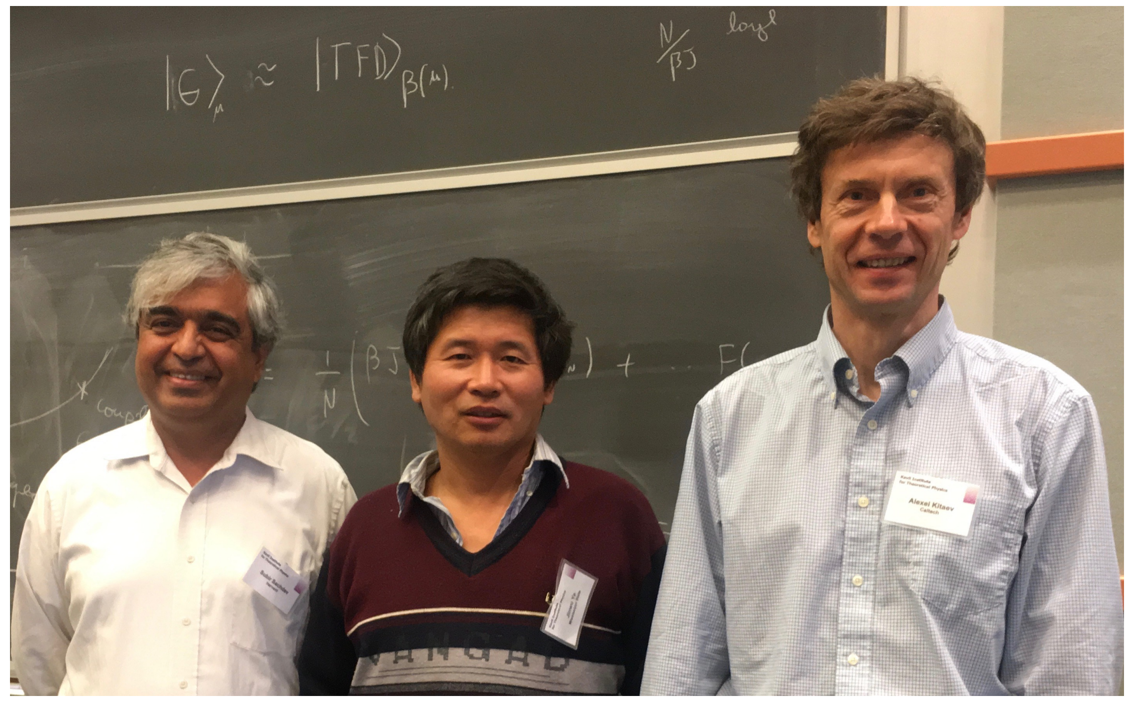


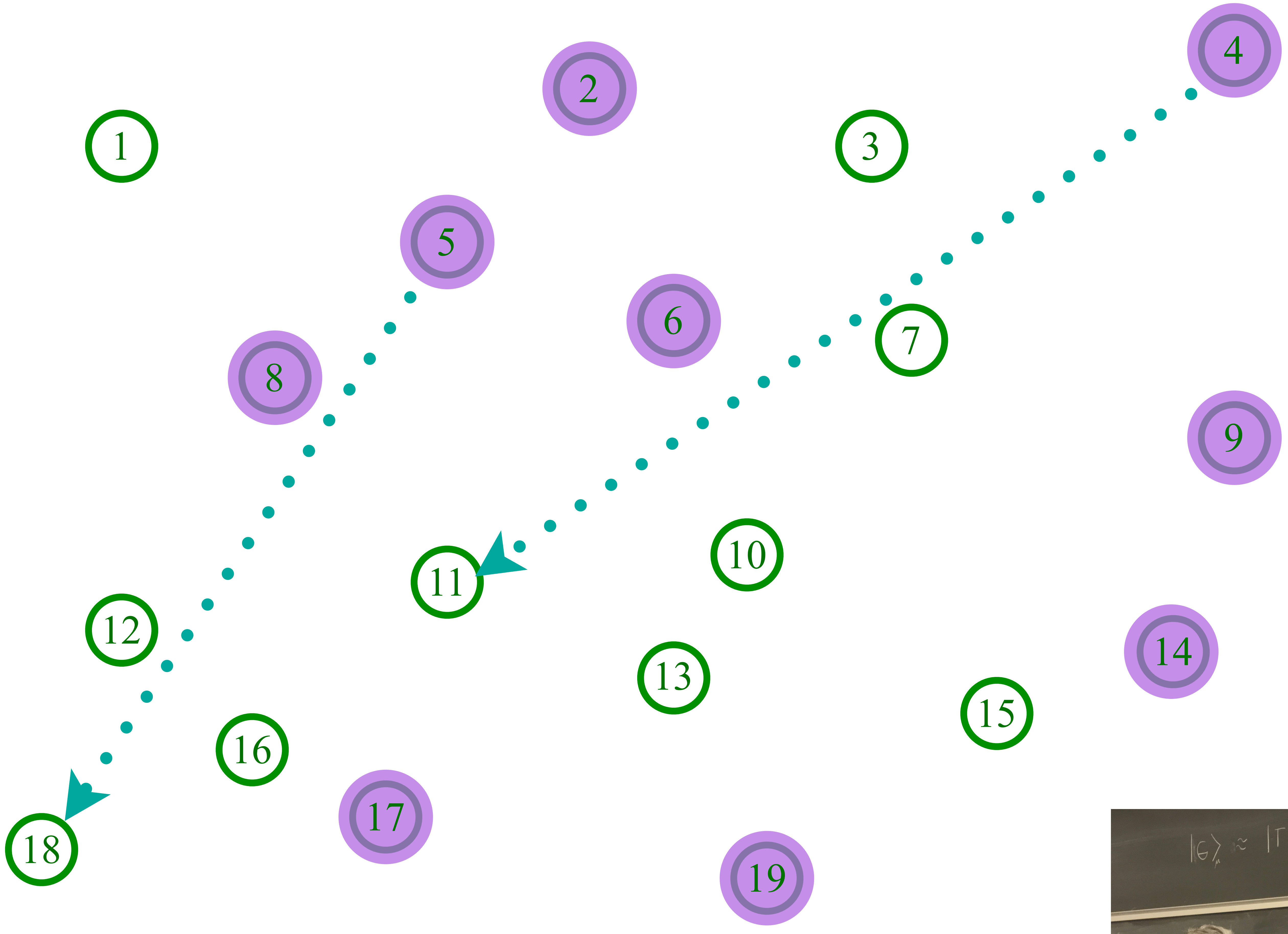
Entangle electrons pairwise randomly



The SYK model

Sachdev, Ye (1993); Kitaev (2015)

$$U_{4,5;11,18}$$



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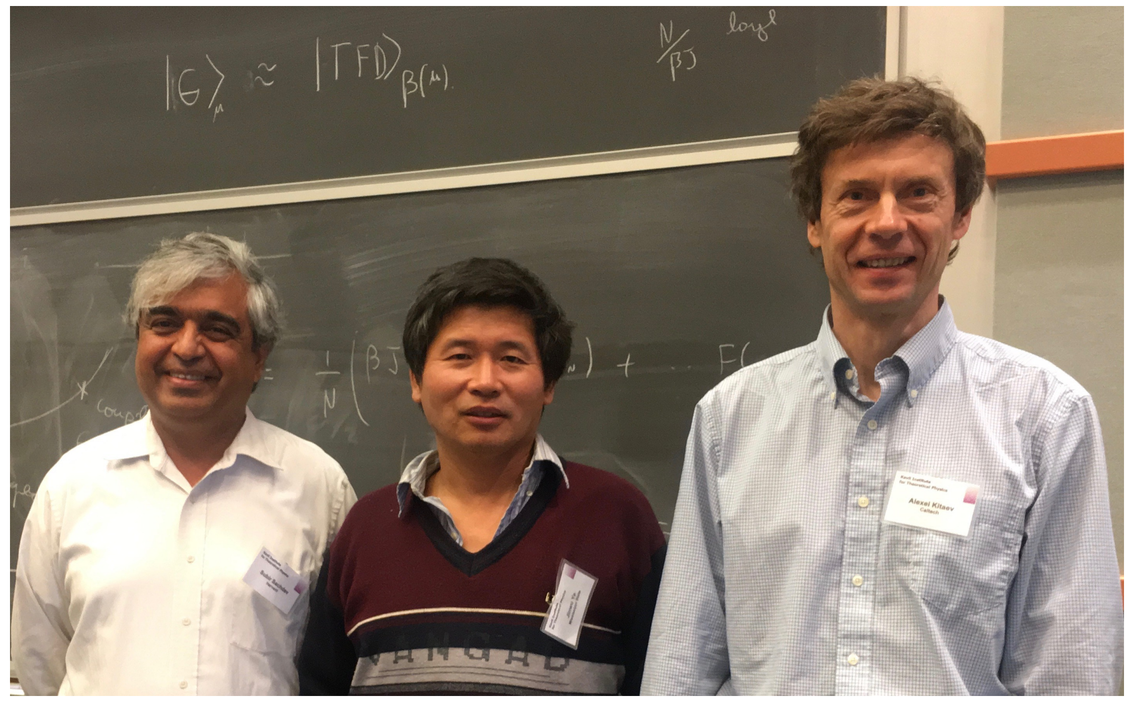

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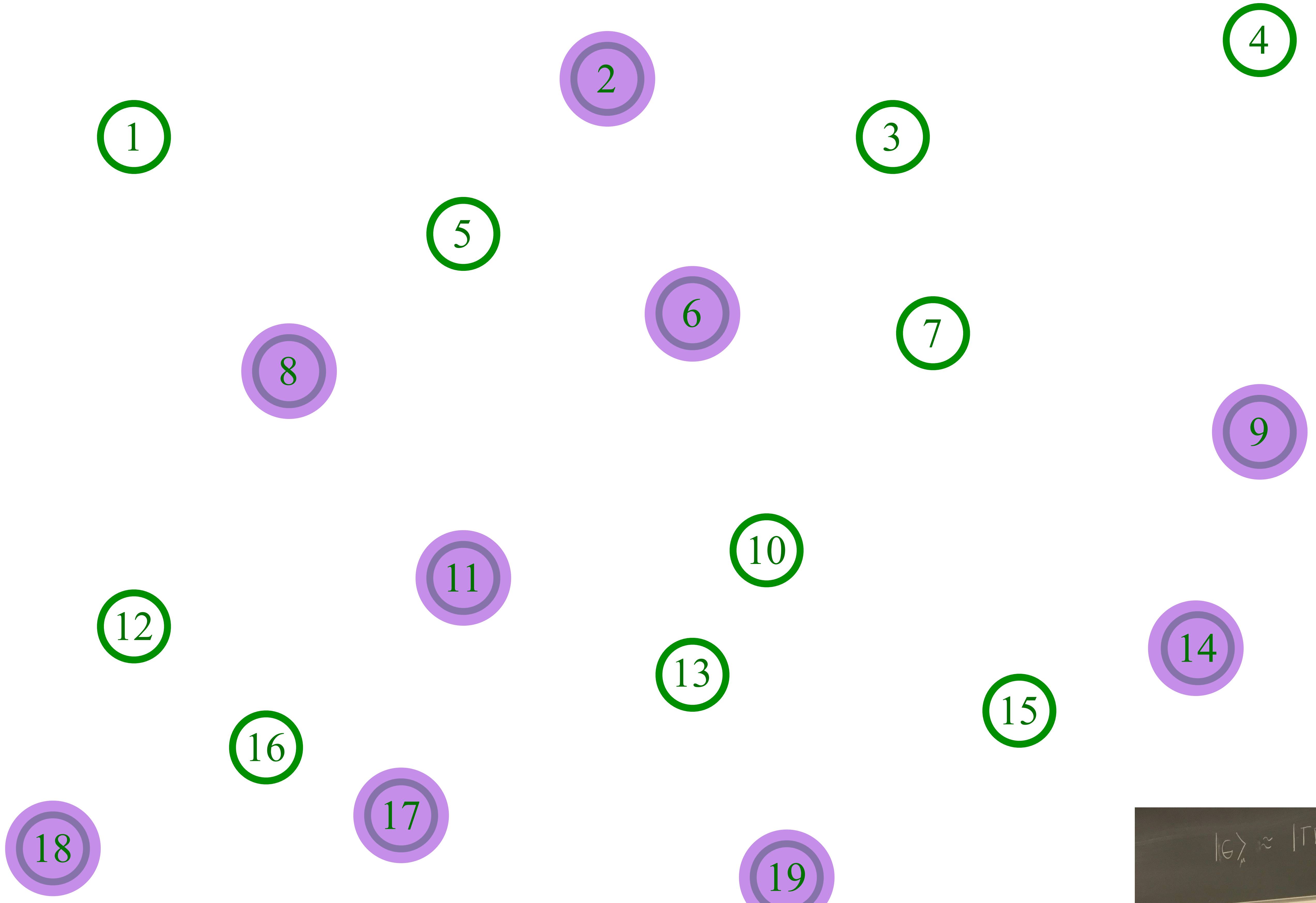

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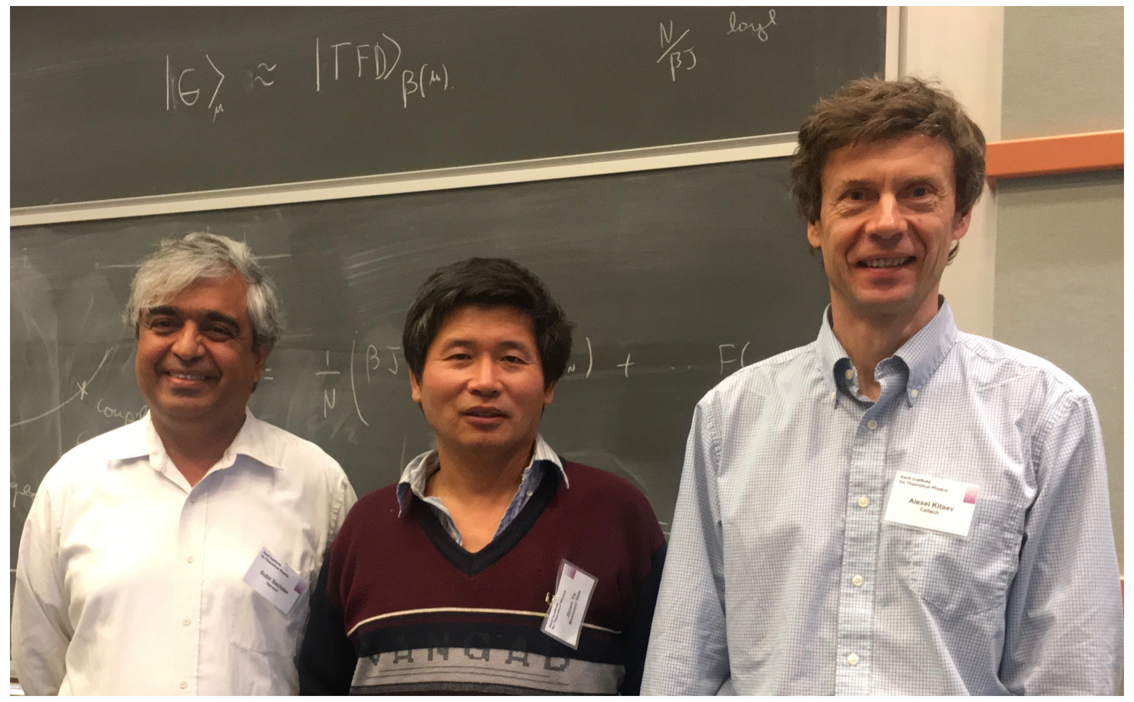

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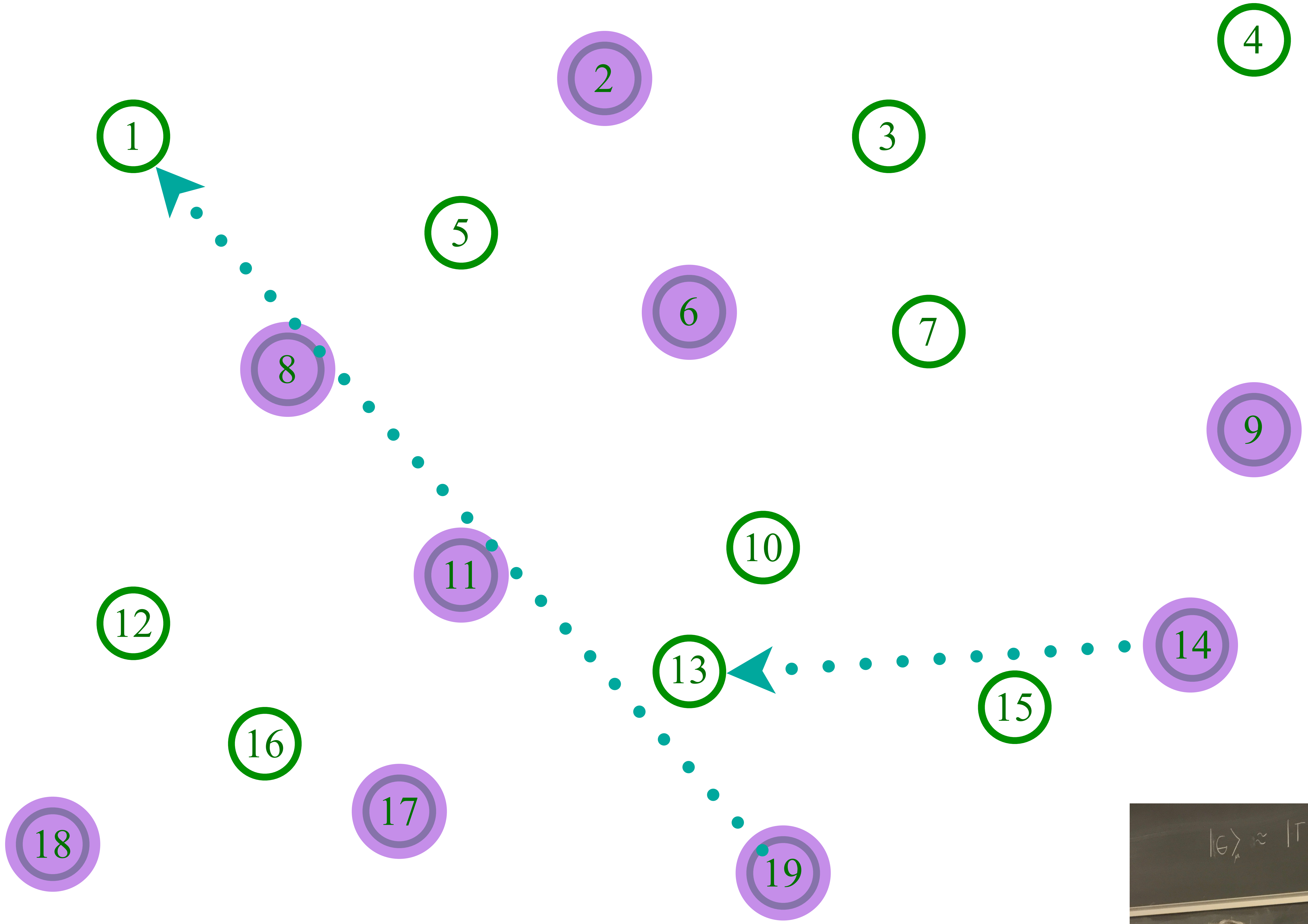
$$U_{14,19;1,13}$$



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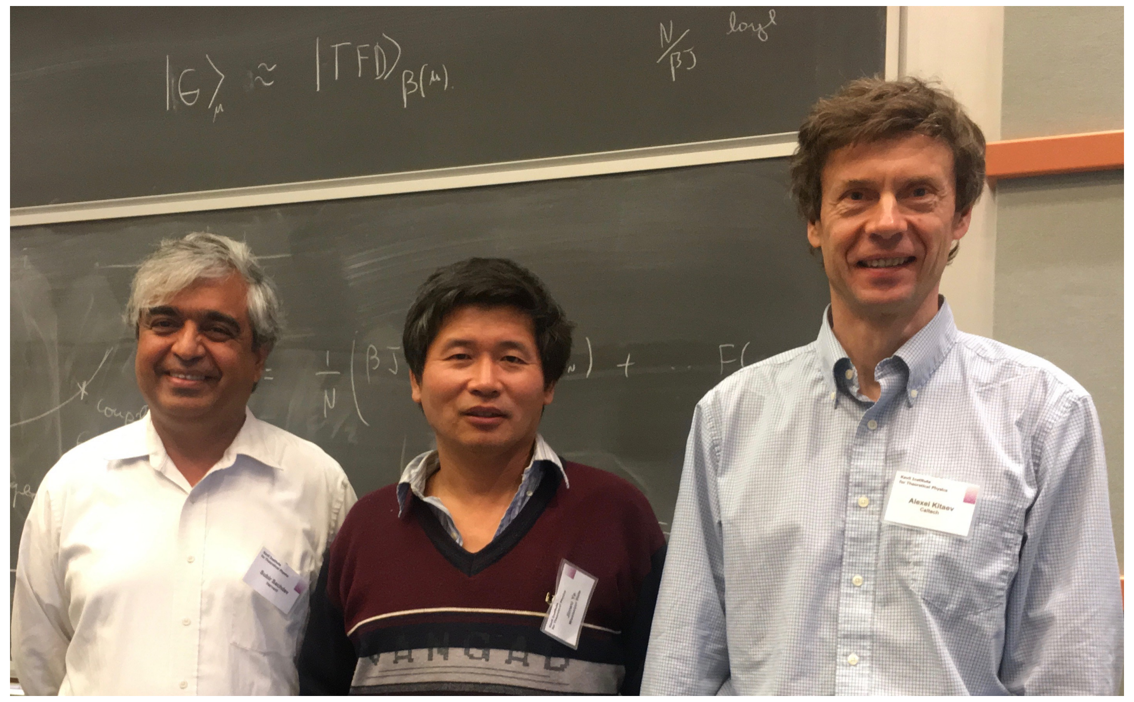

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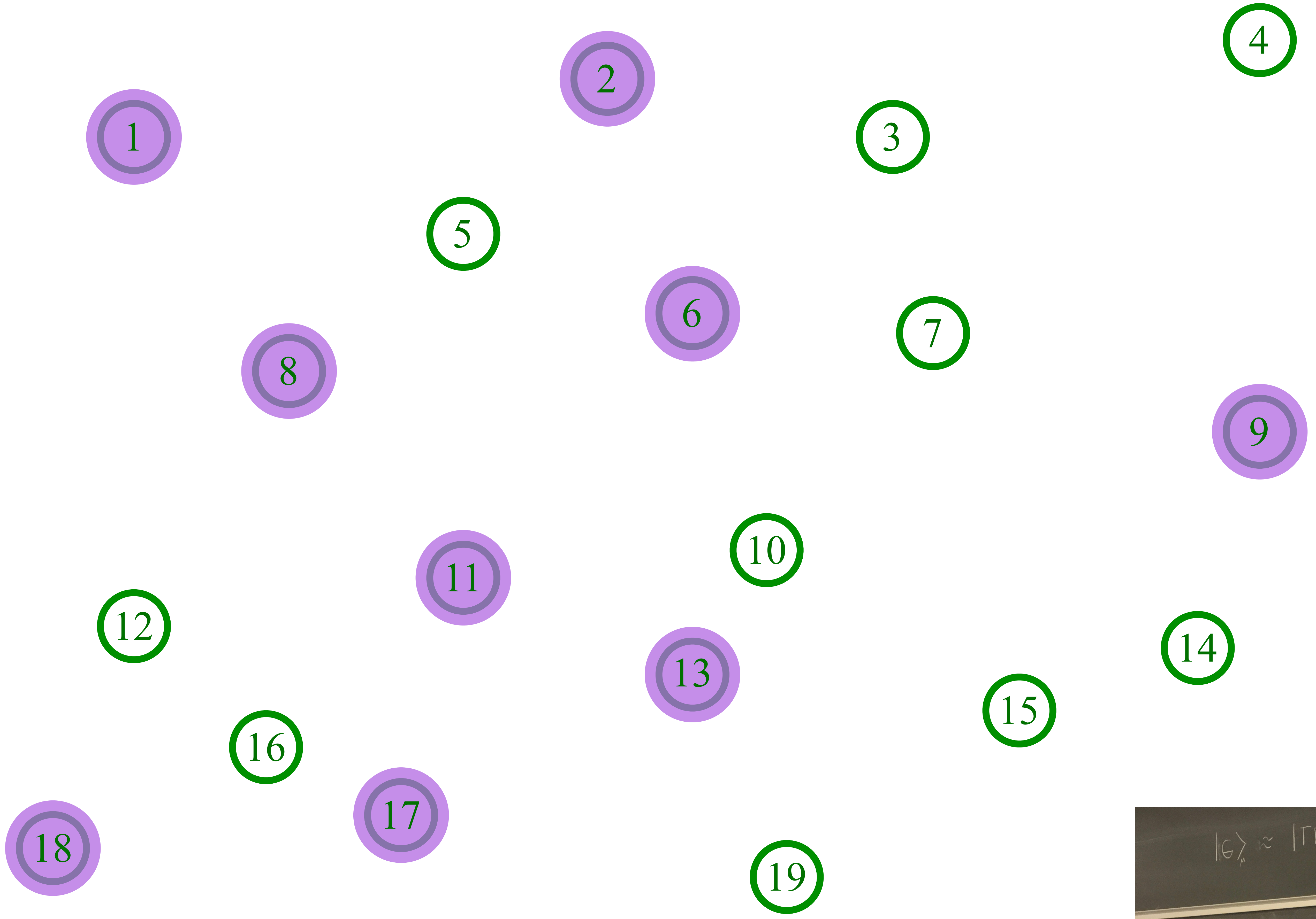

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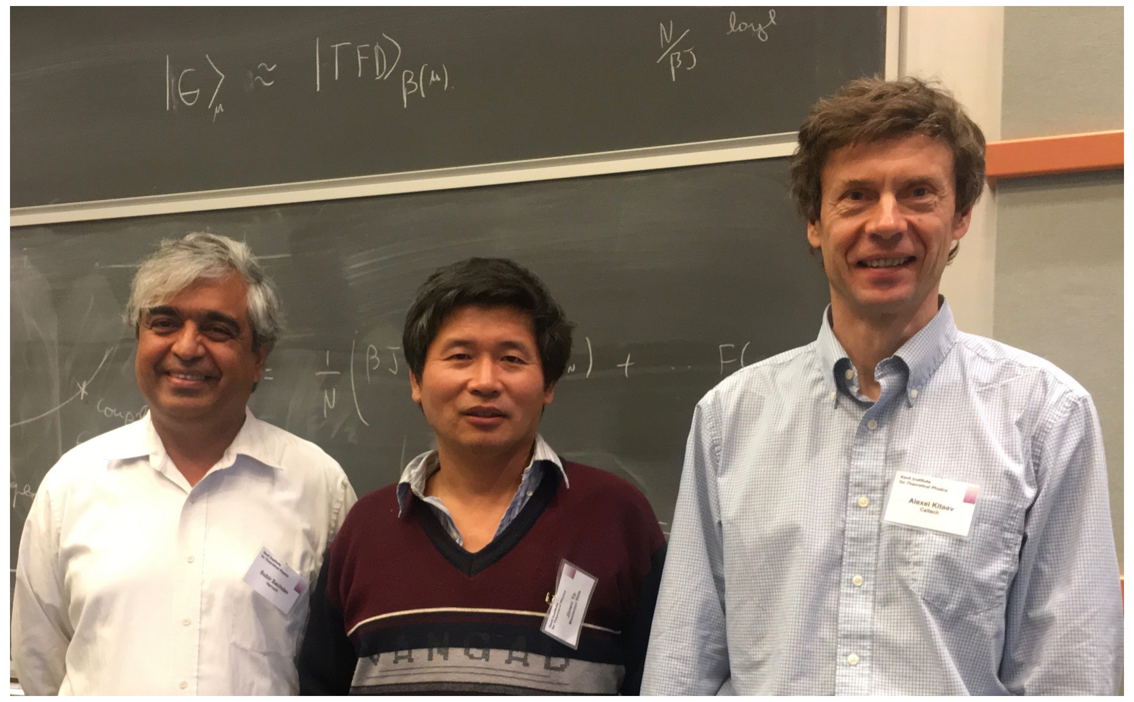

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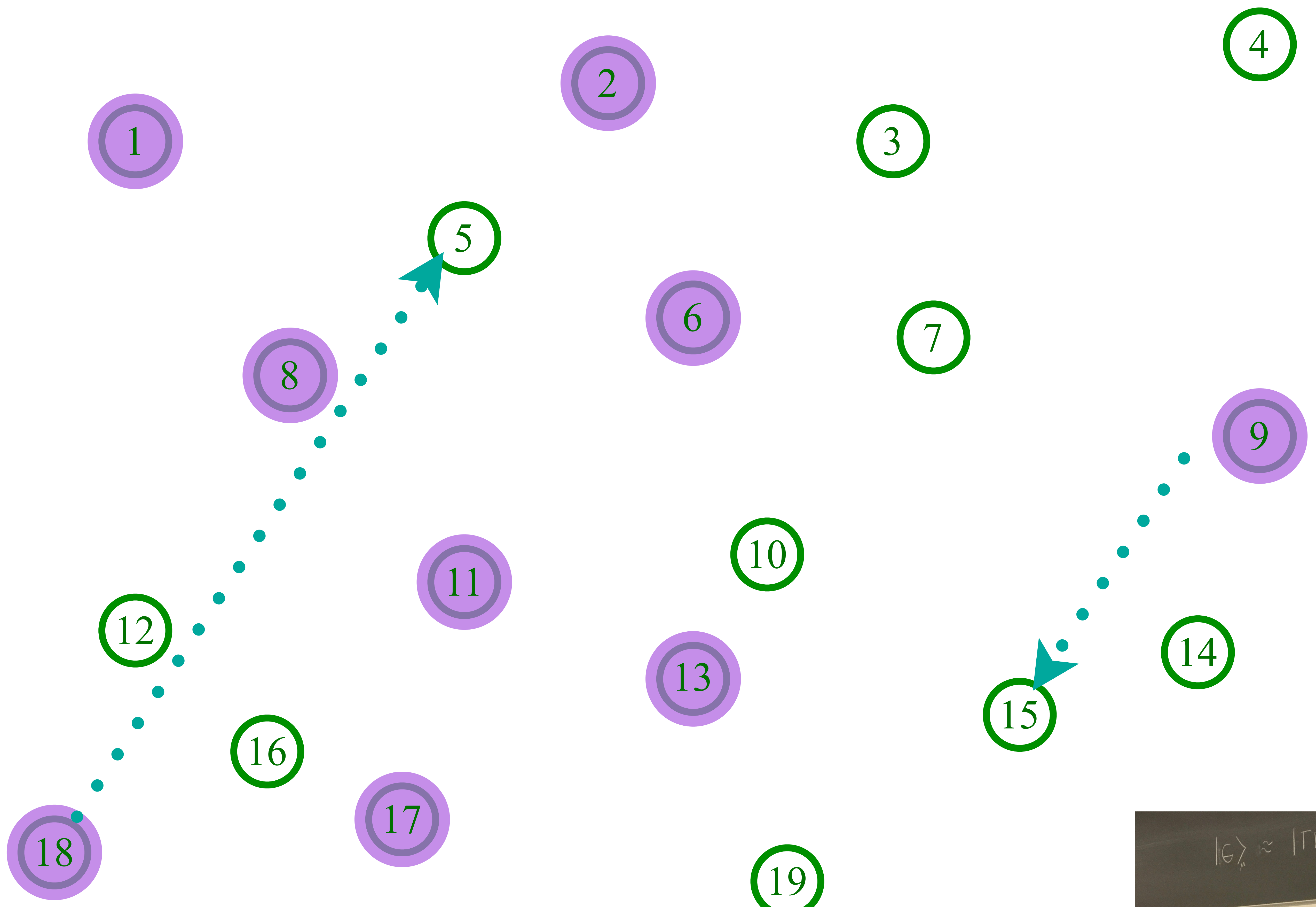
Sachdev, Ye (1993); Kitaev (2015)

$$U_{9,18;5,15}$$



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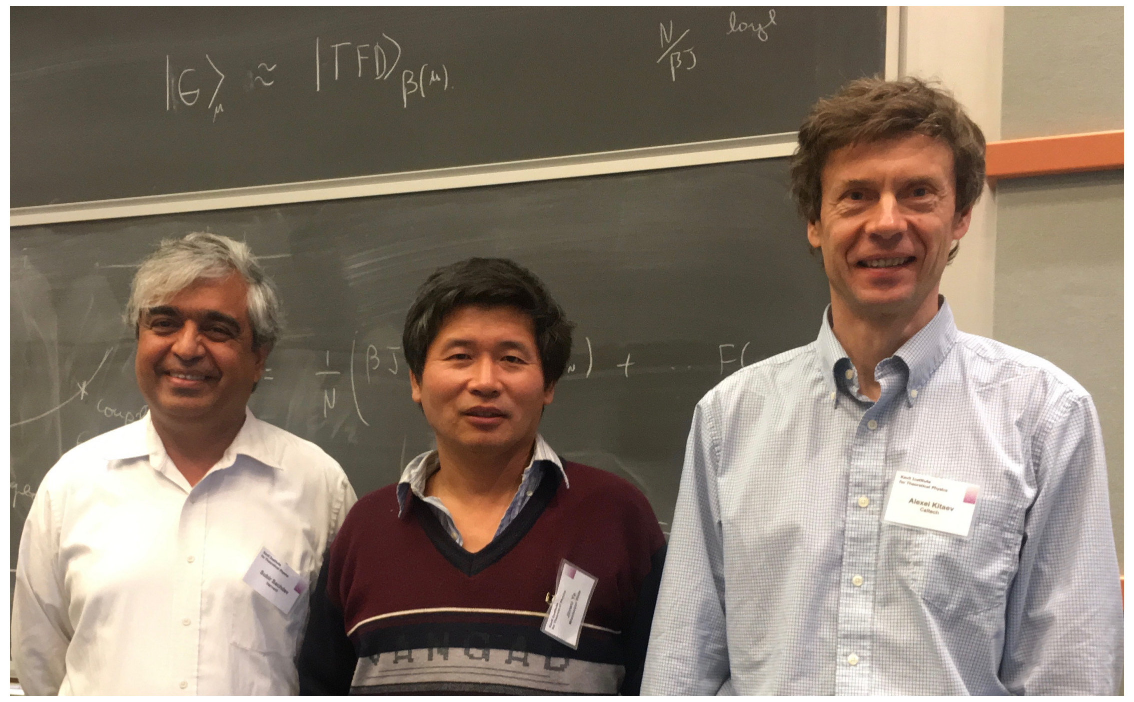

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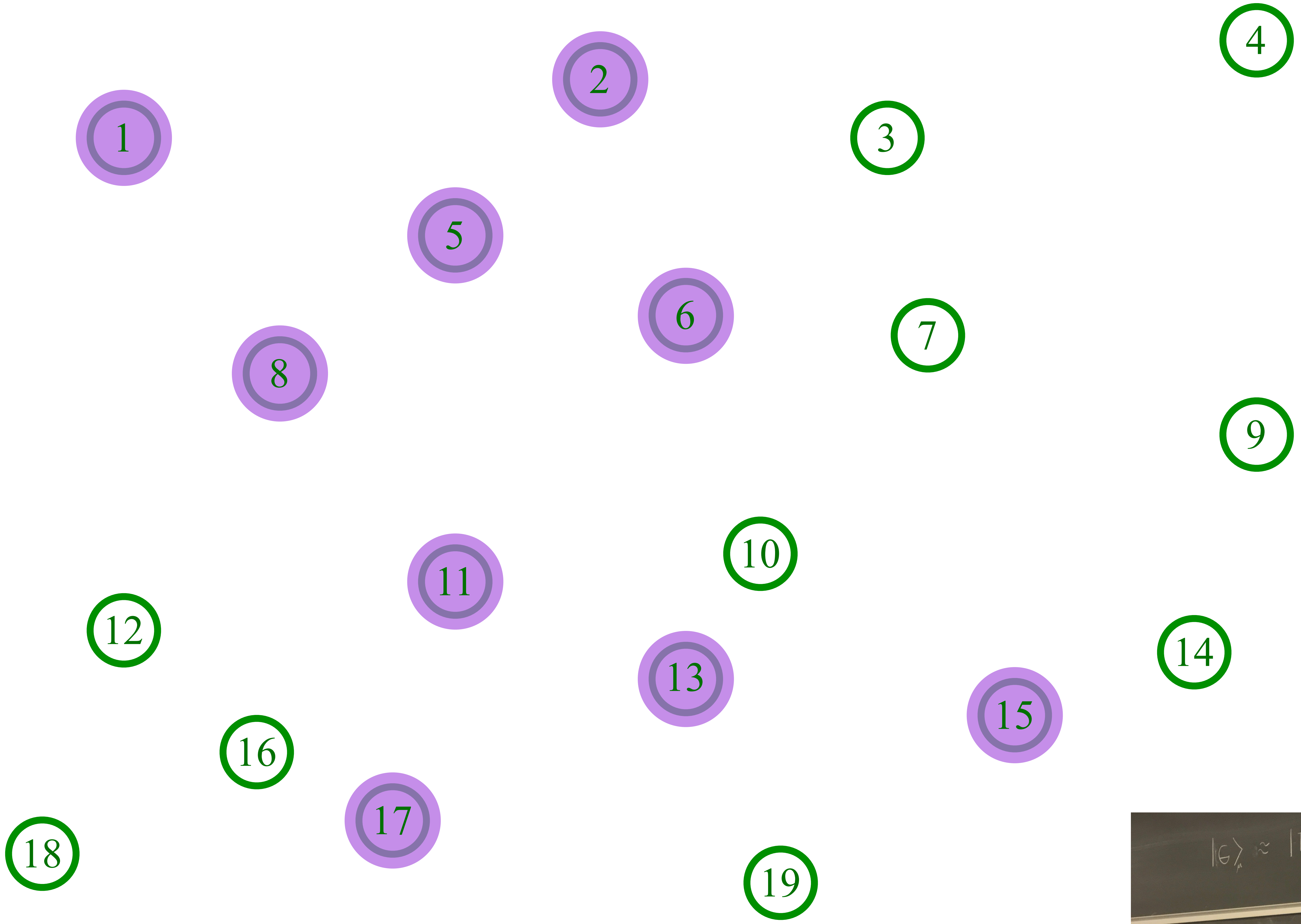

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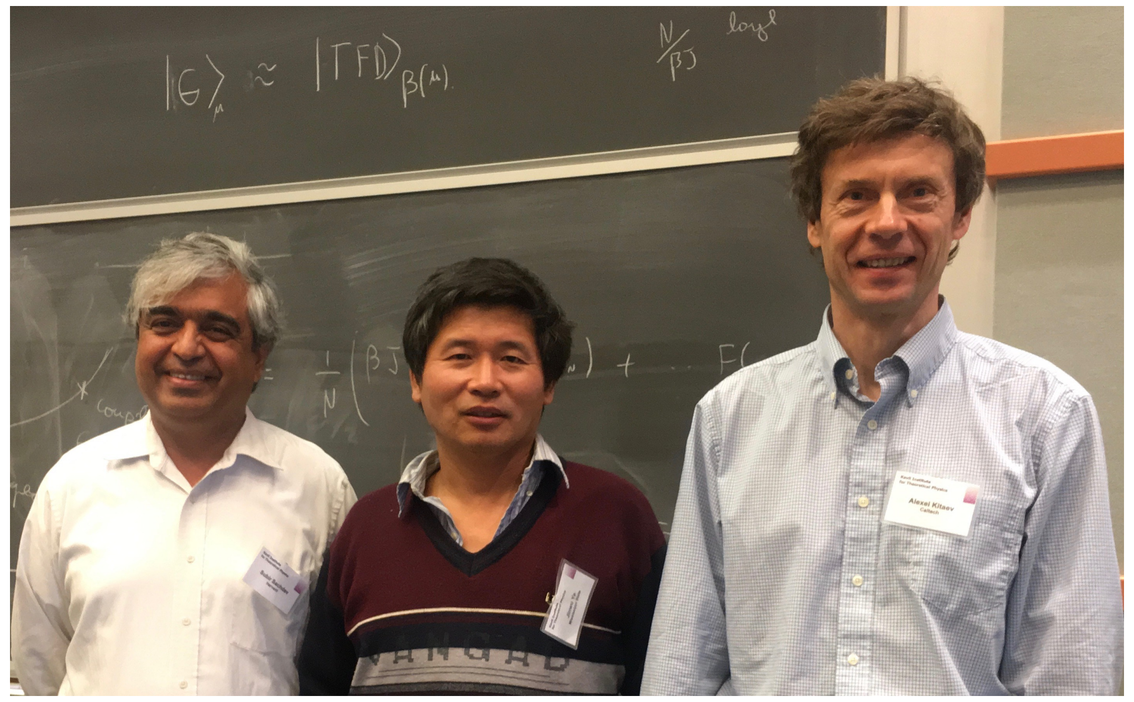
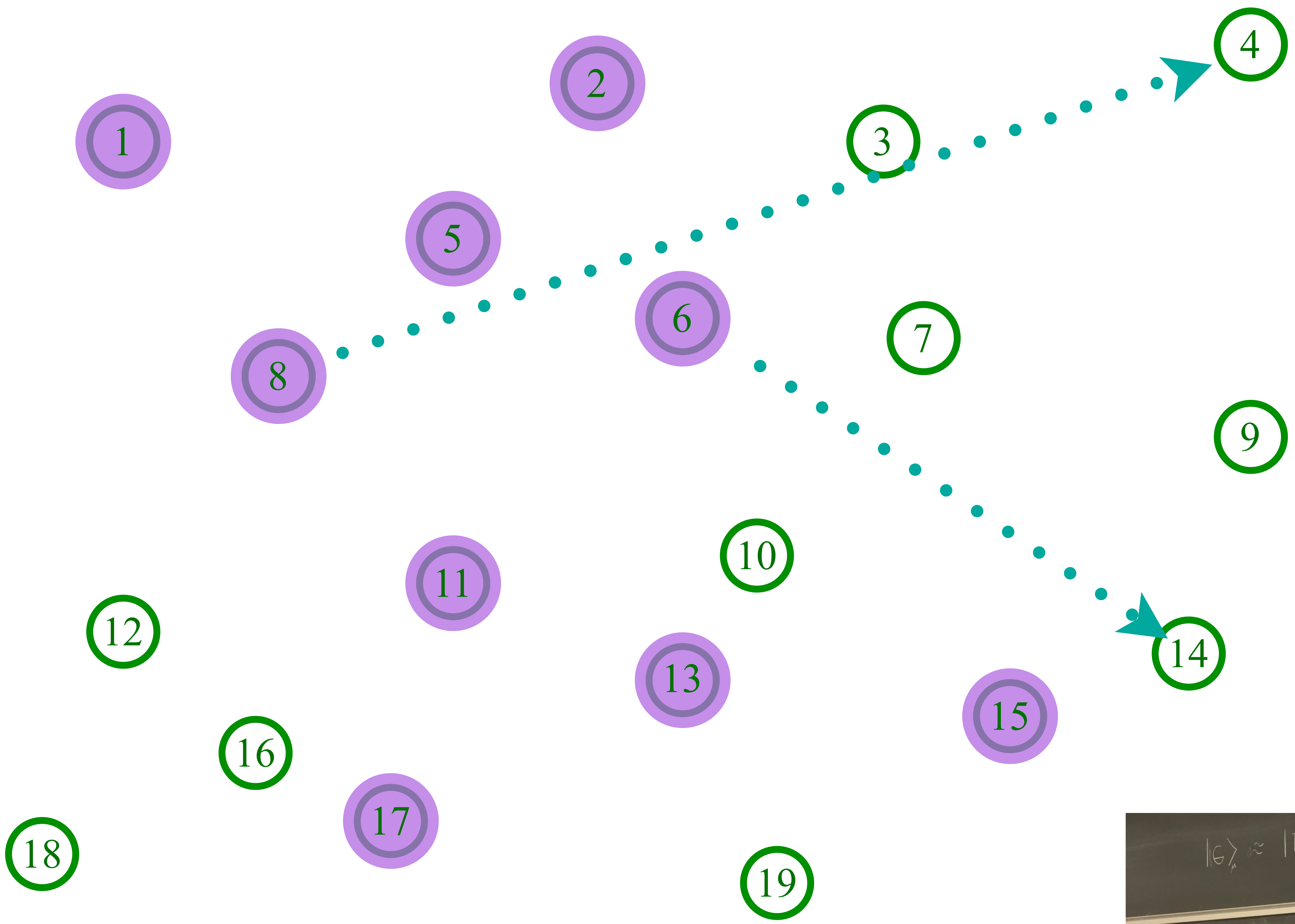


The SYK model

Sachdev, Ye (1993); Kitaev (2015)

$$U_{6,8;4,14}$$



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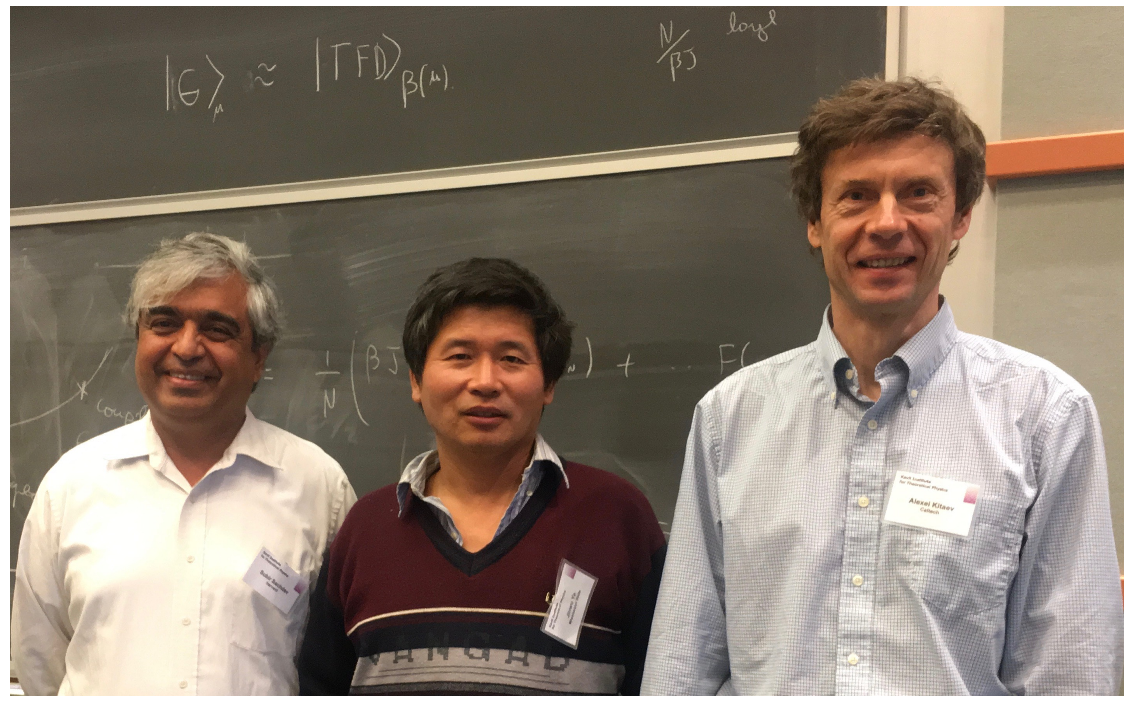

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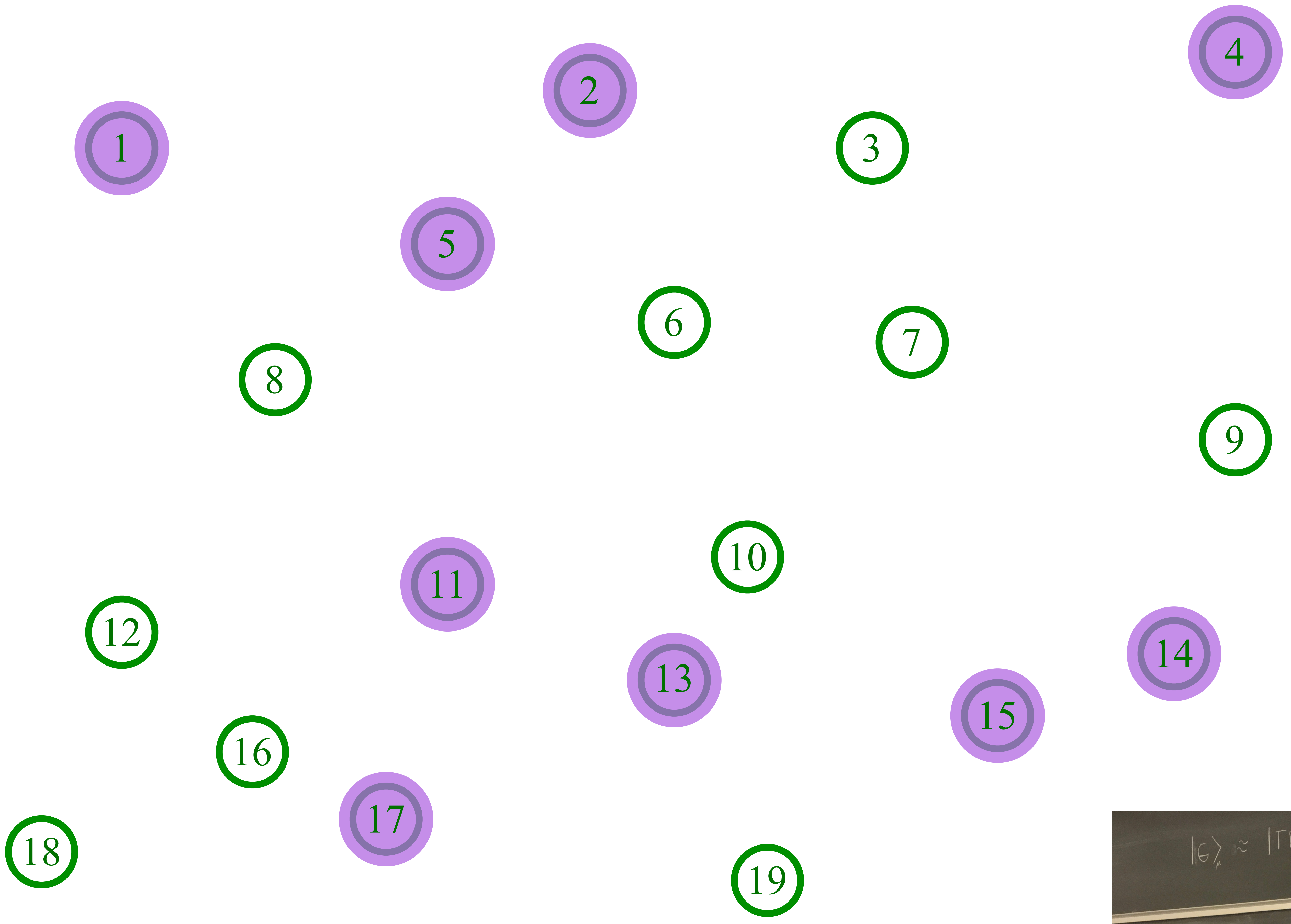

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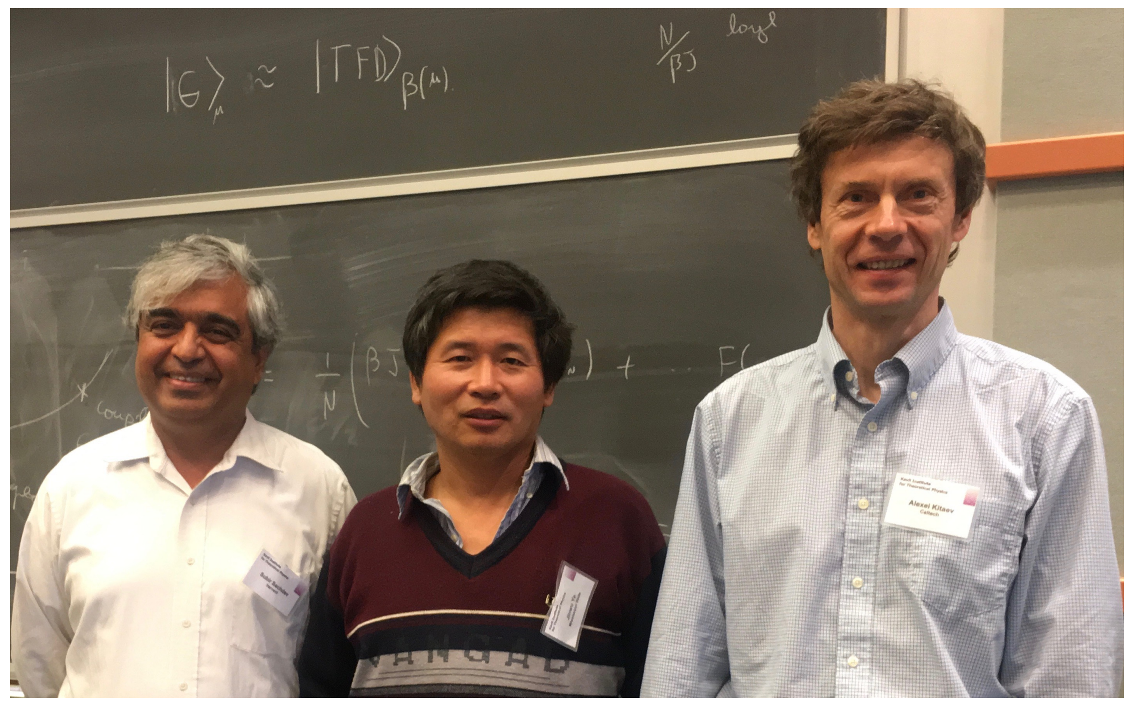


The Sachdev-Ye-Kitaev (SYK) model

(See also: the "2-Body Random Ensemble" in nuclear physics; did not obtain the large *N* limit;
T.A. Brody, J. Flores, J.B. French, P.A. Mello, A. Pandey, and S.S.M. Wong, Rev. Mod. Phys. **53**, 385 (1981))

$$\mathcal{H} = \frac{1}{(2N)^{3/2}} \sum_{\alpha,\beta,\gamma,\delta=1}^{N} U_{\alpha\beta;\gamma\delta}\, c_\alpha^\dagger c_\beta^\dagger c_\gamma c_\delta - \mu \sum_\alpha c_\alpha^\dagger c_\alpha$$

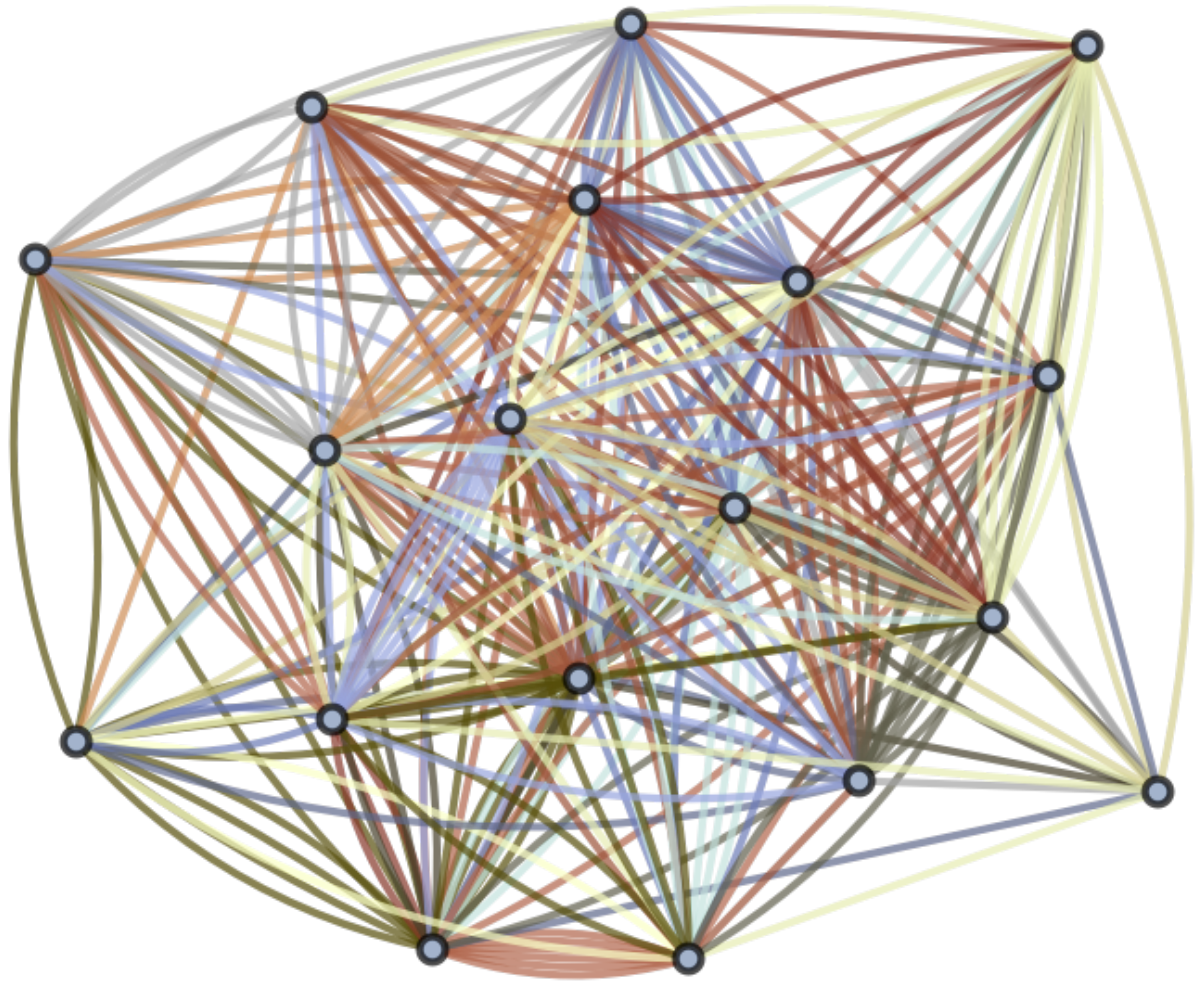
$$c_\alpha c_\beta + c_\beta c_\alpha = 0 \quad , \quad c_\alpha c_\beta^\dagger + c_\beta^\dagger c_\alpha = \delta_{\alpha\beta}$$

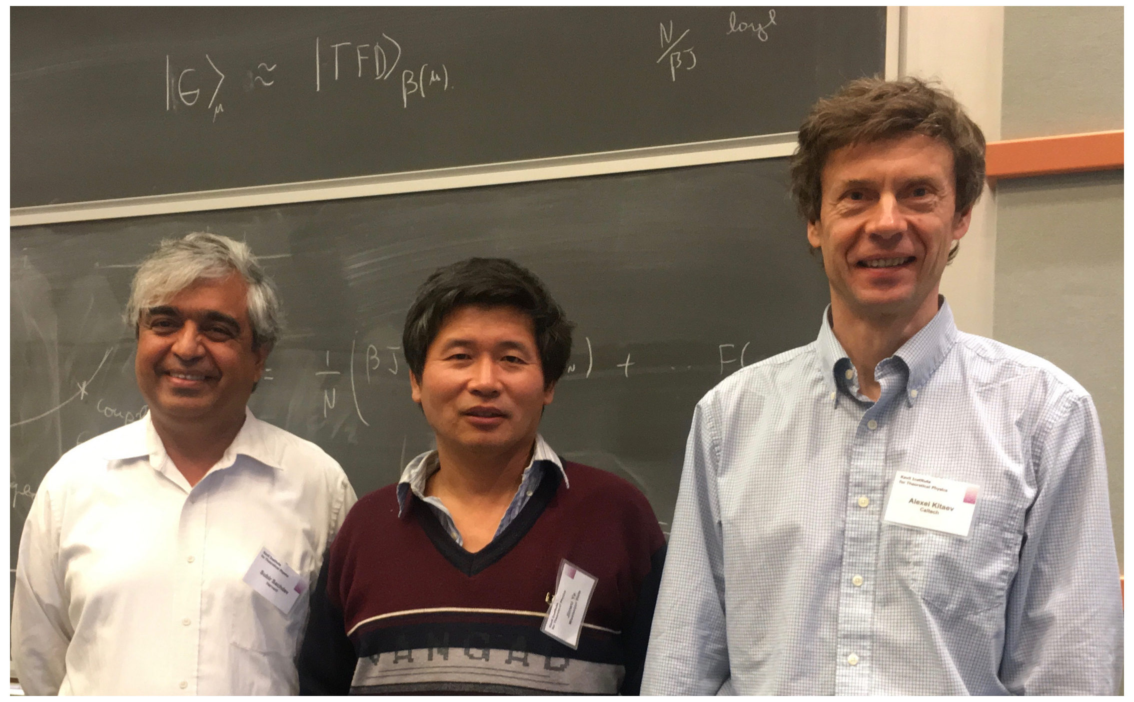
$$\mathcal{Q} = \frac{1}{N} \sum_\alpha c_\alpha^\dagger c_\alpha \,; \quad [\mathcal{H}, \mathcal{Q}] = 0 \,; \quad 0 \leq \mathcal{Q} \leq 1$$

$U_{\alpha\beta;\gamma\delta}$ are independent random variables with $\overline{U_{\alpha\beta;\gamma\delta}} = 0$ and $\overline{|U_{\alpha\beta;\gamma\delta}|^2} = U^2$
$N \to \infty$ yields critical strange metal.

S. Sachdev and J. Ye, PRL **70**, 3339 (1993)

A. Kitaev, unpublished; S. Sachdev, PRX **5**, 041025 (2015)



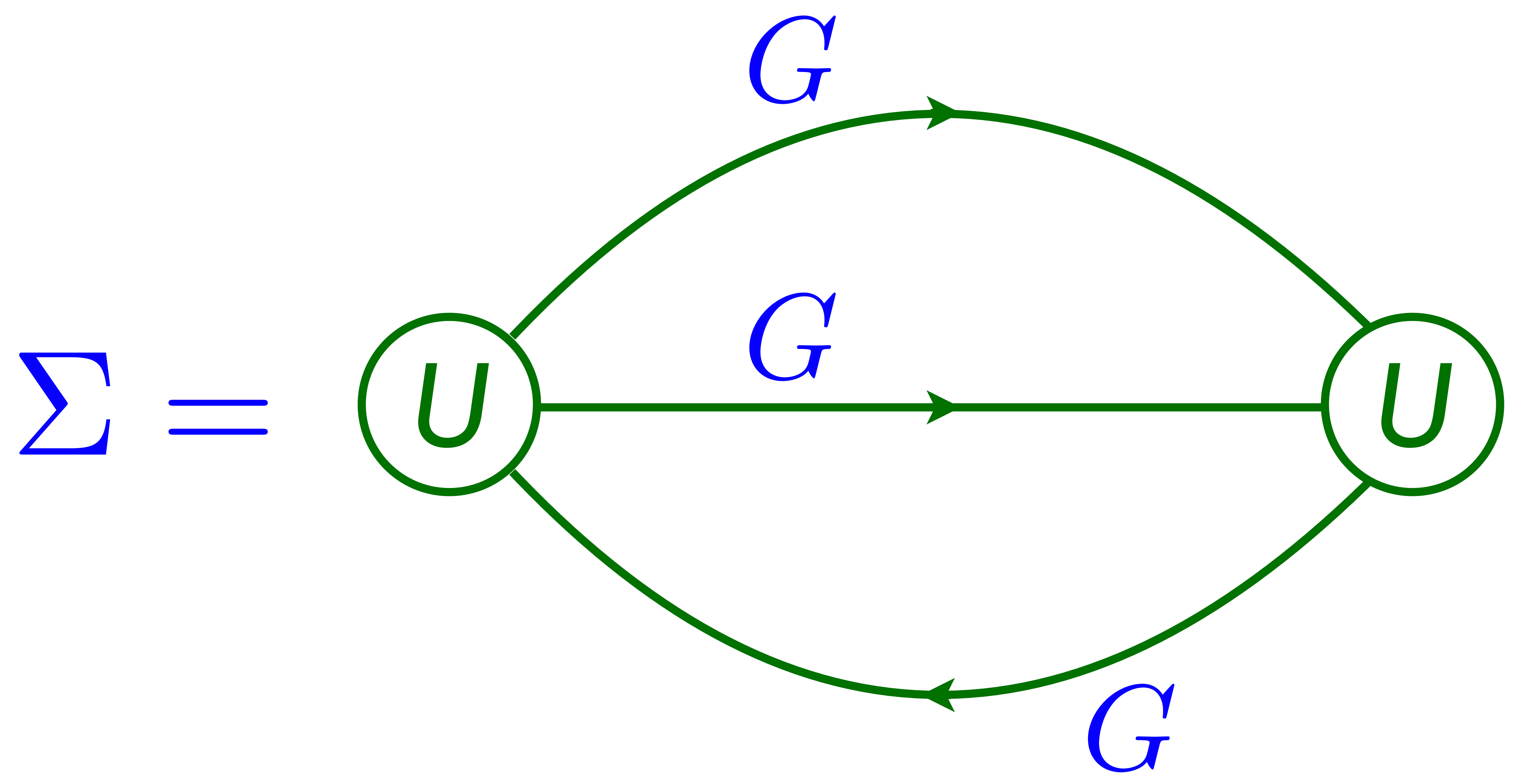


The Sachdev-Ye-Kitaev (SYK) model

Feynman graph expansion in $U_{\alpha\beta;\gamma\delta}$, and graph-by-graph average, yields exact equations in the large N limit:

$$G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = -U^2 G^2(\tau) G(-\tau)$$

$$G(\tau = 0^-) = \mathcal{Q}.$$



S. Sachdev and J. Ye,
PRL **70**, 3339 (1993)

The complex SYK model

Solution of these equations, and of the free energy, yields universal results for the SYK model:

- At long times, and at $T = 0$, $G(\tau) \sim |\tau|^{-1/2}$ ($\Rightarrow$ indication there are no quasiparticles)

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- At long times, and at $T = 0$, $G(\tau) \sim |\tau|^{-1/2}$ ($\Rightarrow$ indication there are no quasiparticles)
- At general charge $\mathcal{Q}$, there is a spectral symmetry determined by a parameter $\mathcal{E}$:

$$G(\tau) \sim \begin{cases} -\tau^{-1/2} & \tau > 0 \\ e^{-2\pi\mathcal{E}}(-\tau)^{-1/2} & \tau < 0 \end{cases} \quad , \quad T = 0$$

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- There is a universal 'Luttinger relation' between $-\infty < \mathcal{E} < \infty$ and the total charge $0 < \mathcal{Q} < 1$

$$\begin{aligned} e^{2\pi\mathcal{E}} &= \frac{\sin(\pi/4 + \theta)}{\sin(\pi/4 - \theta)} \\ \mathcal{Q} &= \frac{1}{2} - \frac{\theta}{\pi} - \frac{\sin(2\theta)}{4} \end{aligned}$$

A. Georges, O. Parcollet, and S. Sachdev, PRB **63**, 134406 (2001)

The complex SYK model

Solution of these equations, and of the free energy, yields universal results for the SYK model:

- At $T > 0$, we obtain a solution with a conformal structure

$$G(\tau) = -A \frac{e^{-2\pi\mathcal{E}T\tau}}{\sqrt{1+e^{-4\pi\mathcal{E}}}} \left(\frac{T}{\sin(\pi T \tau)}\right)^{1/2} \quad , \quad 0 < \tau < 1/T \,,$$

where the 'particle-hole asymmetry' is determined by $\mathcal{E}$

A. Georges and O. Parcollet PRB **59**, 5341 (1999)
S. Sachdev, PRX **5**, 041025 (2015)

The complex SYK model

$$G_*(\tau) = -C \frac{e^{-2\pi\mathcal{E}T\tau}}{\sqrt{1+e^{-4\pi\mathcal{E}}}} \left(\frac{T}{\sin(\pi T \tau)}\right)^{1/2} .$$

$$G^R_*(\omega) =$$

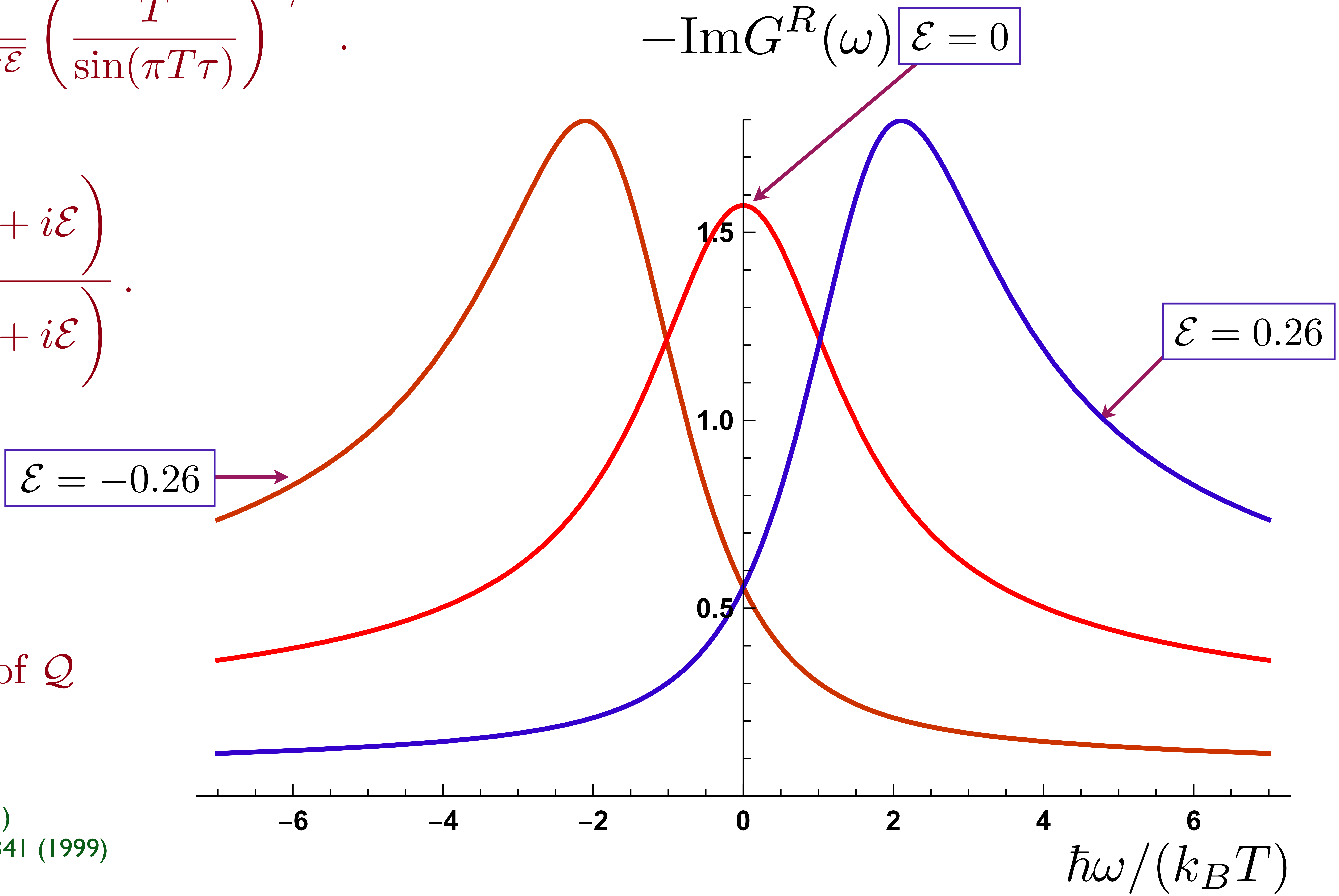
$$\frac{-iCe^{-i\theta}}{(2\pi T)^{1/2}} \frac{\Gamma\left(\frac{1}{4} - \frac{i\omega}{2\pi T} + i\mathcal{E}\right)}{\Gamma\left(\frac{3}{4} - \frac{i\omega}{2\pi T} + i\mathcal{E}\right)} .$$

$$e^{2\pi\mathcal{E}} = \frac{\sin(\pi/4+\theta)}{\sin(\pi/4-\theta)}$$

$$C = \left(\frac{\pi}{U^2\cos(2\theta)}\right)^{1/4}$$

$\mathcal{E}$ is a known function of $\mathcal{Q}$

(Luttinger relation)



S. Sachdev and J. Ye, PRL **70**, 3339 (1993)
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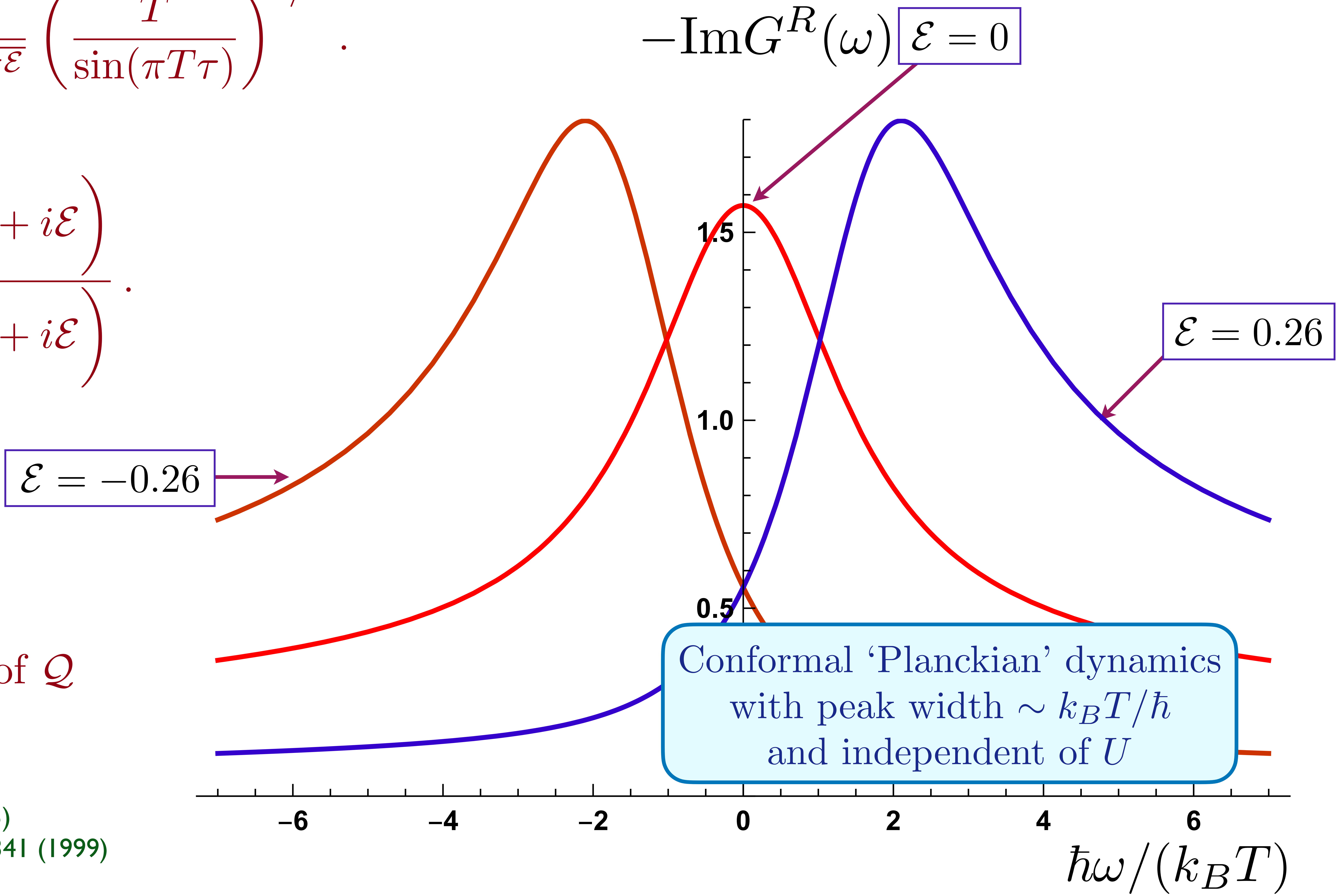
$\mathcal{E}$ is a known function of $\mathcal{Q}$ (Luttinger relation)



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S. Sachdev, PRX **5**, 041025 (2015)

G-Σ path integral

After introducing replicas $a = 1 \ldots n$, and integrating out the disorder, the partition function can be written as

$$
Z = \int \mathcal{D}c_{\alpha a}(\tau) \exp \left[- \sum_{ia} \int_0^\beta d\tau \, c_{\alpha a}^\dagger \left(\frac{\partial}{\partial \tau} - \mu \right) c_{\alpha a} \right.
$$
$$
\left. - \frac{U^2}{4N^3} \sum_{ab} \int_0^\beta d\tau d\tau' \left| \sum_i c_{\alpha a}^\dagger(\tau) c_{\alpha b}(\tau') \right|^4 \right] .
$$

For simplicity, we neglect the replica indices, and introduce the identity

$$
1 = \int \mathcal{D}G(\tau_1, \tau_2) \mathcal{D}\Sigma(\tau_1, \tau_2) \exp \left[-N \int_0^\beta d\tau_1 d\tau_2 \Sigma(\tau_1, \tau_2) \left(G(\tau_2, \tau_1) \right. \right.
$$
$$
\left. \left. + \frac{1}{N} \sum_\alpha c_\alpha(\tau_2) c_\alpha^\dagger(\tau_1) \right) \right] .
$$

Then the partition function can be written as a path integral with an action S analogous to a Luttinger-Ward functional

G-Σ path integral

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$$Z = \int \mathcal{D}G(\tau_1, \tau_2)\mathcal{D}\Sigma(\tau_1, \tau_2) \exp(-NS)$$

$$S = \ln\det\left[\delta(\tau_1 - \tau_2)(\partial_{\tau_1} + \mu) - \Sigma(\tau_1, \tau_2)\right] + \int d\tau_1 d\tau_2 \left[\Sigma(\tau_1, \tau_2)G(\tau_2, \tau_1) + (U^2/2)G^2(\tau_2, \tau_1)G^2(\tau_1, \tau_2)\right]$$

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Saddle-point equations:

$$G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = -U^2 G^2(\tau) G(-\tau)$$

$$G(\tau = 0^-) = \mathcal{Q}.$$

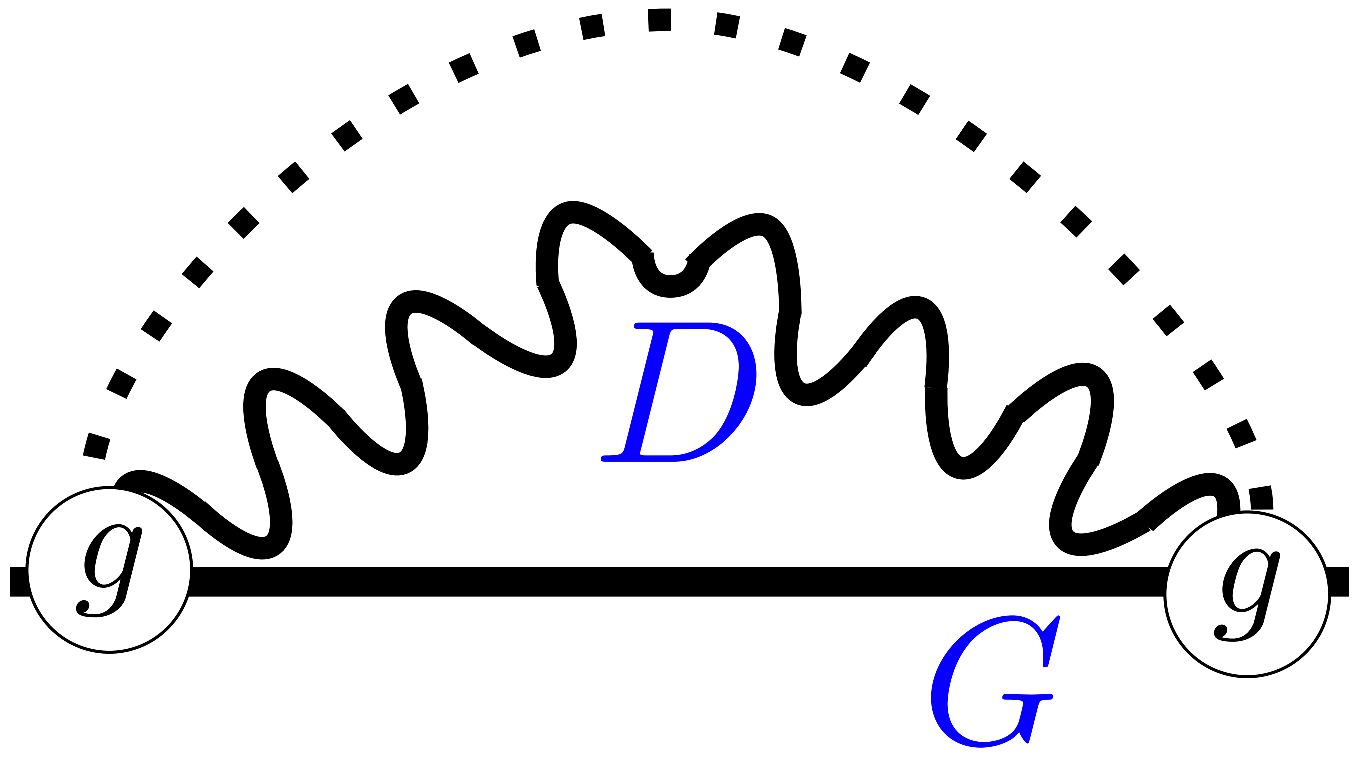
Yukawa-SYK model

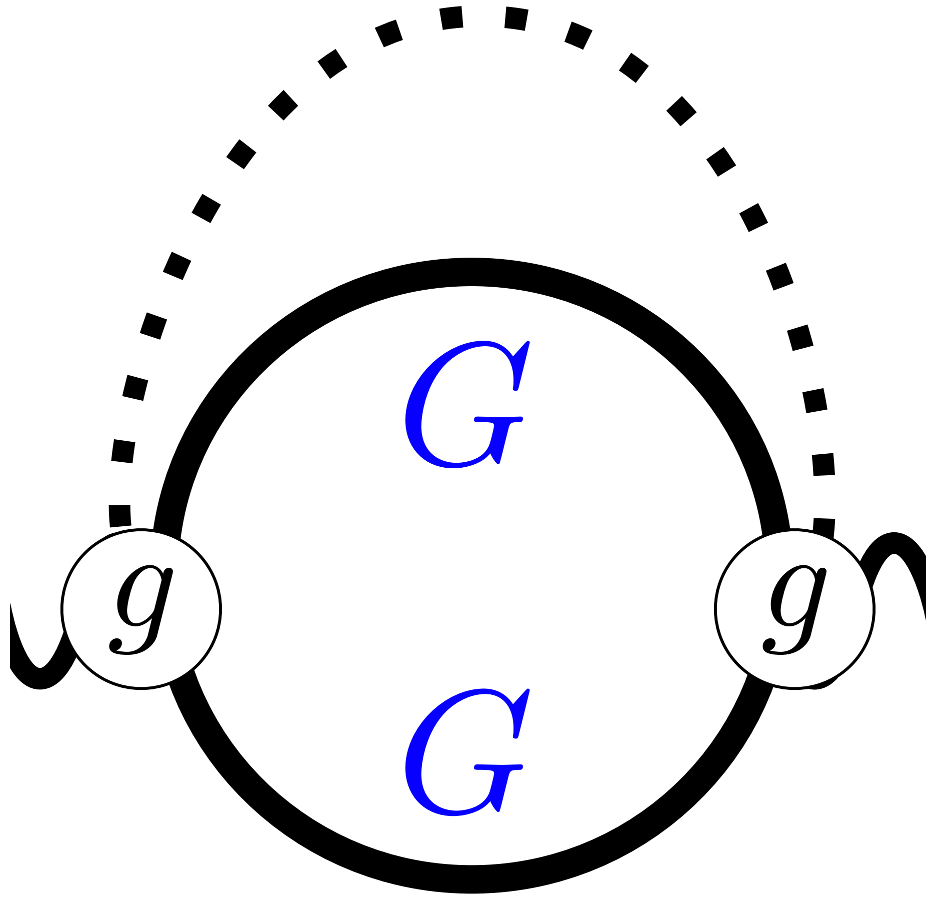
$$\mathcal{H} = -\mu \sum_i \psi_i^\dagger \psi_i + \sum_\ell \frac{1}{2}\left(\pi_\ell^2 + \omega_0^2 \phi_\ell^2\right) + \frac{1}{N} \sum_{ij\ell} g_{ij\ell} \psi_i^\dagger \psi_j \phi_\ell$$

with $g_{ij\ell}$ independent random numbers with zero mean. The large N equations for the Green's functions and self energies of the fermions (G, Σ) and bosons (D, Π) are

$$G(i\omega_n) = \frac{1}{i\omega_n + \mu - \Sigma(i\omega_n)} \quad , \quad D(i\omega_n) = \frac{1}{\omega_n^2 + \omega_0^2 - \Pi(i\omega_n)}$$

$$\Sigma(\tau) = g^2 G(\tau) D(\tau) \quad , \quad \Pi(\tau) = -g^2 G(\tau) G(-\tau)$$

$\Sigma =$ 

$\Pi =$ 

I. Esterlis and J. Schmalian, PRB **100**, 115132 (2019)
See also Yuxuan Wang, PRL **124**, 017002 (2020)

Yukawa-SYK model

These results can also be obtained from the saddle-point of a G-Σ-D-Π action, obtained using replica methods as for the SYK model.

$$\mathcal{Z} = \int \mathcal{D}G \, \mathcal{D}\Sigma \, \mathcal{D}D \, \mathcal{D}\Pi \exp(-N S_{\rm all})$$

$$\begin{aligned} S_{\rm all} = & -\ln\det(\partial_\tau + -\mu + \Sigma) + \frac{1}{2}\ln\det(-\partial_\tau^2 + \omega_0^2 - \Pi) \\ & + \int d\tau \int d\tau' \Big[-\Sigma(\tau';\tau)G(\tau;\tau') + \frac{1}{2}\Pi(\tau';\tau)D(\tau;\tau') \\ & \qquad + \frac{g^2}{2} G(\tau;\tau')G(\tau';\tau)D(\tau;\tau') \Big] \,. \end{aligned}$$

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Saddle-point equations:

$$\Sigma(\tau) = g^2 D(\tau) G(\tau),$$
$$\Pi(\tau) = -g^2 G(-\tau) G(\tau),$$
$$G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)},$$
$$D(i\Omega) = \frac{1}{\Omega^2 + \omega_0^2 - \Pi(i\Omega)}.$$

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$$\mathcal{H} = -\mu \sum_i \psi_i^\dagger \psi_i + \sum_\ell \frac{1}{2}\left(\pi_\ell^2 + \omega_0^2 \phi_\ell^2\right) + \frac{1}{N} \sum_{ij\ell} g_{ij\ell} \psi_i^\dagger \psi_j \phi_\ell$$

with $g_{ij\ell}$ independent random numbers with zero mean. The large N equations for the Green's functions and self energies of the fermions (G, Σ) and bosons (D, Π) are

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$$\Sigma(\tau) = g^2 G(\tau) D(\tau) \quad , \quad \Pi(\tau) = -g^2 G(\tau) G(-\tau)$$

Make the low frequency ansatz

$$G(i\omega) \sim -i\,\mathrm{sgn}(\omega)|\omega|^{-(1-2\Delta)} \quad , \quad D(i\omega) \sim |\omega|^{1-4\Delta} \quad , \quad \frac{1}{4} < \Delta < \frac{1}{2}$$

A consistent solution exists for

$$\frac{4\Delta - 1}{2(2\Delta - 1)[\sec(2\pi\Delta) - 1]} = 1 \quad , \quad \Delta = 0.42037\ldots$$

I. Esterlis and J. Schmalian, PRB **100**, 115132 (2019)
See also Yuxuan Wang, PRL **124**, 017002 (2020)

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$$S = \ln\det\left[\delta(\tau_1-\tau_2)(\partial_{\tau_1}+\mu) - \Sigma(\tau_1,\tau_2)\right] + \int d\tau_1 d\tau_2 \left[\Sigma(\tau_1,\tau_2)G(\tau_2,\tau_1) + (U^2/2)G^2(\tau_2,\tau_1)G^2(\tau_1,\tau_2)\right]$$

At frequencies $\ll U$, the time derivative in the determinant is less important, and without it the path integral is invariant under the reparametrization and gauge transformations

A. Georges and O. Parcollet
PRB **59**, 5341 (1999)
A. Kitaev, 2015
S. Sachdev, PRX **5,** 041025 (2015)

$$\tau = f(\sigma)$$

$$G(\tau_1,\tau_2) = [f'(\sigma_1)f'(\sigma_2)]^{-1/4}\frac{g(\sigma_1)}{g(\sigma_2)}\widetilde{G}(\sigma_1,\sigma_2)$$

$$\Sigma(\tau_1,\tau_2) = [f'(\sigma_1)f'(\sigma_2)]^{-3/4}\frac{g(\sigma_1)}{g(\sigma_2)}\widetilde{\Sigma}(\sigma_1,\sigma_2)$$

where $f(\sigma)$ and $g(\sigma)$ are arbitrary functions.

G-Σ path integral

Then the partition function can be written as a path integral with an action S analogous to a Luttinger-Ward functional

$$Z = \int \mathcal{D}G(\tau_1, \tau_2)\mathcal{D}\Sigma(\tau_1, \tau_2) \exp(-NS)$$

$$S = \ln \det \left[\delta(\tau_1 - \tau_2)(\partial_{\tau_1} + \mu) - \Sigma(\tau_1, \tau_2)\right] + \int d\tau_1 d\tau_2 \left[\Sigma(\tau_1, \tau_2)G(\tau_2, \tau_1) + (U^2/2)G^2(\tau_2, \tau_1)G^2(\tau_1, \tau_2)\right]$$

At frequencies $\ll U$, the time derivative in the determinant is less important, and without it the path integral is invariant under the reparametrization and gauge transformations

A. Georges and O. Parcollet PRB **59**, 5341 (1999)
A. Kitaev, 2015
S. Sachdev, PRX **5,** 041025 (2015)

$$\tau = f(\sigma)$$

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where $f(\sigma)$ and $g(\sigma)$ are arbitrary functions.

We can map the $T = 0$ solution to the $T > 0$ solution by

$$\tau = \frac{1}{\pi T} \tan(\pi T \sigma)$$

$$g(\sigma) = e^{-2\pi \mathcal{E} T \sigma}$$

G-Σ path integral

Reparametrization and phase zero modes

We can write the path integral for the SYK model as

$$\mathcal{Z} = \int \mathcal{D}G(\tau_1, \tau_2)\mathcal{D}\Sigma(\tau_1, \tau_2) e^{-NS[G,\Sigma]}$$

for a known action $S[G, \Sigma]$. We find the saddle point, G_s, Σ_s, and only focus on the "Nambu-Goldstone" modes associated with breaking reparameterization and U(1) gauge symmetries by writing

$$G(\tau_1, \tau_2) = [f'(\tau_1)f'(\tau_2)]^{1/4} G_s(f(\tau_1) - f(\tau_2)) e^{i\phi(\tau_1) - i\phi(\tau_2)}$$

(and similarly for Σ). Then the path integral is approximated by

$$\mathcal{Z} = \int \mathcal{D}f(\tau)\mathcal{D}\phi(\tau) e^{-E_0/T + Ns_0 - NS_{\rm eff}[f,\phi]},$$

where $E_0 \propto N$ is the ground state energy.

J. Maldacena and D. Stanford, arXiv:1604.07818;
R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, arXiv.1612.00849;
S. Sachdev, PRX **5,** 041025 (2015); J. Maldacena, D. Stanford, and Zhenbin Yang, arXiv:1606.01857;
K. Jensen, arXiv:1605.06098; J. Engelsoy, T.G. Mertens, and H. Verlinde, arXiv:1606.03438

Symmetries of the large N saddle point

Let us write the large N saddle point solutions of S as

$$\begin{aligned} G_s(\tau_1 - \tau_2) &\sim (\tau_1 - \tau_2)^{-1/2} \\ \Sigma_s(\tau_1 - \tau_2) &\sim (\tau_1 - \tau_2)^{-3/2}. \end{aligned}$$

The saddle point will be invariant under a reperamaterization $f(\tau)$ when choosing $G(\tau_1, \tau_2) = G_s(\tau_1 - \tau_2)$ leads to a transformed $\widetilde{G}(\sigma_1, \sigma_2) = G_s(\sigma_1 - \sigma_2)$ (and similarly for Σ). It turns out this is true only for the SL(2, R) transformations under which

$$f(\tau) = \frac{a\tau + b}{c\tau + d} \quad , \quad ad - bc = 1.$$

So the (approximate) reparametrization symmetry is spontaneously broken down to SL(2, R) by the saddle point.

Symmetries of the large *N* saddle point

- The saddle-point

$$G(\tau_1 - \tau_2) = -A \frac{e^{-2\pi\mathcal{E}T(\tau_1-\tau_2)}}{\sqrt{1+e^{-4\pi\mathcal{E}}}} \left(\frac{T}{\sin(\pi T(\tau_1 - \tau_2))} \right)^{2\Delta}$$

is invariant only under PSL(2, R) transformations which map the thermal circle onto itself, and an associated gauge transformation

$$\frac{\tan(\pi T f(\tau))}{\pi T} = \frac{a \dfrac{\tan(\pi T \tau)}{\pi T} + b}{c \dfrac{\tan(\pi T \tau)}{\pi T} + d} \quad , \quad ad - bc = 1,$$

$$-i\phi(\tau) = -i\phi_0 + 2\pi\mathcal{E}T(\tau - f(\tau))$$

A. Kitaev, 2015

R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, PRB **95**, 155131 (2017)

G-Σ path integral

Symmetry arguments, and explicit computations, show that the effective action is

$$S_{\rm eff}[f,\phi] = \frac{NK}{2}\int_0^{1/T} d\tau (\partial_\tau \phi + i(2\pi \mathcal{E} T)\partial_\tau f)^2 - \frac{N\gamma}{4\pi^2}\int_0^{1/T} d\tau\, \{\tan(\pi T f(\tau)), \tau\},$$

where $f(\tau)$ is a monotonic map from $[0,1/T]$ to $[0,1/T]$, the couplings K, γ, and $\mathcal{E}$ can be related to thermodynamic derivatives and we have used the Schwarzian:

$$\{g,\tau\} \equiv \frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2 .$$

Specifically, an argument constraining the effective at $T=0$ is

$$S_{\rm eff}\left[f(\tau) = \frac{a\tau + b}{c\tau + d}, \phi(\tau) = 0\right] = 0,$$

and this is origin of the Schwarzian.

J. Maldacena and D. Stanford, arXiv:1604.07818;
R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, PRB **95**, 155131 (2017);
A. Gaikwad, L.K. Joshi, G. Mandal, and S.R. Wadia, arXiv:1802.07746

Low temperature thermodynamics: for $k_B T \ll U$

$$\mathcal{Z} = \mathrm{Tr} \exp\left(-\frac{\mathcal{H}}{k_B T}\right)$$

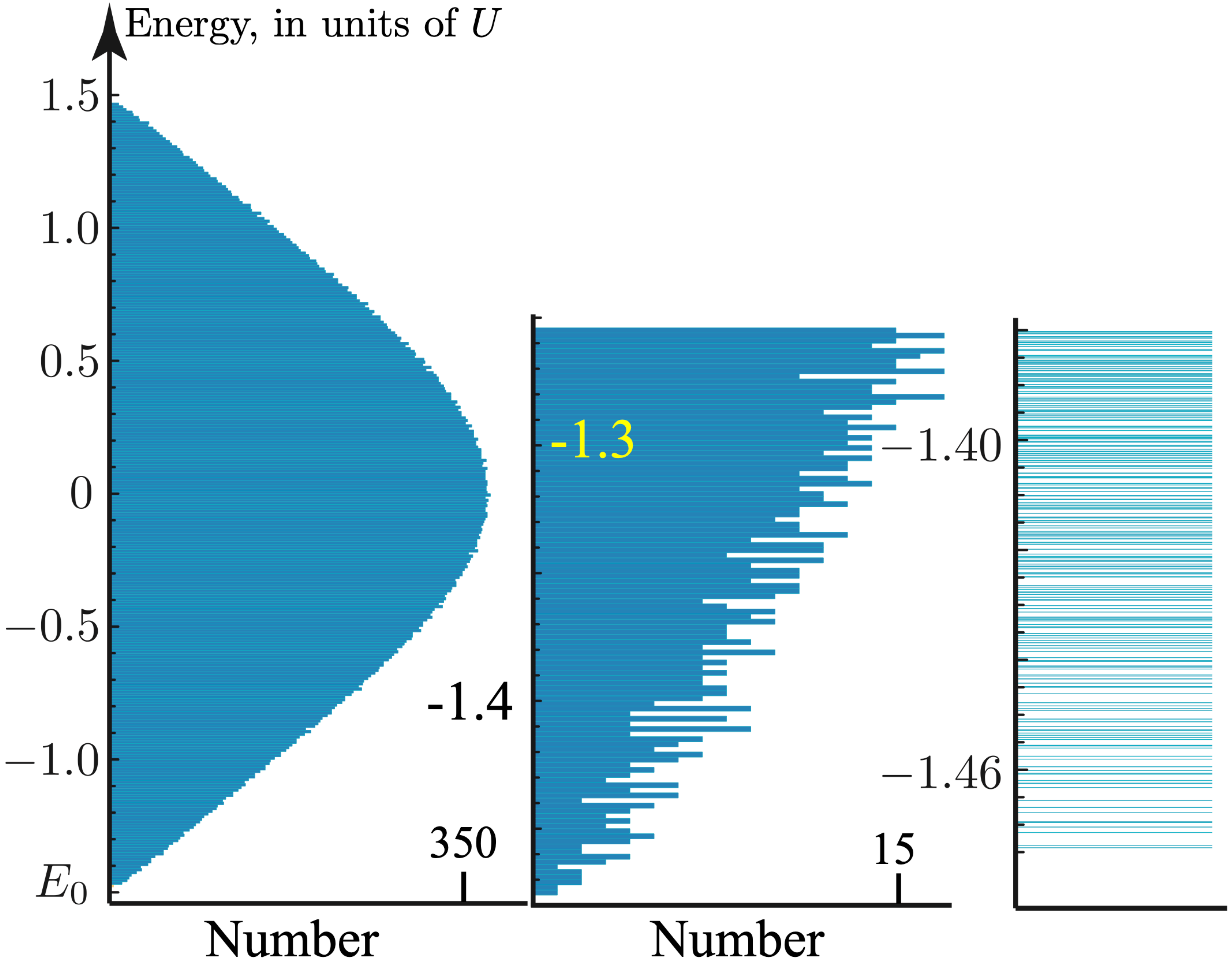
$$= \exp\left(N\frac{S_0}{k_B}\right) \int \frac{\mathcal{D}f(\tau)\mathcal{D}\phi(\tau)}{||\mathrm{SL(2,R)}||} \exp\left(-\frac{1}{\hbar} S_{\mathrm{eff}}\left[f(\tau), \phi(\tau)\right]\right)$$

- Feynman path integral over $f(\tau)$, the reparameterization of the time of the SYK model, and $\phi(\tau)$ a phase conjugate to the total charge $\mathcal{Q}$.

S. Sachdev, Phys. Rev. Lett. **105**, 151602 (2010)
A. Kitaev (2015)
J. Maldacena and D. Stanford, Phys. Rev. D **94,** 106002 (2016)

Many-body density of states

$$D(E) = \sum_i \delta(E - E_i); \quad E_0 + E_i \Rightarrow \text{Many body eigenvalue}$$



Complex SYK model

(Numerics: G. Tarnopolsky)

Many-body density of states

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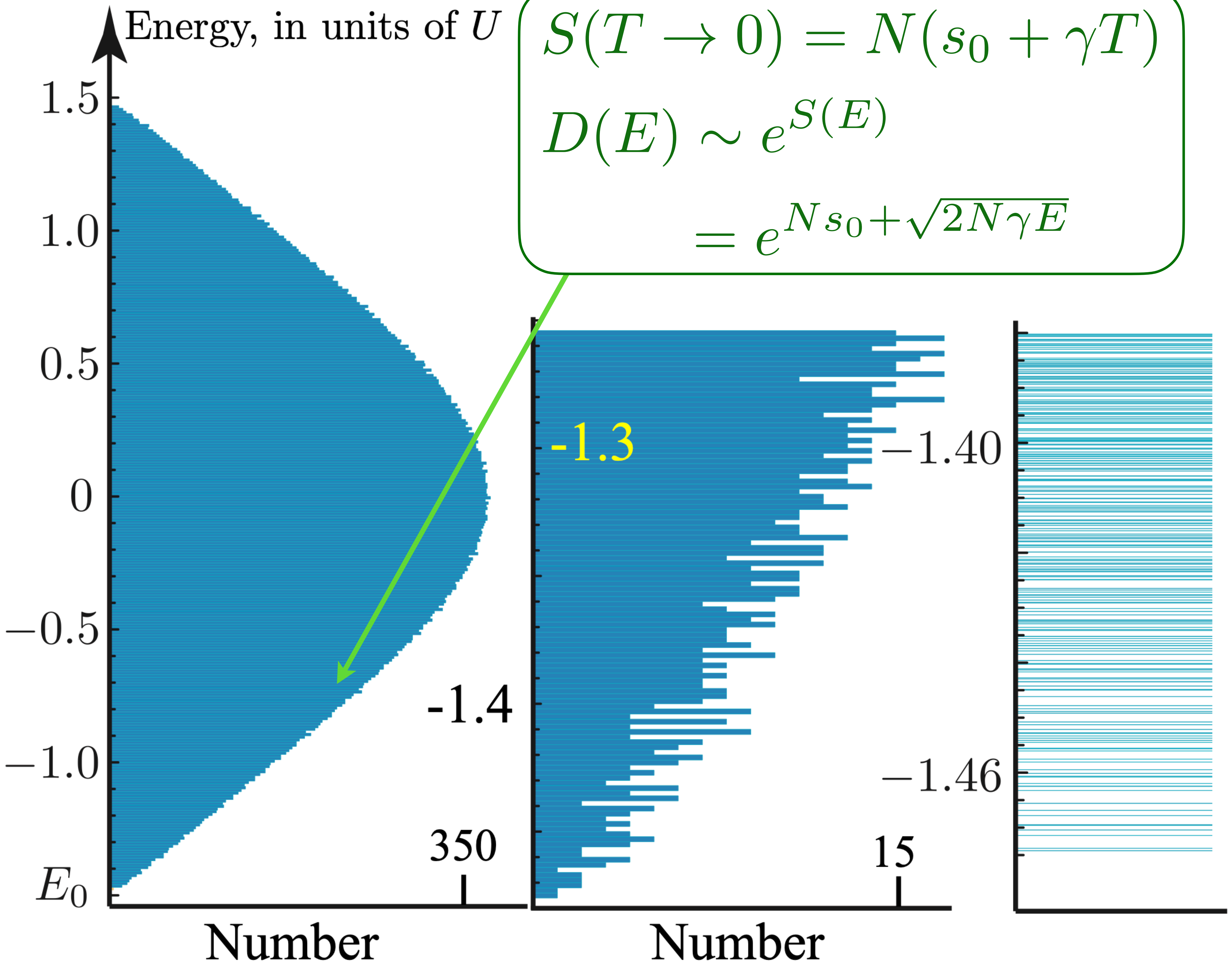

$$S(T \to 0) = N(s_0 + \gamma T)$$
$$D(E) \sim e^{S(E)}$$
$$= e^{N s_0 + \sqrt{2N\gamma E}}$$

At $\mathcal{Q} = 1/2$

$$s_0 = \frac{\text{Catalan}}{\pi} + \frac{\ln 2}{4}$$
$$= 0.46484769917\ldots.$$

A. Georges, O. Parcollet, and S. Sachdev, PRB **63**, 134406 (2001)

(Numerics: G. Tarnopolsky)

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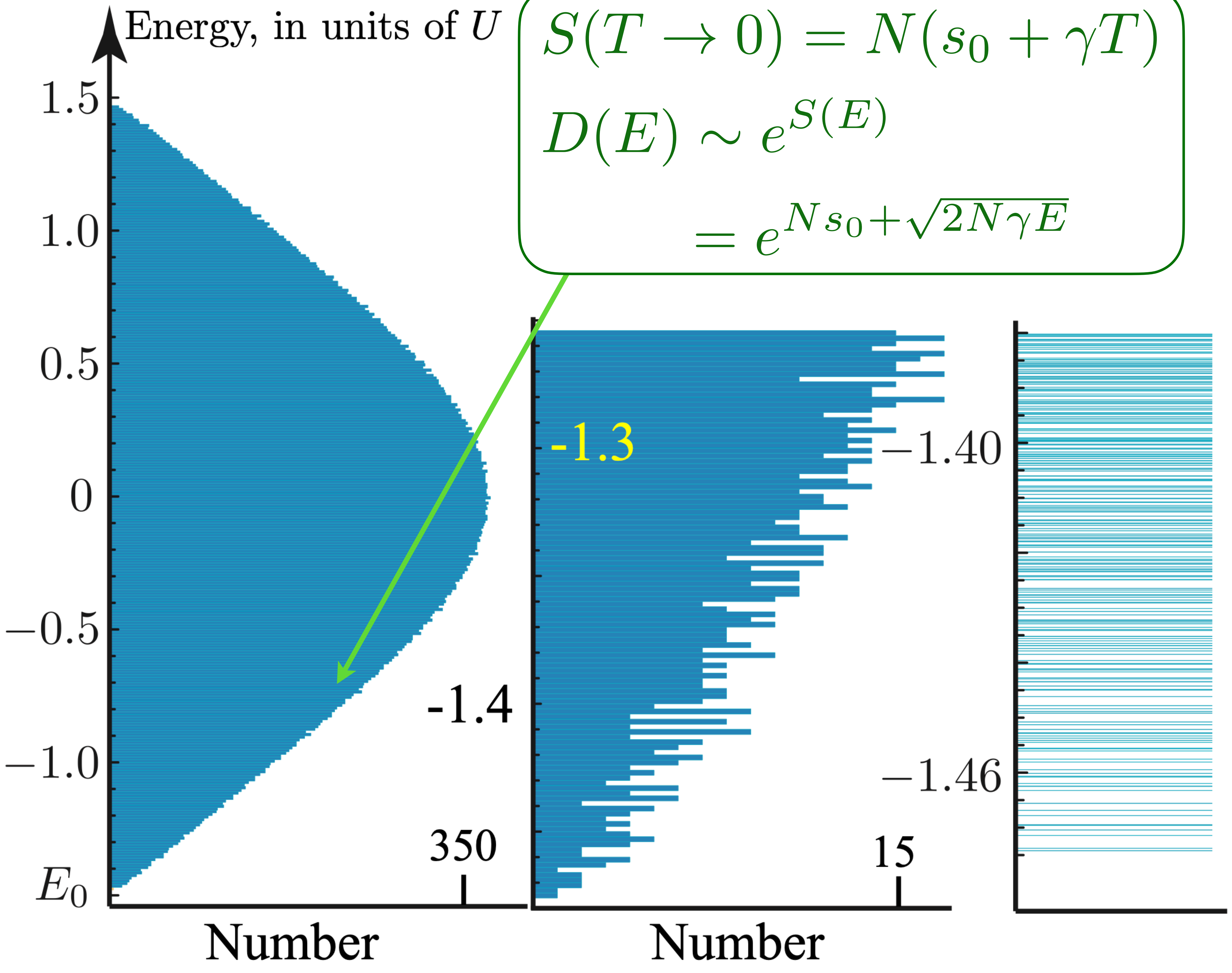

Energy level spacing $\sim e^{-N s_0}$!

No quasiparticle decomposition: wavefunctions change chaotically from one state to the next.

Complex SYK model

(Numerics: G. Tarnopolsky)

Many-body density of states

Beyond Boltzmann

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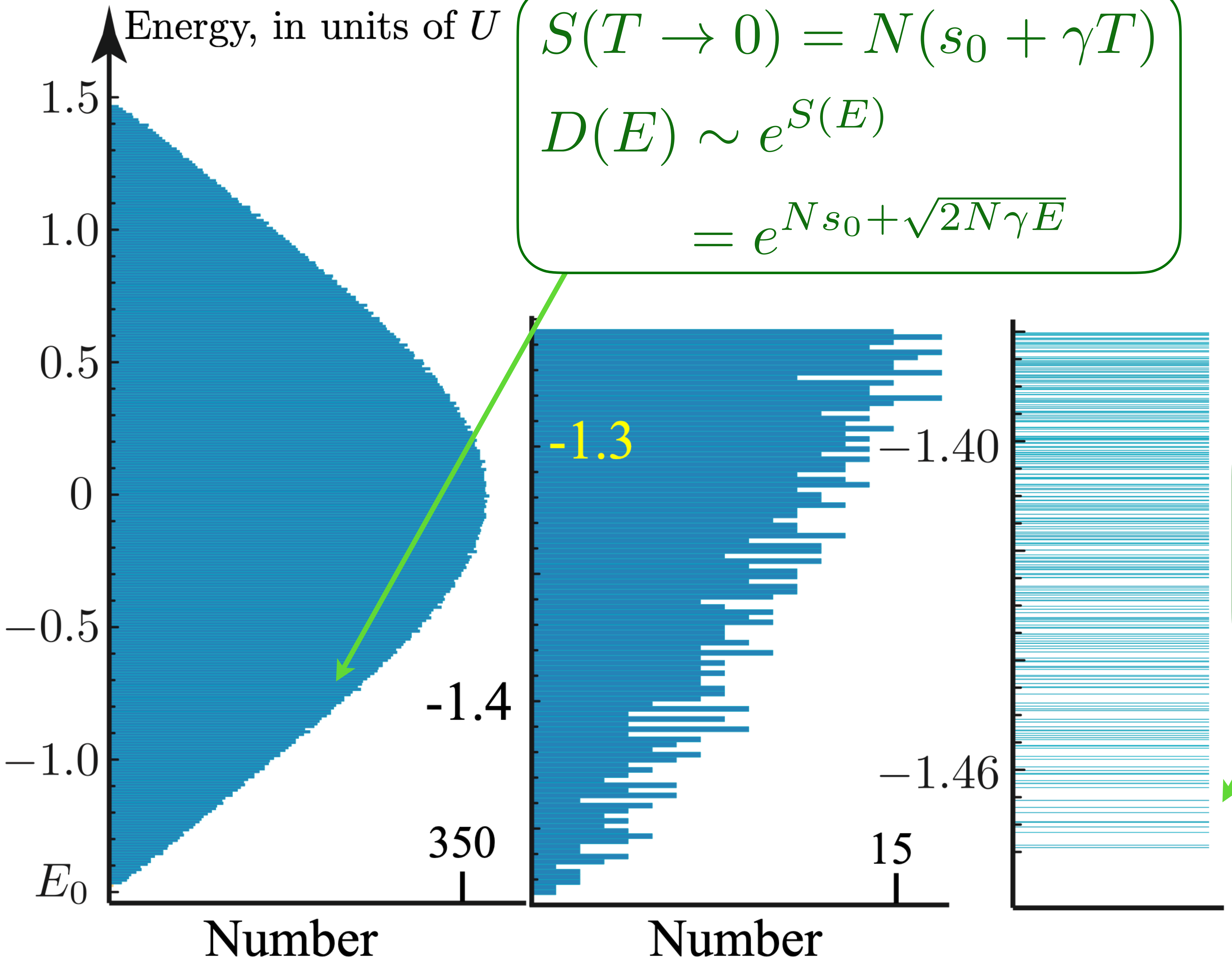

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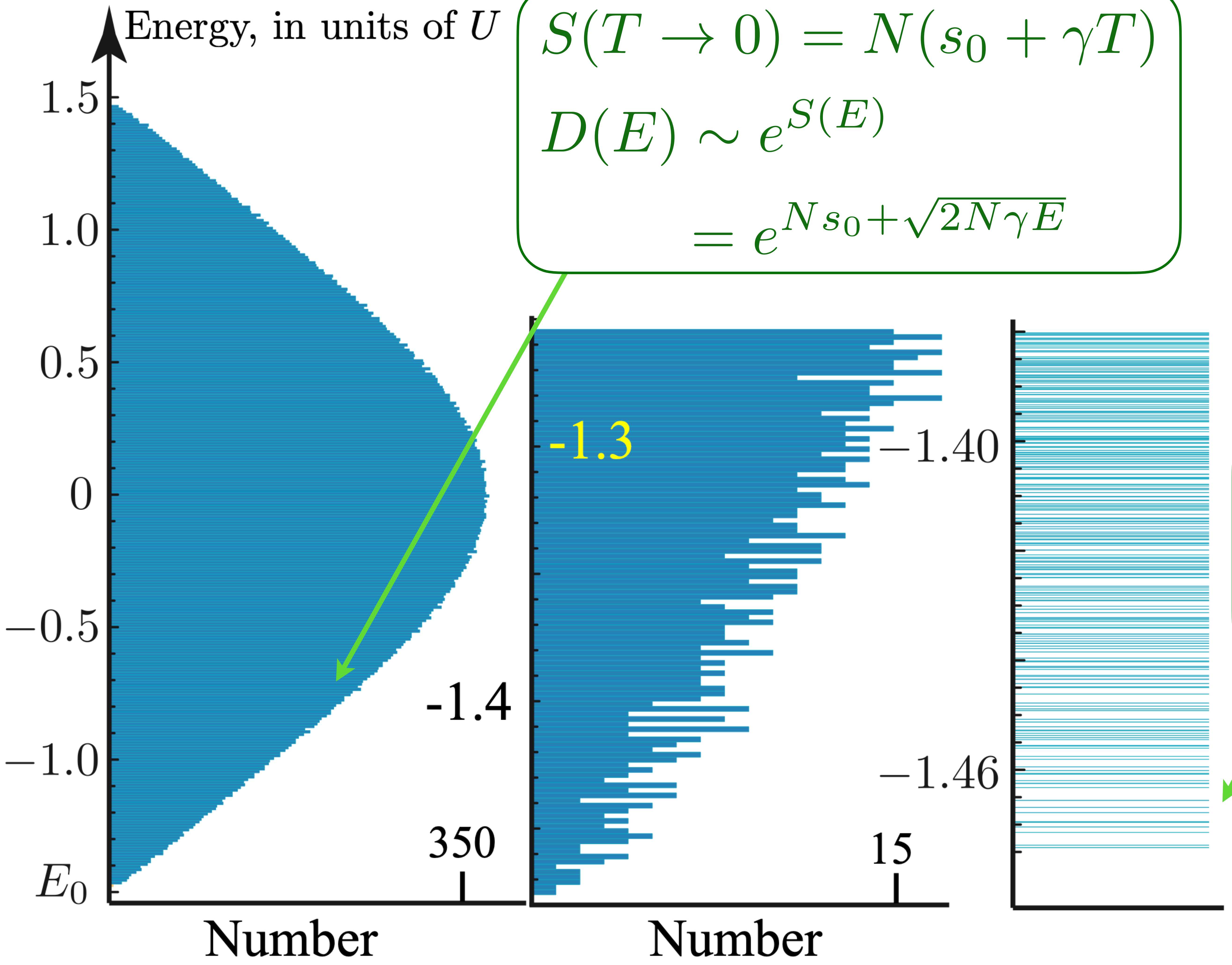

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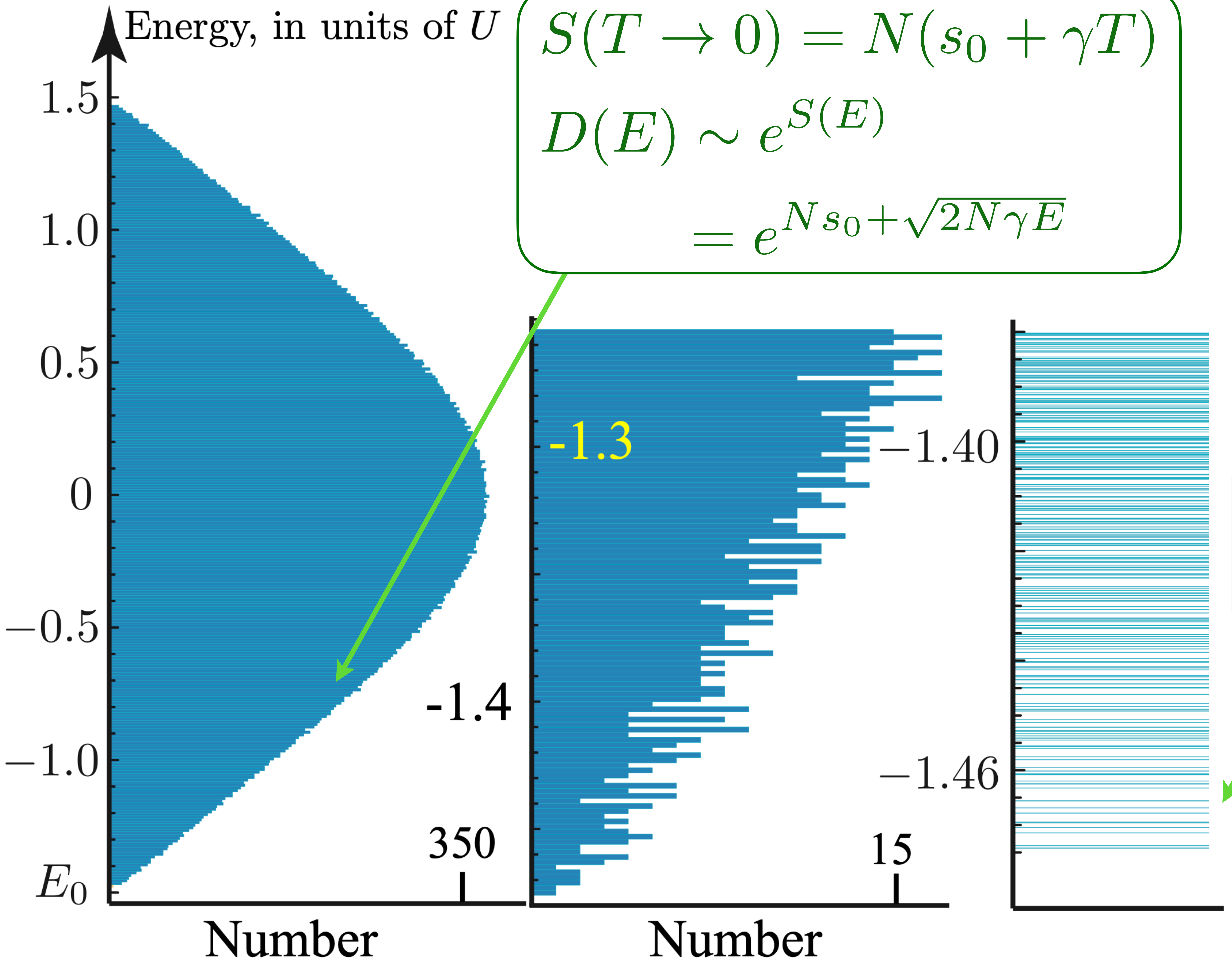
J. S. Cotler et al., JHEP 05 (2017) 118



Complex SYK model

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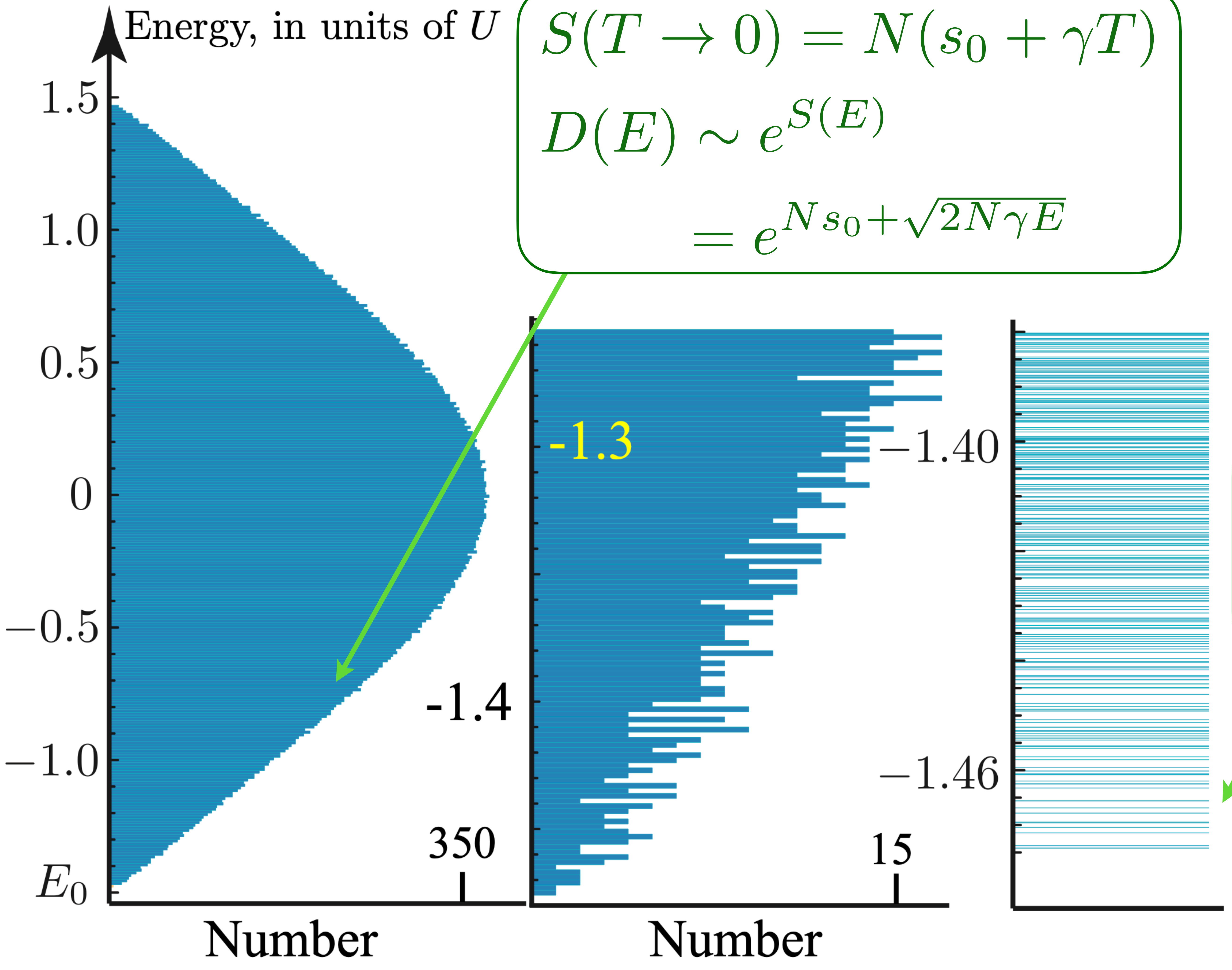

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J. S. Cotler et al., JHEP 05 (2017) 118

Yingfei Gu, A. Kitaev, S. Sachdev, and G. Tarnopolsky, JHEP 02 (2020) 157

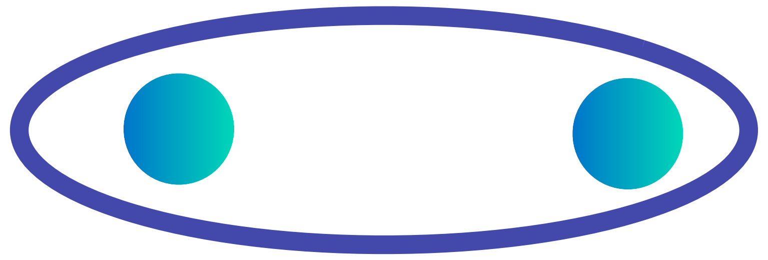
Complex SYK model

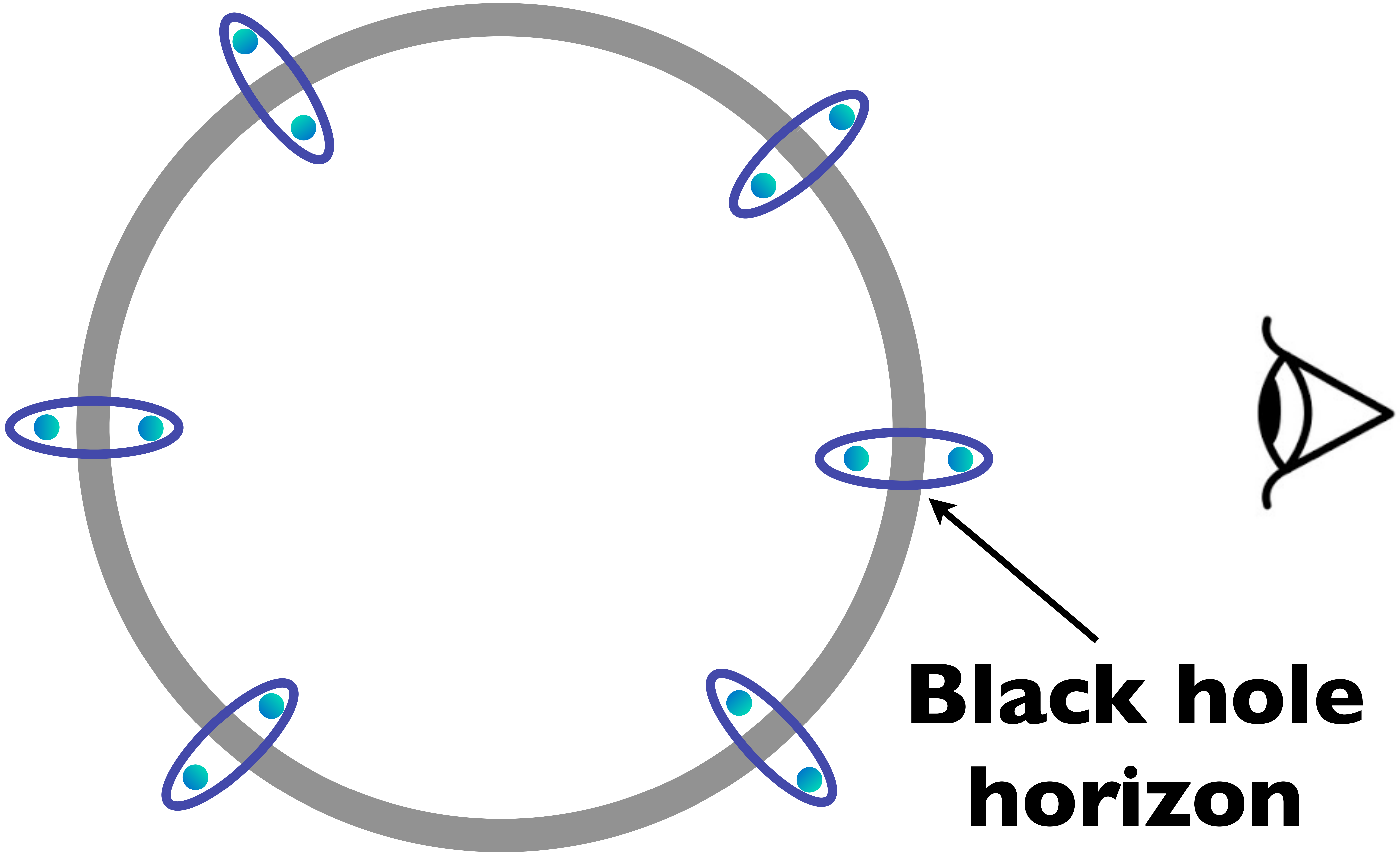
(Numerics: G. Tarnopolsky)

1. Large-*N* theory of the SYK model
2. Finite-*N* theory of the SYK model
3. Quantum Einstein-Maxwell gravity theory of charged black holes
4. Universal theory of strange metals

Quantum Entanglement across a black hole horizon

Quantum entanglement on the surface

 $= |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$



By computations *outside* the black hole, Hawking obtained

$$S = \frac{Ac^3}{4G\hbar}$$

where A is area of the black hole horizon.

All other systems have entropy proportional to their volume.

The Einstein action for gravity in 3+1 dimensions is

$$I_E = \int d^4x\sqrt{g}\left[-\frac{1}{2\kappa^2}\mathcal{R}_4\right] \quad , \quad \mathcal{Z} = \int \mathcal{D}g \exp(-I_E)\,,$$

where $\kappa^2 = 8\pi G_N$ is the gravitational constant, $\mathcal{R}_4$ is the Ricci scalar. The Schwarzschild solution of the saddle-point equations is

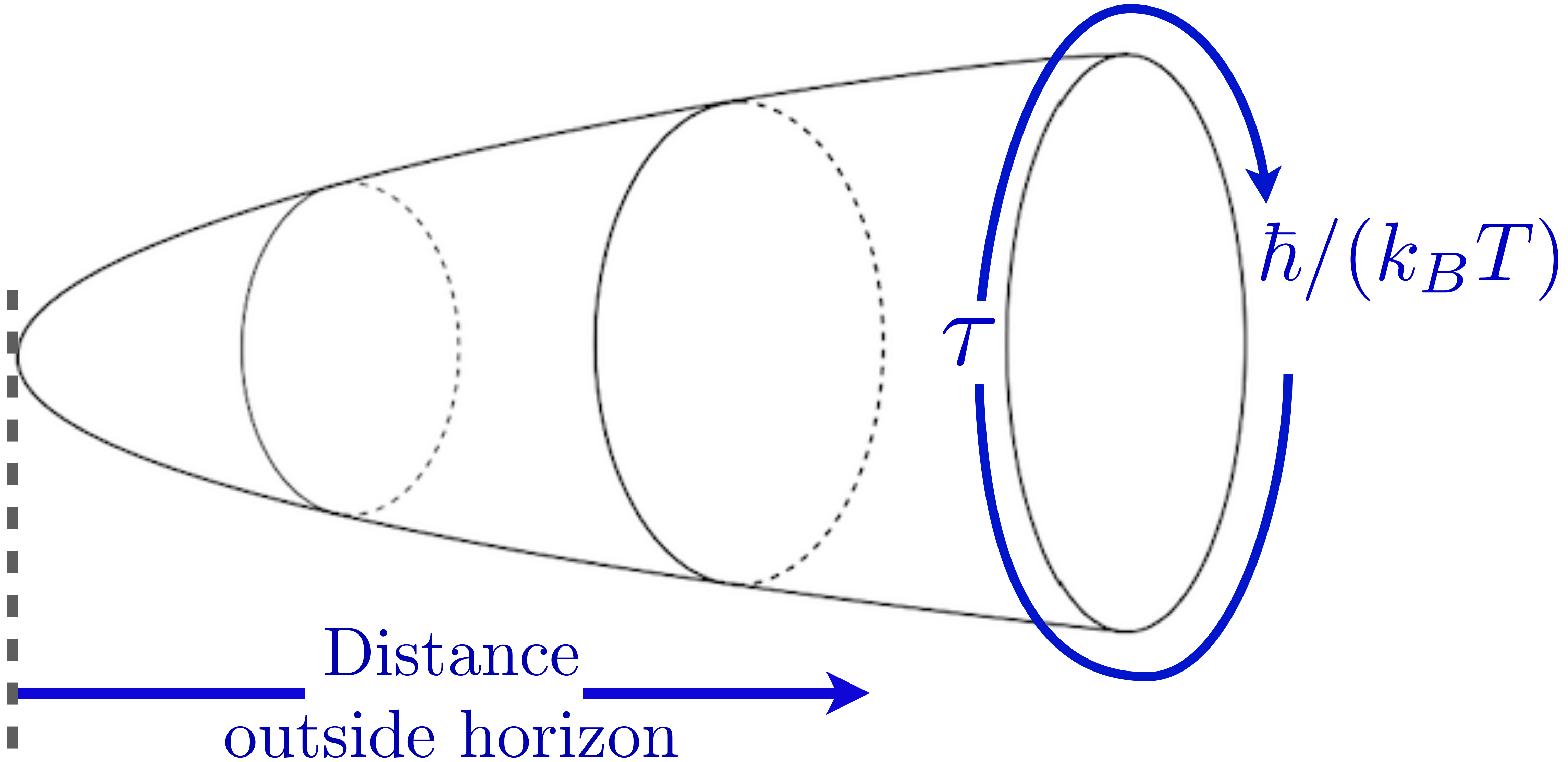
$$ds^2 = V(r)d\tau^2 + r^2 d\Omega_2^2 + \frac{dr^2}{V(r)}$$

where $d\Omega_2^2$ is the metric of the 2-sphere, and

$$V(r) = 1 - \frac{m}{r}.$$

The gravitational mass of the black hole is $M = 2G_N m$. The black hole horizon is at $r = r_0$ where $V(r_0) = 0$; so

$$r_0 = m$$



The $T > 0$ quantum partition function is obtained in a spacetime which is periodic as a function of τ with period $\hbar/(k_B T)$. We have to ensure that there is no singularity at the horizon r_0 where $V(r_0) = 0$. Let us change radial co-ordinates to y, where $r = r_0 + y^2$. Then for small y

$$ds^2 = \frac{4}{V'(r_0)}\left[\frac{(V'(r_0))^2}{4}y^2 d\tau^2 + dy^2\right] + r_0^2 d\Omega_2^2 = \frac{4}{V'(r_0)}\left[y^2 d\theta^2 + dy^2\right] + r_0^2 d\Omega_2^2$$

The expression in the square brackets is the metric of the flat plane in polar co-ordinates, with radial co-ordinate y and angular co-ordinate $\theta = V'(r_0)\tau/2$. Smoothness requires periodicity in θ with period 2π, and so

$$4\pi T = V'(r_0) = \frac{1}{m}\,.$$

The free energy $\beta F = I_E$, where $\beta = 1/T$. So the entropy is

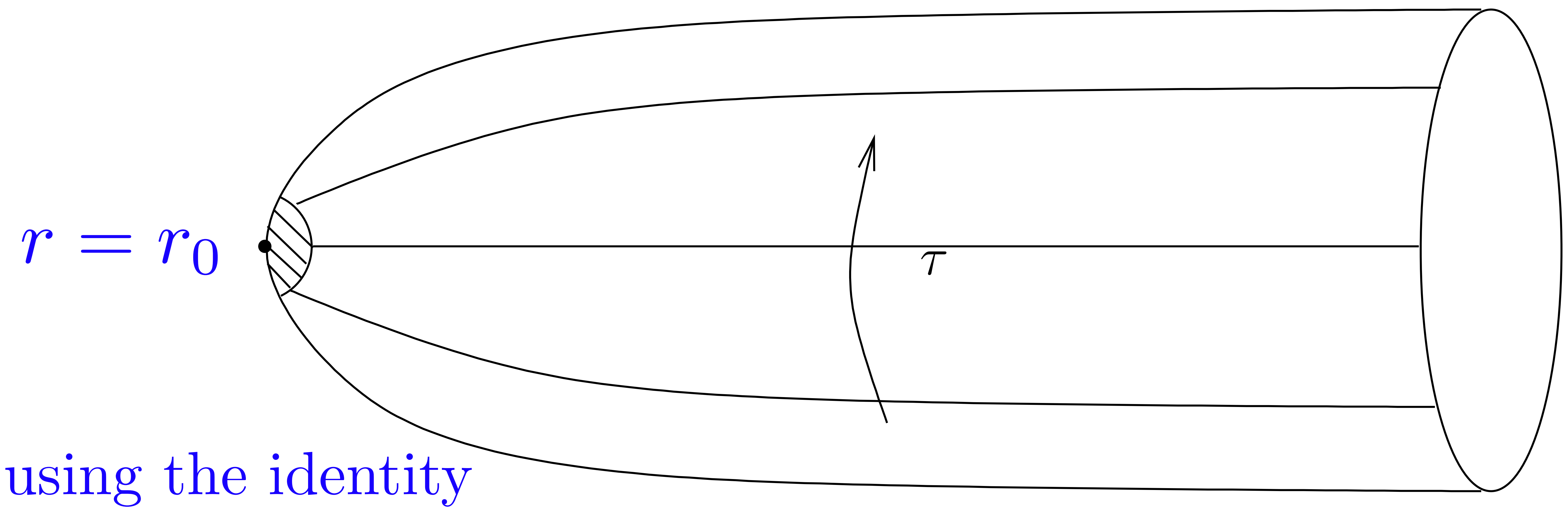
$$S = -\frac{\partial F}{\partial T} = \left(\beta \frac{\partial}{\partial \beta} - 1\right) I_E$$

However, the metric is τ-independent, and the only explicit dependence of the action is via $I_E = \beta H$. Such an action implies $S = 0$.

The entire contribution to the entropy comes from the vicinity of the co-ordinate singularity at $r = r_0$. We evaluate the action is the small region around this point

$$I_{\text{grav}} = I_E + I_{GH} \quad , \quad I_{GH} = \int_\partial d^3x \sqrt{g_b} \left[-\frac{1}{\kappa^2}\mathcal{K}_3\right] \quad , \quad \mathcal{Z} = \int \mathcal{D}g \exp(-I_{\text{grav}}) \, ,$$

where $\mathcal{K}_3$ is the extrinsic scalar curvature of the 3-dimensional boundary of spacetime. I_{GH} is the Gibbons-Hawking boundary term, deduced by the requirement that the Euler-Lagrange equations of I_{grav} co-incide with the Einstein equations, with no additional boundary terms. The entire contribution to the entropy will come from I_{GH}.



We evaluate I_{GH} by using the identity

$$\int_\partial d^3x\sqrt{g_b}\,\mathcal{K}_3 = \frac{\partial}{\partial n}\int_\partial d^3x\sqrt{g_b}$$

where n is the Gaussian normal co-ordinate of the boundary. Evaluating at $y = \epsilon$, we have

$$\int_\partial d^3x\sqrt{g_b} = 2\pi\epsilon\mathcal{A}$$

where $\mathcal{A} = 4\pi r_0^2$ is the area of the horizon. Combining everything, we have the famous result of Hawking

$$S = \frac{2\pi\mathcal{A}}{\kappa^2} = \frac{\mathcal{A}}{4G_N}.$$

Quantum Entanglement across a black hole horizon

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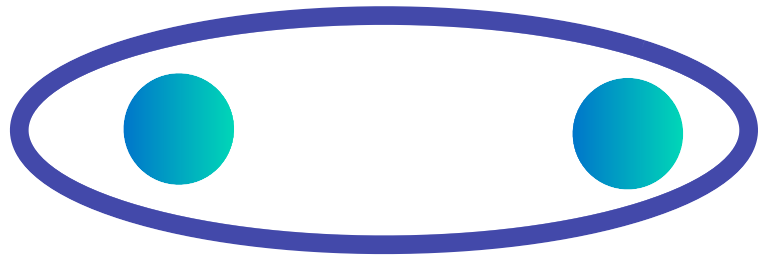
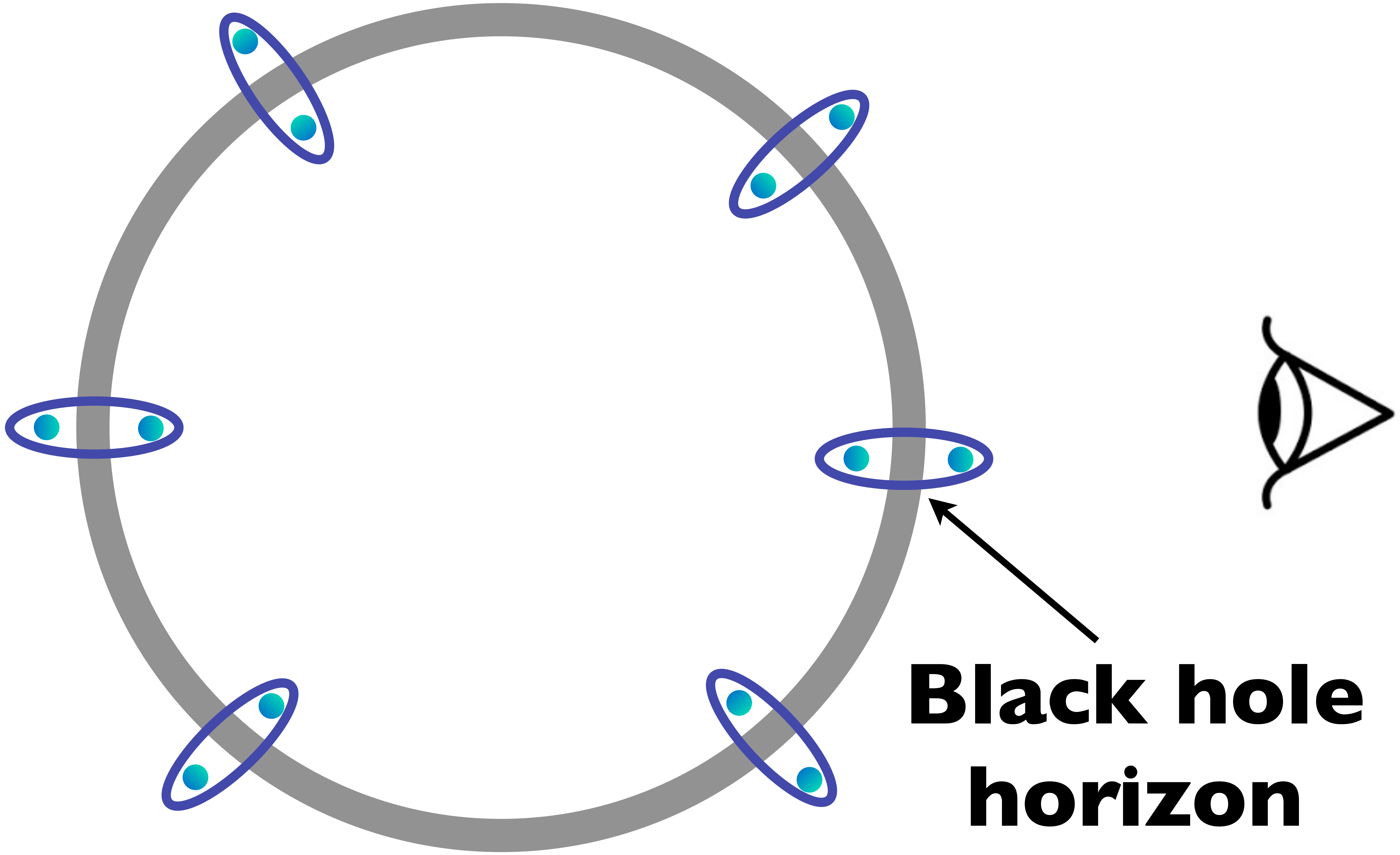
 $= |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$



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where A is area of the black hole horizon.

All other systems have entropy proportional to their volume.

Charged black holes

We consider a charged black hole in Einstein-Maxwell theory of g and a U(1) gauge flux $F = dA$

$$I_{EM} = \int d^4x\sqrt{g}\left[-\frac{1}{2\kappa^2}\mathcal{R}_4 + \frac{1}{4g_F^2}F^2\right] \quad , \quad \mathcal{Z}_\mathcal{Q} = \int \mathcal{D}g\mathcal{D}A\,\exp(-I_{EM} - I_{GH})\,.$$

The saddle-point equations now yield a solution as before with

$$V(r) = 1 + \frac{\Theta^2}{r^2} - \frac{m}{r} \quad ; \quad A_\tau = i\mu\left(1 - \frac{r_0}{r}\right) \quad ; \quad \Theta = \frac{\kappa r_0}{\sqrt{2}g_F}\mu \quad ; \quad \mathcal{Q} = \frac{4\pi\mu r_0}{g_F^2} \quad ; \quad S = \frac{2\pi\mathcal{A}}{\kappa^2}$$

where $\mathcal{Q}$ is the total charge, the chemical potential is μ, and as before the horizon is where $V(r_0) = 0$, the temperature $T = V'(r_0)/(4\pi)$, and $\mathcal{A} = 4\pi r_0^2$.

This defines a two parameter family of charged black hole solutions of I_{EM} determined by T and $\mathcal{Q}$.

Charged black holes

Now we take the limit $T \to 0$ at fixed $\mathcal{Q}$. Then we find the remarkable feature that the horizon radius remains finite

$$R_h \equiv r_0(T \to 0, \mathcal{Q}) = \frac{\mathcal{Q} \kappa g_F}{4\pi}$$

In this limit, entropy becomes

$$S(T \to 0, \mathcal{Q}) = \frac{4\pi R_h^2}{G_N} + \gamma T \quad , \quad \gamma \equiv \frac{4\pi^2 R_h^3}{G_N}$$

For the near-horizon metric, it is useful to introduce the co-ordinate ζ

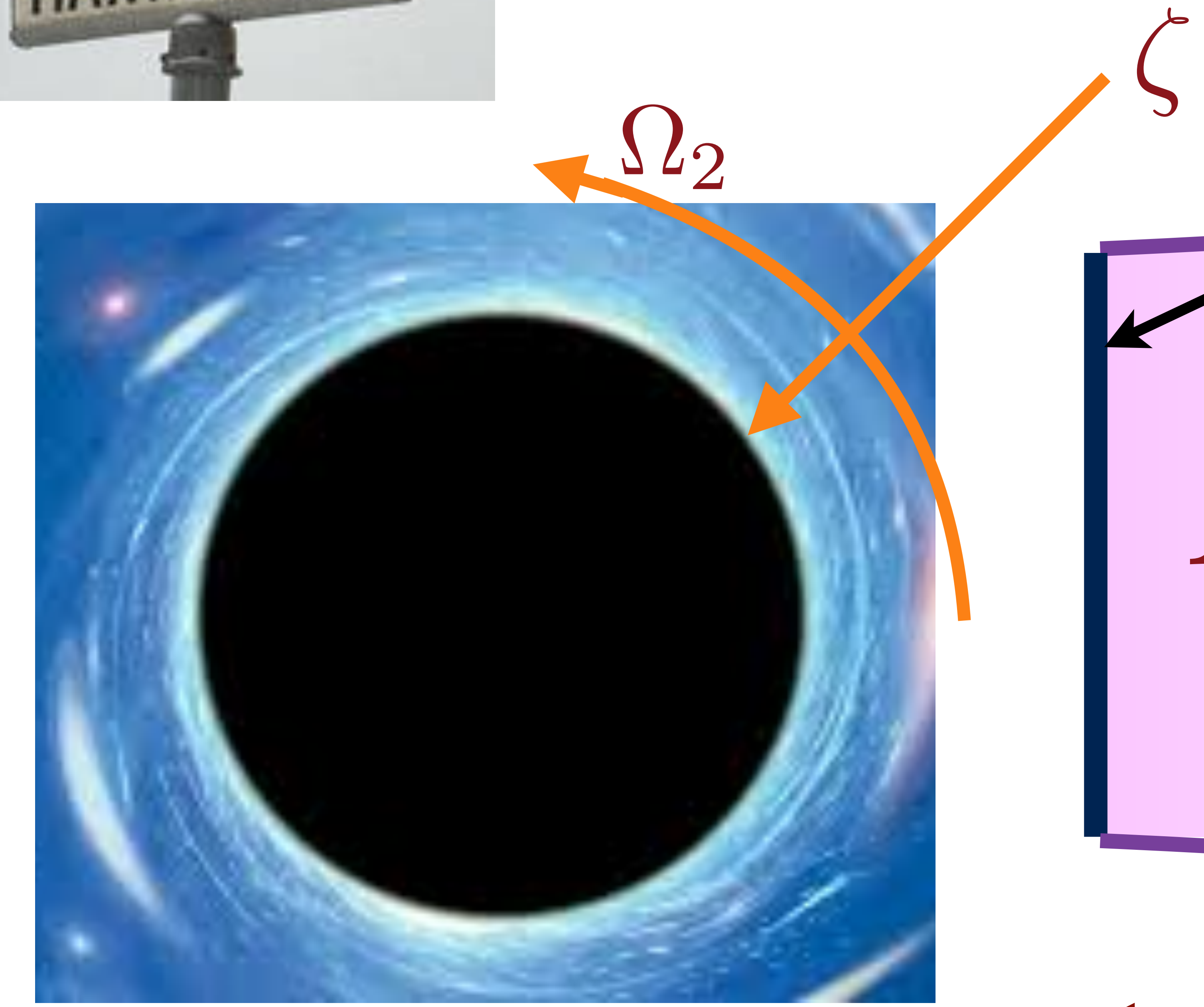
$$r = R_h + \frac{R_h^2}{\zeta}$$

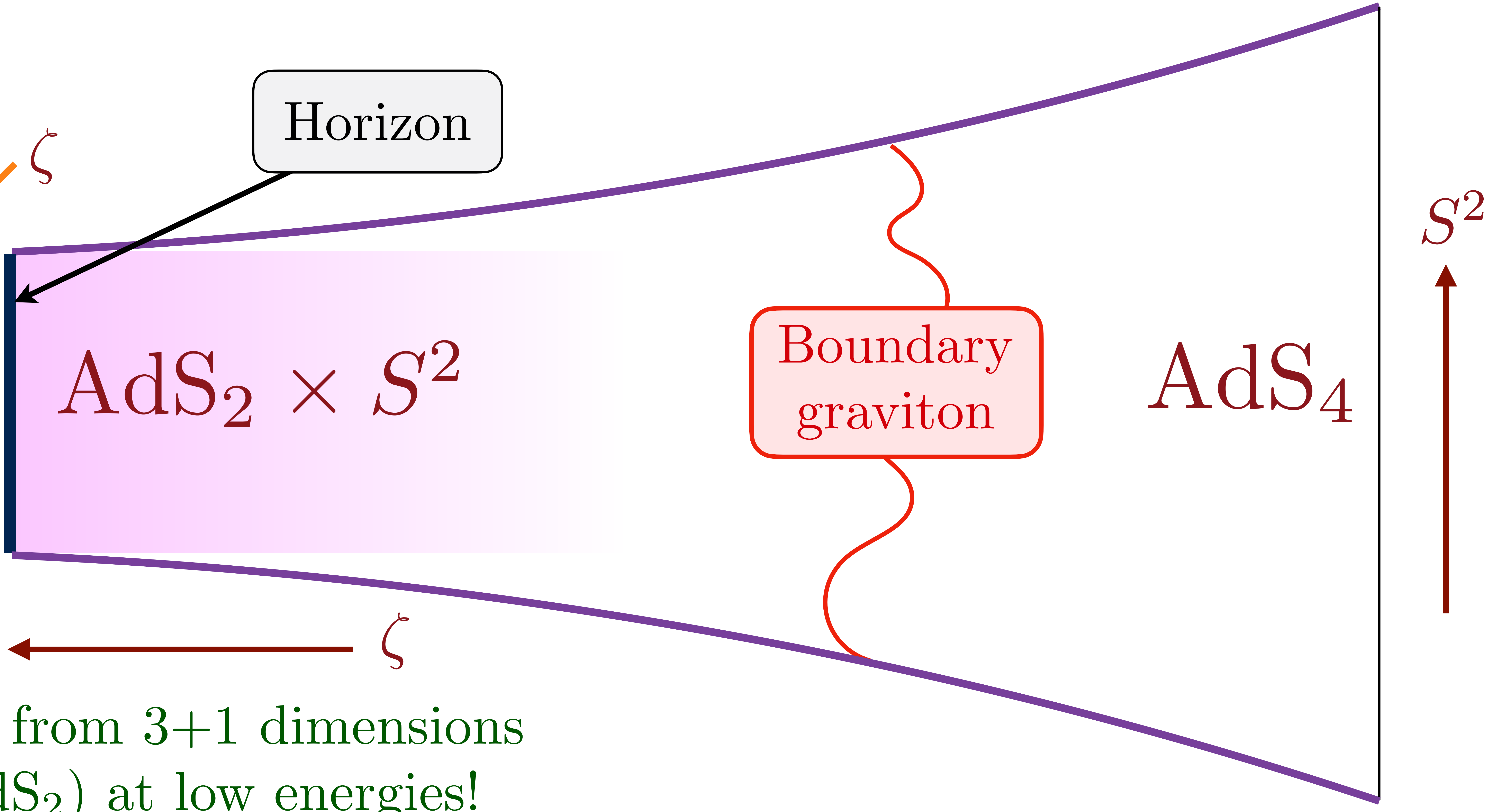
so that the horizon at $T = 0$ is at $\zeta = \infty$. Then in the near-horizon regime $R_h \ll \zeta < \infty$ the $T = 0$ metric is

$$ds^2 = R_h^2 \frac{d\tau^2 + d\zeta^2}{\zeta^2} + R_h^2 d\Omega_2^2$$

This spacetime is $\mathrm{AdS}_2 \times S^2$.

Reissner-Nordstrom black hole of Einstein-Maxwell theory





Dimensional reduction from 3+1 dimensions to 1+1 dimensions (AdS_2) at low energies!
The isometry group of AdS_2 is the 0+1 dimensional conformal group SL(2,R).

Faulkner, Liu, McGreevy, Vegh 2009; Sachdev 2010

Thermodynamics of quantum black holes with charge $\mathcal{Q}$:

$$\mathcal{Z}(\mathcal{Q},T) = \int \mathcal{D}g_{\mu\nu}\mathcal{D}A_\mu \exp\left(-\frac{1}{\hbar}\, I^{(3+1)}_{\text{Einstein gravity+Maxwell EM}}[g_{\mu\nu},A_\mu]\right)$$

Saddle-point:

$$S_{BH}(T\to 0,\mathcal{Q}) = \frac{A(T)c^3}{4G\hbar} = \frac{A_0c^3}{4G\hbar}\left(1+\frac{2(\pi A_0)^{1/2}T}{\hbar c}+\ldots\right)$$

$A_0 = 2G\mathcal{Q}^2/c^4$ is the area of the charged black hole horizon at $T=0$.

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$$\approx \exp\left(\frac{A_0 c^3}{4\hbar G}\right)\int \mathcal{D}g_{\mu\nu}\mathcal{D}A_\mu \exp\left(-\frac{1}{\hbar}\, I^{(1+1)}_{\text{JT gravity of AdS}_2\text{+boundary graviton}}[g_{\mu\nu},A_\mu]\right)$$

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Maldacena, Stanford, Yang (2016)

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$$= \int \mathcal{D}f(\tau)\mathcal{D}\phi(\tau)\exp\left(-\frac{1}{\hbar} I_{\text{SYK}}[\text{time reparameterizations } f(\tau), \text{ phase rotations } \phi(\tau)]\right)$$

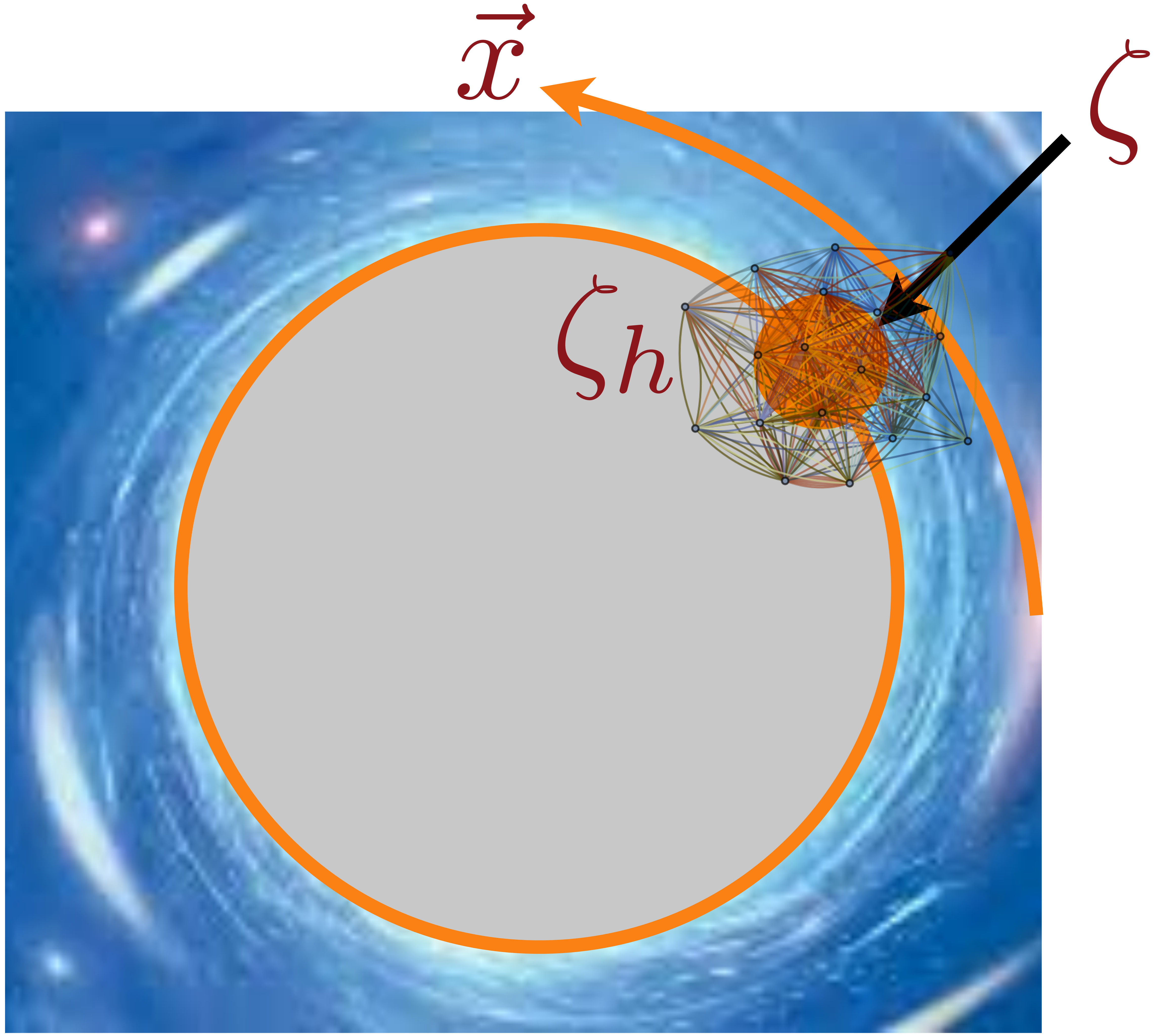
Saddle-point:

$$S_{BH}(T\to 0,\mathcal{Q}) = \frac{A(T)c^3}{4G\hbar} = \frac{A_0 c^3}{4G\hbar}\left(1+\frac{2(\pi A_0)^{1/2}T}{\hbar c}+\ldots\right)$$

$A_0 = 2G\mathcal{Q}^2/c^4$ is the area of the charged black hole horizon at $T=0$.

Sachdev (2010); Kitaev (2015); Sachdev (2015); Bagrets, Altland, Kamenev (2016); Maldacena, Stanford, Yang (2016); Moitra, Trivedi, Vishal (2018) ; Gaikwad, Joshi, Mandal, Wadia (2018); Sachdev (2019); Iliesiu, Turaci (2020)

Maxwell's electromagnetism and Einstein's general relativity allow black hole solutions with a net charge



The quantum versions of Maxwell's and Einstein's equations in this two-dimensional spacetime are also the equations describing electron entanglement in the SYK model!

D. Chowdhury, A. Georges, O. Parcollet, and S. S., Rev. Mod. Phys. **94**, 035004 (2022)

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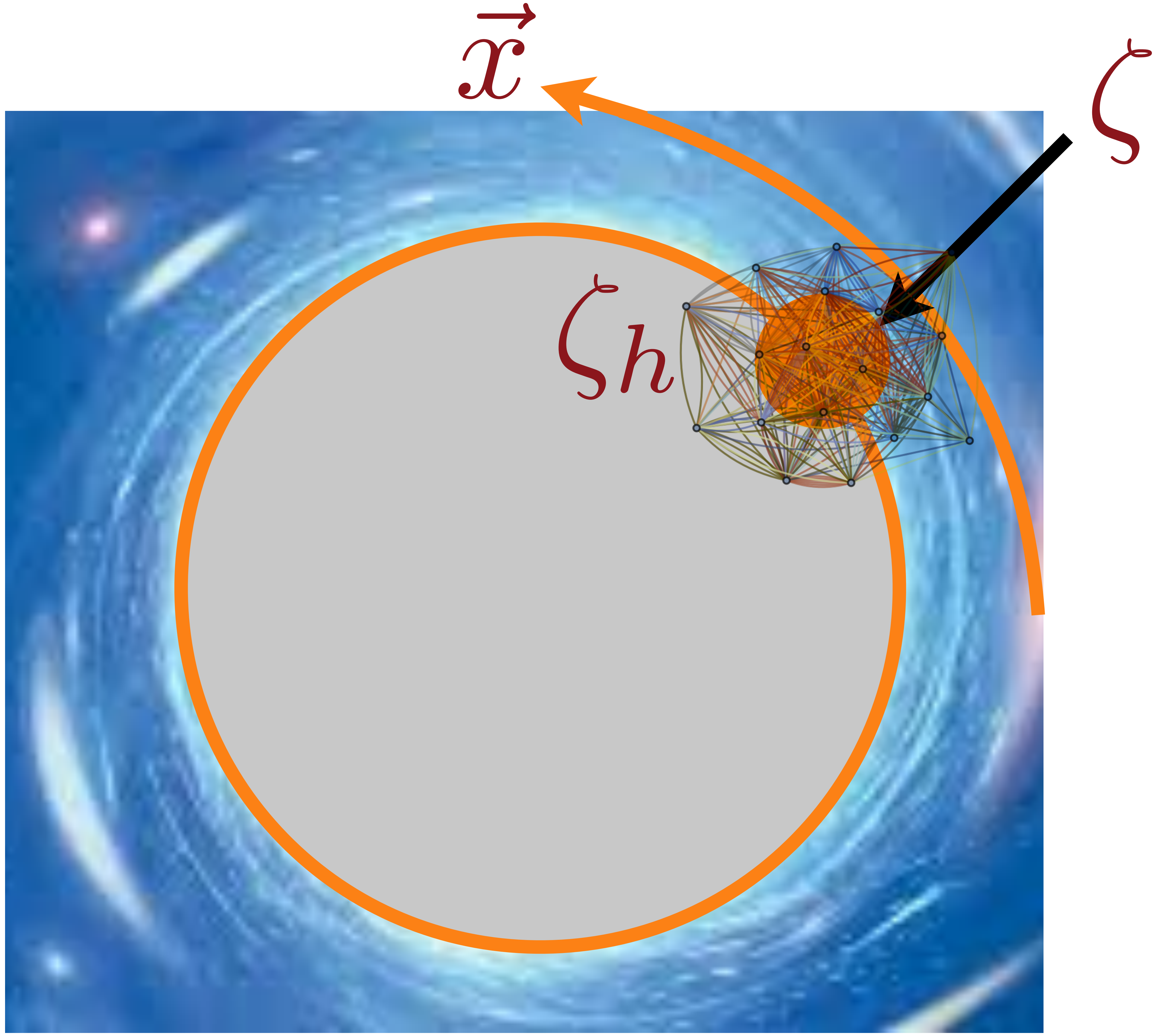

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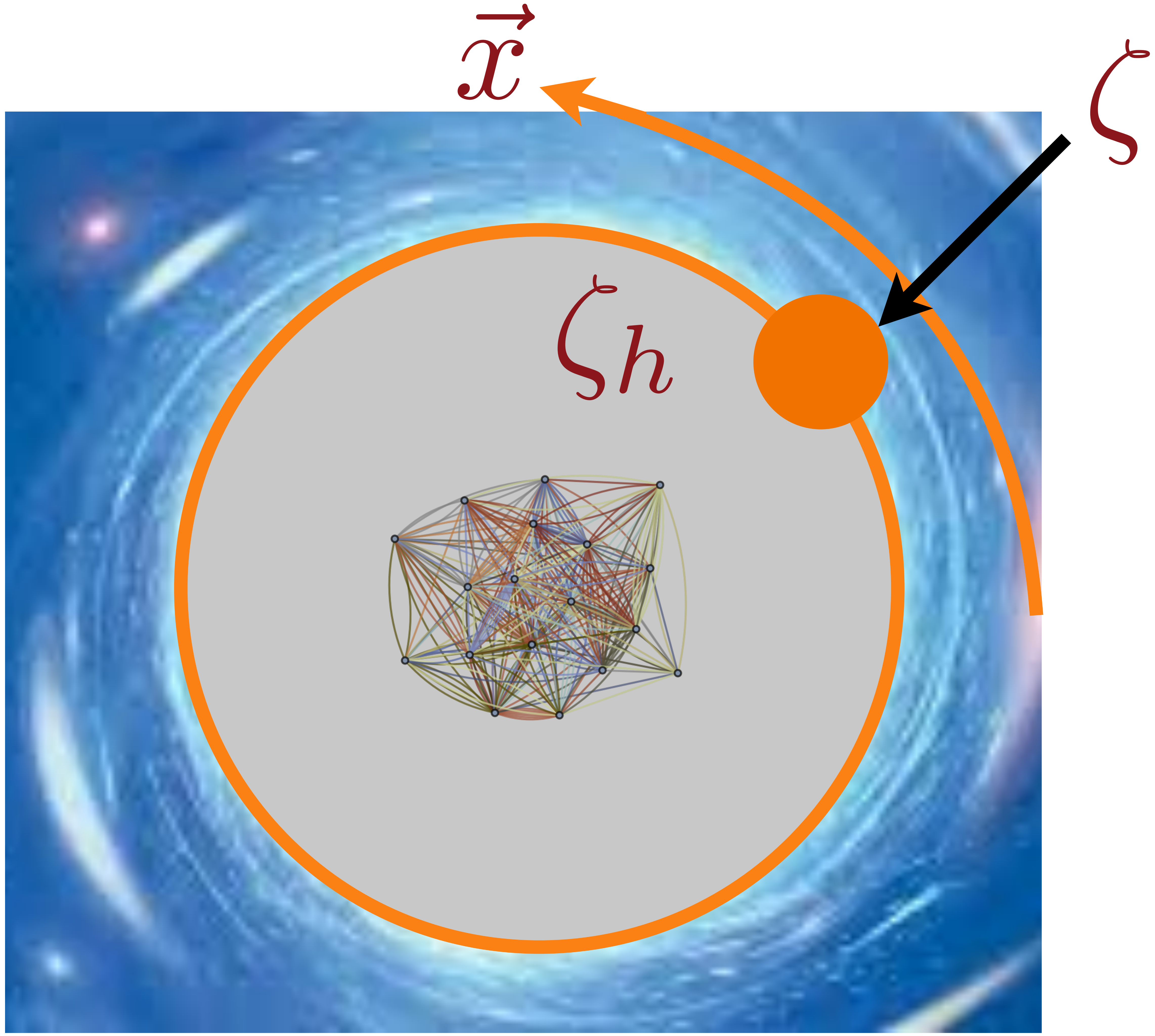

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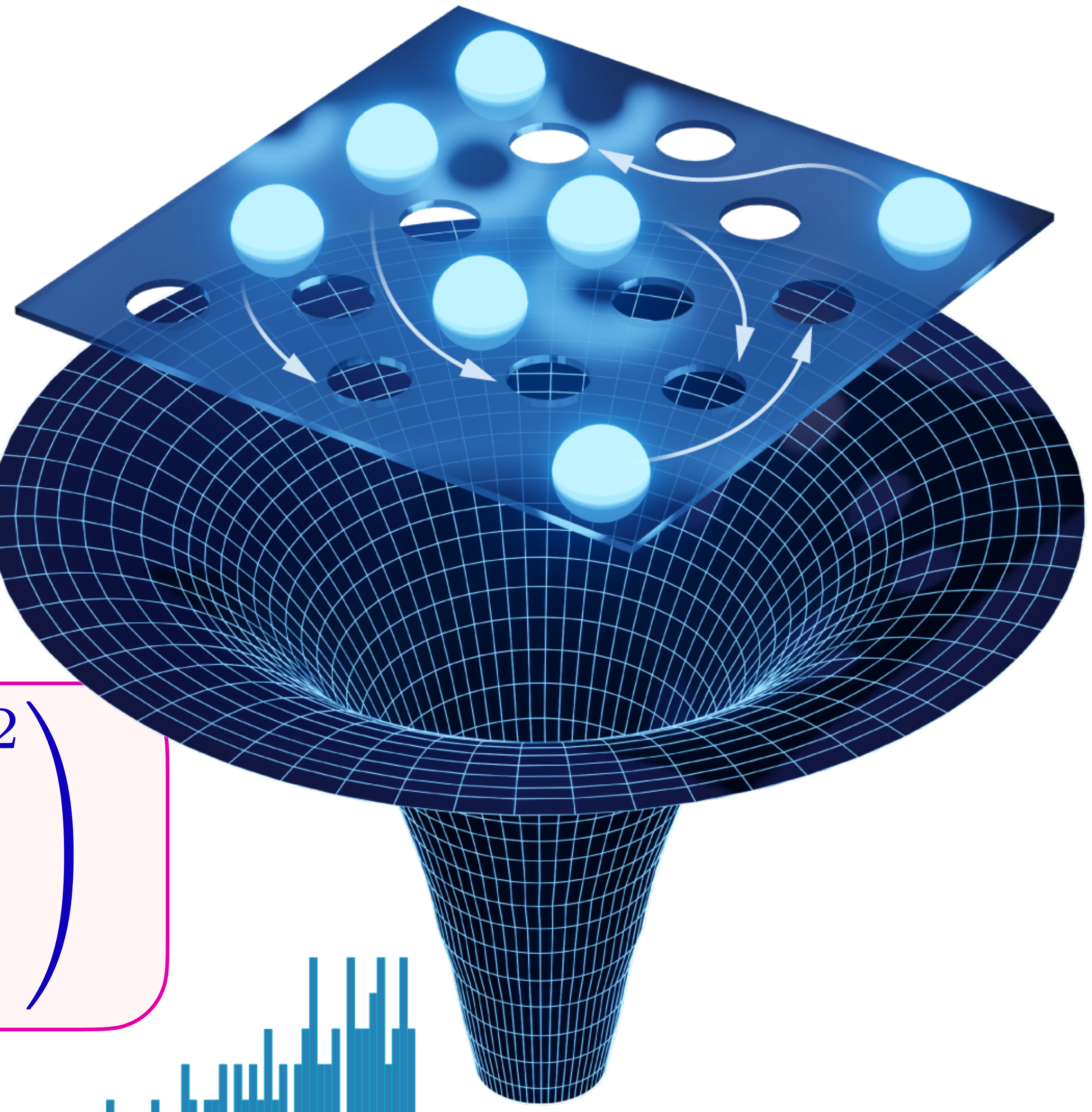
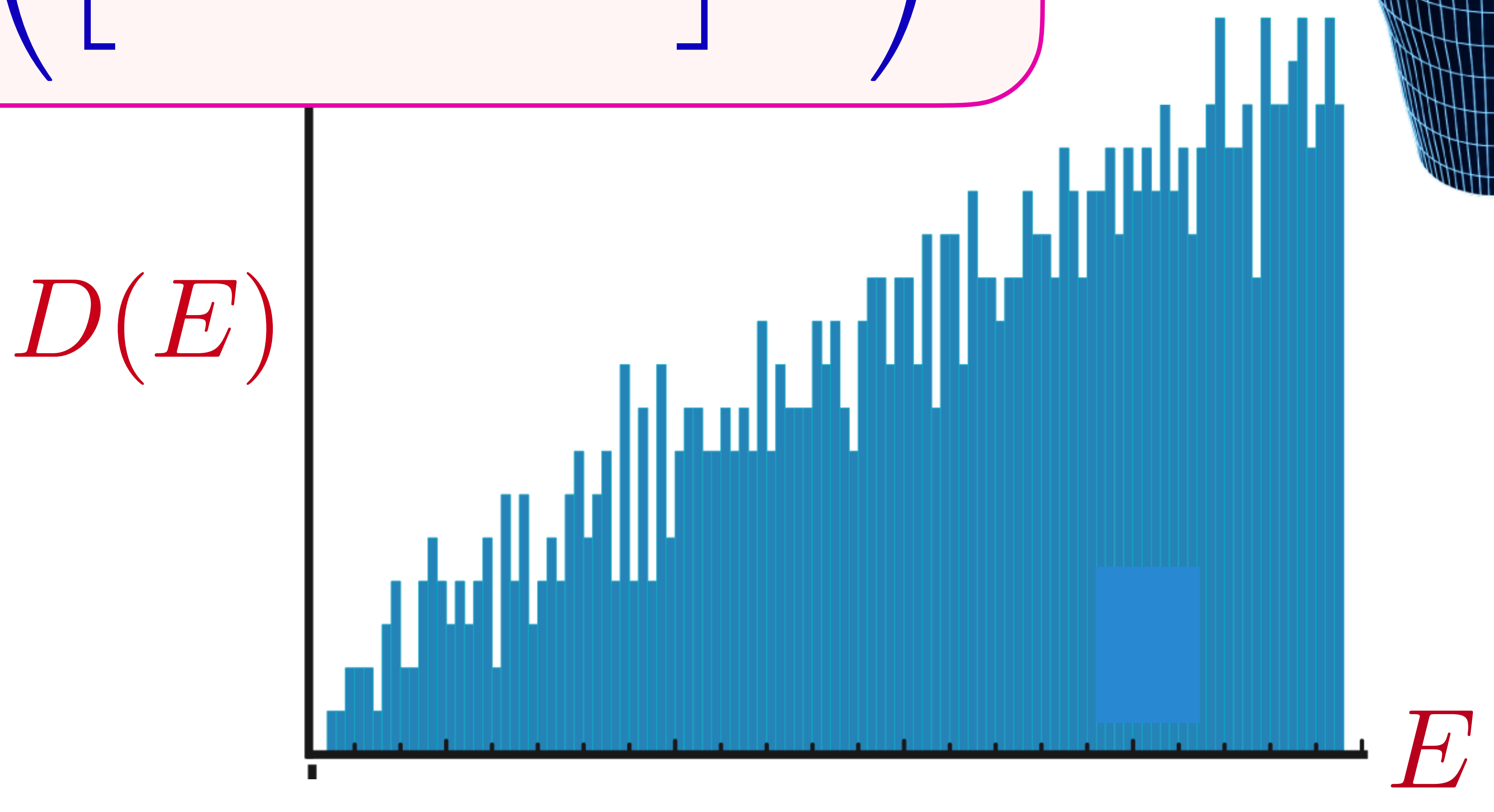
Quantum simulation of charged black holes by the SYK model

- For generic charged black holes in 3+1 dimensions, the SYK model yields, in terms of $A_0 = 2GQ^2/c^4$ the horizon area at $T = 0$:

$$D(E) \sim \left(\frac{A_0 c^3}{\hbar G}\right)^{-347/90} \exp\left(\frac{A_0 c^3}{4\hbar G}\right) \sinh\left(\left[\frac{\sqrt{\pi} A_0^{3/2} c^2}{\hbar^2 G} E\right]^{1/2}\right)$$

There is no degeneracy, but an exponentially small level spacing down to the ground state.





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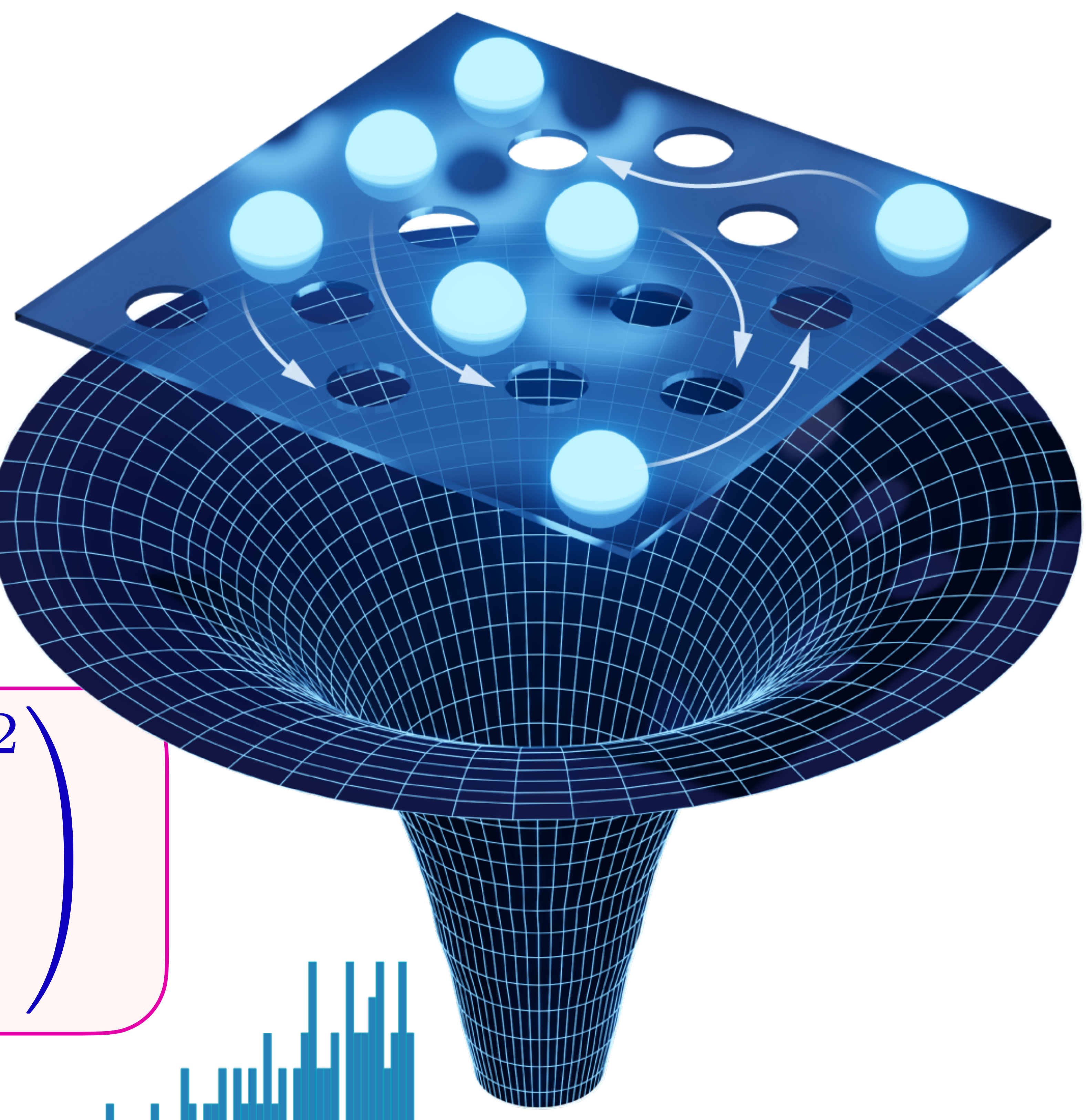

Bekenstein-Hawking

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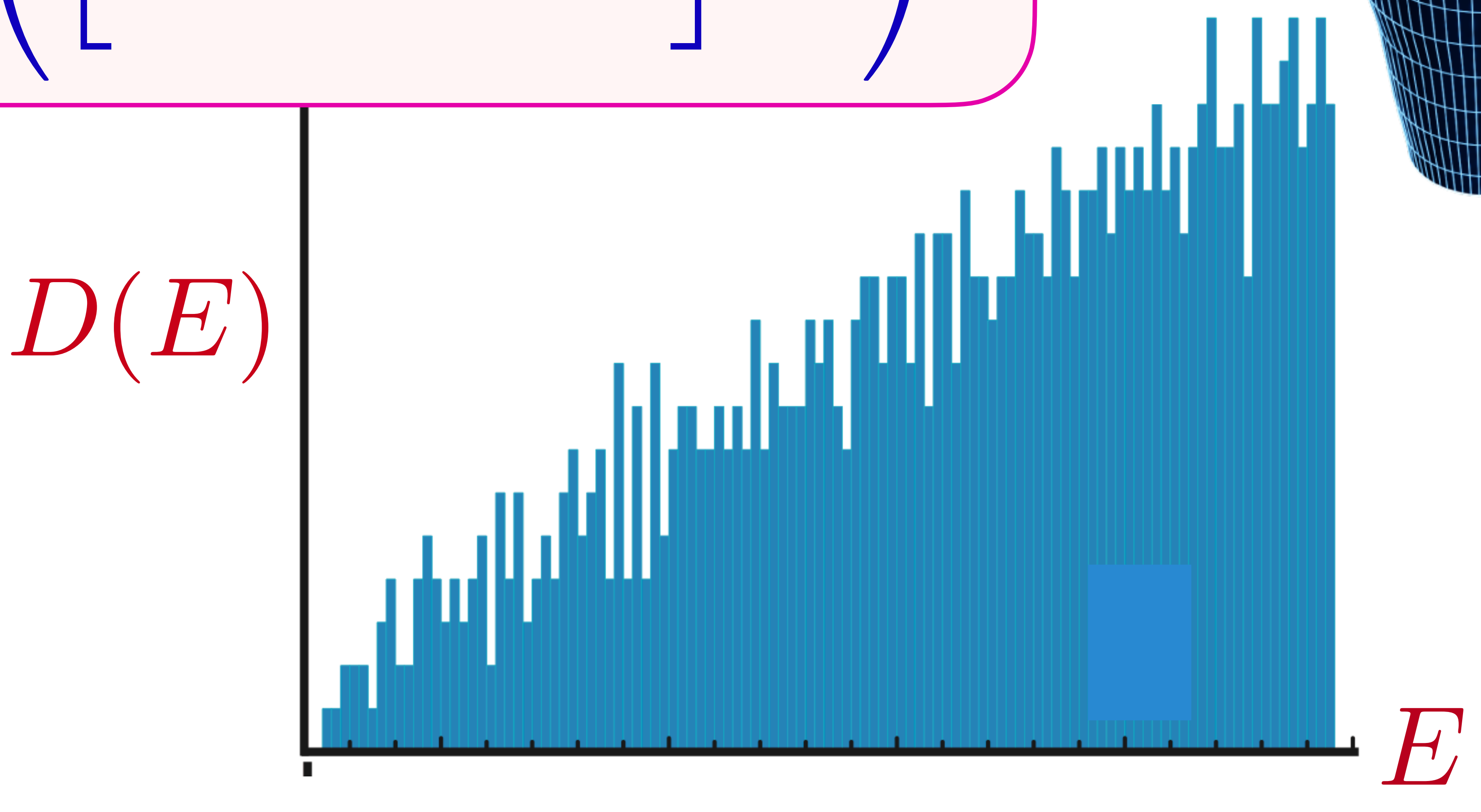

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Developments from the SYK model

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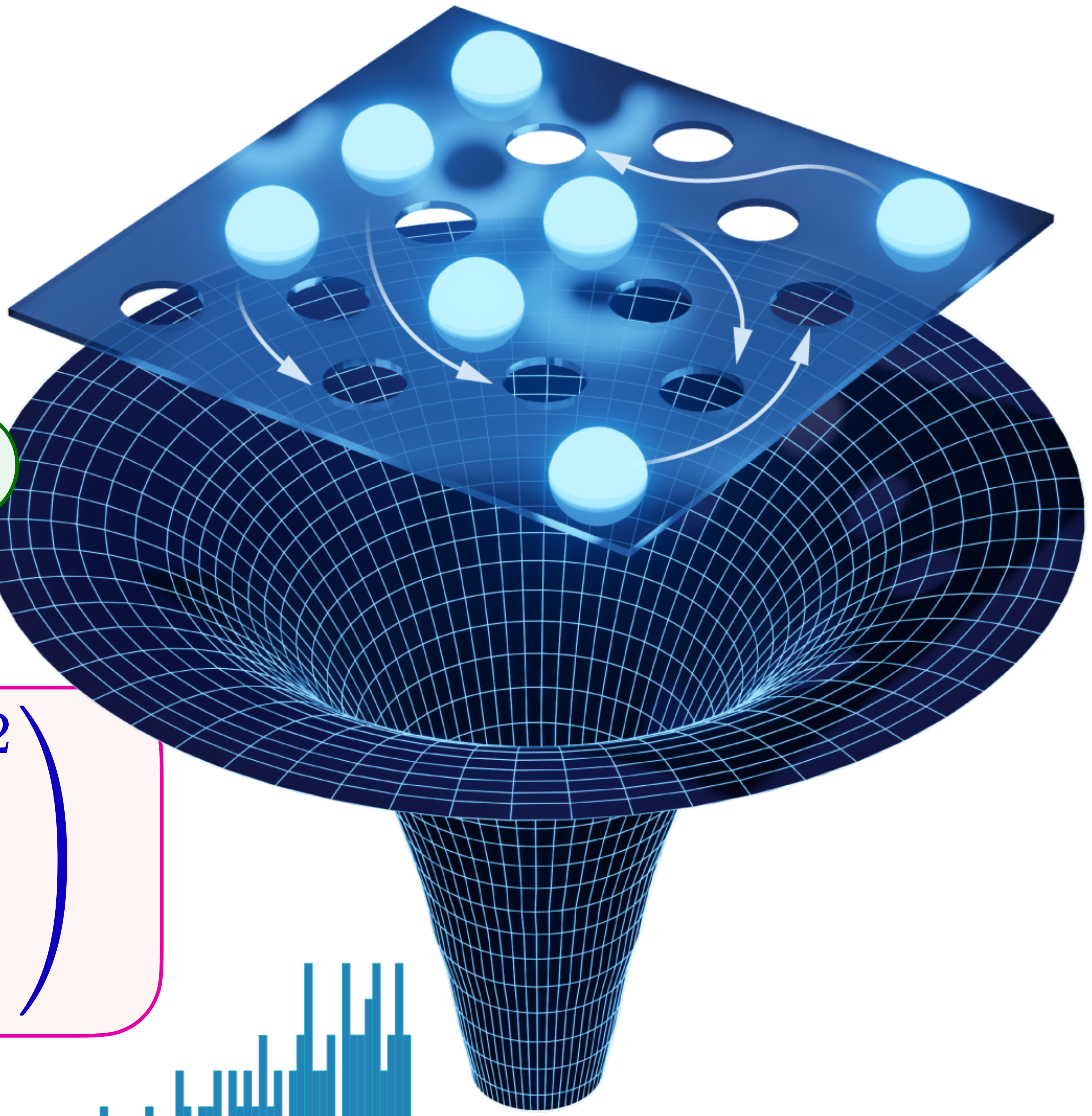
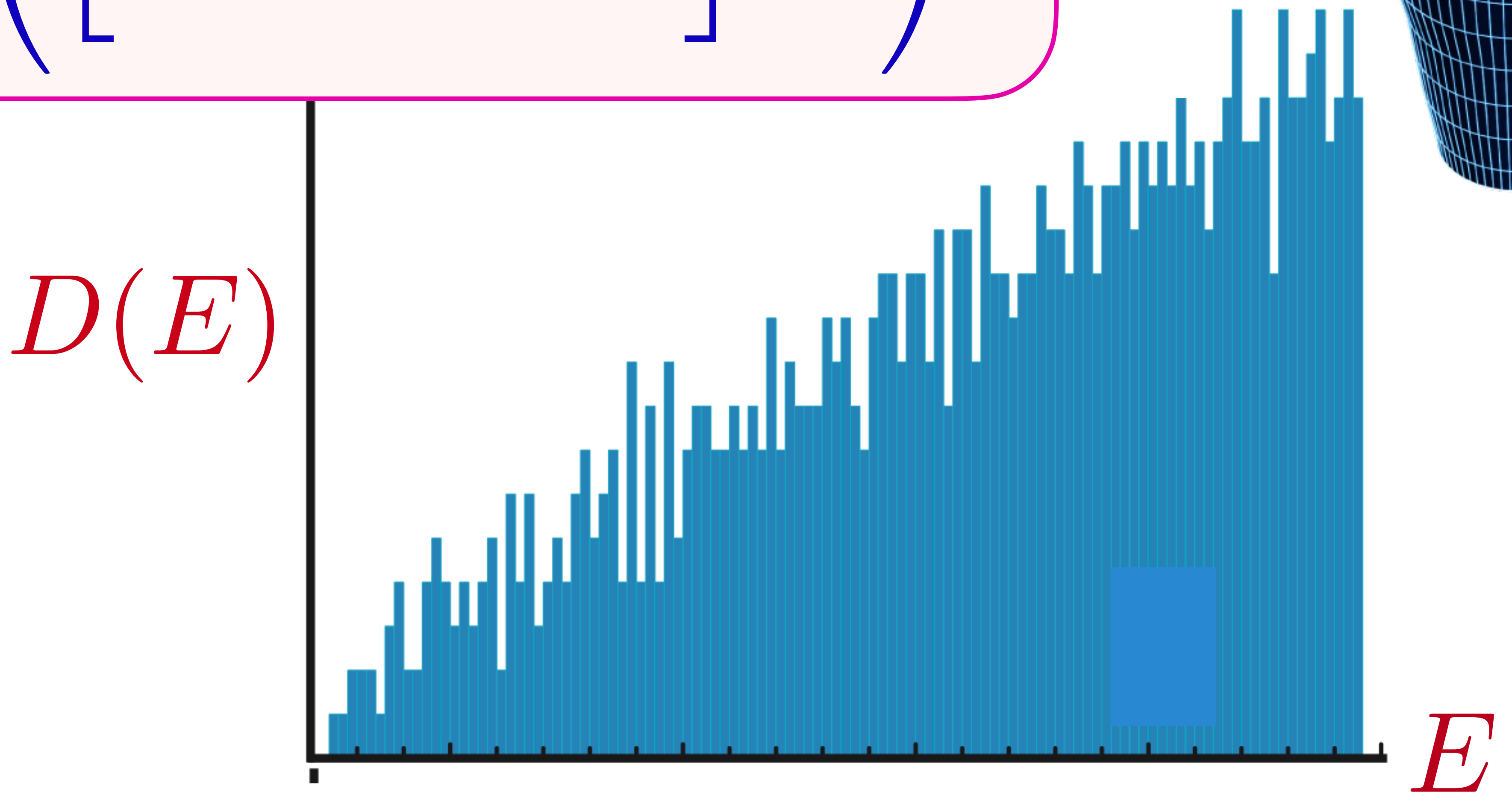
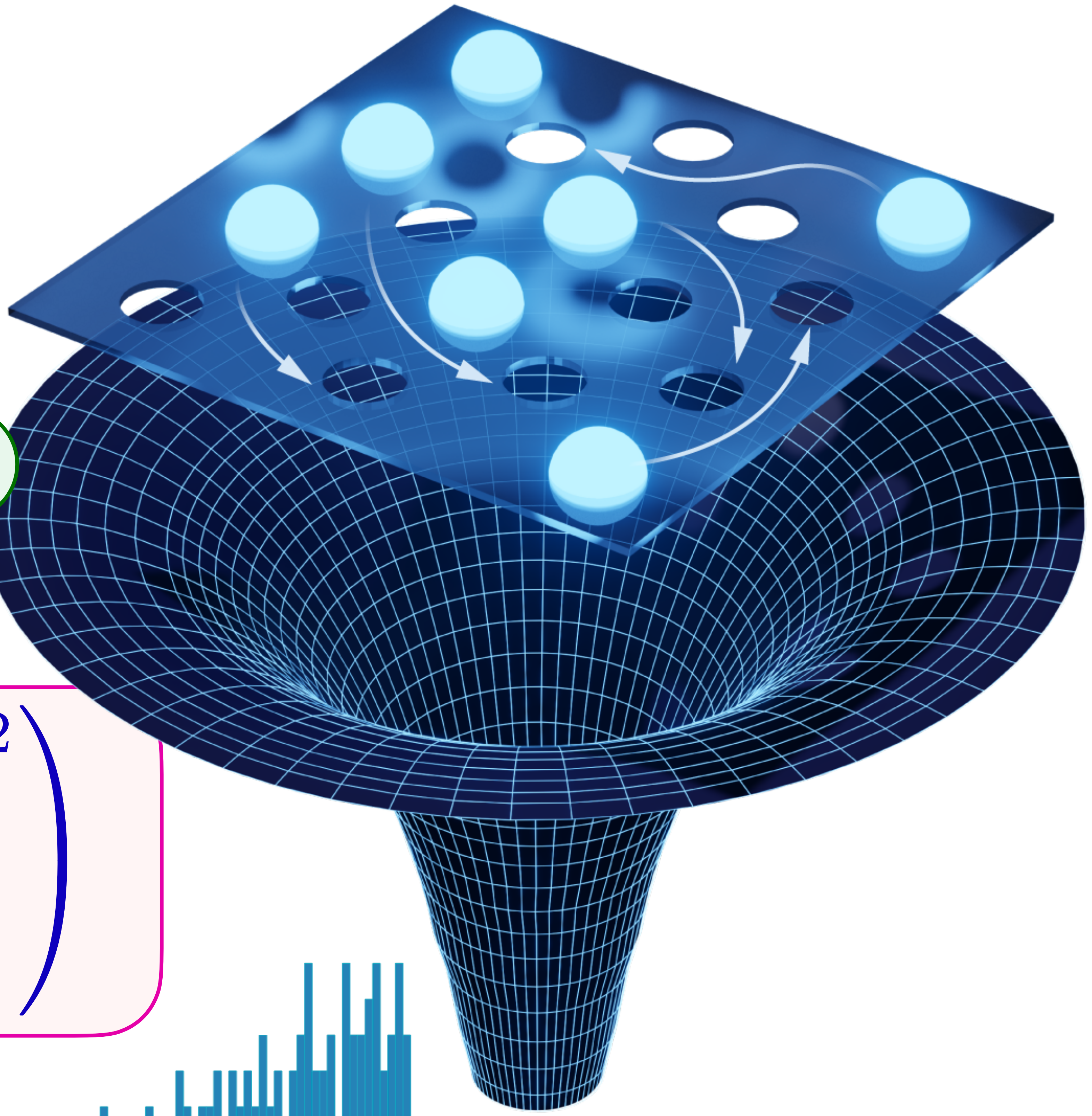




Quantum simulation of charged black holes by the SYK model



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Iliesiu, Murthy, Turiaci (2022)

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Developments from the SYK model

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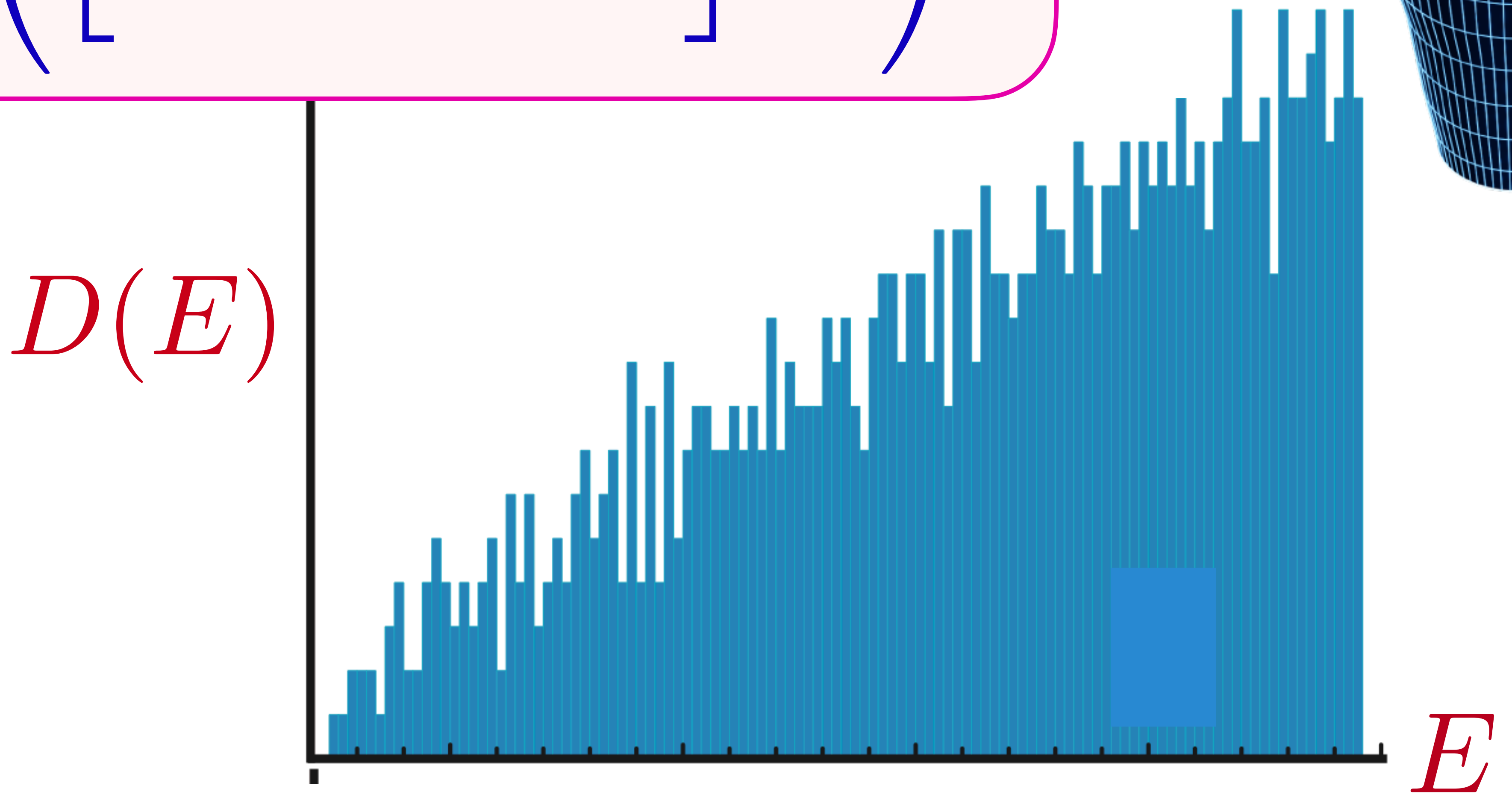


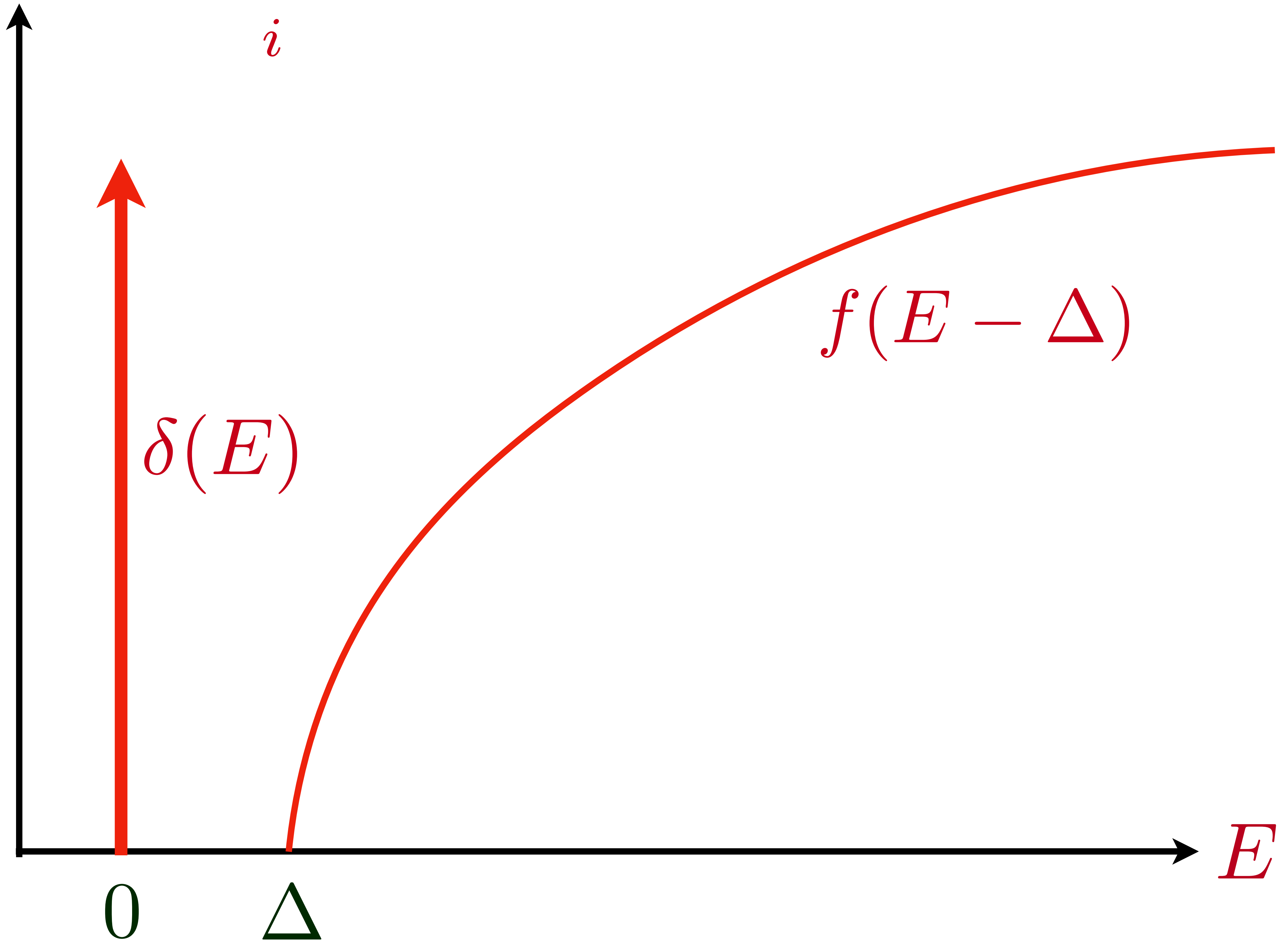
String theory of charged black holes

- With sufficient low energy supersymmetry, string theory yields:

$$D(E) = \exp\left(\frac{A_0 c^3}{4\hbar G}\right)\delta(E) + \theta(E-\Delta) f(E-\Delta) + \ldots$$

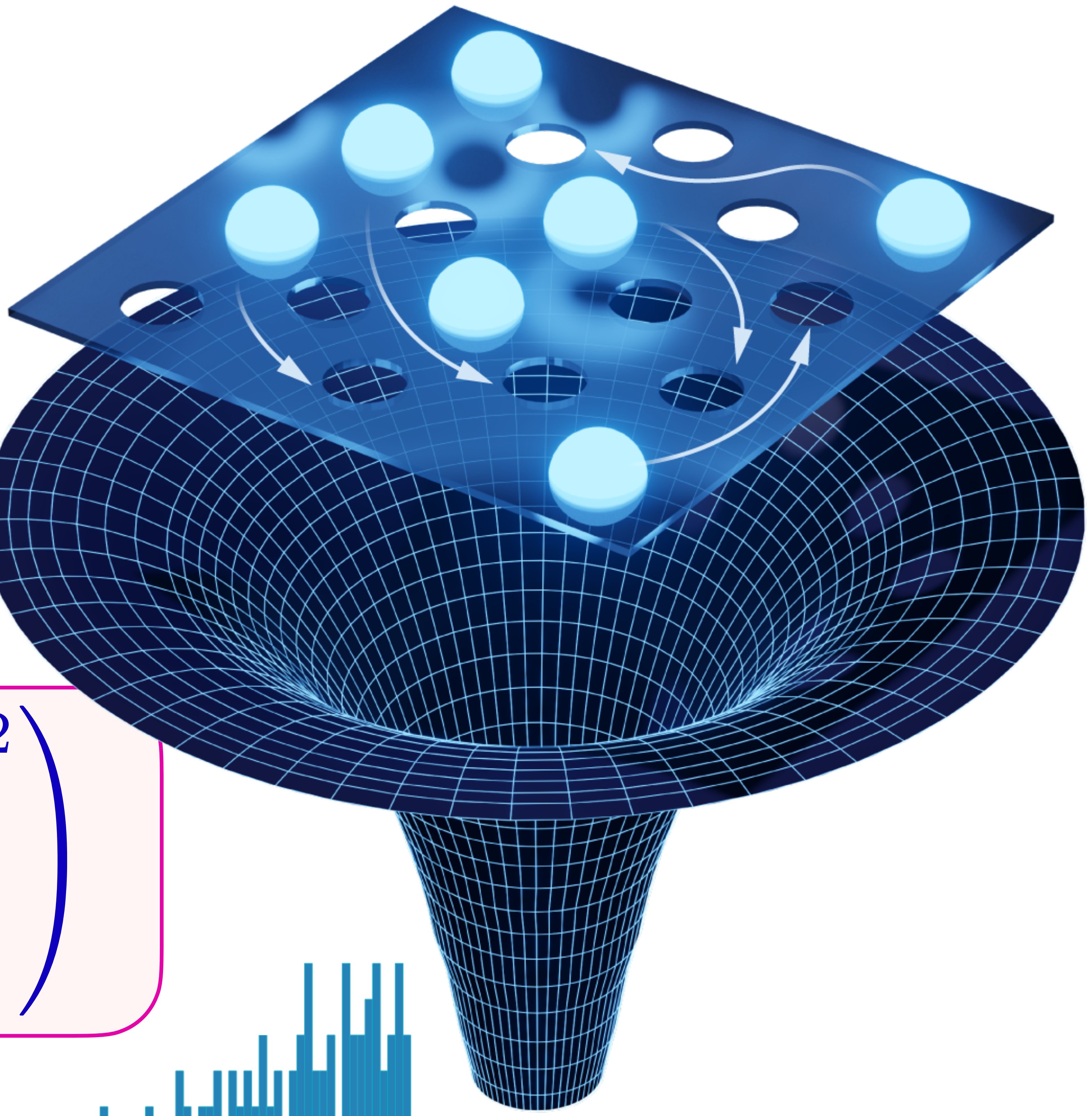
There are exponentially many degenerate BPS ground states, and an energy gap Δ above the ground state.

$$D(E) = \sum_i \delta(E - E_i)$$



M. Heydeman, L.V. Iliesiu, G. J. Turiaci, and W. Zhao, 2020
L.V. Iliesiu, S. Murthy, G. J. Turiaci, 2022

Quantum simulation of charged black holes by the SYK model



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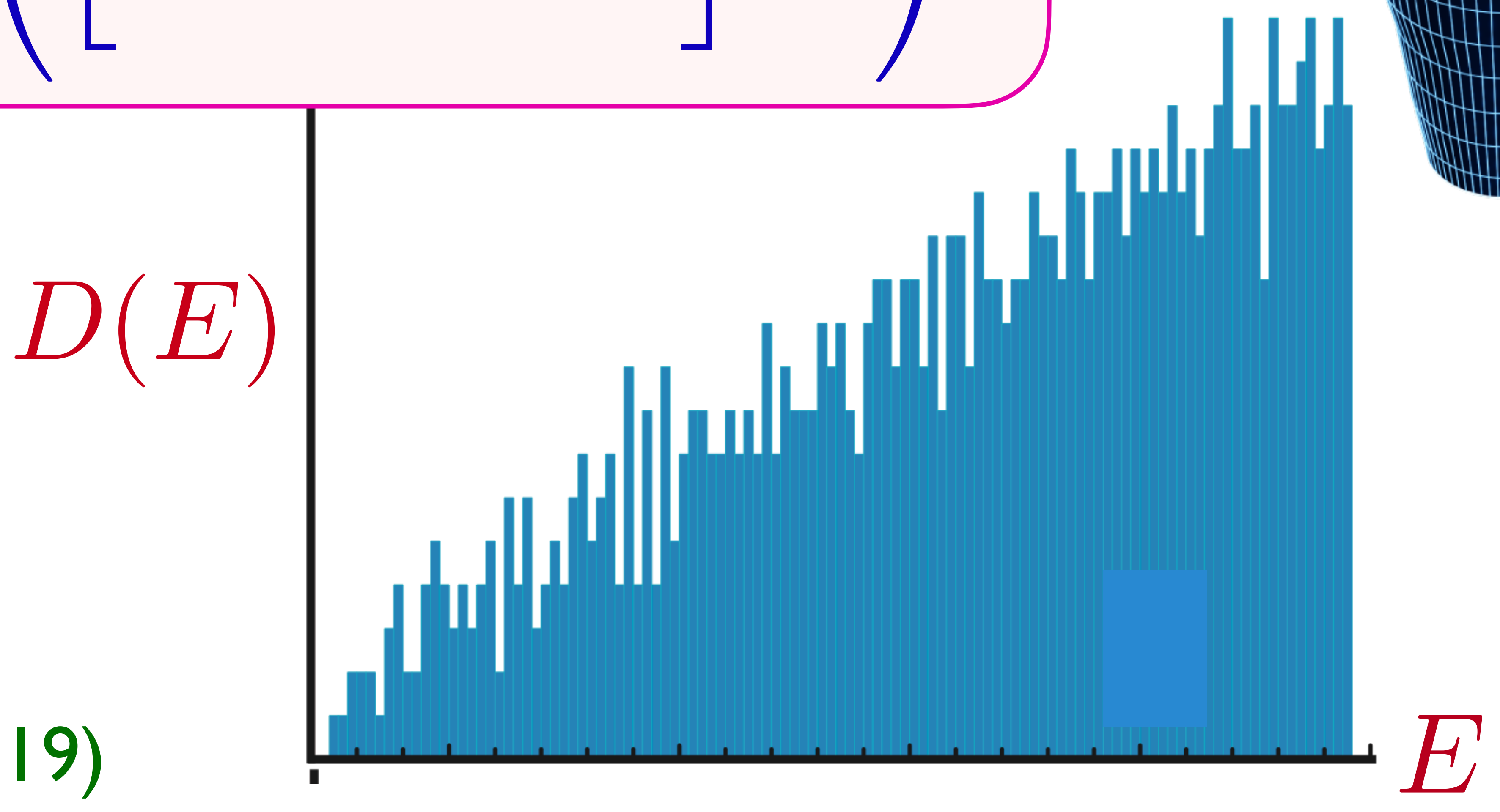
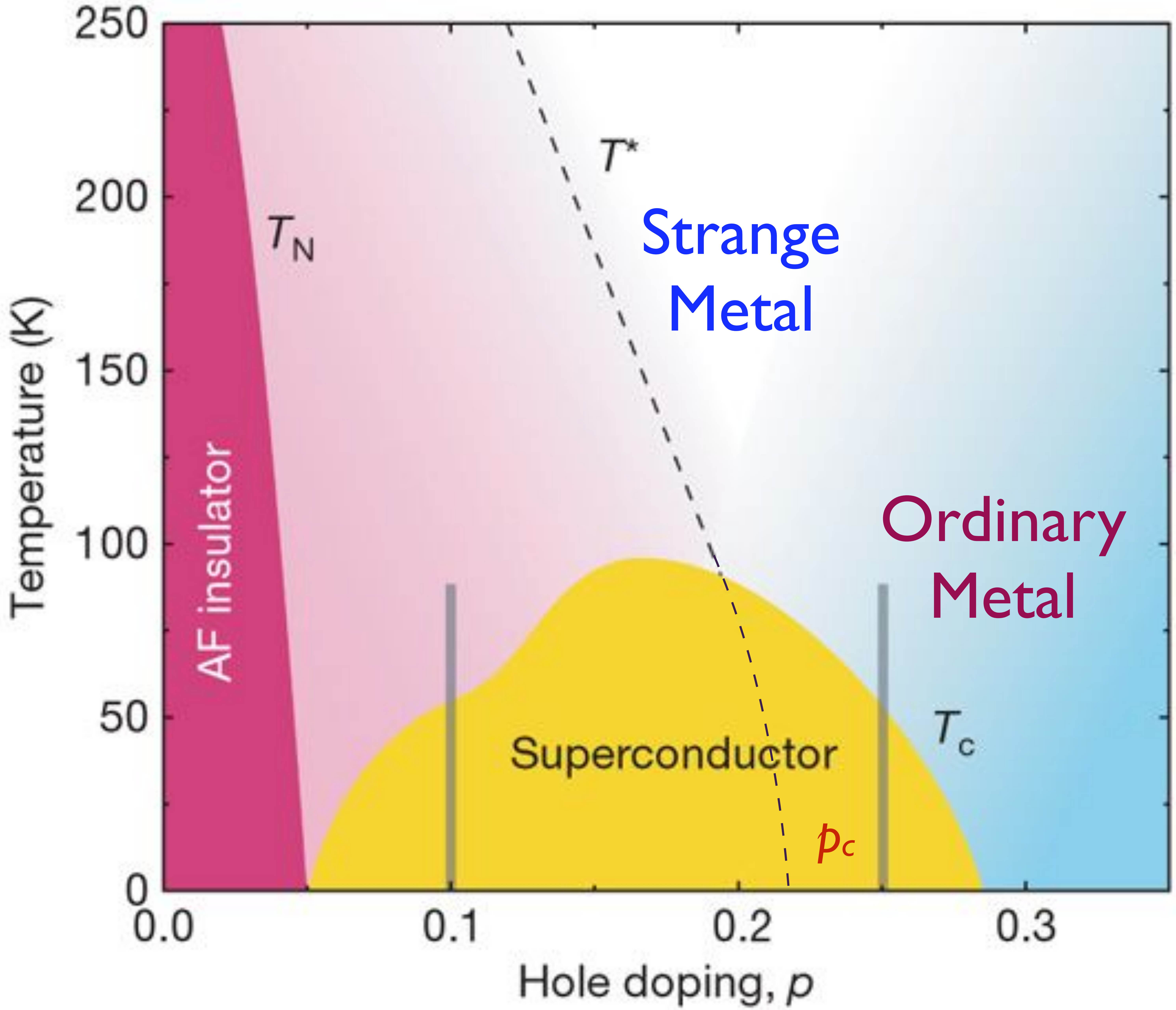
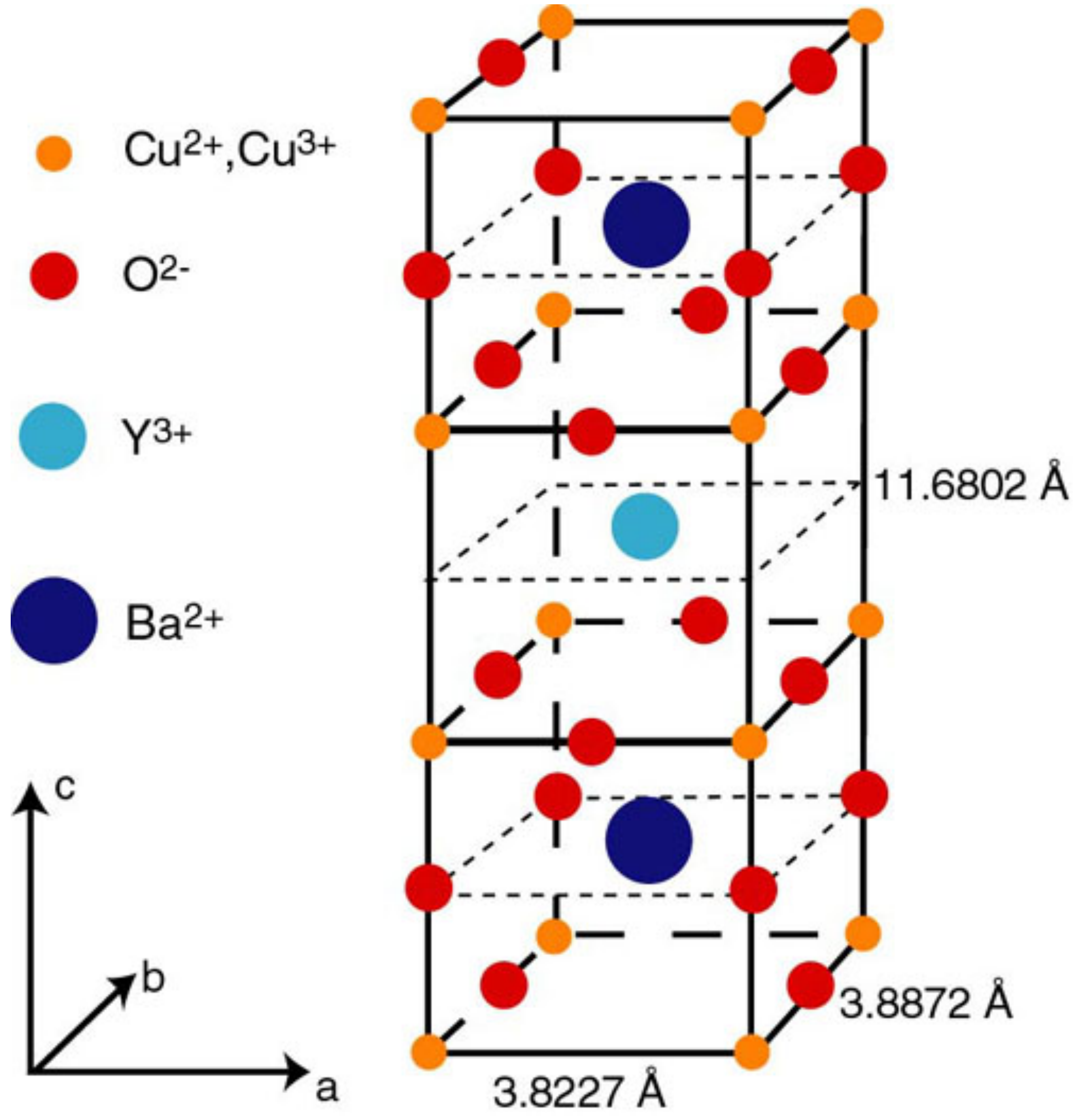
- 'Wormhole' contributions to this quantum simulation have led to an understanding of the Page curve of entanglement entropy of evaporating black holes.



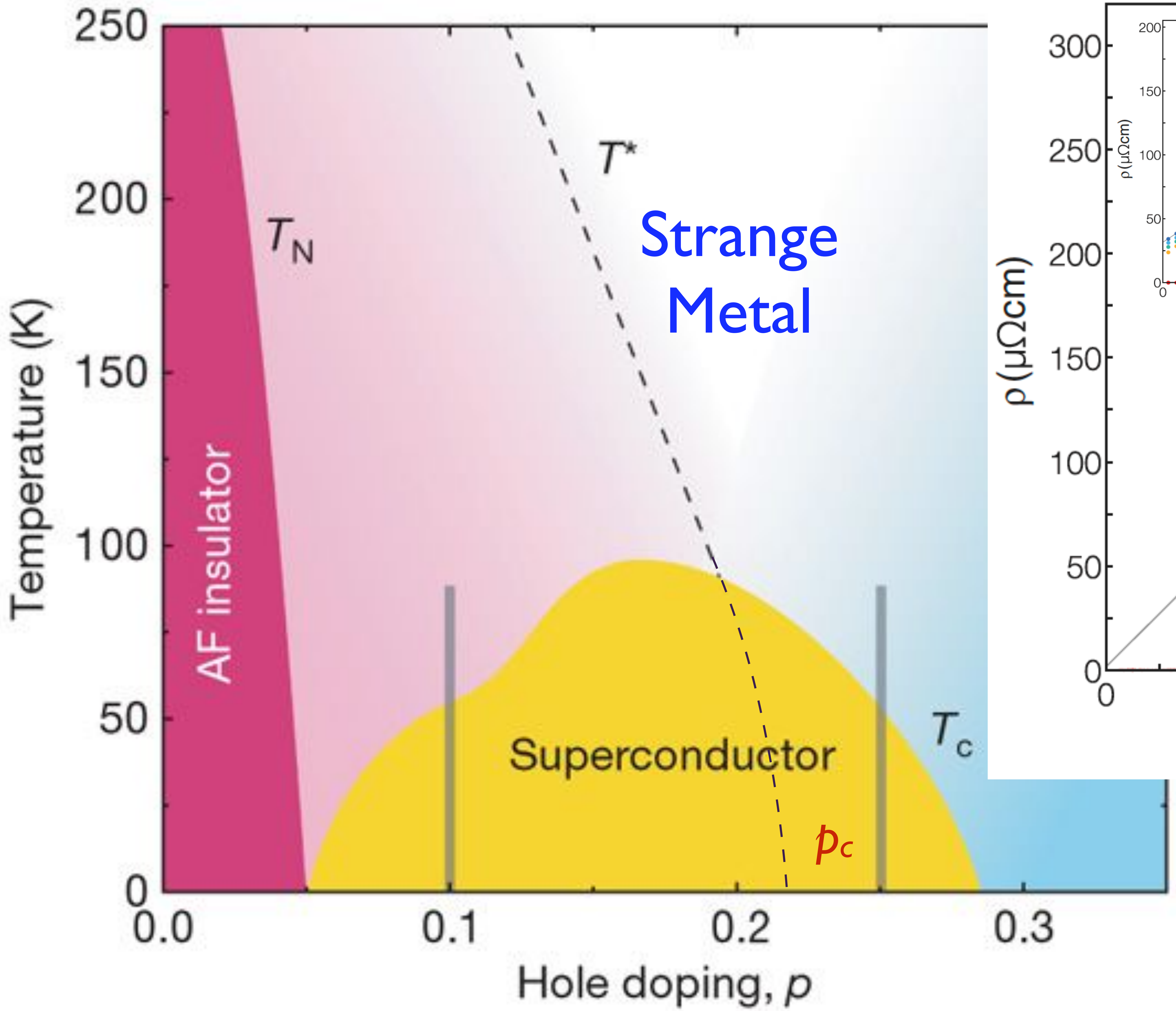
Saad, Shenker, Stanford (2019)

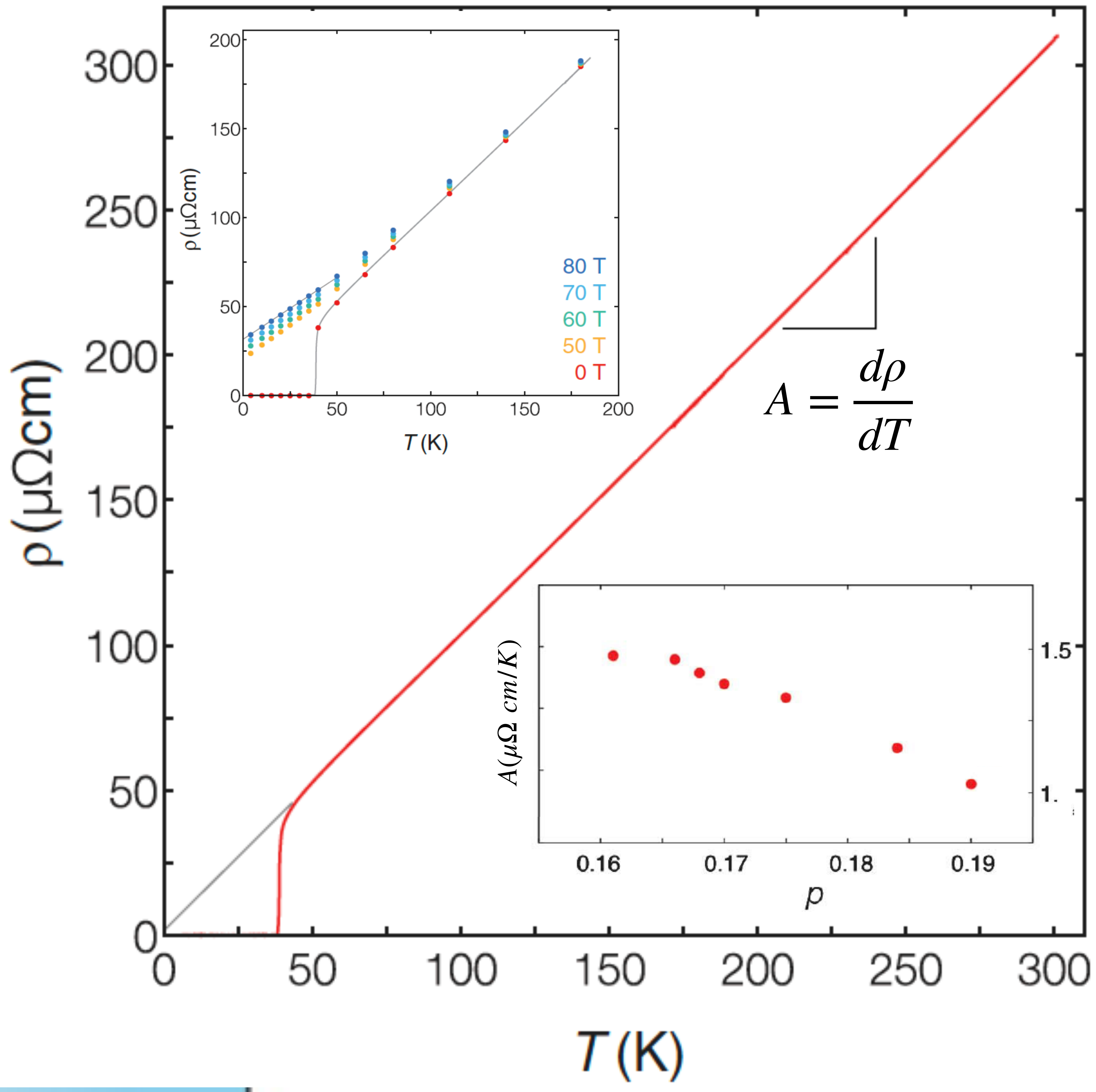
1. Large-N theory of the SYK model
2. Finite-N theory of the SYK model
3. Quantum Einstein-Maxwell gravity theory of charged black holes
4. Universal theory of strange metals


250
200
150
100
50
0
0.0
0.1
0.2
0.3
Temperature (K)
Hole doping, p
AF insulator
T_N
T*
Strange Metal
Ordinary Metal
Superconductor
T_c
p_c


Cu2+,Cu3+
O2-
Y3+
Ba2+
c
b
a
11.6802 Å
3.8872 Å
3.8227 Å

$YBa_2Cu_3O_{6+x}$





LSCO: Giraldo-Gallo et al. 2018

Transport properties of a strange metal:

1. Resistivity $\rho(T) = \rho_0 + AT + \ldots$ as $T \to 0$
   and $\rho(T) < h/e^2$ (in $d = 2$).
   Metals with $\rho(T) > h/e^2$ are bad metals.

2. Optical conductivity

$$\sigma(\omega) = \frac{K}{\dfrac{1}{\tau_{\rm trans}(\omega)} - i\omega \dfrac{m^*_{\rm trans}(\omega)}{m}} \quad ; \quad \frac{1}{\tau_{\rm trans}(\omega)} \sim |\omega| \Phi_\sigma \left(\frac{\hbar\omega}{k_B T} \right)$$

B. Michon……A. Georges, arXiv:2205.04030

Electronic properties of a marginal Fermi liquid:

1. Photoemission: nearly marginal Fermi liquid electron spectral density:

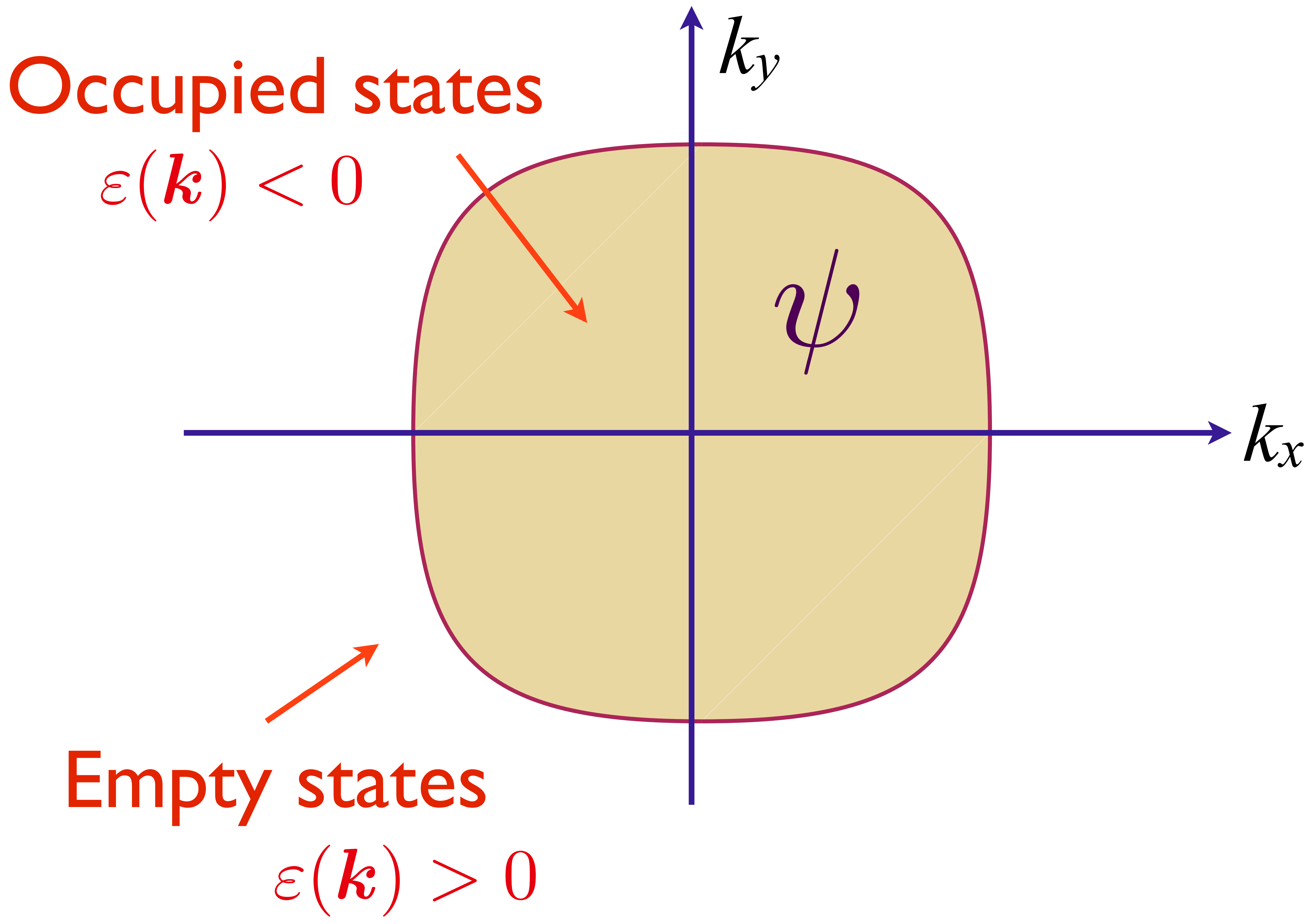
$$\text{Im}\Sigma(\omega) \sim |\omega|^{2\alpha} \Phi_\Sigma \left(\frac{\hbar\omega}{k_B T} \right) \quad \text{with } \alpha \approx 1/2 \quad ; \quad \frac{1}{\tau_{\rm in}(\omega)} \sim |\omega| \Phi_\Sigma \left(\frac{\hbar\omega}{k_B T} \right)$$

T.J. Reber….D. Dessau, Nature Communications **10**, 5737 (2019)

2. Specific heat $\sim T \ln(1/T)$ as $T \to 0$.

S.A. Hartnoll and A.P. MacKenzie, RMP (2022)

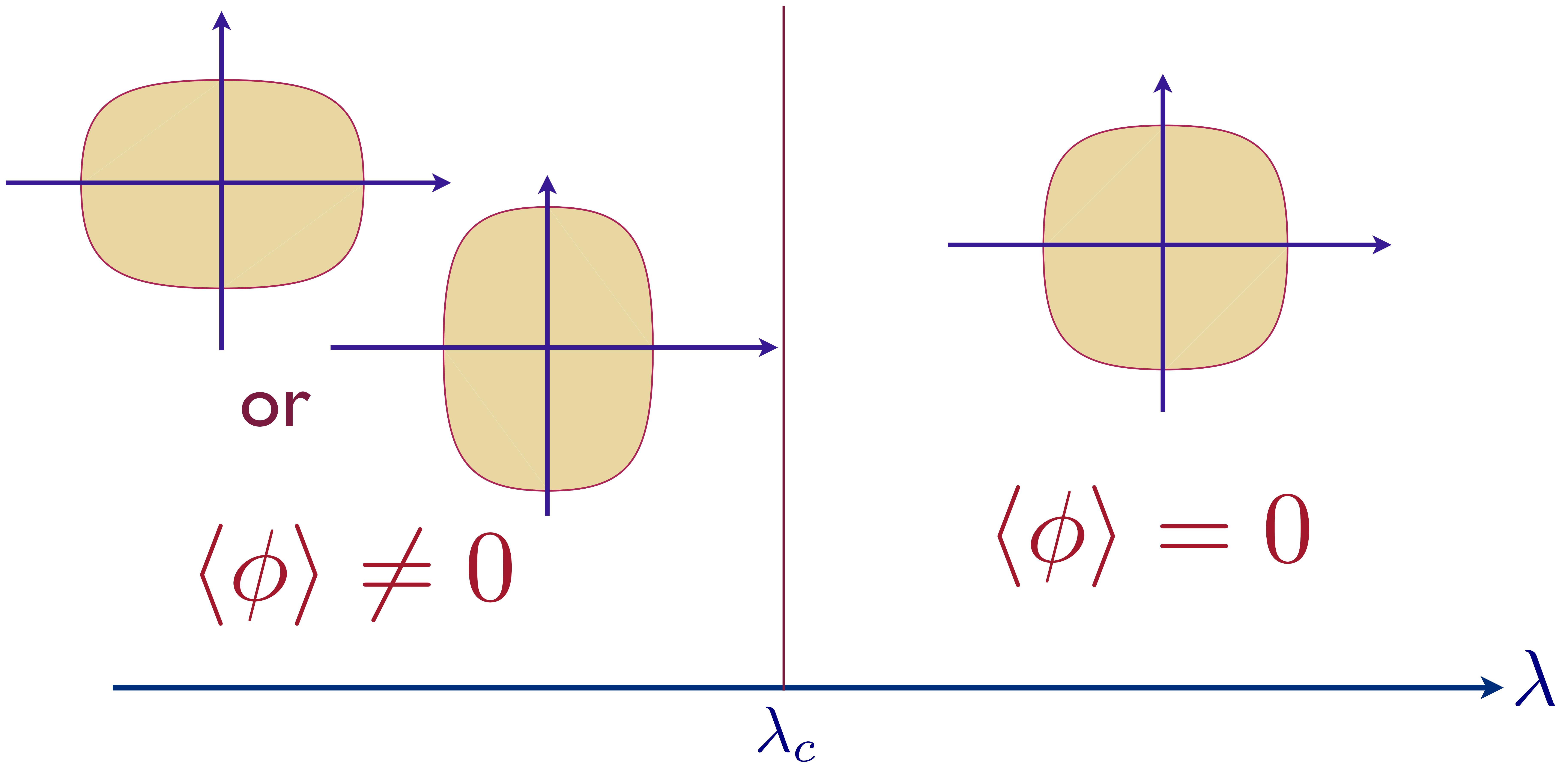
Fermi surface coupled to a critical boson



+

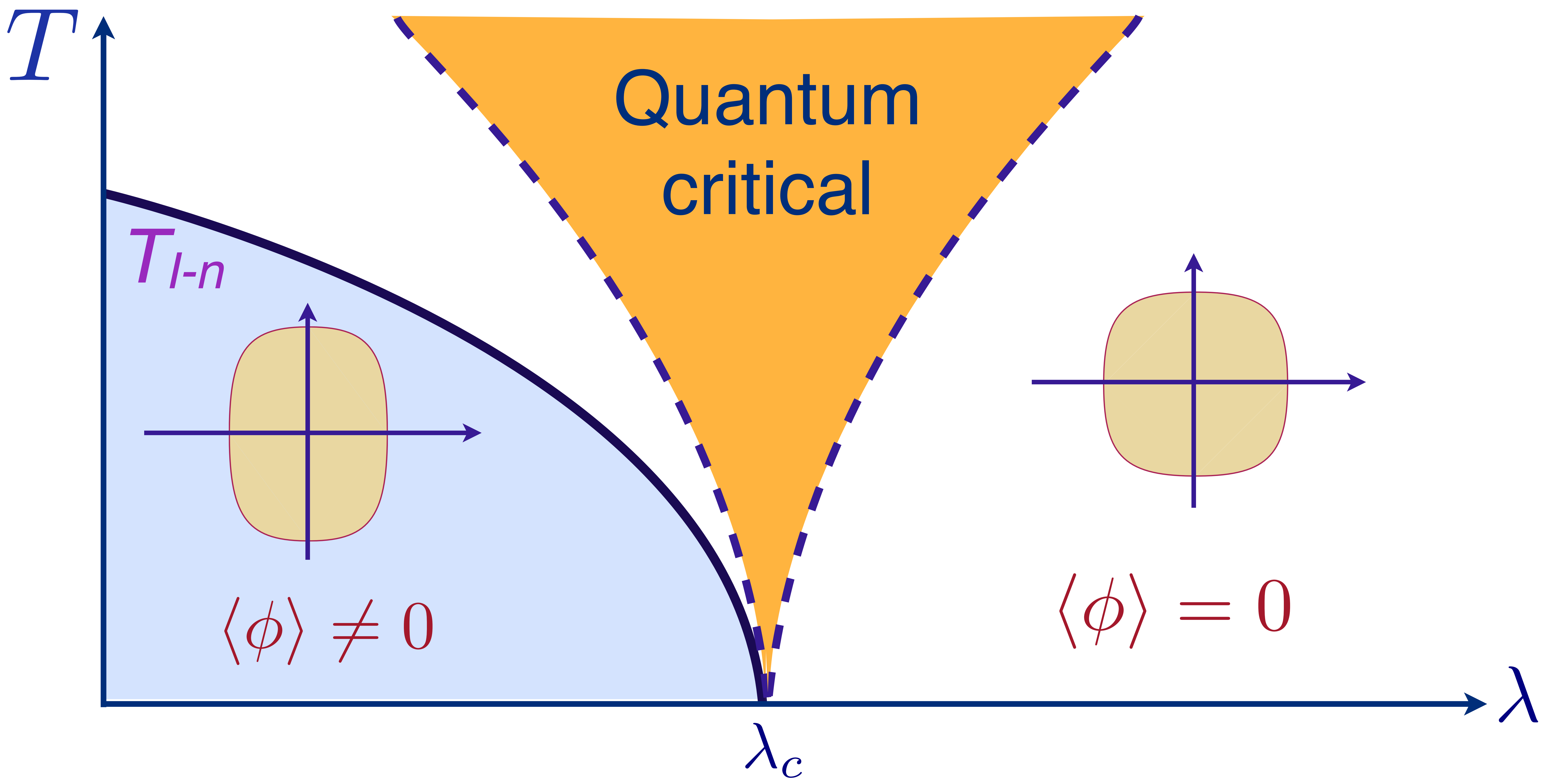
a
critical
boson
ϕ

Quantum criticality of Ising-nematic ordering in a metal



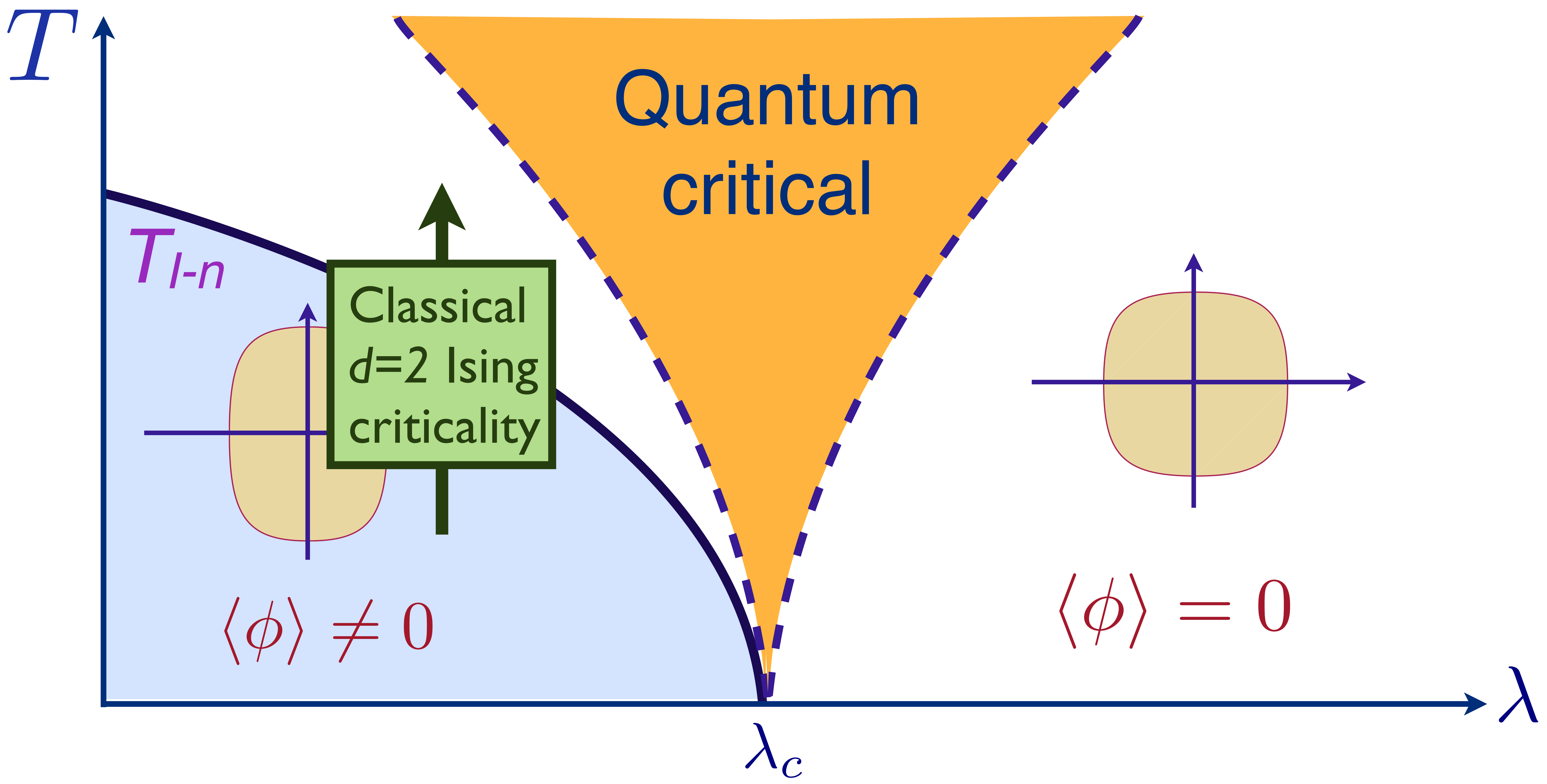
Pomeranchuk instability as a function of coupling λ

Quantum criticality of Ising-nematic ordering in a metal



Phase diagram as a function of T and λ

Quantum criticality of Ising-nematic ordering in a metal



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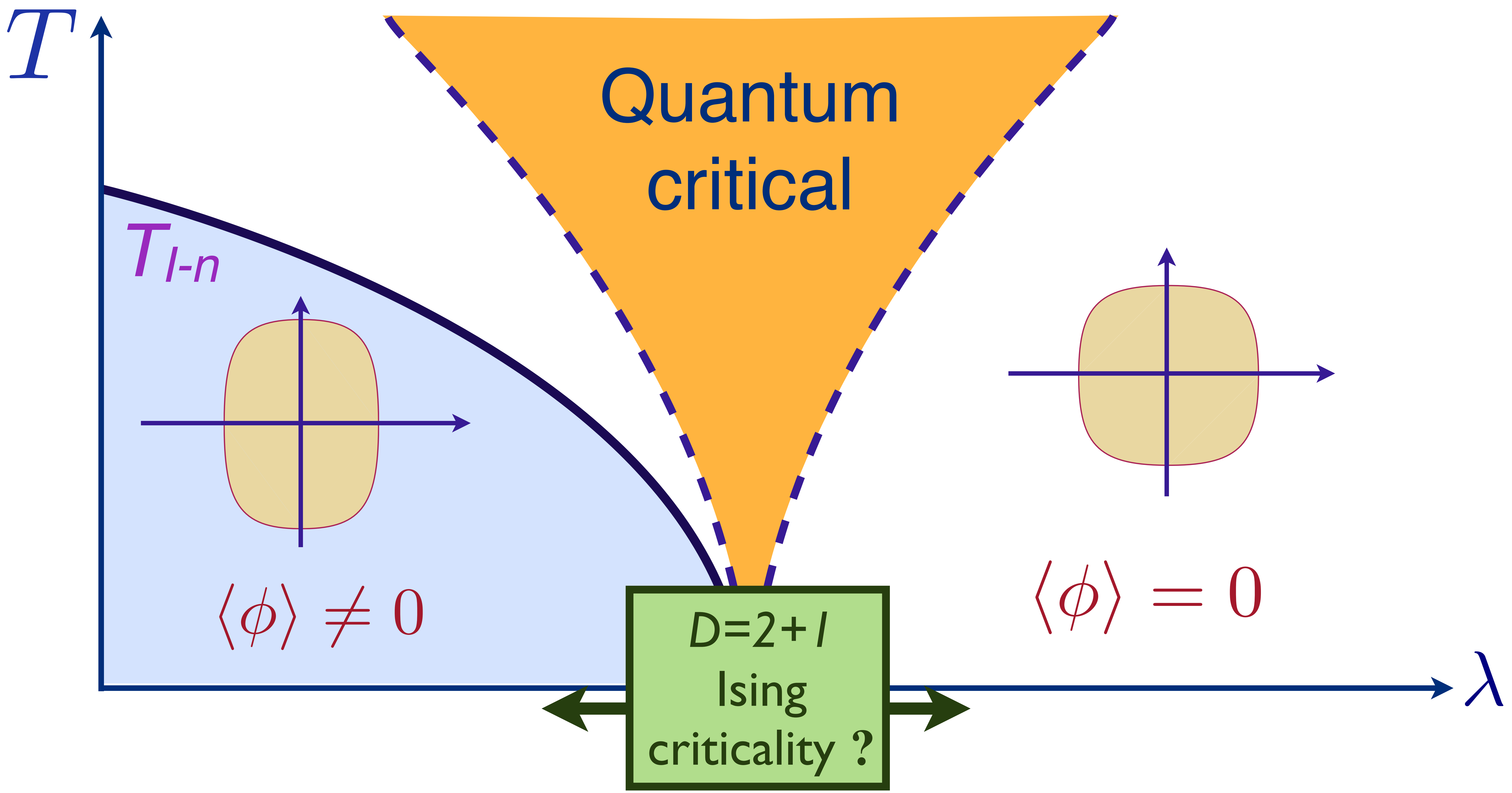

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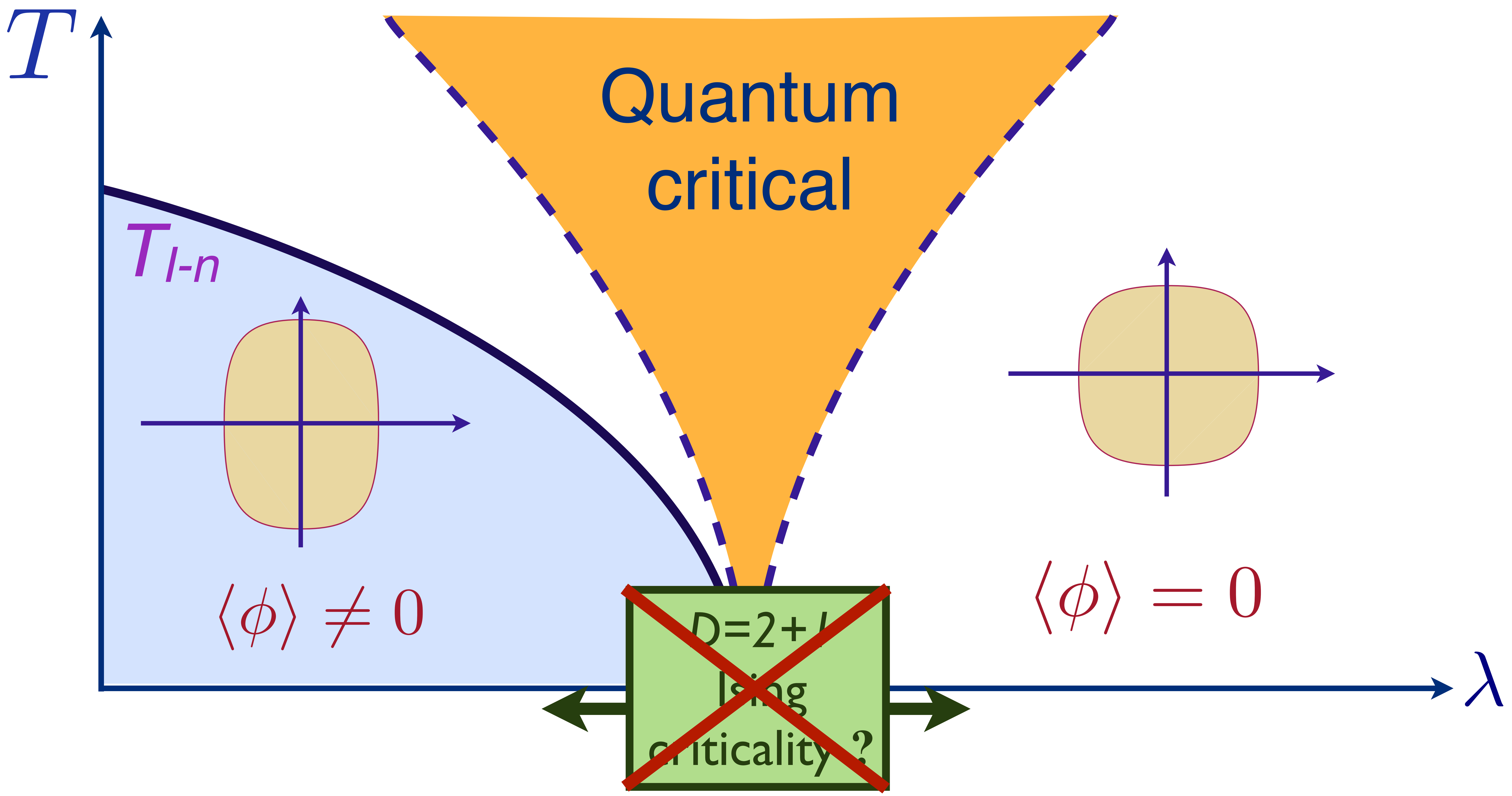

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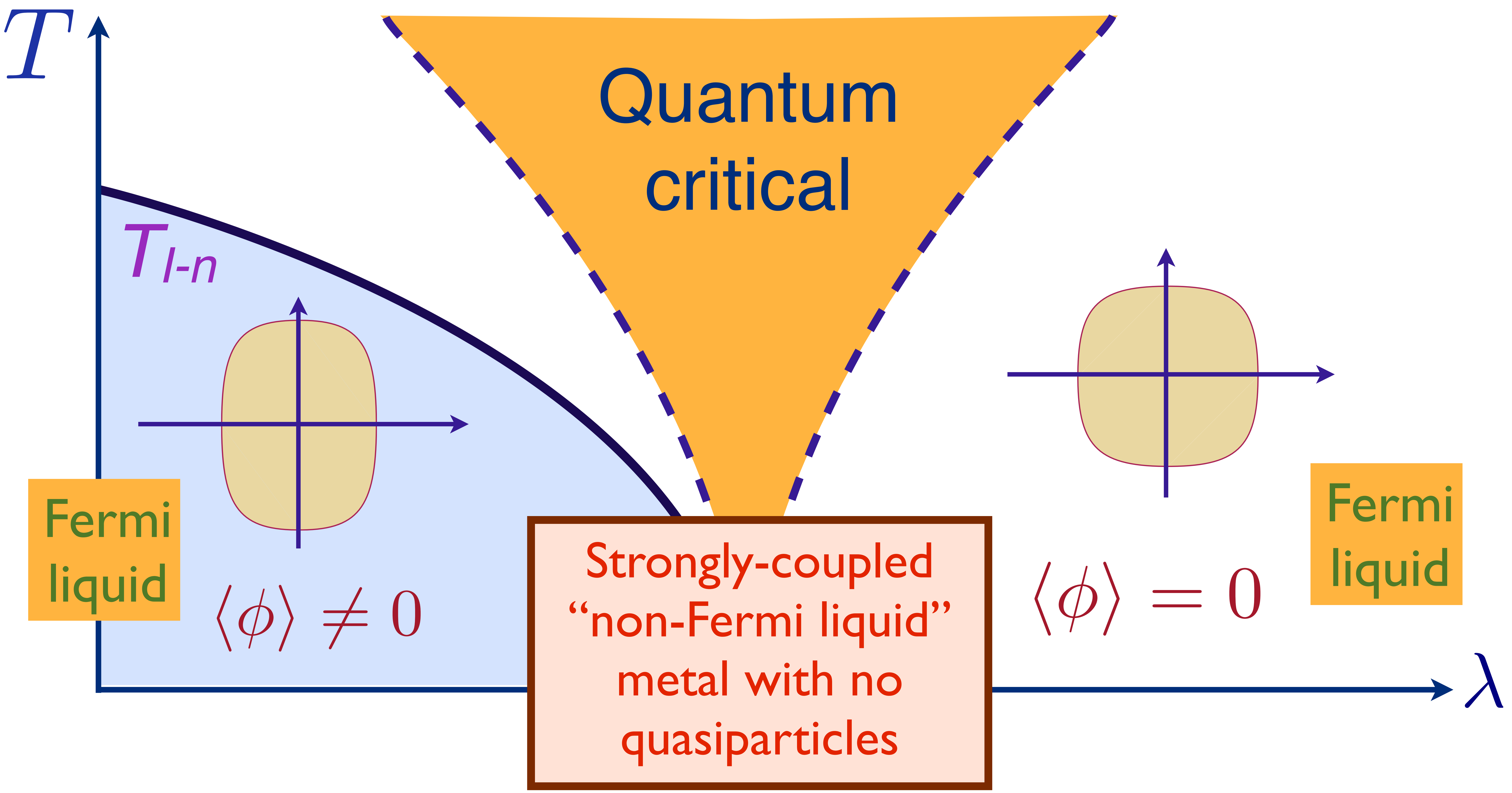

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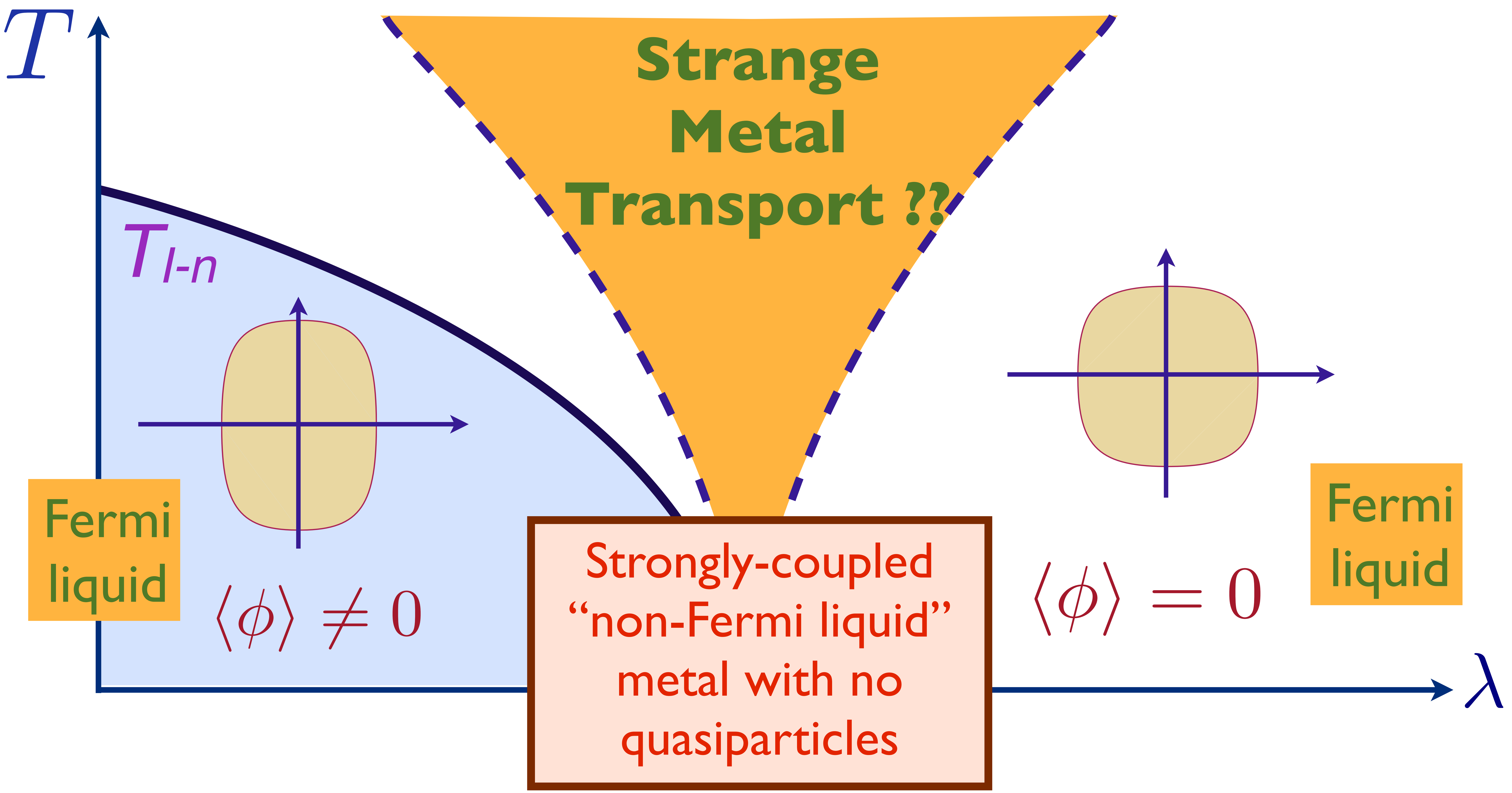

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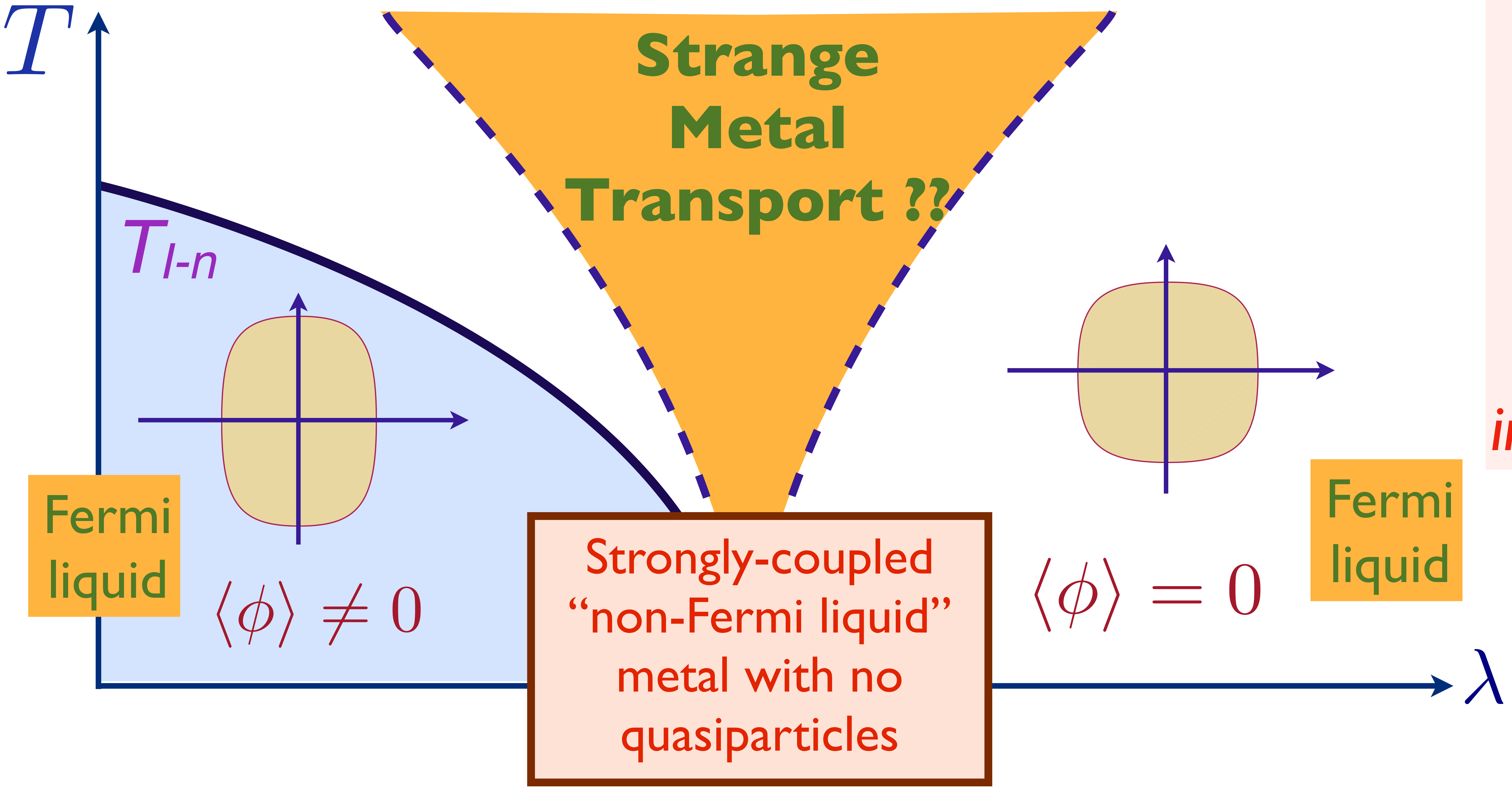


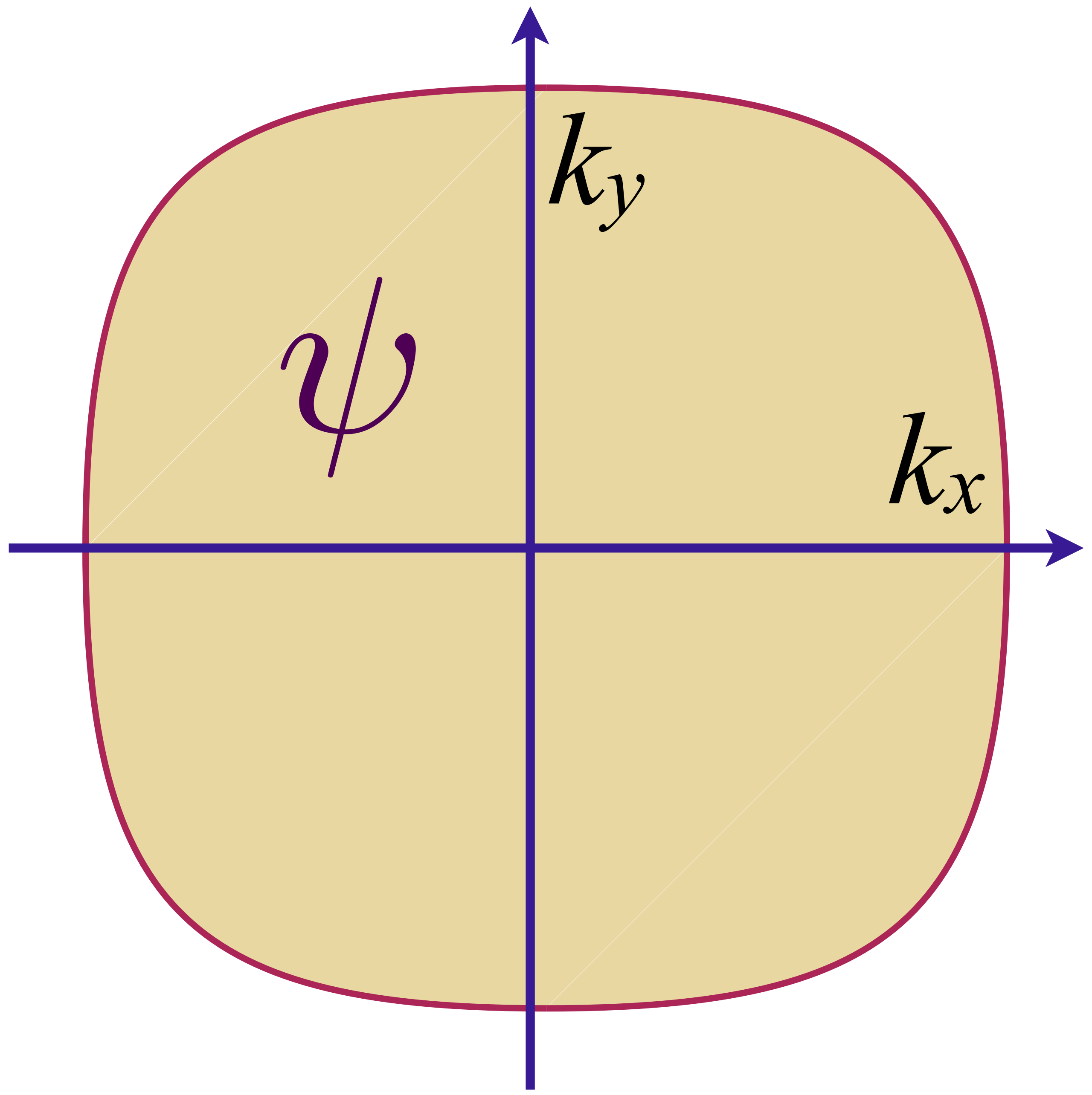
No.

Need spatial *disorder* in the *interactions*

Phase diagram as a function of T and λ

Fermi surface

$$\mathcal{L}_\psi = \psi_{\boldsymbol{k}}^\dagger \left(\frac{\partial}{\partial \tau} + \varepsilon(\boldsymbol{k}) \right) \psi_{\boldsymbol{k}}$$



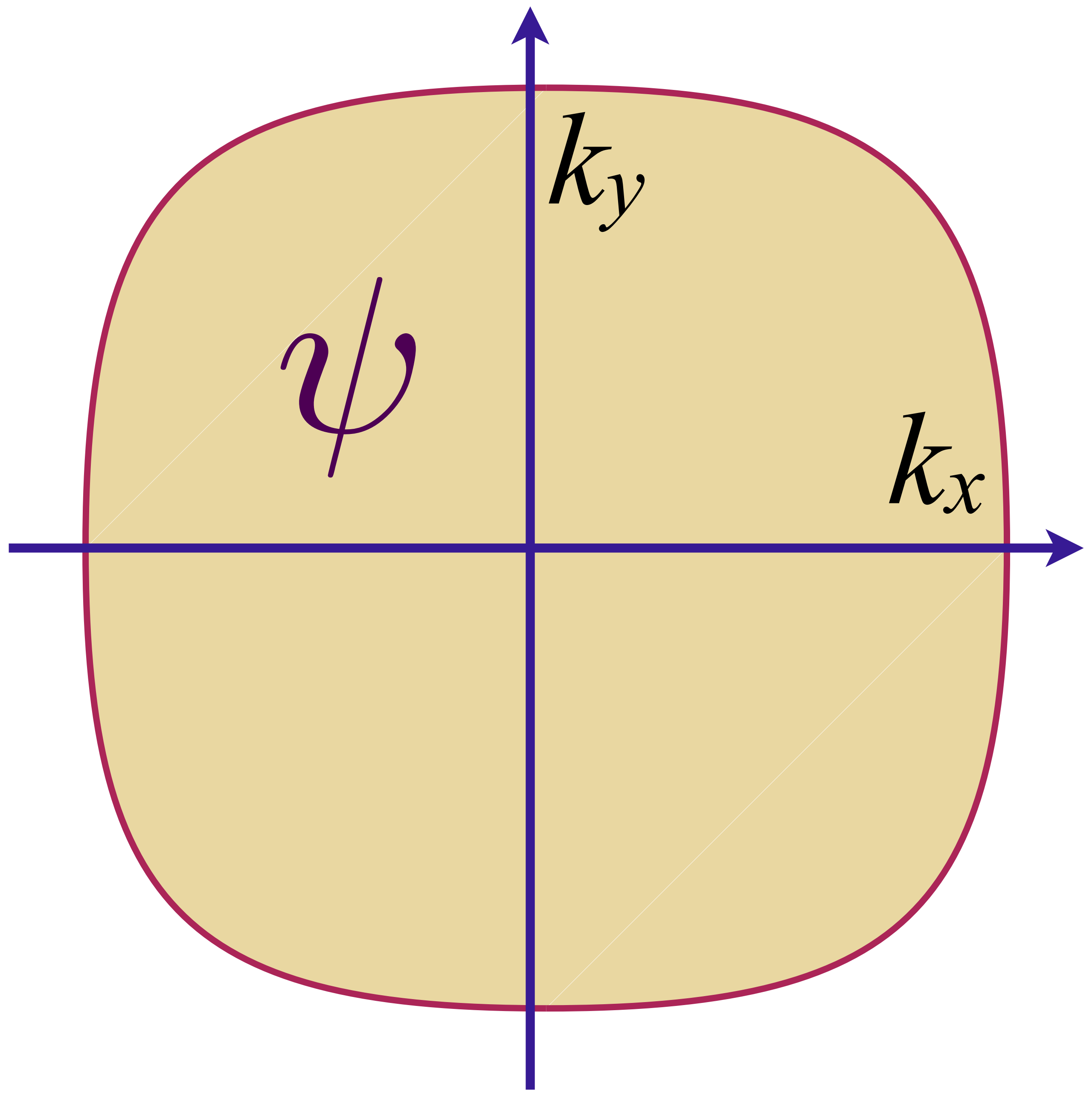
$$- J \, \psi^\dagger(\boldsymbol{r}) \psi^\dagger(\boldsymbol{r}) \psi(\boldsymbol{r}) \psi(\boldsymbol{r})$$

Fermi surface + critical boson

$$\mathcal{L}_\psi = \psi^\dagger_{\boldsymbol{k}} \left(\frac{\partial}{\partial \tau} + \varepsilon(\boldsymbol{k}) \right) \psi_{\boldsymbol{k}}$$

a critical boson ϕ

e.g. Ising-nematic order

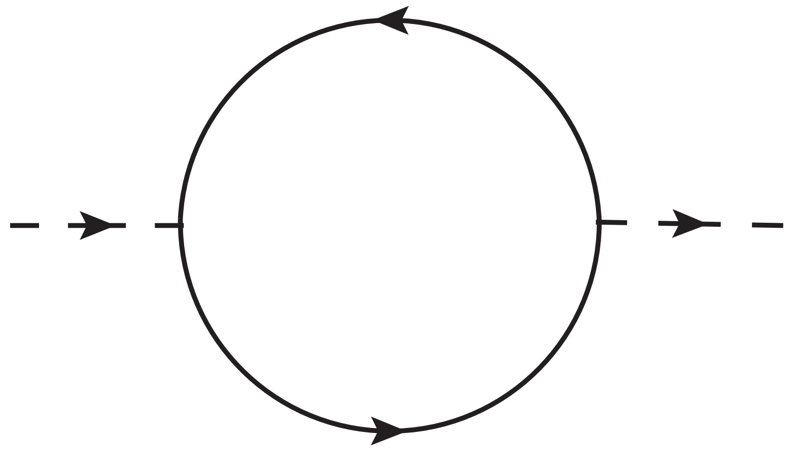


$$\frac{[\phi(\boldsymbol{r})]^2}{J} + \psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})\,\phi(\boldsymbol{r})$$

Fermi surface coupled to a critical boson

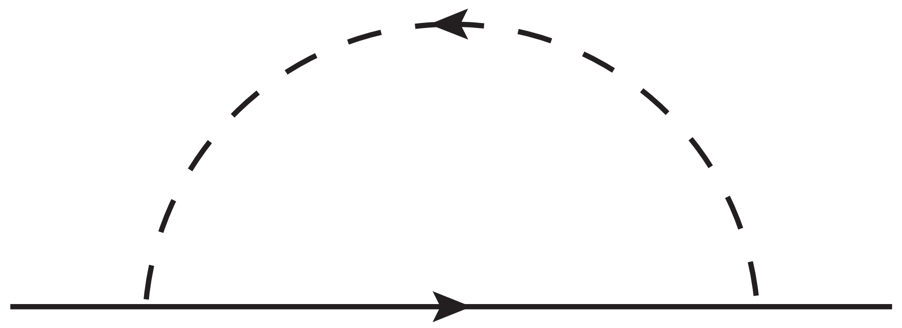
"Yukawa" coupling: $g \int d^2r d\tau \, \psi^\dagger(r,\tau)\psi(r,\tau)\phi(r,\tau)$

Boson self energy $\Pi(q, i\Omega) \sim -g^2 \dfrac{|\Omega|}{q}$ (Landau damping)



Boson Green's function $D(q, i\Omega) = \dfrac{1}{q^2 + \gamma|\Omega|/q}$

Fermion self energy $\Sigma(\hat{\boldsymbol{k}}, i\omega) \sim -i\mathrm{sgn}(\omega)|\omega|^{2/3}$



Fermion Green's function $G(\boldsymbol{k}, i\omega) = \dfrac{1}{i\omega - \varepsilon(\boldsymbol{k}) - \Sigma(\hat{\boldsymbol{k}}, i\omega)}$

P.A. Lee (1989)

Yields a state without quasiparticle excitations, but the theory is not systematic at large N

Sung-Sik Lee (2009)

Fermi surface + critical boson

These results can also be obtained from the saddle-point and response functions of a G-Σ-D-Π action. Such an action can formally be obtained in a Yukawa-SYK-like large-N limit of a theory with couplings which are random in an additional (fictitious) flavor space.

$$\mathcal{Z} = \int \mathcal{D}G\,\mathcal{D}\Sigma\,\mathcal{D}D\,\mathcal{D}\Pi\,\exp(-N S_{\rm all})$$

$$\begin{aligned} S_{\rm all} = & -\ln\det(\partial_\tau + \varepsilon(\mathbf{k}) - \mu + \Sigma) + \frac{1}{2}\ln\det(-\partial_\tau^2 + \mathbf{q}^2 + m_b^2 - \Pi) \\ & + \int d\tau d^2r \int d\tau' d^2r' \Big[-\Sigma(\tau',\mathbf{r}';\tau,\mathbf{r})G(\tau,\mathbf{r};\tau',\mathbf{r}') + \frac{1}{2}\Pi(\tau',\mathbf{r}';\tau,\mathbf{r})D(\tau,\mathbf{r};\tau',\mathbf{r}') \\ & + \frac{g^2}{2}G(\tau,\mathbf{r};\tau',\mathbf{r}')G(\tau',\mathbf{r}';\tau,\mathbf{r})D(\tau,\mathbf{r};\tau',\mathbf{r}')\Big]\,. \end{aligned}$$

Fermi surface + critical boson

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$$\mathcal{Z} = \int \mathcal{D}G\, \mathcal{D}\Sigma\, \mathcal{D}D\, \mathcal{D}\Pi \exp(-N S_{\rm all})$$

Saddle-point equations:

Migdal-Eliashberg

$$\Sigma(\tau, \mathbf{r}) = g^2 D(\tau, \mathbf{r}) G(\tau, \mathbf{r}),$$

$$\Pi(\tau, \mathbf{r}) = -g^2 G(-\tau, -\mathbf{r}) G(\tau, \mathbf{r}),$$

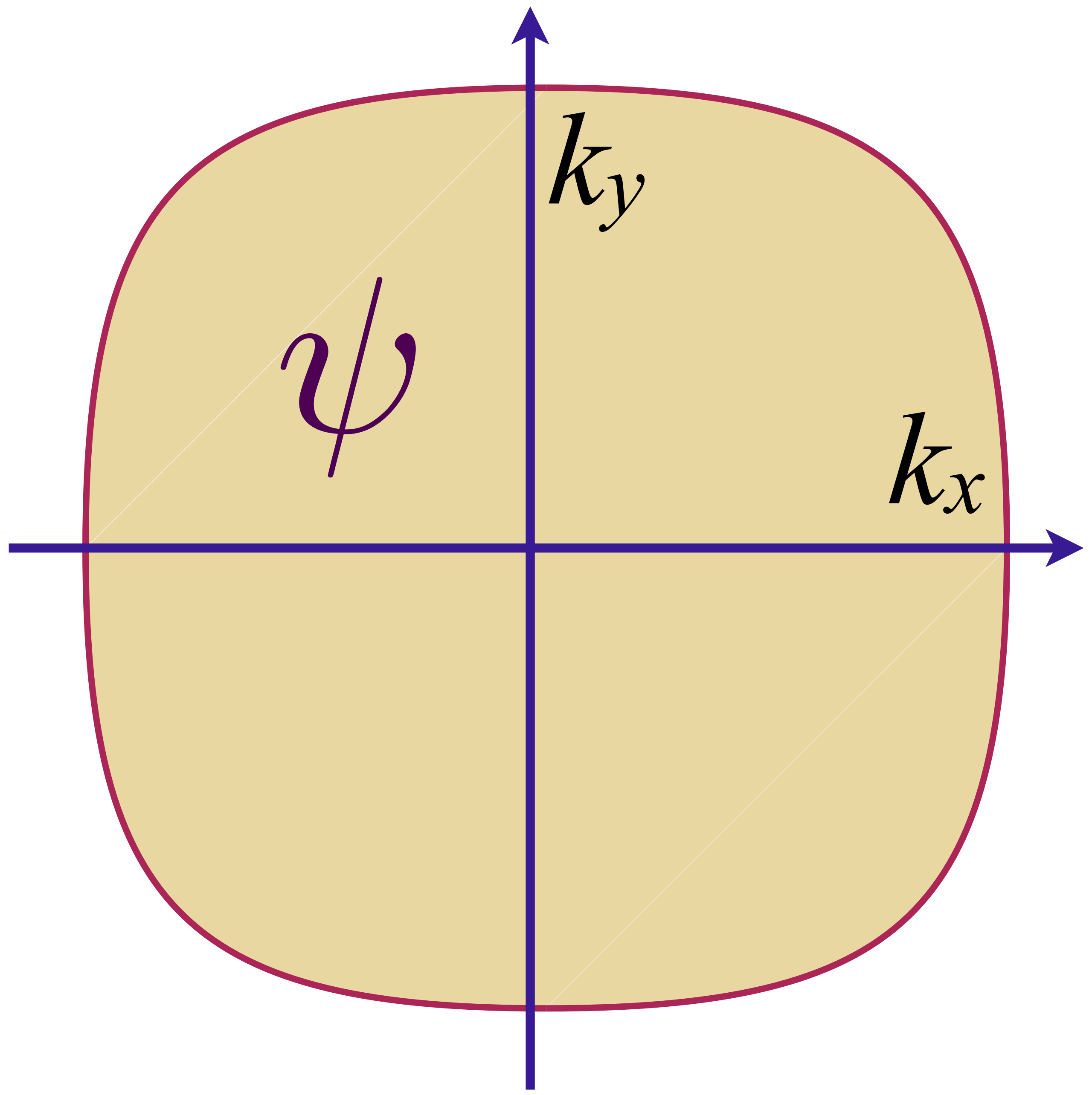
$$G(i\omega, \mathbf{k}) = \frac{1}{i\omega - \varepsilon(\mathbf{k}) + \mu - \Sigma(i\omega, \mathbf{k})},$$

$$D(i\Omega, \mathbf{q}) = \frac{1}{\Omega^2 + \mathbf{q}^2 + m_b^2 - \Pi(i\Omega, \mathbf{q})}.$$

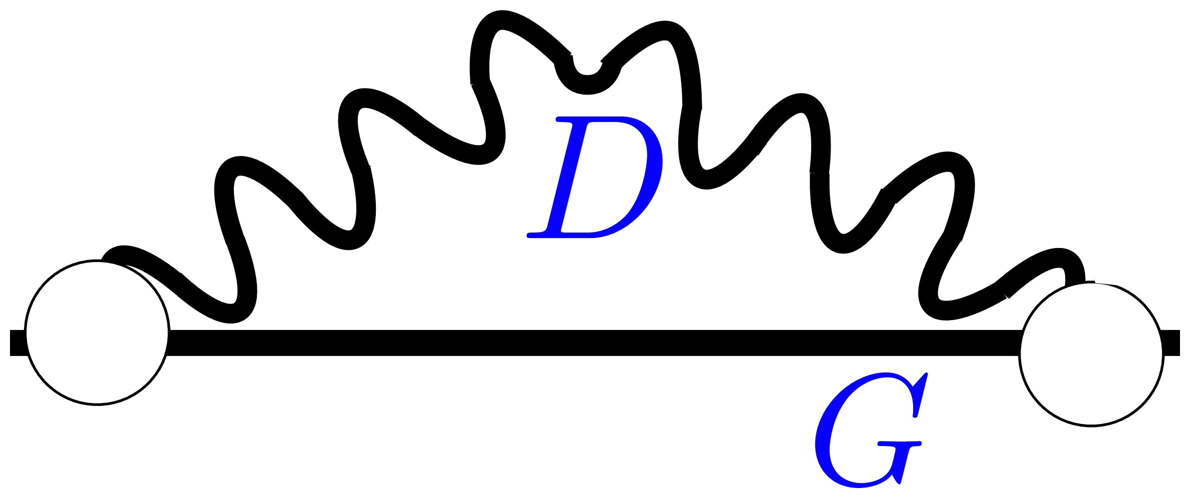
Fermi surface + critical boson

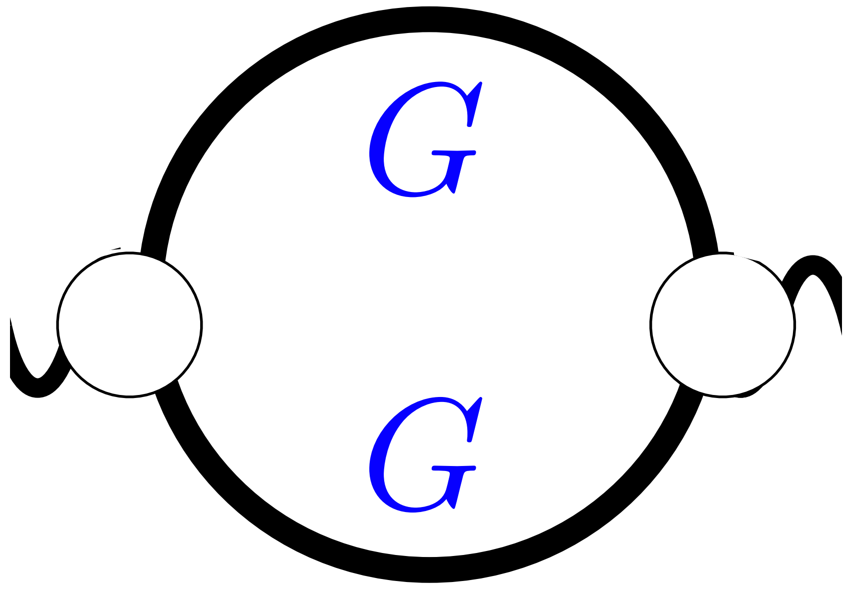
$$\mathcal{L}_\psi = \psi^\dagger_{\boldsymbol{k}} \left(\frac{\partial}{\partial \tau} + \varepsilon(\boldsymbol{k}) \right) \psi_{\boldsymbol{k}}$$

a critical boson ϕ
e.g. Ising-nematic order



$$\frac{[\phi(\boldsymbol{r})]^2}{J} + \psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})\,\phi(\boldsymbol{r})$$

$\Sigma =$ 

$\Pi =$ 

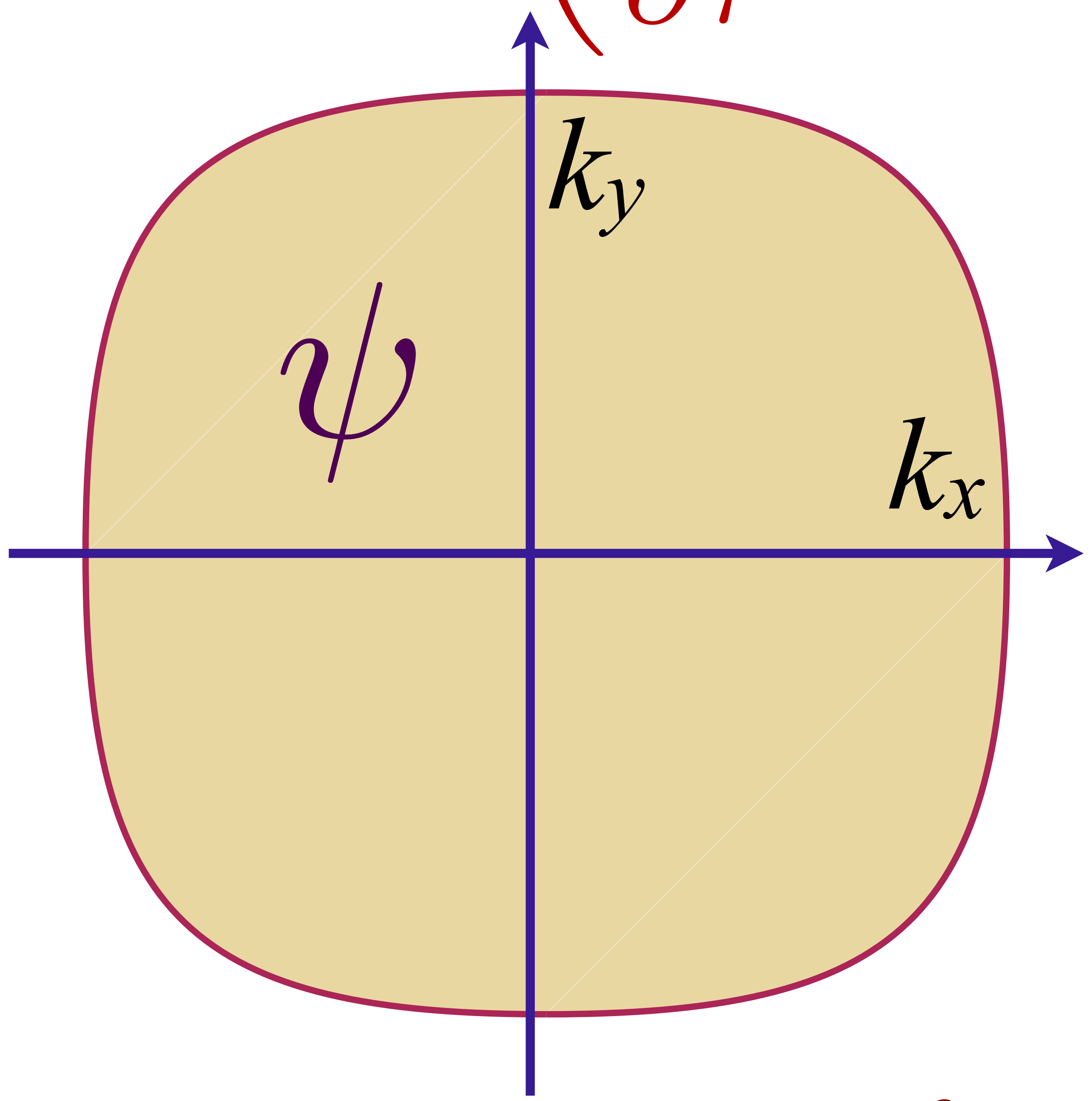
Solution of Migdal-Eliashberg equations for electron (G) and boson (D) Green's functions at small ω:

P.A. Lee (1989)

$$\Sigma(\hat{\boldsymbol{k}}, i\omega) \sim -i\mathrm{sgn}(\omega)|\omega|^{2/3}, \quad G(\boldsymbol{k}, i\omega) = \frac{1}{i\omega - \varepsilon(\boldsymbol{k}) - \Sigma(\hat{\boldsymbol{k}}, i\omega)}, \quad D(\boldsymbol{q}, i\Omega) = \frac{1}{\Omega^2 + q^2 + \gamma|\Omega|/q}$$

Fermi surface + critical boson

$$\mathcal{L}_\psi = \psi^\dagger_{\boldsymbol{k}} \left(\frac{\partial}{\partial \tau} + \varepsilon(\boldsymbol{k}) \right) \psi_{\boldsymbol{k}}$$



a critical boson ϕ

e.g. Ising-nematic order

$$\frac{[\phi(\boldsymbol{r})]^2}{J} + \psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})\,\phi(\boldsymbol{r})$$

Transport—a perfect metal!

Conservation of momentum and fermion-boson drag imply:

$$\text{Re}\left[\sigma(\omega)\right] = D\delta(\omega) + \ldots$$

S. A. Hartnoll, P. K. Kovtun, M. Muller, and S.S. PRB **76**, 144502 (2007)

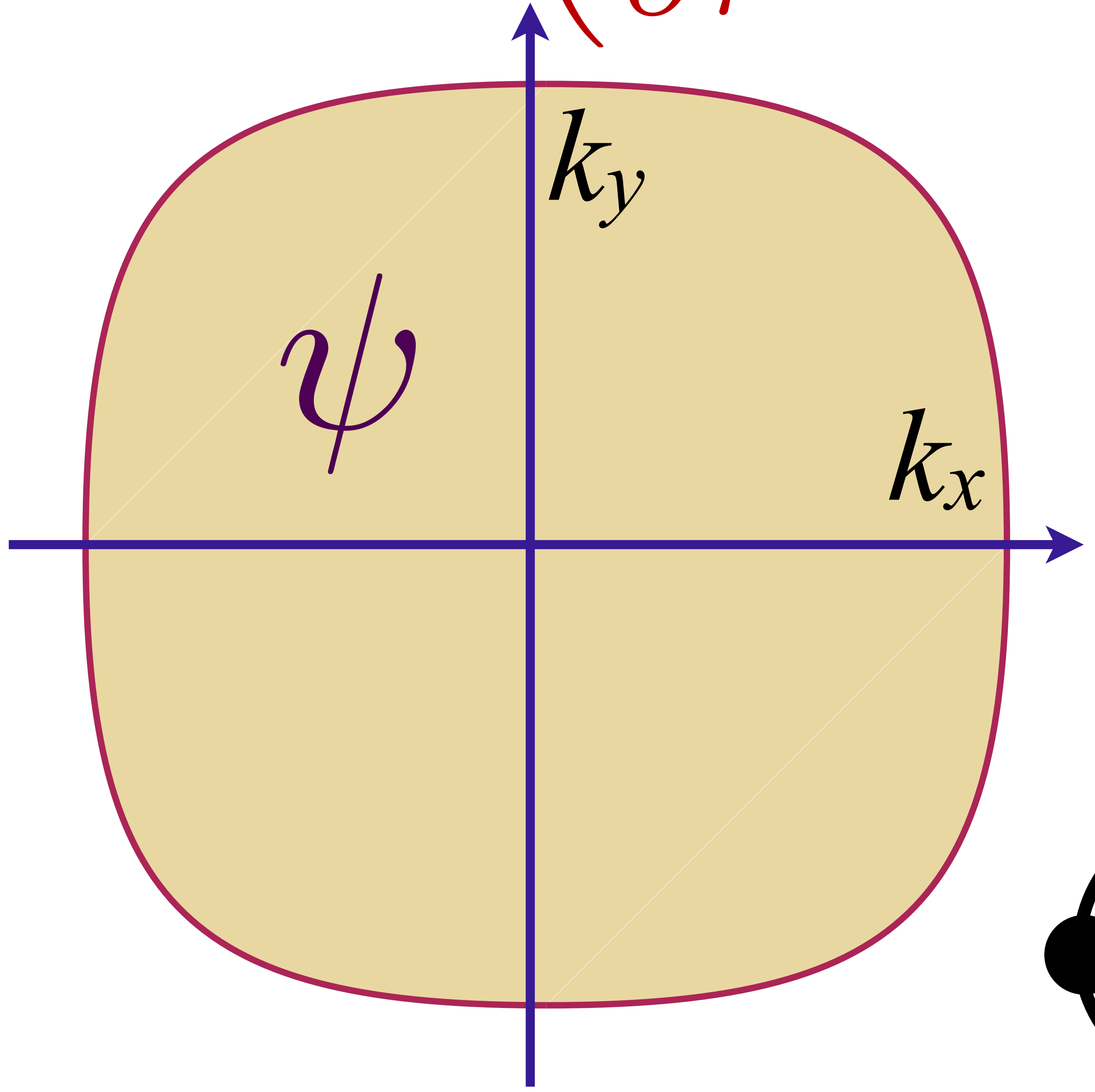
D. L. Maslov, V. I. Yudson, and A. V. Chubukov PRL **106**, 106403 (2011)

S. A. Hartnoll, R. Mahajan, M. Punk, and S.S. PRB **89**, 155130 (2014)

A. Eberlein, I. Mandal, and S.S. PRB **94**, 045133 (2016)

Fermi surface + critical boson

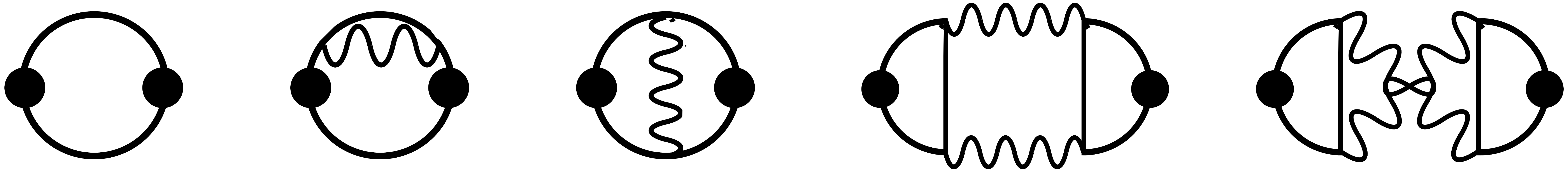
$$\mathcal{L}_\psi = \psi_{\boldsymbol{k}}^\dagger \left(\frac{\partial}{\partial \tau} + \varepsilon(\boldsymbol{k}) \right) \psi_{\boldsymbol{k}}$$



a critical boson ϕ
e.g. Ising-nematic order

$$\frac{[\phi(\boldsymbol{r})]^2}{J} + \psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})\,\phi(\boldsymbol{r})$$

Optical conductivity—Diagrams

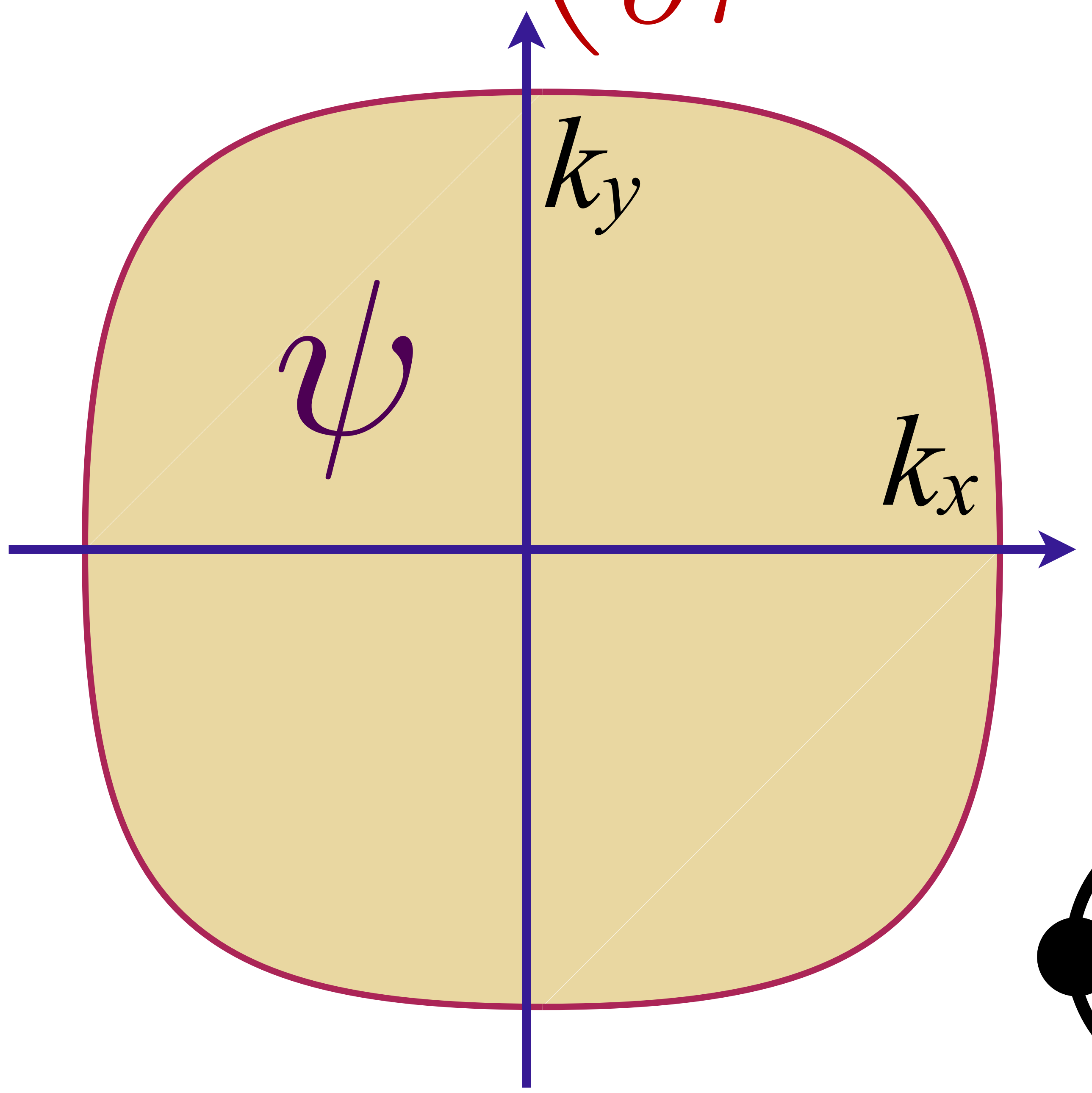


$$\text{Re}\left[\sigma(\omega)\right] = C\,|\omega|^{-2/3}$$

Yong Baek Kim, A. Furusaki, Xiao-Gang Wen, and P. A. Lee, PRB **50**, 17917 (1994).

Fermi surface + critical boson

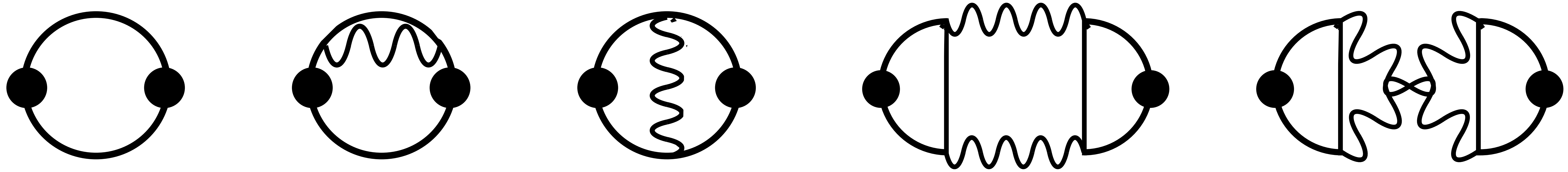
$$\mathcal{L}_\psi = \psi^\dagger_{\boldsymbol{k}} \left(\frac{\partial}{\partial \tau} + \varepsilon(\boldsymbol{k}) \right) \psi_{\boldsymbol{k}}$$



a critical boson ϕ
e.g. Ising-nematic order

$$\frac{[\phi(\boldsymbol{r})]^2}{J} + \psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})\,\phi(\boldsymbol{r})$$

Optical conductivity—Diagrams



$$\text{Re}\,[\sigma(\omega)] = C\,|\omega|^{-2/3}$$

Yong Baek Kim, A. Furusaki, Xiao-Gang Wen, and P. A. Lee, PRB **50**, 17917 (1994).

$$C = 0\,; \quad \sigma(\omega) \sim i/(\omega) + \omega^0 + \ldots$$

Haoyu Guo, Aavishkar Patel, Ilya Esterlis, S.S. PRB **106**, 115151 (2022)
Z. Darius Shi, D.V. Else, H. Goldman and T. Senthil, arXiv:2208.04328

Fermi surface coupled to a critical boson: No spatial disorder

A non-Fermi liquid but NO strange metal transport

Fermi surface coupled to a critical boson:

No spatial disorder

A non-Fermi liquid but NO strange metal transport

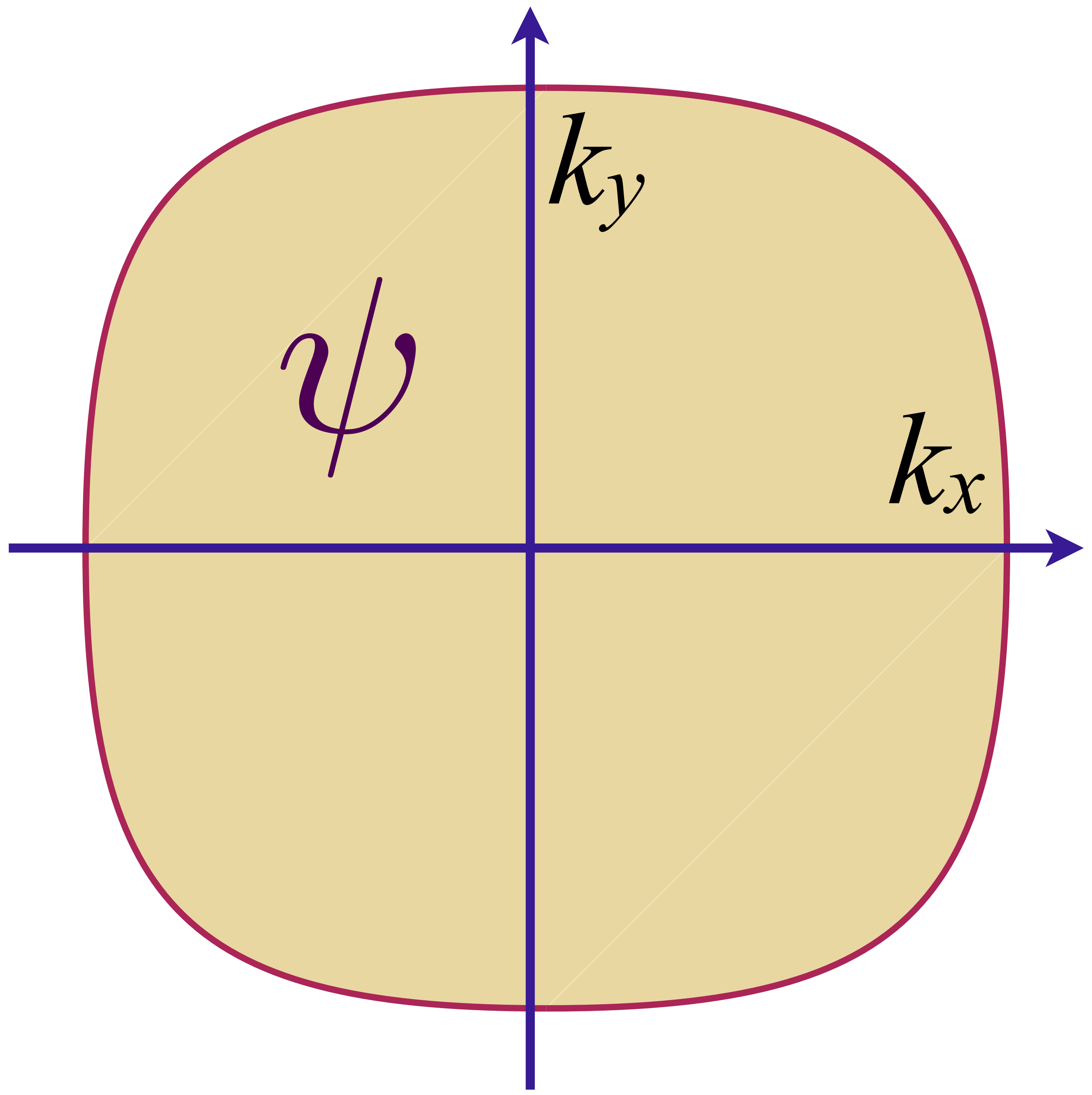
Fermi surface coupled to a critical boson:

Potential disorder v

A marginal Fermi liquid but NO strange metal transport

Fermi surface + critical boson with potential disorder

$$\mathcal{L}_\psi = \psi_{\boldsymbol{k}}^\dagger \left(\frac{\partial}{\partial \tau} + \varepsilon(\boldsymbol{k}) \right) \psi_{\boldsymbol{k}}$$



a critical boson ϕ

e.g. Ising-nematic order

$$\frac{[\phi(\boldsymbol{r})]^2}{J} + \psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})\,\phi(\boldsymbol{r})$$

$$+v(\boldsymbol{r})\psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})$$

Spatially random potential $v(\boldsymbol{r})$ with $\overline{v(\boldsymbol{r})} = 0$, $\overline{v(\boldsymbol{r})v(\boldsymbol{r}')} = v^2\delta(\boldsymbol{r} - \boldsymbol{r}')$

Fermi surface + critical boson with potential disorder

All results are obtained from the large N saddle-point and response functions of this G-Σ-D-Π theory:

$$\mathcal{Z} = \int \mathcal{D}G\,\mathcal{D}\Sigma\,\mathcal{D}D\,\mathcal{D}\Pi \exp(-N S_{\text{all}})$$

$$S_{\text{all}} = -\ln\det(\partial_\tau + \varepsilon(\mathbf{k}) - \mu + \Sigma) + \frac{1}{2}\ln\det(-\partial_\tau^2 + \mathbf{q}^2 + m_b^2 - \Pi)$$
$$+ \int d\tau d^2r \int d\tau' d^2r' \Big[-\Sigma(\tau', \mathbf{r}'; \tau, \mathbf{r}) G(\tau, \mathbf{r}; \tau', \mathbf{r}') + \frac{1}{2}\Pi(\tau', \mathbf{r}'; \tau, \mathbf{r}) D(\tau, \mathbf{r}; \tau', \mathbf{r}')$$
$$+ \frac{g^2}{2} G(\tau, \mathbf{r}; \tau', \mathbf{r}') G(\tau', \mathbf{r}'; \tau, \mathbf{r}) D(\tau, \mathbf{r}; \tau', \mathbf{r}') + \frac{v^2}{2} G(\tau, \mathbf{r}; \tau', \mathbf{r}') G(\tau', \mathbf{r}'; \tau, \mathbf{r}) \delta(\mathbf{r} - \mathbf{r}') \Big]$$

Fermi surface + critical boson with potential disorder

All results are obtained from the large N saddle-point and response functions of this G-Σ-D-Π theory:

$$\mathcal{Z} = \int \mathcal{D}G\,\mathcal{D}\Sigma\,\mathcal{D}D\,\mathcal{D}\Pi\,\exp(-NS_{\rm all})$$

Saddle-point equations

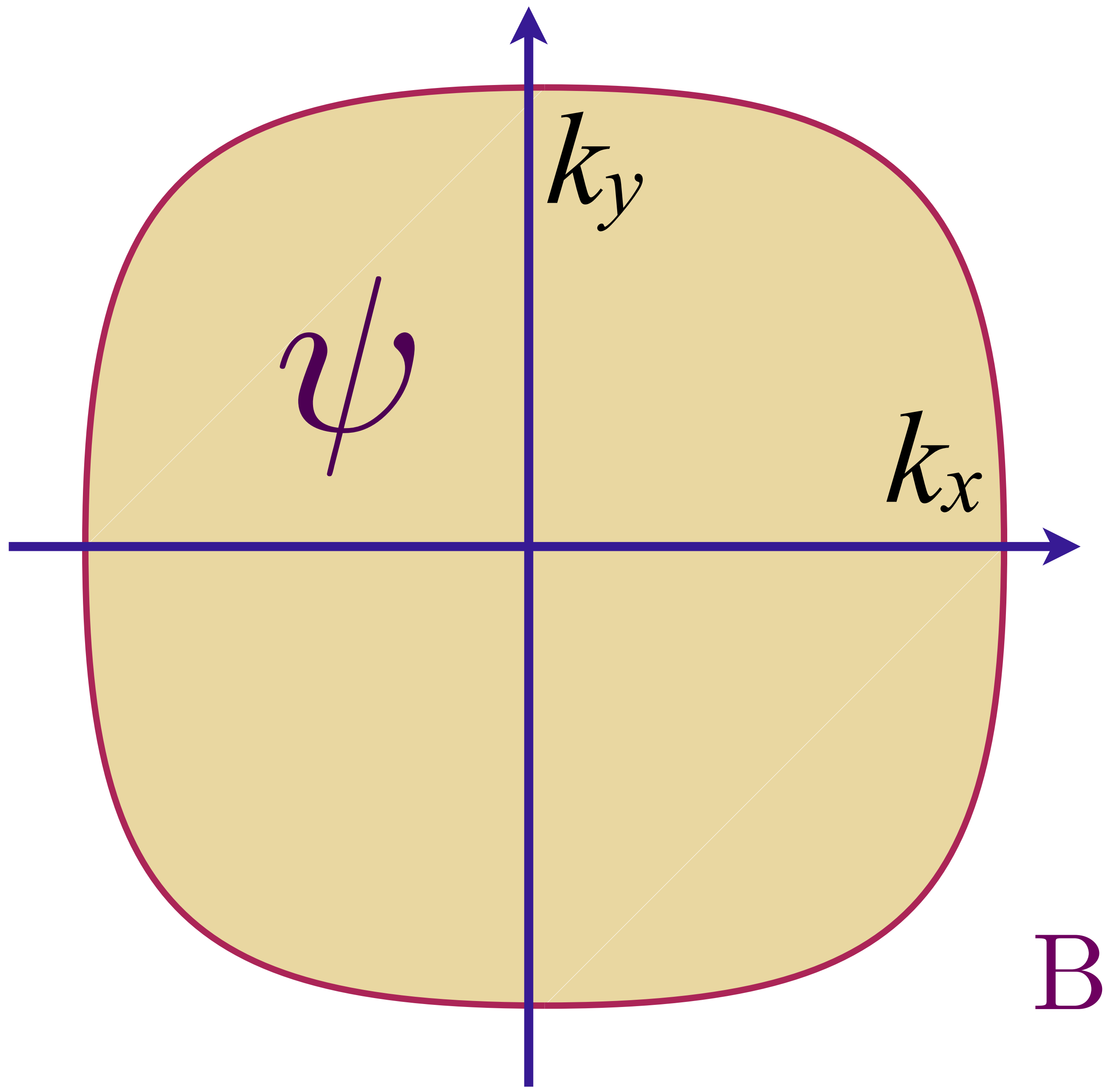
$$\Sigma(\tau,\mathbf{r}) = g^2 D(\tau,\mathbf{r})G(\tau,\mathbf{r}) + v^2 G(\tau,\mathbf{r})\delta^2(\mathbf{r}),$$
$$\Pi(\tau,\mathbf{r}) = -g^2 G(-\tau,-\mathbf{r})G(\tau,\mathbf{r}),$$
$$G(i\omega,\mathbf{k}) = \frac{1}{i\omega - \varepsilon(\mathbf{k}) + \mu - \Sigma(i\omega,\mathbf{k})},$$
$$D(i\Omega,\mathbf{q}) = \frac{1}{\Omega^2 + \mathbf{q}^2 + m_b^2 - \Pi(i\Omega,\mathbf{q})}.$$

Fermi surface + critical boson with potential disorder

$$\mathcal{L}_\psi = \psi^\dagger_{\boldsymbol{k}} \left(\frac{\partial}{\partial \tau} + \varepsilon(\boldsymbol{k}) \right) \psi_{\boldsymbol{k}}$$

a critical boson ϕ

e.g. Ising-nematic order



$$\frac{[\phi(\boldsymbol{r})]^2}{J} + \psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})\,\phi(\boldsymbol{r})$$
$$+v(\boldsymbol{r})\psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})$$

Boson self energy: $\Pi \sim -\dfrac{g^2}{v^2}|\Omega|$, $\quad D(q, i\Omega) = \dfrac{1}{q^2 + \gamma|\Omega|}$

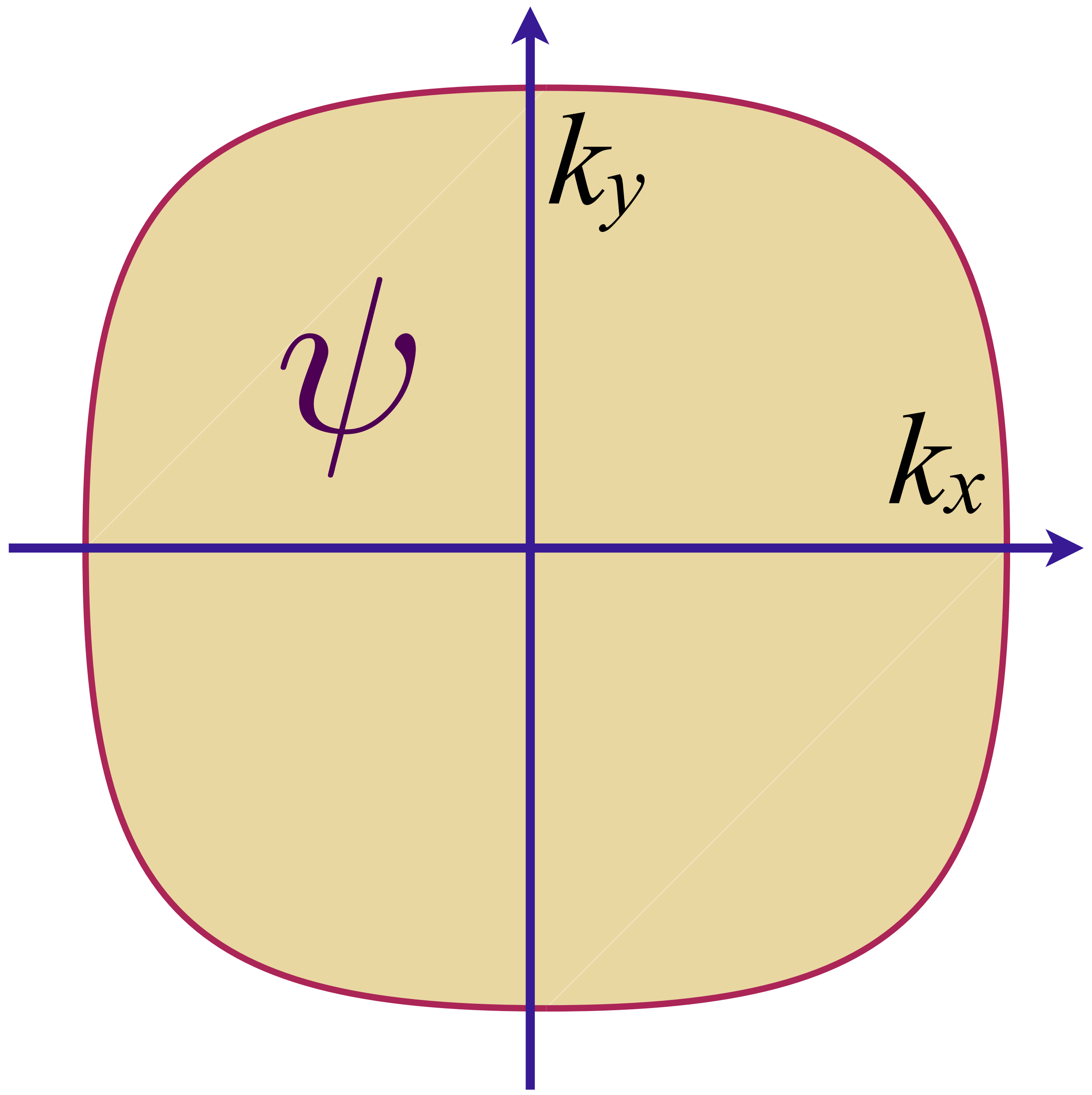
Fermion self energy: $\Sigma(i\omega) \sim -iv^2 \text{sgn}(\omega) - i\dfrac{g^2}{v^2}\omega \ln(1/|\omega|)$; $\quad \dfrac{1}{\tau_{\text{in}}(\varepsilon)} \sim |\varepsilon|$

Marginal Fermi liquid self energy and $T \ln(1/T)$ specific heat

Fermi surface + critical boson with potential disorder

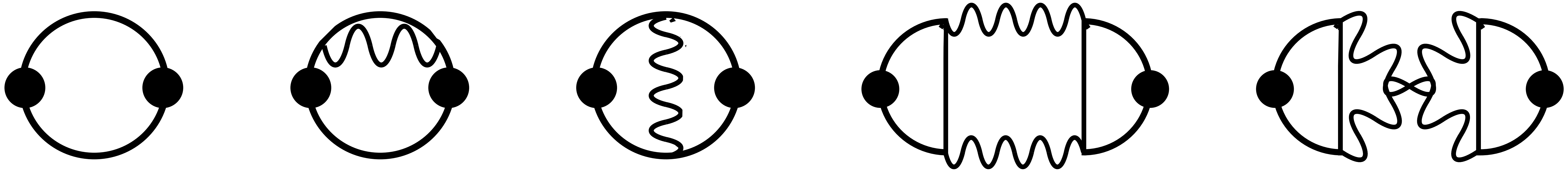
$$\mathcal{L}_\psi = \psi^\dagger_{\boldsymbol{k}} \left(\frac{\partial}{\partial \tau} + \varepsilon(\boldsymbol{k}) \right) \psi_{\boldsymbol{k}}$$

a critical boson ϕ
e.g. Ising-nematic order



$$\frac{[\phi(\boldsymbol{r})]^2}{J} + \psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})\,\phi(\boldsymbol{r})$$

$$+ v(\boldsymbol{r})\psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})$$



Conductivity: $\sigma(\omega) \sim \dfrac{1}{\dfrac{1}{\tau_{\rm trans}} - i\omega}$; $\dfrac{1}{\tau_{\rm trans}} \sim v^2$

MFL self-energy cancels in transport.

Fermi surface coupled to a critical boson:

<u>No spatial disorder</u>

A non-Fermi liquid but NO strange metal transport

Fermi surface coupled to a critical boson:

<u>Potential disorder v</u>

A marginal Fermi liquid but NO strange metal transport

g'

Fermi surface coupled to a critical boson:

No spatial disorder

A non-Fermi liquid but NO strange metal transport

Fermi surface coupled to a critical boson:

Potential disorder v

A marginal Fermi liquid but NO strange metal transport

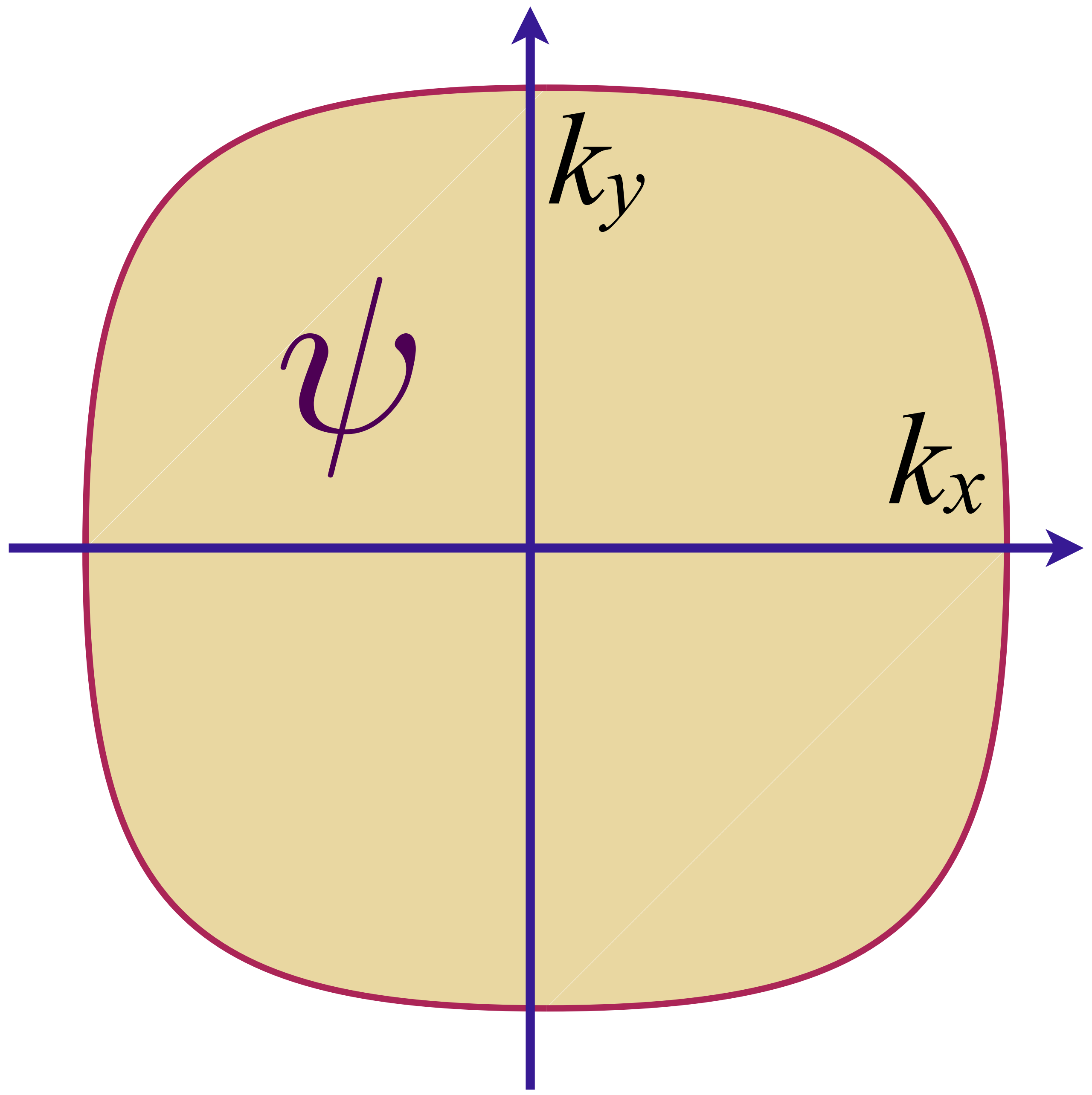
Fermi surface coupled to a critical boson:

Interaction disorder g'

A marginal Fermi liquid AND strange metal transport

Fermi surface + critical boson with potential disorder

$$\mathcal{L}_\psi = \psi_{\boldsymbol{k}}^\dagger \left(\frac{\partial}{\partial\tau} + \varepsilon(\boldsymbol{k}) \right) \psi_{\boldsymbol{k}}$$



a critical boson ϕ
e.g. Ising-nematic order

$$\frac{[\phi(\boldsymbol{r})]^2}{J} + \psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})\,\phi(\boldsymbol{r})$$

$$+v(\boldsymbol{r})\psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})$$

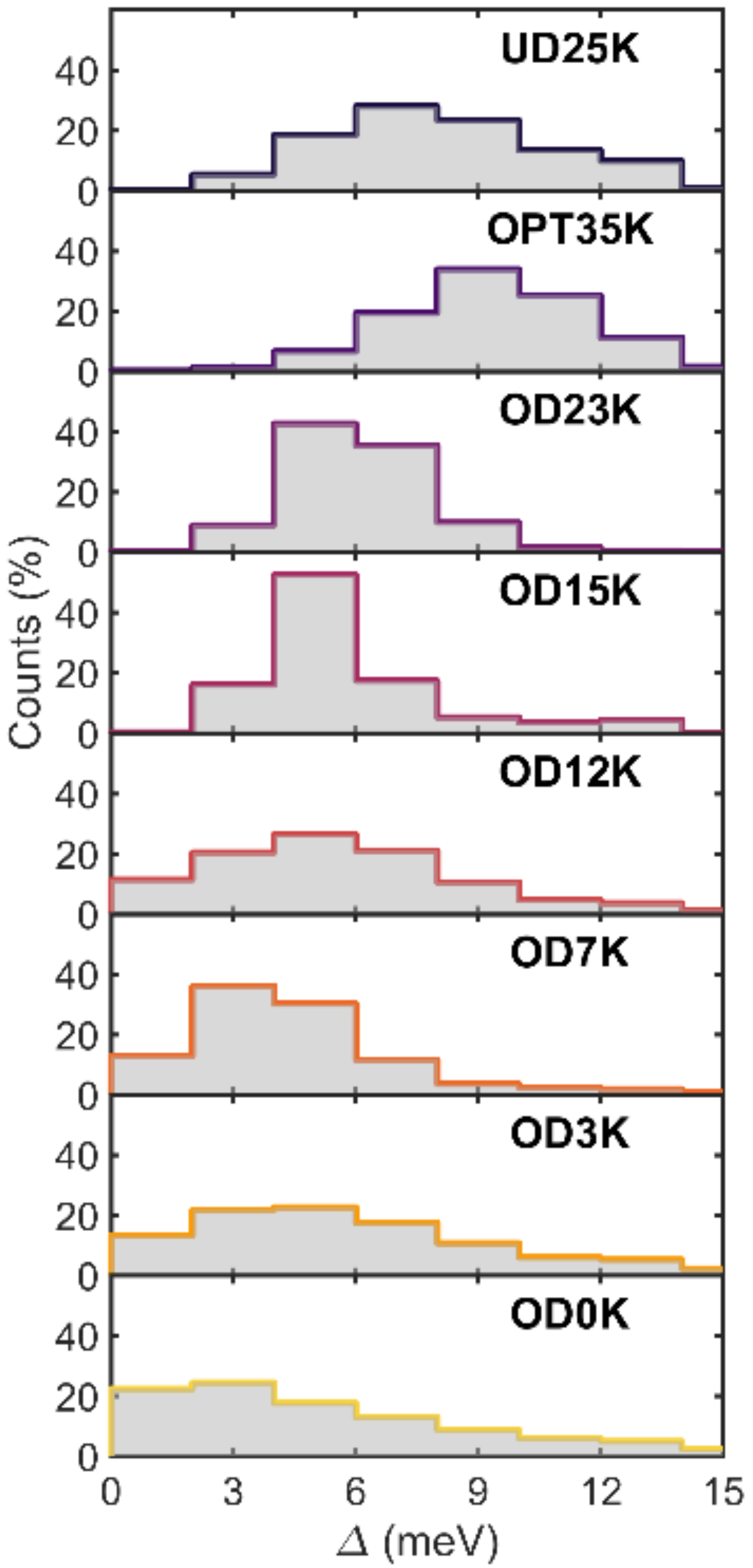
Spatially random interactions!

Puddle formation, persistent gaps, and non-mean-field breakdown of superconductivity in overdoped $(Pb,Bi)_2Sr_2CuO_{6+\delta}$

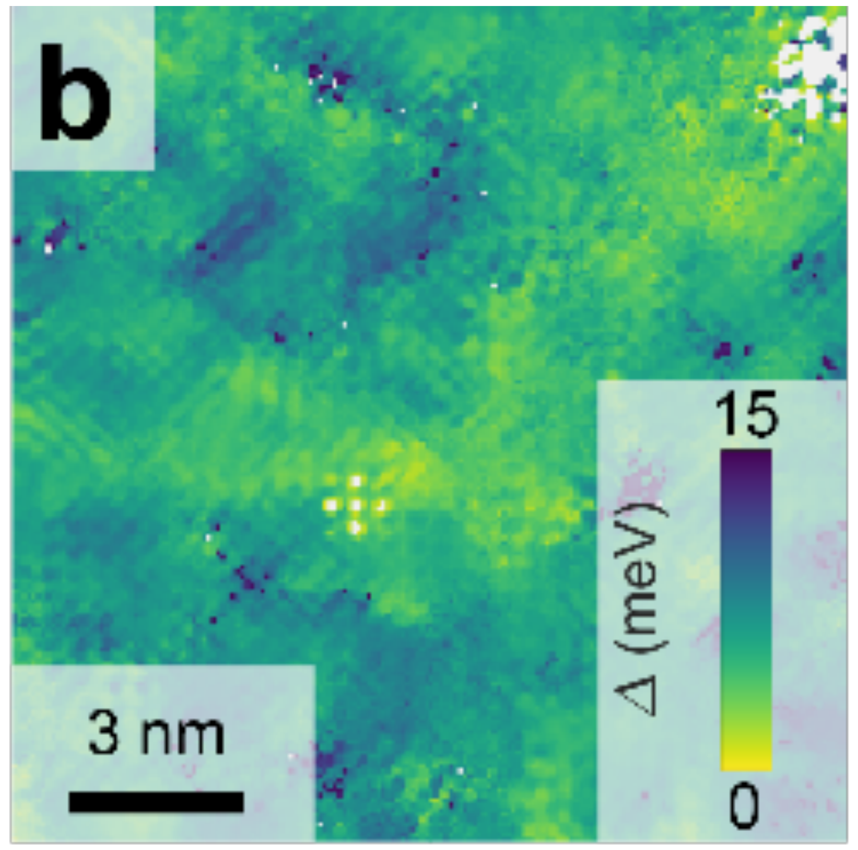
Willem O. Tromp, Tjerk Benschop, Jian-Feng Ge, Irene Battisti, Koen M. Bastiaans, Damianos Chatzopoulos, Amber Vervloet, Steef Smit, Erik van Heumen, Mark S. Golden, Yinkai Huang, Takeshi Kondo, Yi Yin, Jennifer E. Hoffman, Miguel Antonio Sulangi, Jan Zaanen, Milan P. Allan

Our scanning tunneling spectroscopy measurements in the overdoped regime of the $(Pb,Bi)_2Sr_2CuO_{6+\delta}$ high-temperature superconductor show the emergence of puddled superconductivity, featuring nanoscale superconducting islands in a metallic matrix

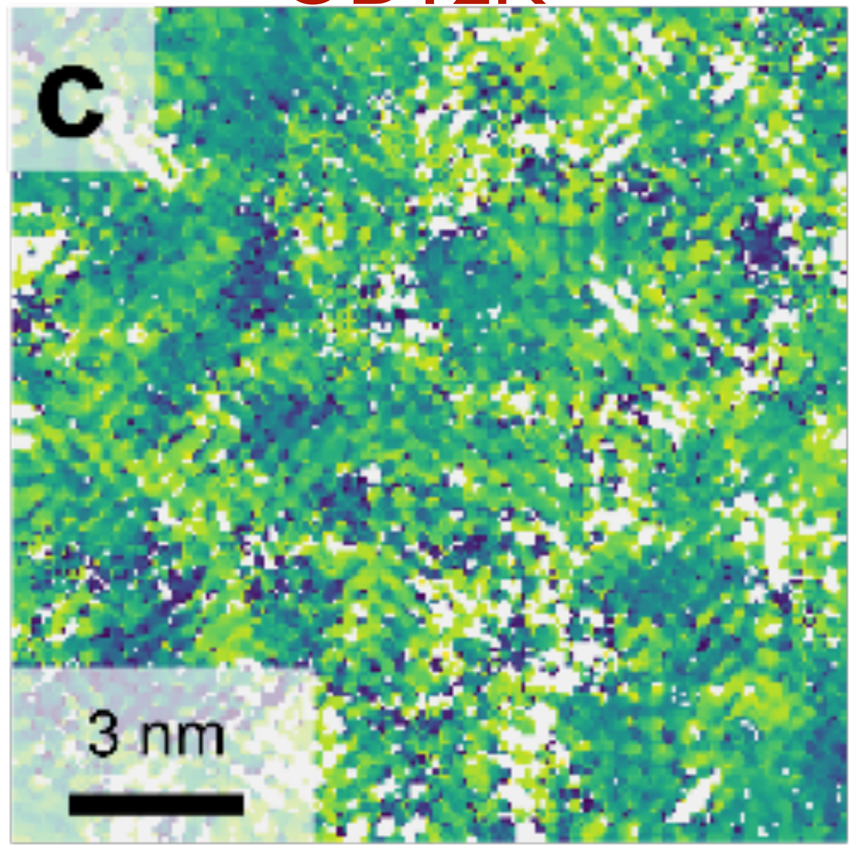
arXiv:2205.09740



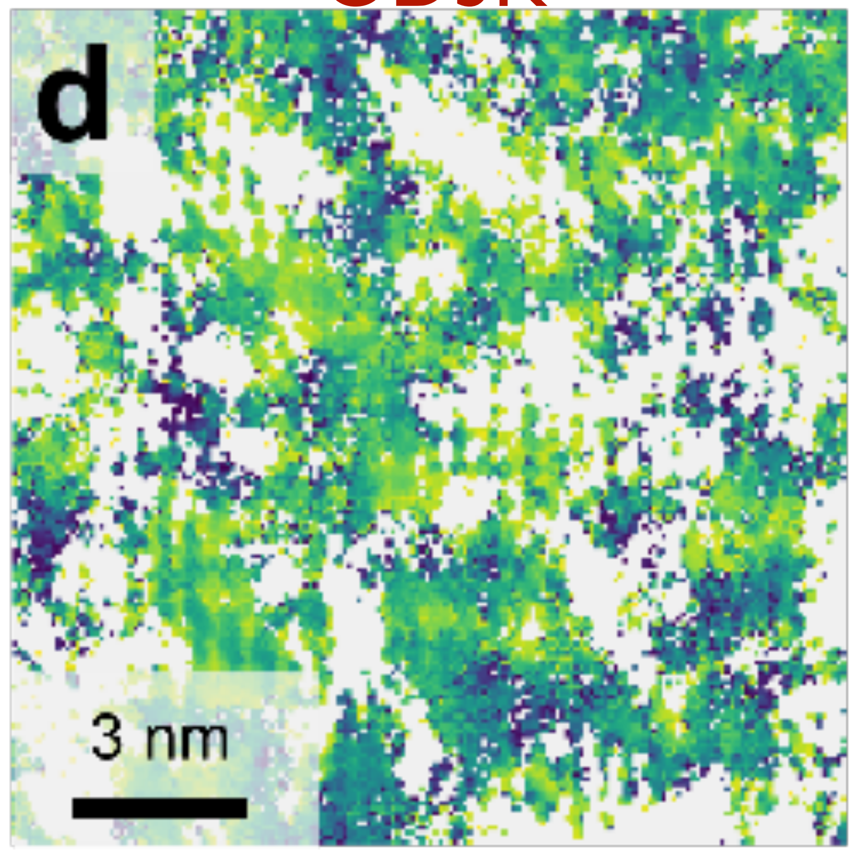
OD23K



OD12K



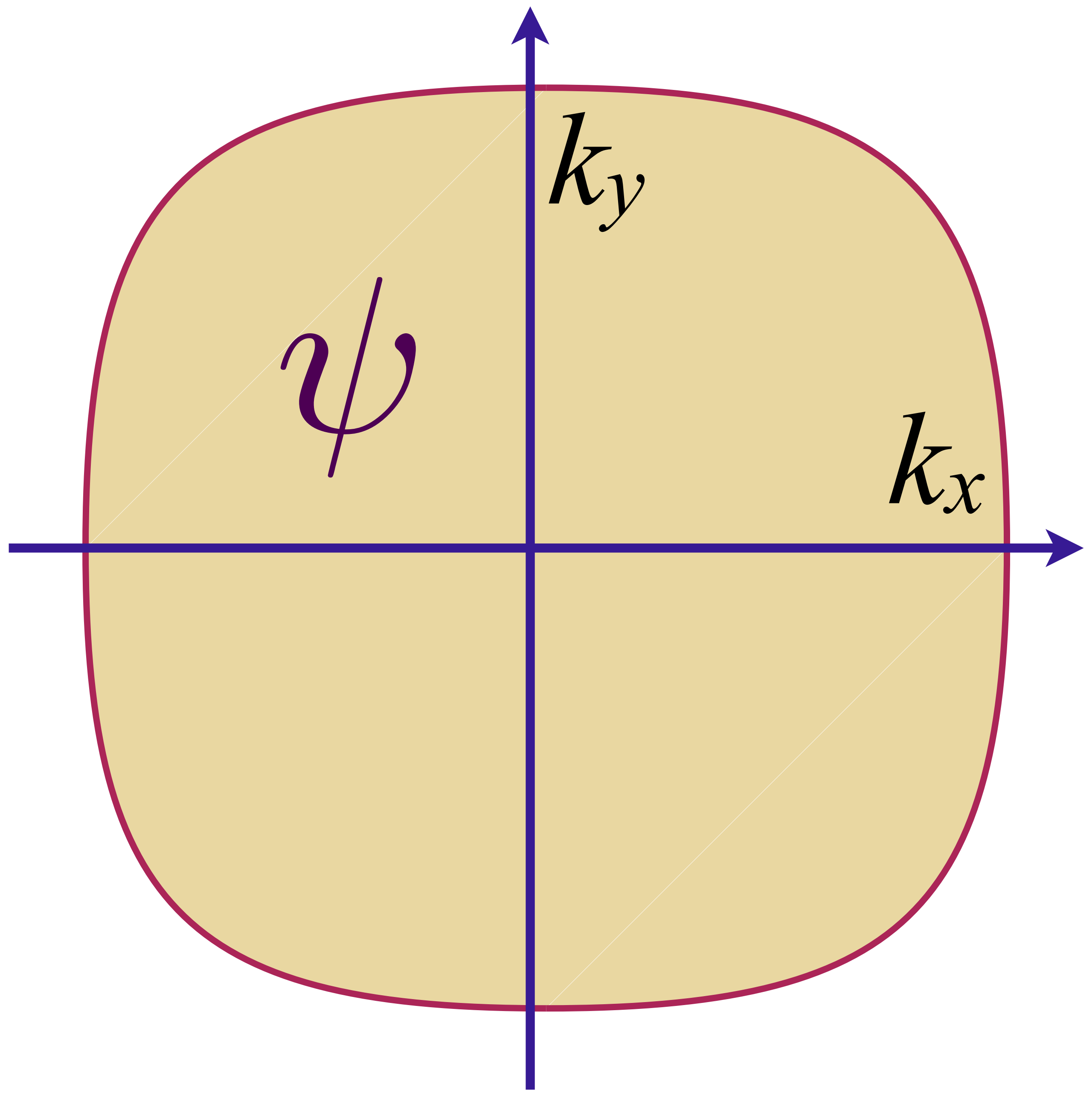
OD3K



Fermi surface + critical boson with potential and interaction disorder

$$\mathcal{L}_\psi = \psi_{\boldsymbol{k}}^\dagger \left(\frac{\partial}{\partial \tau} + \varepsilon(\boldsymbol{k}) \right) \psi_{\boldsymbol{k}}$$

a critical boson ϕ
e.g. Ising-nematic order



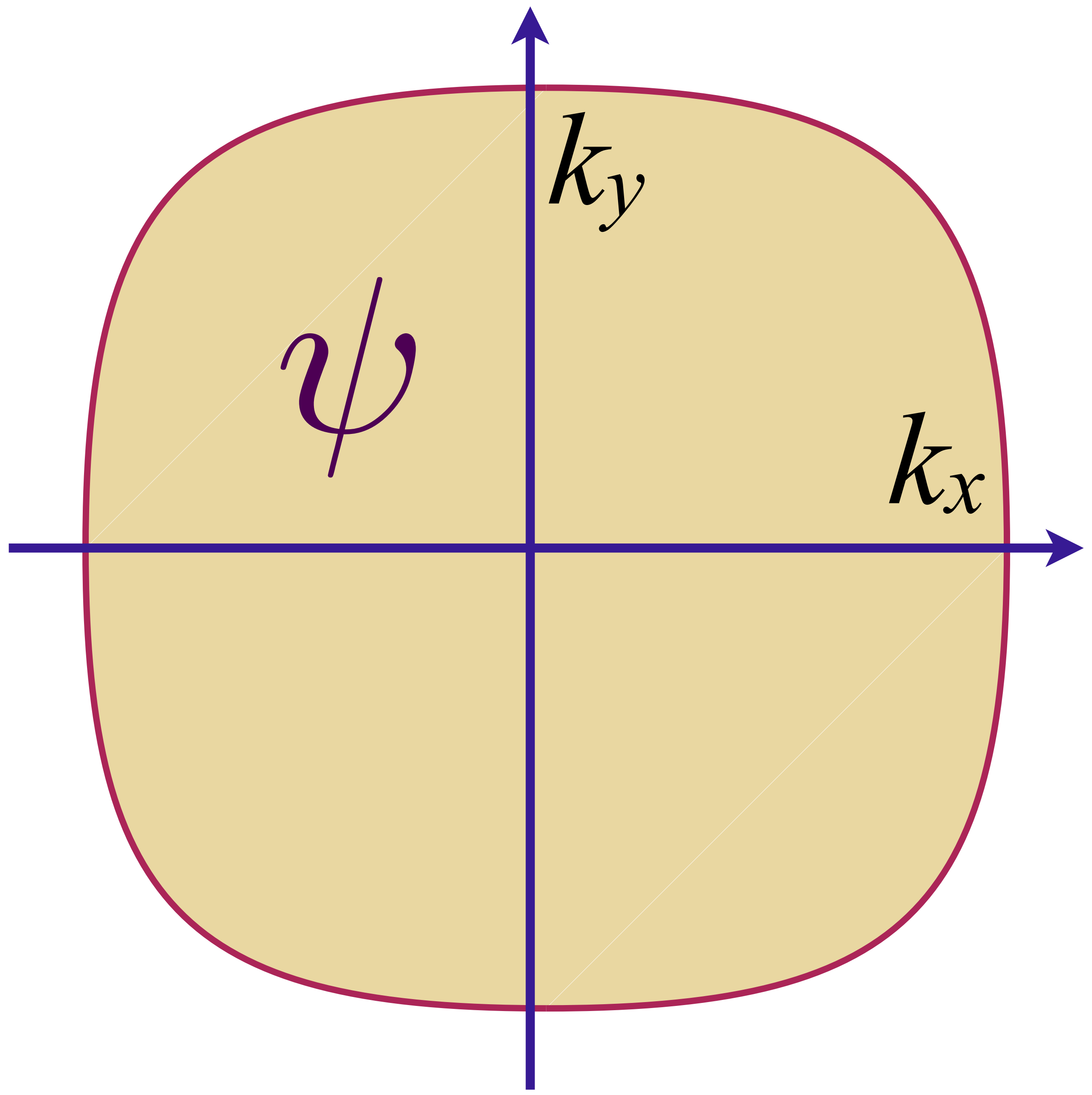
$$\frac{[\phi(\boldsymbol{r})]^2}{J + J'(\boldsymbol{r})} + \psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})\,\phi(\boldsymbol{r})$$

$$+v(\boldsymbol{r})\psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})$$

Fermi surface + critical boson with potential and interaction disorder

$$\mathcal{L}_\psi = \psi_{\boldsymbol{k}}^\dagger \left(\frac{\partial}{\partial \tau} + \varepsilon(\boldsymbol{k}) \right) \psi_{\boldsymbol{k}}$$

a critical boson ϕ
e.g. Ising-nematic order



$$[\phi(\boldsymbol{r})]^2 + [g + g'(\boldsymbol{r})]\, \psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})\, \phi(\boldsymbol{r})$$

$$+ v(\boldsymbol{r})\psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})$$

ϕ^2 "mass" disorder $J'(\boldsymbol{r})$ is strongly relevant;
rescale ϕ to move disorder to the Yukawa coupling;

Spatially random Yukawa coupling $g'(\boldsymbol{r})$ with $\overline{g'(\boldsymbol{r})} = 0$, $\overline{g'(\boldsymbol{r})g'(\boldsymbol{r}')} = g'^2\delta(\boldsymbol{r} - \boldsymbol{r}')$

Spatially random potential $v(\boldsymbol{r})$ with $\overline{v(\boldsymbol{r})} = 0$, $\overline{v(\boldsymbol{r})v(\boldsymbol{r}')} = v^2\delta(\boldsymbol{r} - \boldsymbol{r}')$

Fermi surface coupled to a critical boson with disorder

All results are obtained from the large N saddle-point and response functions of this G-Σ-D-Π theory:

$$\mathcal{Z} = \int \mathcal{D}G\,\mathcal{D}\Sigma\,\mathcal{D}D\,\mathcal{D}\Pi \exp(-NS_{\rm all})$$

$$\begin{aligned} S_{\rm all} &= -\ln\det(\partial_\tau + \varepsilon(\mathbf{k}) - \mu + \Sigma) + \frac{1}{2}\ln\det(-\partial_\tau^2 + \mathbf{q}^2 + m_b^2 - \Pi) \\ &+ \int d\tau d^2r \int d\tau' d^2r' \Big[-\Sigma(\tau',\mathbf{r}';\tau,\mathbf{r})G(\tau,\mathbf{r};\tau',\mathbf{r}') + \frac{1}{2}\Pi(\tau',\mathbf{r}';\tau,\mathbf{r})D(\tau,\mathbf{r};\tau',\mathbf{r}') \\ &+ \frac{g^2}{2}G(\tau,\mathbf{r};\tau',\mathbf{r}')G(\tau',\mathbf{r}';\tau,\mathbf{r})D(\tau,\mathbf{r};\tau',\mathbf{r}') + \frac{v^2}{2}G(\tau,\mathbf{r};\tau',\mathbf{r}')G(\tau',\mathbf{r}';\tau,\mathbf{r})\delta(\mathbf{r}-\mathbf{r}') \\ &+ \frac{g'^2}{2}G(\tau,\mathbf{r};\tau',\mathbf{r}')G(\tau',\mathbf{r}';\tau,\mathbf{r})D(\tau,\mathbf{r};\tau',\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}') \Big] . \end{aligned}$$

Fermi surface coupled to a critical boson with disorder

All results are obtained from the large N saddle-point and response functions of this G-Σ-D-Π theory:

$$\mathcal{Z} = \int \mathcal{D}G\,\mathcal{D}\Sigma\,\mathcal{D}D\,\mathcal{D}\Pi\,\exp(-NS_{\text{all}})$$

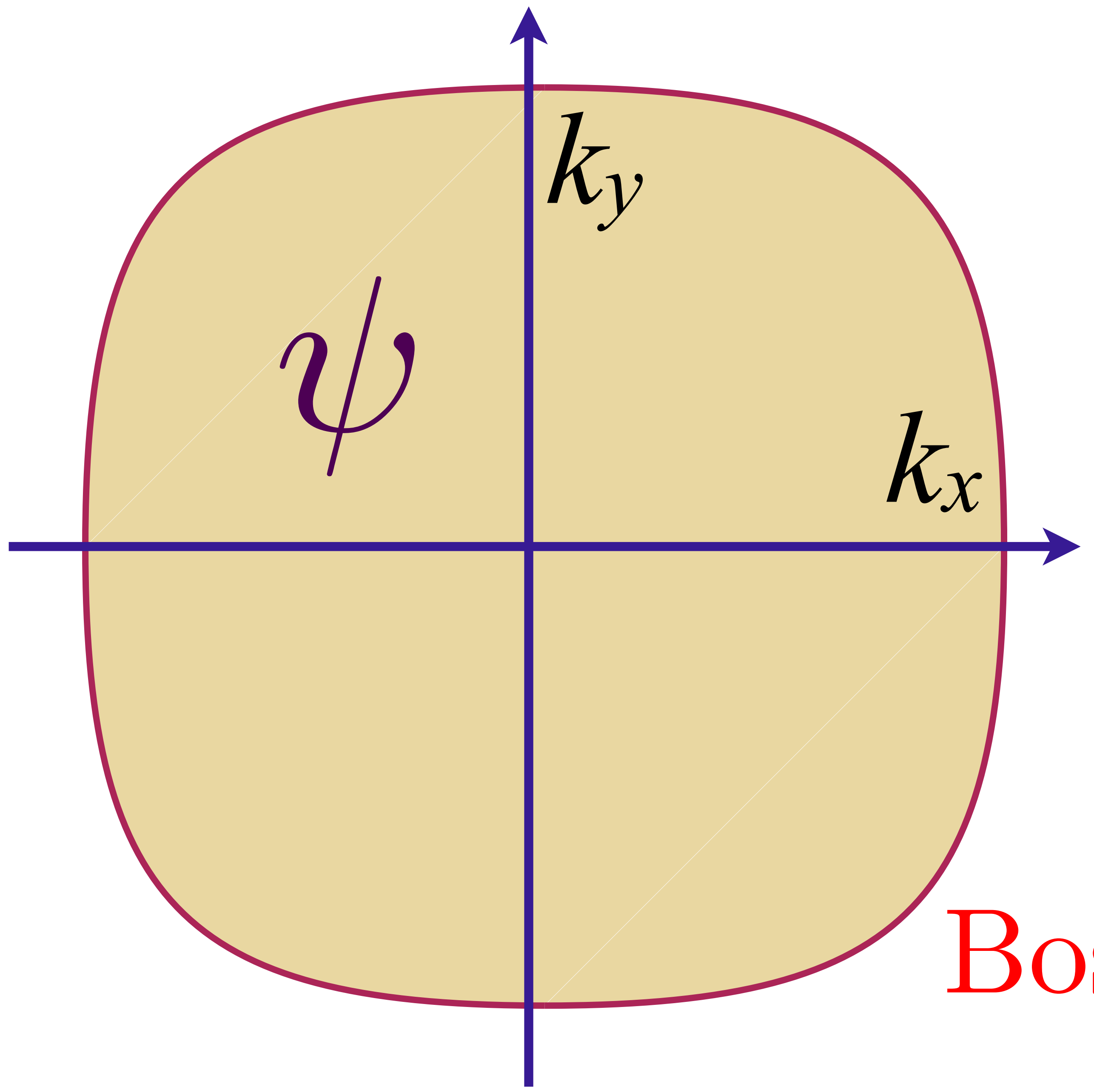
Saddle-point equations

$$\Sigma(\tau,\mathbf{r}) = g^2 D(\tau,\mathbf{r})G(\tau,\mathbf{r}) + v^2 G(\tau,\mathbf{r})\delta^2(\mathbf{r}) + {g'}^2 G(\tau,\mathbf{r})D(\tau,\mathbf{r})\delta^2(\mathbf{r}),$$
$$\Pi(\tau,\mathbf{r}) = -g^2 G(-\tau,-\mathbf{r})G(\tau,\mathbf{r}) - {g'}^2 G(-\tau,\mathbf{r})G(\tau,\mathbf{r})\delta^2(\mathbf{r}),$$
$$G(i\omega,\mathbf{k}) = \frac{1}{i\omega - \varepsilon(\mathbf{k}) + \mu - \Sigma(i\omega,\mathbf{k})},$$
$$D(i\Omega,\mathbf{q}) = \frac{1}{\Omega^2 + \mathbf{q}^2 + m_b^2 - \Pi(i\Omega,\mathbf{q})}.$$

Fermi surface coupled to a critical boson with disorder

$$\mathcal{L}_\psi = \psi_{\boldsymbol{k}}^\dagger \left(\frac{\partial}{\partial \tau} + \varepsilon(\boldsymbol{k}) \right) \psi_{\boldsymbol{k}}$$

a critical boson ϕ
e.g. Ising-nematic order



$$[\phi(\boldsymbol{r})]^2 + [g + g'(\boldsymbol{r})]\, \psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})\, \phi(\boldsymbol{r})$$

$$+ v(\boldsymbol{r})\psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})$$

Boson Green's function: $D(q, i\Omega) \sim 1/(q^2 + \gamma|\Omega|)$

Fermion self energy:

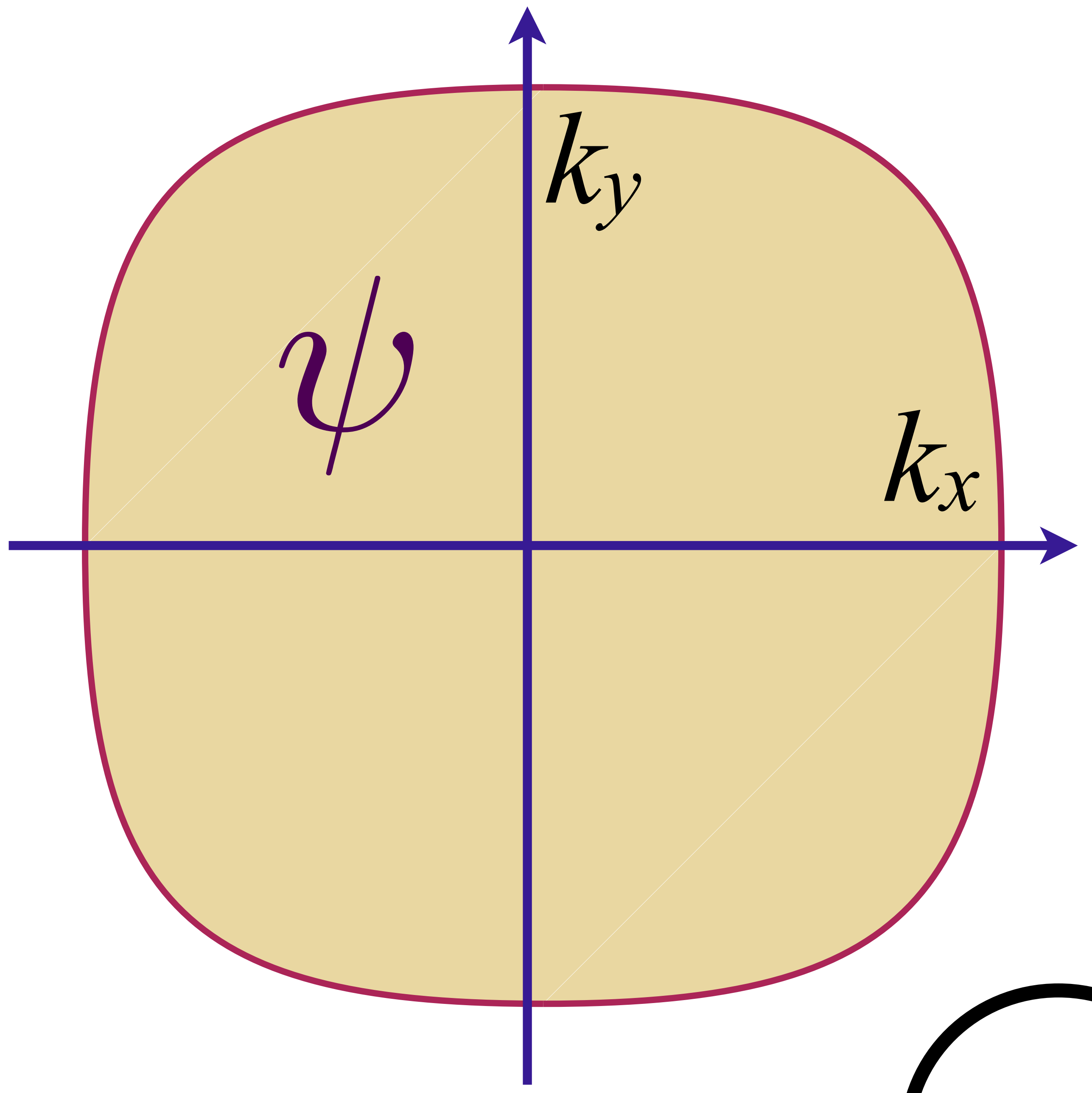
$$\Sigma(i\omega) \sim -iv^2 \mathrm{sgn}(\omega) - i\left(\frac{g^2}{v^2} + g'^2\right) \omega \ln(1/|\omega|); \quad \frac{1}{\tau_{\rm in}(\omega)} \sim \left(\frac{g^2}{v^2} + g'^2\right) |\omega|$$

Marginal Fermi liquid self energy and $T\ln(1/T)$ specific heat

Fermi surface coupled to a critical boson with disorder

$$\mathcal{L}_\psi = \psi^\dagger_{\boldsymbol{k}} \left(\frac{\partial}{\partial \tau} + \varepsilon(\boldsymbol{k}) \right) \psi_{\boldsymbol{k}}$$

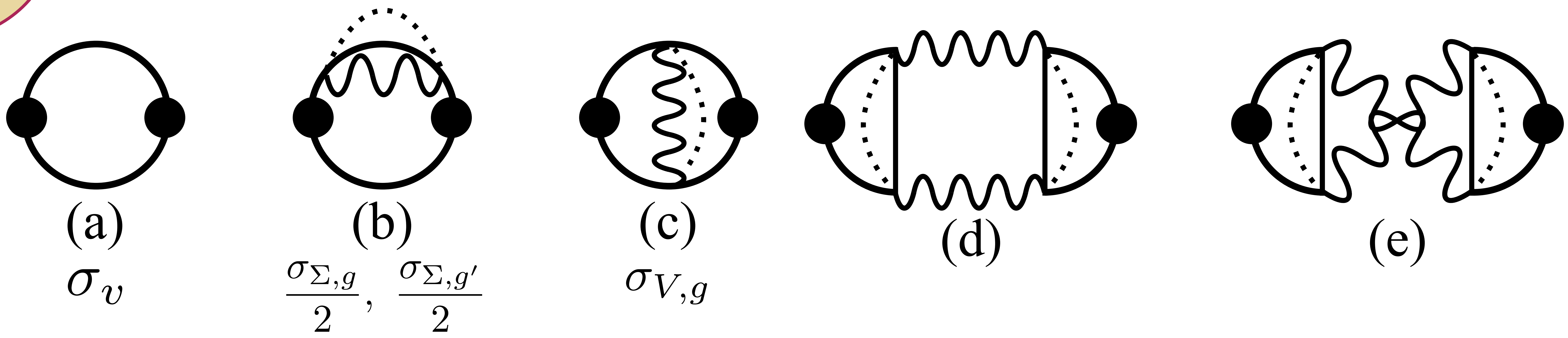
a critical boson ϕ
e.g. Ising-nematic order



$$[\phi(\boldsymbol{r})]^2 + [g + g'(\boldsymbol{r})]\, \psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})\, \phi(\boldsymbol{r})$$

$$+ v(\boldsymbol{r})\psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})$$

Conductivity:



+ all ladders and bubbles…..

Fermi surface coupled to a critical boson with disorder

$$\text{Conductivity: } \sigma(\omega) \sim \frac{1}{\dfrac{1}{\tau_{\rm trans}(\omega)} - i\omega\, \dfrac{m^*_{\rm trans}(\omega)}{m}}$$

$$\frac{1}{\tau_{\rm trans}(\omega)} \sim v^2 + g'^2|\omega| \quad ; \quad \frac{m^*_{\rm trans}(\omega)}{m} \sim \frac{2g'^2}{\pi}\ln(\Lambda/\omega)$$

$$\text{Electron Green's function: } G(\omega) \sim \frac{1}{\omega\, \dfrac{m^*(\omega)}{m} - \varepsilon(\boldsymbol{k}) + i\left(\dfrac{1}{\tau_e} + \dfrac{1}{\tau_{\rm in}(\omega)}\right)\mathrm{sgn}(\omega)}$$

$$\frac{1}{\tau_e} \sim v^2 \quad ; \quad \frac{1}{\tau_{\rm in}(\omega)} \sim \left(\frac{g^2}{v^2} + g'^2\right)|\omega| \quad ; \quad \frac{m^*(\omega)}{m} \sim \frac{2}{\pi}\left(\frac{g^2}{v^2} + g'^2\right)\ln(\Lambda/\omega)$$

Residual resistivity is determined by v^2; Linear-in-T resistivity determined by g'^2; Transport insensitive to g; Marginal Fermi liquid self energy and $T\ln(1/T)$ specific heat.

Fermi surface coupled to a critical boson:

No spatial disorder

A non-Fermi liquid but NO strange metal transport

Fermi surface coupled to a critical boson:

Potential disorder v

A marginal Fermi liquid but NO strange metal transport

Fermi surface coupled to a critical boson:

Interaction disorder g'

A marginal Fermi liquid AND strange metal transport

Transport properties of a strange metal:

1. Resistivity $\rho(T) = \rho_0 + AT + \ldots$ as $T \to 0$
   and $\rho(T) < h/e^2$ (in $d = 2$).
   Metals with $\rho(T) > h/e^2$ are bad metals.

2. Optical conductivity

$$\sigma(\omega) = \frac{K}{\dfrac{1}{\tau_{\text{trans}}(\omega)} - i\omega\dfrac{m^*_{\text{trans}}(\omega)}{m}} \quad ; \quad \frac{1}{\tau_{\text{trans}}(\omega)} \sim |\omega|\Phi_\sigma\left(\frac{\hbar\omega}{k_B T}\right)$$

B. Michon……A. Georges, arXiv:2205.04030

Electronic properties of a marginal Fermi liquid:

1. Photoemission: nearly marginal Fermi liquid electron spectral density:

$$\text{Im}\Sigma(\omega) \sim |\omega|^{2\alpha}\Phi_\Sigma\left(\frac{\hbar\omega}{k_B T}\right) \quad \text{with } \alpha \approx 1/2 \quad ; \quad \frac{1}{\tau_{\text{in}}(\omega)} \sim |\omega|\Phi_\Sigma\left(\frac{\hbar\omega}{k_B T}\right)$$

T.J. Reber….D. Dessau, Nature Communications **10**, 5737 (2019)

2. Specific heat $\sim T\ln(1/T)$ as $T \to 0$.

S.A. Hartnoll and A.P. MacKenzie, RMP (2022)