

Quantum matter and gauge-gravity duality

Subir Sachdev

Talk online at sachdev.physics.harvard.edu



“Complex entangled” states of
quantum matter,
not adiabatically connected to independent particle states

Gapped quantum matter

Z_2 Spin liquids, quantum Hall states

Conformal quantum matter

Graphene, ultracold atoms, antiferromagnets

Compressible quantum matter

Strange metals, Bose metals

S. Sachdev, 100th anniversary Solvay conference (2011), arXiv:1203.4565

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Conformal quantum matter

A. Field theory:

Honeycomb lattice

Hubbard model

B. Gauge-gravity duality

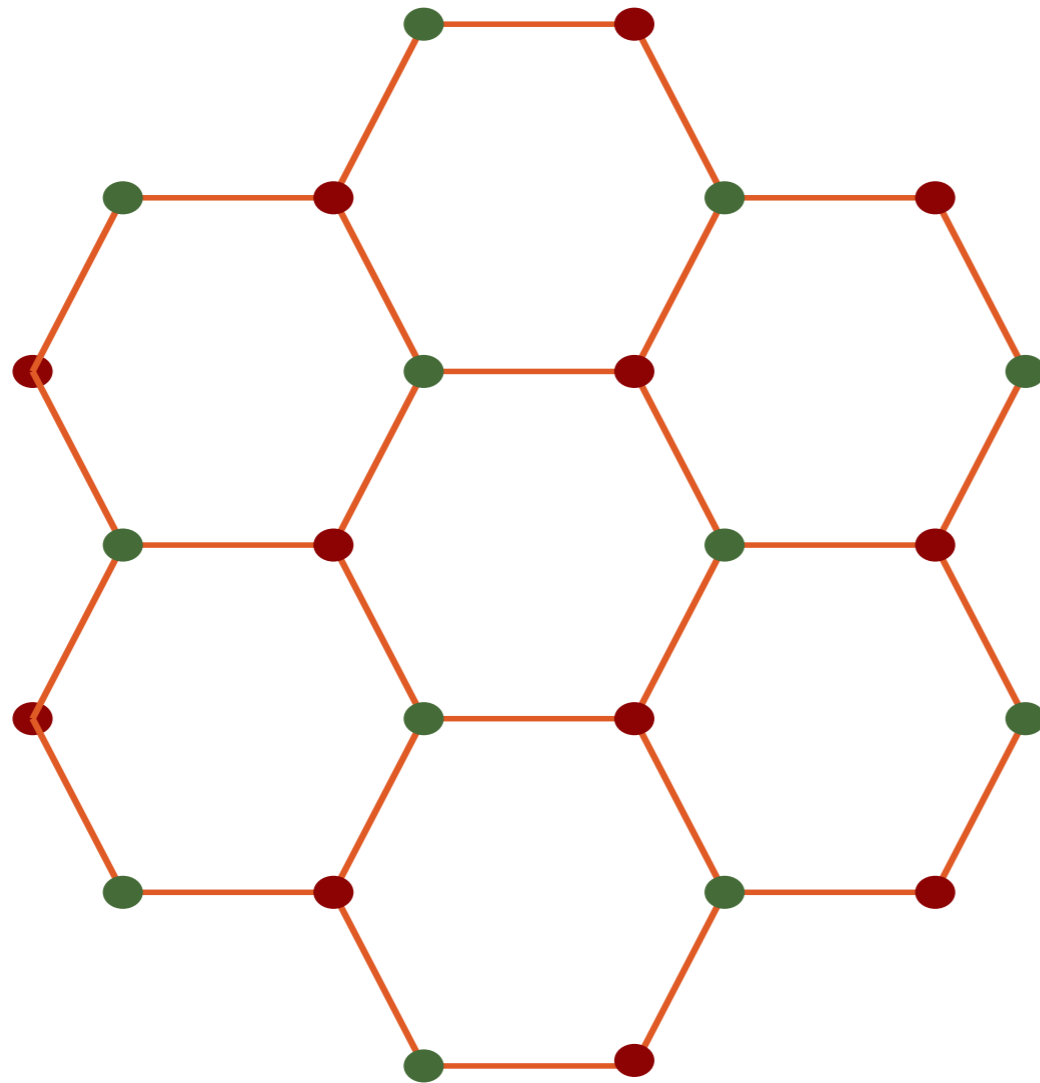
Conformal quantum matter

*A. Field theory:
Honeycomb lattice
Hubbard model*

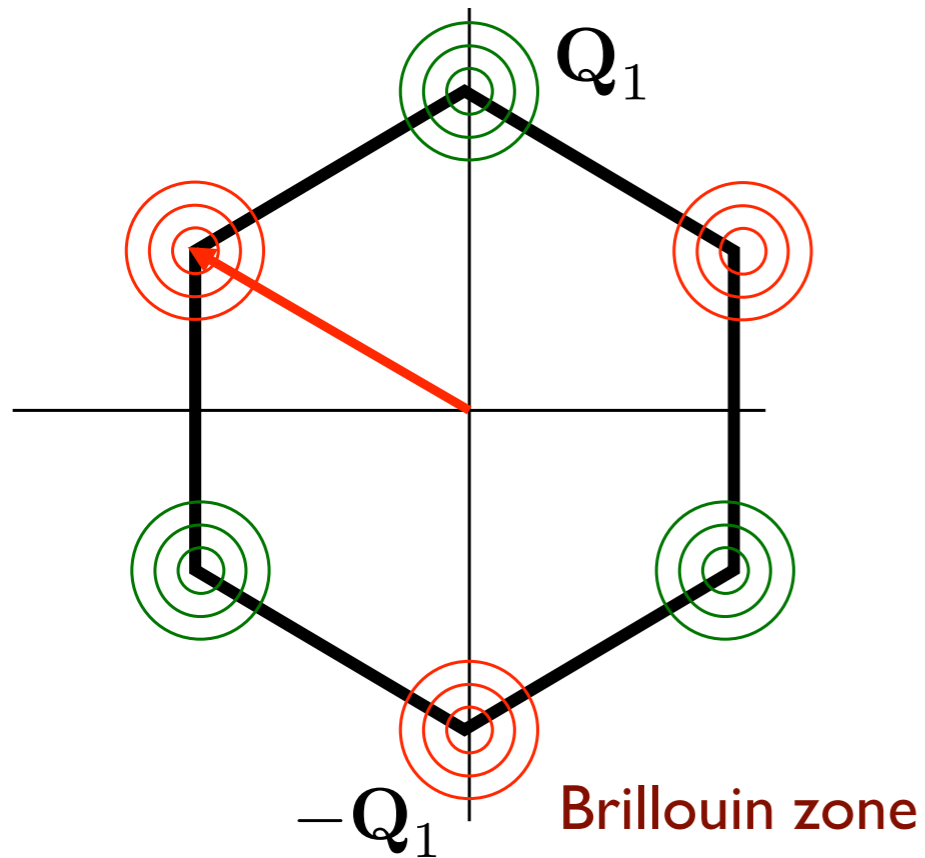
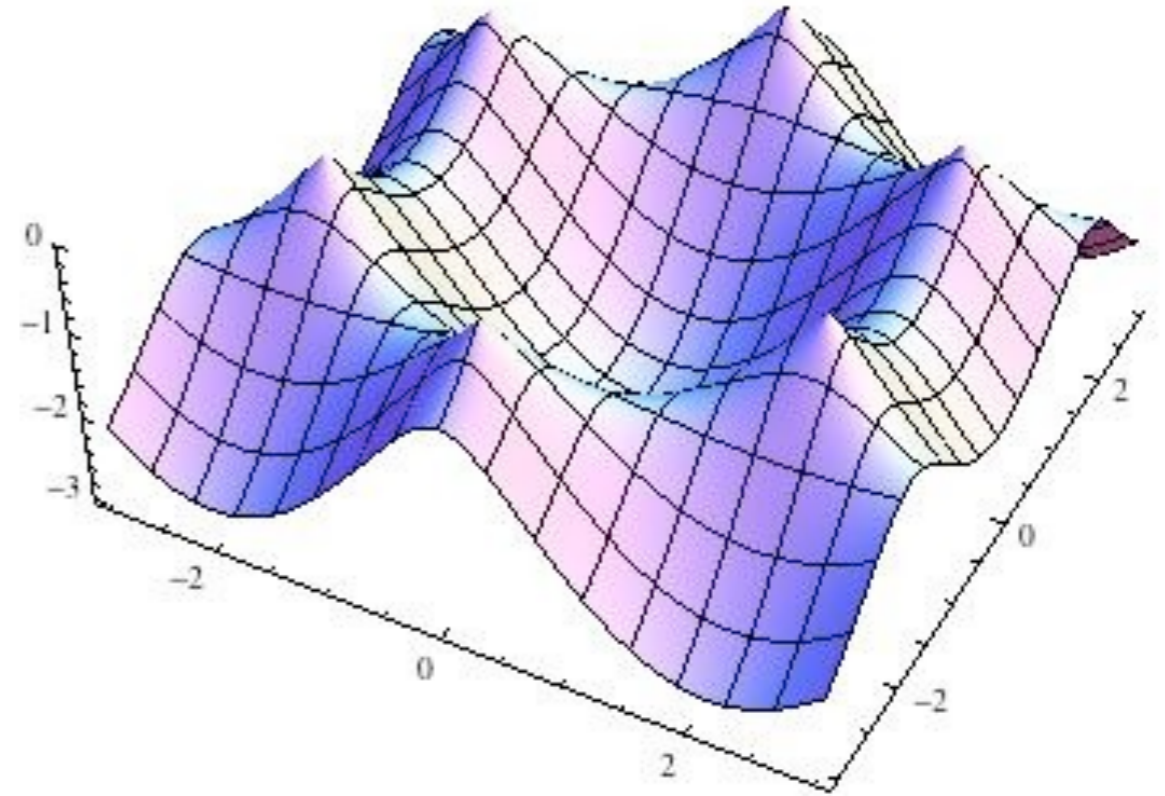
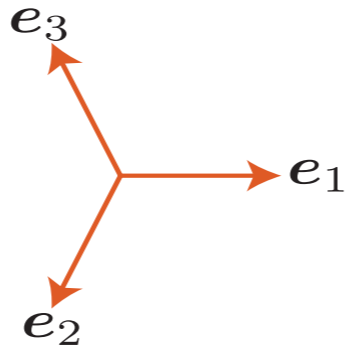
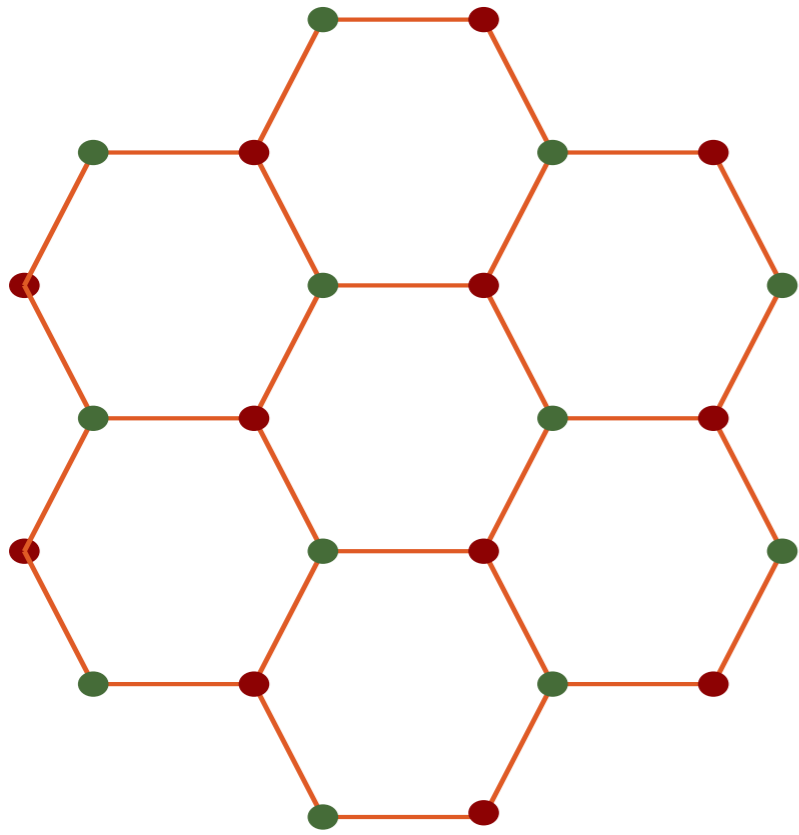
B. Gauge-gravity duality

Honeycomb lattice

(describes graphene after adding long-range Coulomb interactions)



$$H = -t \sum_{\langle ij \rangle} c_{i\alpha}^\dagger c_{j\alpha} + U \sum_i \left(n_{i\uparrow} - \frac{1}{2} \right) \left(n_{i\downarrow} - \frac{1}{2} \right)$$



**Semi-metal with
massless Dirac fermions
at small U/t**

We define the Fourier transform of the fermions by

$$c_A(\mathbf{k}) = \sum_{\mathbf{r}} c_A(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} \quad (4)$$

and similarly for c_B . **A** and **B** are sublattice indices. The hopping Hamiltonian is

$$H_0 = -t \sum_{\langle ij \rangle} \left(c_{Ai\alpha}^\dagger c_{Bj\alpha} + c_{Bj\alpha}^\dagger c_{Ai\alpha} \right) \quad (5)$$

where α is a spin index. If we introduce Pauli matrices τ^a in sublattice space ($a = x, y, z$), this Hamiltonian can be written as

$$H_0 = \int \frac{d^2k}{4\pi^2} c^\dagger(\mathbf{k}) \left[-t \left(\cos(\mathbf{k} \cdot \mathbf{e}_1) + \cos(\mathbf{k} \cdot \mathbf{e}_2) + \cos(\mathbf{k} \cdot \mathbf{e}_3) \right) \tau^x + t \left(\sin(\mathbf{k} \cdot \mathbf{e}_1) + \sin(\mathbf{k} \cdot \mathbf{e}_2) + \sin(\mathbf{k} \cdot \mathbf{e}_3) \right) \tau^y \right] c(\mathbf{k}) \quad (6)$$

The low energy excitations of this Hamiltonian are near $\mathbf{k} \approx \pm \mathbf{Q}_1$.

In terms of the fields near \mathbf{Q}_1 and $-\mathbf{Q}_1$, we define

$$\begin{aligned}
 \Psi_{A1\alpha}(\mathbf{k}) &= c_{A\alpha}(\mathbf{Q}_1 + \mathbf{k}) \\
 \Psi_{A2\alpha}(\mathbf{k}) &= c_{A\alpha}(-\mathbf{Q}_1 + \mathbf{k}) \\
 \Psi_{B1\alpha}(\mathbf{k}) &= c_{B\alpha}(\mathbf{Q}_1 + \mathbf{k}) \\
 \Psi_{B2\alpha}(\mathbf{k}) &= c_{B\alpha}(-\mathbf{Q}_1 + \mathbf{k})
 \end{aligned} \tag{7}$$

We consider Ψ to be a 8 component vector, and introduce Pauli matrices ρ^a which act in the 1, 2 valley space. Then the Hamiltonian is

$$H_0 = \int \frac{d^2k}{4\pi^2} \Psi^\dagger(\mathbf{k}) \left(v\tau^y k_x + v\tau^x \rho^z k_y \right) \Psi(\mathbf{k}), \tag{8}$$

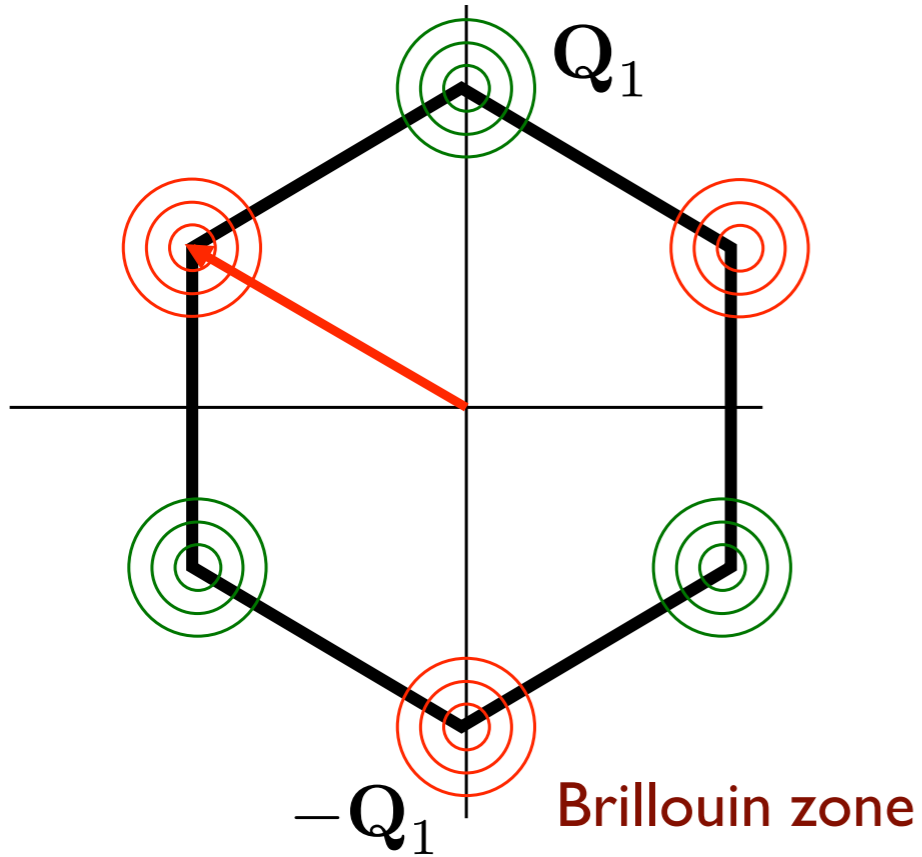
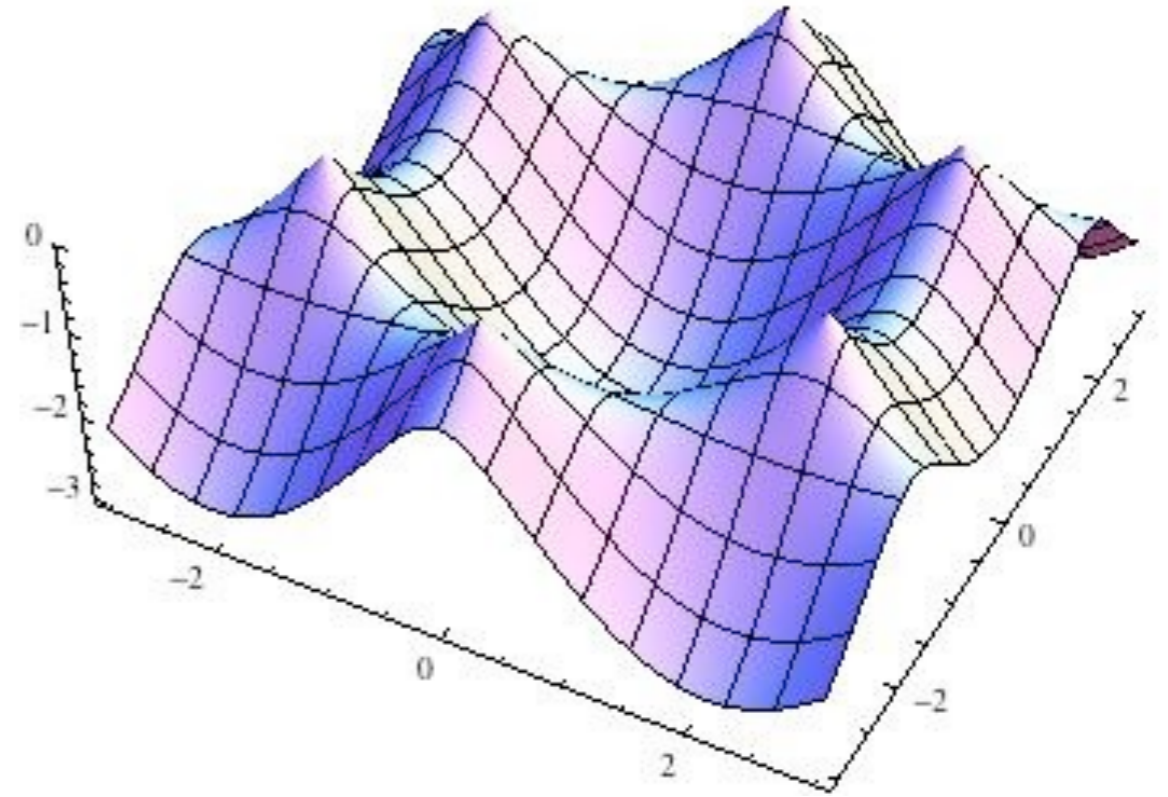
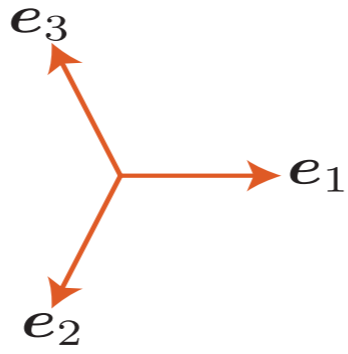
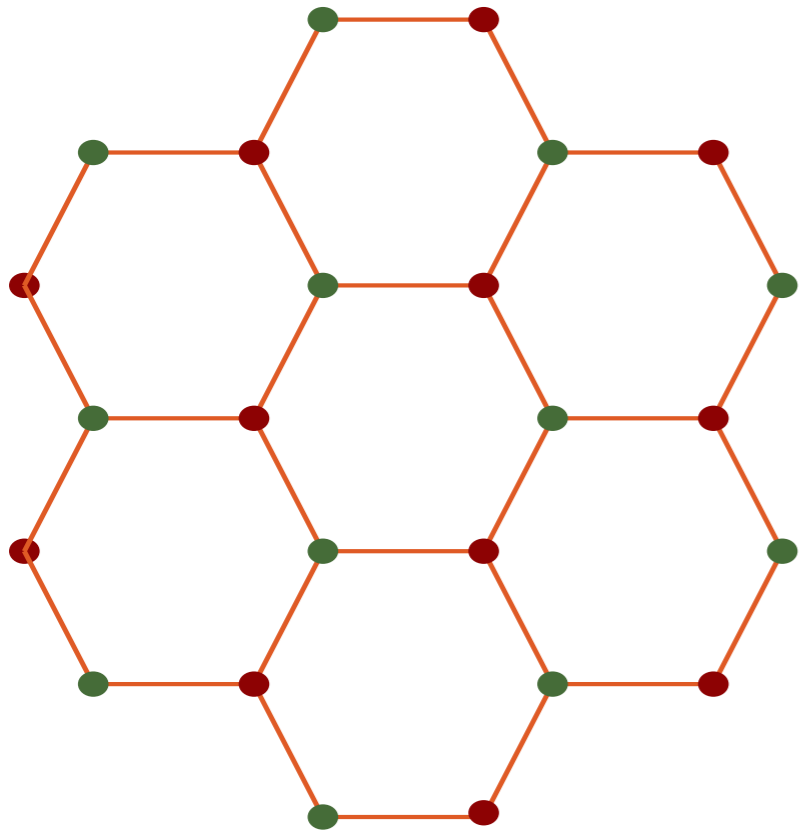
where $v = 3t/2$; below we set $v = 1$. Now define $\bar{\Psi} = \Psi^\dagger \rho^z \tau^z$. Then we can write the imaginary time Lagrangian as

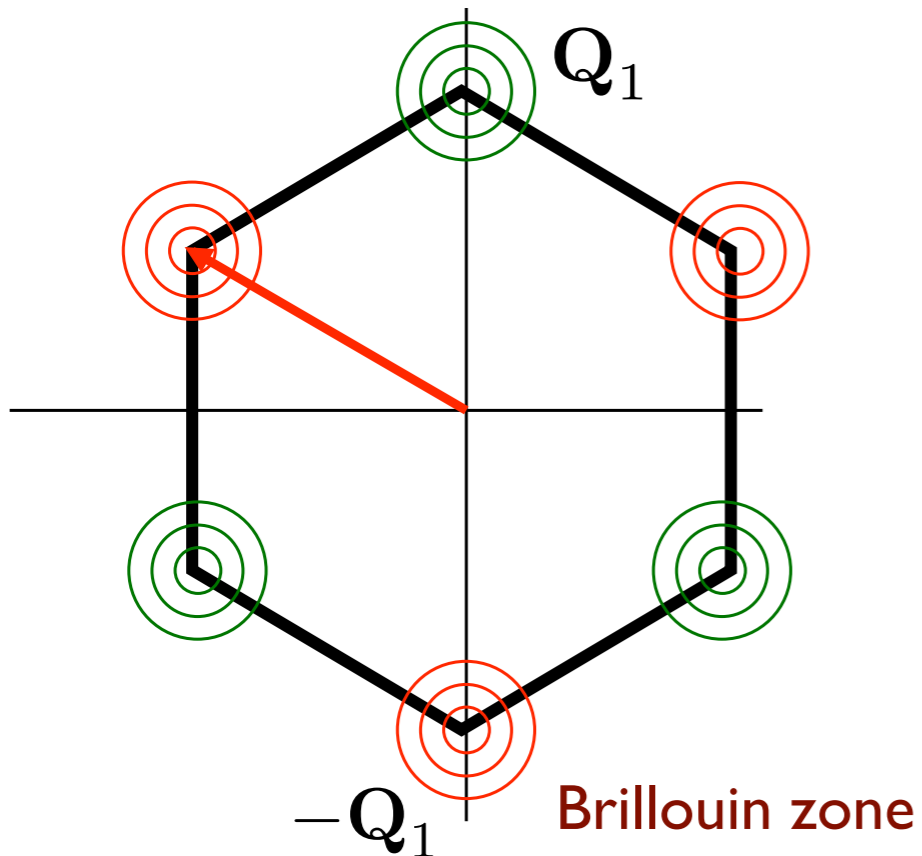
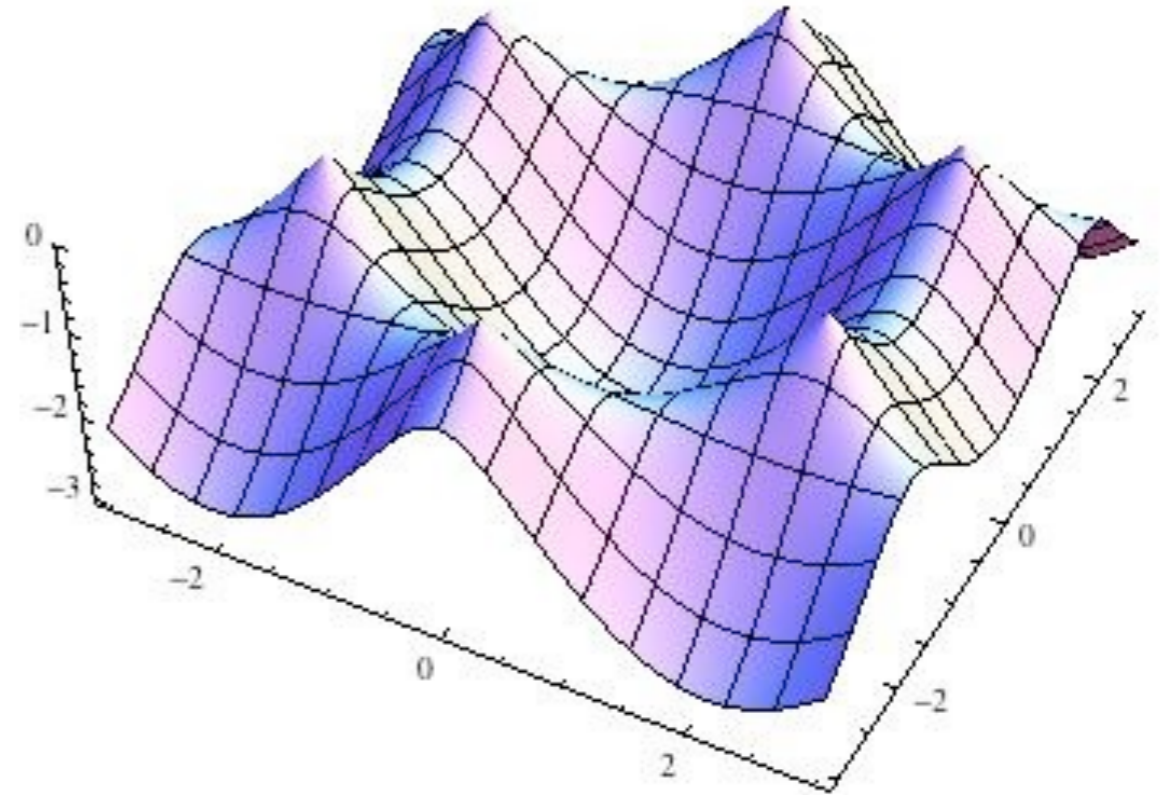
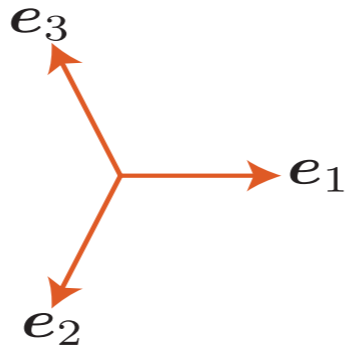
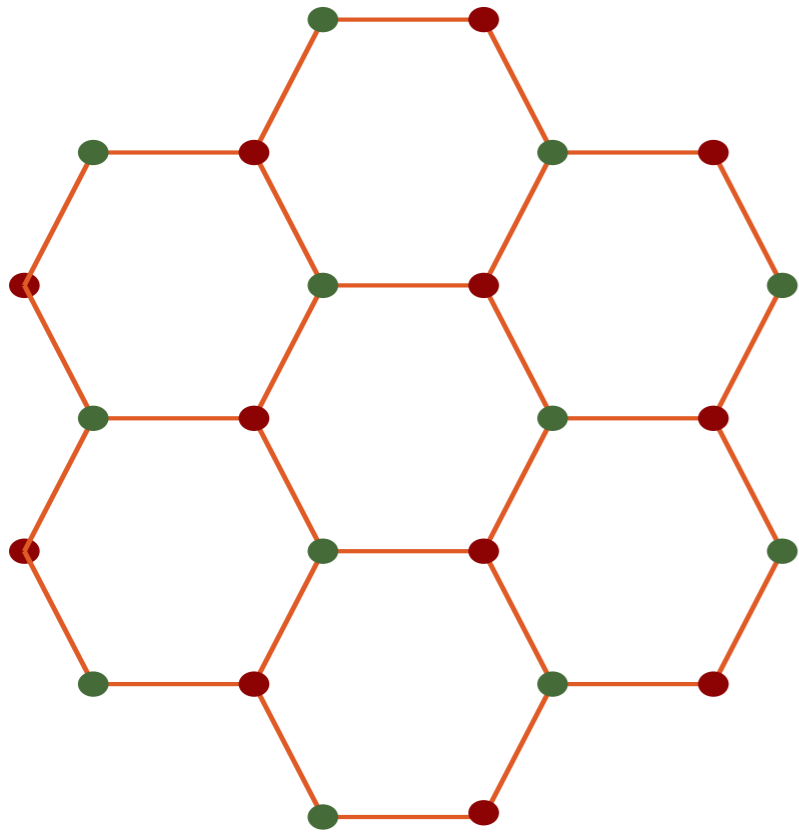
$$\mathcal{L}_0 = -i\bar{\Psi} (\omega\gamma_0 + k_x\gamma_1 + k_y\gamma_2) \Psi \tag{9}$$

where

$$\gamma_0 = -\rho^z \tau^z \quad \gamma_1 = \rho^z \tau^x \quad \gamma_2 = -\tau^y \tag{10}$$

Exercise: Observe that \mathcal{L}_0 is invariant under the scaling transformation $x' = xe^{-\ell}$ and $\tau' = \tau e^{-\ell}$. Write the Hubbard interaction U in terms of the Dirac fermions, and show that it has the tree-level scaling transformation $U' = Ue^{-\ell}$. So argue that all short-range interactions are *irrelevant* in the Dirac semi-metal phase.





The theory of free Dirac fermions is invariant under conformal transformations of spacetime. This is a realization of a simple conformal field theory in $2+1$ dimensions: a CFT3

The Hubbard Model at large U

$$H = - \sum_{i,j} t_{ij} c_{i\alpha}^\dagger c_{j\alpha} + U \sum_i \left(n_{i\uparrow} - \frac{1}{2} \right) \left(n_{i\downarrow} - \frac{1}{2} \right) - \mu \sum_i c_{i\alpha}^\dagger c_{i\alpha}$$

In the limit of large U , and at a density of one particle per site, this maps onto the Heisenberg antiferromagnet

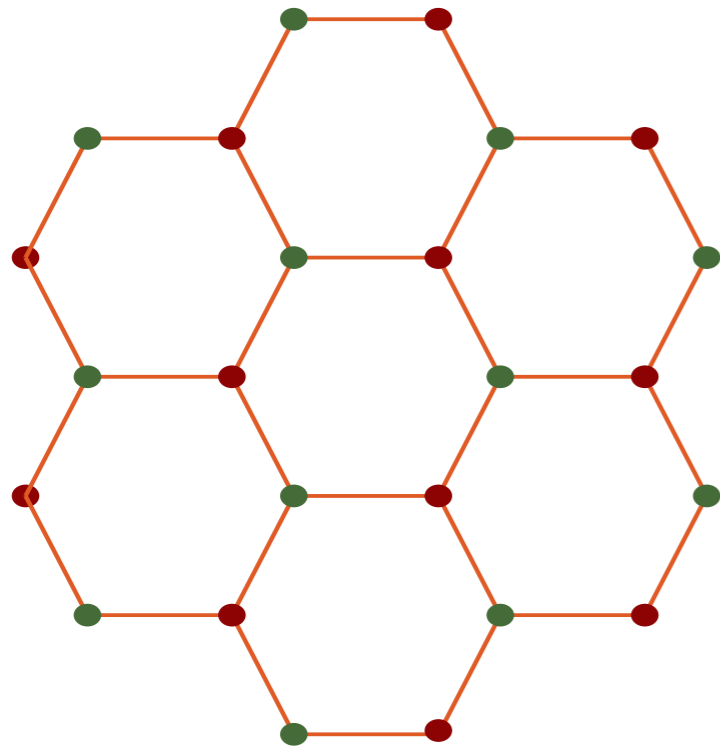
$$H_{AF} = \sum_{i < j} J_{ij} S_i^a S_j^a$$

where $a = x, y, z$,

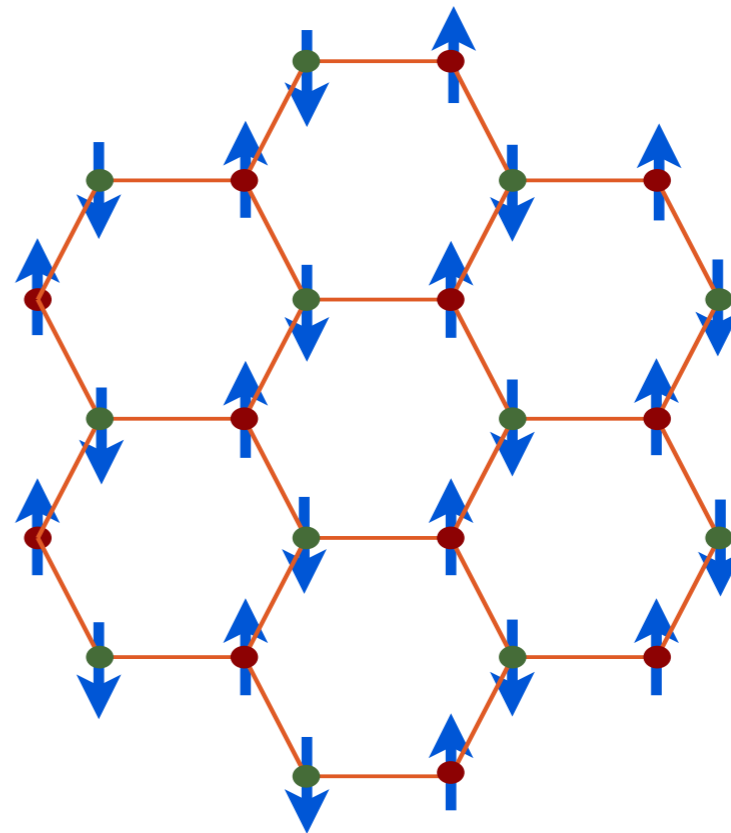
$$S_i^a = \frac{1}{2} c_{i\alpha}^{a\dagger} \sigma_{\alpha\beta}^a c_{i\beta},$$

with σ^a the Pauli matrices and

$$J_{ij} = \frac{4t_{ij}^2}{U}$$



Dirac
semi-metal



Insulating
antiferromagnet
with Neel order

U/t

Antiferromagnetism

We use the operator equation (valid on each site i):

$$U \left(n_{\uparrow} - \frac{1}{2} \right) \left(n_{\downarrow} - \frac{1}{2} \right) = -\frac{2U}{3} S_i^{a2} + \frac{U}{4} \quad (11)$$

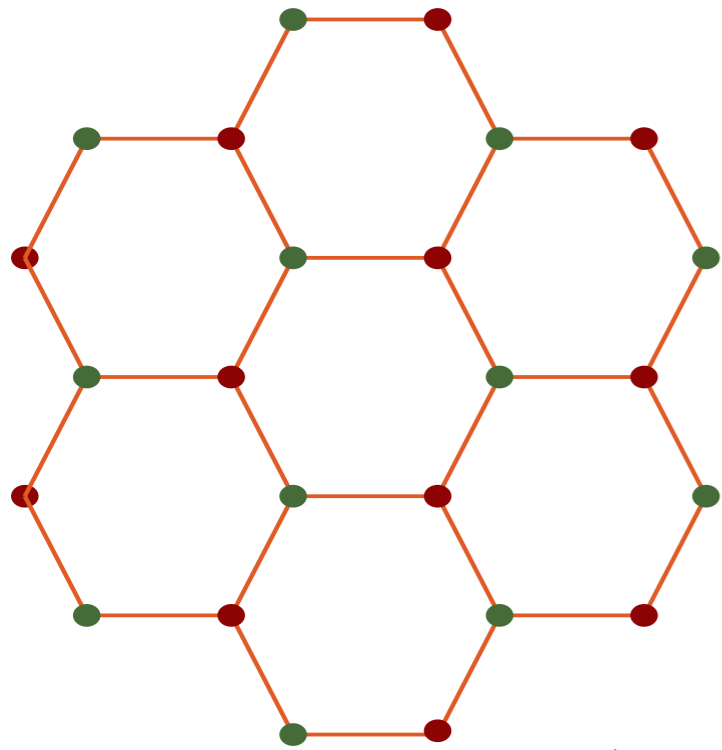
Then we decouple the interaction via

$$\exp \left(\frac{2U}{3} \sum_i \int d\tau S_i^{a2} \right) = \int \mathcal{D} J_i^a(\tau) \exp \left(- \sum_i \int d\tau \left[\frac{3}{8U} J_i^{a2} - J_i^a S_i^a \right] \right) \quad (12)$$

We now integrate out the fermions, and look for the saddle point of the resulting effective action for J_i^a .

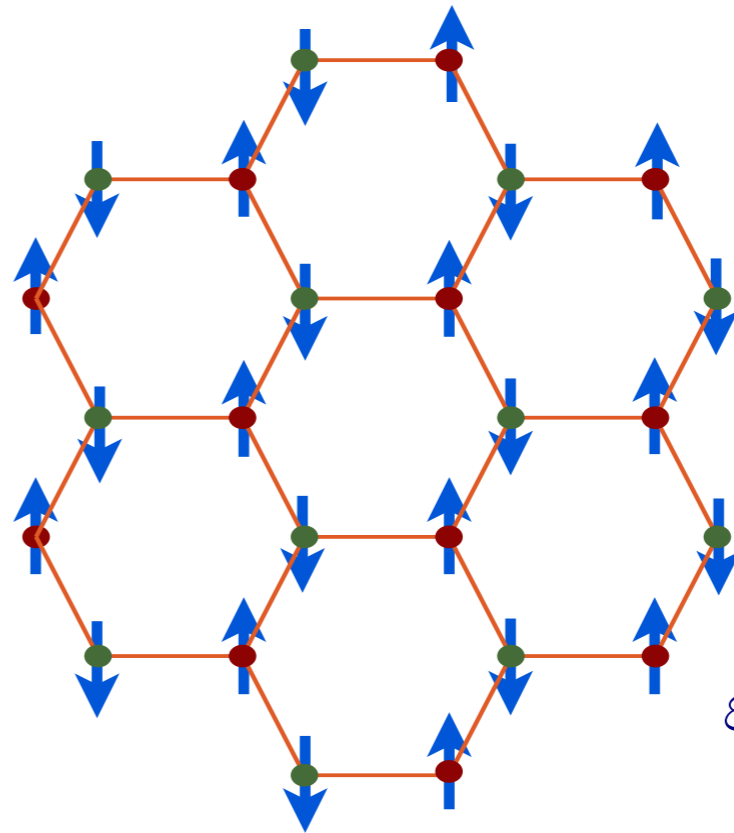
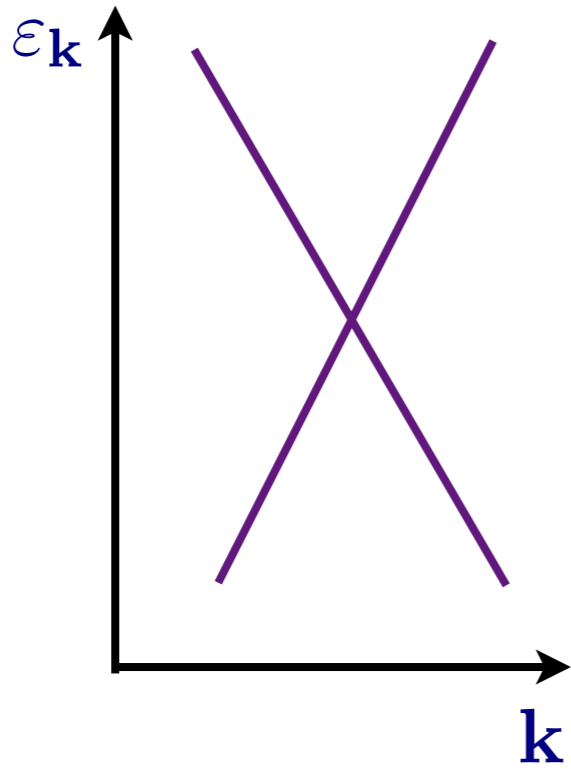
Long wavelength fluctuations about this saddle point are described by a field theory of the Néel order parameter, φ^a , coupled to the Dirac fermions in the **Gross-Neveu** model.

$$\mathcal{L} = \bar{\Psi} \gamma_{\mu} \partial_{\mu} \Psi + \frac{1}{2} \left[(\partial_{\mu} \varphi^a)^2 + s \varphi^{a2} \right] + \frac{U}{24} (\varphi^{a2})^2 - \lambda \varphi^a \bar{\Psi} \rho^z \sigma^a \Psi$$



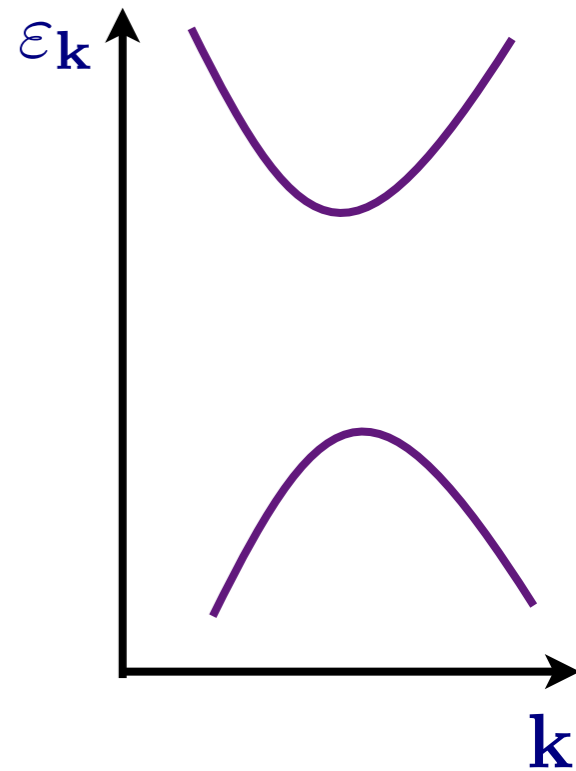
Dirac
semi-metal

$$\langle \varphi^a \rangle = 0$$

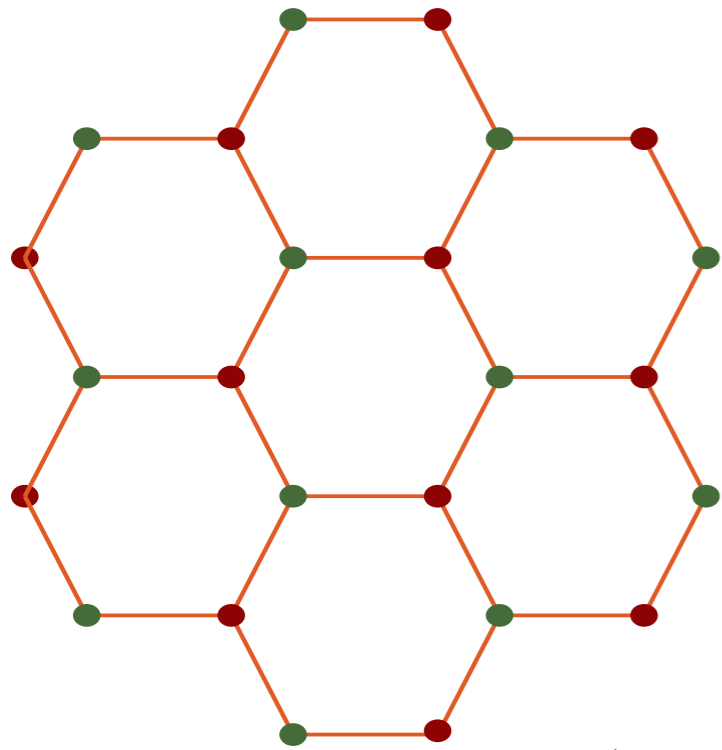


Insulating
antiferromagnet
with Neel order

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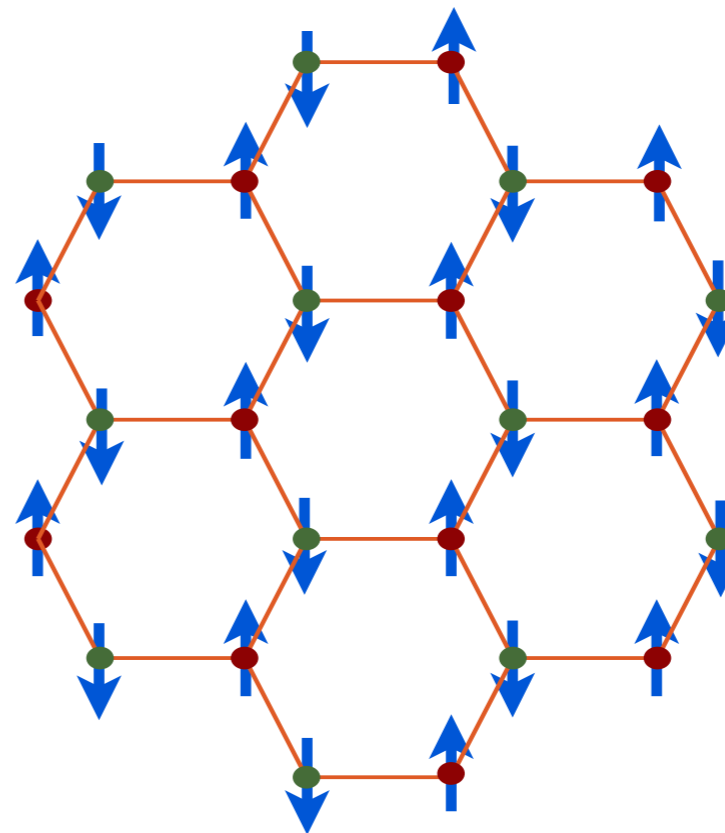
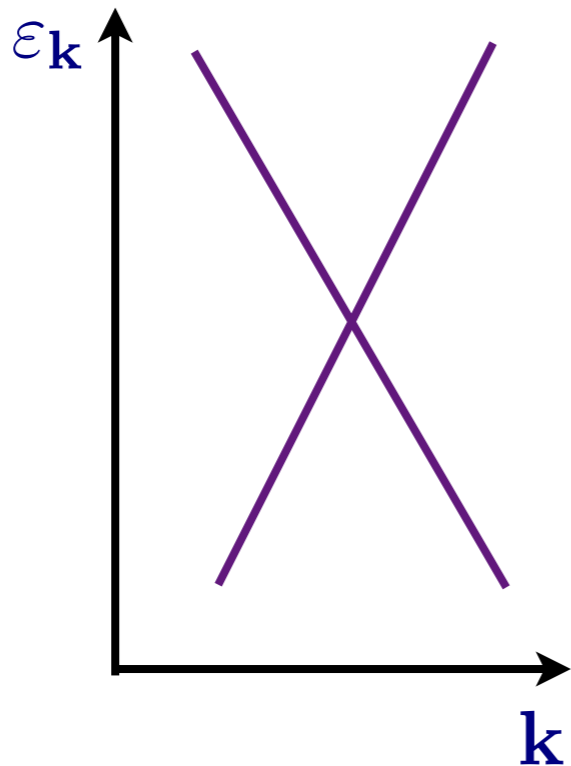


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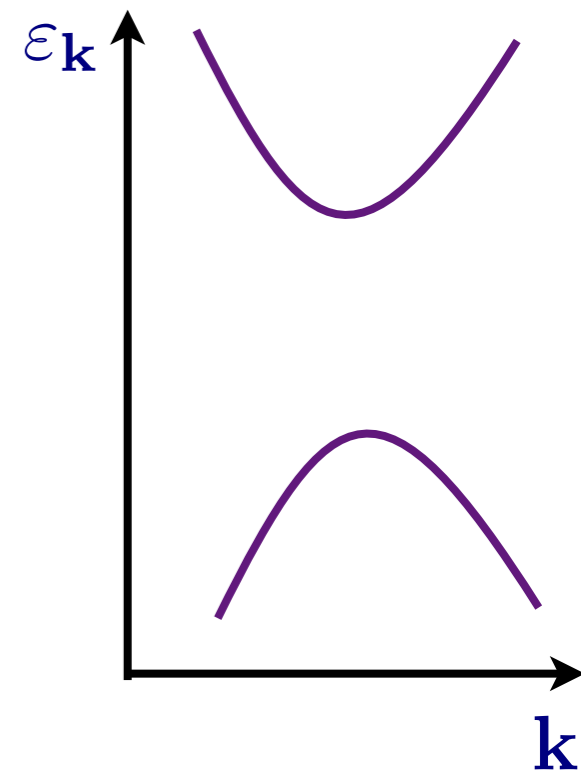
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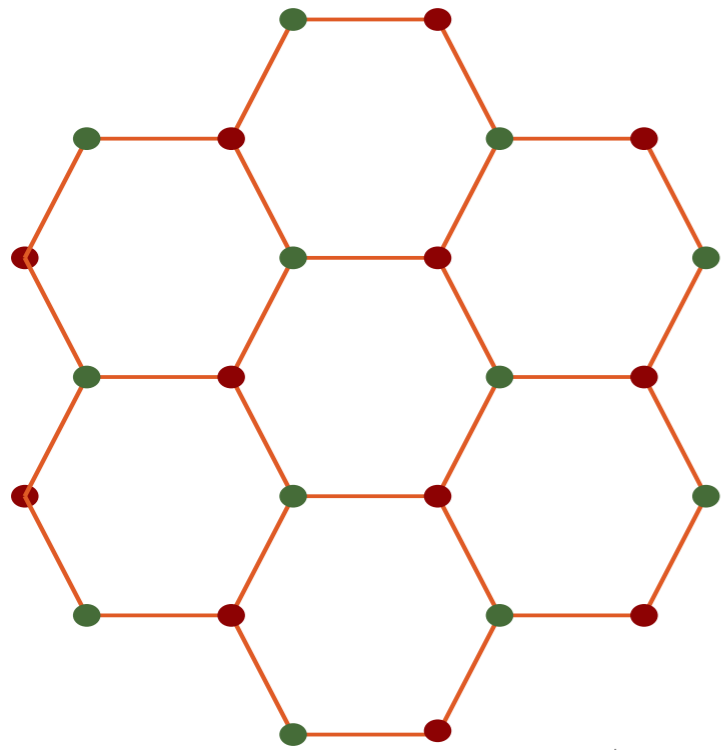
Insulating
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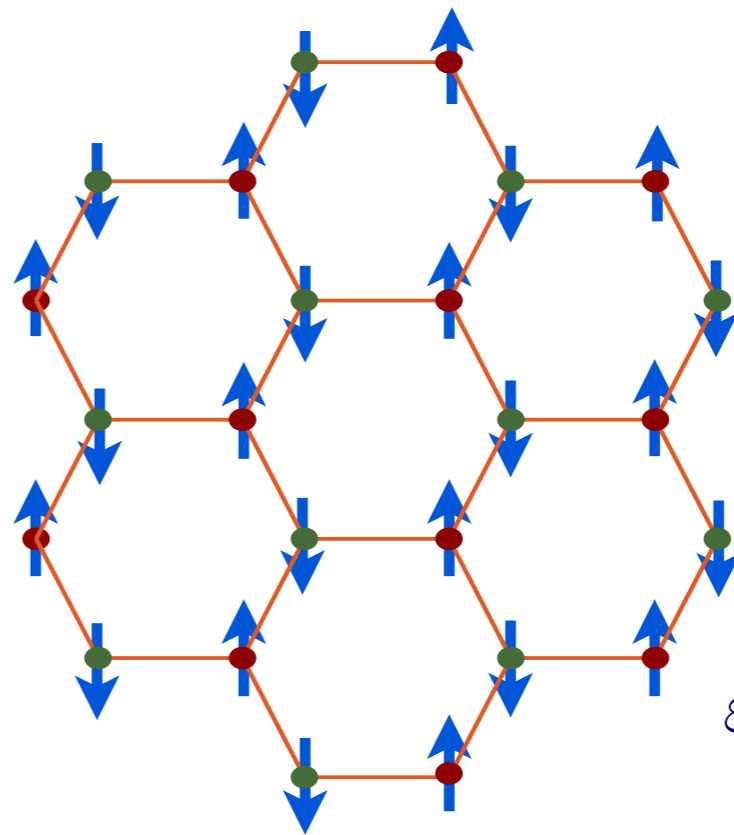
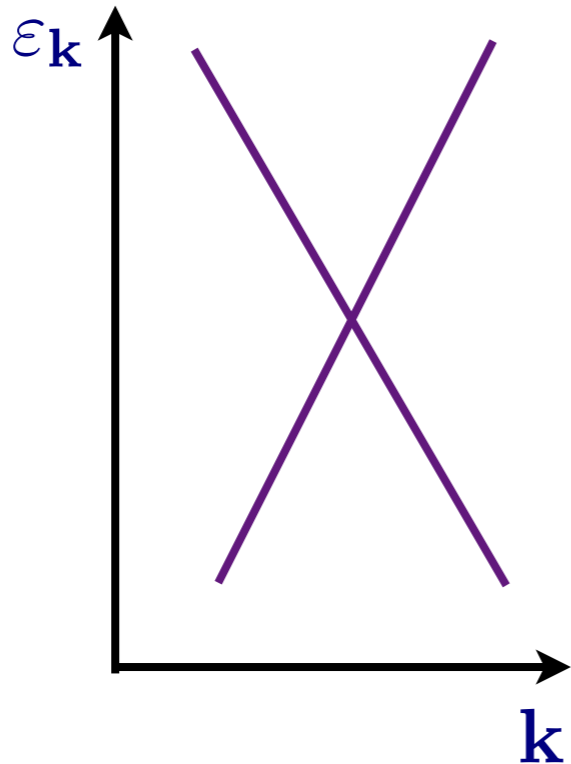
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At the quantum critical point, the non-linear couplings λ and u in the Gross-Neveu model reach non-zero fixed-point values under the renormalization group flow. The critical theory is an *interacting* CFT₃



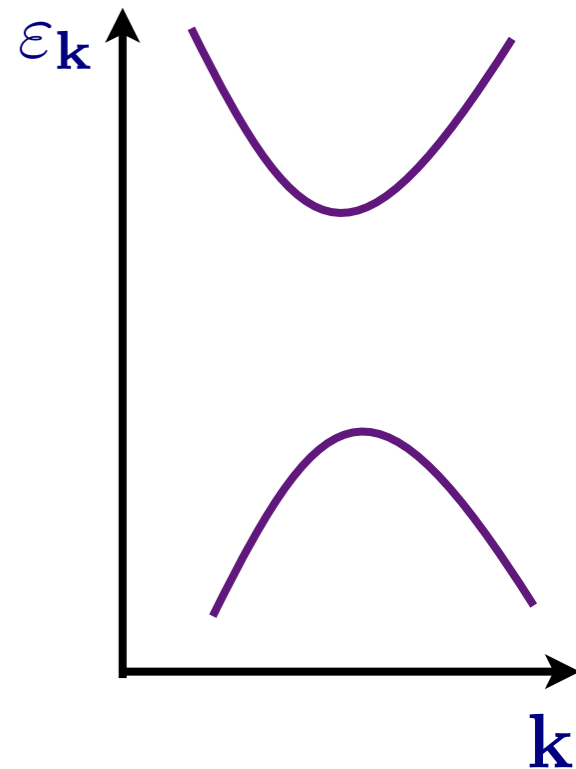
Dirac
semi-metal

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Insulating
antiferromagnet
with Neel order

$$\langle \varphi^a \rangle \neq 0$$



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Free CFT3

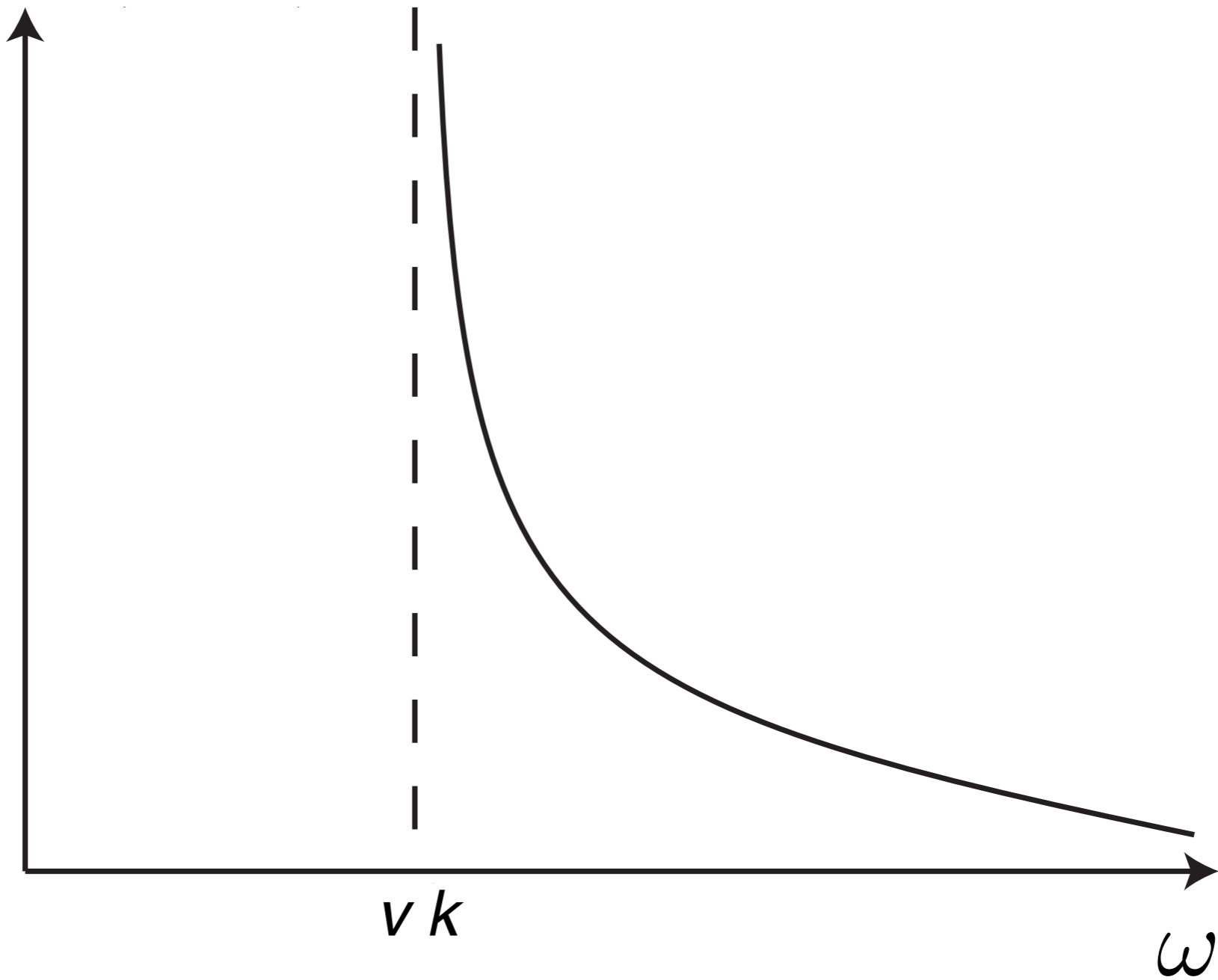
Interacting CFT3
with many-body entanglement

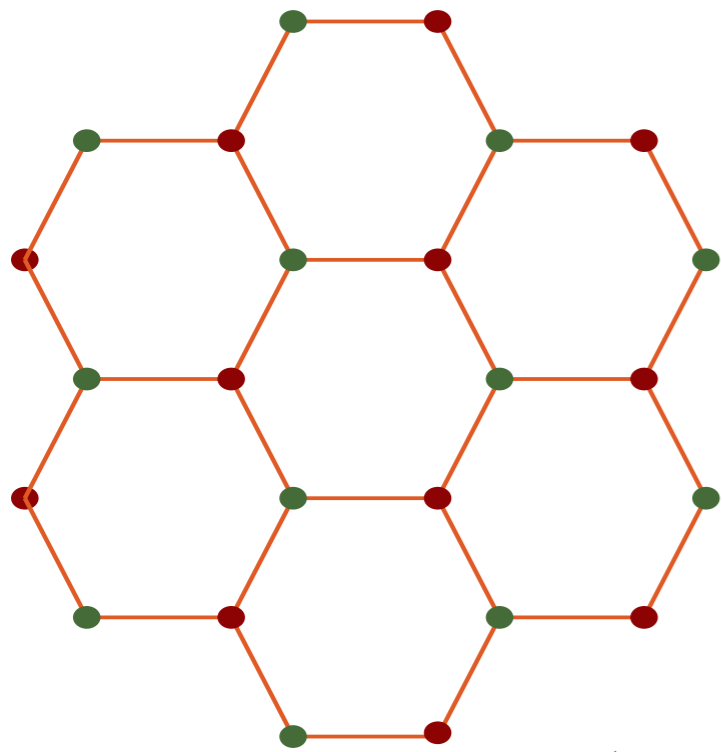
Electron Green's function for the interacting CFT3

$$G(k, \omega) = \langle \Psi(k, \omega); \Psi^\dagger(k, \omega) \rangle \sim \frac{i\omega + vk_x\tau^y + vk_y\tau^x\rho^z}{(\omega^2 + v^2k_x^2 + v^2k_y^2)^{1-\eta/2}}$$

where $\eta > 0$ is the *anomalous dimension* of the fermion. Note that this leads to a fermion spectral density which has no quasiparticle pole: thus the quantum critical point has no well-defined quasiparticle excitations.

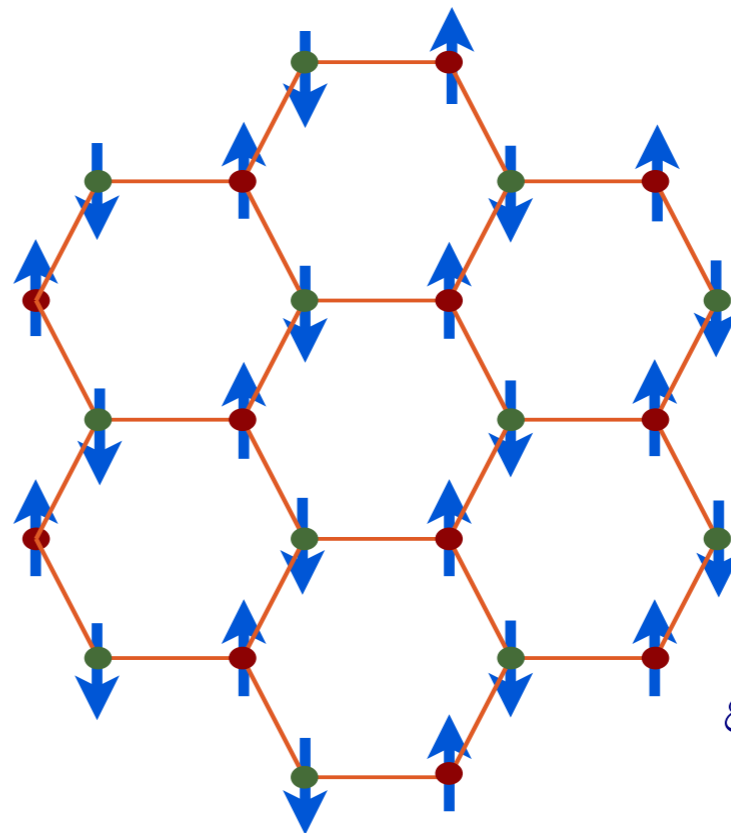
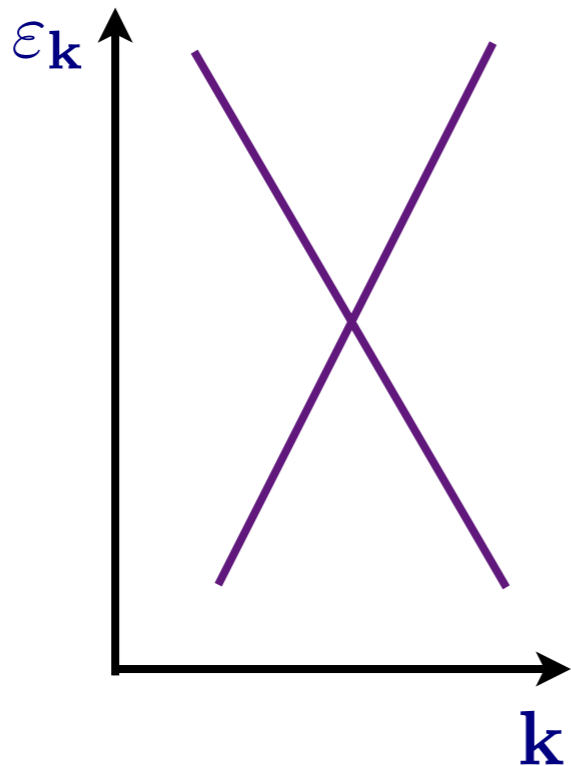
$\text{Im}G(k, \omega)$





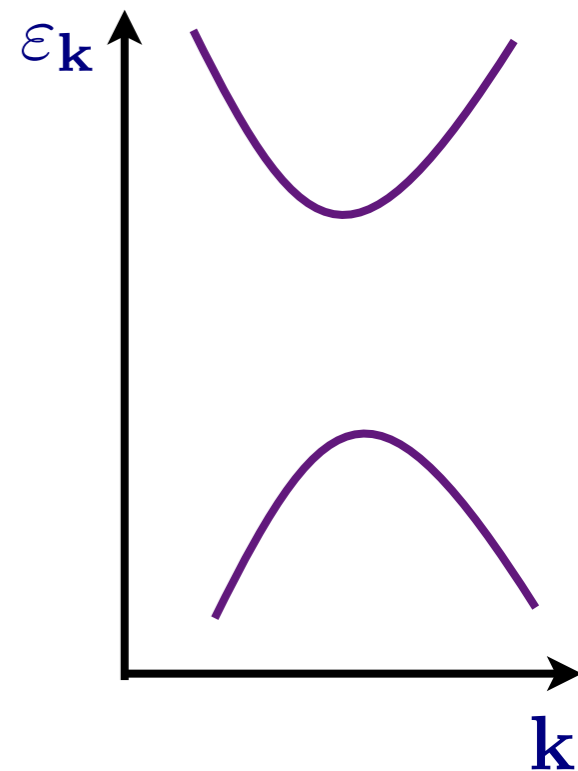
Dirac
semi-metal

$$\langle \varphi^a \rangle = 0$$



Insulating
antiferromagnet
with Neel order

$$\langle \varphi^a \rangle \neq 0$$



S

Quantum phase transition described by a strongly-coupled conformal field theory without well-defined quasiparticles

Electrical transport

The conserved electrical current is

$$J_\mu = -i\bar{\Psi}\gamma_\mu\Psi. \quad (1)$$

Let us compute its two-point correlator, $K_{\mu\nu}(k)$ at a spacetime momentum k_μ at $T = 0$. At leading order, this is given by a one fermion loop diagram which evaluates to

$$\begin{aligned} K_{\mu\nu}(k) &= \int \frac{d^3p}{8\pi^3} \frac{\text{Tr} [\gamma_\mu (i\gamma_\lambda p_\lambda + m\rho^z \sigma^z) \gamma_\nu (i\gamma_\delta (k_\delta + p_\delta) + m\rho^z \sigma^z)]}{(p^2 + m^2)((p+k)^2 + m^2)} \\ &= -\frac{2}{\pi} \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \int_0^1 dx \frac{k^2 x(1-x)}{\sqrt{m^2 + k^2 x(1-x)}}, \end{aligned} \quad (2)$$

where the mass $m = 0$ in the semi-metal and at the quantum critical point, while $m = |\lambda N_0|$ in the insulator. Note that the current correlation is purely transverse, and this follows from the requirement of current conservation

$$k_\mu K_{\mu\nu} = 0. \quad (3)$$

Of particular interest to us is the K_{00} component, after analytic continuation to Minkowski space where the spacetime momentum k_μ is replaced by (ω, k) . The conductivity is obtained from this correlator via the Kubo formula

$$\sigma(\omega) = \lim_{k \rightarrow 0} \frac{-i\omega}{k^2} K_{00}(\omega, k). \quad (4)$$

In the insulator, where $m > 0$, analysis of the integrand in Eq. (2) shows that the spectral weight of the density correlator has a gap of $2m$ at $k = 0$, and the conductivity in Eq. (4) vanishes.

These properties are as expected in any insulator.

In the metal, and at the critical point, where $m = 0$, the fermionic spectrum is gapless, and so is that of the charge correlator. The density correlator in Eq. (2) and the conductivity in Eq. (4) evaluate to the simple universal results

$$\begin{aligned} K_{00}(\omega, k) &= \frac{1}{4} \frac{k^2}{\sqrt{k^2 - \omega^2}} \\ \sigma(\omega) &= 1/4. \end{aligned} \quad (5)$$

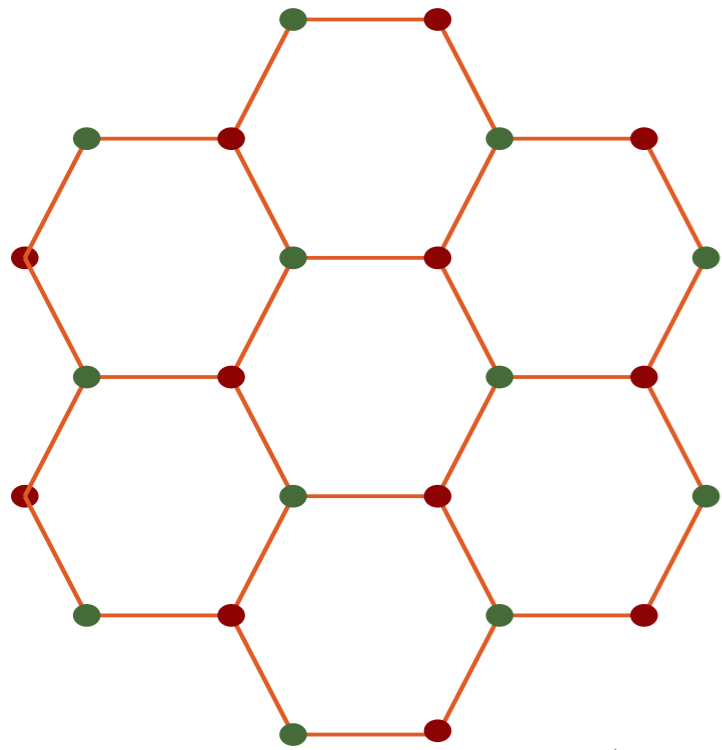
Going beyond one-loop, we find *no change* in these results in the

semi-metal to all orders in perturbation theory. At the quantum critical point, there are no anomalous dimensions for the conserved current, but the amplitude does change yielding

$$\begin{aligned} K_{00}(\omega, k) &= \mathcal{K} \frac{k^2}{\sqrt{k^2 - \omega^2}} \\ \sigma(\omega) &= \mathcal{K}, \end{aligned} \tag{6}$$

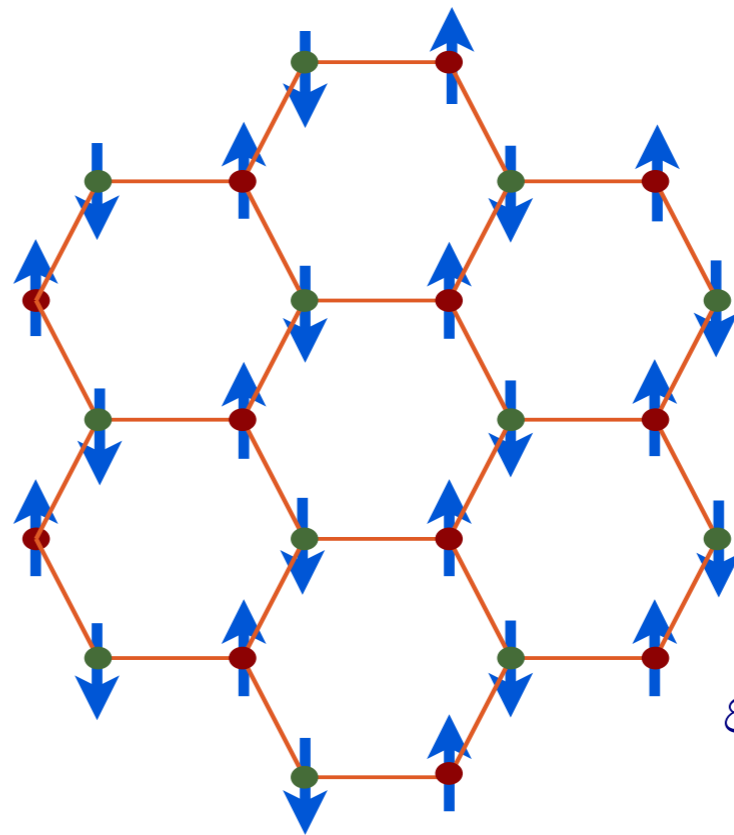
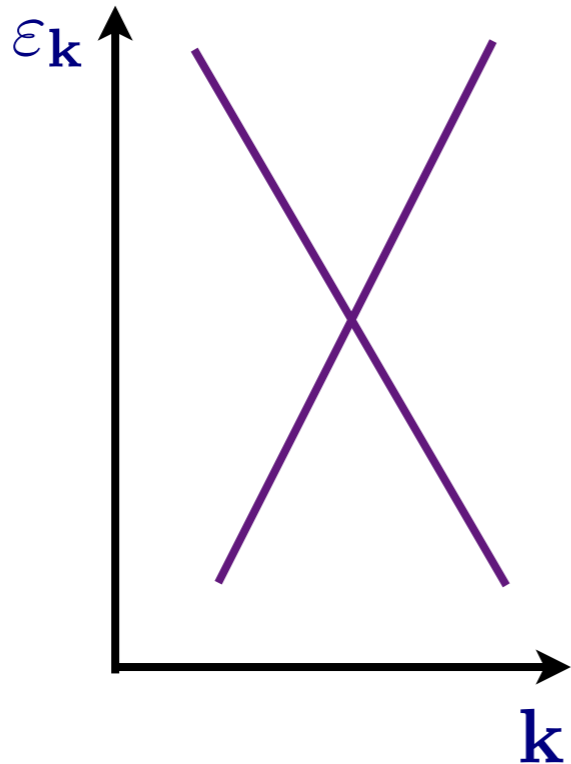
where \mathcal{K} is a universal number dependent only upon the universality class of the quantum critical point. The value of the \mathcal{K} for the Gross-Neveu model is not known exactly, but can be estimated by computations in the $(3 - d)$ or $1/N$ expansions.

$$\text{Also note } K_{\mu\nu} = \mathcal{K}|k| \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right)$$



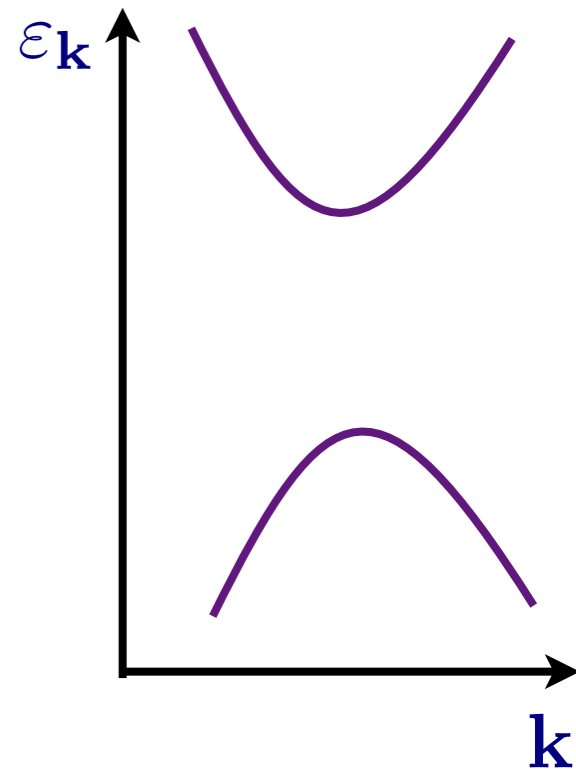
Dirac
semi-metal

$$\langle \varphi^a \rangle = 0$$



Insulating
antiferromagnet
with Neel order

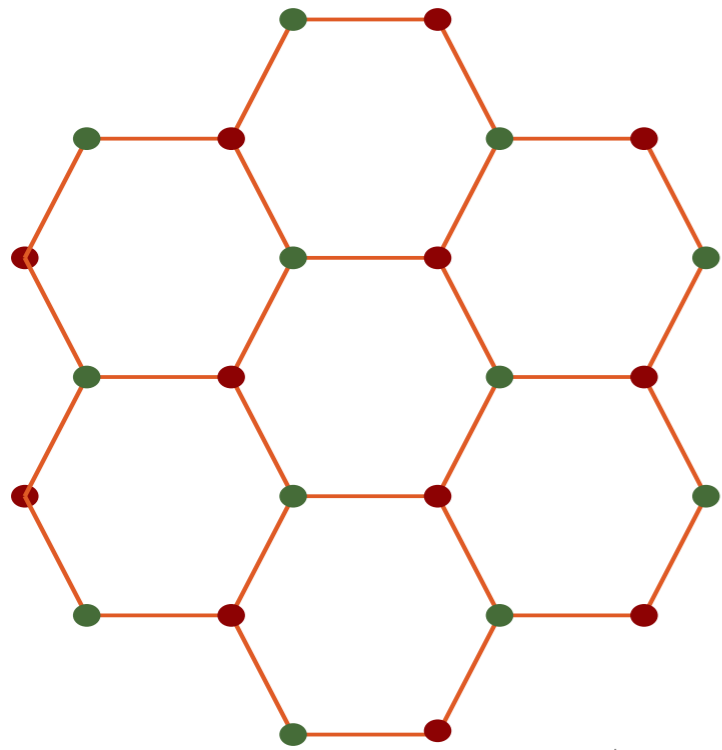
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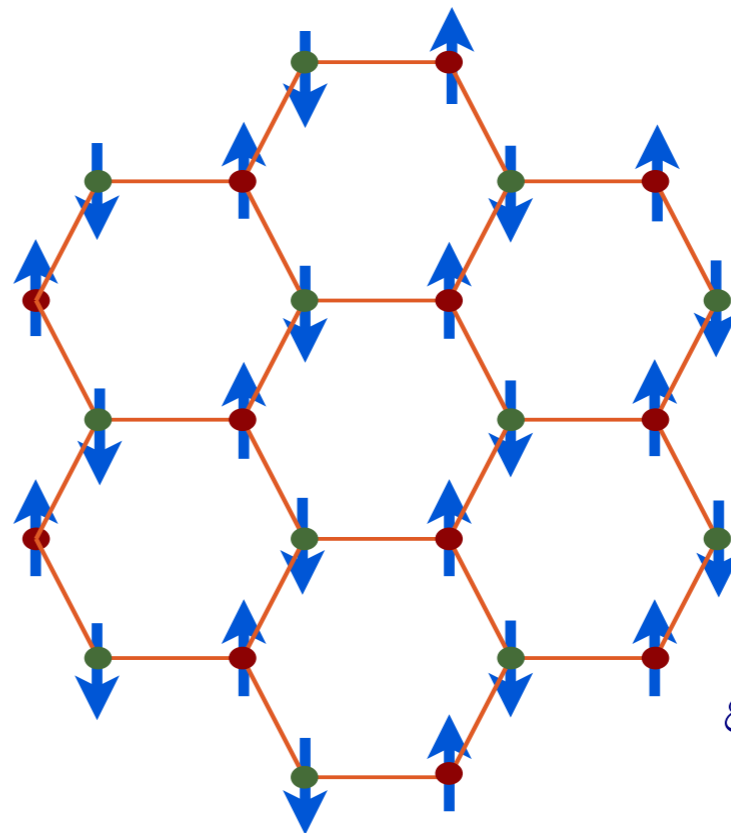
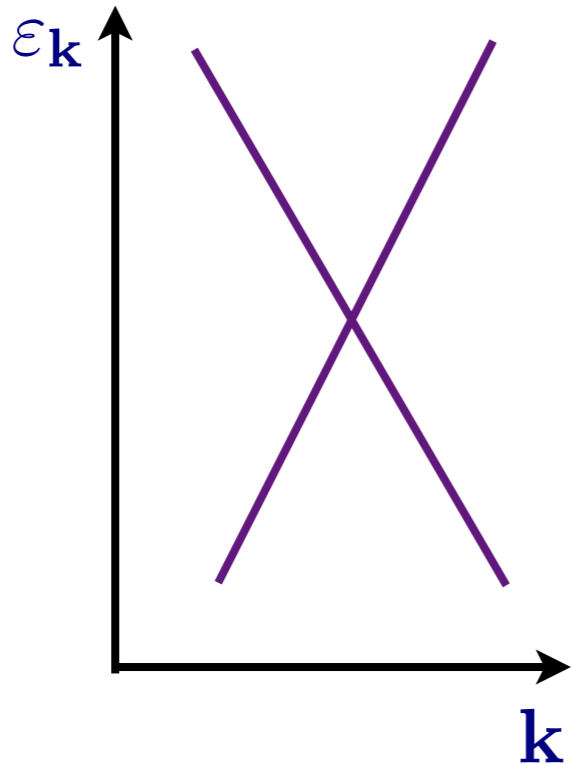
Free CFT3

Interacting CFT3
with long-range entanglement



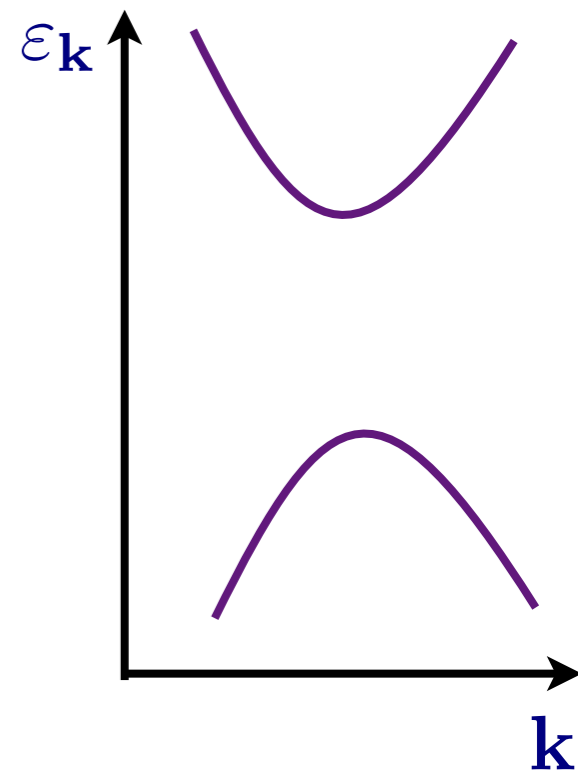
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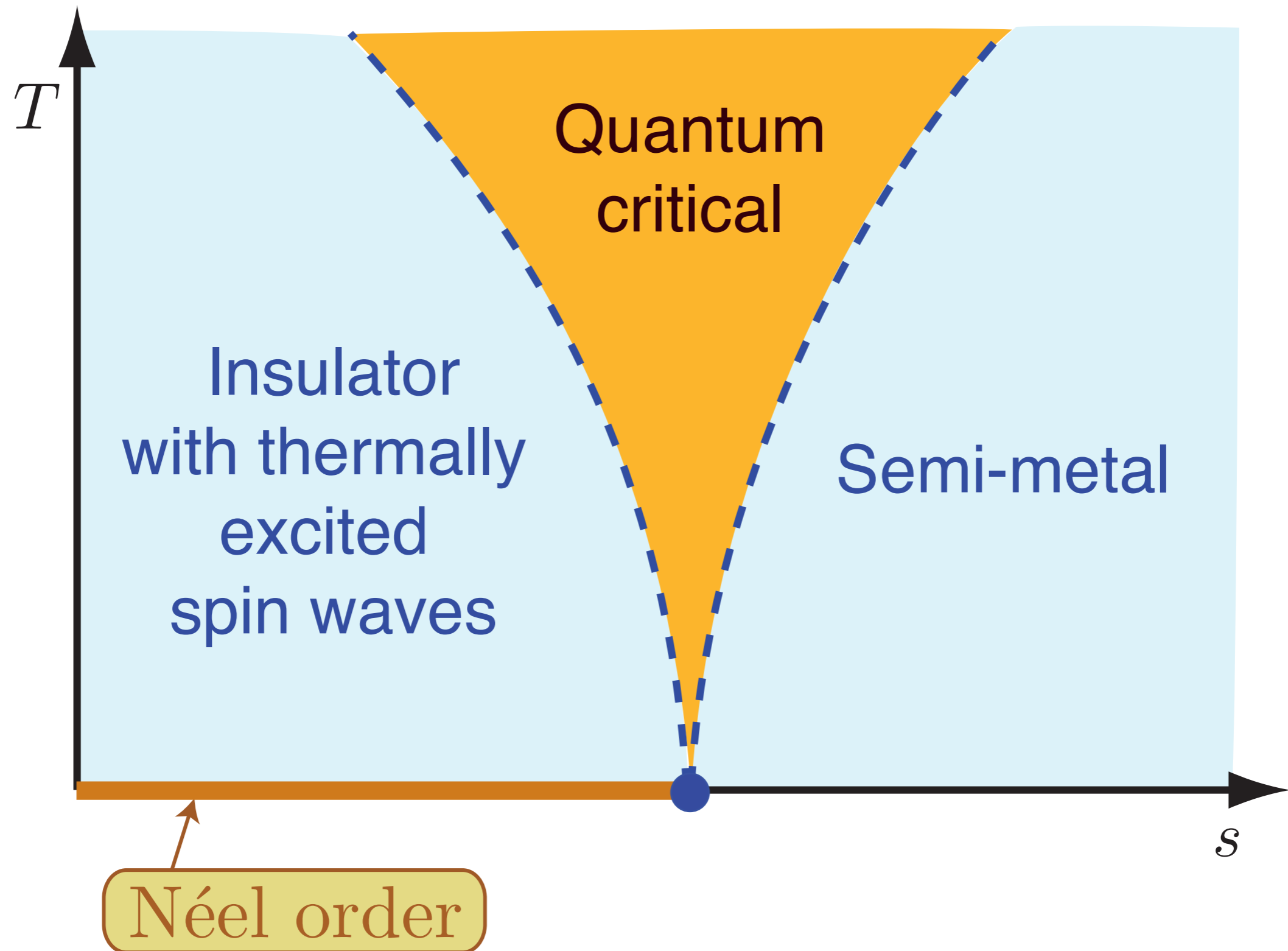


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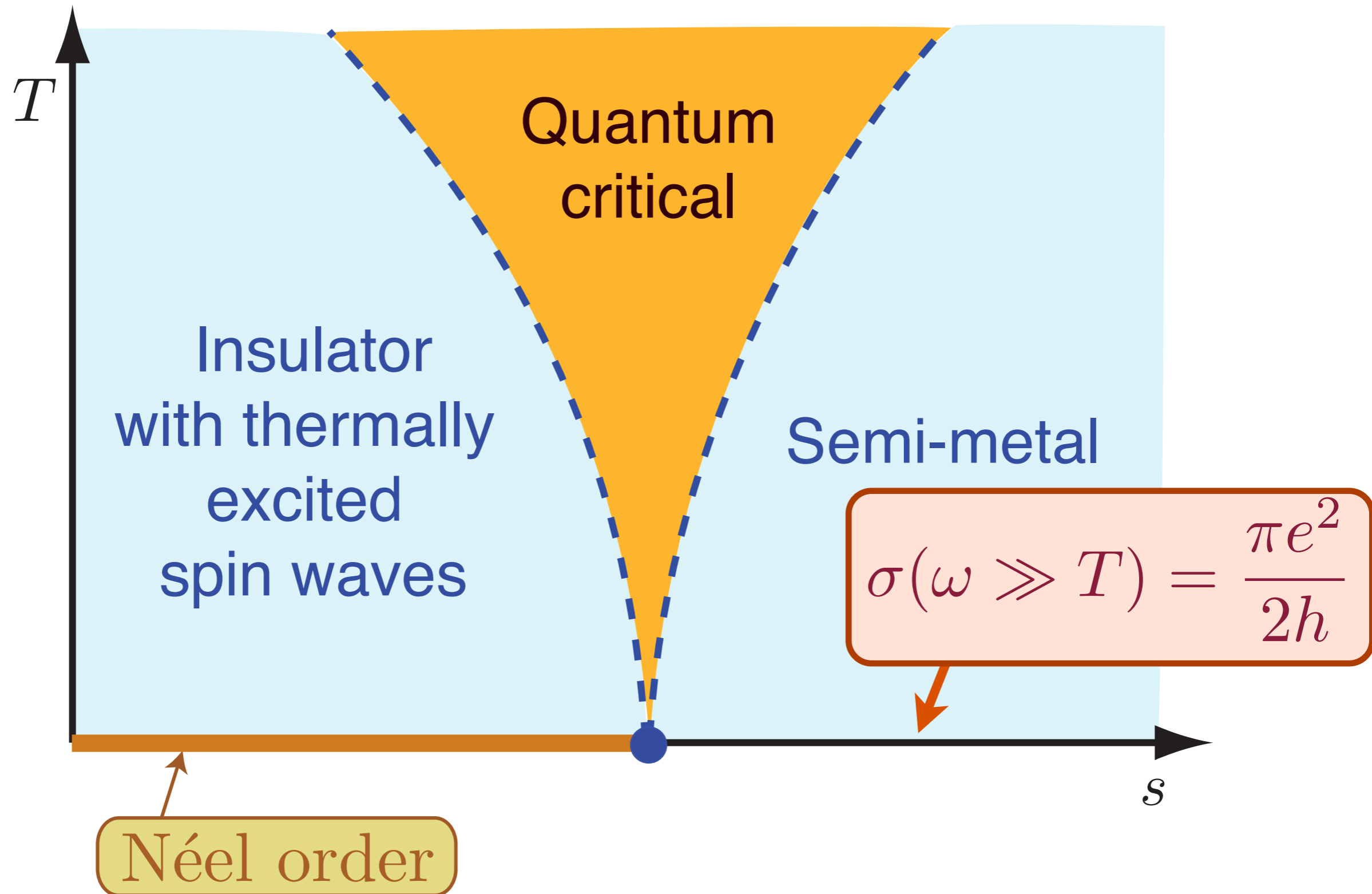
$$\sigma(\omega) = \frac{\pi e^2}{2h}$$

$$\sigma(\omega) = \frac{\mathcal{K} e^2}{h}$$

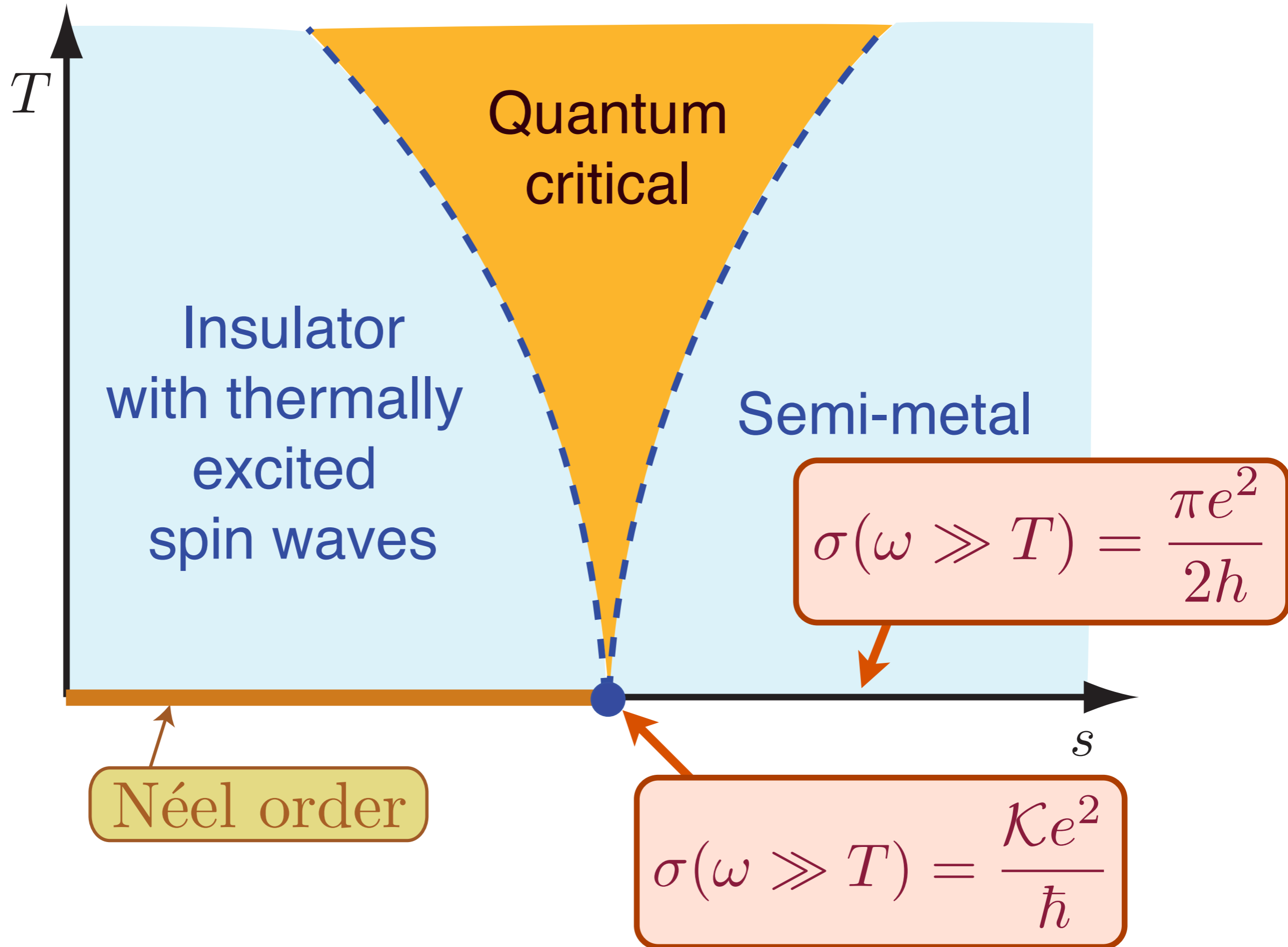
Phase diagram at non-zero temperatures



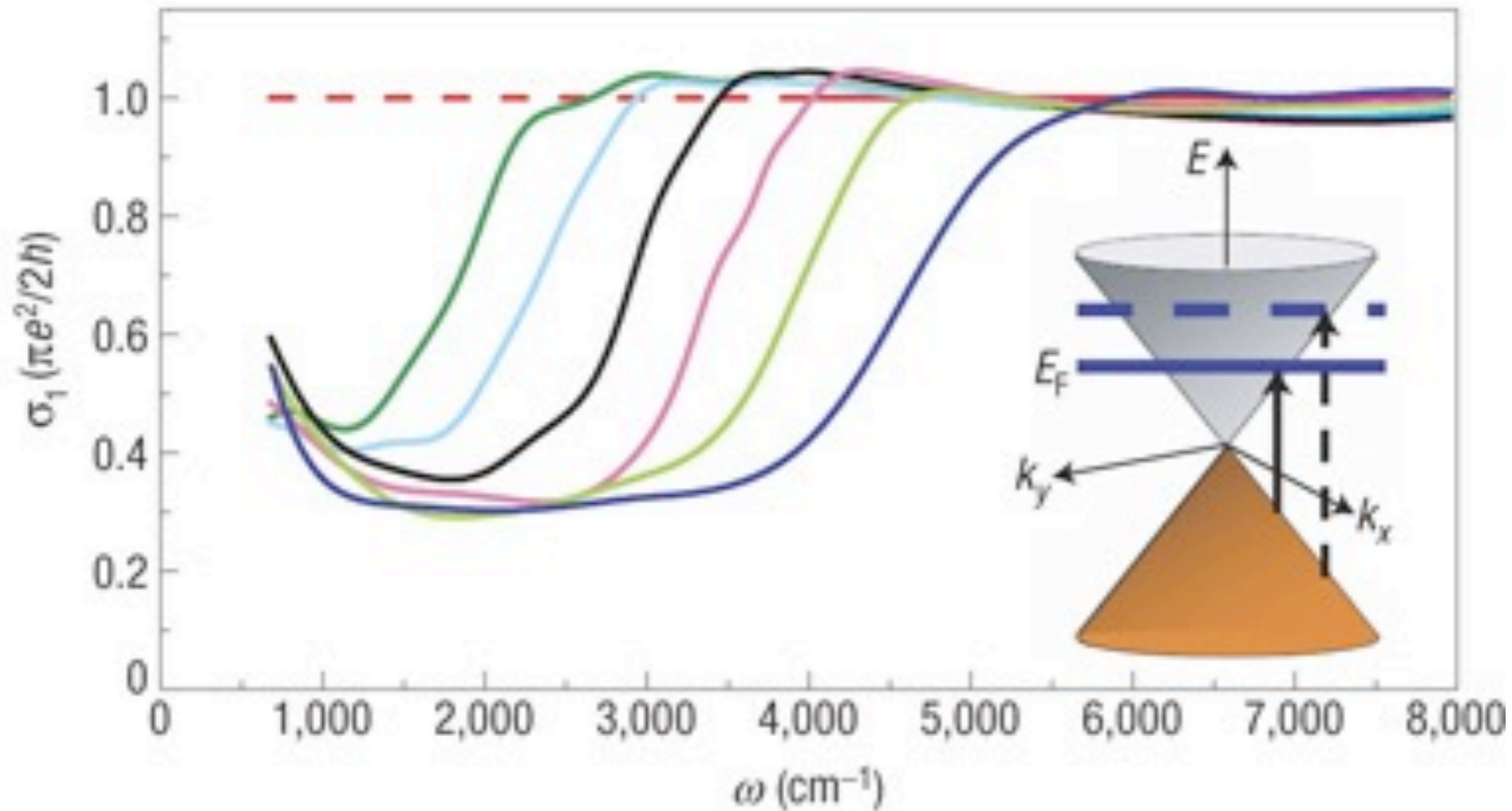
Phase diagram at non-zero temperatures



Phase diagram at non-zero temperatures



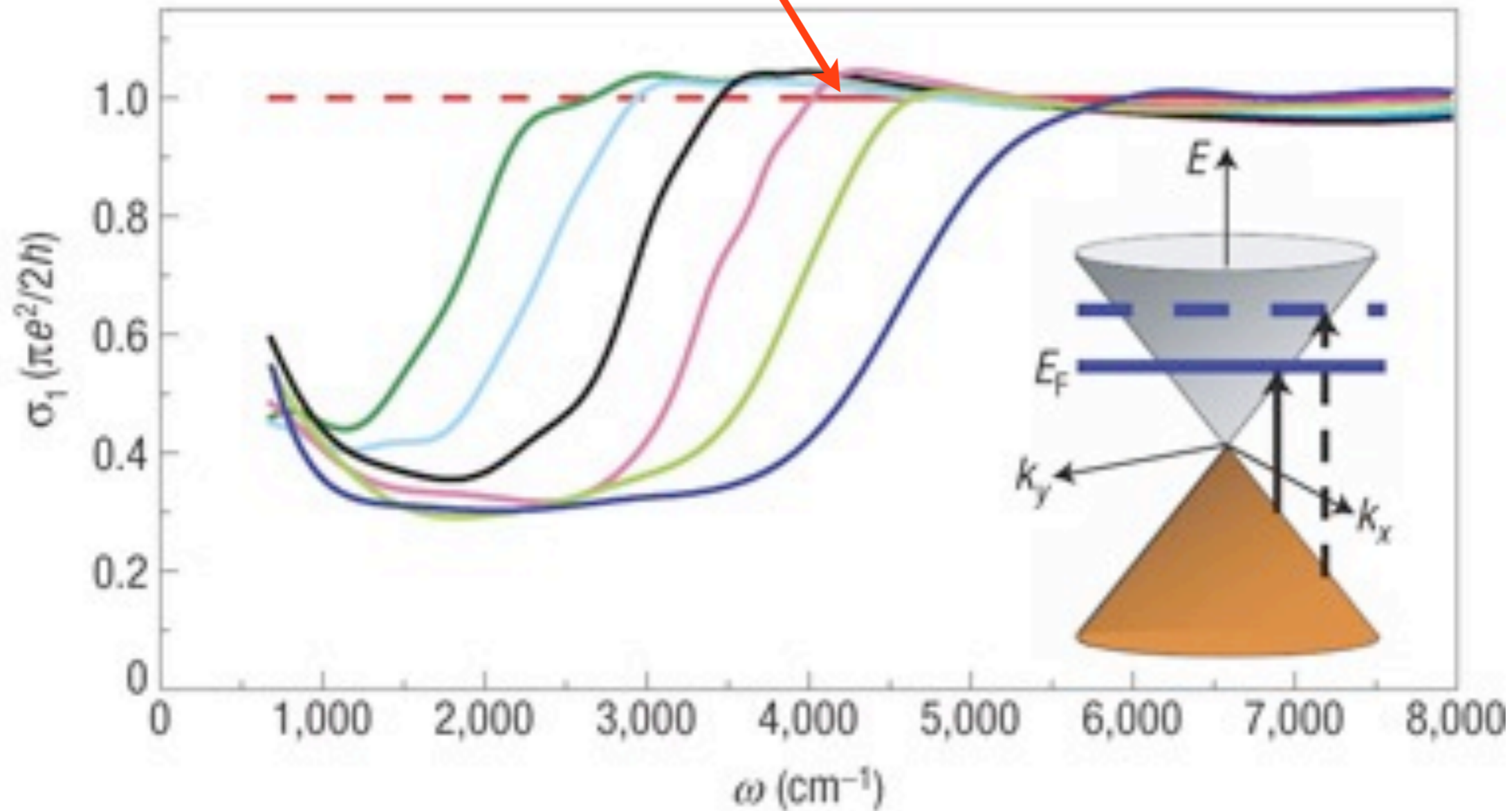
Optical conductivity of graphene



Z. Q. Li, E. A. Henriksen, Z. Jiang, Z. Hao, M. C. Martin, P. Kim, H. L. Stormer, and D. N. Basov, *Nature Physics* 4, 532 (2008).

Optical conductivity of graphene

Undoped graphene



Z. Q. Li, E. A. Henriksen, Z. Jiang, Z. Hao, M. C. Martin, P. Kim, H. L. Stormer, and D. N. Basov, *Nature Physics* **4**, 532 (2008).

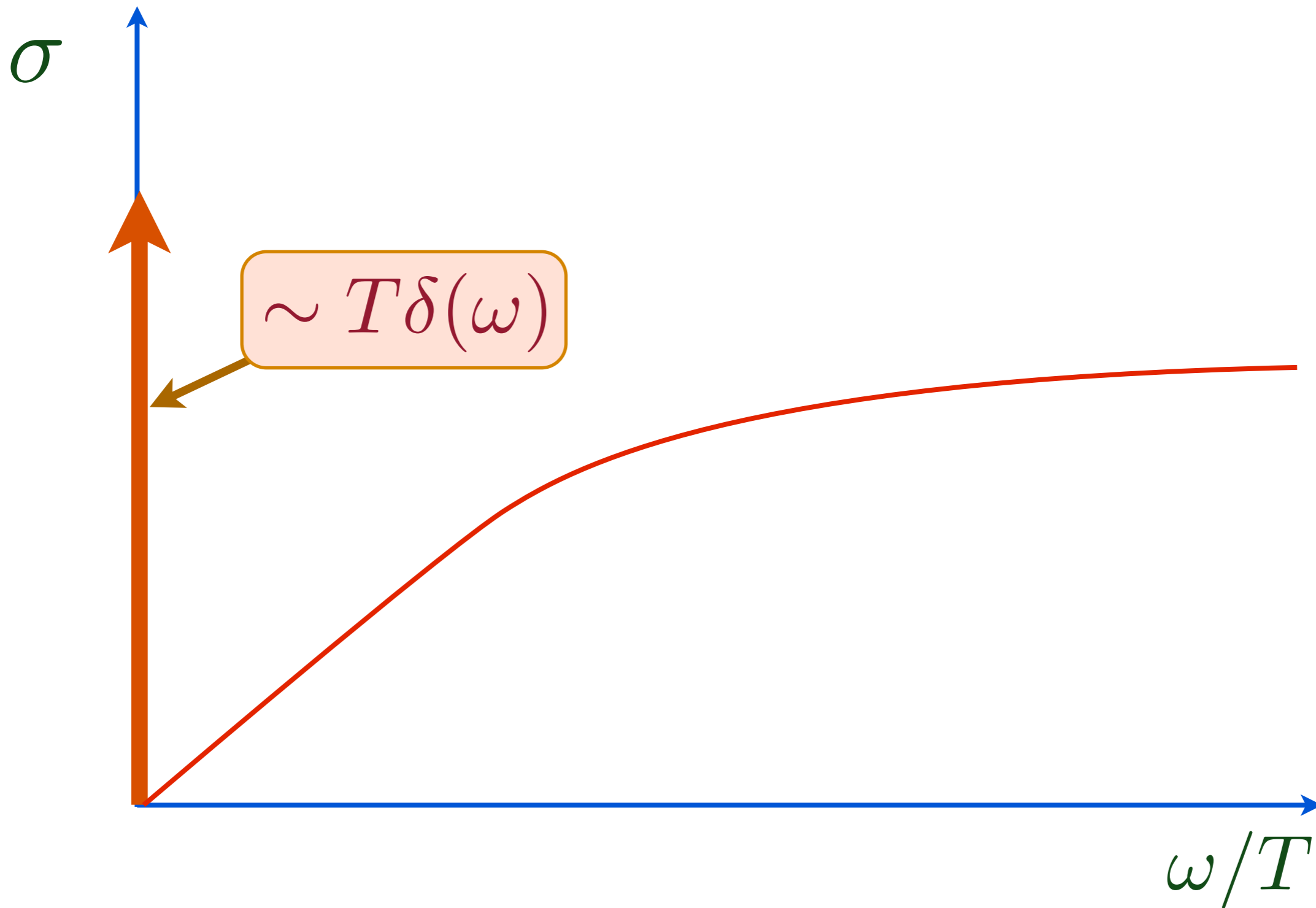
Non-zero temperatures

At the quantum-critical point at one-loop order, we can set $m = 0$, and then repeat the computation in Eq. (2) at $T > 0$. This only requires replacing the integral over the loop frequency by a summation over the Matsubara frequencies, which are quantized by odd multiples of πT . Such a computation, via Eq. (4) leads to the conductivity

$$\text{Re}[\sigma(\omega)] = (2T \ln 2) \delta(\omega) + \frac{1}{4} \tanh\left(\frac{|\omega|}{4T}\right); \quad (7)$$

the imaginary part of $\sigma(\omega)$ is the Hilbert transform of $\text{Re}[\sigma(\omega)] - 1/4$. Note that this reduces to Eq. (5) in the limit $\omega \gg T$. However, the most important new feature of Eq. (7) arises for $\omega \ll T$, where we find a delta function at zero frequency in the real part. Thus the d.c. conductivity is infinite at this order, arising from the collisionless transport of thermally excited carriers.

Electrical transport in a free CFT3 for $T > 0$



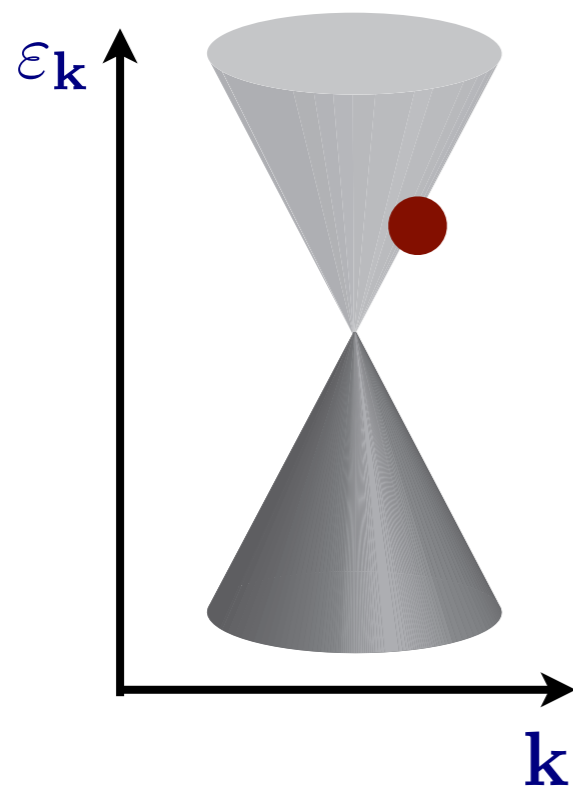
Particles



Momentum



Current



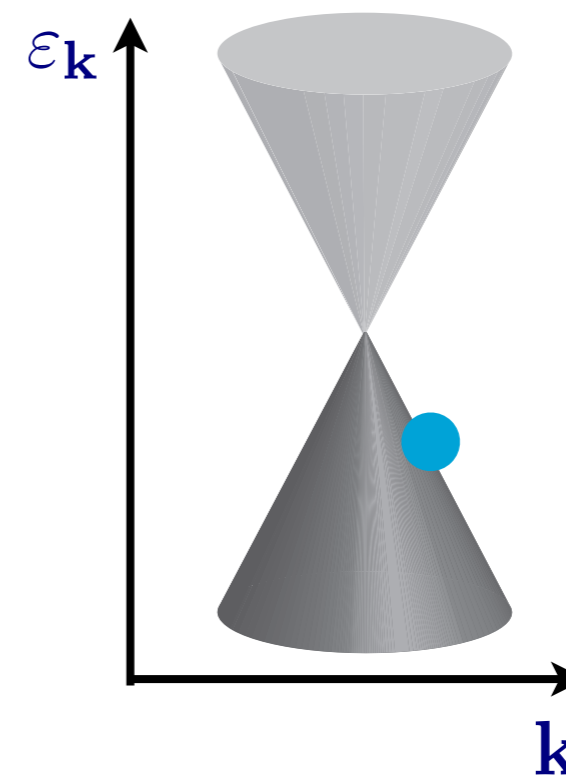
Holes



Momentum



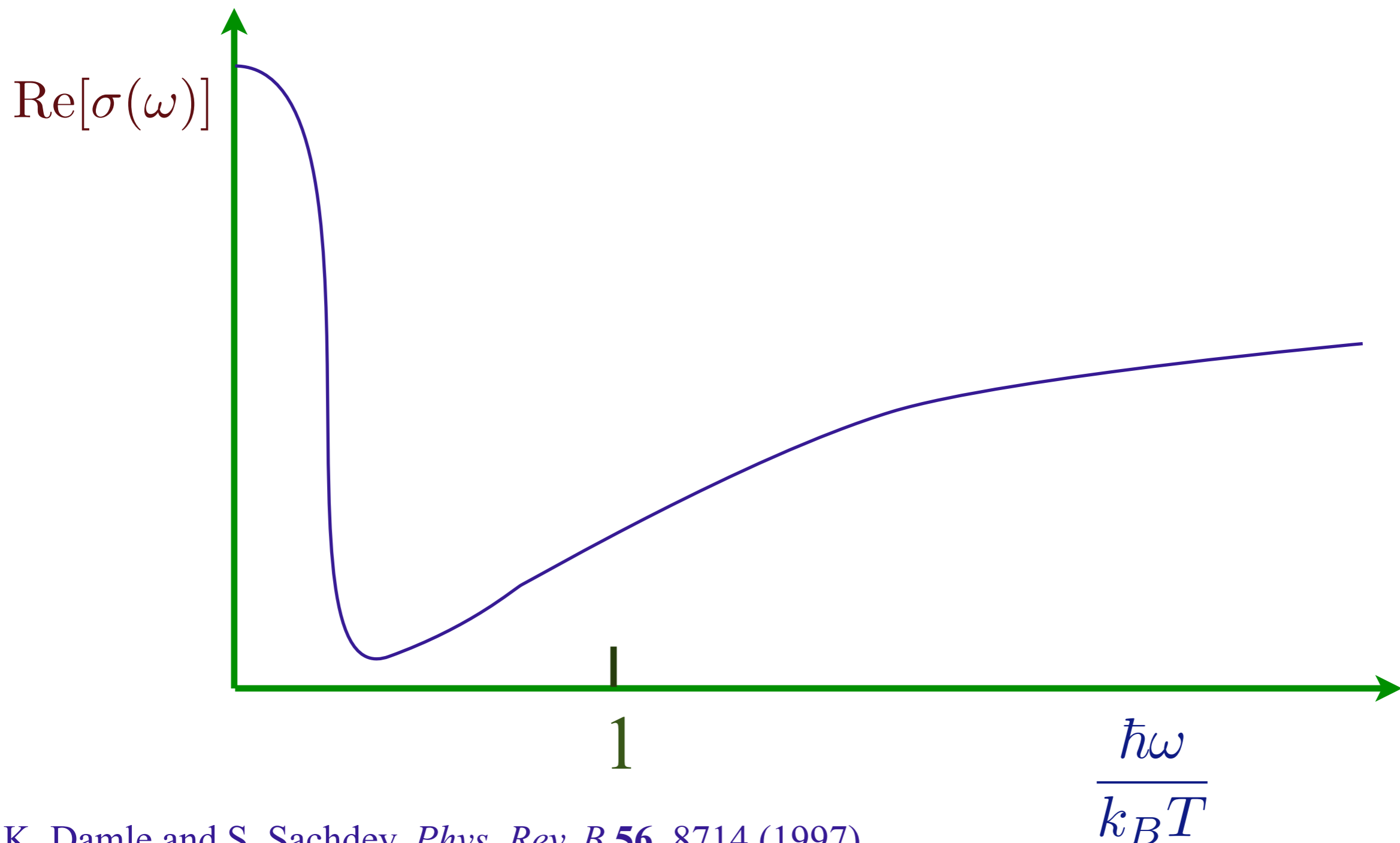
Current



Particle hole symmetry: current carrying state has zero momentum, and collisions can relax current to zero

Electrical transport for a (weakly) interacting CFT3

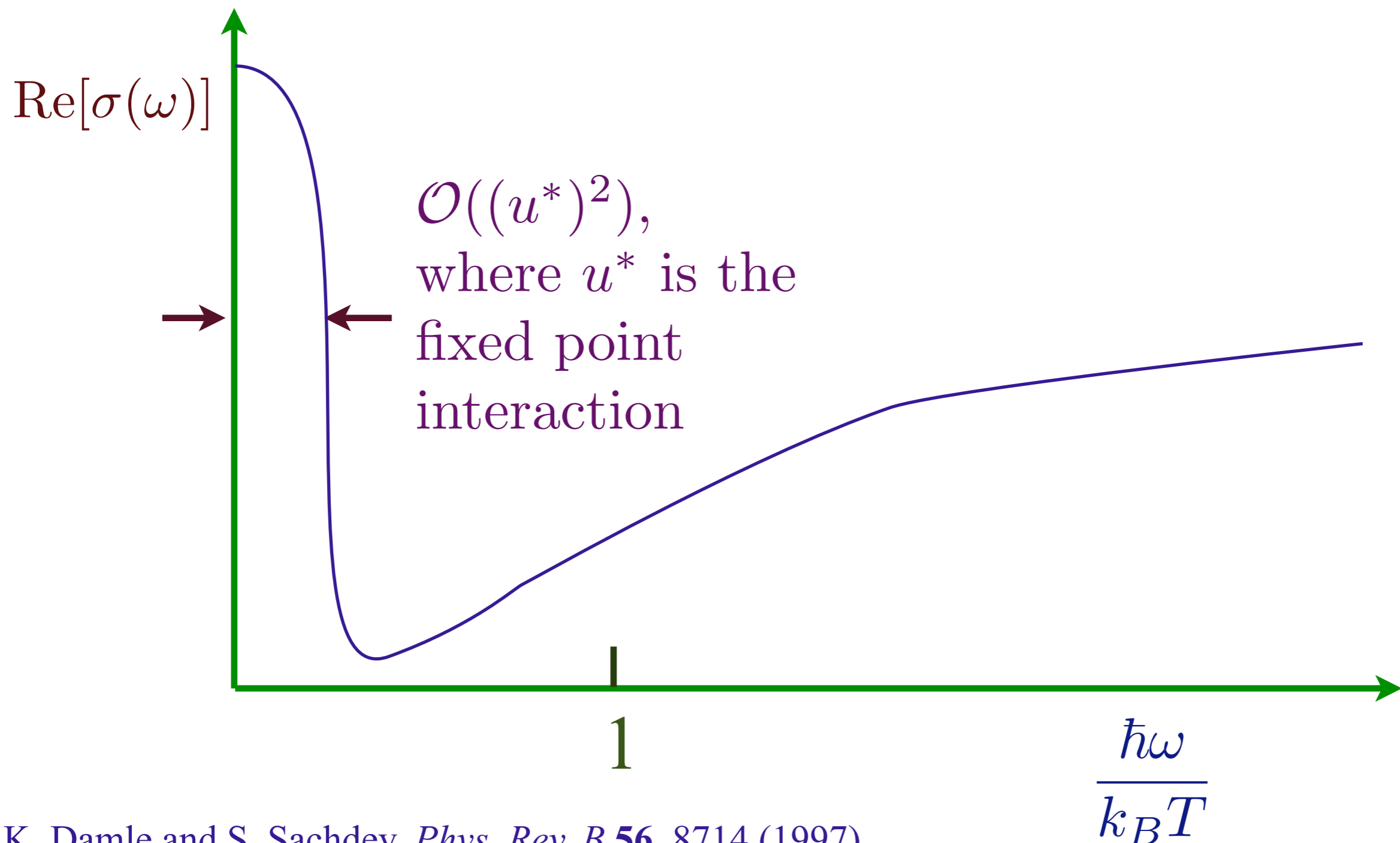
$$\sigma(\omega, T) = \frac{e^2}{h} \Sigma \left(\frac{\hbar\omega}{k_B T} \right) ; \quad \Sigma \rightarrow \text{a universal function}$$



K. Damle and S. Sachdev, *Phys. Rev. B* **56**, 8714 (1997).

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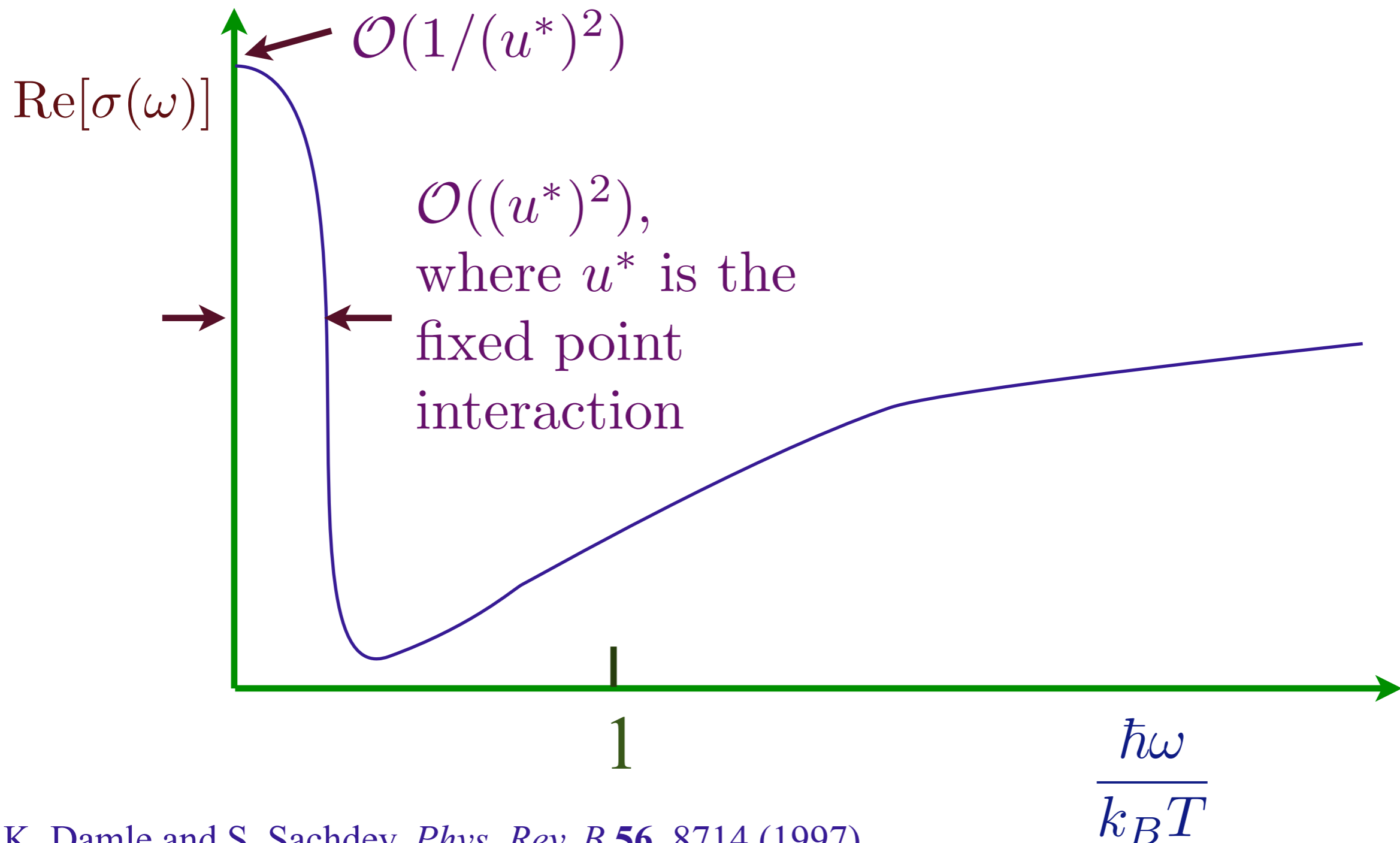
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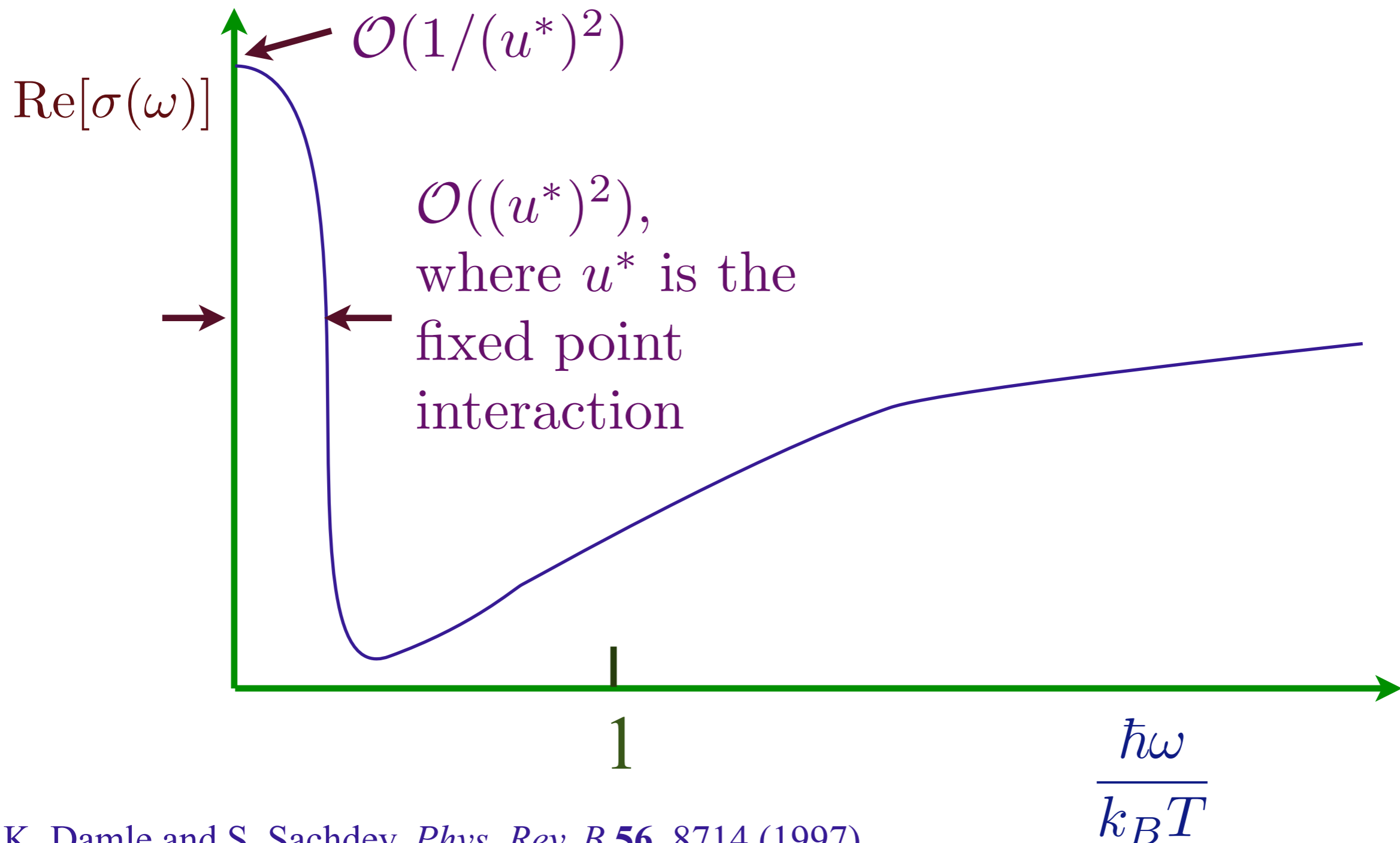
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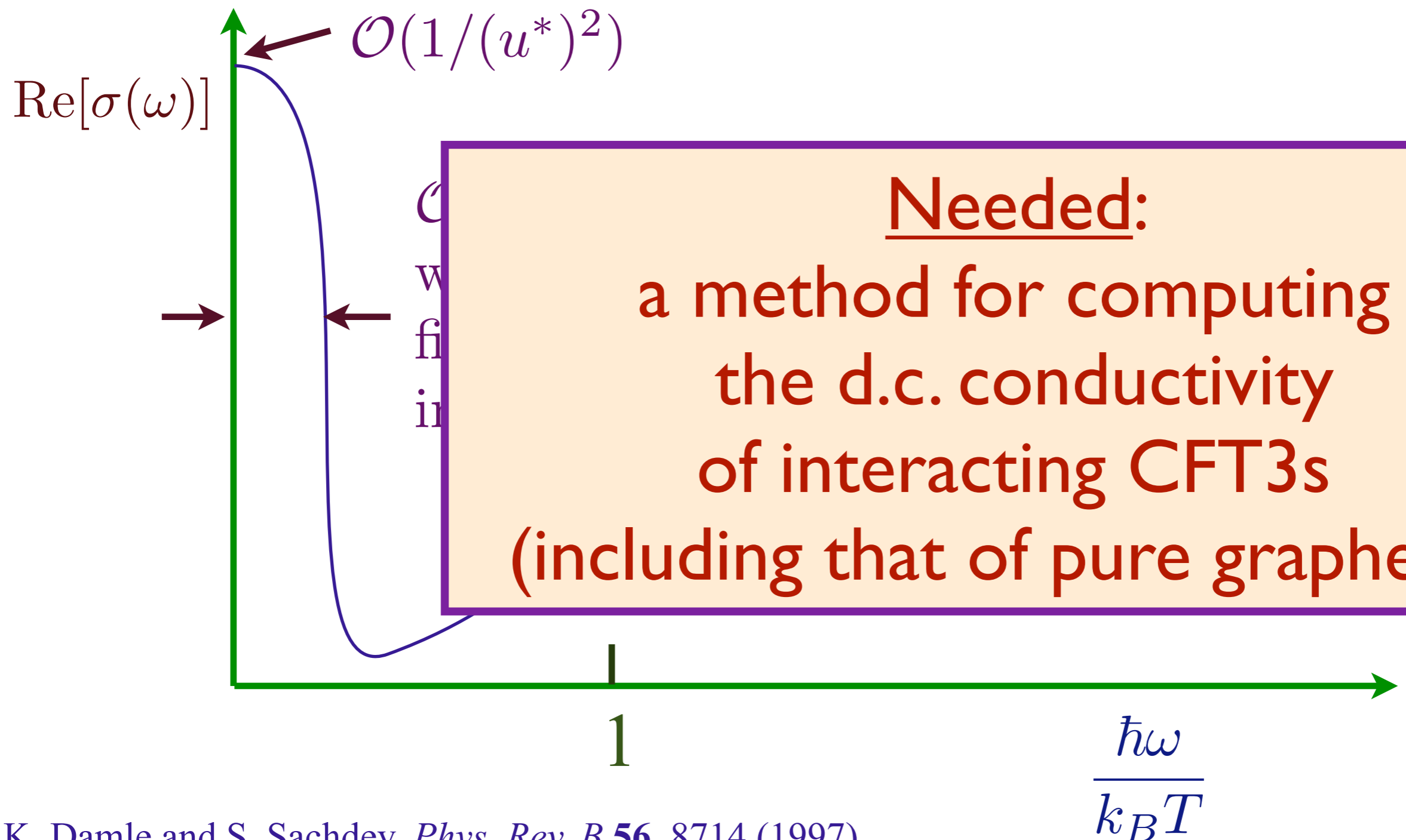
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Quantum critical transport

Quantum “*nearly perfect fluid*”
with shortest possible
equilibration time, τ_{eq}

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where \mathcal{C} is a *universal* constant

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Zaanen: Planckian dissipation

S. Sachdev, *Quantum Phase Transitions*, Cambridge (1999).

Quantum critical transport

Transport co-efficients not determined
by collision rate, but by
universal constants of nature

Conductivity

$$\sigma = \frac{Q^2}{h} \times [\text{Universal constant } \mathcal{O}(1)]$$

(Q is the “charge” of one particle)

M.P.A. Fisher, G. Grinstein, and S.M. Girvin, *Phys. Rev. Lett.* **64**, 587 (1990)

K. Damle and S. Sachdev, *Phys. Rev. B* **56**, 8714 (1997).

Quantum critical transport

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Momentum transport

$$\frac{\eta}{s} \equiv \frac{\text{viscosity}}{\text{entropy density}}$$
$$= \frac{\hbar}{k_B} \times [\text{Universal constant } \mathcal{O}(1)]$$