The SYK model: a window into random quantum spin liquids and black holes

Mathematical Physics Seminar
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ADVANCED STUDY

PHYSICS
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Talk online: sachdev.physics.harvard.edu

1. SYK: a solvable model without quasiparticles

2. Spin liquid and spin glass of $S=1/2$ SU(2) spins

3. Time reparameterization soft mode

4. Charged black holes

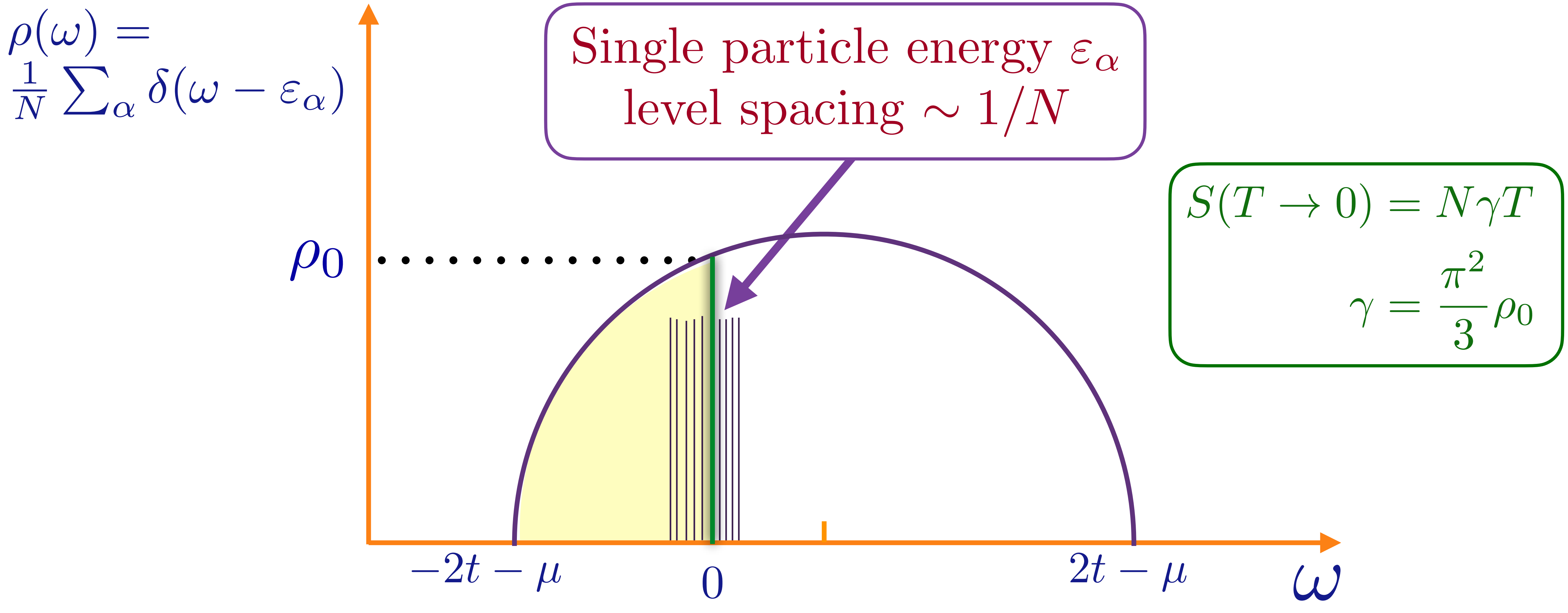
A simple model of a metal with quasiparticles

$$H = \frac{1}{(N)^{1/2}} \sum_{i,j=1}^{N} t_{ij} c_i^\dagger c_j - \mu \sum_i c_i^\dagger c_i$$

$$c_i c_j + c_j c_i = 0 \quad , \quad c_i c_j^\dagger + c_j^\dagger c_i = \delta_{ij}$$

$$\frac{1}{N} \sum_i c_i^\dagger c_i = \mathcal{Q}$$

t_{ij} are independent random variables with $\overline{t_{ij}} = 0$ and $\overline{|t_{ij}|^2} = t^2$



$\rho(\omega) =$
$\frac{1}{N} \sum_\alpha \delta(\omega - \varepsilon_\alpha)$

Single particle energy ε_α
level spacing $\sim 1/N$

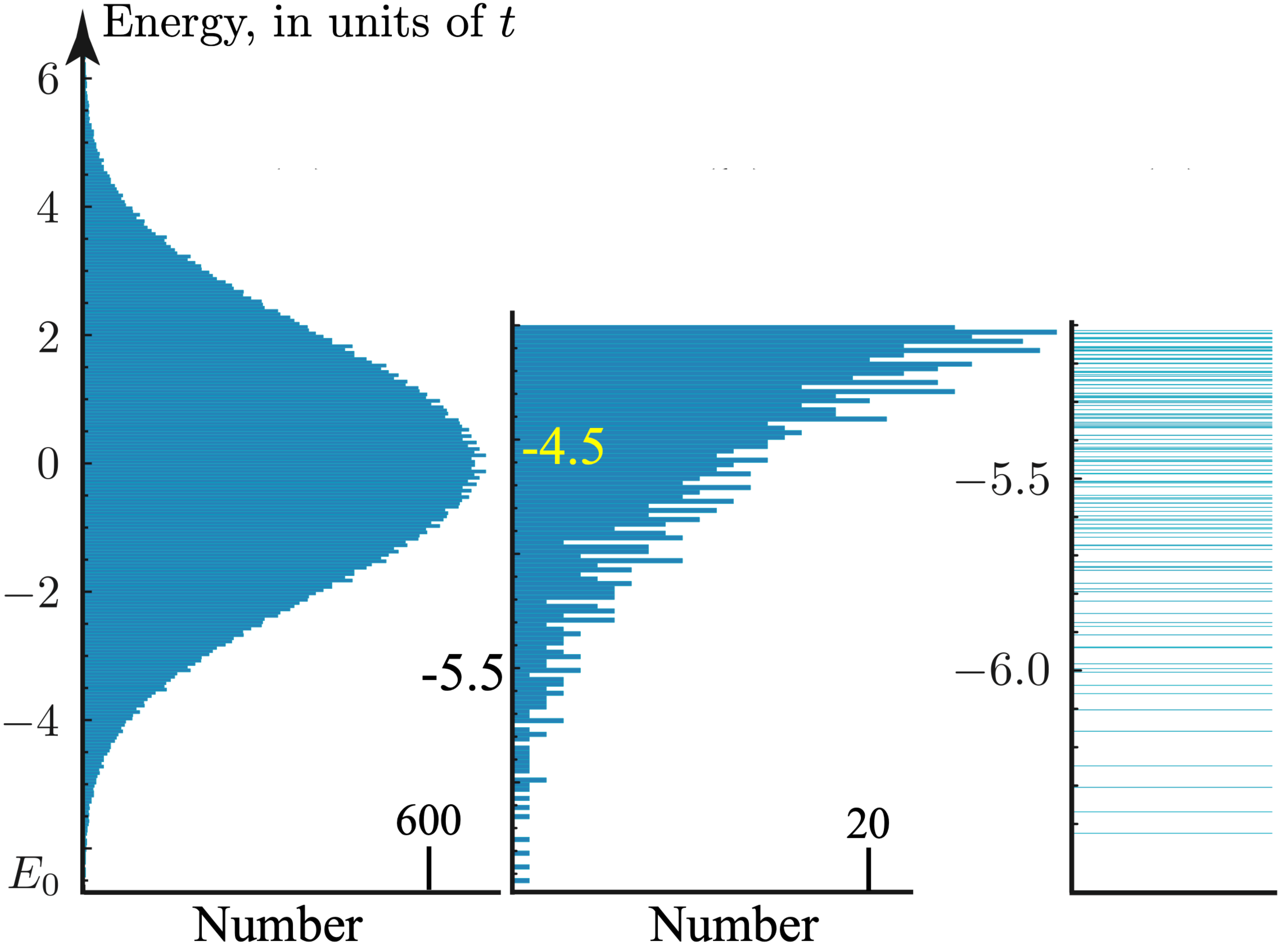
$$S(T \to 0) = N\gamma T$$

$$\gamma = \frac{\pi^2}{3} \rho_0$$

Many-body density of states

$$D(E) = \sum_i \delta(E - E_i); \quad E_0 + E_i \Rightarrow \text{Many body eigenvalue}$$

For random matrix model: $E_0 + E_i = \sum_\alpha n_\alpha \varepsilon_\alpha$ $n_\alpha = 0, 1,$ occupation number



Random matrix model

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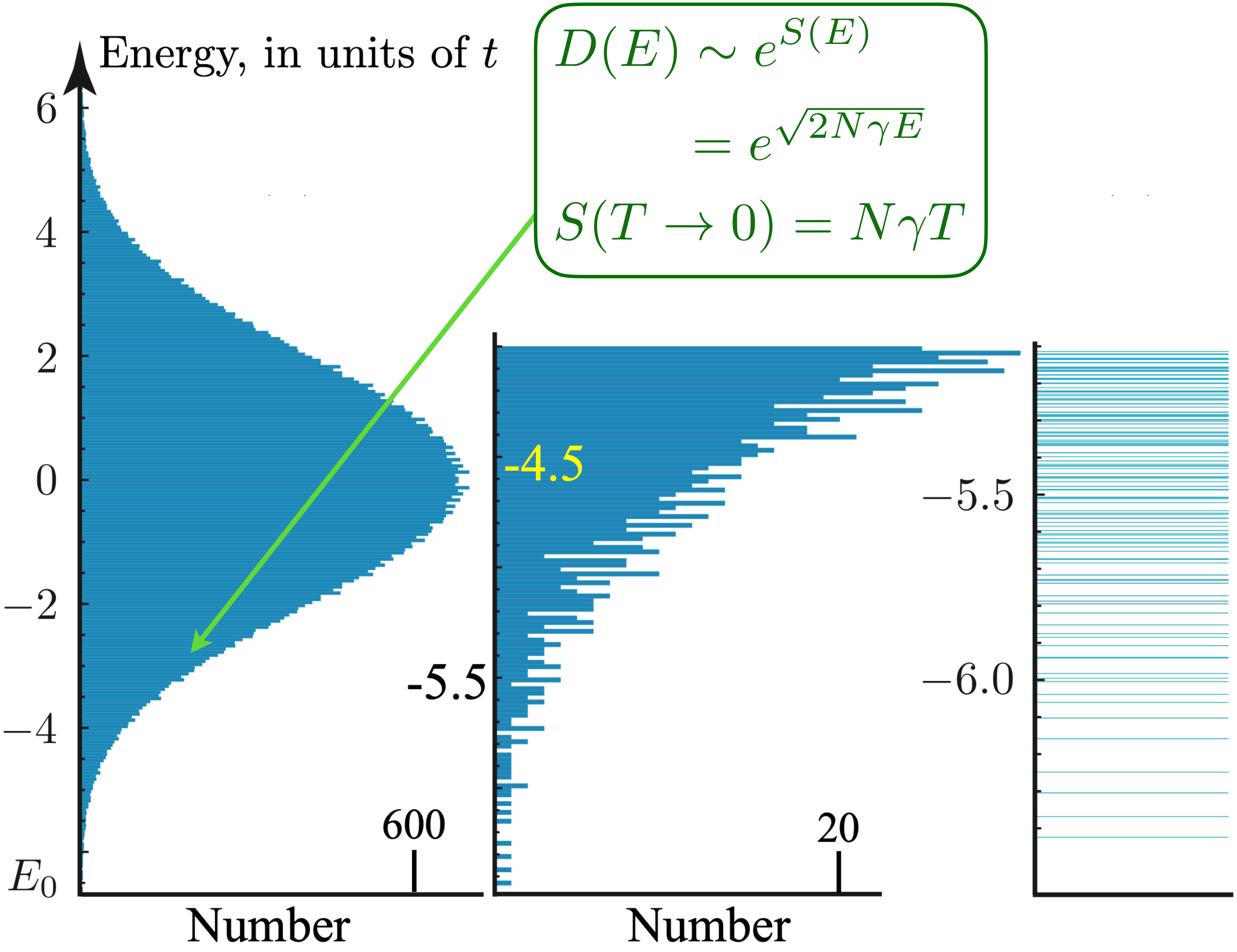

$$D(E) \sim e^{S(E)}$$

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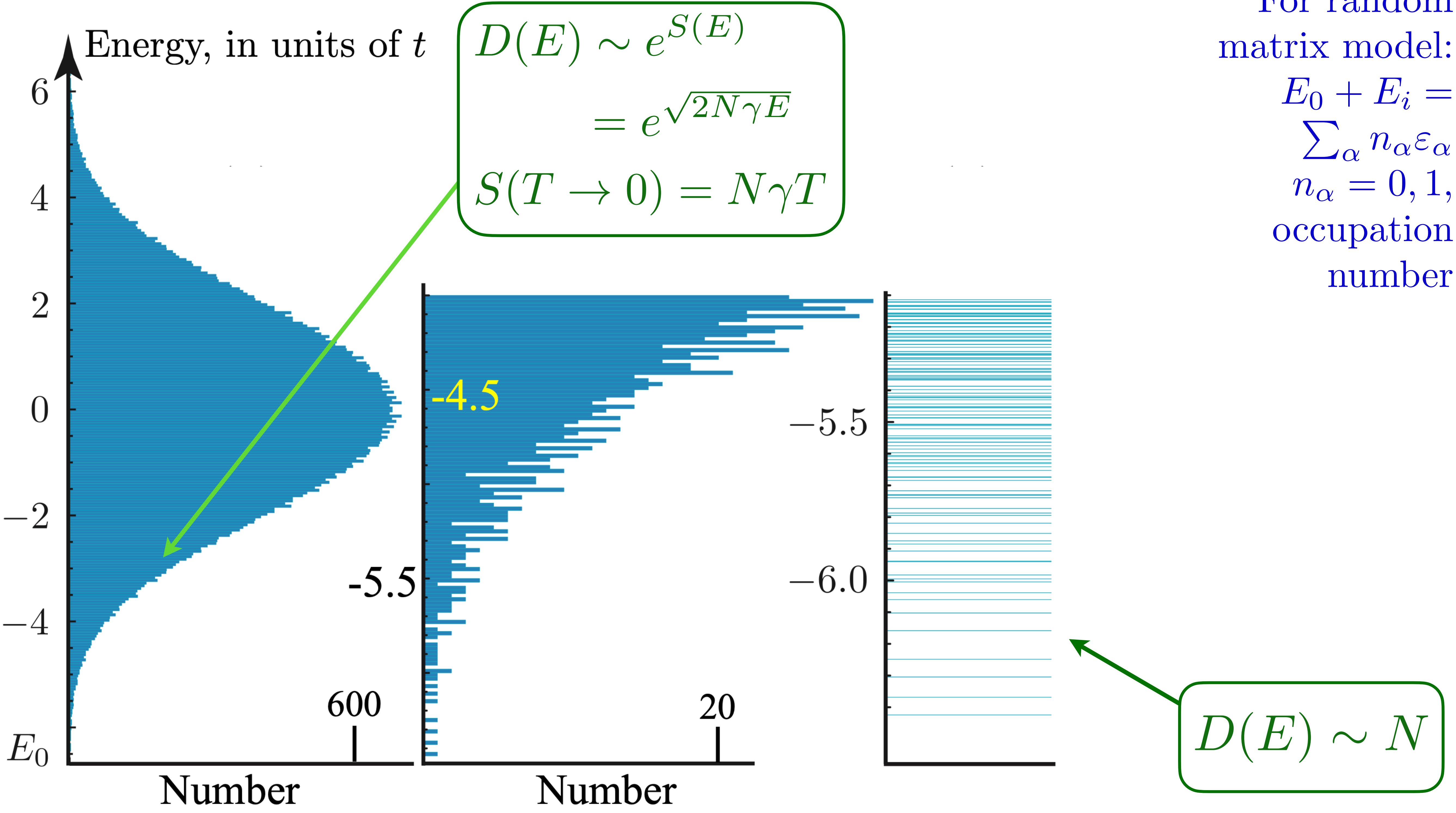

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$$D(E) \sim N$$

Random matrix model

The Sachdev-Ye-Kitaev (SYK) model

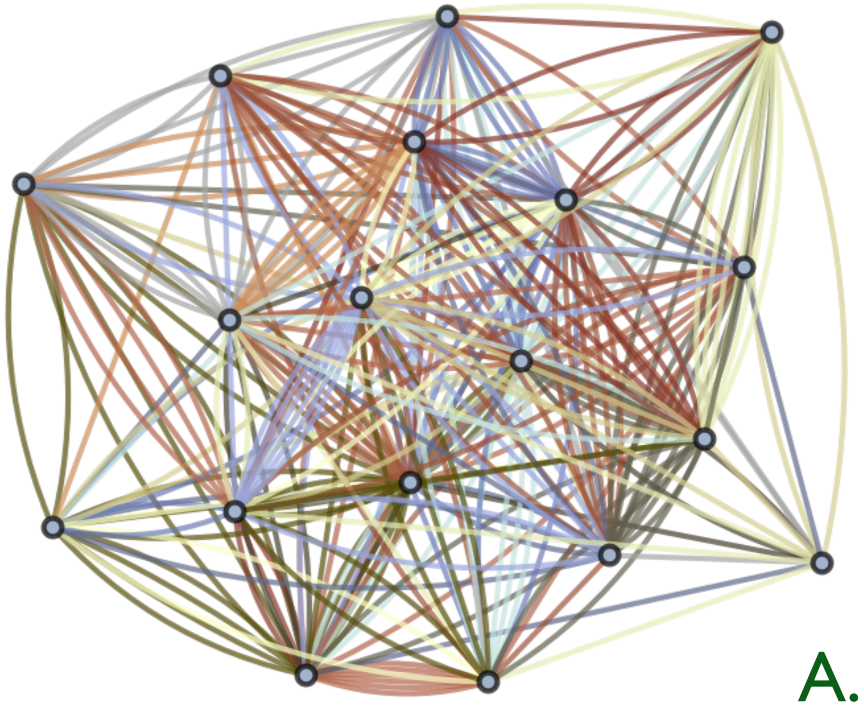
(See also: the "2-Body Random Ensemble" in nuclear physics; did not obtain the large *N* limit;
T.A. Brody, J. Flores, J.B. French, P.A. Mello, A. Pandey, and S.S.M. Wong, Rev. Mod. Phys. **53**, 385 (1981))

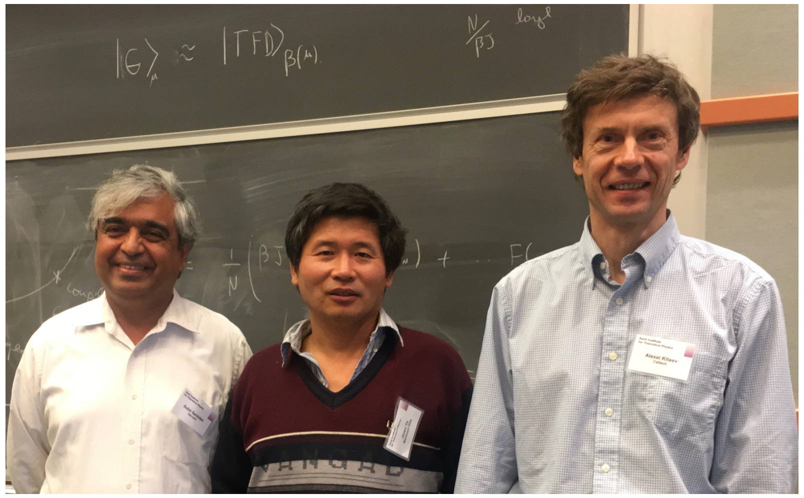
$$H = \frac{1}{(2N)^{3/2}} \sum_{\alpha,\beta,\gamma,\delta=1}^{N} U_{\alpha\beta;\gamma\delta}\, c_\alpha^\dagger c_\beta^\dagger c_\gamma c_\delta - \mu \sum_\alpha c_\alpha^\dagger c_\alpha$$

$$c_\alpha c_\beta + c_\beta c_\alpha = 0 \quad , \quad c_\alpha c_\beta^\dagger + c_\beta^\dagger c_\alpha = \delta_{\alpha\beta}$$

$$\mathcal{Q} = \frac{1}{N} \sum_\alpha c_\alpha^\dagger c_\alpha$$

$U_{\alpha\beta;\gamma\delta}$ are independent random variables with $\overline{U_{\alpha\beta;\gamma\delta}} = 0$ and $\overline{|U_{\alpha\beta;\gamma\delta}|^2} = U^2$
$N \to \infty$ yields critical strange metal.





S. Sachdev and J. Ye, PRL **70**, 3339 (1993)

A. Kitaev, unpublished; S. Sachdev, PRX **5**, 041025 (2015)

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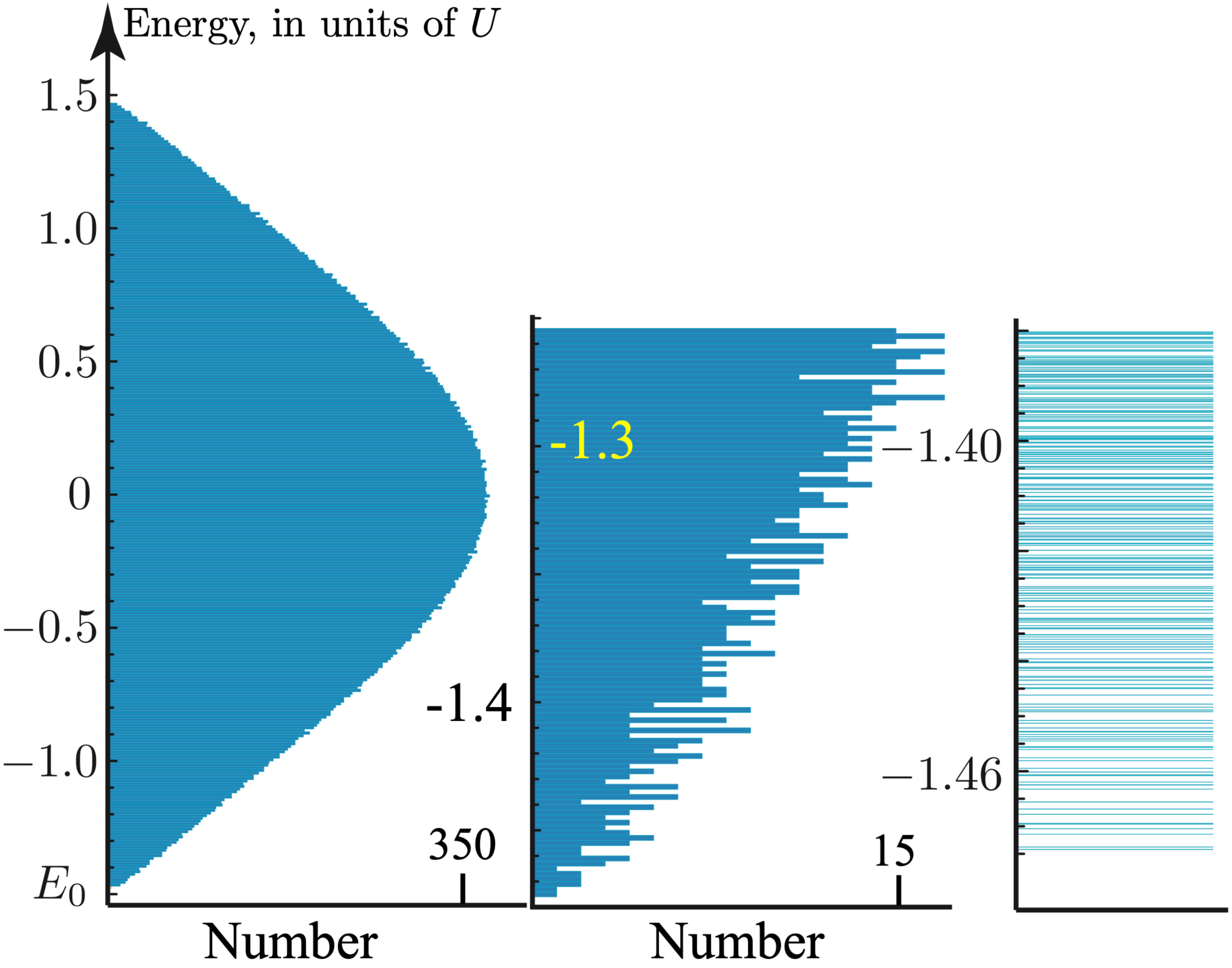

Complex SYK model

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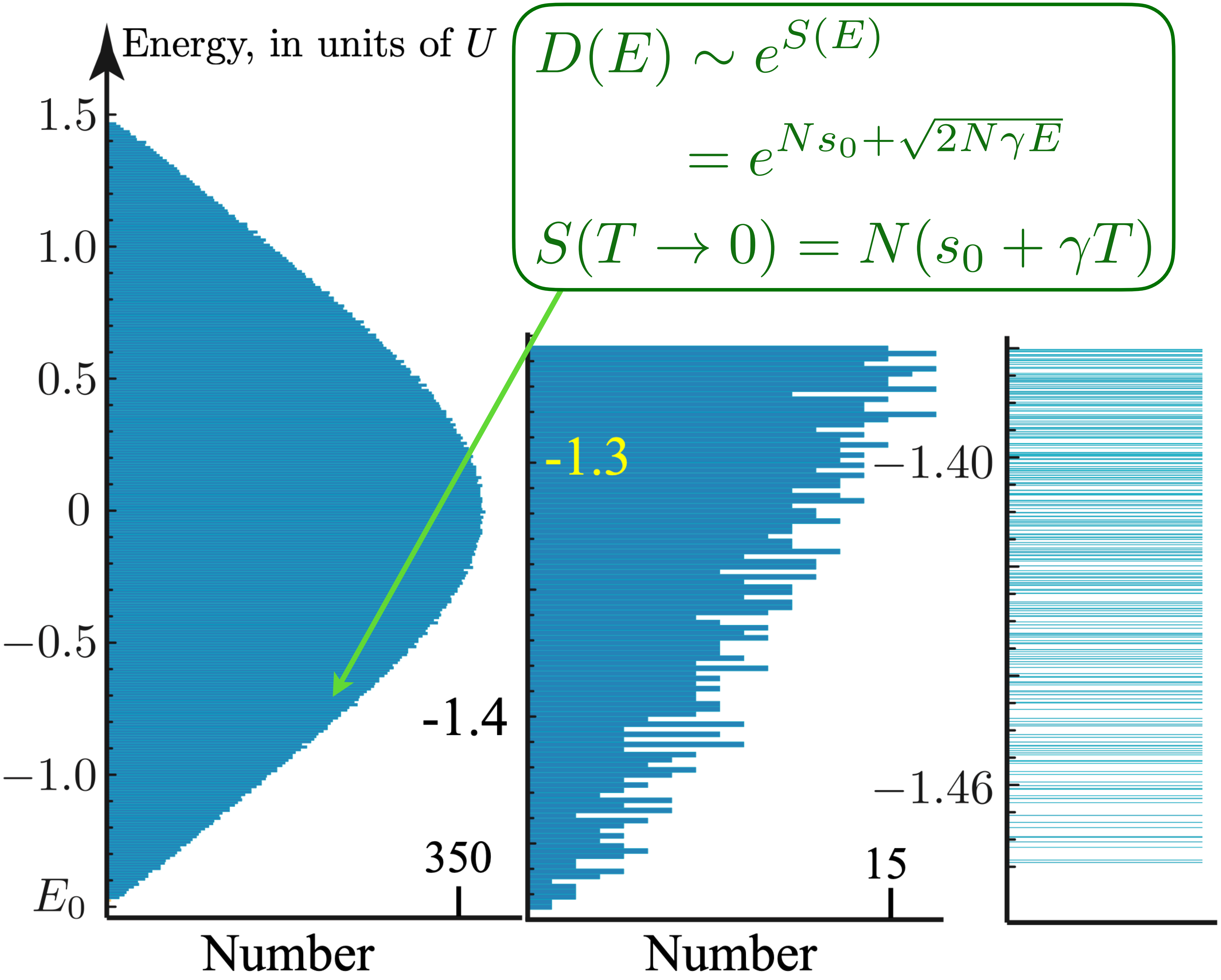

$$D(E) \sim e^{S(E)}$$
$$= e^{Ns_0 + \sqrt{2N\gamma E}}$$
$$S(T \to 0) = N(s_0 + \gamma T)$$

$$s_0 = 0.464848\ldots$$

A. Georges, O. Parcollet, and S. Sachdev,
PRB **63**, 134406 (2001)

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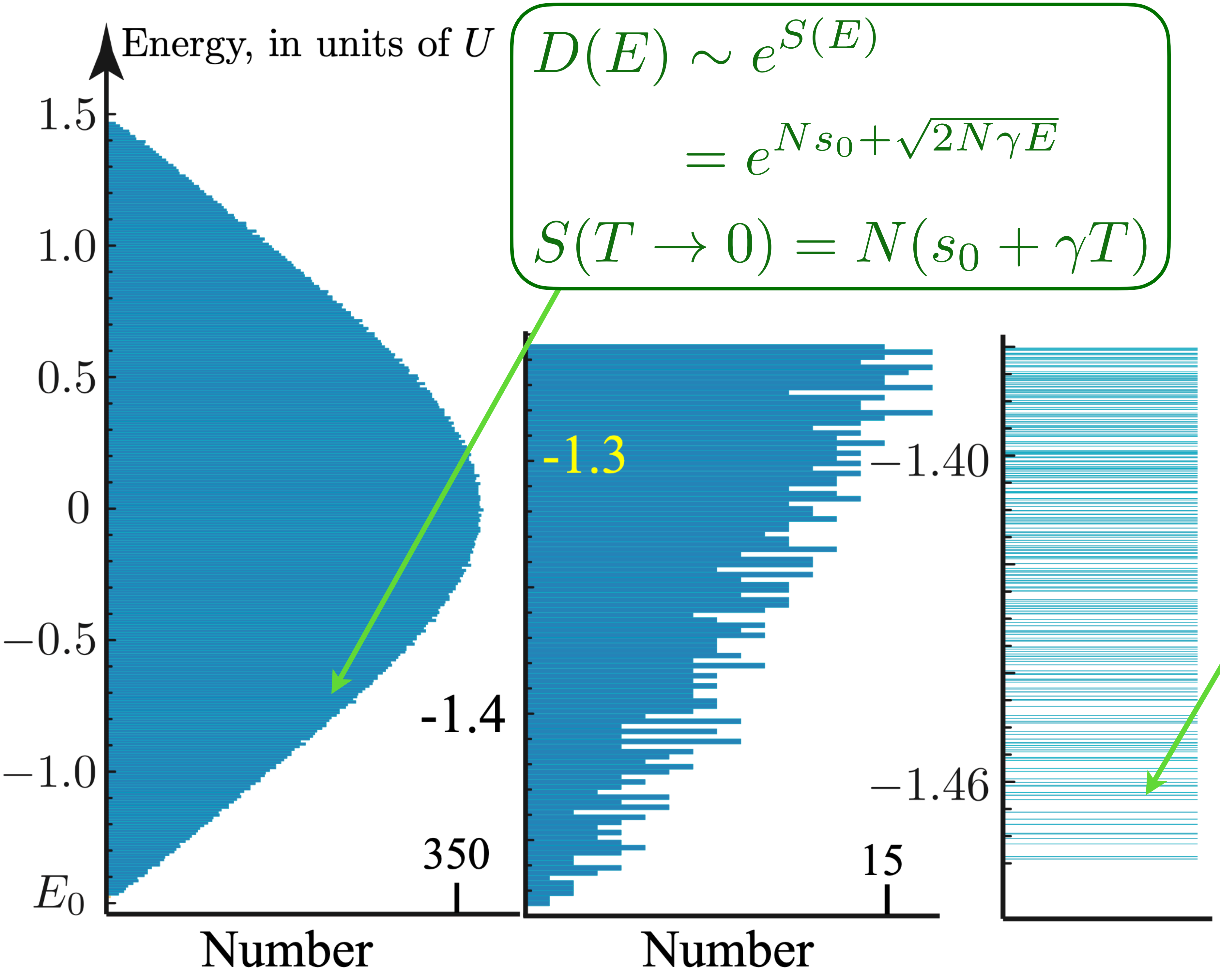

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$$D(E) \sim 2\, e^{N s_0} \sqrt{2N\gamma E}$$

No quasiparticle decomposition of many-body states

$$s_0 = 0.464848\ldots$$

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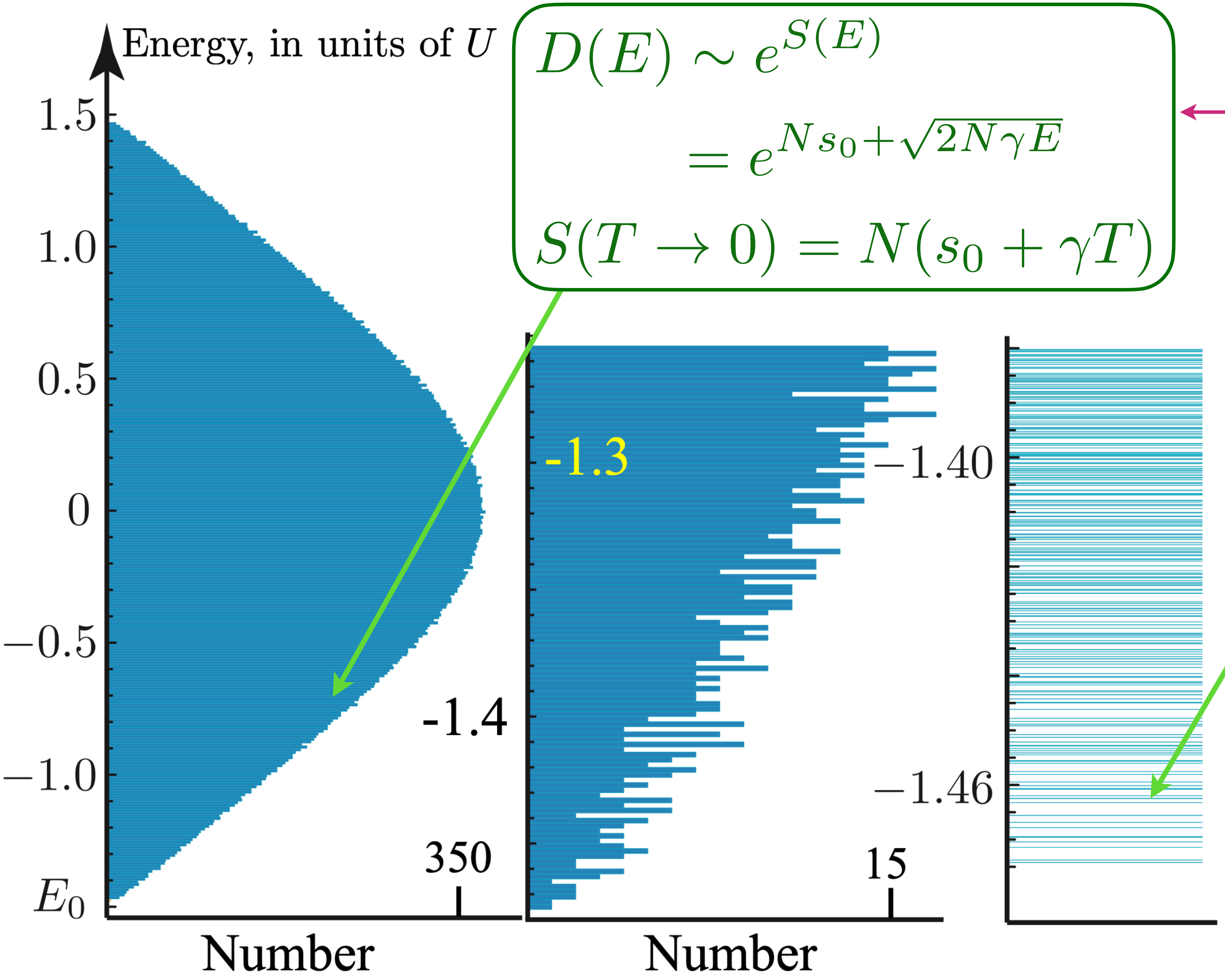

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$$D(E) \sim 2\, e^{Ns_0} \sinh(\sqrt{2N\gamma E})$$
$$S(T) = N(s_0 + \gamma T) - \frac{3}{2} \ln\left(\frac{U}{T}\right)$$

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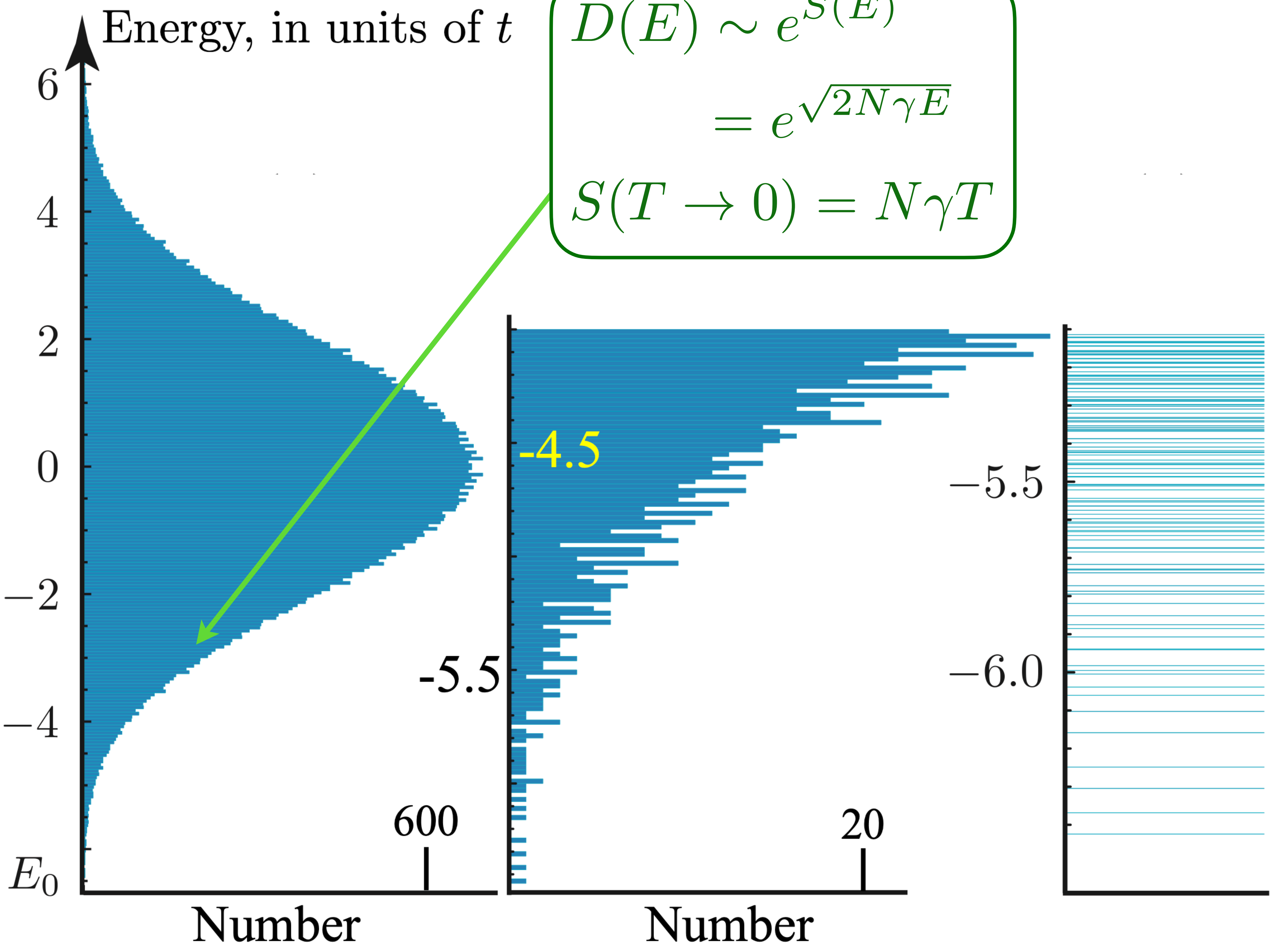

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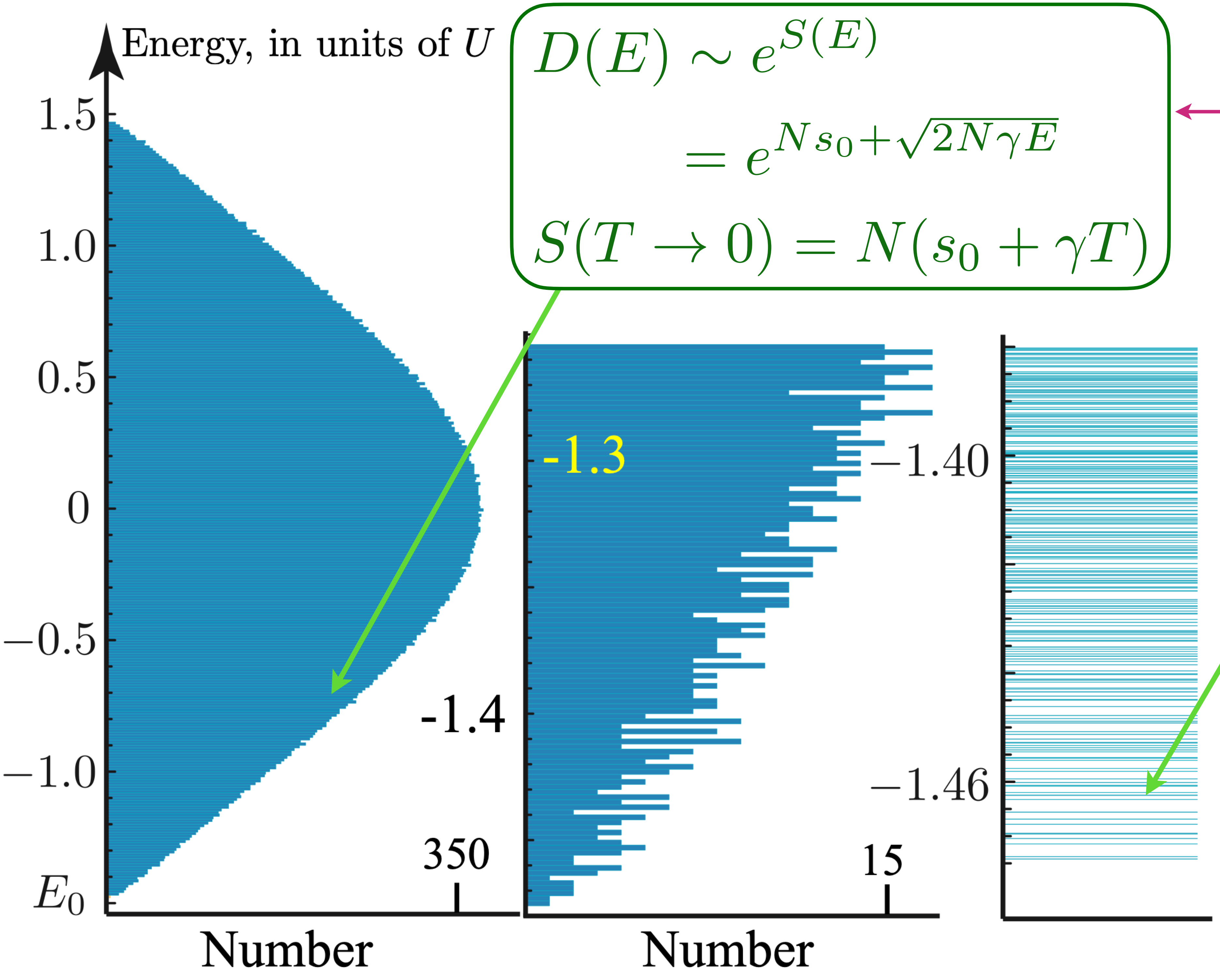

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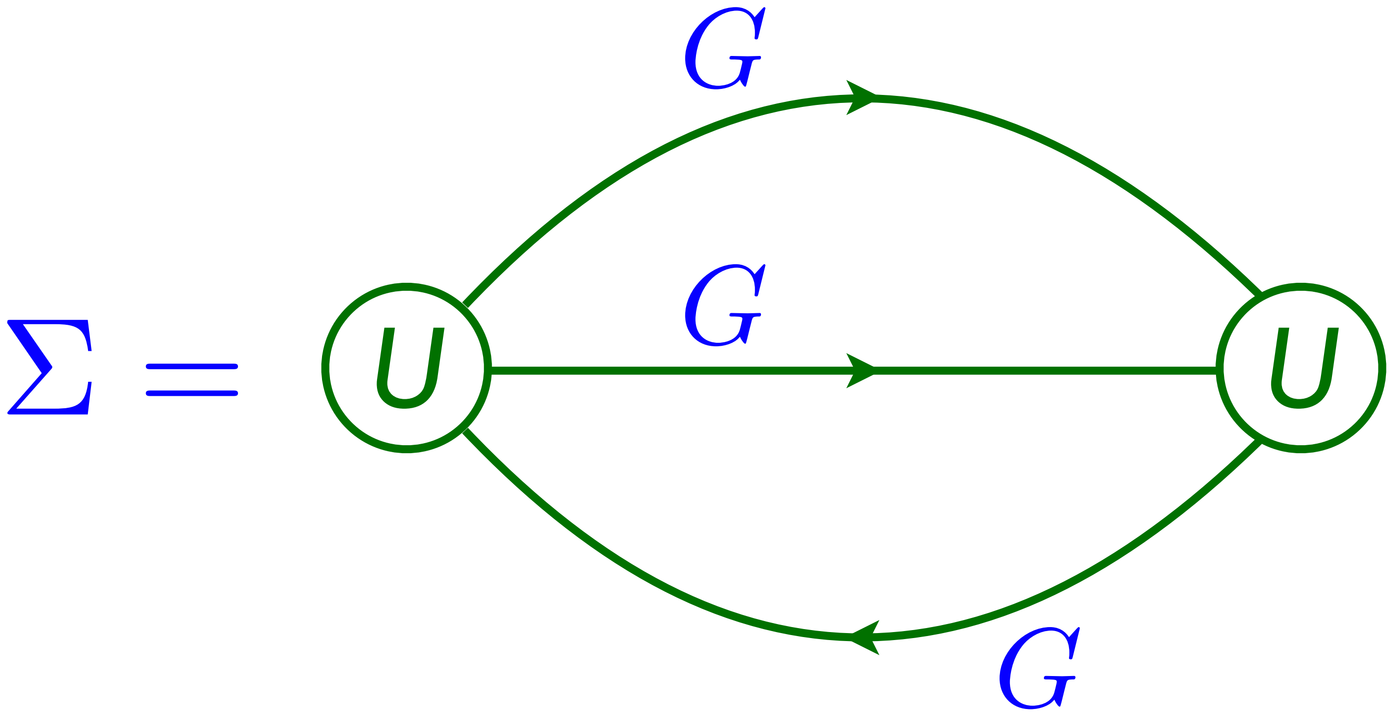
Complex SYK model

The complex SYK model

Feynman graph expansion in $U_{\alpha\beta;\gamma\delta}$, and graph-by-graph average, yields exact equations in the large N limit:

$$\boxed{\begin{gathered} G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = -U^2 G^2(\tau) G(-\tau) \\ G(\tau = 0^-) = \mathcal{Q}. \end{gathered}}$$



S. Sachdev and J. Ye,
PRL **70**, 3339 (1993)

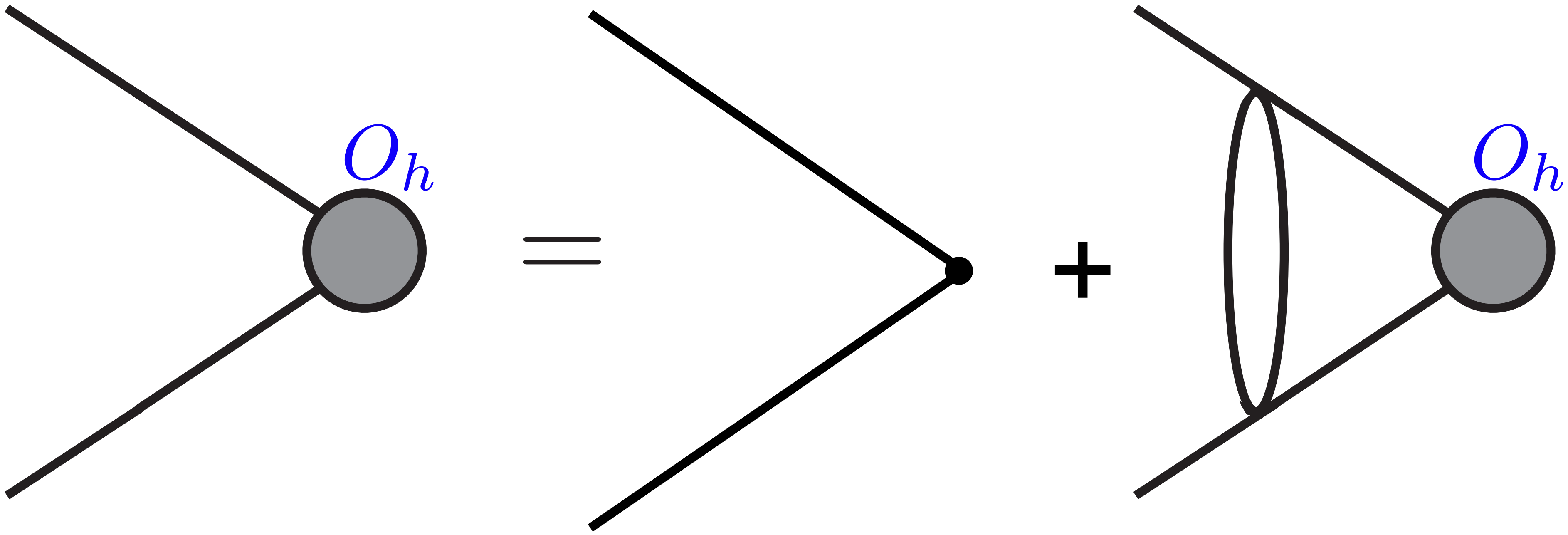
Conformal solution at $\mu = 0$, $G(\tau) \sim \dfrac{\text{sgn}(\tau)}{\sqrt{|\tau|}}$.

The SYK model

Conformal Perturbation theory

$$S = S_{\text{CFT}} + \sum_h g_h \int_0^\beta d\tau O_h(\tau)$$

where $G_{\text{CFT}} \sim \text{sgn}(\tau)/\sqrt{|\tau|}$ and $\langle O_h(\tau) O_h(0)\rangle \sim 1/|\tau|^{2h}$



Gross, Rosenhaus (2017)
Klebanov, Tarnopolsky (2017)

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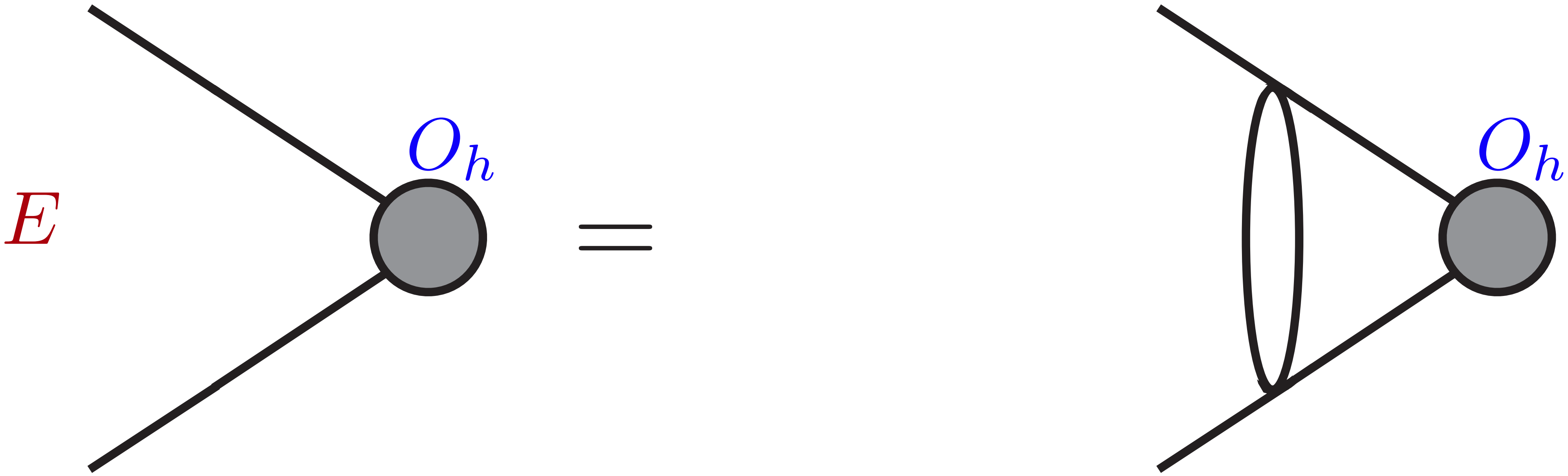

Solution of eigenvalue equation with $E = 1$ yields a tower of operators O_h with scaling dimensions h. Smallest non-trivial value is $h = 2$, and O_2 is the 'boundary graviton'.

$$G(\tau) \sim \frac{{\rm sgn}(\tau)}{\sqrt{|\tau|}} \left(1 + \sum_h \frac{c_h g_h}{|\tau|^{h-1}} + \dots \right)$$

Gross, Rosenhaus (2017)
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Sherrington-Kirkpatrick model

$$H = \sum_{i<j} J_{ij}\sigma_i\,\sigma_j$$

$$\sigma_i = \pm 1$$

$\overline{J_{ij}} = 0, \quad \overline{J_{ij}^2} = J^2,$ Different J_{ij} uncorrelated.

Edwards-Anderson order parameter $q_{EA} = \overline{\langle\sigma_i\rangle^2}$

Quantum generalization of the Sherrington-Kirkpatrick model to $S = 1/2$ spins with SU(2) symmetry

$$H = \sum_{i<j} J_{ij} \boldsymbol{S}_i \cdot \boldsymbol{S}_j$$

$$[S_{i\mu}, S_{j\nu}] = i\delta_{ij}\epsilon_{\mu\nu\lambda} S_{i\lambda} \quad , \quad \boldsymbol{S}_i^2 = 3/4$$

$\overline{J_{ij}} = 0, \quad \overline{J_{ij}^2} = J^2, \quad$ Different J_{ij} uncorrelated.

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Two possible ground states

I. Gapless spin liquid

$$\lim_{\tau\to\infty} \langle \boldsymbol{S}_i(\tau) \cdot \boldsymbol{S}_i(0) \rangle \sim \frac{1}{|\tau|^a}$$

II. Spin glass order

$$\lim_{\tau\to\infty} \langle \boldsymbol{S}_i(\tau) \cdot \boldsymbol{S}_i(0) \rangle = q_{EA} > 0$$

Introduce $n \to 0$ replicas, average over J_{ij}, and take the large N limit. Then we have to determine the saddle points of the action $\mathcal{S}[Q]$ for the quantum spin glass order parameter: $Q_{ab}(\tau)$, $a, b = 1 \ldots n$.

Q_{ab} is τ independent for $a \neq b$.

$$Q_{ab}(\tau - \tau') = \frac{1}{N} \sum_i \boldsymbol{S}_{ia}(\tau) \cdot \boldsymbol{S}_{ib}(\tau')$$

$$q_{EA} = \overline{\langle \boldsymbol{S}_i(\tau) \rangle \cdot \langle \boldsymbol{S}_i(0) \rangle} = \lim_{n \to 0} \frac{1}{n(n-1)} \sum_{a \neq b} Q_{ab}$$

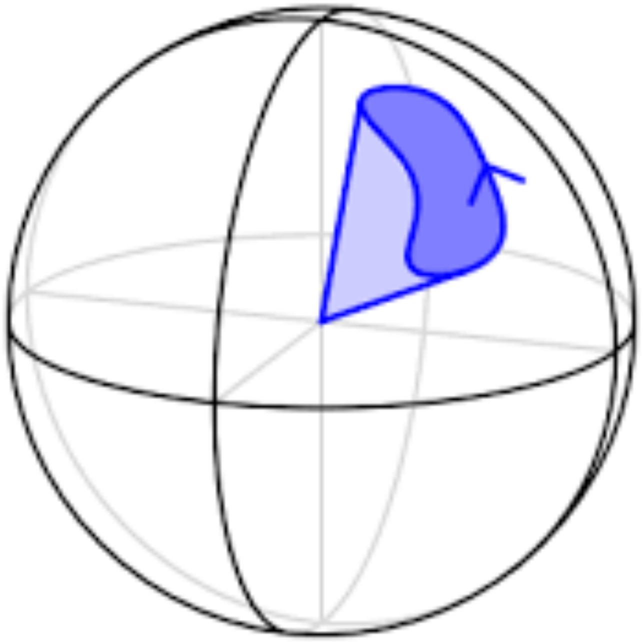
$$q_{EA} = \lim_{\tau \to \infty} \overline{\langle \boldsymbol{S}_i(\tau) \cdot \boldsymbol{S}_i(0) \rangle} = \lim_{n \to 0} \lim_{\tau \to \infty} \frac{1}{n} \sum_a Q_{aa}(\tau), \quad T = 0$$

Action for quantum spin glass order $Q_{ab}(\tau)$

$$\mathcal{S}[Q] = \frac{\beta J^2}{2} \int d\tau \left[Q_{ab}(\tau)\right]^2 - \ln \mathcal{Z}_f[Q]$$

$$\mathcal{Z}_f[Q] = \int \mathcal{D}\boldsymbol{S}_a(\tau)\delta(\boldsymbol{S}_a^2 - 1)\exp\Bigg[-\frac{i}{2}\int d\tau \boldsymbol{A}_a(\boldsymbol{S}_a)\cdot\partial_\tau \boldsymbol{S}_a - J^2 \int d\tau d\tau' Q_{ab}(\tau-\tau')\boldsymbol{S}_a(\tau)\cdot\boldsymbol{S}_b(\tau')\Bigg]$$

where $\boldsymbol{\nabla}_{\boldsymbol{S}_a} \times \boldsymbol{A}_a(\boldsymbol{S}_a) = \boldsymbol{S}_a$.



Generalize to SU(M) spins by writing $S_{\alpha\beta} = f_\alpha^\dagger f_\beta$, $f_\alpha^\dagger f_\alpha = M/2$, $\alpha = 1, \ldots, M$. In the limit $M \to \infty$, the saddle point equations for the fermion Green's function, self energy and Q become

$$\Sigma_{ab}(\tau) = J^2 Q_{ab}(\tau) G_{ab}(\tau)$$

$$G_{ab}(i\omega) = \left[i\omega\delta_{ab} - \Sigma_{ab}(i\omega) \right]^{-1}$$

$$Q_{ab}(\tau) = -G_{ab}(\tau) G_{ba}(-\tau)$$

S. Sachdev and J. Ye, PRL **70**, 3339 (1993)

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$$\begin{aligned} \Sigma(\tau) &= J^2 Q_{aa}(\tau) G(\tau) \\ G(i\omega) &= \left[i\omega - \Sigma(i\omega) \right]^{-1} \\ Q_{ab}(\tau) &= -G(\tau) G(-\tau) \delta_{ab} \end{aligned}$$

It is not possible for a fermion Green's function to have non-zero replica off-diagonal components. Then Q_{ab} must also be replica diagonal, and these equations are precisely those of the SYK model!

Solution of these equations yield a spin liquid ground state.

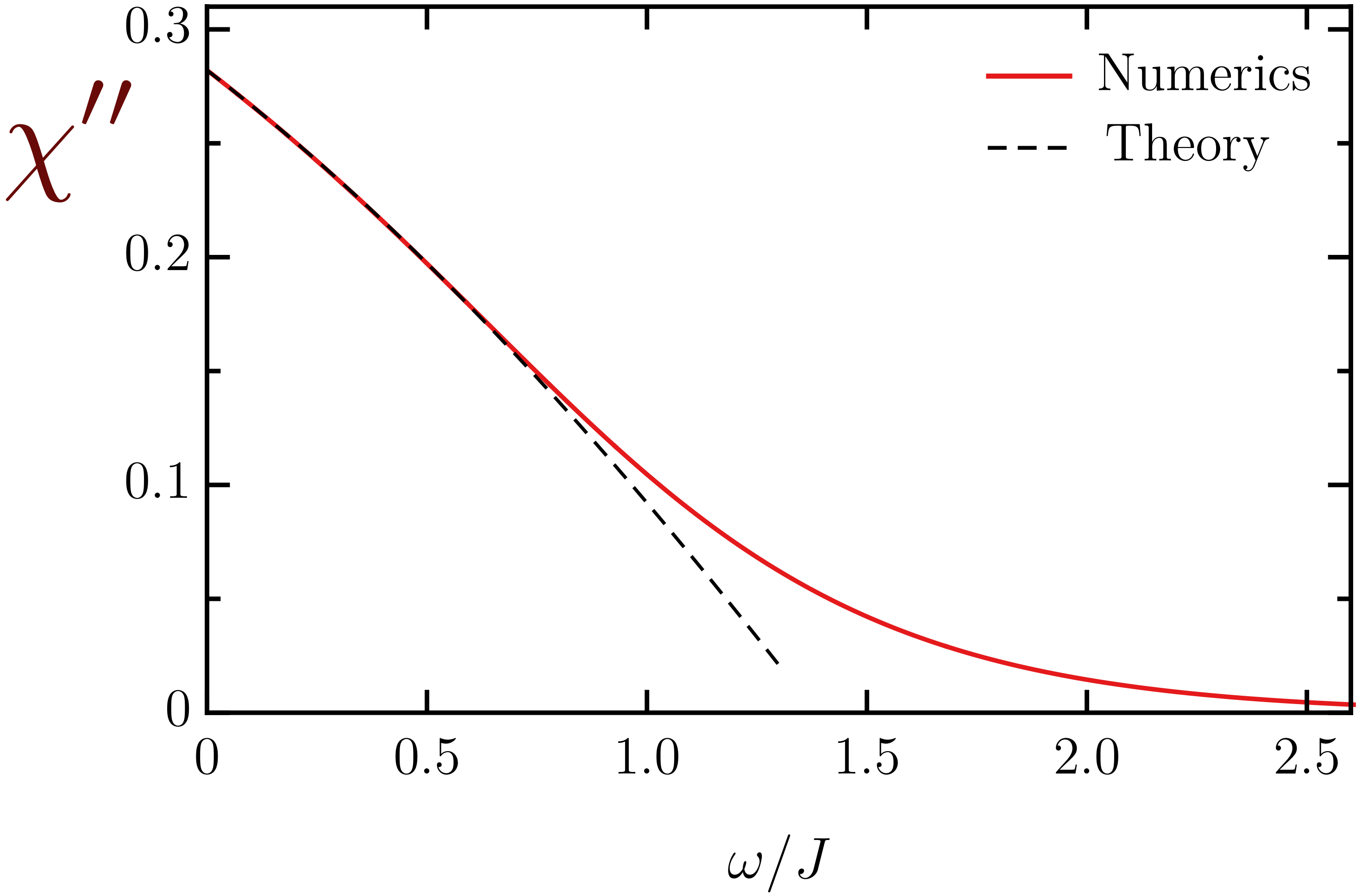
S. Sachdev and J. Ye, PRL **70**, 3339 (1993)

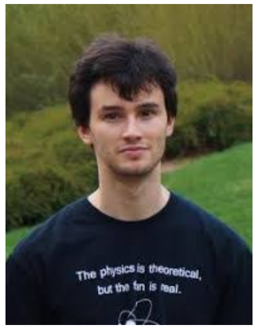
Dynamic spin susceptibility of the spin liquid at $M = \infty$

$$Q(\tau) = \int_0^\infty \frac{d\omega}{\pi} \chi''(\omega) e^{-\omega\tau}$$

$$\chi''(\omega) \sim \text{sgn}(\omega) \left[1 - \mathcal{C}\gamma|\omega| - \frac{7}{16}(\mathcal{C}\gamma)^2|\omega|^2 - \mathcal{C}'|\omega|^{2.77354\ldots} + \frac{37}{48}(\mathcal{C}\gamma)^3|\omega|^3 - \ldots \right]$$

Numerical solution of SYK equations (SY, PRL 1993), compared with conformal perturbation theory. $\mathcal{C}$ is a known number, and γ is the co-efficient of the action for the 'boundary graviton' in holographic dual.





Maria Tikhanovskaya, Haoyu Guo, S. Sachdev, G. Tarnopolsky, arXiv: 2010.09742, 2012.14449

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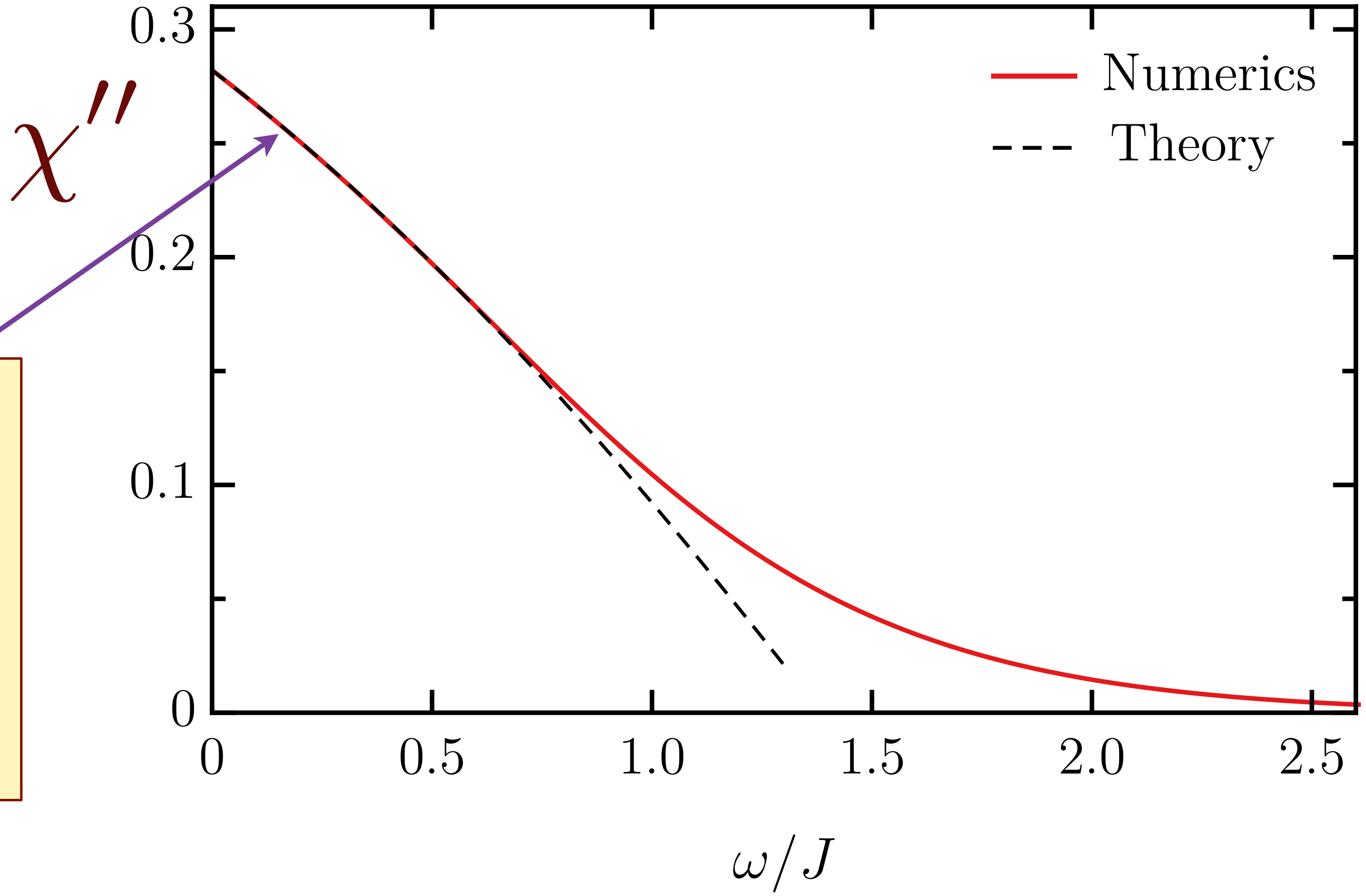
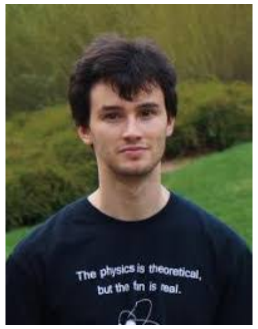
Correction from the boundary graviton





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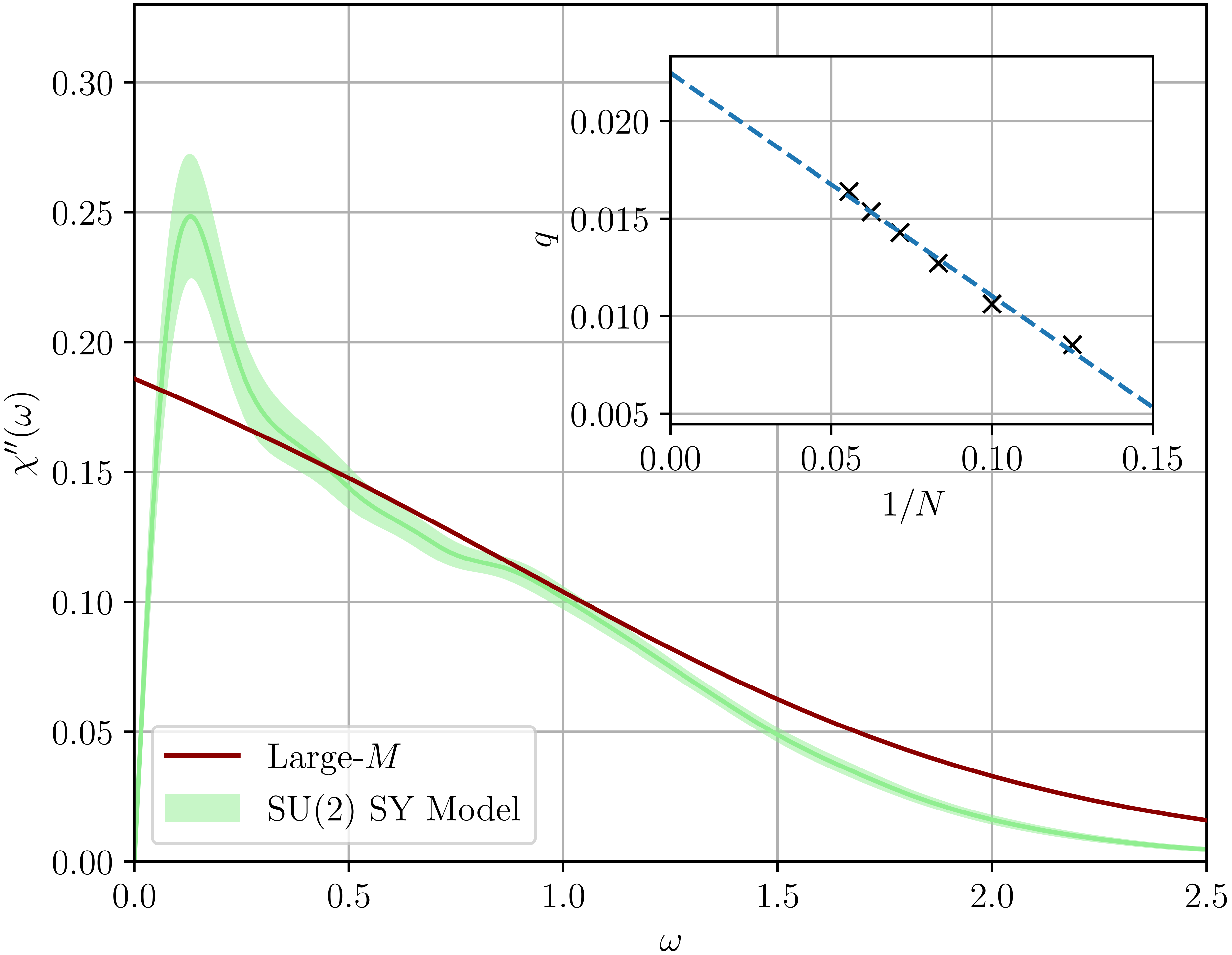
Exact diagonalization of clusters of SU(2) spins



H. Shackleton, A. Wietek, A. Georges, and S. Sachdev, PRL **126**, 136602 (2021)

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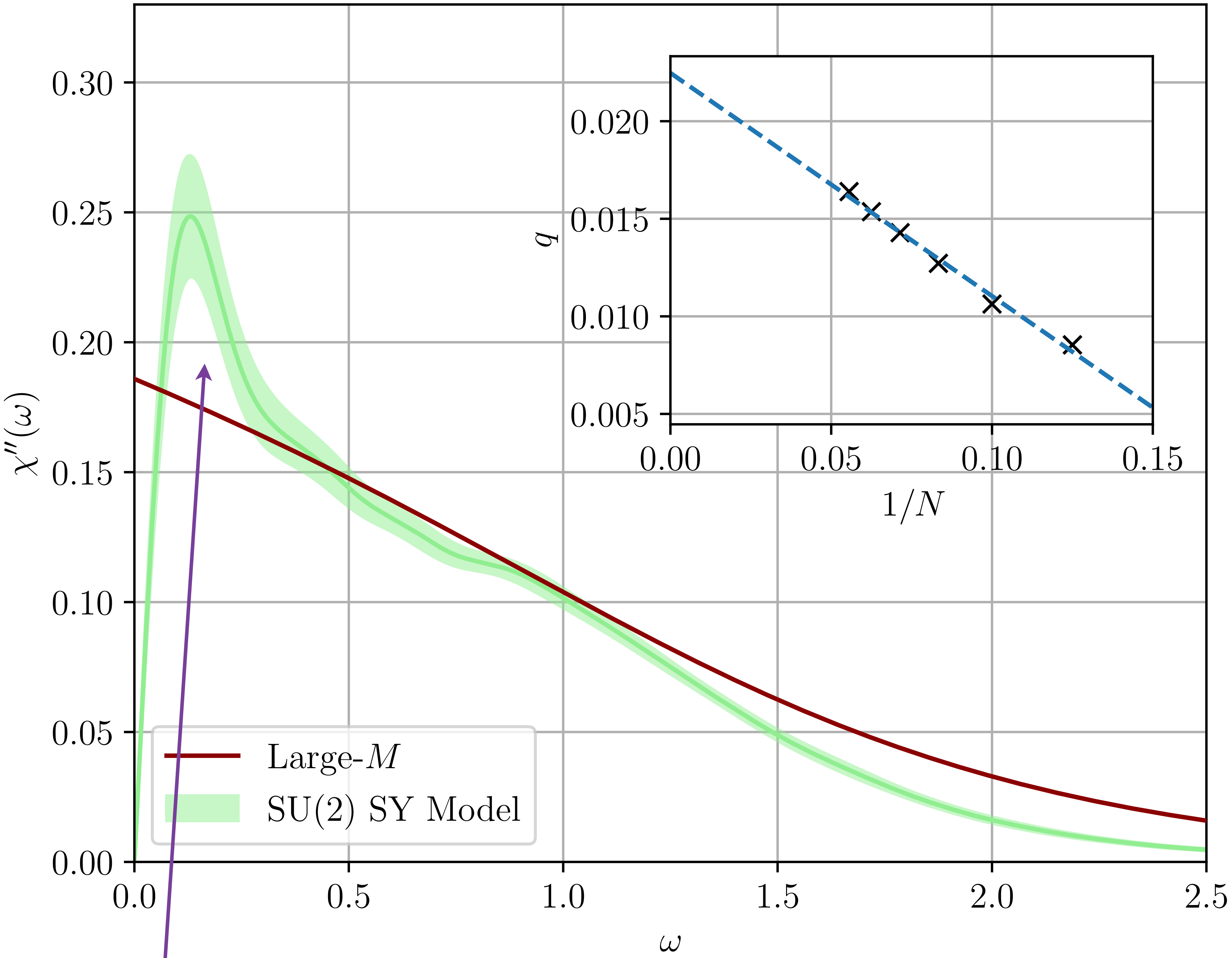

Spin glass

H. Shackleton, A. Wietek, A. Georges, and S. Sachdev, PRL **126**, 136602 (2021)

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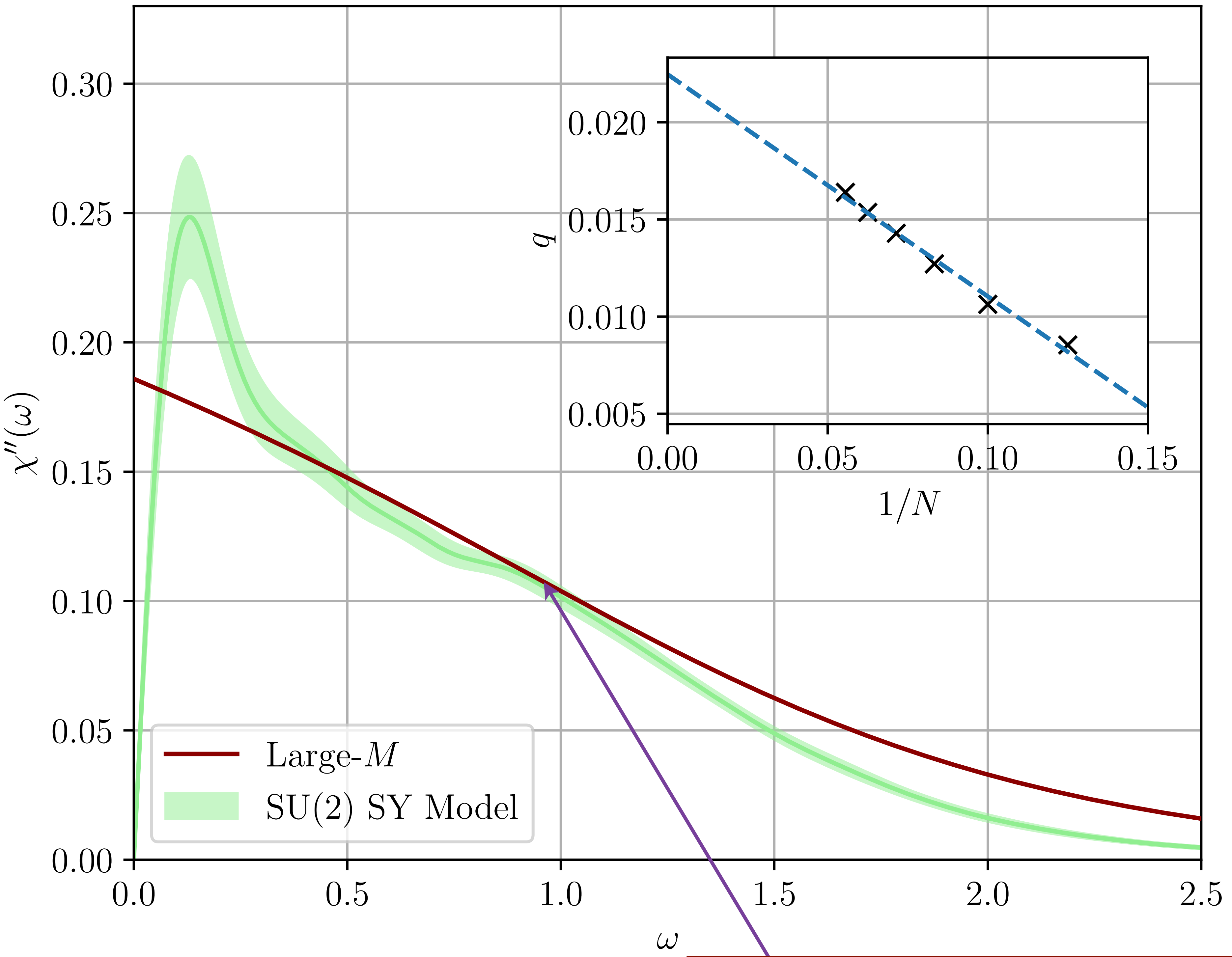


SY spin liquid (boundary graviton)

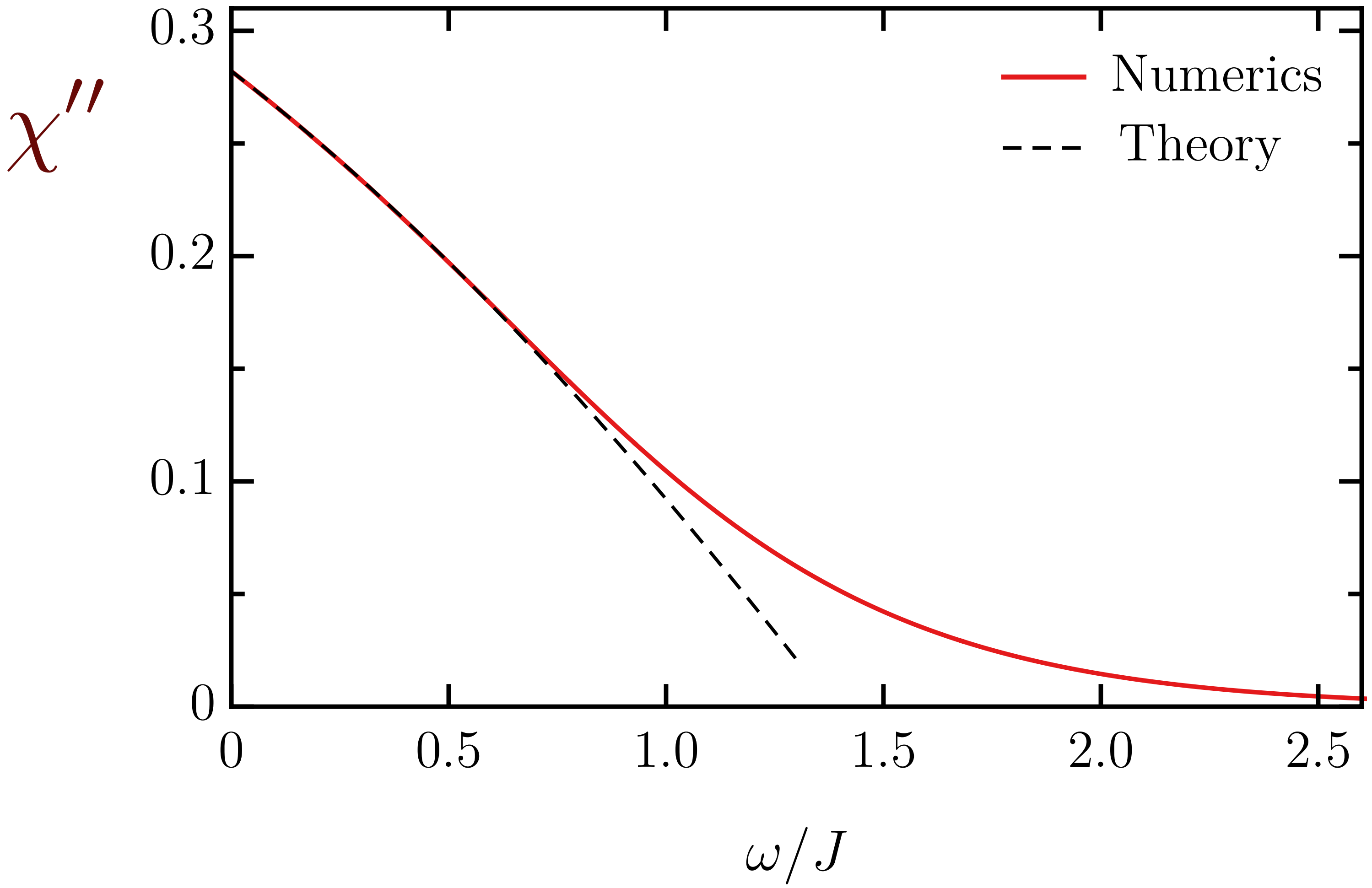
H. Shackleton, A. Wietek, A. Georges, and S. Sachdev, PRL **126**, 136602 (2021)

Adding spin glass order to the SU($M \to \infty$) equations:

$$\Sigma(\tau) = J^2 Q_{aa}(\tau) G(\tau)$$

$$G(i\omega) = \left[i\omega - \Sigma(i\omega) \right]^{-1}$$

$$Q_{ab}(\tau) = -G(\tau) G(-\tau) \delta_{ab}$$



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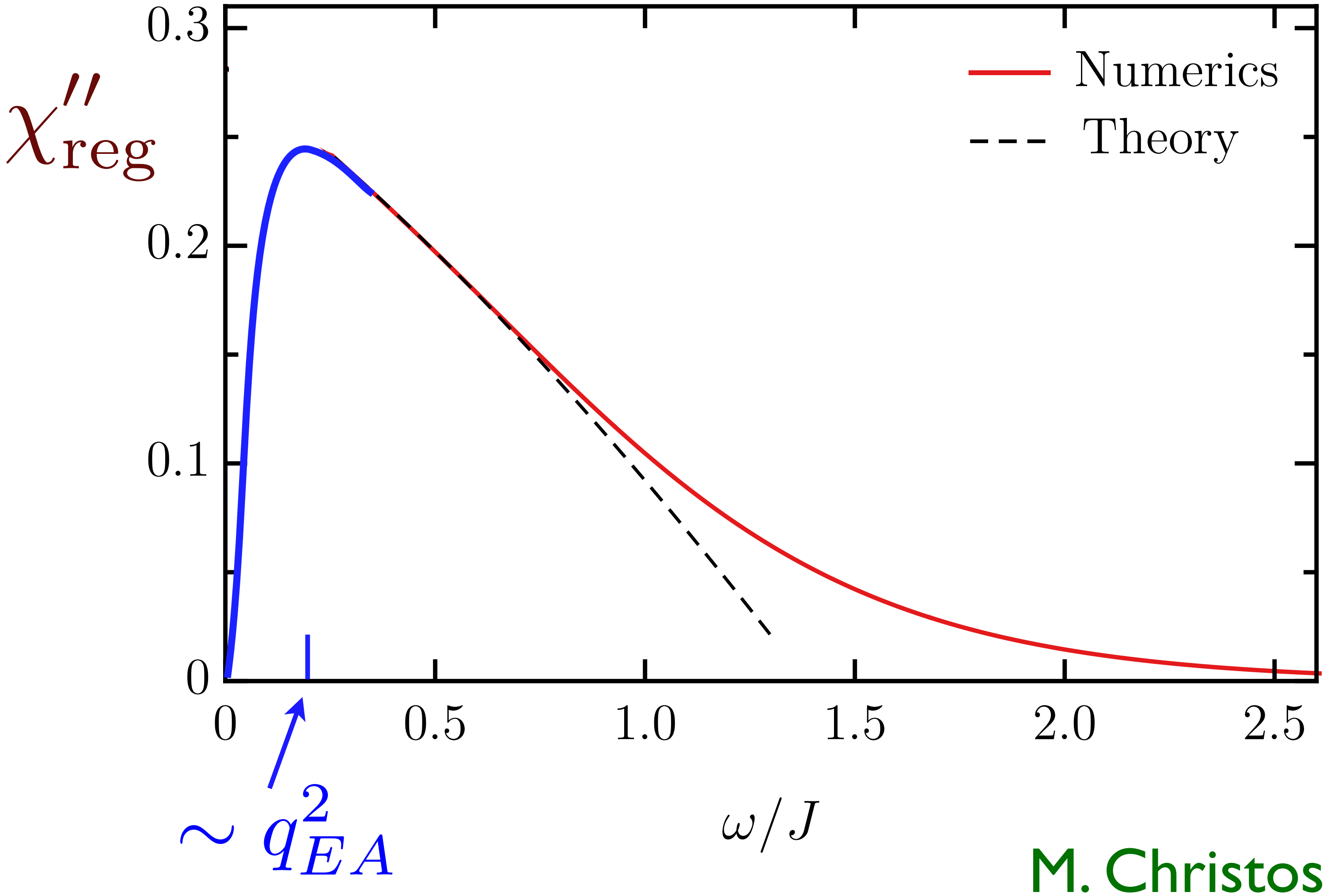
$$G(i\omega) = \left[i\omega - \Sigma(i\omega) \right]^{-1}$$

$$Q_{ab}(\tau) = -G(\tau)G(-\tau)\delta_{ab} + q_{ab}$$

Need only add the static spin glass order parameter q_{ab}, which is determined by the $1/M$ corrections.



$$\chi''(\omega) = \pi \beta q_{EA}\, \omega \delta(\omega) + \chi''_{\rm reg}(\omega)$$

Entropy $S(T \to 0) = 0$.

M. Christos, F. Haehl, S. Sachdev, to appear

Dope the quantum Sherrington-Kirkpatrick model with mobile electrons

$$H = \sum_{i<j} \left[-t_{ij} c_{i\alpha}^\dagger c_{j\alpha} + \text{H.c.} + J_{ij} \boldsymbol{S}_i \cdot \boldsymbol{S}_j \right]$$

$$\boldsymbol{S}_i = \frac{1}{2} c_{i\alpha}^\dagger \boldsymbol{\sigma}_{\alpha\beta} c_{i\beta}$$

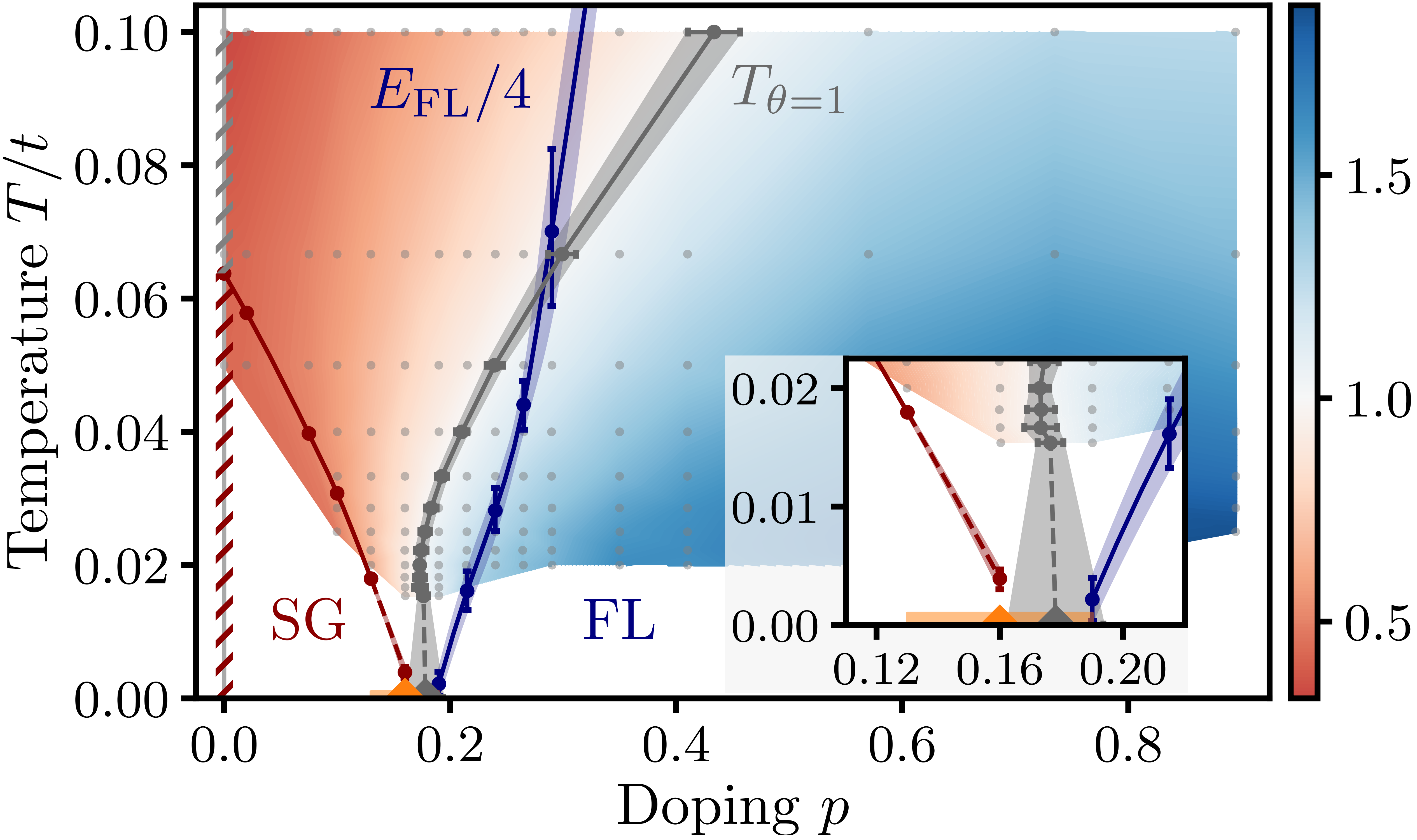
$$[c_{i\alpha}, c_{j\beta}^\dagger]_+ = \delta_{ij}\delta_{\alpha\beta} \quad , \quad \sum_{\alpha} c_{i\alpha}^\dagger c_{i\alpha} \leq 1$$

$$\frac{1}{N} \sum_{i\alpha} c_{i\alpha}^\dagger c_{i\alpha} = 1 - p$$

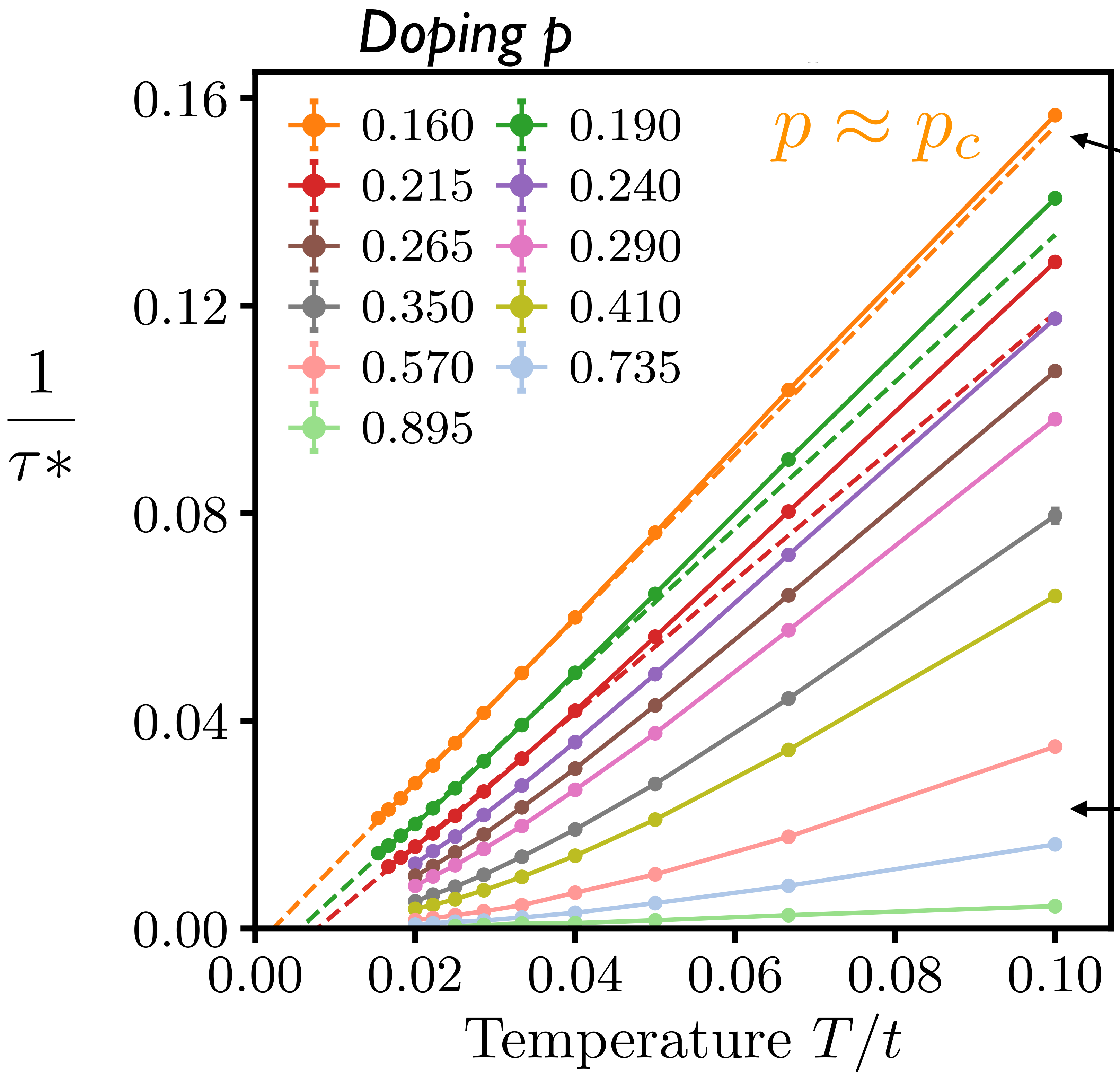
$\overline{J_{ij}} = 0, \quad \overline{J_{ij}^2} = J^2,$ Different J_{ij} uncorrelated.

$\overline{t_{ij}} = 0, \quad \overline{t_{ij}^2} = t^2,$ Different t_{ij} uncorrelated.

D. Chowdhury, A. Georges, O. Parcollet, S. Sachdev, review article, to appear



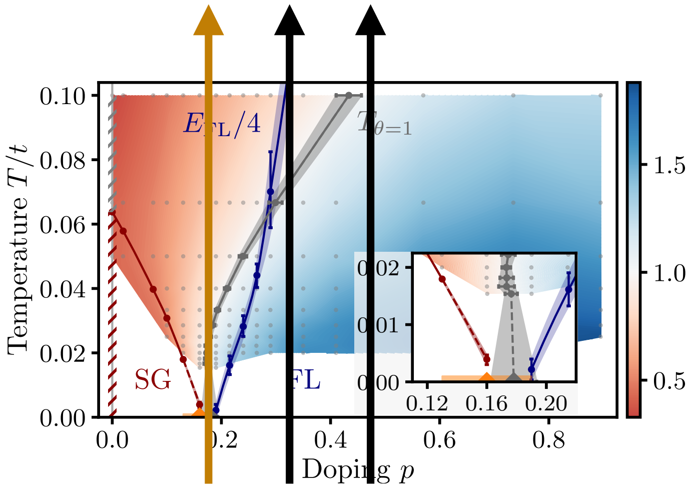
P. T. Dumitrescu, N. Wentzell, A. Georges, O. Parcollet, arXiv:2103.08607;
also H. Shackleton, A. Wietek, A. Georges, and S. Sachdev, PRL **126**, 136602 (2021)



$$\frac{1}{\tau *} \simeq c\,\frac{k_B T}{\hbar}$$

Planckian metal for $p \approx p_c$

$$\frac{1}{\tau *} \propto T^2$$



P. T. Dumitrescu, N. Wentzell, A. Georges, O. Parcollet, arXiv:2103.08607

1. SYK: a solvable model without quasiparticles

2. Spin liquid and spin glass of $S=1/2$ SU(2) spins

3. Time reparameterization soft mode

4. Charged black holes

Time reparameterization symmetry and 2D gravity

After introducing replicas $a = 1 \ldots n$, and integrating out the disorder, the partition function can be written as

$$Z = \int \mathcal{D}c_{\alpha a}(\tau) \exp \Bigg[- \sum_{ia} \int_0^\beta d\tau \, c_{\alpha a}^\dagger \left(\frac{\partial}{\partial \tau} - \mu \right) c_{\alpha a} \\ - \frac{U^2}{4N^3} \sum_{ab} \int_0^\beta d\tau d\tau' \left| \sum_i c_{\alpha a}^\dagger(\tau) c_{\alpha b}(\tau') \right|^4 \Bigg] .$$

For simplicity, we neglect the replica indices, and introduce the identity

$$1 = \int \mathcal{D}G(\tau_1, \tau_2) \mathcal{D}\Sigma(\tau_1, \tau_2) \exp \Bigg[-N \int_0^\beta d\tau_1 d\tau_2 \Sigma(\tau_1, \tau_2) \Bigg(G(\tau_2, \tau_1) \\ + \frac{1}{N} \sum_i c_\alpha(\tau_2) c_\alpha^\dagger(\tau_1) \Bigg) \Bigg] .$$

Time reparameterization symmetry and 2D gravity

Then the partition function can be written as a path integral with an action S analogous to a Luttinger-Ward functional

$$Z = \int \mathcal{D}G(\tau_1, \tau_2)\mathcal{D}\Sigma(\tau_1, \tau_2) \exp(-NS)$$

$$\begin{aligned} S = &\ln\det\left[\delta(\tau_1 - \tau_2)(\partial_{\tau_1} + \mu) - \Sigma(\tau_1, \tau_2)\right] \\ &+ \int d\tau_1 d\tau_2 \left[\Sigma(\tau_1, \tau_2)G(\tau_2, \tau_1) + (U^2/2)G^2(\tau_2, \tau_1)G^2(\tau_1, \tau_2)\right] \end{aligned}$$

Time reparameterization symmetry and 2D gravity

Then the partition function can be written as a path integral with an action S analogous to a Luttinger-Ward functional

$$Z = \int \mathcal{D}G(\tau_1,\tau_2)\mathcal{D}\Sigma(\tau_1,\tau_2)\exp(-NS)$$

$$S = \ln\det\left[\xcancel{\delta(\tau_1-\tau_2)(\partial_{\tau_1}+\mu)} - \Sigma(\tau_1,\tau_2)\right]$$

$$+ \int d\tau_1 d\tau_2 \left[\Sigma(\tau_1,\tau_2)G(\tau_2,\tau_1) + (U^2/2)G^2(\tau_2,\tau_1)G^2(\tau_1,\tau_2)\right]$$

At frequencies $\ll U$, the time derivative in the determinant is less important, and without it the path integral is invariant under the reparametrization and gauge transformations

A. Georges and O. Parcollet
PRB **59**, 5341 (1999)
A. Kitaev, 2015
S. Sachdev, PRX **5,** 041025 (2015)

$$\tau = f(\sigma)$$

$$G(\tau_1,\tau_2) = [f'(\sigma_1)f'(\sigma_2)]^{-1/4}\,\frac{g(\sigma_1)}{g(\sigma_2)}\,\widetilde{G}(\sigma_1,\sigma_2)$$

$$\Sigma(\tau_1,\tau_2) = [f'(\sigma_1)f'(\sigma_2)]^{-3/4}\,\frac{g(\sigma_1)}{g(\sigma_2)}\,\widetilde{\Sigma}(\sigma_1,\sigma_2)$$

where $f(\sigma)$ and $g(\sigma)$ are arbitrary functions.

Time reparameterization symmetry and 2D gravity

Reparametrization and phase zero modes

We can write the path integral for the SYK model as

$$\mathcal{Z} = \int \mathcal{D}G(\tau_1, \tau_2)\mathcal{D}\Sigma(\tau_1, \tau_2)e^{-NS[G,\Sigma]}$$

for a known action $S[G, \Sigma]$. We find the saddle point, G_s, Σ_s, and only focus on the "Nambu-Goldstone" modes associated with breaking reparameterization and U(1) gauge symmetries by writing

$$G(\tau_1, \tau_2) = [f'(\tau_1)f'(\tau_2)]^{1/4}G_s(f(\tau_1) - f(\tau_2))e^{i\phi(\tau_1)-i\phi(\tau_2)}$$

(and similarly for Σ). Then the path integral is approximated by

$$\mathcal{Z} = \int \mathcal{D}f(\tau)\mathcal{D}\phi(\tau)e^{-E_0/T+NS(E_0)-NS_{\text{eff}}[f,\phi]},$$

where $E_0 \propto N$ is the ground state energy.

J. Maldacena and D. Stanford, arXiv:1604.07818;
R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, arXiv.1612.00849;
S. Sachdev, PRX **5,** 041025 (2015); J. Maldacena, D. Stanford, and Zhenbin Yang, arXiv:1606.01857;
K. Jensen, arXiv:1605.06098; J. Engelsoy, T.G. Mertens, and H. Verlinde, arXiv:1606.03438

Time reparameterization symmetry and 2D gravity

Symmetry arguments, and explicit computations, show that the effective action is

$$\boxed{S_{\text{eff}}[f,\phi]=\frac{NK}{2}\int_0^{1/T}d\tau(\partial_\tau\phi+i(2\pi\mathcal{E}T)\partial_\tau f)^2-\frac{N\gamma}{4\pi^2}\int_0^{1/T}d\tau\,\{\tan(\pi Tf(\tau)),\tau\},}$$

where $f(\tau)$ is a monotonic map from $[0,1/T]$ to $[0,1/T]$, the couplings K, γ, and $\mathcal{E}$ can be related to thermodynamic derivatives and we have used the Schwarzian:

$$\{g,\tau\}\equiv\frac{g'''}{g'}-\frac{3}{2}\left(\frac{g''}{g'}\right)^2.$$

Specifically, an argument constraining the effective at $T=0$ is

$$S_{\text{eff}}\left[f(\tau)=\frac{a\tau+b}{c\tau+d},\phi(\tau)=0\right]=0,$$

and this is origin of the Schwarzian.

J. Maldacena and D. Stanford, arXiv:1604.07818;
R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, PRB **95**, 155131 (2017);
A. Gaikwad, L.K. Joshi, G. Mandal, and S.R. Wadia, arXiv:1802.07746

Consequences of 2D-gravity

- Exact evaluation of the path integral over $f(\tau)$ and $\phi(\tau)$ leads to the many-body density of states

$$D(E) \sim 2e^{S_0} \sinh(\sqrt{2N\gamma E})$$

- Saddle-point shift leads to a correction to the Green's function:

$$G(\tau) \sim \frac{\text{sgn}(\tau)}{\sqrt{|\tau|}} \left(1 + \frac{\alpha_G}{|\tau|} + \ldots\right)$$

  There is a universal relationship between α_G and γ, the coefficient of the Schwarzian and the linear-T entropy. This is due to the connection between the $h = 2$ O_h operator and the 'boundary graviton'.

Maldacena, Stanford (2016); Bagrets *et al.* (2017); Cotler *et al.* (2017); Kitaev, Suh (2017); Stanford, Witten (2017)

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Thermodynamics of quantum black holes:

$$\int \mathcal{D}g_{\mu\nu} \exp\left(-\frac{1}{\hbar}\, \mathcal{S}^{(3+1)}_{\text{Einstein gravity}}[g_{\mu\nu}]\right)$$

Metric of spacetime

In general, this summation is not well defined, because to the uncontrollably large number of spacetime configurations.

Thermodynamics of quantum black holes:

$$\int \mathcal{D}g_{\mu\nu} \exp\left(-\frac{1}{\hbar}\,\mathcal{S}^{(3+1)}_{\text{Einstein gravity}}[g_{\mu\nu}]\right)$$

$$= \exp\left(S_{BH}\right) \times \left(\quad \ldots.????\ldots \quad \right)$$

Metric of spacetime

Gibbons, Hawking (1977)

$$S_{BH} = \frac{A(T)c^3}{4G\hbar}$$

$(\hbar/(k_B T)$ is the length of the Euclidean time circle)
$A(T)$ is the area of the black hole horizon at a temperature T.
Interpretation: Black holes have finite number of quantum degrees of freedom, and black hole entropy is their entanglement entropy across the horizon.

Thermodynamics of quantum black holes with charge $\mathcal{Q}$:

$$\int \mathcal{D}g_{\mu\nu}\mathcal{D}A_\mu \exp\left(-\frac{1}{\hbar}\,\mathcal{S}^{(3+1)}_{\text{Einstein gravity+Maxwell EM}}[g_{\mu\nu}, A_\mu]\right)$$

$$= \exp\Big(S_{BH}\Big) \times \Bigg(\quad \text{....????...} \quad \Bigg)$$

Gibbons, Hawking (1977)

$$S_{BH}(T \to 0, \mathcal{Q}) = \frac{A(T)c^3}{4G\hbar} = \frac{A_0 c^3}{4G\hbar}\left(1 + \frac{2(\pi A_0)^{1/2}T}{\hbar c}\right)$$

A_0 is the area of the charged black hole horizon at $T = 0$, $\mathcal{Q}$ is the black hole charge.

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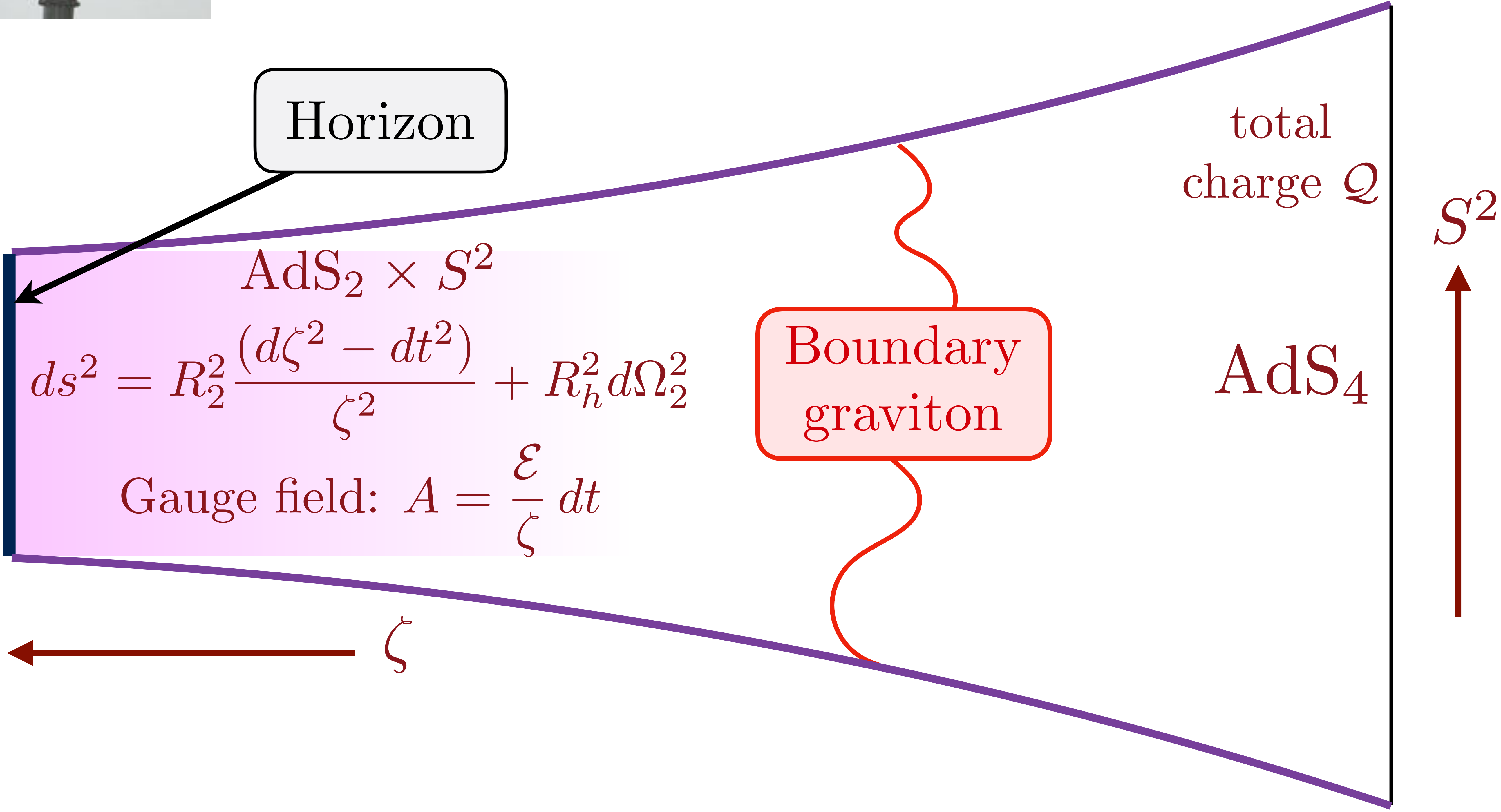
Note the similarity to the large *N* entropy of the SYK model !

Sachdev PRL 2010

Reissner-Nordstrom black hole of Einstein-Maxwell theory



$\mathrm{AdS}_2 \times S^2$

$$ds^2 = R_2^2 \frac{(d\zeta^2 - dt^2)}{\zeta^2} + R_h^2 d\Omega_2^2$$

Gauge field: $A = \dfrac{\mathcal{E}}{\zeta}\, dt$

Thermodynamics of quantum black holes with charge $\mathcal{Q}$:

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$$\approx \int \mathcal{D}g_{\mu\nu}\mathcal{D}A_\mu \exp\left(-\frac{1}{\hbar}\,\mathcal{S}^{(1+1)}_{\text{JTgravity of AdS}_2\text{ and boundary}}[g_{\mu\nu}, A_\mu]\right)$$

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$$= \int \mathcal{D}f(\tau)\mathcal{D}\phi(\tau) \exp\left(-\text{Schwarzian boundary graviton} + \text{rotor action}[f, \phi]\right)$$

$$S(T \to 0, \mathcal{Q}) = S_{BH} - \frac{3}{4}\ln\left(\frac{\hbar c^5}{G T^2}\right)$$

$$S_{BH} = \frac{A(T)c^3}{4G\hbar} = \frac{A_0 c^3}{4G\hbar}\left(1 + \frac{2(\pi A_0)^{1/2}T}{\hbar c}\right)$$

A_0 is the area of the charged black hole horizon at $T = 0$, $\mathcal{Q}$ is the black hole charge. The $\ln T$ term is the contribution of the boundary graviton.

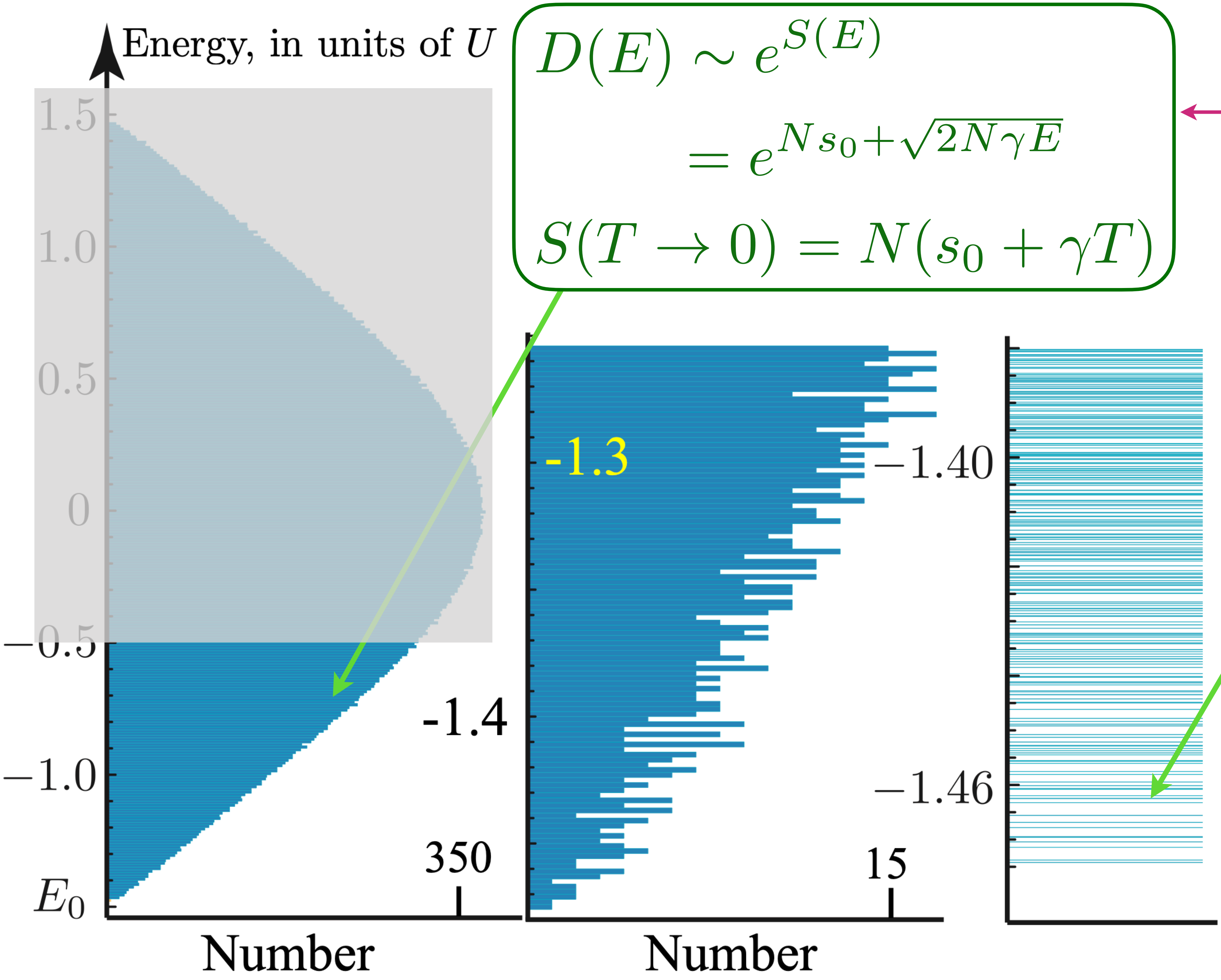
Sachdev (2010); Kitaev (2015); Sachdev (2015); Bagrets, Altland, Kamenev (2016); Maldacena, Stanford, Yang (2016); Moitra, Trivedi, Vishal (2018) ; Gaikwad, Joshi, Mandal, Wadia (2018); Iliesiu, Turaci (2020)

Many-body density of states

$$D(E) = \sum_i \delta(E - E_i); \quad E_0 + E_i \Rightarrow \text{Many body eigenvalue}$$

$$D(E) \sim e^{S(E)}$$
$$= e^{N s_0 + \sqrt{2N\gamma E}}$$
$$S(T \to 0) = N(s_0 + \gamma T)$$

$$D(E) \sim 2\, e^{N s_0} \sinh(\sqrt{2N\gamma E})$$
$$S(T) = N(s_0 + \gamma T) - \frac{3}{2} \ln \left(\frac{U}{T}\right)$$



$$D(E) \sim 2\, e^{N s_0} \sqrt{2N\gamma E}$$

No quasiparticle decomposition of many-body states

$$s_0 = 0.464848\ldots$$

A. Georges, O. Parcollet, and S. Sachdev, PRB **63**, 134406 (2001)

Complex SYK model

Many-body density of states

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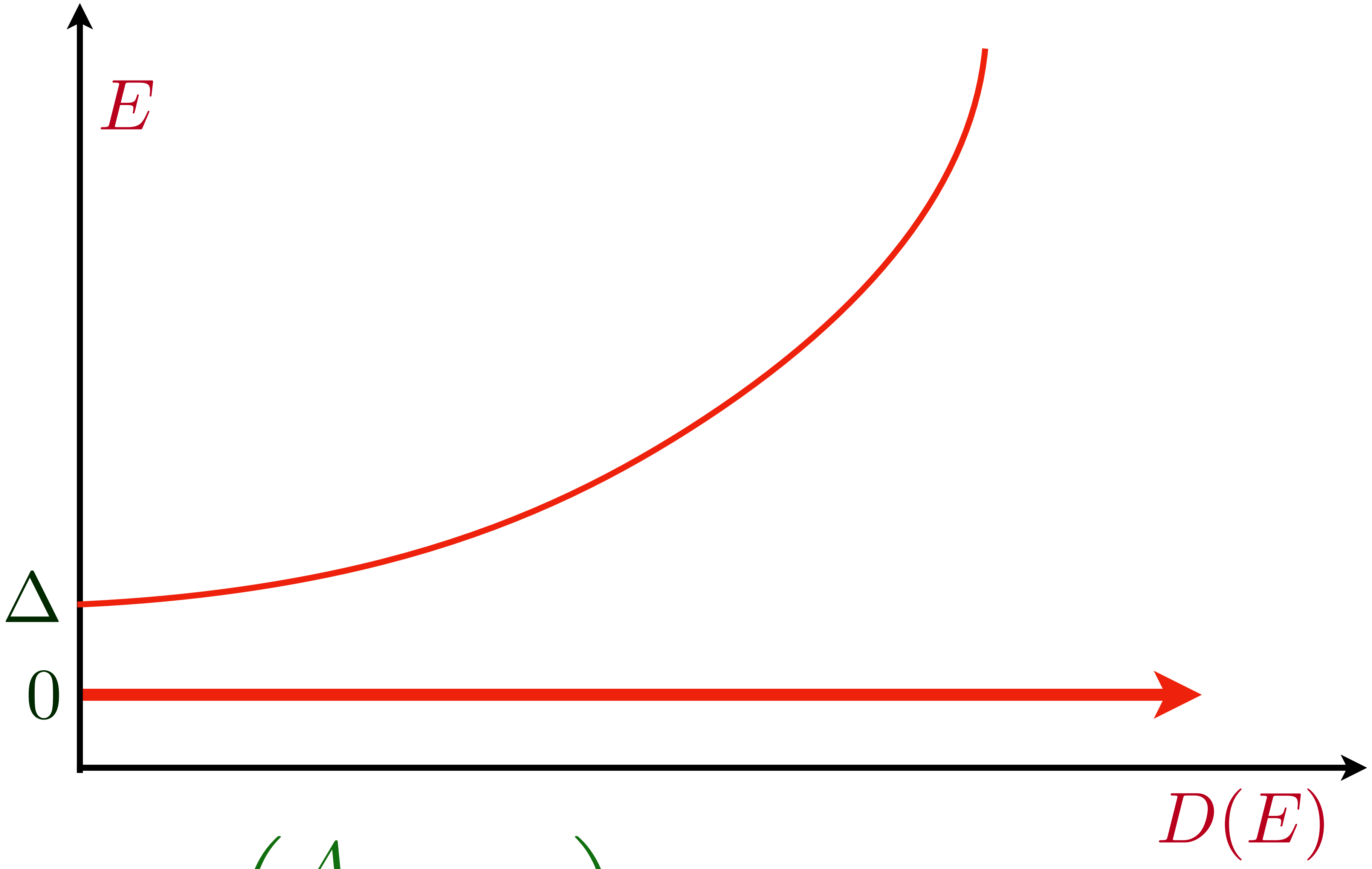

$$D(E) \sim \exp\left(\frac{A_0}{4G} + \ldots\right)\delta(E) + f_{\text{reg}}(E - \Delta), \quad \Delta \sim R_h^{-1}$$

Supersymmetric black holes and SYK models

Fu, Gaiotto, Maldacena, Sachdev (2017); Stanford, Witten (2017); Heydeman, Iliesiu, Turiaci, Zhao (2020)

Summary

- SYK: a solvable model without quasiparticle excitations, exhibiting thermalization and many-body chaos in a time of order $\hbar/(k_B T)$, independent of microscopic energy scales.

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Summary

- SYK: a solvable model without quasiparticle excitations, exhibiting thermalization and many-body chaos in a time of order $\hbar/(k_B T)$, independent of microscopic energy scales.

- Low energy theory of time reparameterizations is the theory of the boundary graviton in 2D quantum gravity on AdS$_2$.

- Boundary graviton leads to:
  - Dynamic spin susceptibility $\sim \text{sgn}(\omega)\,[1 - c|\omega| + \ldots]$
  - Universal $-3/2 \ln(1/T)$ correction to Bekenstein-Hawking entropy of low T charged black holes in Einstein gravity.

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- Spin $S = 1/2$ SU(2) quantum Sherrington-Kirkpatrick model display co-existence of spin glass order and SY(K) spin liquid behavior

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- Random t-J model has a quantum phase transition where spin glass order vanishes, with Planckian metal criticality.