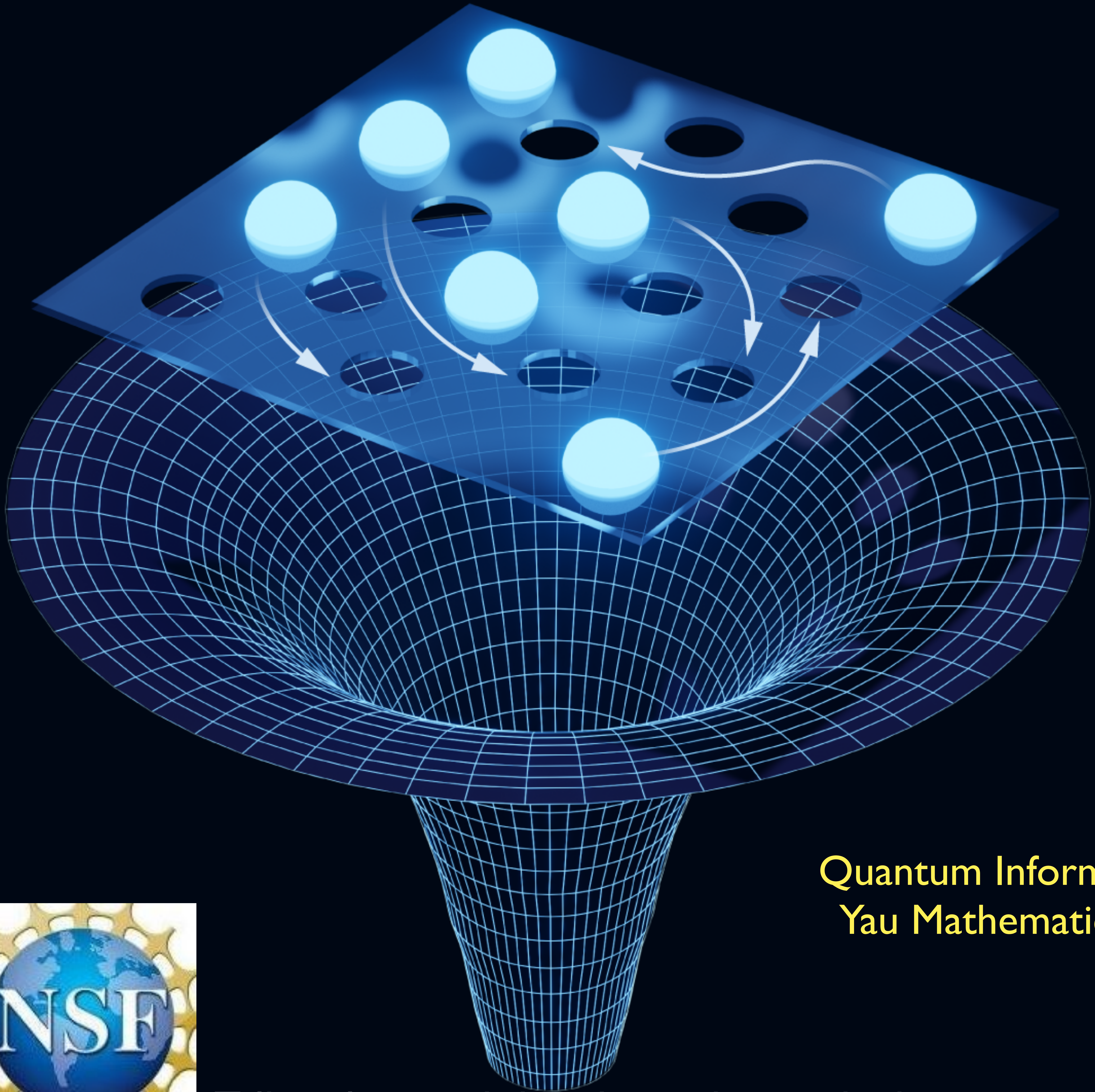

# Quantum statistical mechanics of charged black holes and strange metals

QuIST webinar
Quantum Information, Spacetime, and Topological Matter
Yau Mathematical Sciences Center, Tsinghua University
October 27, 2022

Subir Sachdev

Talk online: sachdev.physics.harvard.edu

PHYSICS
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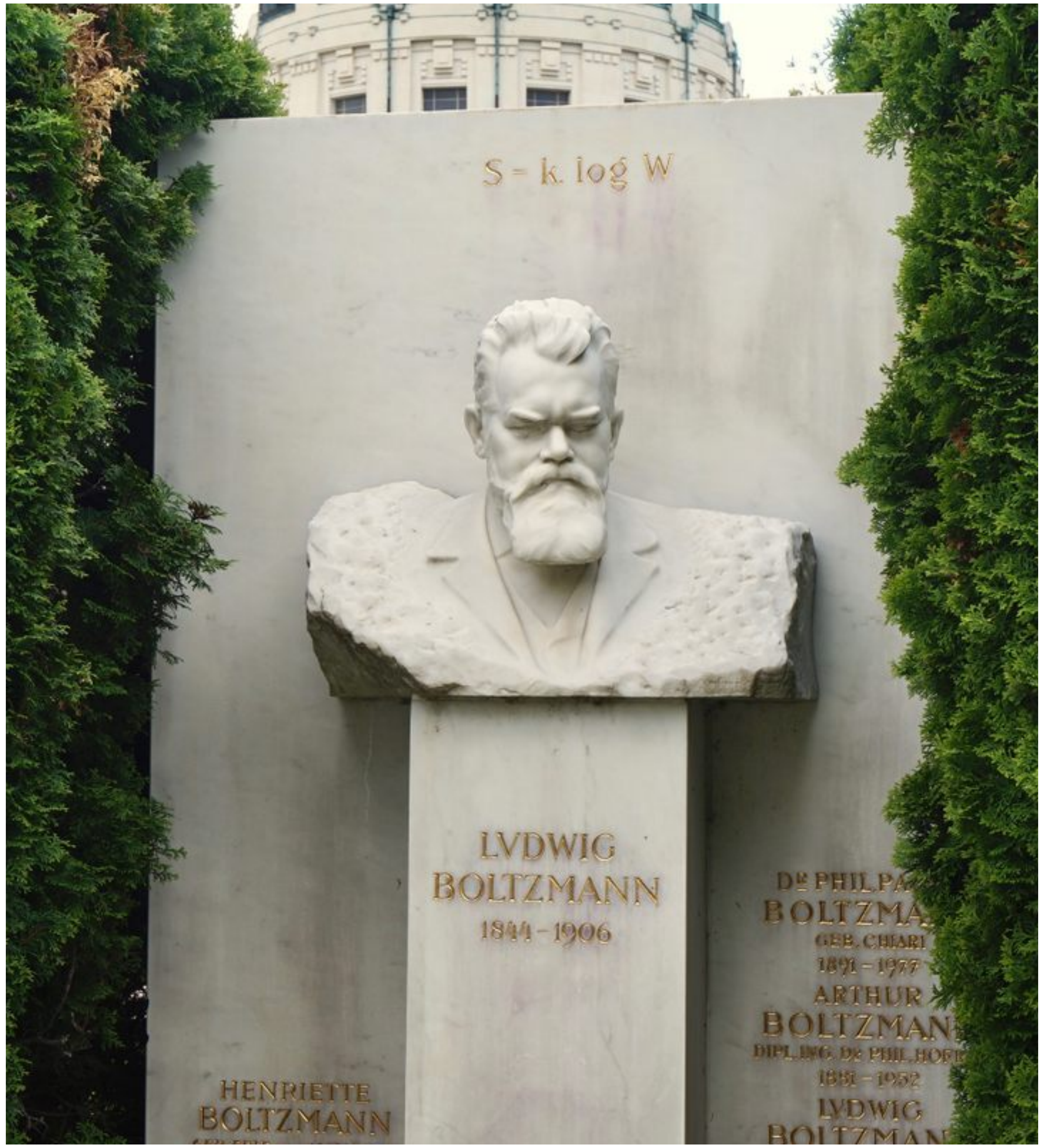


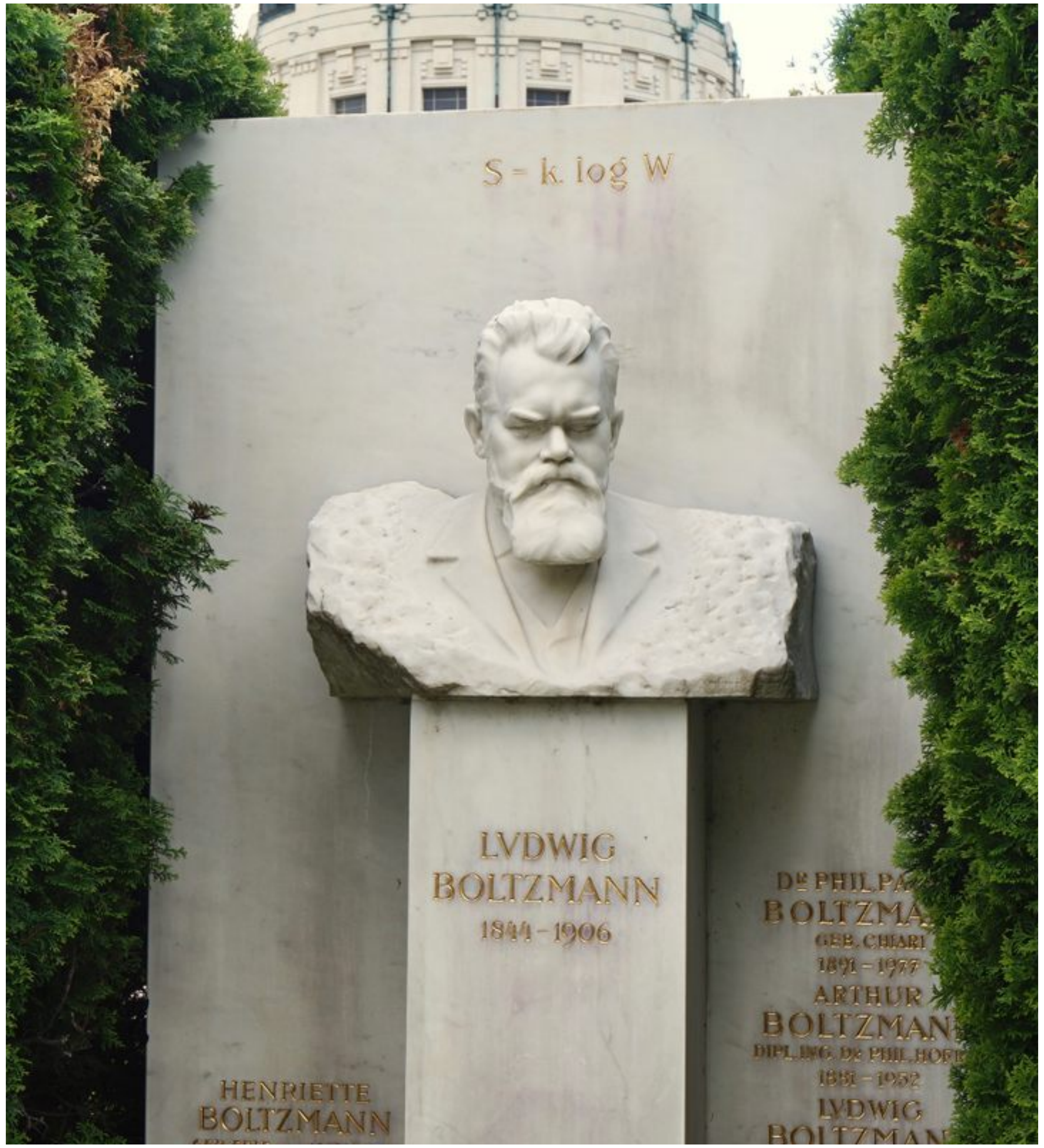
Ludwig Boltzmann
20 February 1844 - September 5, 1906
Vienna, Austria

# Statistical interpretation of entropy (1870)

$$S = k_B \log W$$

Density of quantum states $D(E) = \exp(S(E)/k_B)$

# Boltzmann equation (1872)
# Dilute classical gas



Ludwig Boltzmann
20 February 1844 - September 5, 1906
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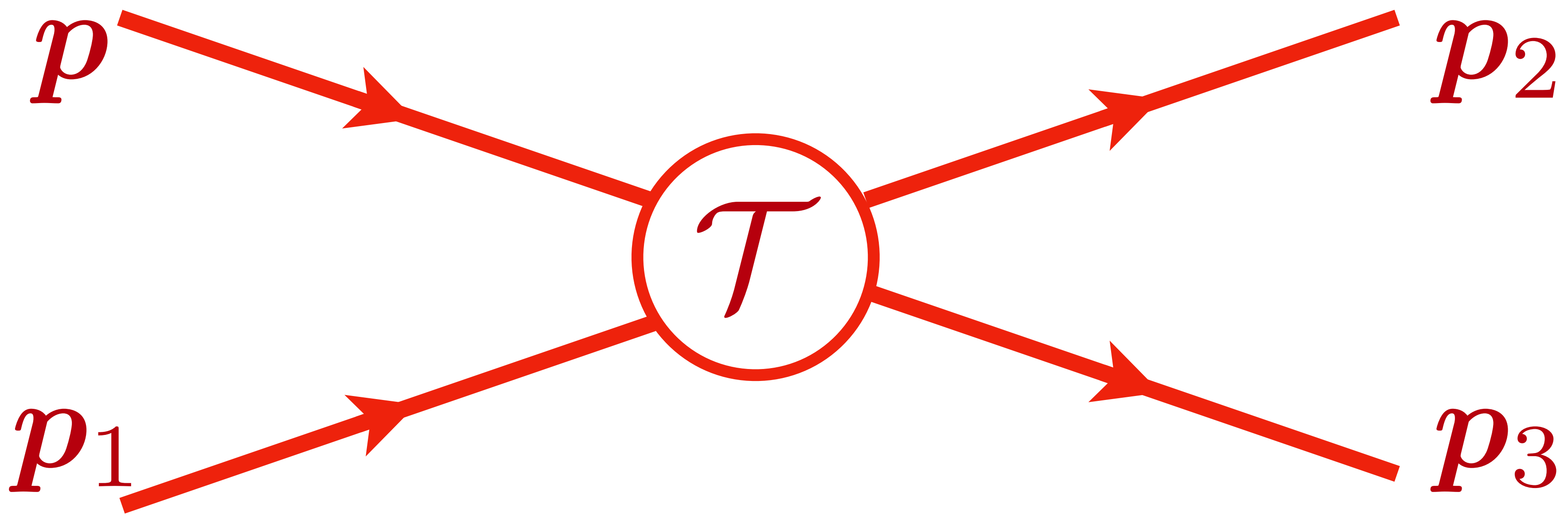
Molecular chaos: successive collisions are statistically independent

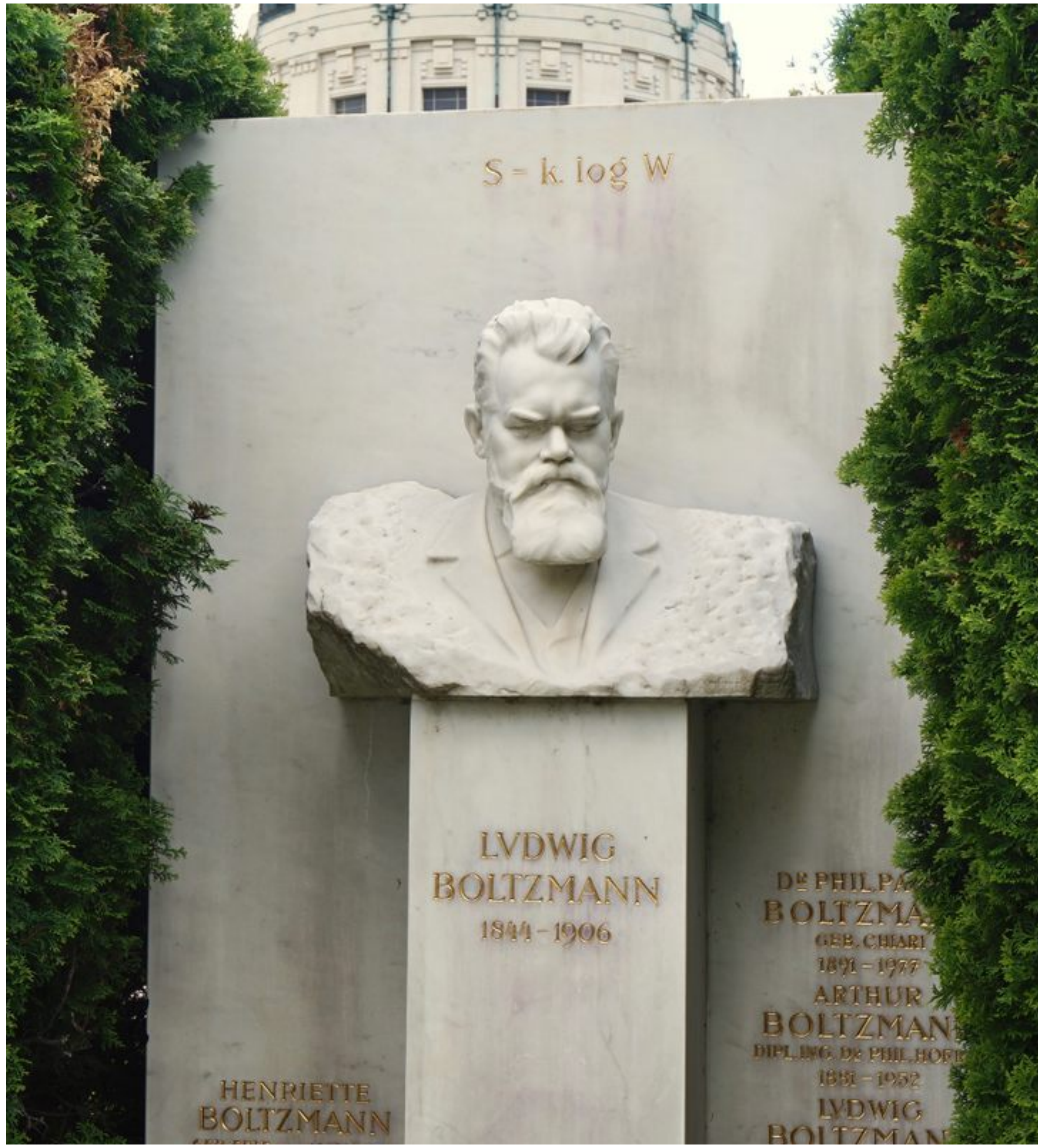
$$\frac{\partial f_{\boldsymbol{p}}}{\partial t} + \frac{\partial \varepsilon_{\boldsymbol{p}}}{\partial \boldsymbol{p}} \cdot \nabla_{\boldsymbol{r}} f_{\boldsymbol{p}} + \boldsymbol{F} \cdot \nabla_{\boldsymbol{p}} f_{\boldsymbol{p}} =$$

$$- 2\pi \int_{\boldsymbol{p}_{1,2,3}} |\mathcal{T}|^2 \delta(\varepsilon_{\boldsymbol{p}} + \varepsilon_{\boldsymbol{p}_1} - \varepsilon_{\boldsymbol{p}_2} - \varepsilon_{\boldsymbol{p}_3}) \delta(\boldsymbol{p} + \boldsymbol{p}_1 - \boldsymbol{p}_2 - \boldsymbol{p}_3)$$

$$\times \left[ f_{\boldsymbol{p}} f_{\boldsymbol{p}_1} - f_{\boldsymbol{p}_2} f_{\boldsymbol{p}_3} \right]$$



# Quantum Boltzmann equation (Landau)
# Dense gas of electrons



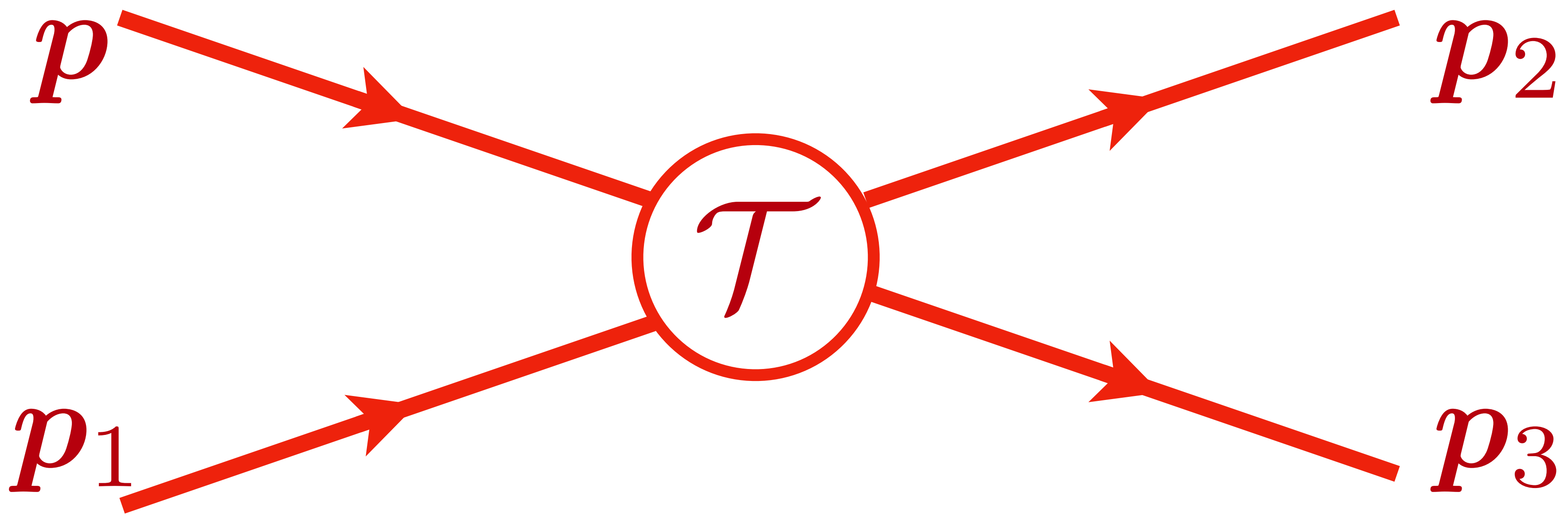
Ludwig Boltzmann
20 February 1844 - September 5, 1906
Vienna, Austria

Neglects quantum interference (entanglement)
between successive collisions

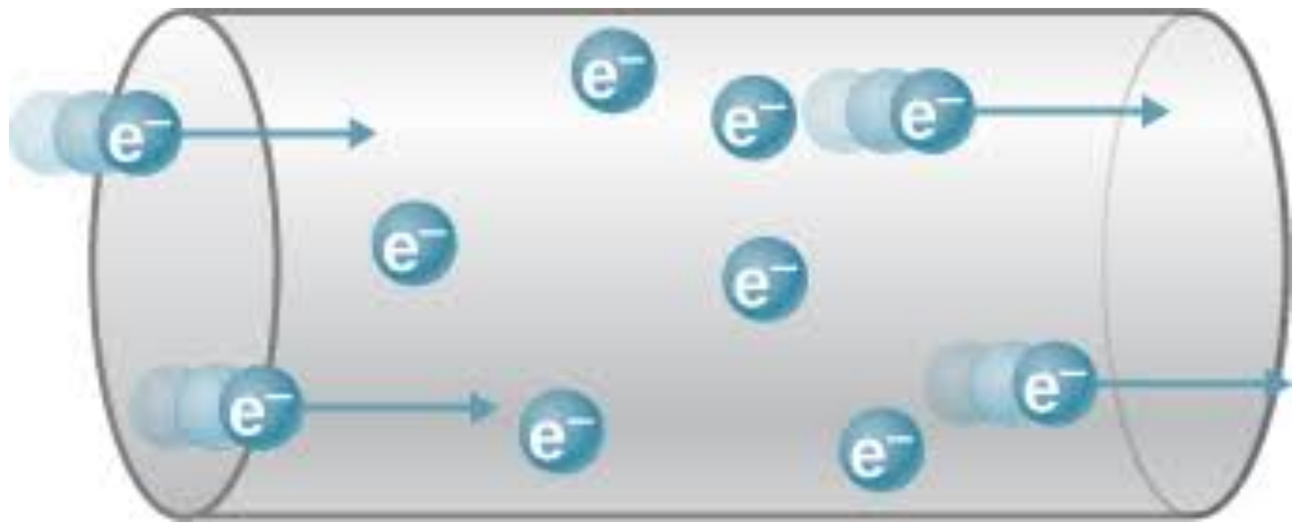
$$\frac{\partial f_{\boldsymbol{p}}}{\partial t} + \frac{\partial \varepsilon_{\boldsymbol{p}}}{\partial \boldsymbol{p}} \cdot \nabla_{\boldsymbol{r}} f_{\boldsymbol{p}} + \boldsymbol{F} \cdot \nabla_{\boldsymbol{p}} f_{\boldsymbol{p}} =$$

$$-2\pi \int_{\boldsymbol{p}_{1,2,3}} |\mathcal{T}|^2 \delta(\varepsilon_{\boldsymbol{p}} + \varepsilon_{\boldsymbol{p}_1} - \varepsilon_{\boldsymbol{p}_2} - \varepsilon_{\boldsymbol{p}_3}) \delta(\boldsymbol{p} + \boldsymbol{p}_1 - \boldsymbol{p}_2 - \boldsymbol{p}_3)$$

$$\times \left[ f_{\boldsymbol{p}} f_{\boldsymbol{p}_1} (1 - f_{\boldsymbol{p}_2})(1 - f_{\boldsymbol{p}_3}) - f_{\boldsymbol{p}_2} f_{\boldsymbol{p}_3} (1 - f_{\boldsymbol{p}})(1 - f_{\boldsymbol{p}_1}) \right]$$



# *Current flow with electrons in Copper*

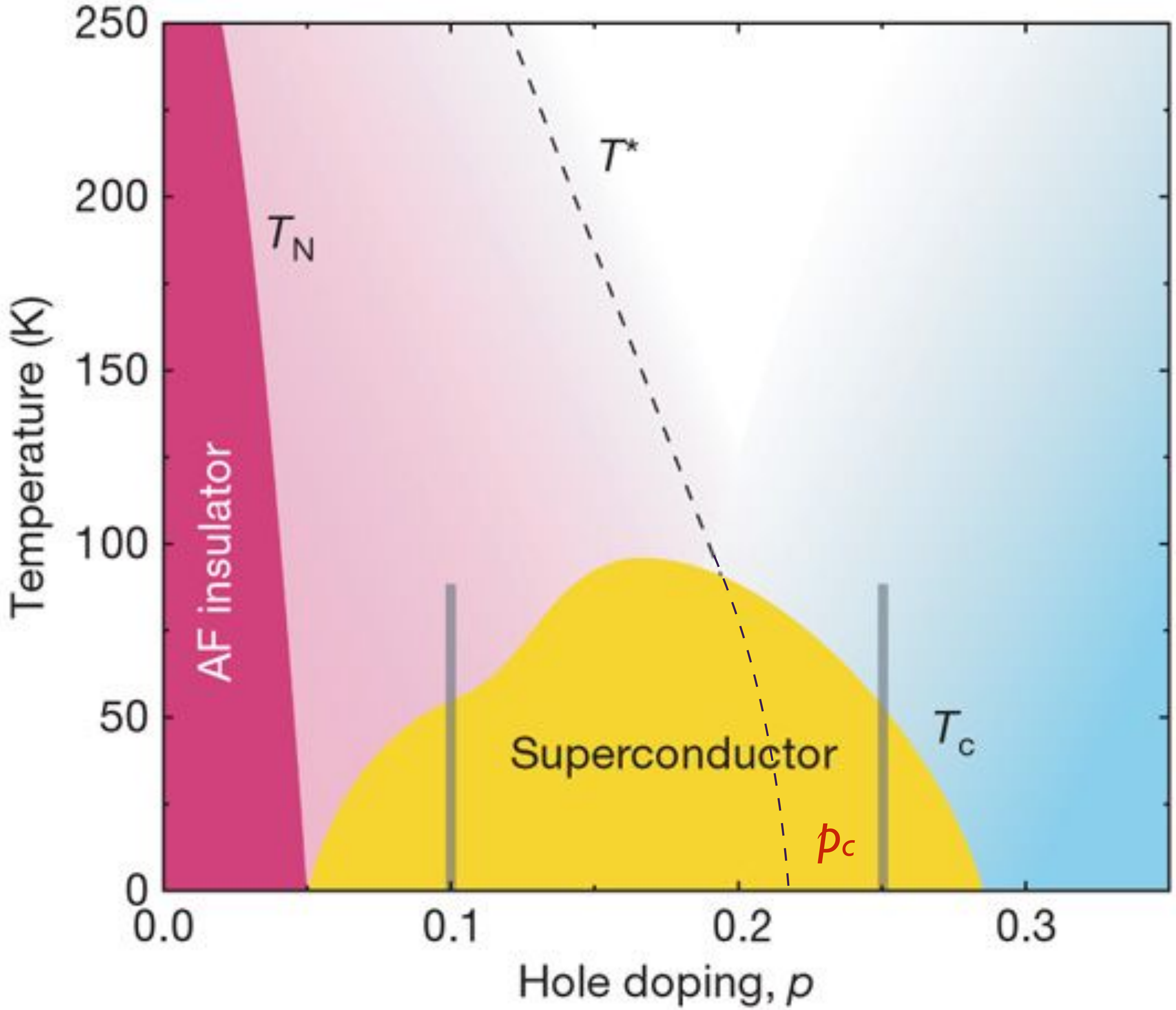


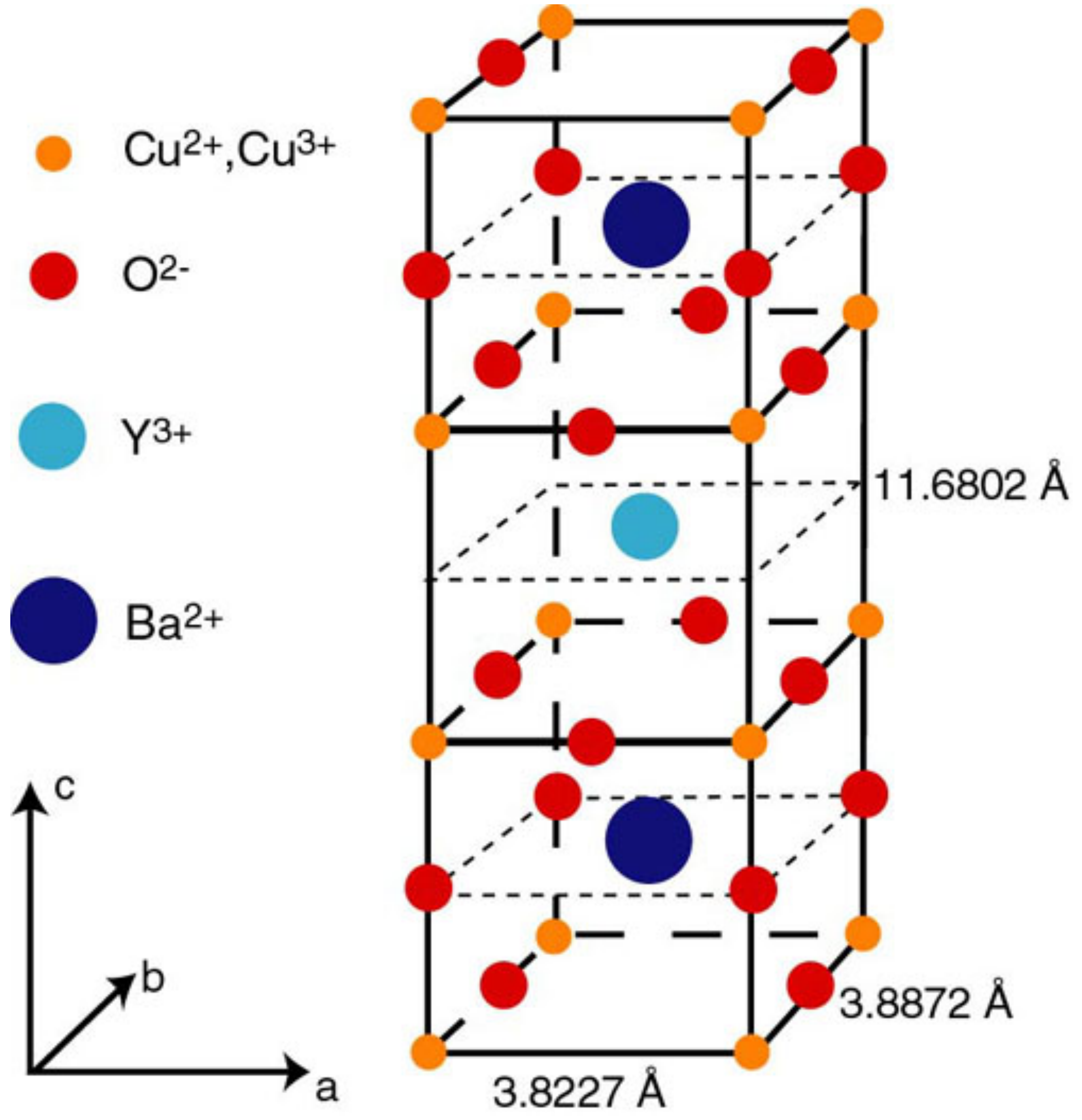
Flow of electrons described by Boltzmann equation $\Rightarrow$
typical scattering time $\tau \sim 1/T^2$, resistivity $\rho(T) = \rho(0) + AT^2$

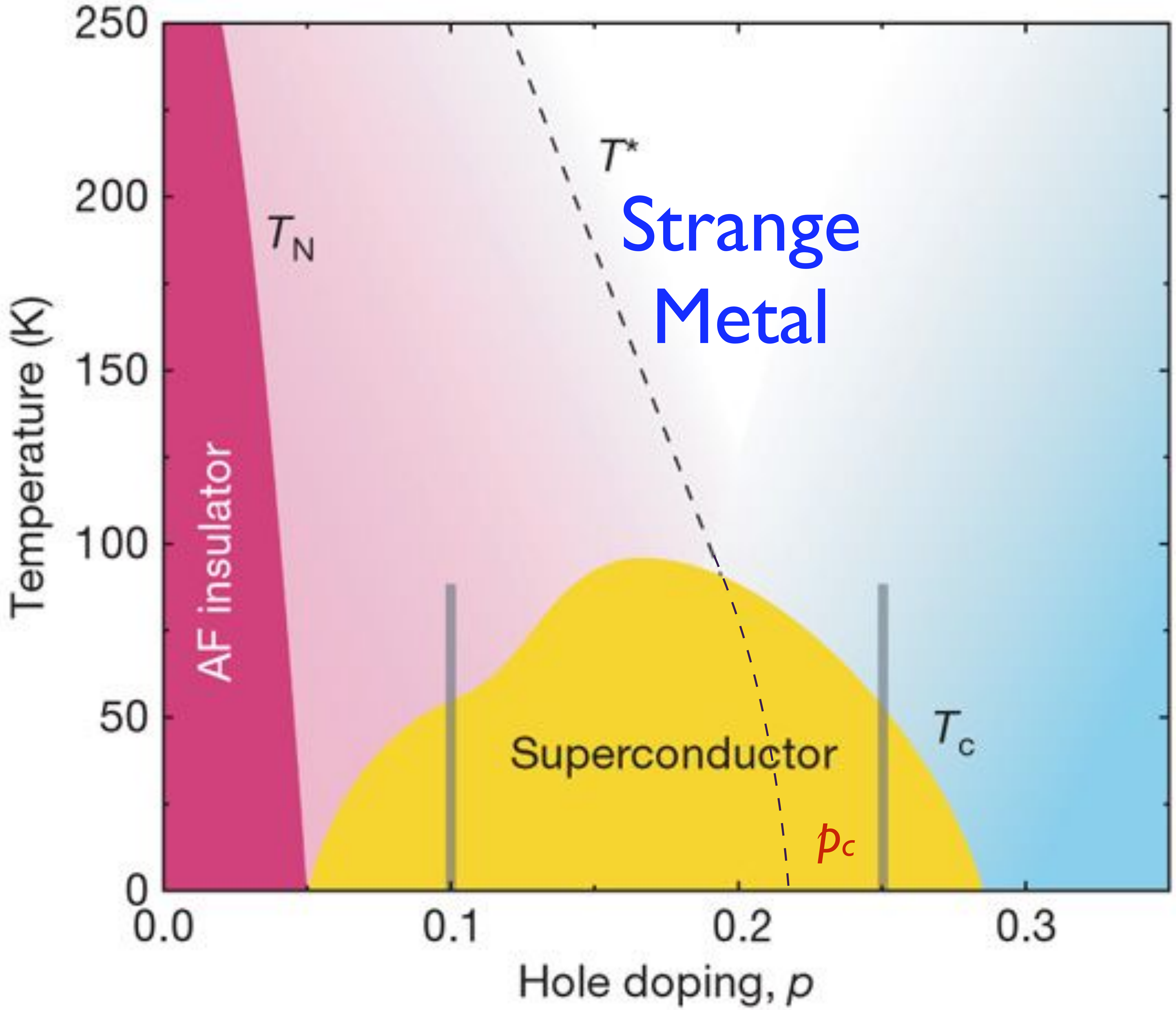
The time $\tau$ is much longer than a limiting 'Planckian time' $\dfrac{\hbar}{k_B T}$.

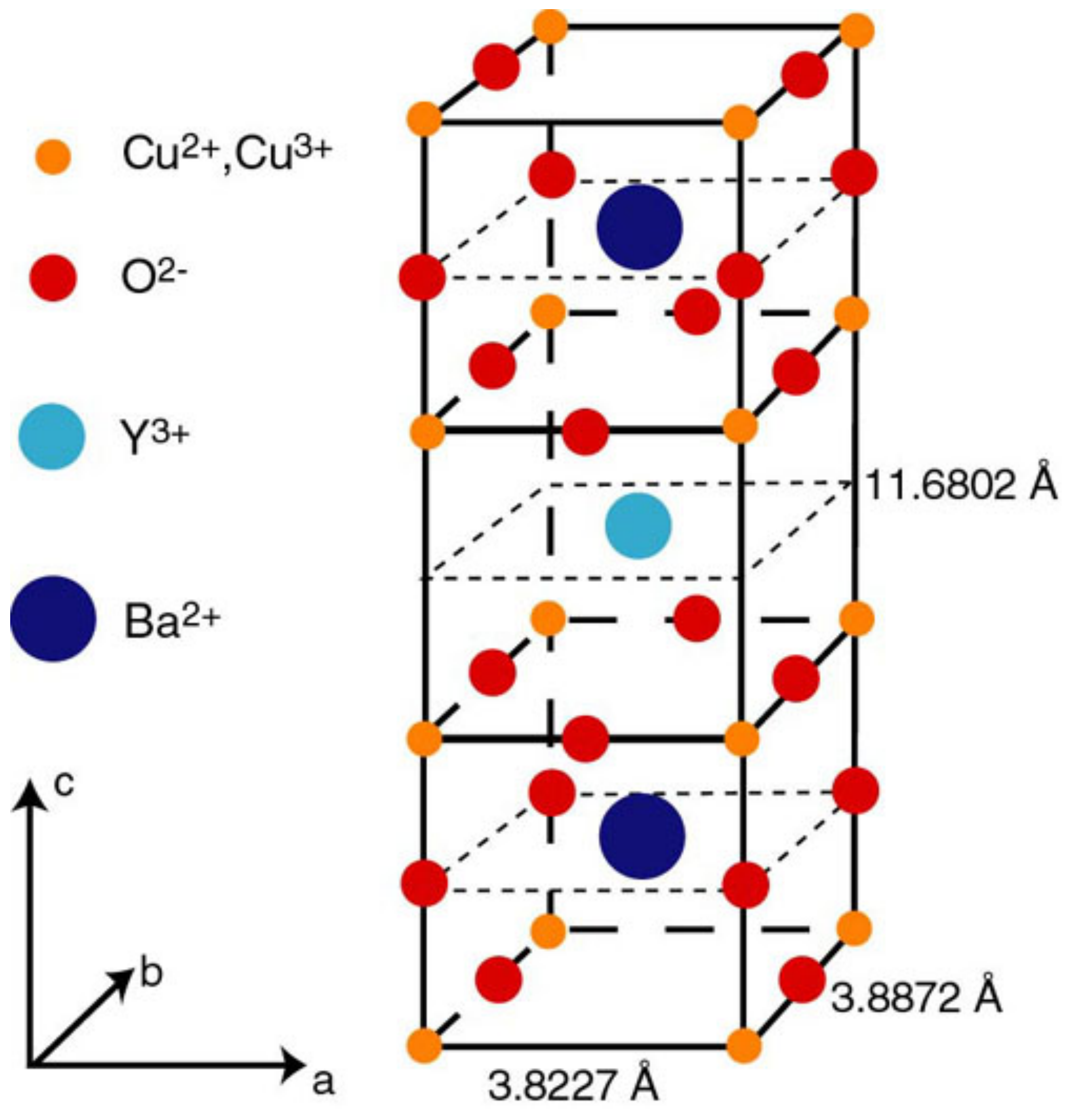
The long scattering time implies that individual electrons are well-defined.

The motion of electrons is 'ballistic' or 'integrable'
up to the long time $\tau$, after which it is chaotic.


250
200
150
100
50
0
Temperature (K)
0.0
0.1
0.2
0.3
Hole doping, p
AF insulator
$T_N$
$T^*$
Superconductor
$T_c$
$p_c$


$Cu^{2+}, Cu^{3+}$
$O^{2-}$
$Y^{3+}$
$Ba^{2+}$
c
b
a
11.6802 Å
3.8872 Å
3.8227 Å


250
200
150
100
50
0
Temperature (K)
0.0
0.1
0.2
0.3
Hole doping, p
AF insulator
T_N
T*
Strange
Metal
Superconductor
T_c
p_c


Cu2+,Cu3+
O2-
Y3+
Ba2+
c
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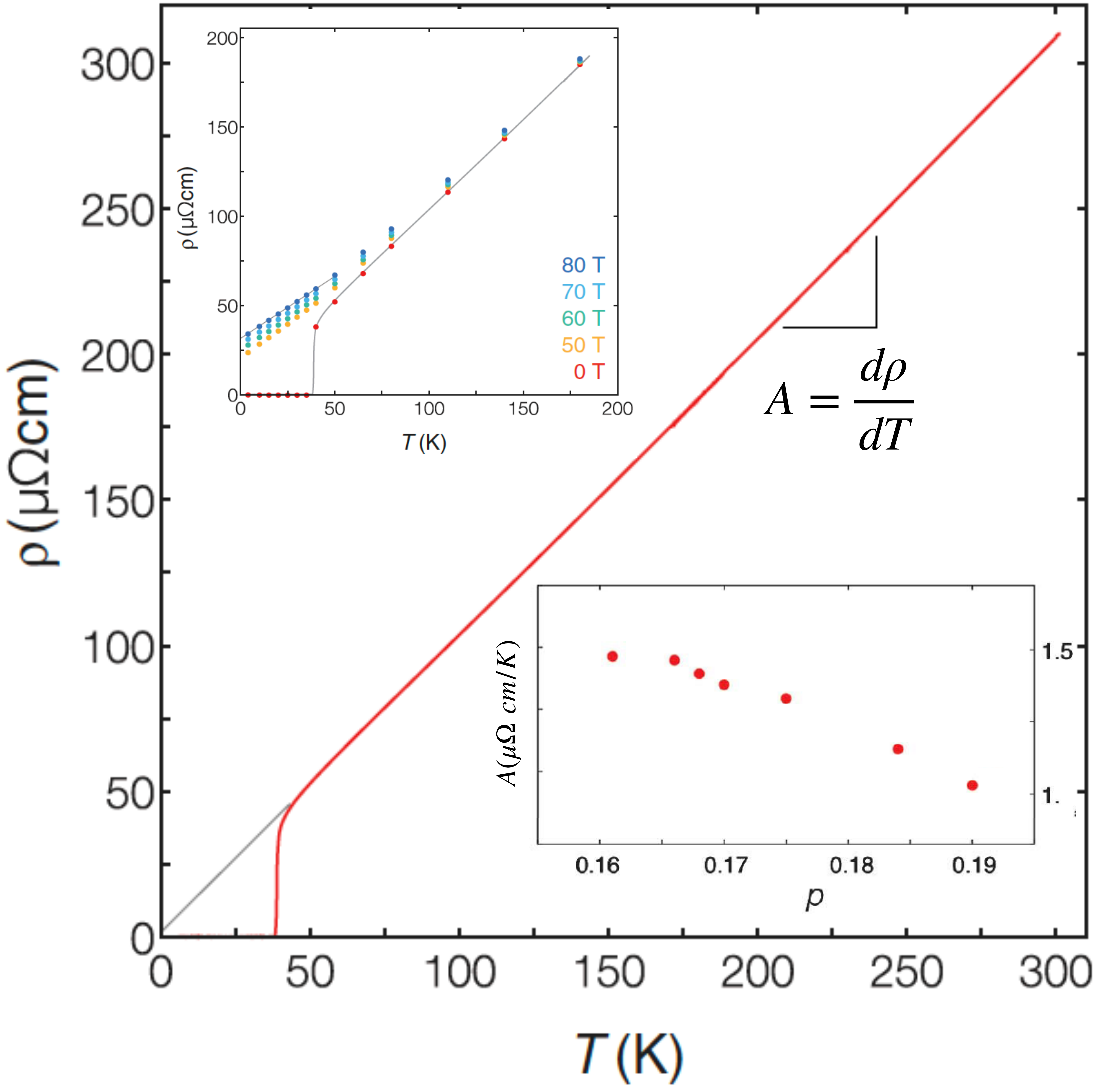
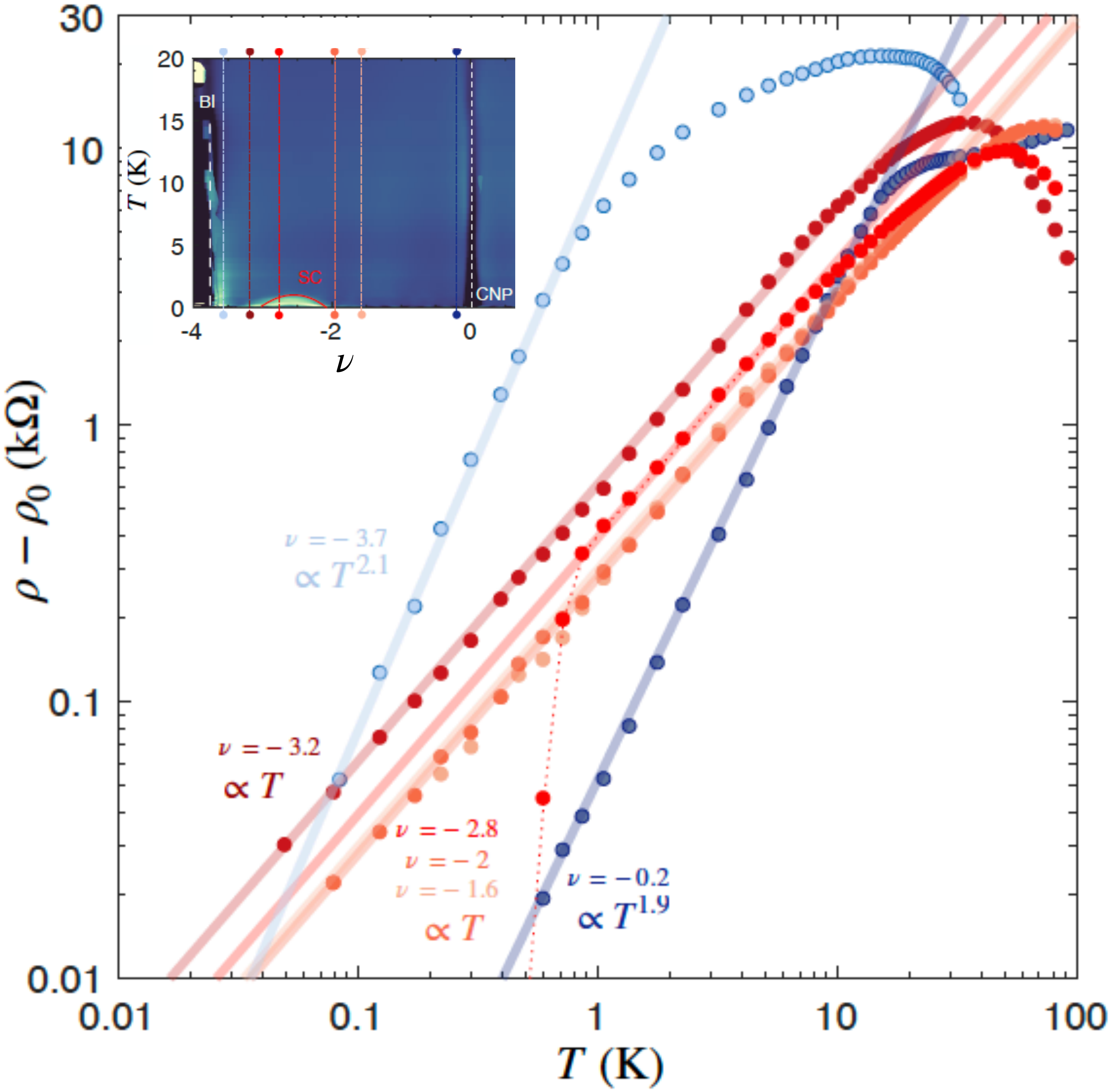


LSCO: Giraldo-Gallo et al. 2018

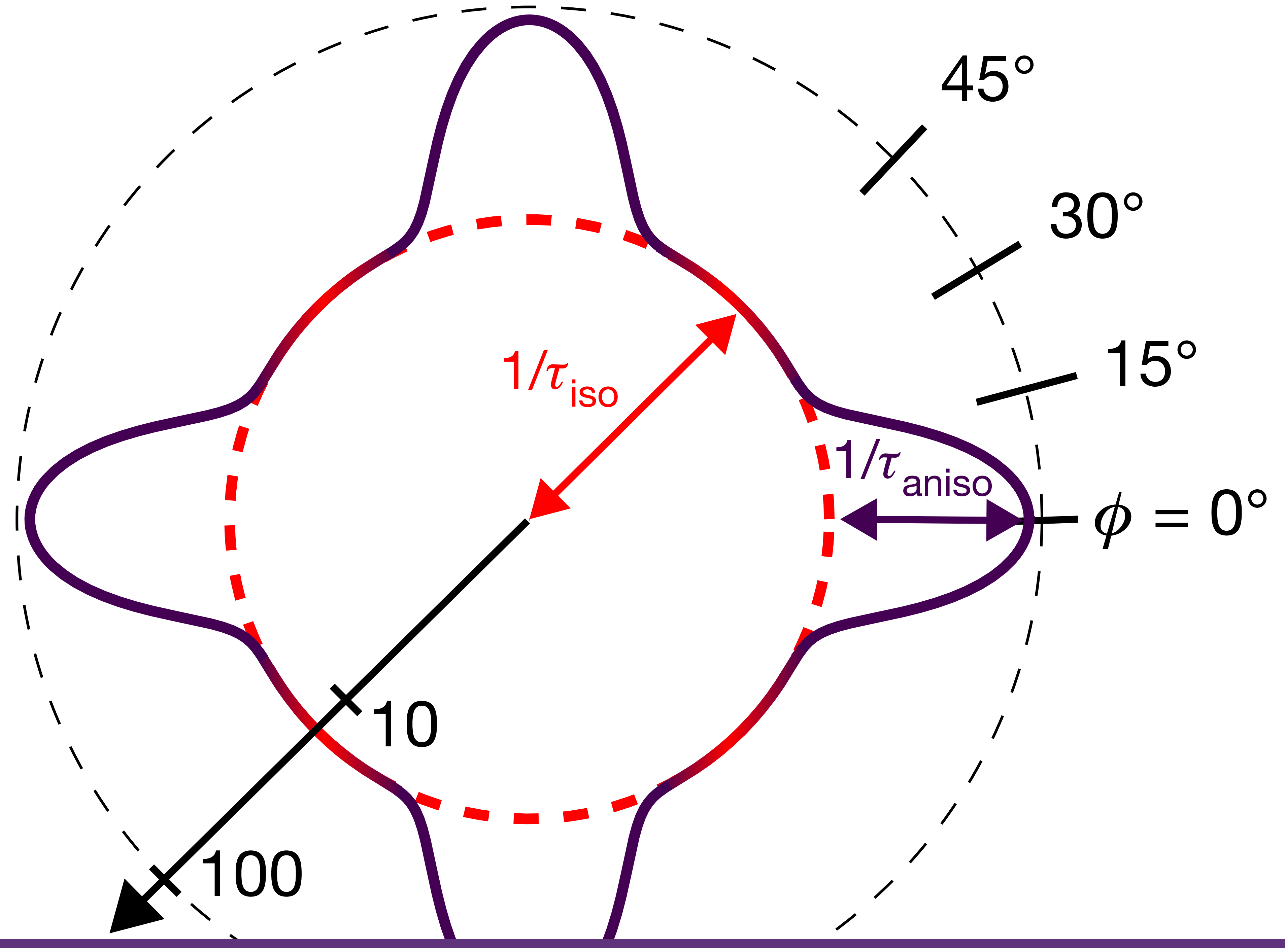


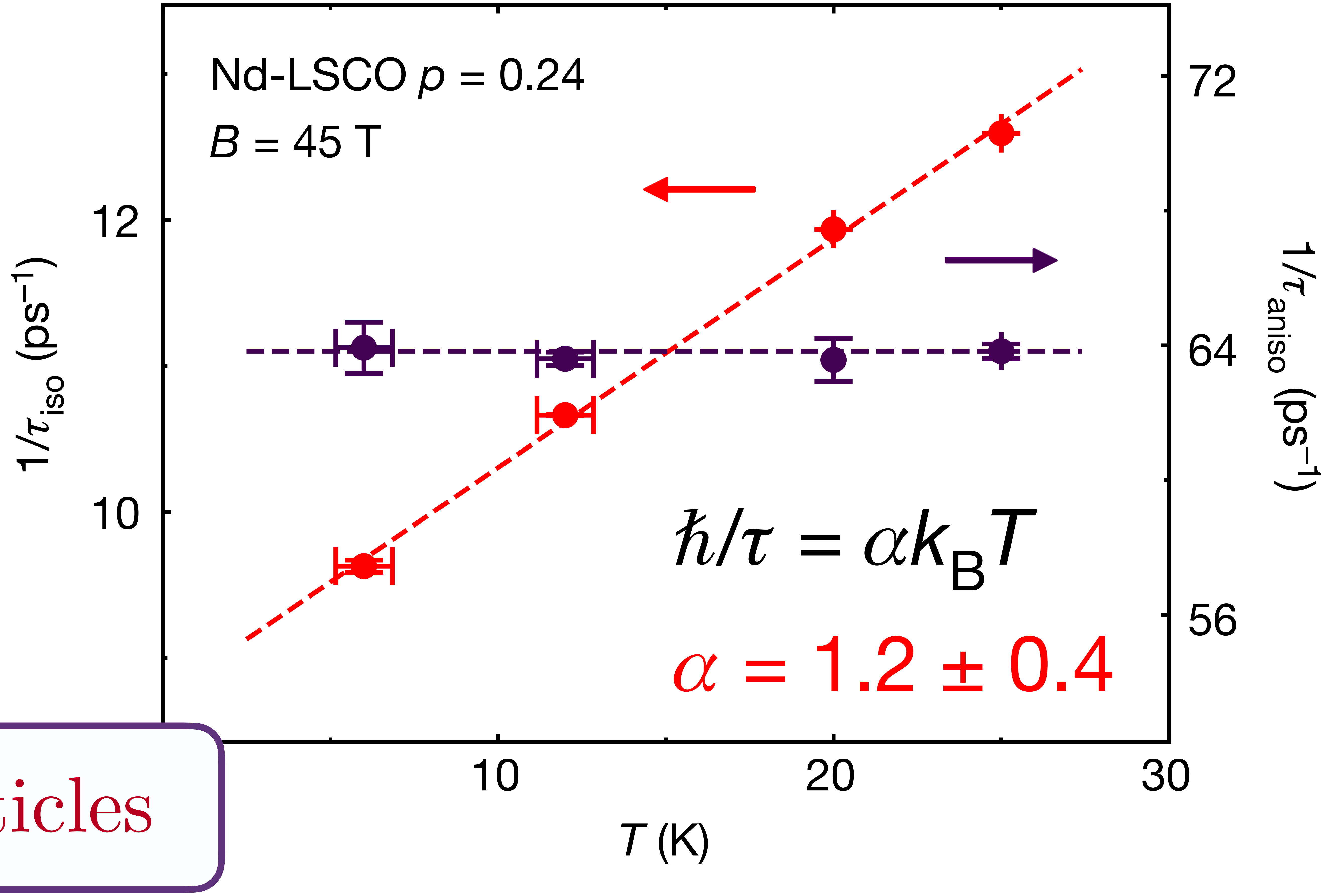
MATBG: Jaoui et al. 2021

# Linear-in temperature resistivity from an isotropic Planckian scattering rate

Nature **595**, 667-672 (2021)

G. Grissonnanche, Y. Fang, A. Legros, S. Verret, F. Laliberté, C. Collignon, J. Zhou, D. Graf, P. Goddard, L. Taillefer, B. J. Ramshaw

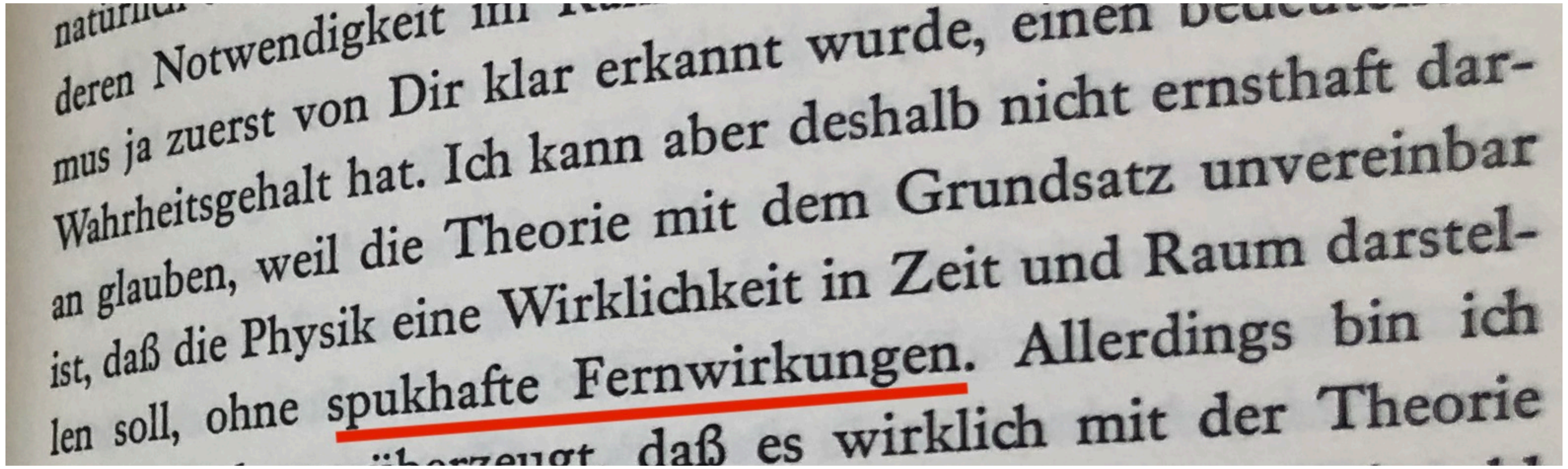




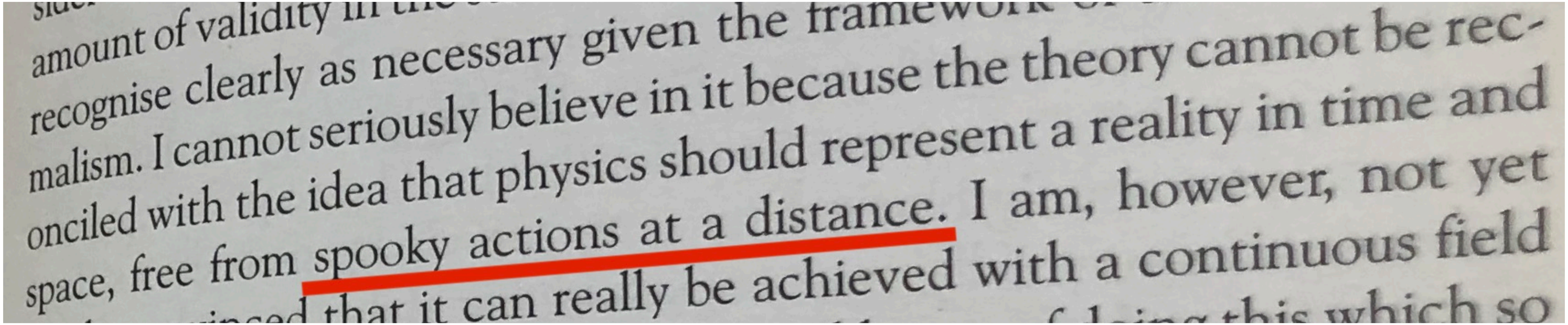
Current flow without quasiparticles

No Boltzmann-Landau quasiparticle description $\Rightarrow$
Many particle quantum entanglement
from quantum interference between “collisions”

# No Boltzmann-Landau quasiparticle description ⇒ Many particle quantum entanglement from quantum interference between “collisions”



natürlich … deren Notwendigkeit im … mus ja zuerst von Dir klar erkannt wurde, einen … Wahrheitsgehalt hat. Ich kann aber deshalb nicht ernsthaft daran glauben, weil die Theorie mit dem Grundsatz unvereinbar ist, daß die Physik eine Wirklichkeit in Zeit und Raum darstellen soll, ohne spukhafte Fernwirkungen. Allerdings bin ich … daß es wirklich mit der Theorie



amount of validity in the … recognise clearly as necessary given the framework … malism. I cannot seriously believe in it because the theory cannot be reconciled with the idea that physics should represent a reality in time and space, free from spooky actions at a distance. I am, however, not yet … that it can really be achieved with a continuous field … this which so

Acknowledgement: Ahmed Almheiri

Albert Einstein to Max Born, 3 March 1947

1. SYK model: a solvable and generic 0+1 dimensional CFT
2. Charged black holes
3. Strange metals
4. From Yukawa-SYK models to a universal theory of strange metals

D. Chowdhury, A. Georges, O. Parcollet, and S. S.,
Reviews of Modern Physics **94**, 035004 (2022)

# SYK model:
# A solvable and generic
# 0+1 dimensional CFT

Operators in a generic $\mathrm{CFT}_1$ are similar to heavy (‘black hole’) operators in $\mathrm{CFT}_D$ with $D \geq 2$

# The Sachdev-Ye-Kitaev (SYK) model

(See also: the "2-Body Random Ensemble" in nuclear physics; did not obtain the large *N* limit;
T.A. Brody, J. Flores, J.B. French, P.A. Mello, A. Pandey, and S.S.M. Wong, Rev. Mod. Phys. **53**, 385 (1981))

$$\mathcal{H} = \frac{1}{(2N)^{3/2}} \sum_{\alpha,\beta,\gamma,\delta=1}^{N} U_{\alpha\beta;\gamma\delta}\, c_\alpha^\dagger c_\beta^\dagger c_\gamma c_\delta - \mu \sum_\alpha c_\alpha^\dagger c_\alpha$$

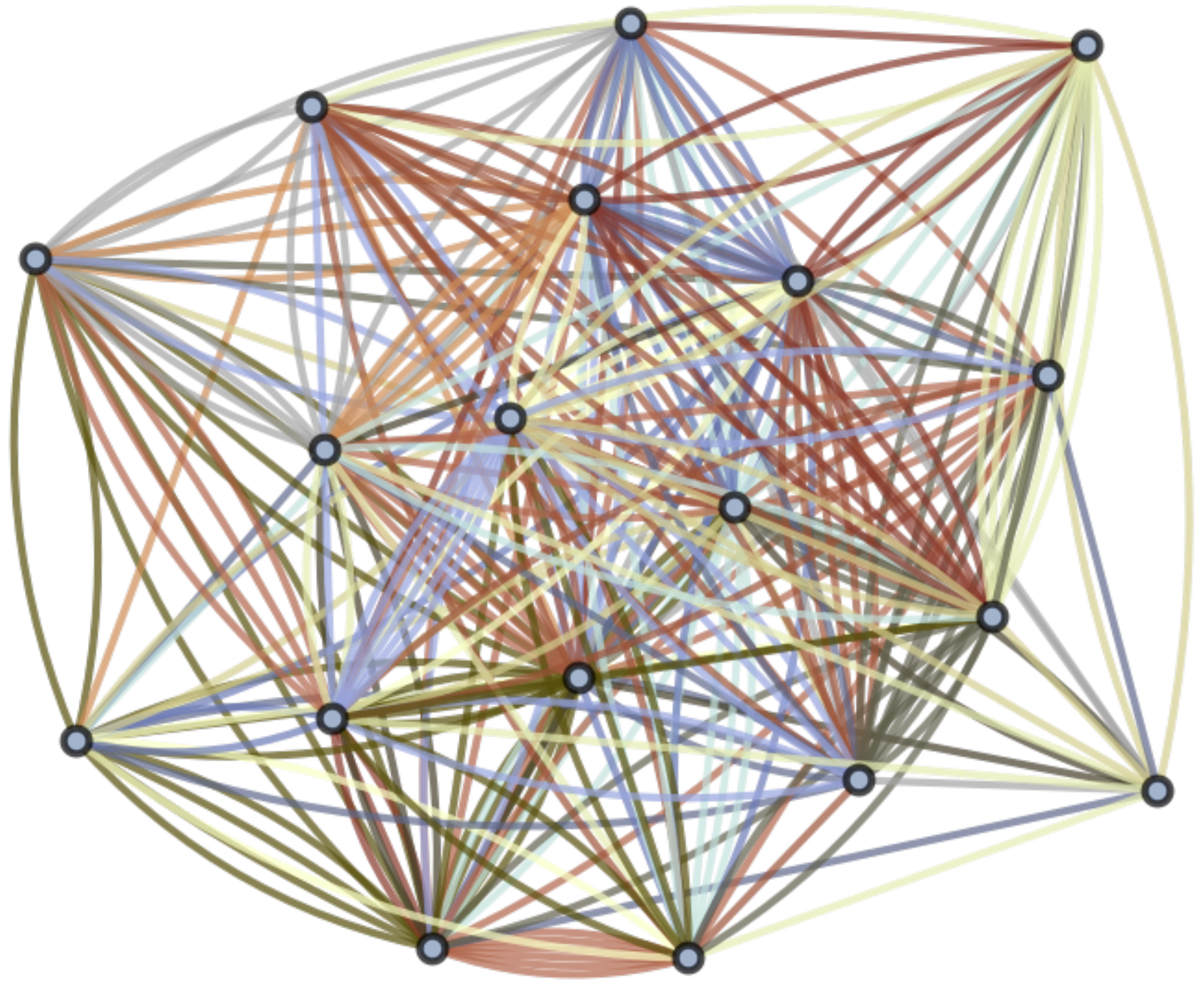
$$c_\alpha c_\beta + c_\beta c_\alpha = 0 \quad , \quad c_\alpha c_\beta^\dagger + c_\beta^\dagger c_\alpha = \delta_{\alpha\beta}$$

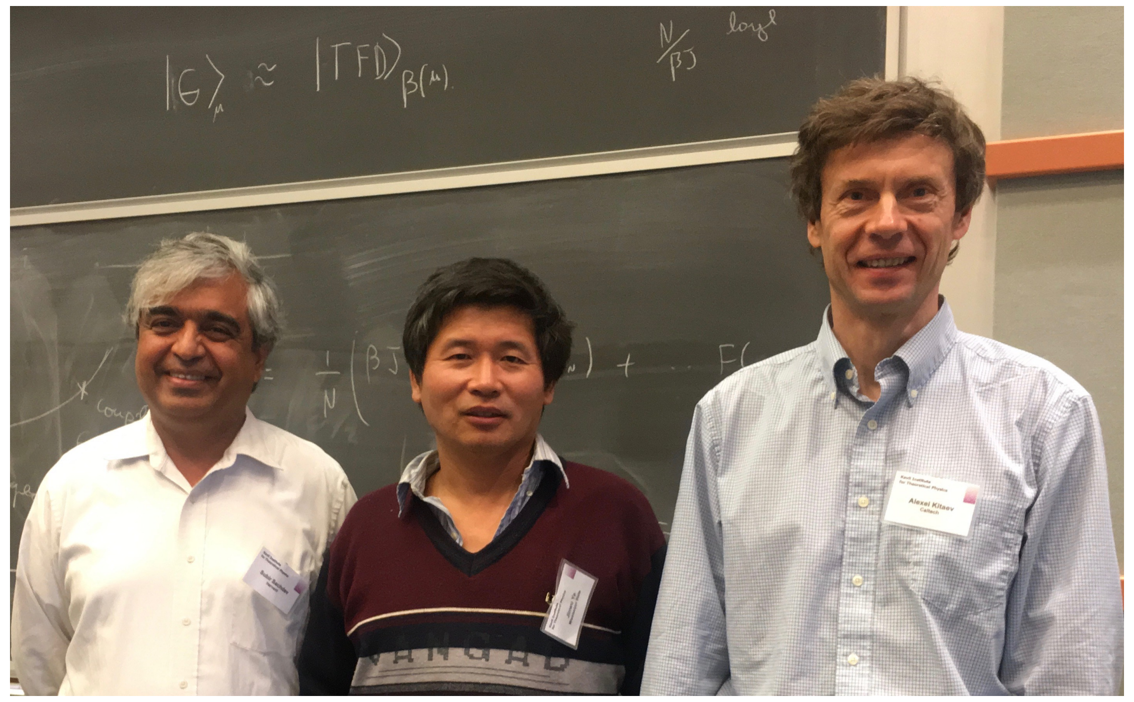
$$\mathcal{Q} = \frac{1}{N} \sum_\alpha c_\alpha^\dagger c_\alpha \, ; \quad [\mathcal{H}, \mathcal{Q}] = 0 \, ; \quad 0 \le \mathcal{Q} \le 1$$

$U_{\alpha\beta;\gamma\delta}$ are independent random variables with $\overline{U_{\alpha\beta;\gamma\delta}} = 0$ and $\overline{|U_{\alpha\beta;\gamma\delta}|^2} = U^2$
$N \to \infty$ yields critical strange metal.

S. Sachdev and J. Ye, PRL **70**, 3339 (1993)

A. Kitaev, unpublished; S. Sachdev, PRX **5**, 041025 (2015)



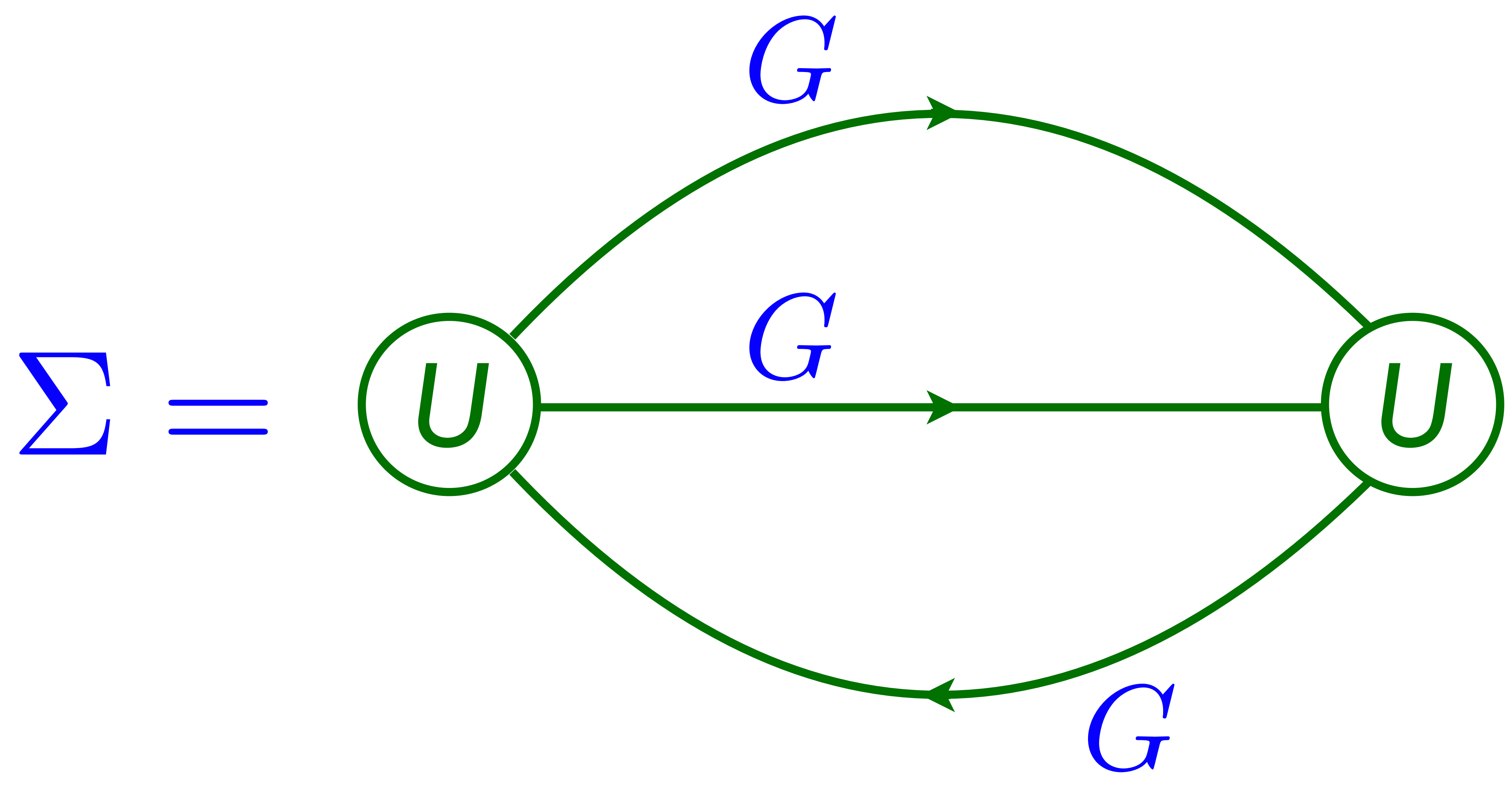


# The Sachdev-Ye-Kitaev (SYK) model

Feynman graph expansion in $U_{\alpha\beta;\gamma\delta}$, and graph-by-graph average, yields exact equations in the large $N$ limit:

$$G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = -U^2 G^2(\tau) G(-\tau)$$

$$G(\tau = 0^-) = \mathcal{Q}.$$



S. Sachdev and J. Ye,
PRL **70**, 3339 (1993)

# The complex SYK model

$$G_*(\tau) = -C \frac{e^{-2\pi \mathcal{E} T \tau}}{\sqrt{1+e^{-4\pi\mathcal{E}}}} \left( \frac{T}{\sin(\pi T \tau)} \right)^{1/2} .$$

$$G^R_*(\omega) =$$

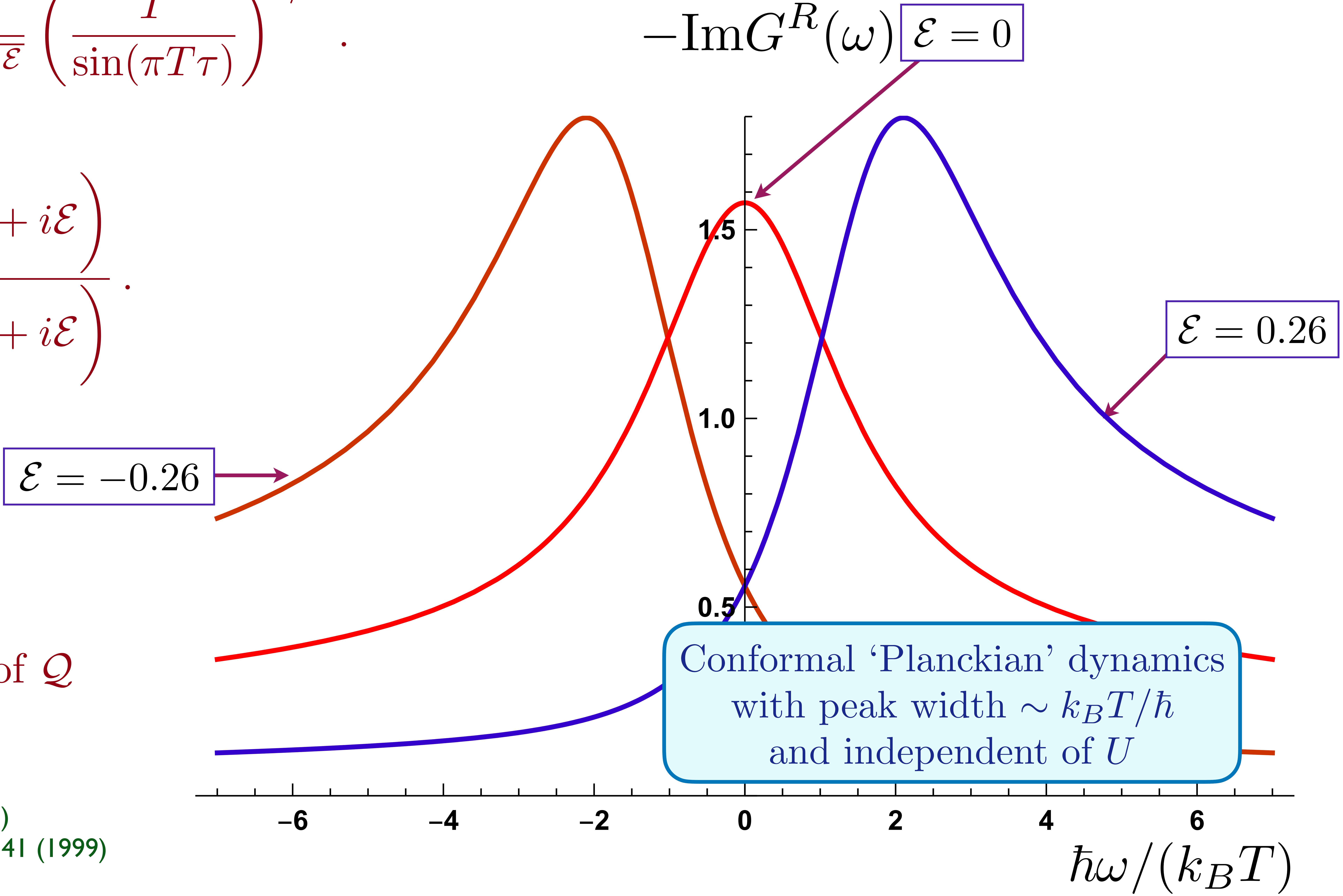
$$\frac{-iCe^{-i\theta}}{(2\pi T)^{1/2}} \frac{\Gamma\left(\frac{1}{4} - \frac{i\omega}{2\pi T} + i\mathcal{E}\right)}{\Gamma\left(\frac{3}{4} - \frac{i\omega}{2\pi T} + i\mathcal{E}\right)} .$$

$$e^{2\pi\mathcal{E}} = \frac{\sin(\pi/4+\theta)}{\sin(\pi/4-\theta)}$$

$$C = \left( \frac{\pi}{U^2 \cos(2\theta)} \right)^{1/4}$$

$\mathcal{E}$ is a known function of $\mathcal{Q}$

(Luttinger relation)



Conformal ‘Planckian’ dynamics with peak width $\sim k_B T/\hbar$ and independent of $U$

S. Sachdev and J. Ye, PRL **70**, 3339 (1993)
A. Georges and O. Parcollet PRB **59**, 5341 (1999)
S. Sachdev, PRX **5**, 041025 (2015)

# The Sachdev-Ye-Kitaev (SYK) model

$$
\begin{aligned}
\mathcal{Z}(\mathcal{Q},T) &= \mathrm{Tr}_{\mathcal{Q}} \exp\left(-\frac{\mathcal{H}}{T}\right) = \exp(-F/T)\,;\ \text{Entropy } S = -\frac{\partial F}{\partial T}. \\
&\equiv \int_{E_0^-}^{\infty} D(E) e^{-E/T}\,; \quad D(E) = \sum_i \delta(E - E_i)\,; \quad \mathcal{H}\left|\Psi_i\right\rangle = E_i \left|\Psi_i\right\rangle
\end{aligned}
$$

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\end{aligned}
$$

A CFT in $d$ spatial dimensions has an entropy density $S \sim T^d$. The SYK model is a 0+1 dimensional CFT, and we obtain a $T$-independent entropy:

$$
\lim_{T \to 0} \lim_{N \to \infty} \frac{S}{N} = s_0
$$

A. Georges, O. Parcollet, and S. Sachdev,
PRB **63**, 134406 (2001)

The constant $s_0$ is a known universal number. At $\mathcal{Q} = 1/2$, we have

$$
s_0 = \frac{\text{Catalan}}{\pi} + \frac{\ln 2}{4} = 0.464847699170805107492692486833\ldots\,.
$$

# The Sachdev-Ye-Kitaev (SYK) model

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$$\lim_{T\to 0} \lim_{N\to\infty} \frac{S}{N} = s_0 \quad \boxed{\Rightarrow \quad D(E) \overset{?}{=} e^{N s_0} \delta(E - E_0)}$$

A. Georges, O. Parcollet, and S. Sachdev, PRB **63**, 134406 (2001)

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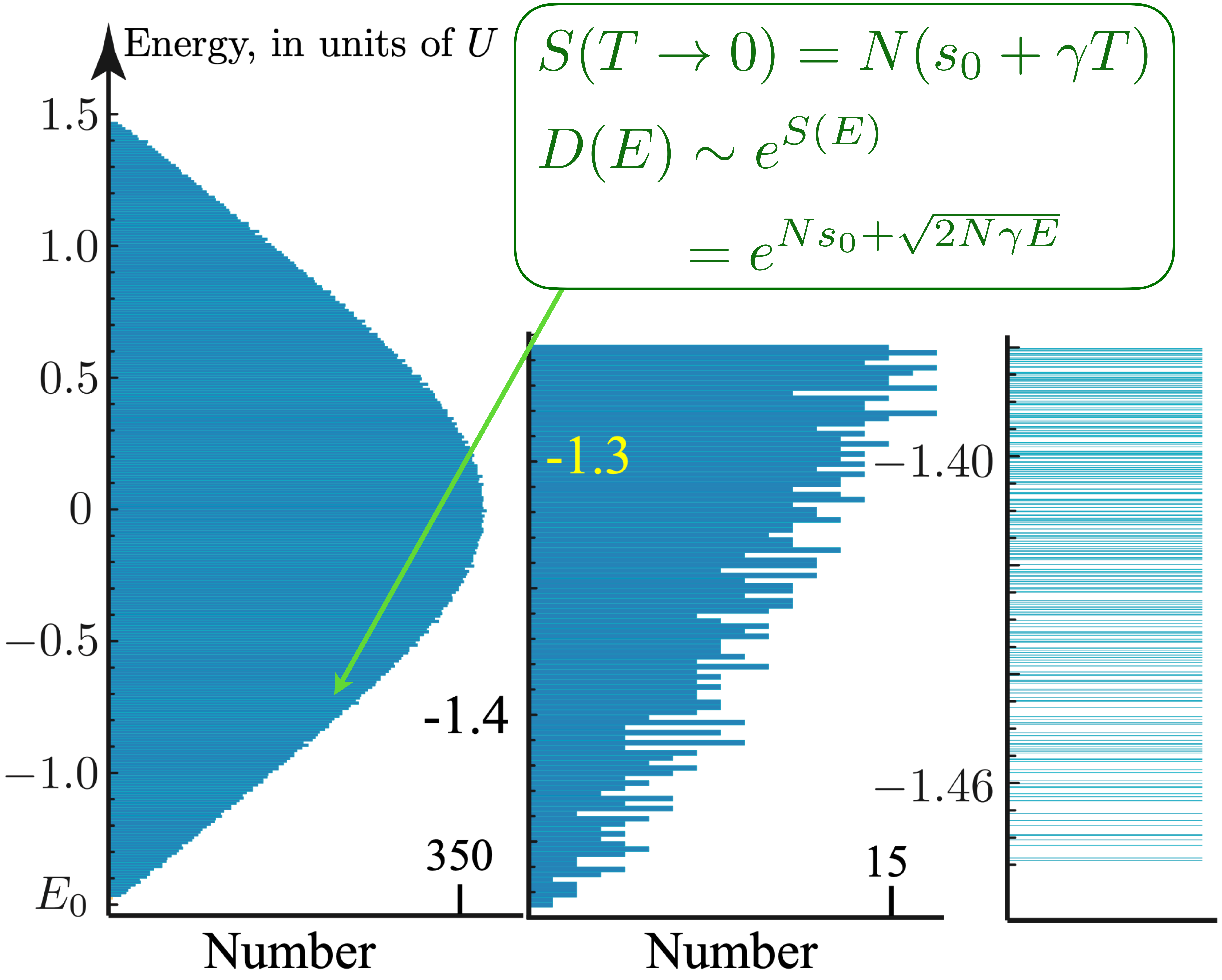
Conformal perturbations of saddle-point theory:

$$\boxed{S = N(s_0 + \gamma T) \quad \Rightarrow \quad D(E) \sim \exp\left(N s_0 + \sqrt{2N\gamma E}\right)}$$

$\gamma = \#/U$ is non-universal.

# Many-body density of states

$$D(E) = \sum_i \delta(E - E_i); \quad E_0 + E_i \Rightarrow \text{Many body eigenvalue}$$



$$S(T \to 0) = N(s_0 + \gamma T)$$

$$D(E) \sim e^{S(E)}$$

$$= e^{N s_0 + \sqrt{2 N \gamma E}}$$

# Complex SYK model

# Many-body density of states

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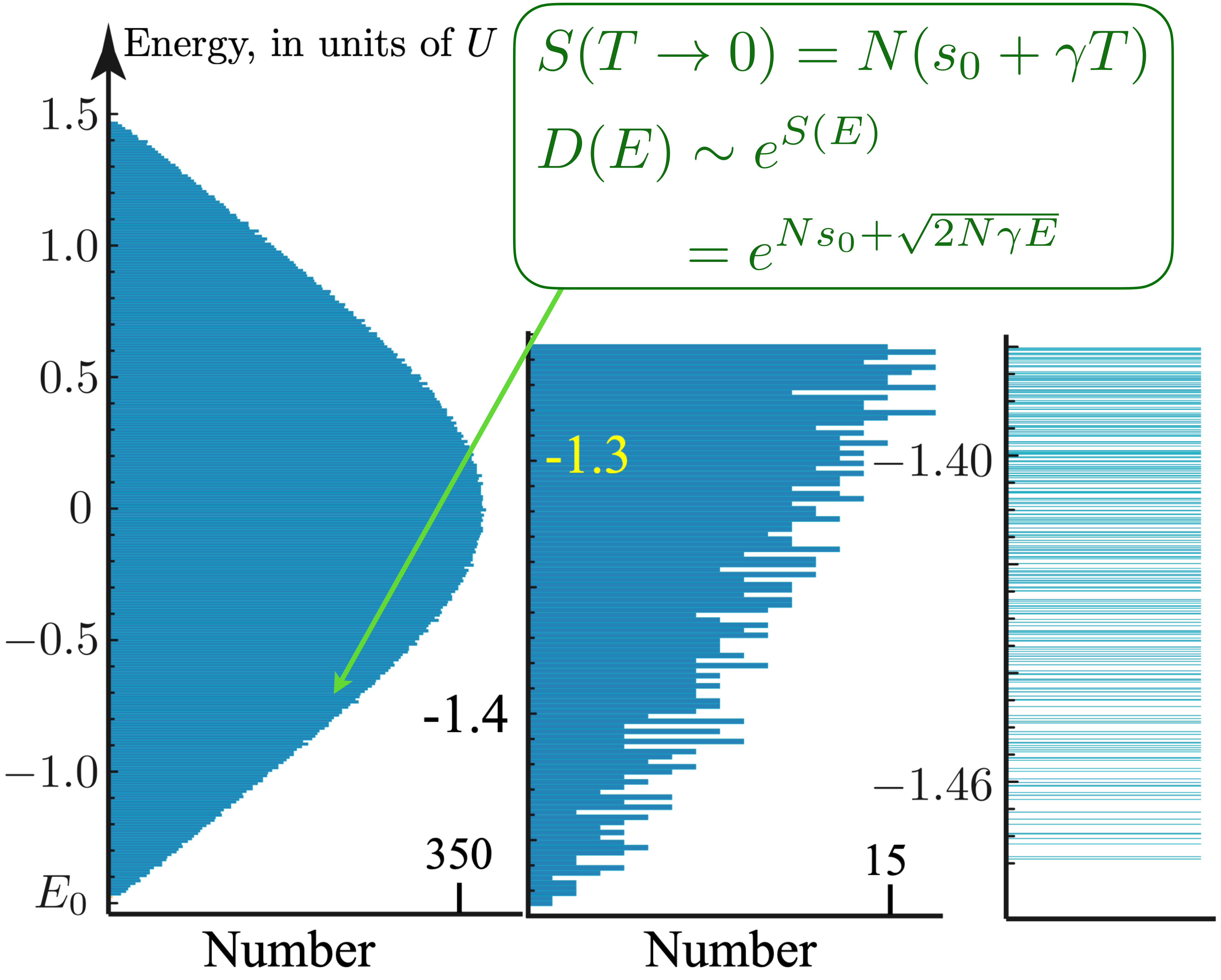


$$S(T \to 0) = N(s_0 + \gamma T)$$

$$D(E) \sim e^{S(E)}$$

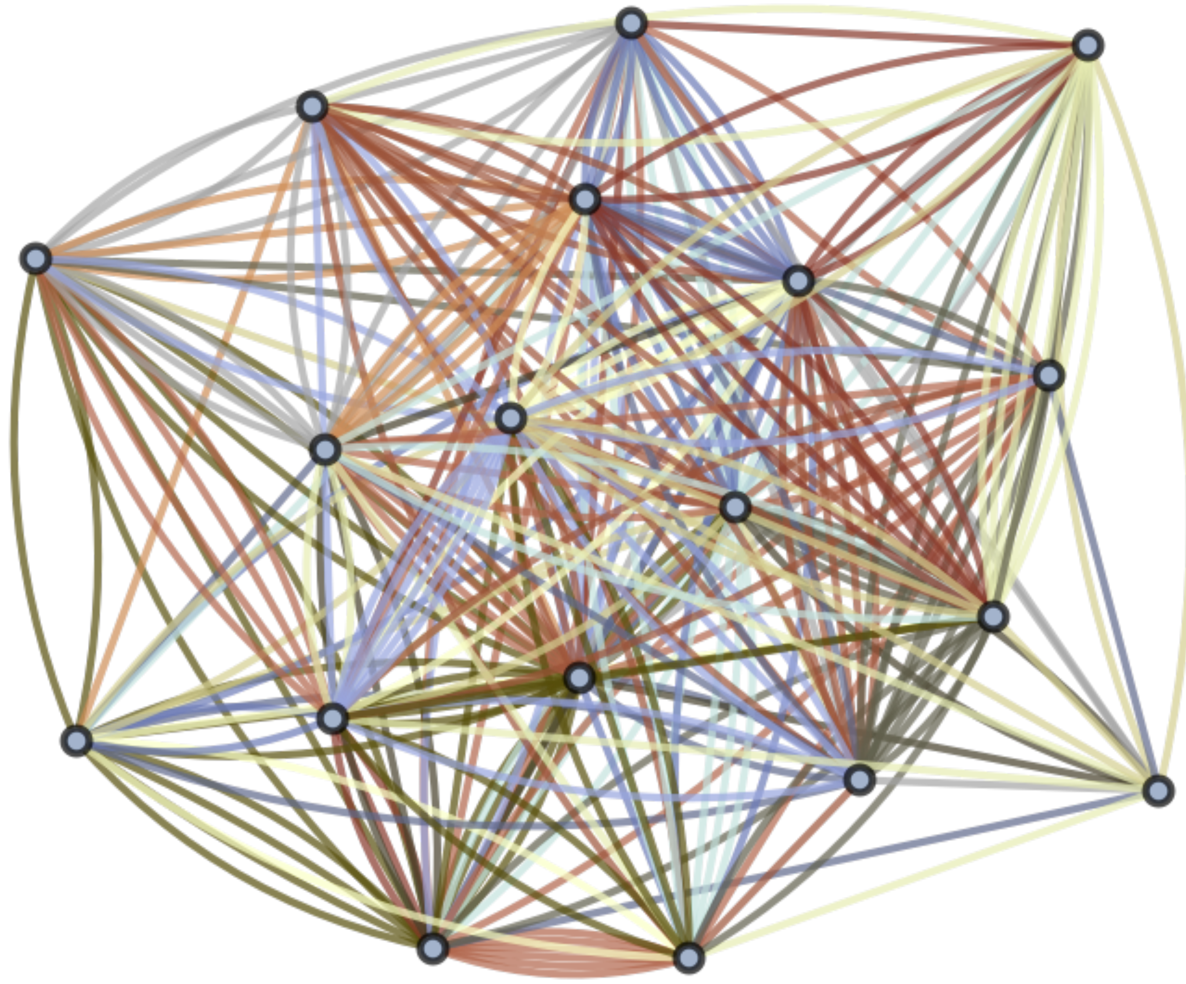
$$= e^{Ns_0 + \sqrt{2N\gamma E}}$$

Energy level spacing $\sim e^{-Ns_0}$ !

## Complex SYK model

# The Sachdev-Ye-Kitaev (SYK) model

The disorder-averaged partition is exactly this $G$-$\Sigma$ theory:



$$\mathcal{Z} = \int \mathcal{D}G(\tau_1,\tau_2)\mathcal{D}\Sigma(\tau_1,\tau_2)\exp(-NI)$$

$$I = \ln\det\left[\delta(\tau_1-\tau_2)(\partial_{\tau_1}+\mu) - \Sigma(\tau_1,\tau_2)\right] + \int d\tau_1 d\tau_2 \left[\Sigma(\tau_1,\tau_2)G(\tau_2,\tau_1) + (U^2/2)G^2(\tau_2,\tau_1)G^2(\tau_1,\tau_2)\right]$$

Saddle-point equations for $G(\tau_1-\tau_2)$ and $\Sigma(\tau_1-\tau_2)$:

$$G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = -J^2G^2(\tau)G(-\tau)$$

$$G(\tau = 0^-) = \mathcal{Q}.$$

# $G$-$\Sigma$ path integral

At frequencies $\ll U$, the time derivative in the determinant is less important, and without it the path integral is invariant under the reparametrization and gauge transformations

$$
\begin{aligned}
\tau &= f(\sigma) \\
G(\tau_1, \tau_2) &= \left[f'(\sigma_1) f'(\sigma_2)\right]^{-1/4} e^{-i\phi(\sigma_1) + i\phi(\sigma_2)}\, G(\sigma_1, \sigma_2) \\
\Sigma(\tau_1, \tau_2) &= \left[f'(\sigma_1) f'(\sigma_2)\right]^{-3/4} e^{-i\phi(\sigma_1) + i\phi(\sigma_2)}\, \Sigma(\sigma_1, \sigma_2)
\end{aligned}
$$

where $f(\sigma)$ and $\phi(\sigma)$ are arbitrary functions.

A. Georges and O. Parcollet
PRB **59**, 5341 (1999)
A. Kitaev, 2015
S. Sachdev, PRX **5,** 041025 (2015)

# Reparametrization and phase zero modes

We find the conformal saddle point, $G_*$, $\Sigma_*$, and only focus on the "Nambu-Goldstone" modes associated with breaking reparameterization and U(1) gauge symmetries by writing

$$G(\tau_1, \tau_2) = [f'(\tau_1) f'(\tau_2)]^{1/4} G_*(f(\tau_1) - f(\tau_2)) e^{i\phi(\tau_1) - i\phi(\tau_2)}$$

(and similarly for $\Sigma$). Then the path integral is approximated by

$$\mathcal{Z} = \int \mathcal{D}f(\tau) \mathcal{D}\phi(\tau) e^{-E_0/T + N s_0 - N I_{\text{eff}}[f, \phi]},$$

where $E_0 \propto N$ is the ground state energy.

A. Kitaev (2015); J. Maldacena and D. Stanford, arXiv:1604.07818;
R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, arXiv.1612.00849;
S. Sachdev, PRX **5,** 041025 (2015)

$$
\begin{aligned}
\mathcal{Z} &= \mathrm{Tr}\exp\left(-\frac{\mathcal{H}}{k_B T}\right) \\
&\approx \exp\left(N\frac{s_0}{k_B}\right)\int\frac{\mathcal{D}f(\tau)\mathcal{D}\phi(\tau)}{||\mathrm{SL(2,R)}||}\exp\left(-\frac{1}{\hbar}I_{\text{eff}}\left[f(\tau),\phi(\tau)\right]\right) \\
I_{\text{eff}}[f,\phi] &= \frac{NK}{2}\int_0^{1/T}d\tau(\partial_\tau\phi+i(2\pi\mathcal{E}T)\partial_\tau f)^2-\frac{N\gamma}{4\pi^2}\int_0^{1/T}d\tau\left\{\tan(\pi T f(\tau)),\tau\right\},
\end{aligned}
$$

where $f(\tau)$ is a monotonic map from $[0,1/T]$ to $[0,1/T]$. The conformal group is $d$ spatial dimensions is SO($d$+2, 1), and PSL(2,R)≅SO(2,1), and the Schwarzian

$$
\{g,\tau\}\equiv\frac{g'''}{g'}-\frac{3}{2}\left(\frac{g''}{g'}\right)^2,
$$

vanishes for $g(\tau)=(a\tau+b)/(c\tau+d)$ a SL(2,R) transformation $(ad-bc=1)$. The couplings $K$, $\gamma$, and $\mathcal{E}$ can be related to thermodynamic derivatives.

A. Kitaev (2015); J. Maldacena and D. Stanford, arXiv:1604.07818;
R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, arXiv.1612.00849;
S. Sachdev, PRX **5,** 041025 (2015)

# The Sachdev-Ye-Kitaev (SYK) model

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A. Georges, O. Parcollet, and S. Sachdev,
PRB **63**, 134406 (2001)

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J. S. Cotler et al., JHEP 05 (2017) 118

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Yingfei Gu, A. Kitaev, S. Sachdev, and G. Tarnopolsky, JHEP 02 (2020) 157

A. Georges, O. Parcollet, and S. Sachdev, PRB **63**, 134406 (2001)

J. S. Cotler et al., JHEP 05 (2017) 118

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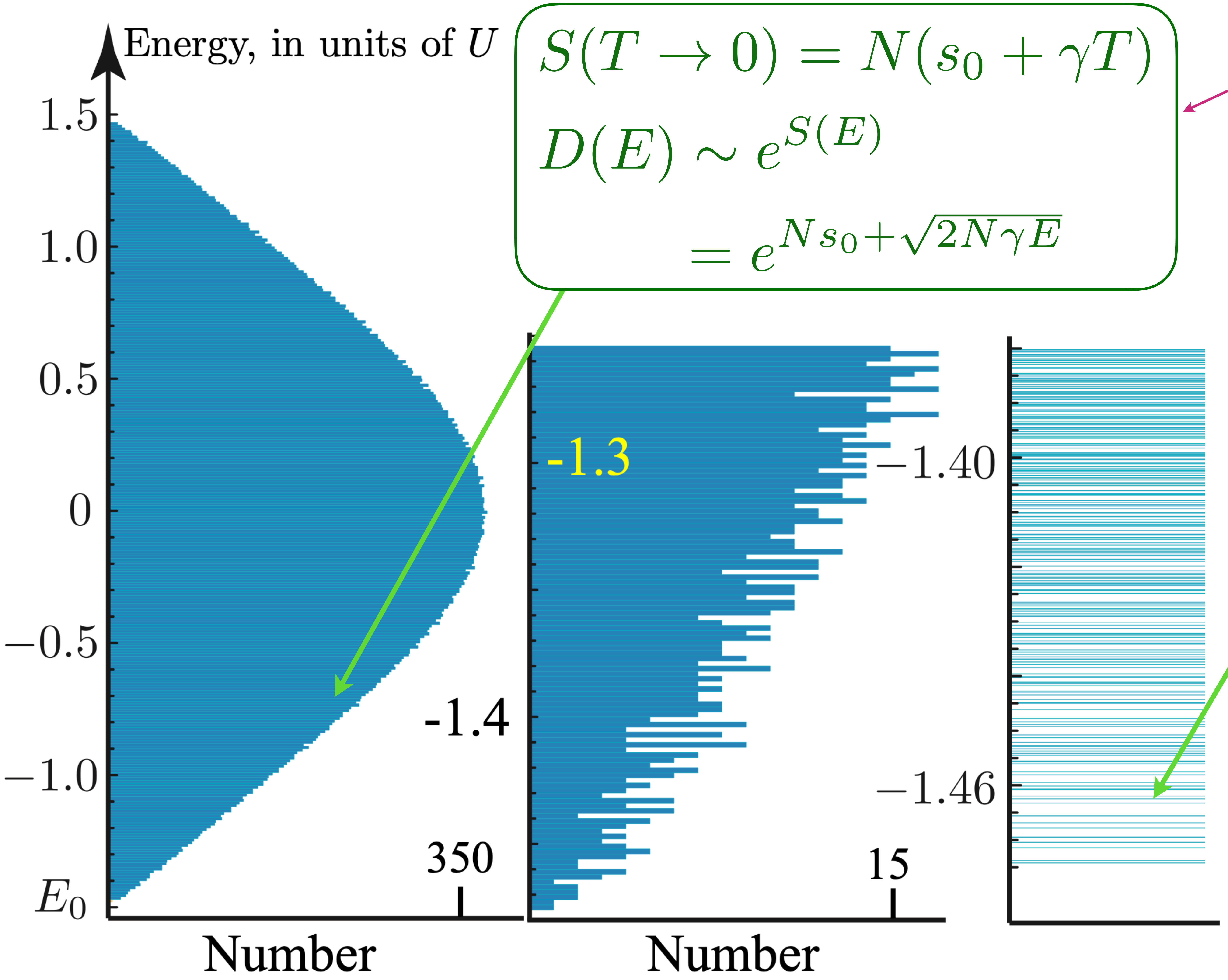
# Many-body density of states

$$D(E) = \sum_i \delta(E - E_i); \quad E_0 + E_i \Rightarrow \text{Many body eigenvalue}$$



$$S(T \to 0) = N(s_0 + \gamma T)$$

$$D(E) \sim e^{S(E)}$$

$$= e^{N s_0 + \sqrt{2 N \gamma E}}$$

$$D(E) \sim N^{-1} \, e^{N s_0} \sinh(\sqrt{2 N \gamma E})$$

$$D(E) \sim e^{N s_0} \sqrt{2 \gamma E / N}$$

## Complex SYK model

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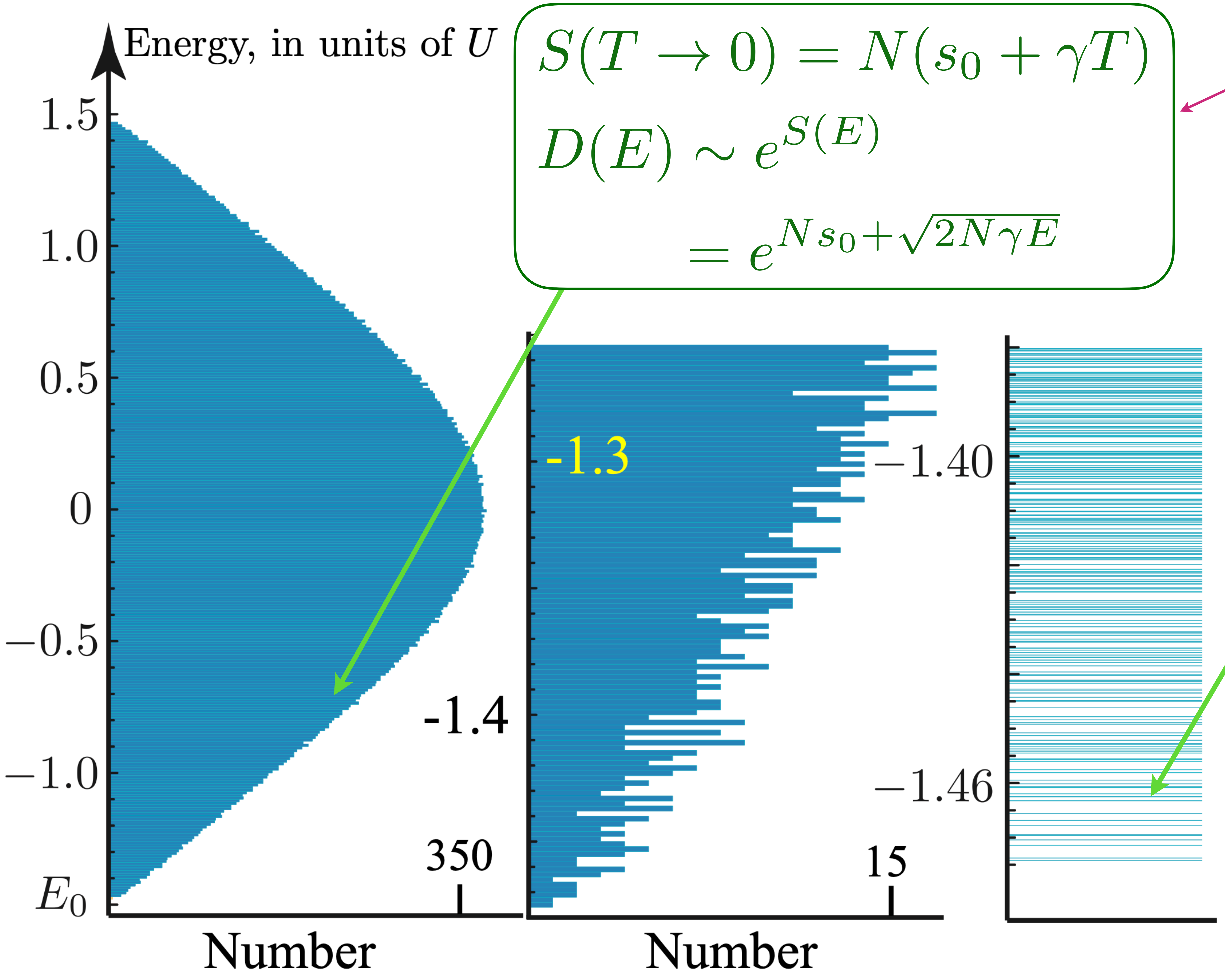

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$$D(E) \sim e^{N s_0} \sqrt{2\gamma E/N}$$

No exponentially large degeneracy, but exponentially small level spacing!

No quasiparticle decomposition: wavefunctions change chaotically from one state to the next.

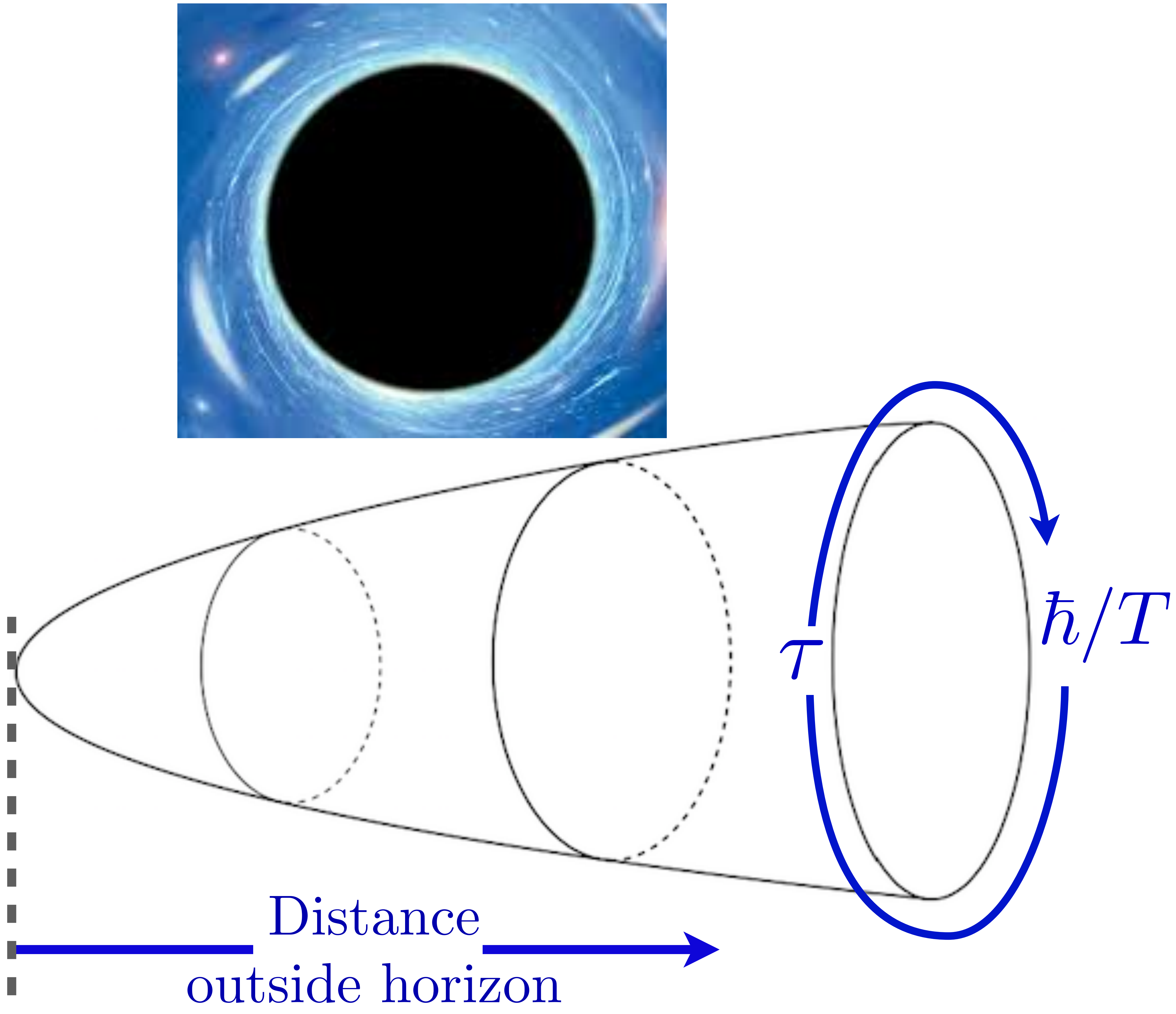
# Complex SYK model

# Charged black holes

S.S., arXiv:2205.02285

# Thermodynamics of quantum black holes with charge $\mathcal{Q}$:

$$\mathcal{Z}(\mathcal{Q},T) = \int \mathcal{D}g_{\mu\nu}\mathcal{D}A_\mu \exp\left(-\frac{1}{\hbar}\, I^{(3+1)}_{\text{Einstein gravity+Maxwell EM}}[g_{\mu\nu}, A_\mu]\right)$$



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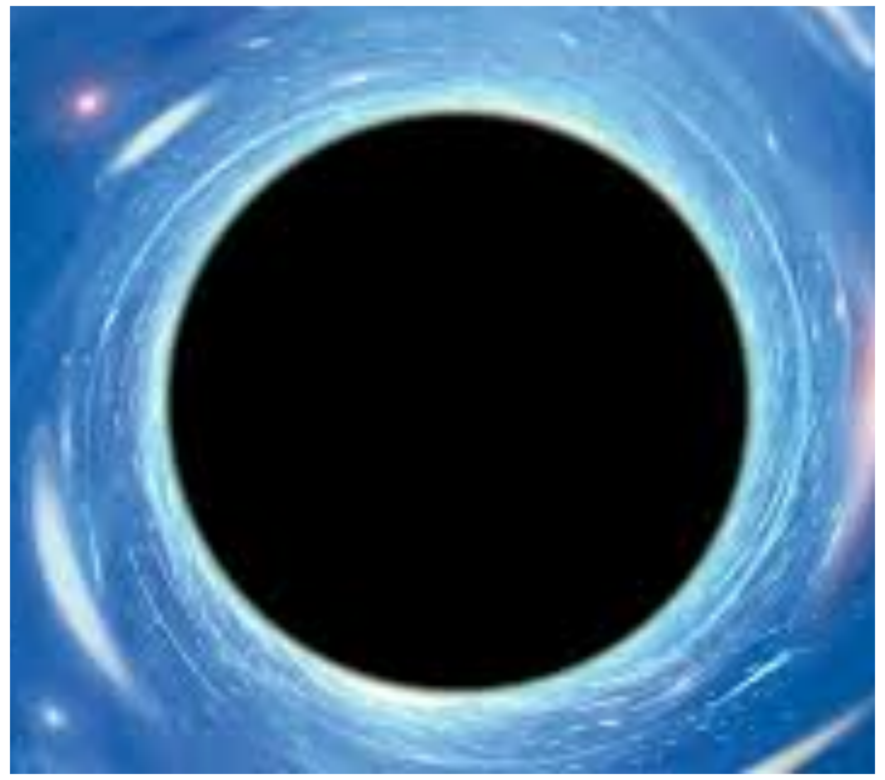
$$= \exp\Big(S_{BH}\Big) \times \Bigg( \quad ....????... \quad \Bigg)$$

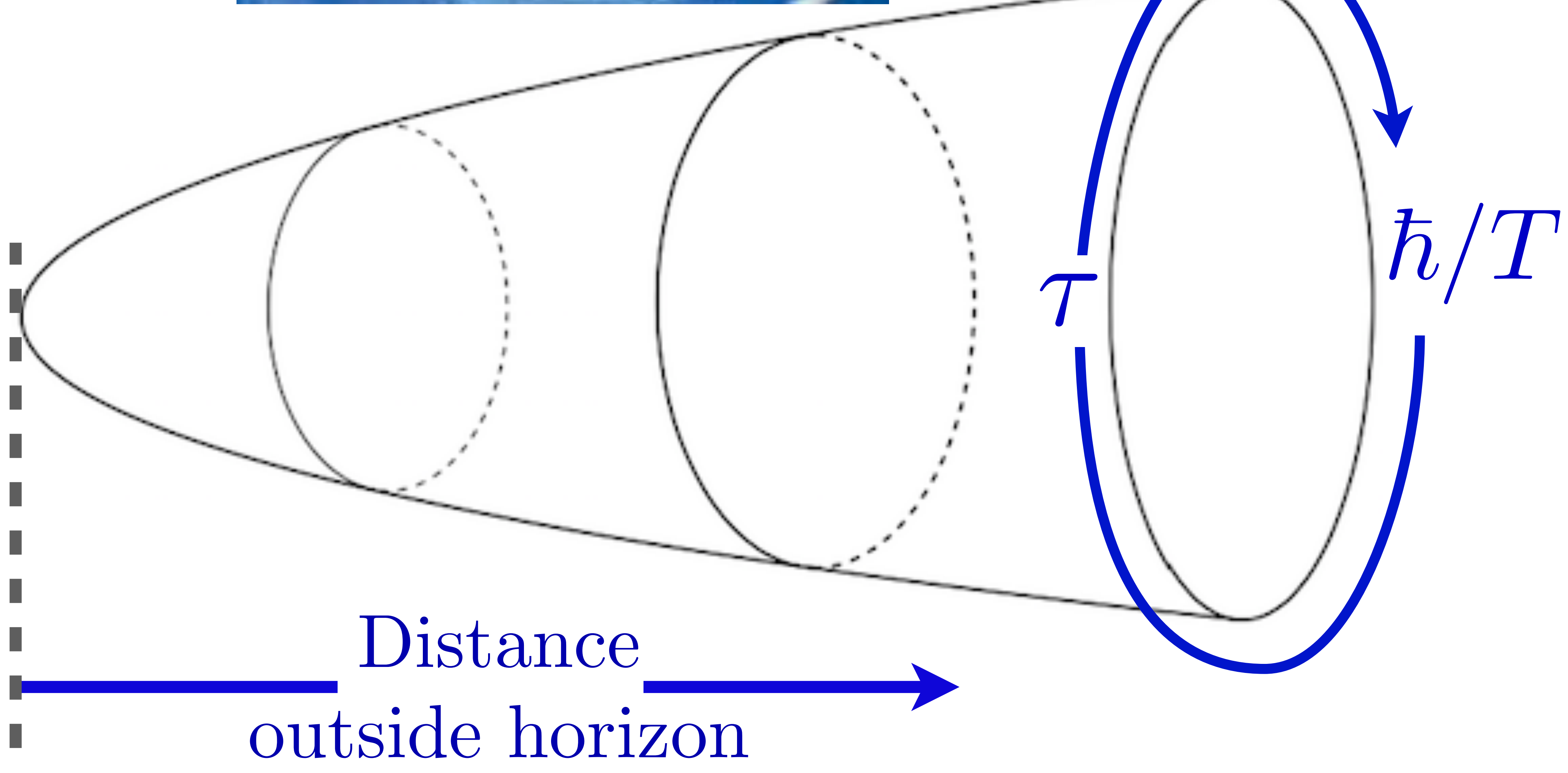
Gibbons, Hawking (1977)
Chambin, Emparan, Johnson, Myers (1999)



$$S_{BH}(T \to 0, \mathcal{Q}) = \frac{A(T)c^3}{4G\hbar} = \frac{A_0 c^3}{4G\hbar}\left(1 + \frac{2(\pi A_0)^{1/2}T}{\hbar c}\right)$$

$A_0 = 2G\mathcal{Q}^2/c^4$ is the area of the charged black hole horizon at $T = 0$.

Obtained from the saddle-point of the gravity path integral in the imaginary time spacetime outside the black hole.



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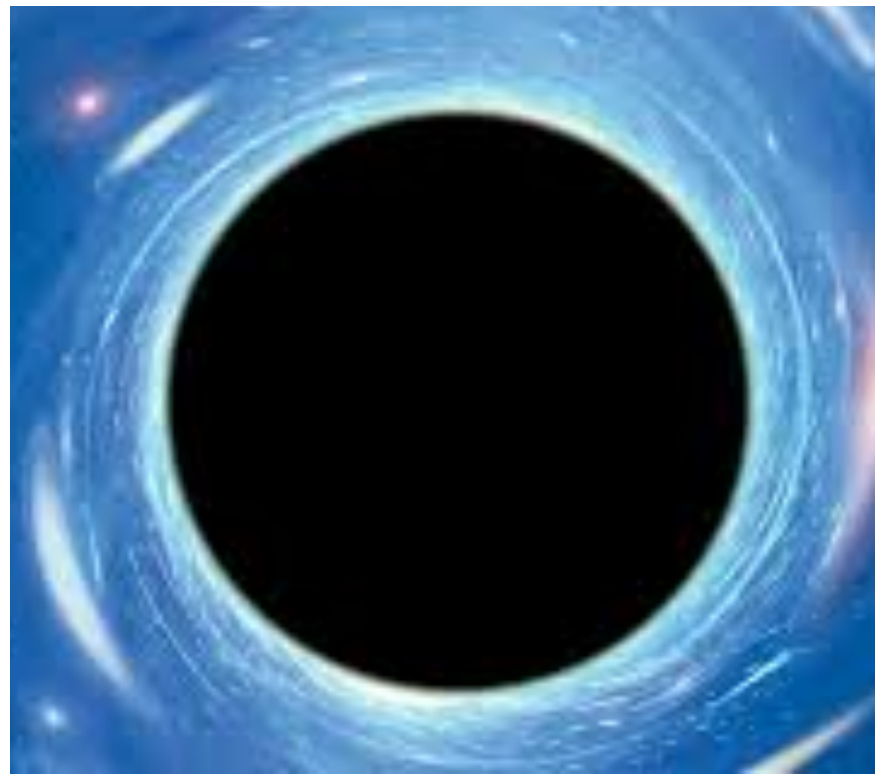

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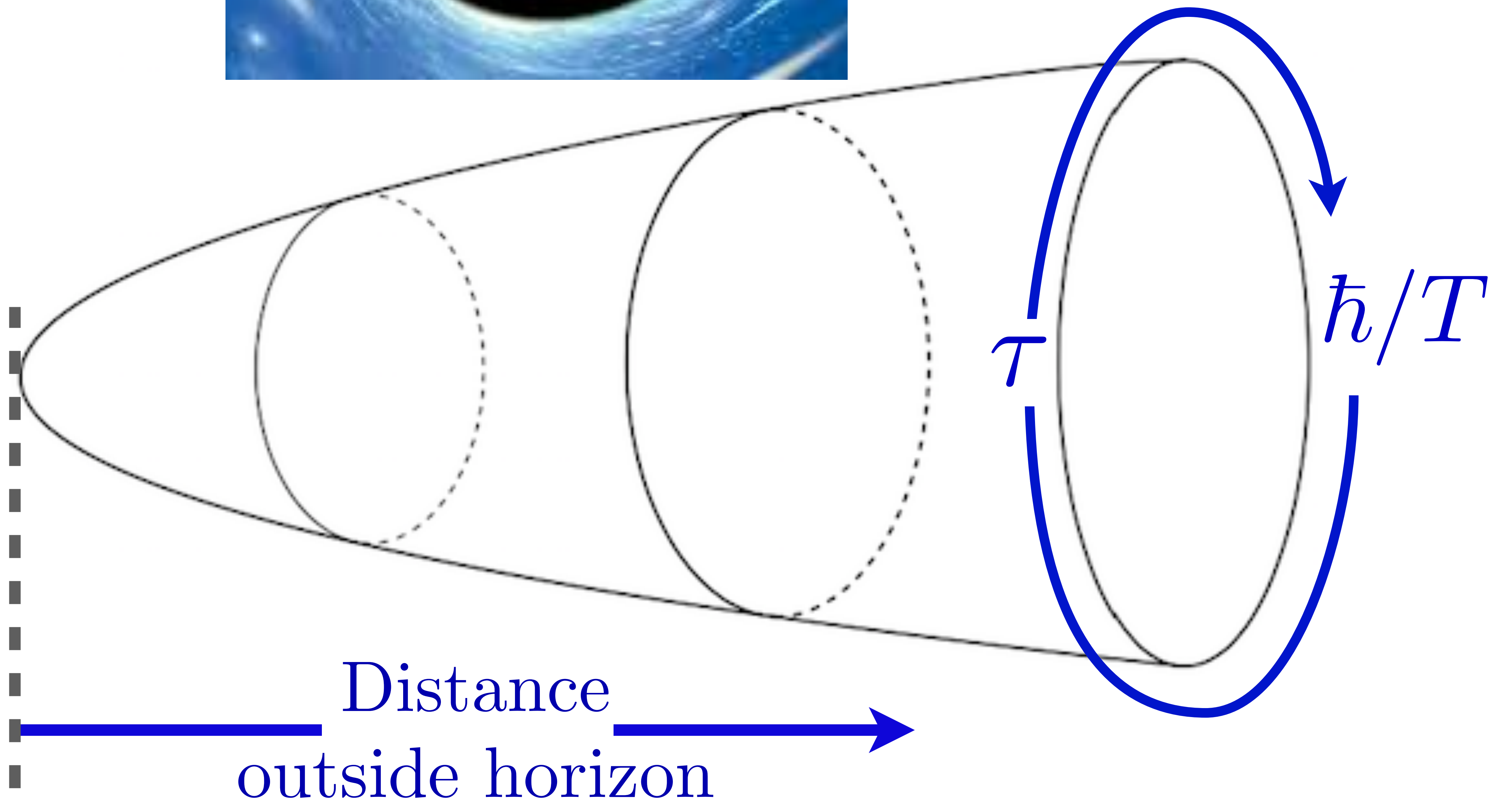
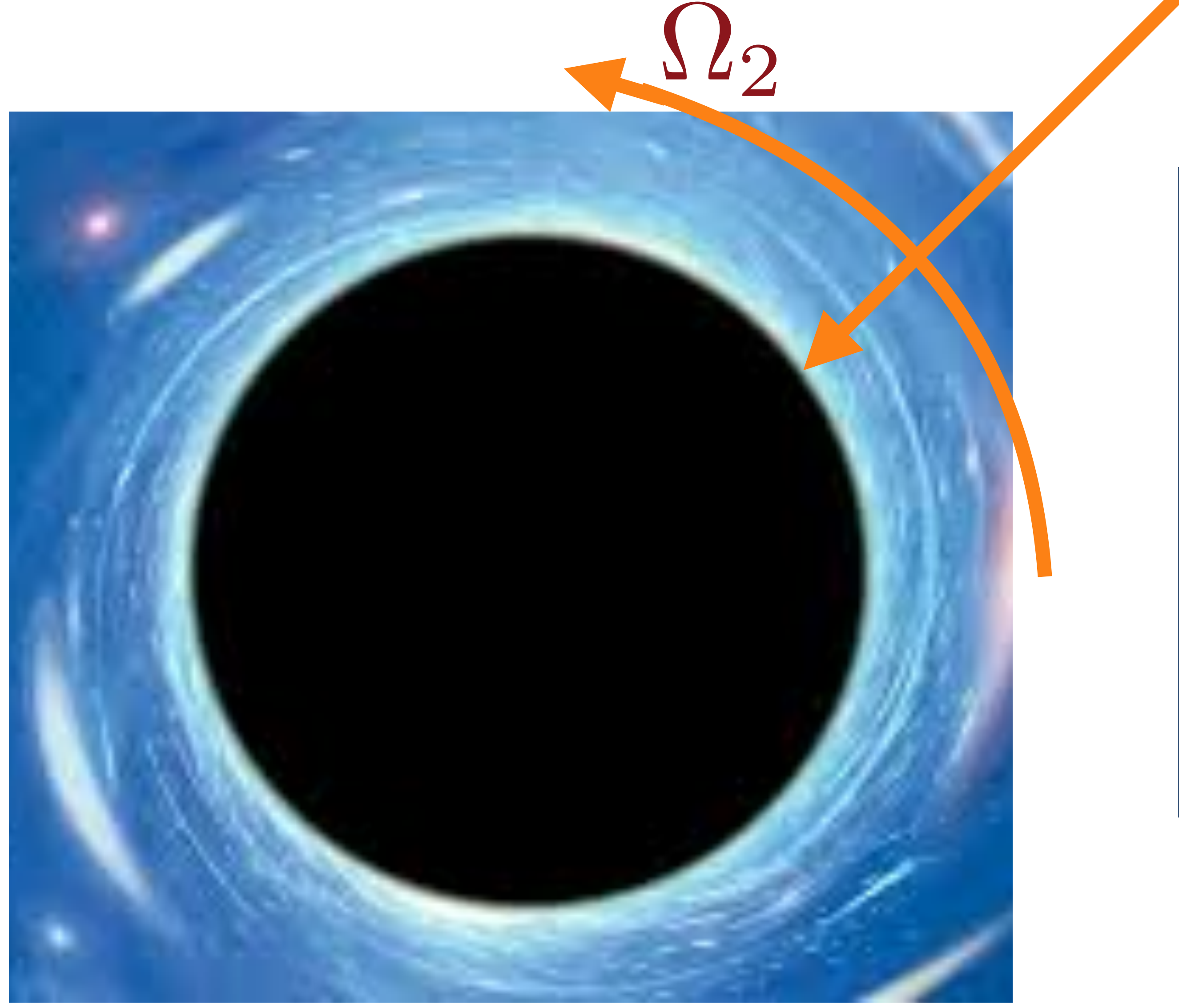
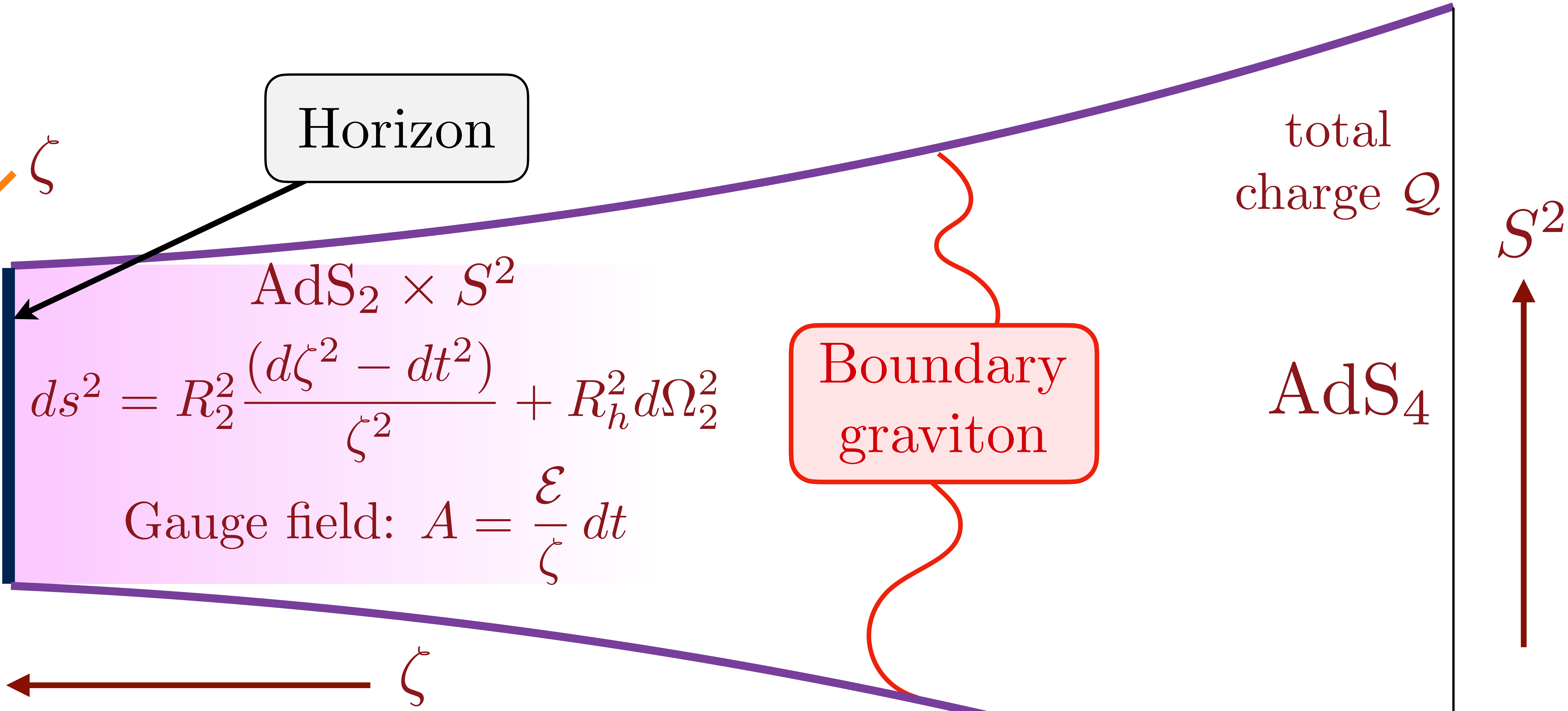


Note the similarity to the large *N* entropy of the SYK model ! (along with other similarities)

Sachdev PRL 2010

# Reissner-Nordstrom black hole of Einstein-Maxwell theory





AdS$_2 \times S^2$

$$ds^2 = R_2^2 \frac{(d\zeta^2 - dt^2)}{\zeta^2} + R_h^2 d\Omega_2^2$$

Gauge field: $A = \frac{\mathcal{E}}{\zeta}\, dt$

Dimensional reduction from 3+1 dimensions to 1+1 dimensions (AdS$_2$) at low energies!

The isometry group of AdS$_2$ is the 0+1 dimensional conformal group SL(2,R).

Faulkner, Liu, McGreevy, Vegh 2009; Sachdev 2010

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Sachdev (2010); Kitaev (2015); Sachdev (2015); Bagrets, Altland, Kamenev (2016); Maldacena, Stanford, Yang (2016); Moitra, Trivedi, Vishal (2018) ; Gaikwad, Joshi, Mandal, Wadia (2018); Sachdev (2019); Iliesiu, Turaci (2020)

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$$S(T\to 0,\mathcal{Q}) = \frac{A(T)c^3}{4G\hbar} - \frac{3}{2}\ln\left(\frac{(\hbar c^5/G)^{1/2}}{T}\right)$$

The $\ln T$ term is the SYK/boundary-graviton correction to Bekenstein-Hawking.

There is also a

$$-\frac{559}{180}\ln\left(\frac{A_0 c^3}{\hbar G}\right)$$

term from other massless modes; Sen (2011) Iliesiu, Murthy, Turiaci (2022)

Sachdev (2010); Kitaev (2015); Sachdev (2015); Bagrets, Altland, Kamenev (2016); Maldacena, Stanford, Yang (2016); Moitra, Trivedi, Vishal (2018) ; Gaikwad, Joshi, Mandal, Wadia (2018); Sachdev (2019); Iliesiu, Turaci (2020)

# Black hole questions and answers

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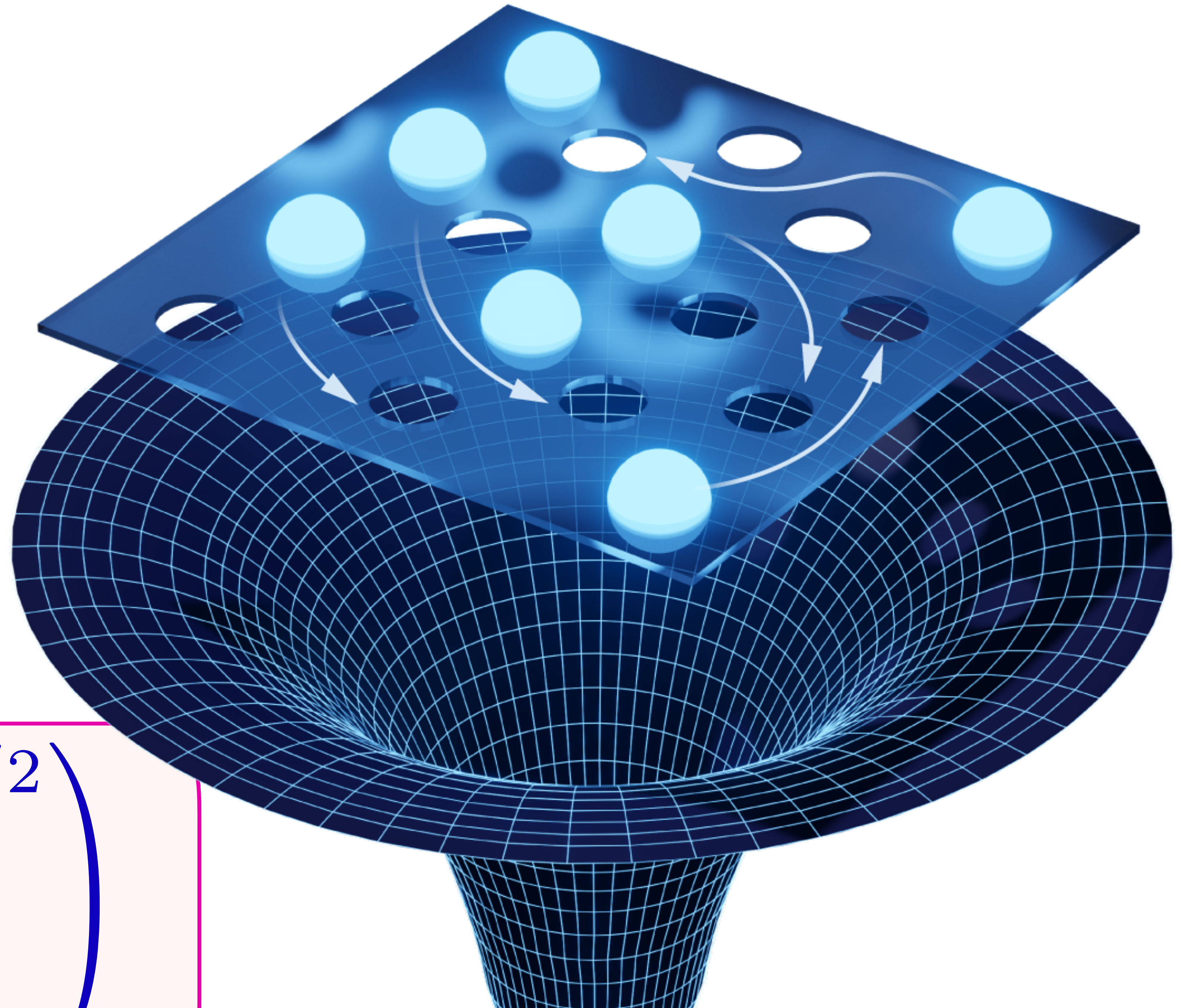
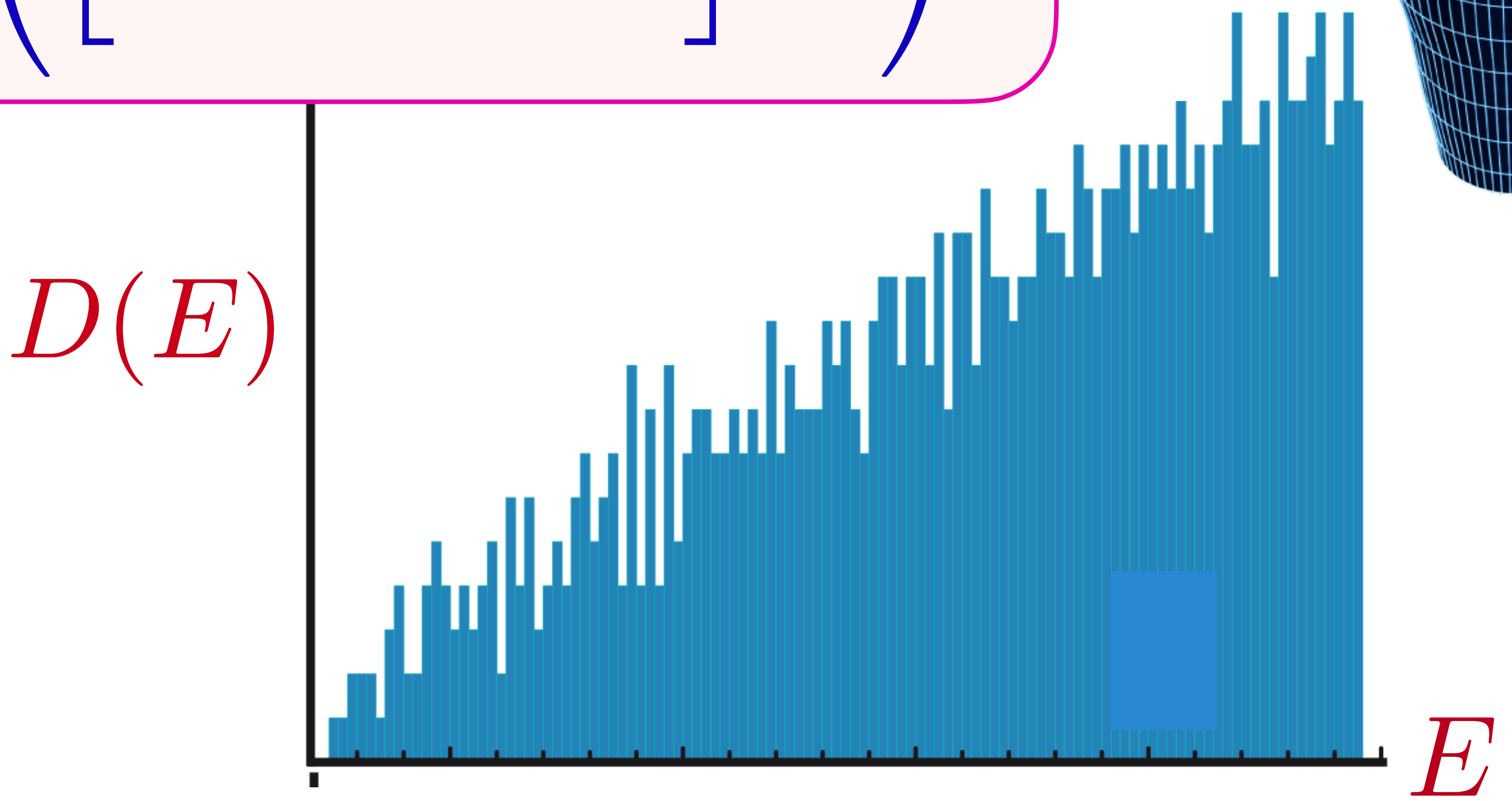


- For generic charged black holes in 3+1 dimensions, the SYK model yields, in terms of $A_0 = 2G\mathcal{Q}^2/c^4$ the horizon area at $T = 0$:

$$\boxed{D(E) \sim \left(\frac{A_0 c^3}{\hbar G}\right)^{-347/90} \exp\left(\frac{A_0 c^3}{4\hbar G}\right) \sinh\left(\left[\frac{\sqrt{\pi} A_0^{3/2} c^2}{\hbar^2 G} E\right]^{1/2}\right)}$$

There is no degeneracy, but an exponentially small level spacing down to the ground state.



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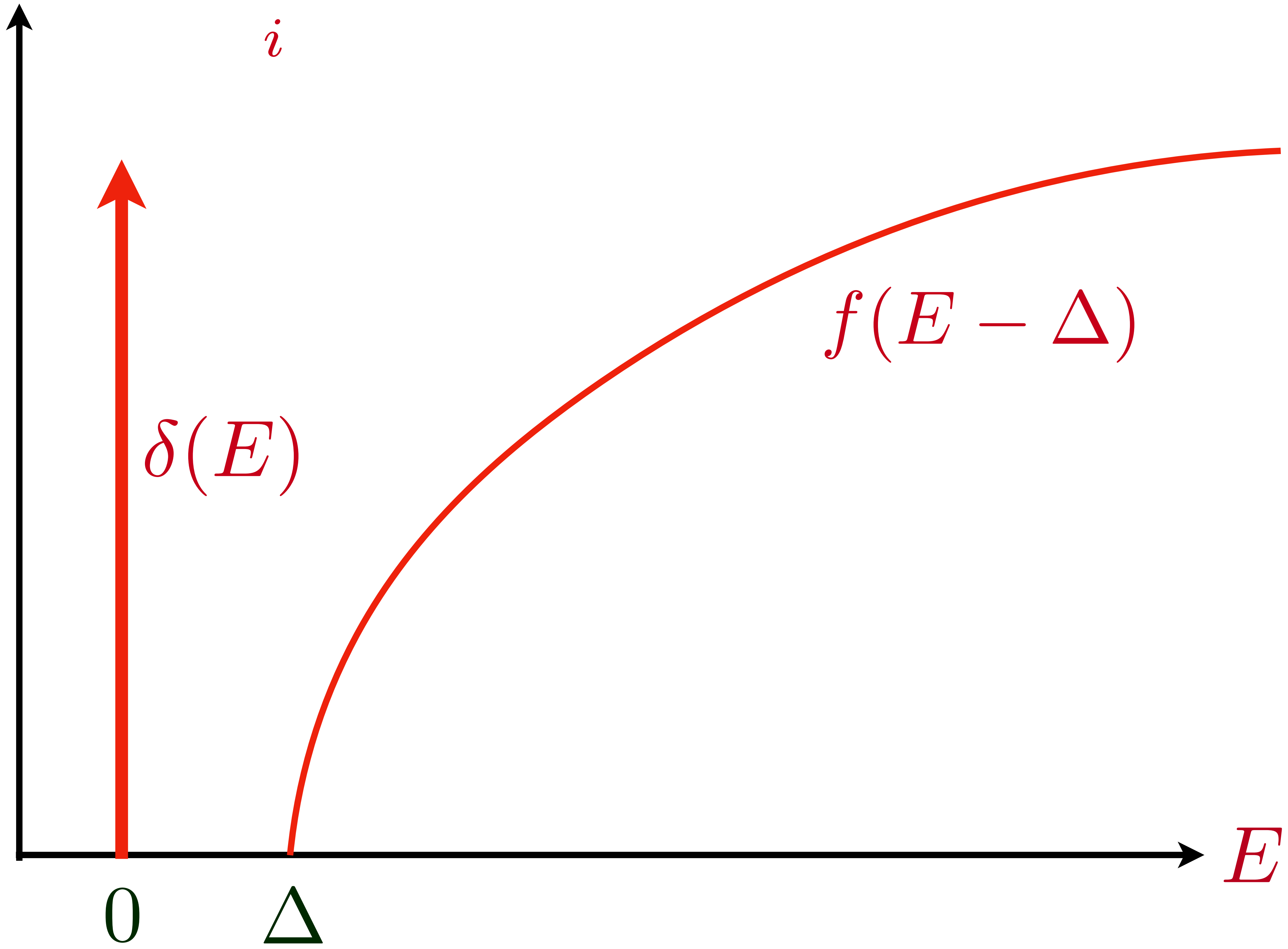
Yes, for charged black holes:

- With sufficient low energy supersymmetry, string theory yields:

$$D(E) = \exp\left(\frac{A_0 c^3}{4\hbar G}\right)\delta(E) + \theta(E-\Delta) f(E-\Delta) + \ldots$$

There are exponentially many degenerate BPS ground states, and an energy gap $\Delta$ above the ground state.

$$D(E) = \sum_i \delta(E - E_i)$$



M. Heydeman, L.V. Iliesiu, G. J. Turiaci, and W. Zhao, 2020
L.V. Iliesiu, S. Murthy, G. J. Turiaci, 2022

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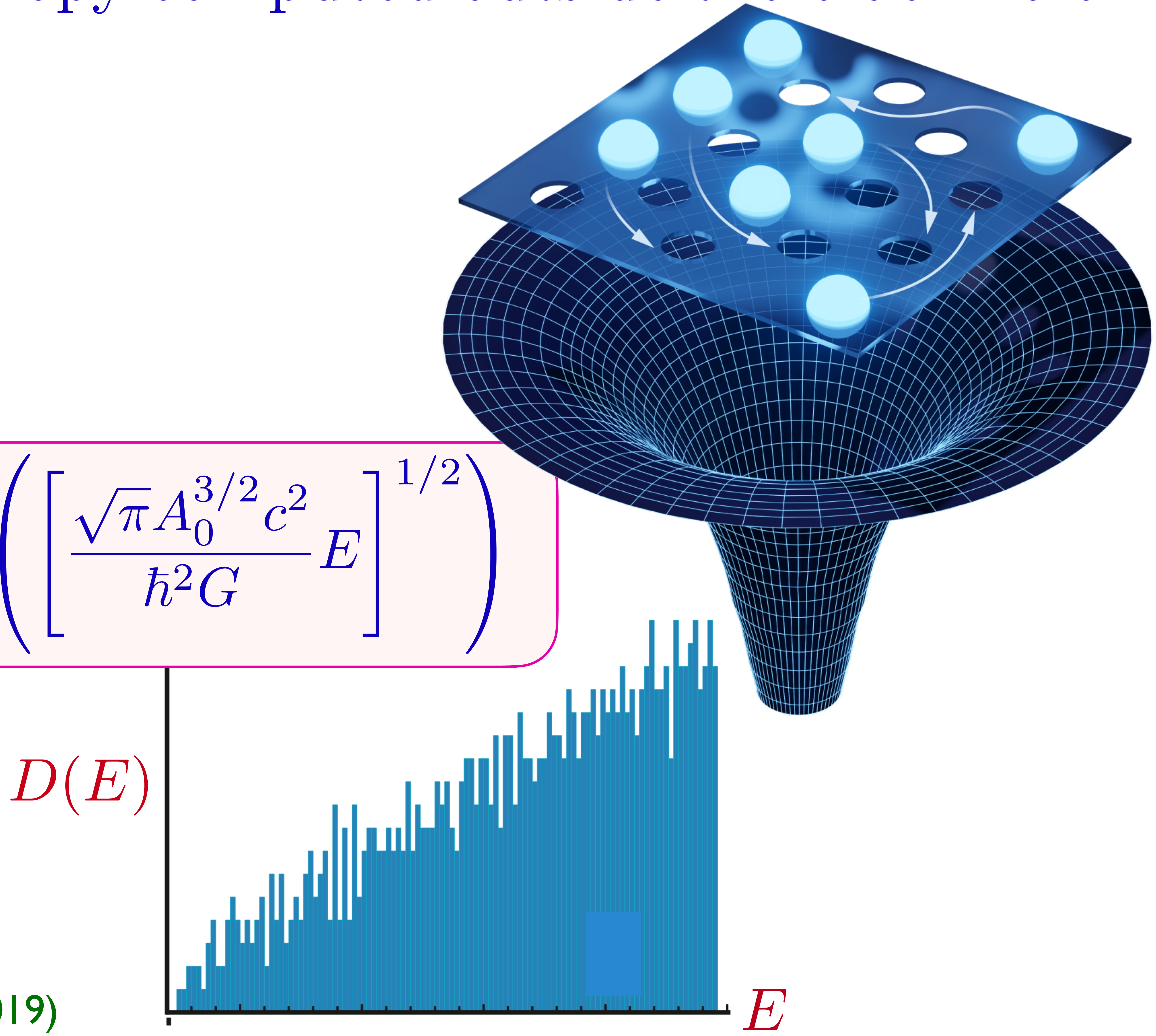
- ‘Wormhole’ contributions to this quantum simulation have led to an understanding of the Page curve of entanglement entropy of evaporating black holes.



Saad, Shenker, Stanford (2019)

# Strange metals

# Properties of a strange metal:

1. Resistivity $\rho(T) = \rho_0 + AT + \ldots$ as $T \to 0$
   and $\rho(T) < h/e^2$ (in $d = 2$).
   Metals with $\rho(T) > h/e^2$ are bad metals.

2. Specific heat $\sim T \ln(1/T)$ as $T \to 0$.

S.A. Hartnoll and A.P. MacKenzie, arXiv:2107.07802

3. Optical conductivity

$$\sigma(\omega) = \frac{K}{\dfrac{1}{\tau_{\rm trans}(\omega)} - i\omega \dfrac{m^*_{\rm trans}(\omega)}{m}} \quad ; \quad \frac{1}{\tau_{\rm trans}(\omega)} \sim |\omega| \Phi_\sigma \left( \frac{\hbar \omega}{k_B T} \right)$$

B. Michon……A. Georges, arXiv:2205.04030

4. Photoemission: nearly "marginal Fermi liquid" electron spectral density:

$$\mathrm{Im}\Sigma(\omega) \sim |\omega|^{2\alpha} \Phi_\Sigma \left( \frac{\hbar \omega}{k_B T} \right) \quad \text{with } \alpha \approx 1/2 \quad ; \quad \frac{1}{\tau(\omega)} \sim |\omega| \Phi_\Sigma \left( \frac{\hbar \omega}{k_B T} \right)$$

T.J. Reber….D. Dessau, Nature Communications **10**, 5737 (2019)

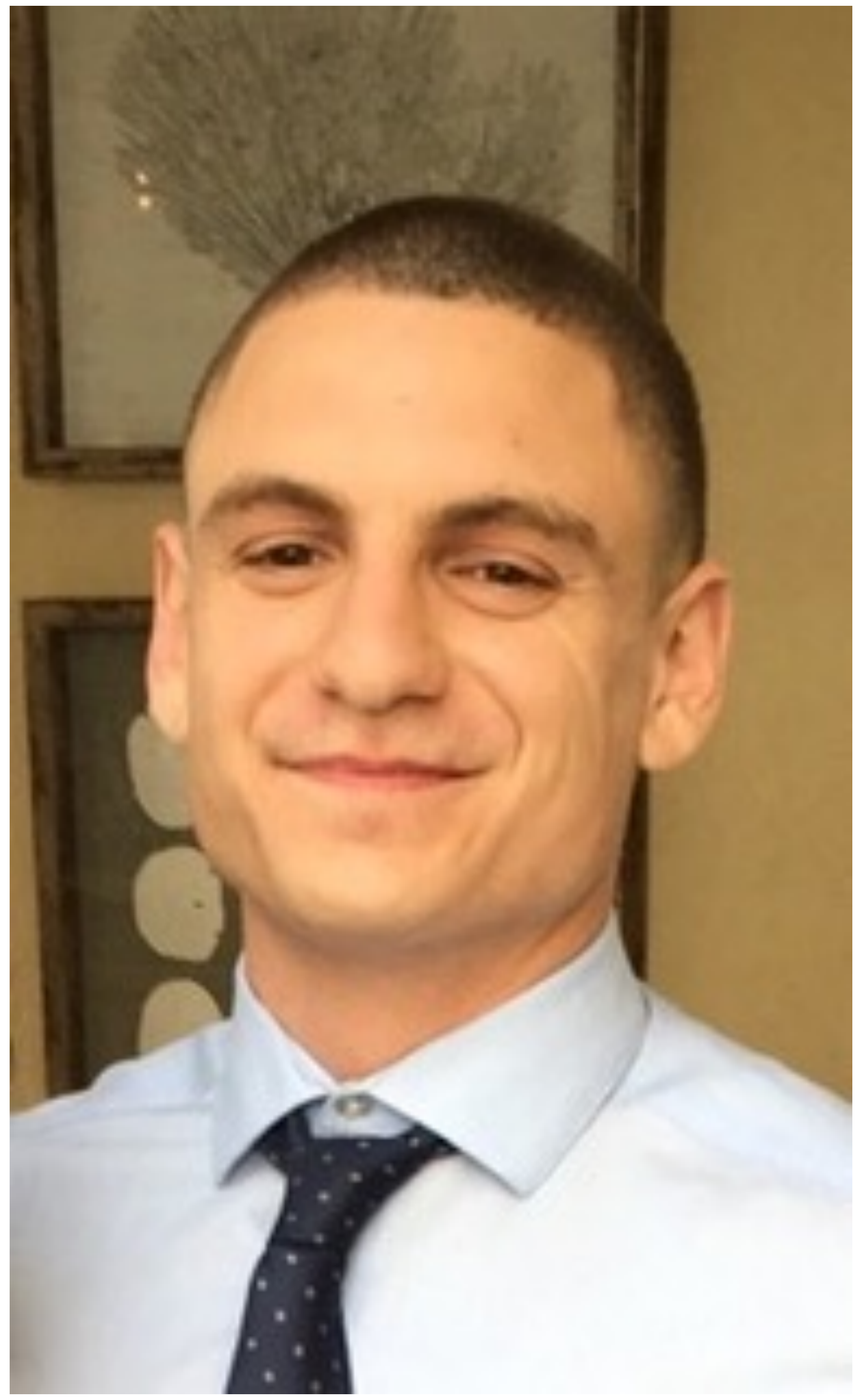
# From
# Yukawa-SYK models
# to a universal theory of
# strange metals

Aavishkar Patel

Flatiron Institute, NYC

Haoyu Guo

Harvard



Ilya Esterlis

Harvard → Wisconsin

arXiv: 2103.08615, 2203.04990, 2207.08841

E. E. Aldape, T. Cookmeyer, Aavishkar A. Patel, and Ehud Altman, arXiv:2012.00763

# Yukawa-SYK models

$$\mathcal{H} = -\mu \sum_i \psi_i^\dagger \psi_i + \sum_\ell \frac{1}{2}\left(\pi_\ell^2 + \omega_0^2 \phi_\ell^2\right) + \frac{1}{N}\sum_{ij\ell} g_{ij\ell}\psi_i^\dagger \psi_j \phi_\ell \,,$$

$g_{ij\ell}$ independent random numbers with zero mean. Large $N$ limit leads to Migdal-Eliashberg equations $\Sigma_\psi \sim g^2 G_\psi G_\phi$, $\Sigma_\phi \sim g^2 G_\psi G_\psi$.

W. Fu, D. Gaiotto, J. Maldacena, and S. Sachdev, PRD **95**, 026009 (2017)
J. Murugan, D. Stanford, and E. Witten, JHEP 08, 146 (2017)
A. A. Patel and S. Sachdev, PRB **98**, 125134 (2018)
E. Marcus and S. Vandoren, JHEP 01, 166 (2018)
Yuxuan Wang, PRL **124**, 017002 (2020)
I. Esterlis and J. Schmalian, PRB **100**, 115132 (2019)
Yuxuan Wang and A. V. Chubukov, PRR **2**, 033084 (2020)
E. E. Aldape, T. Cookmeyer, A. A. Patel, and E. Altman, arXiv:2012.00763
Jaewon Kim, E. Altman, and Xiangyu Cao, PRB **103**, 081113 (2021)
W. Wang, A. Davis, G. Pan, Yuxuan Wang, and Zi Yang Meng, PRB **103**, 195108 (2021)
I. Esterlis, H. Guo, A. A. Patel, and S. Sachdev, PRB **103**, 235129 (2021).

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$$\mathcal{H} = -\mu \sum_i \psi_i^\dagger \psi_+ \sum_\ell \frac{1}{2}\left(\pi_\ell^2 + \omega_0^2 \phi_\ell^2\right) + \frac{1}{N}\sum_{ij\ell} g_{ij\ell} \psi_i^\dagger \psi_j \phi_\ell$$

with $g_{ij\ell}$ independent random numbers with zero mean.

The disorder-averaged partition function is a $G$-$\Sigma$-$D$-$\Pi$ theory:

$$\mathcal{Z} = \int \mathcal{D}G\, \mathcal{D}\Sigma\, \mathcal{D}D\, \mathcal{D}\Pi \exp(-N S_{\rm all})$$

$$\begin{aligned} S_{\rm all} = &-\ln\det(\partial_\tau - \mu + \Sigma) + \frac{1}{2}\ln\det(-\partial_\tau^2 + \omega_0^2 - \Pi) \\ &+ \int d\tau \int d\tau' \left[ -\Sigma(\tau';\tau) G(\tau,\tau') + \frac{1}{2}\Pi(\tau'\tau) D(\tau,\tau') \right. \\ &\qquad\qquad \left. + \frac{g^2}{2} G(\tau,\tau') G(\tau',\tau) D(\tau,\tau') \right] . \end{aligned}$$

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with $g_{ij\ell}$ independent random numbers with zero mean. The large $N$ saddle point equations are

$$\boxed{\begin{aligned} G(i\omega_n) &= \frac{1}{i\omega_n + \mu - \Sigma(i\omega_n)} \quad , \quad D(i\omega_n) = \frac{1}{\omega_n^2 + \omega_0^2 - \Pi(i\omega_n)} \\ \Sigma(\tau) &= g^2 G(\tau) D(\tau) \quad , \quad \Pi(\tau) = -g^2 G(\tau) G(-\tau) \end{aligned}}$$

Make the low frequency ansatz

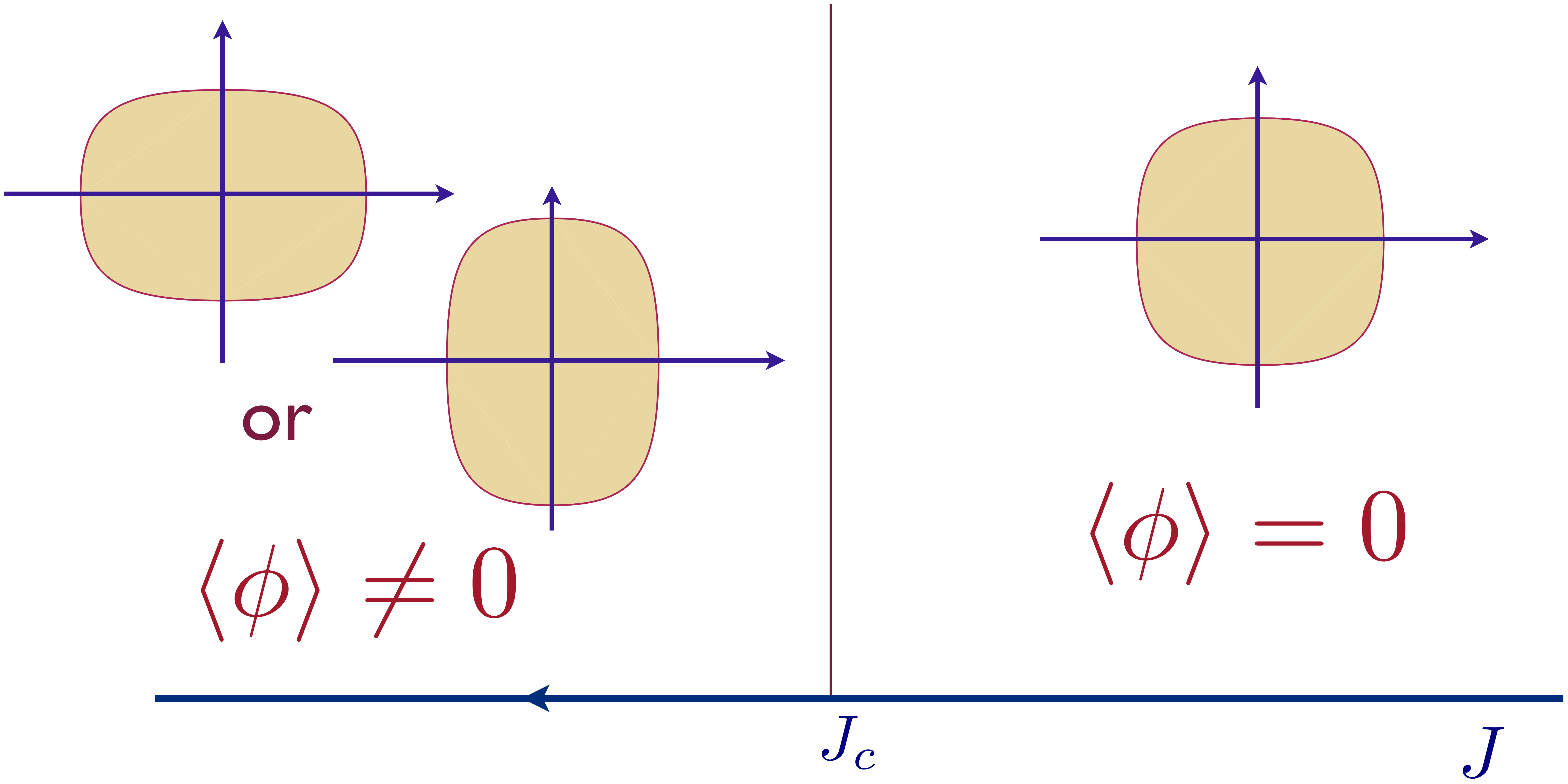
$$G(i\omega) \sim -i\,\mathrm{sgn}(\omega)|\omega|^{-(1-2\Delta)} \quad , \quad D(i\omega) \sim |\omega|^{1-4\Delta} \quad , \quad \frac{1}{4} < \Delta < \frac{1}{2}$$

A consistent solution exists for

$$\frac{4\Delta - 1}{2(2\Delta - 1)[\sec(2\pi\Delta) - 1]} = 1 \quad , \quad \Delta = 0.42037\ldots$$

I. Esterlis and J. Schmalian, PRB **100**, 115132 (2019)
See also Yuxuan Wang, PRL **124**, 017002 (2020)

# Quantum criticality of Ising-nematic ordering in a metal



or

$\langle \phi \rangle \neq 0$

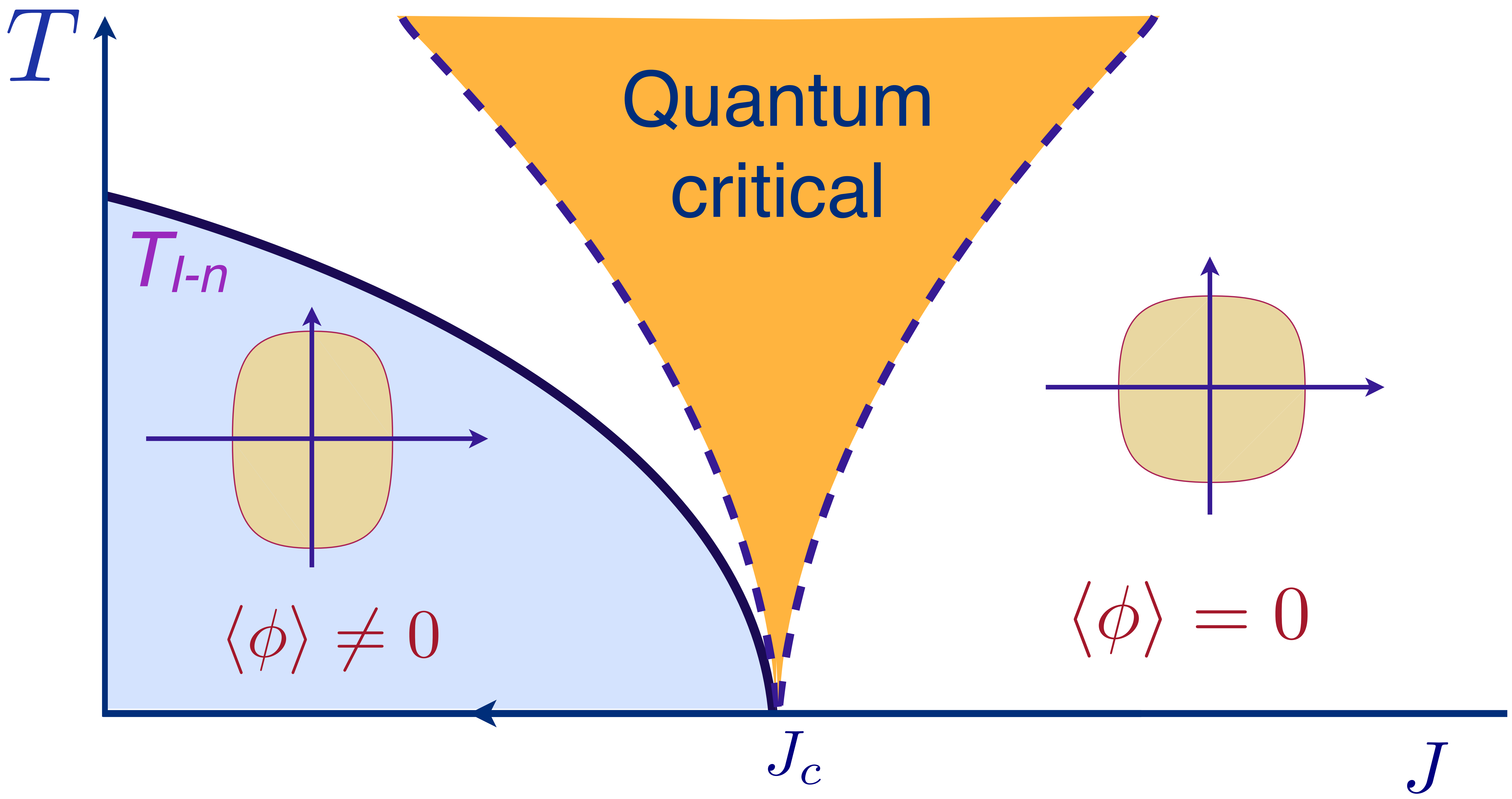
$\langle \phi \rangle = 0$

$J_c$

$J$

Pomeranchuk instability as a function of coupling $J$

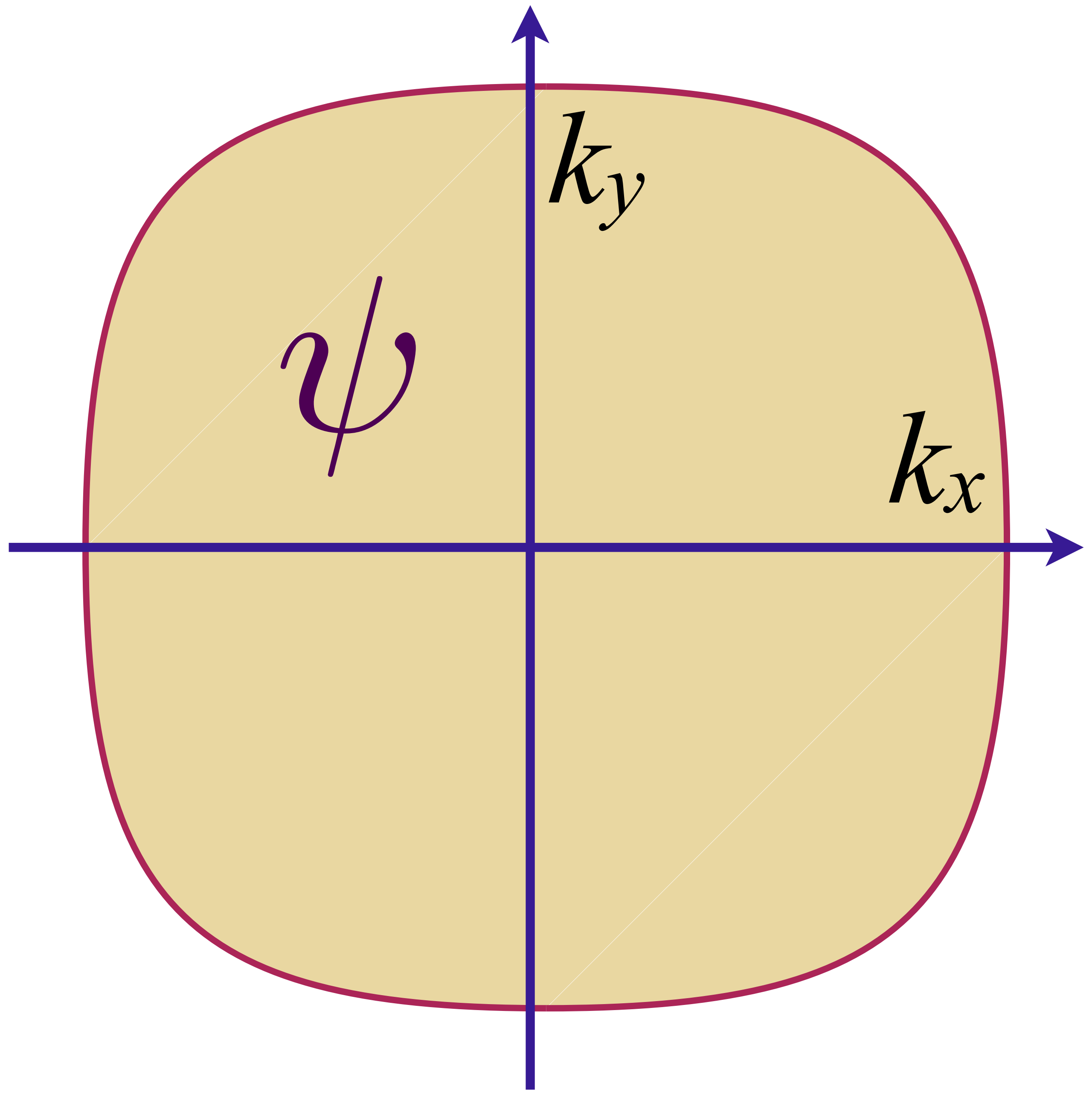
# Quantum criticality of Ising-nematic ordering in a metal



Phase diagram as a function of $T$ and $J$

# Fermi surface

$$\mathcal{L}_\psi = \psi_{\boldsymbol{k}}^\dagger \left( \frac{\partial}{\partial \tau} + \varepsilon(\boldsymbol{k}) \right) \psi_{\boldsymbol{k}}$$

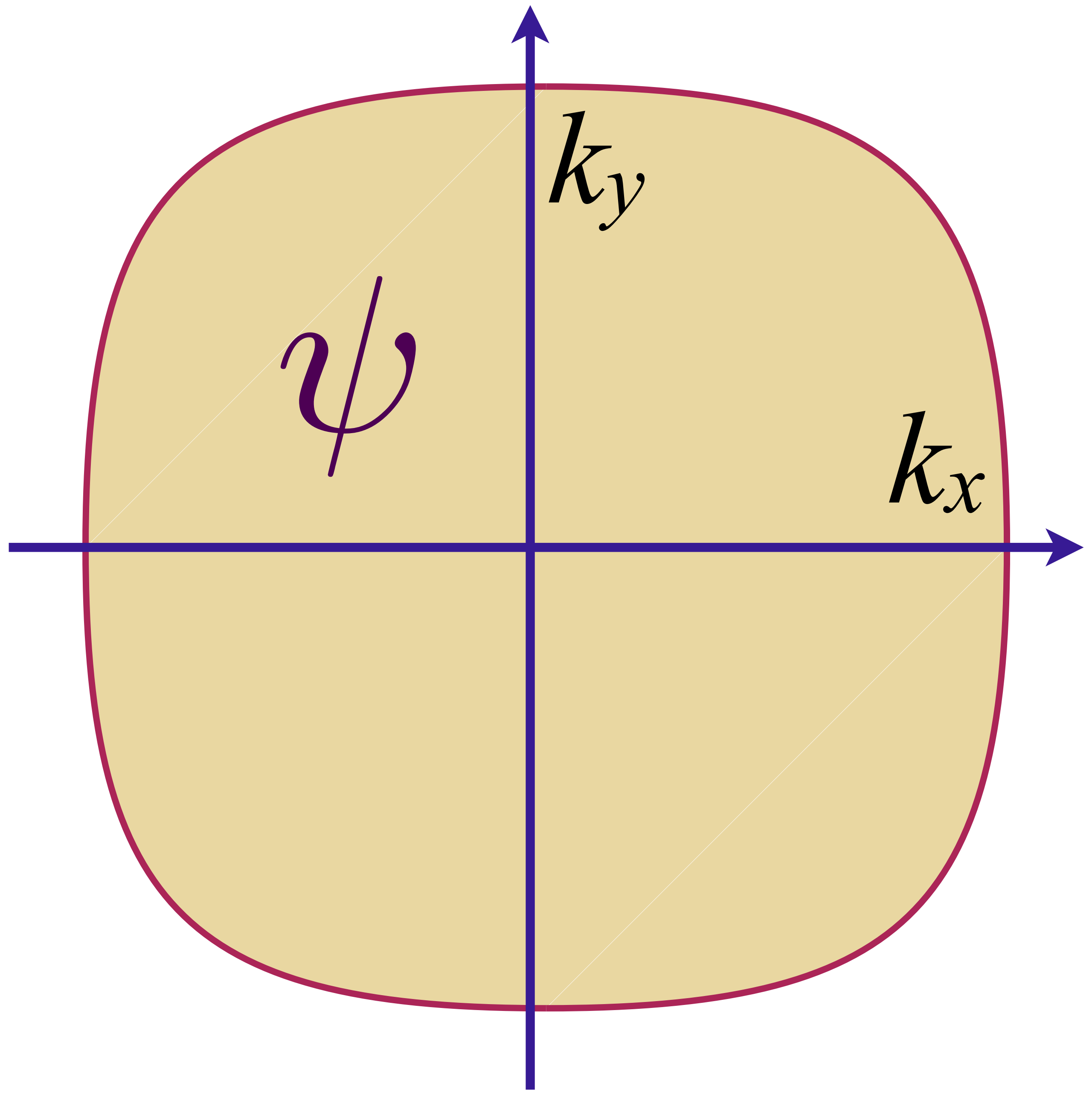


$$-J\, \psi^\dagger(\boldsymbol{r})\psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})\psi(\boldsymbol{r})$$

# Fermi surface coupled to a critical boson

$$\mathcal{L}_\psi = \psi^\dagger_{\boldsymbol{k}} \left( \frac{\partial}{\partial \tau} + \varepsilon(\boldsymbol{k}) \right) \psi_{\boldsymbol{k}}$$

a critical boson $\phi$
*e.g.* Ising-nematic order



$$\frac{[\phi(\boldsymbol{r})]^2}{J} + \psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})\,\phi(\boldsymbol{r})$$

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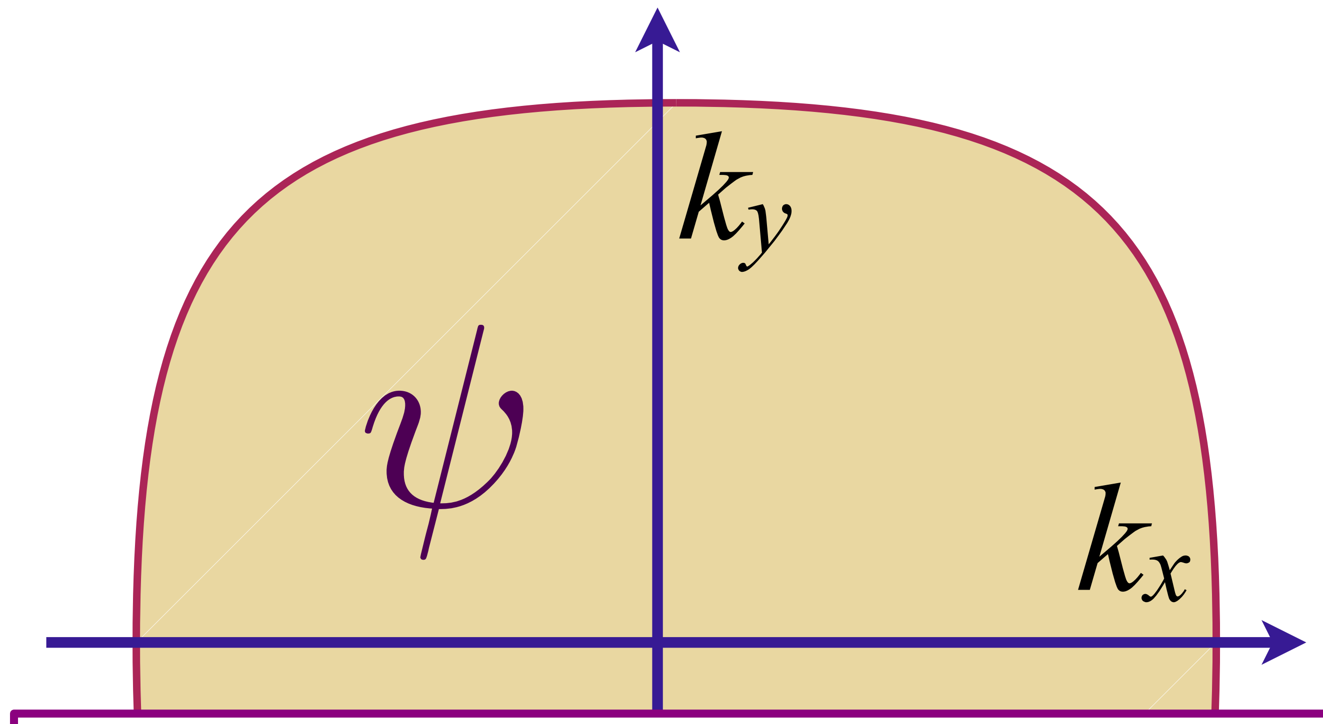

$$\frac{[\phi(\boldsymbol{r})]^2}{J} + \psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})\,\phi(\boldsymbol{r})$$

Solve in a large $N$ limit with Yukawa coupling

$$\frac{g_{ij\ell}}{N} \int d^2r d\tau\, \psi^\dagger_i(r,\tau)\psi_j(r,\tau)\phi_l(r,\tau) \quad , \quad \overline{g_{ijl}} = 0 \quad , \quad \overline{|g_{ijl}|^2} = g^2$$

to obtain Eliashberg solution for electron $(G)$ and boson $(D)$ Green's functions at small $\omega$:

$$\Sigma(\hat{\boldsymbol{k}}, i\omega) \sim -i\mathrm{sgn}(\omega)|\omega|^{2/3}, \quad G(\boldsymbol{k}, i\omega) = \frac{1}{i\omega - \varepsilon(\boldsymbol{k}) - \Sigma(\hat{\boldsymbol{k}}, i\omega)}, \quad D(\boldsymbol{q}, i\Omega) = \frac{1}{\Omega^2 + q^2 + \gamma|\Omega|/q}$$

P.A. Lee (1989)

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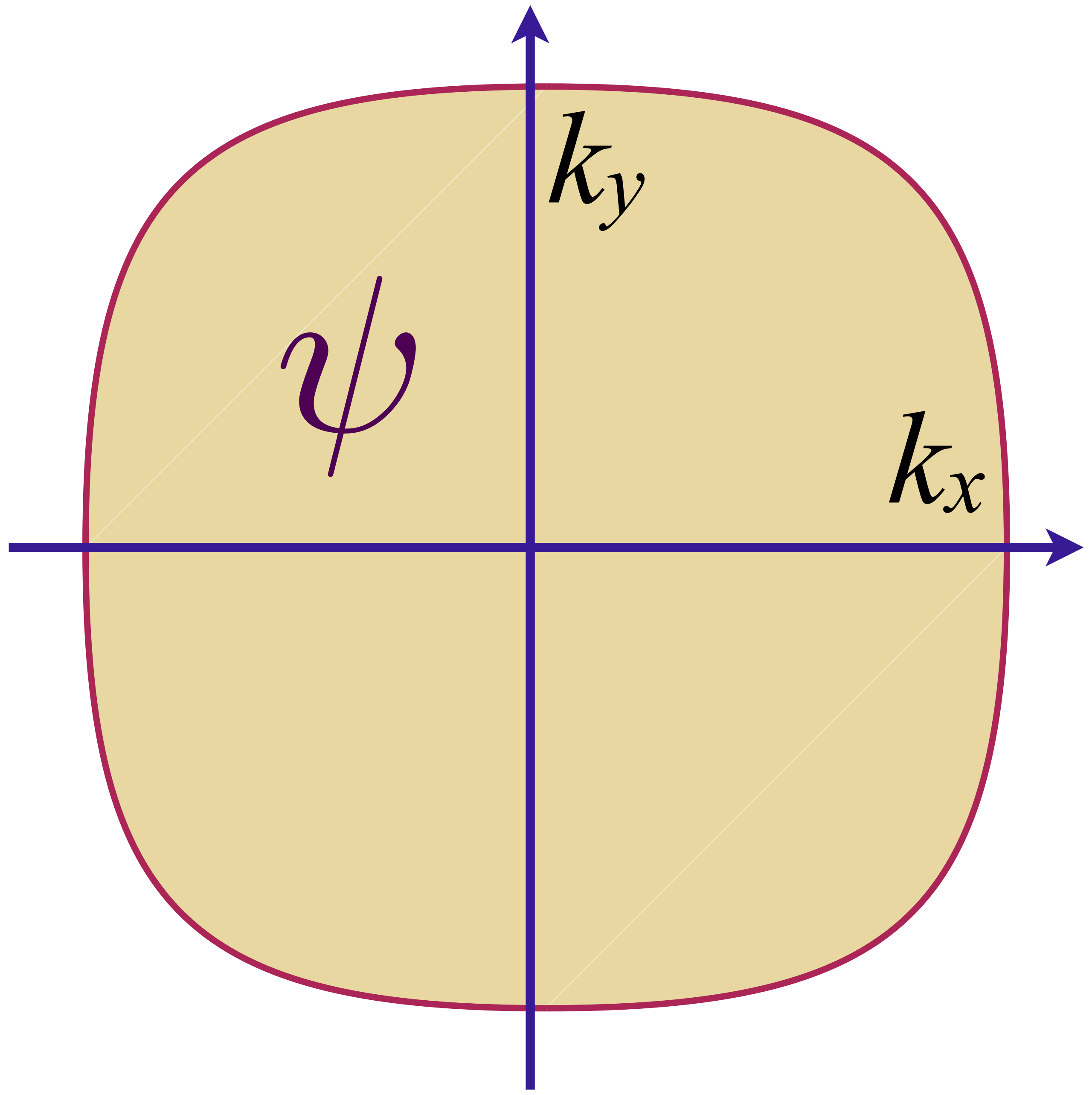

a critical boson $\phi$
*e.g.* Ising-nematic order

$$\frac{[\phi(\boldsymbol{r})]^2}{J} + \psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})\,\phi(\boldsymbol{r})$$

Transport—a perfect metal!

Conservation of momentum and fermion-boson drag imply:

$$\mathrm{Re}\,[\sigma(\omega)] = D\delta(\omega) + \ldots$$

S. A. Hartnoll, P. K. Kovtun, M. Muller, and S.S. PRB **76**, 144502 (2007)
D. L. Maslov, V. I. Yudson, and A. V. Chubukov PRL **106**, 106403 (2011)
S. A. Hartnoll, R. Mahajan, M. Punk, and S.S. PRB **89**, 155130 (2014)
A. Eberlein, I. Mandal, and S.S. PRB **94**, 045133 (2016)

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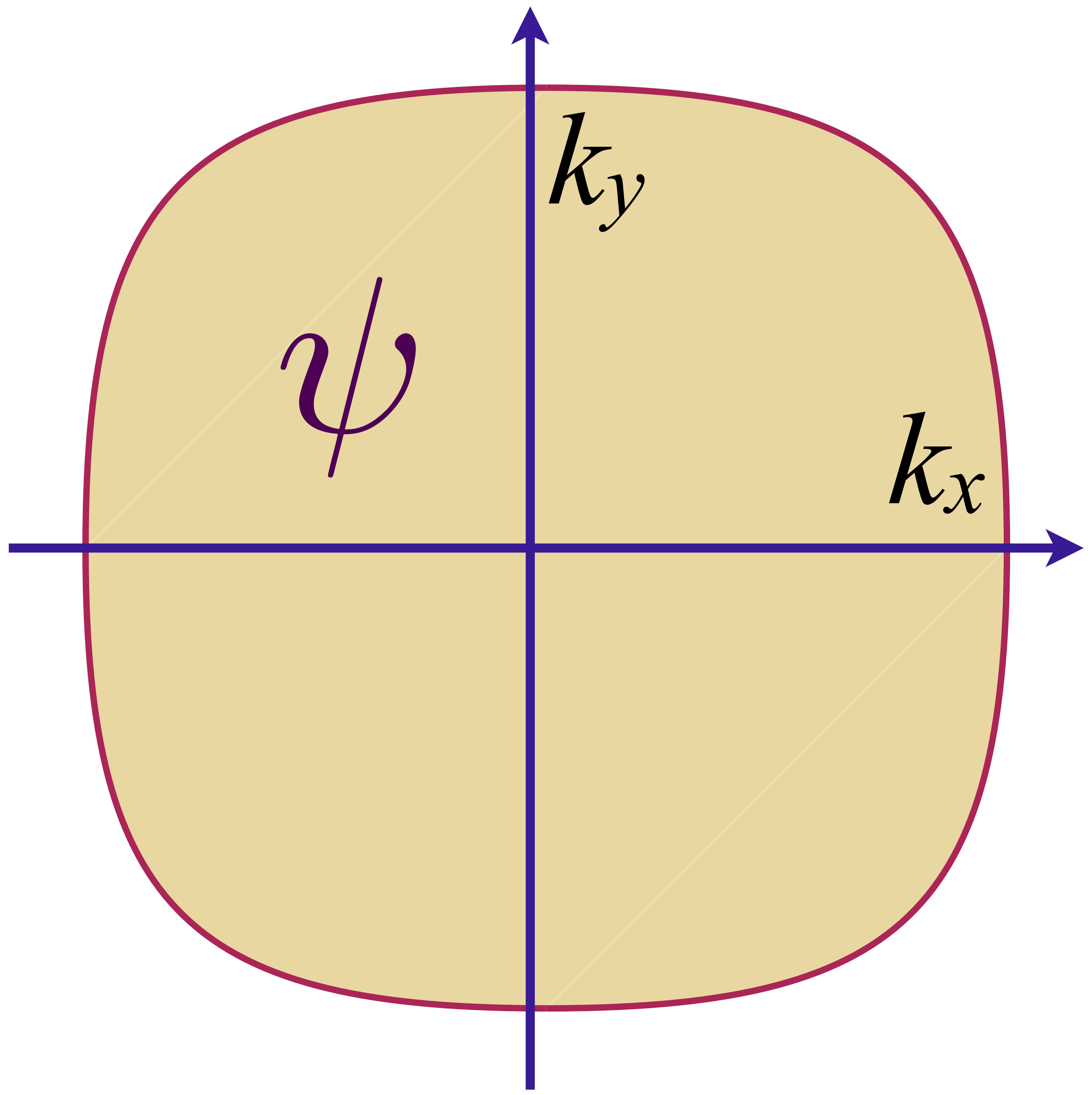


Transport—a perfect metal!

Conservation of momentum and fermion-boson drag imply:

$$\sigma(\omega) \sim \frac{1}{-i\omega} + |\omega|^0 + \cdots \quad (\omega^{-2/3} \text{ term has vanishing co-efficient})$$

**Haoyu Guo**, Aavishkar Patel, Ilya Esterlis, S.S. PRB **106**, 115151 (2022)

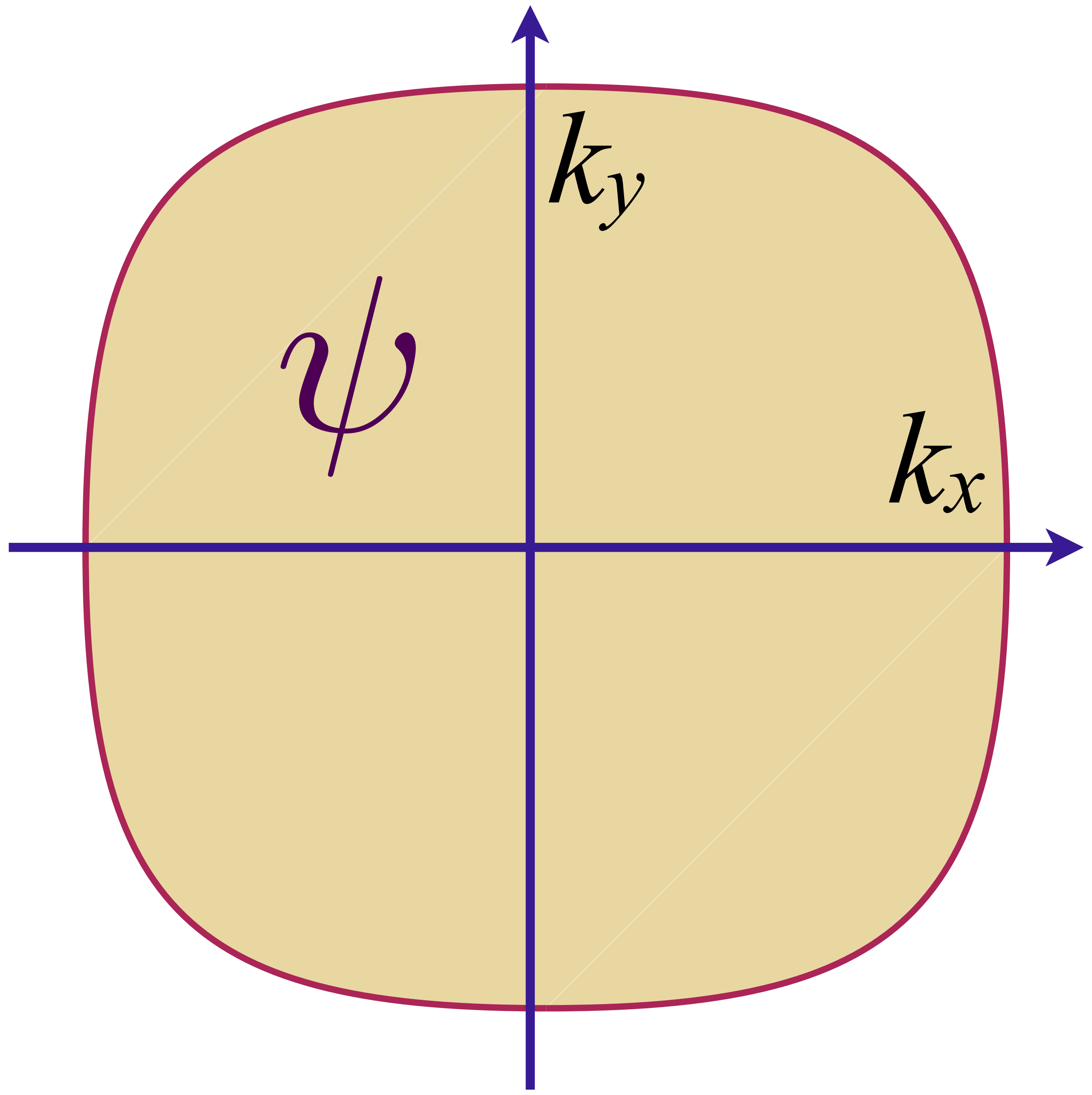
# Fermi surface coupled to a critical boson:

## No spatial disorder

*A non-Fermi liquid but NOT a strange metal*

# Fermi surface coupled to a critical boson with disorder

$$\mathcal{L}_\psi = \psi_{\boldsymbol{k}}^\dagger \left( \frac{\partial}{\partial \tau} + \varepsilon(\boldsymbol{k}) \right) \psi_{\boldsymbol{k}}$$



a critical boson $\phi$
*e.g.* Ising-nematic order

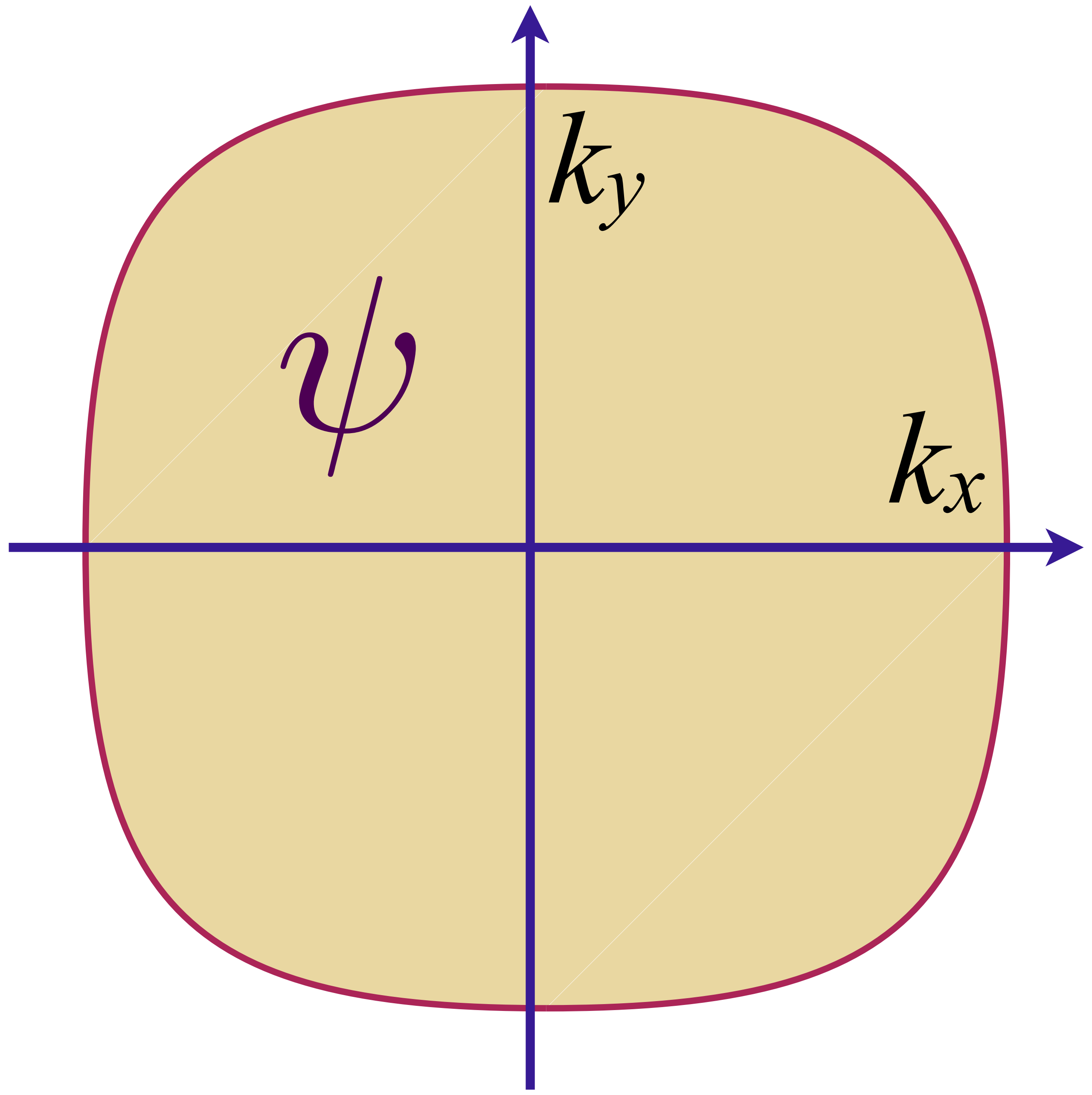
$$\frac{[\phi(\boldsymbol{r})]^2}{J} + \psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})\,\phi(\boldsymbol{r})$$

$$+v(\boldsymbol{r})\psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})$$

Spatially random potential $v(\boldsymbol{r})$ with $\overline{v(\boldsymbol{r})} = 0$, $\overline{v(\boldsymbol{r})v(\boldsymbol{r}')} = v^2\delta(\boldsymbol{r} - \boldsymbol{r}')$

# Fermi surface coupled to a critical boson with disorder

$$\mathcal{L}_\psi = \psi^\dagger_{\boldsymbol{k}} \left( \frac{\partial}{\partial \tau} + \varepsilon(\boldsymbol{k}) \right) \psi_{\boldsymbol{k}}$$



a critical boson $\phi$

*e.g.* Ising-nematic order

$$\frac{[\phi(\boldsymbol{r})]^2}{J + J'(\boldsymbol{r})} + \psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})\,\phi(\boldsymbol{r})$$

$$+ v(\boldsymbol{r})\psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})$$

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$$\mathcal{L}_\psi = \psi_{\boldsymbol{k}}^\dagger \left( \frac{\partial}{\partial \tau} + \varepsilon(\boldsymbol{k}) \right) \psi_{\boldsymbol{k}}$$

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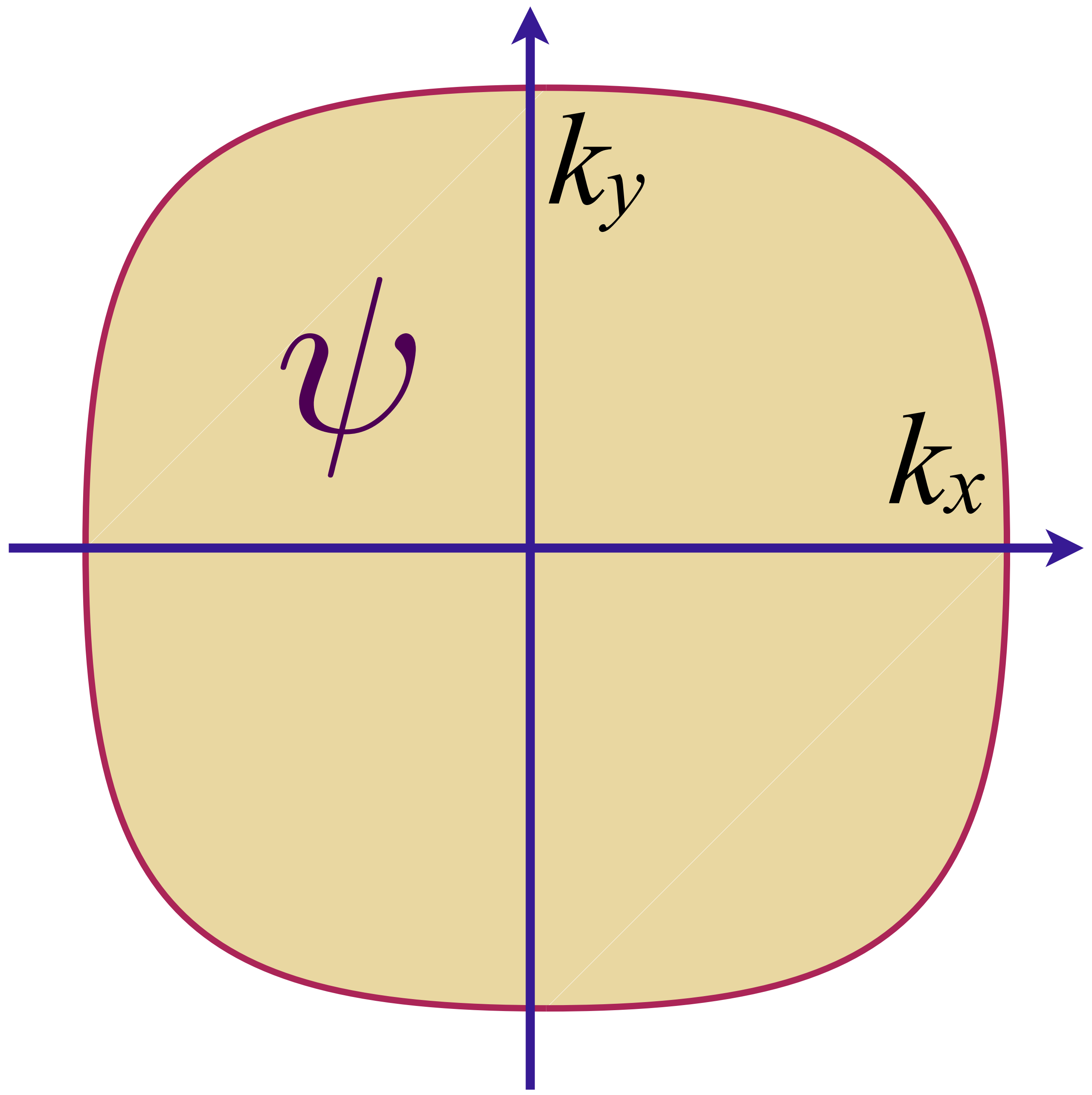


$$[\phi(\boldsymbol{r})]^2 + [g + g'(\boldsymbol{r})]\, \psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})\, \phi(\boldsymbol{r})$$

$$+ v(\boldsymbol{r})\psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})$$

$\phi^2$ "mass" disorder $J'(\boldsymbol{r})$ is strongly relevant;
rescale $\phi$ to move disorder to the Yukawa coupling;

Spatially random Yukawa coupling $g'(\boldsymbol{r})$ with $\overline{g'(\boldsymbol{r})} = 0$, $\overline{g'(\boldsymbol{r})g'(\boldsymbol{r}')} = g'^2 \delta(\boldsymbol{r} - \boldsymbol{r}')$

Spatially random potential $v(\boldsymbol{r})$ with $\overline{v(\boldsymbol{r})} = 0$, $\overline{v(\boldsymbol{r})v(\boldsymbol{r}')} = v^2 \delta(\boldsymbol{r} - \boldsymbol{r}')$

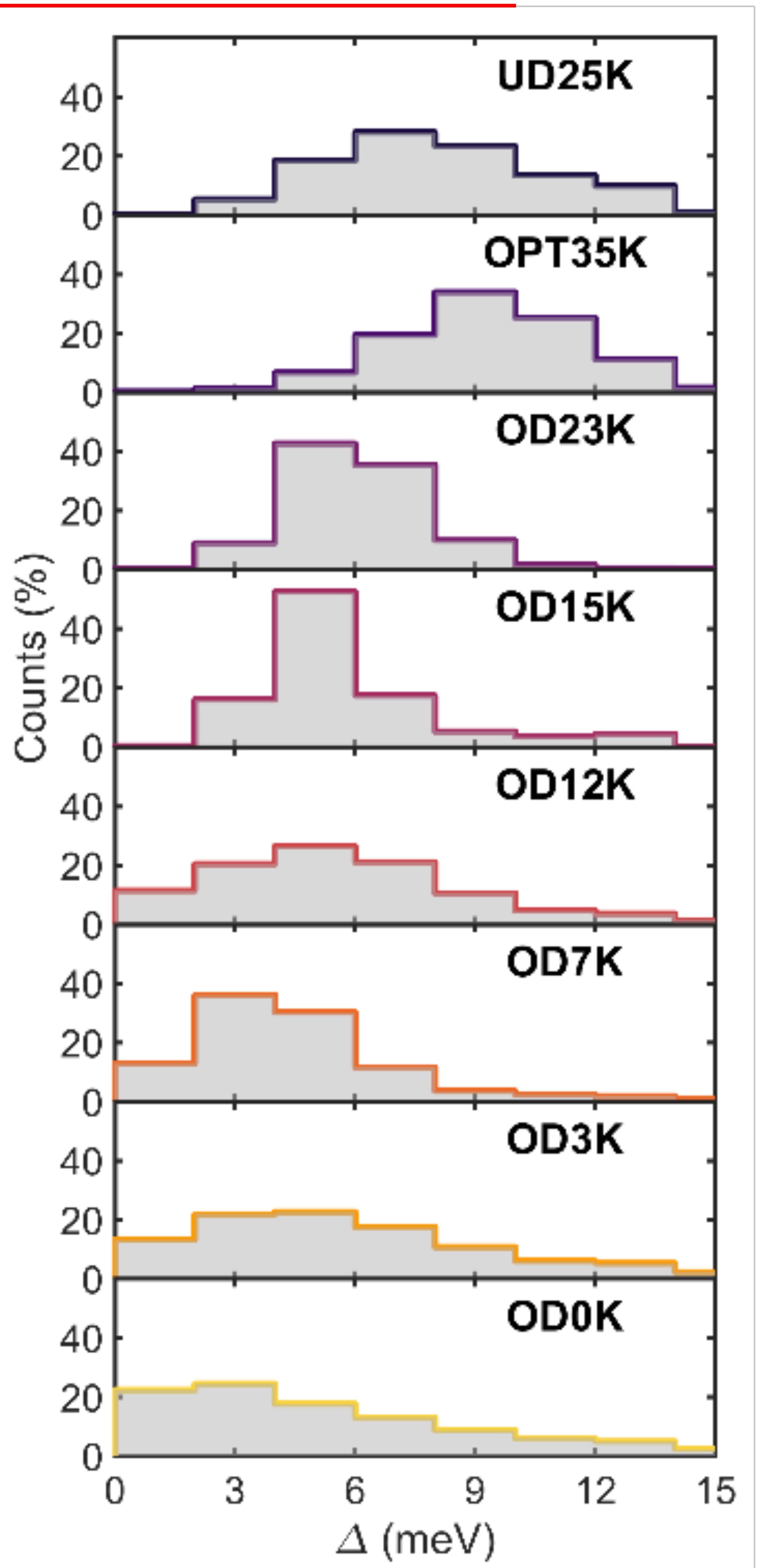
# Spatially random interactions!

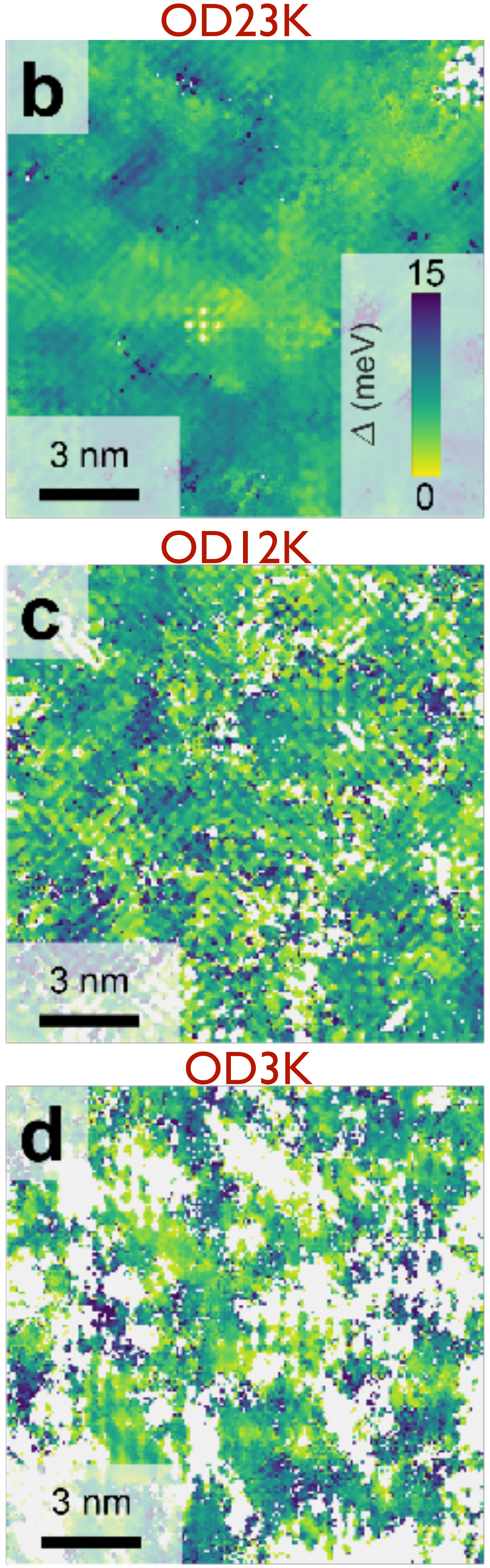
## Puddle formation, persistent gaps, and non-mean-field breakdown of superconductivity in overdoped $(Pb,Bi)_2Sr_2CuO_{6+\delta}$

Willem O. Tromp, Tjerk Benschop, Jian-Feng Ge,
Irene Battisti, Koen M. Bastiaans, Damianos Chatzopoulos,
Amber Vervloet, Steef Smit, Erik van Heumen,
Mark S. Golden, Yinkai Huang, Takeshi Kondo, Yi Yin,
Jennifer E. Hoffman, Miguel Antonio Sulangi, Jan Zaanen,
Milan P. Allan

Our scanning tunneling spectroscopy measurements in the overdoped regime of the $(Pb,Bi)_2Sr_2CuO_{6+\delta}$ high-temperature superconductor show the emergence of puddled superconductivity, featuring nanoscale superconducting islands in a metallic matrix

arXiv:2205.09740





# Fermi surface coupled to a critical boson with disorder

All results are obtained from the large $N$ saddle-point and response functions of this $G$-$\Sigma$-$D$-$\Pi$ theory:

$$\mathcal{Z} = \int \mathcal{D}G\, \mathcal{D}\Sigma\, \mathcal{D}D\, \mathcal{D}\Pi \exp(-N S_{\rm all})$$

$$\begin{aligned} S_{\rm all} = & -\ln\det(\partial_\tau + \varepsilon(\mathbf{k}) - \mu + \Sigma) + \frac{1}{2}\ln\det(-\partial_\tau^2 + \mathbf{q}^2 + m_b^2 - \Pi) \\ & + \int d\tau d^2r \int d\tau' d^2r' \Big[ -\Sigma(\tau',\mathbf{r}';\tau,\mathbf{r}) G(\tau,\mathbf{r};\tau',\mathbf{r}') + \frac{1}{2}\Pi(\tau',\mathbf{r}';\tau,\mathbf{r}) D(\tau,\mathbf{r};\tau',\mathbf{r}') \\ & + \frac{g^2}{2} G(\tau,\mathbf{r};\tau',\mathbf{r}') G(\tau',\mathbf{r}';\tau,\mathbf{r}) D(\tau,\mathbf{r};\tau',\mathbf{r}') + \frac{v^2}{2} G(\tau,\mathbf{r};\tau',\mathbf{r}') G(\tau',\mathbf{r}';\tau,\mathbf{r}) \delta(\mathbf{r}-\mathbf{r}') \\ & + \frac{g'^2}{2} G(\tau,\mathbf{r};\tau',\mathbf{r}') G(\tau',\mathbf{r}';\tau,\mathbf{r}) D(\tau,\mathbf{r};\tau',\mathbf{r}') \delta(\mathbf{r}-\mathbf{r}') \Big] \,. \end{aligned}$$

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All results are obtained from the large $N$ saddle-point and response functions of this $G$-$\Sigma$-$D$-$\Pi$ theory:

$$\mathcal{Z} = \int \mathcal{D}G\,\mathcal{D}\Sigma\,\mathcal{D}D\,\mathcal{D}\Pi\,\exp(-NS_{\rm all})$$

Saddle-point equations

$$\begin{aligned}
\Sigma(\tau,\mathbf{r}) &= g^2 D(\tau,\mathbf{r})G(\tau,\mathbf{r}) + v^2 G(\tau,\mathbf{r})\delta^2(\mathbf{r}) + {g'}^2 G(\tau,\mathbf{r})D(\tau,\mathbf{r})\delta^2(\mathbf{r}),\\
\Pi(\tau,\mathbf{r}) &= -g^2 G(-\tau,-\mathbf{r})G(\tau,\mathbf{r}) - {g'}^2 G(-\tau,\mathbf{r})G(\tau,\mathbf{r})\delta^2(\mathbf{r}),\\
G(i\omega,\mathbf{k}) &= \frac{1}{i\omega - \varepsilon(\mathbf{k}) + \mu - \Sigma(i\omega,\mathbf{k})},\\
D(i\Omega,\mathbf{q}) &= \frac{1}{\Omega^2 + \mathbf{q}^2 + m_b^2 - \Pi(i\Omega,\mathbf{q})}.
\end{aligned}$$

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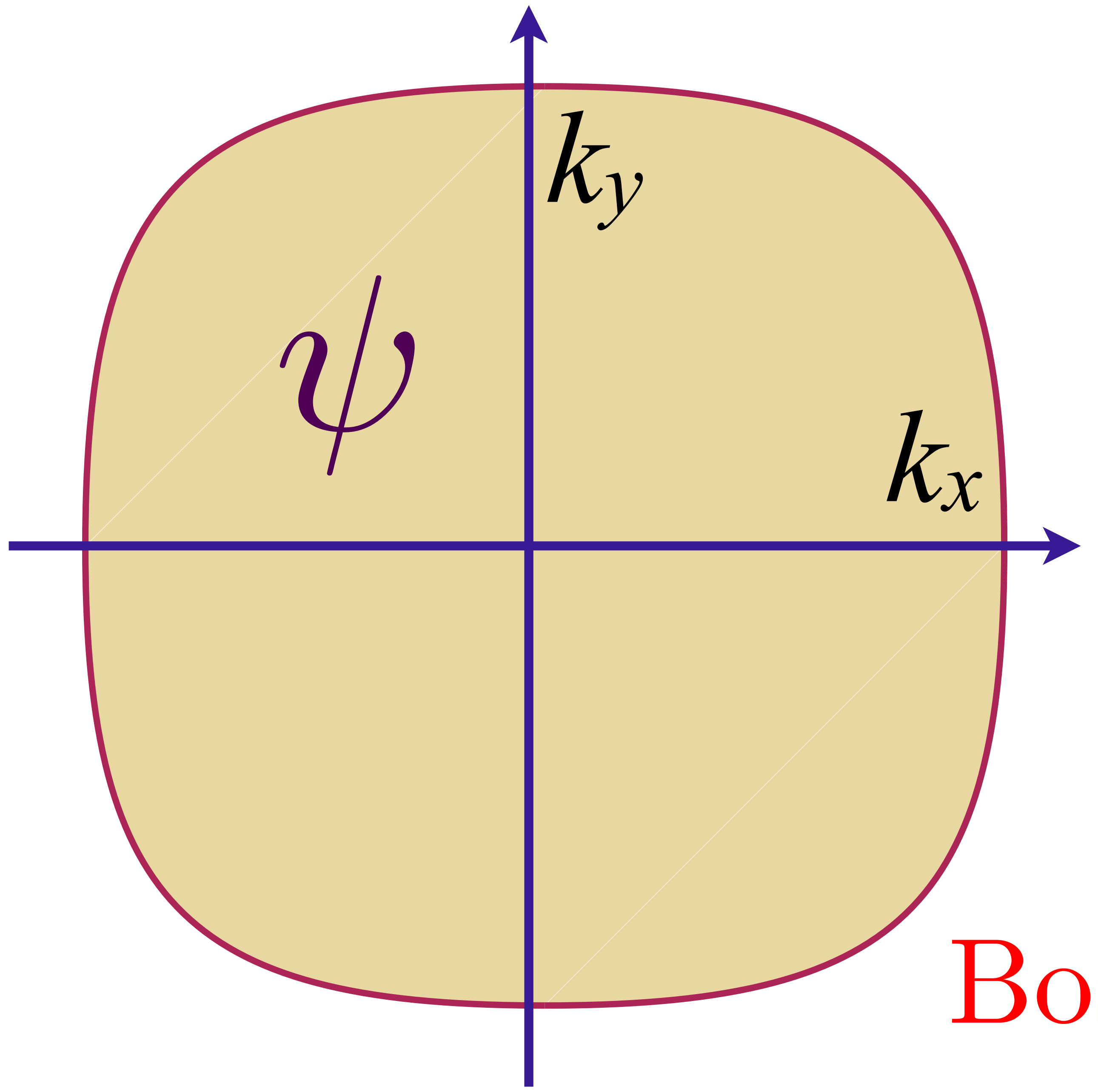
a critical boson $\phi$
*e.g.* Ising-nematic order



$$[\phi(\boldsymbol{r})]^2 + [g + g'(\boldsymbol{r})]\, \psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})\, \phi(\boldsymbol{r})$$

$$+ v(\boldsymbol{r})\psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})$$

Boson Green's function: $D(q, i\Omega) \sim 1/(q^2 + \gamma|\Omega|)$

Fermion self energy:

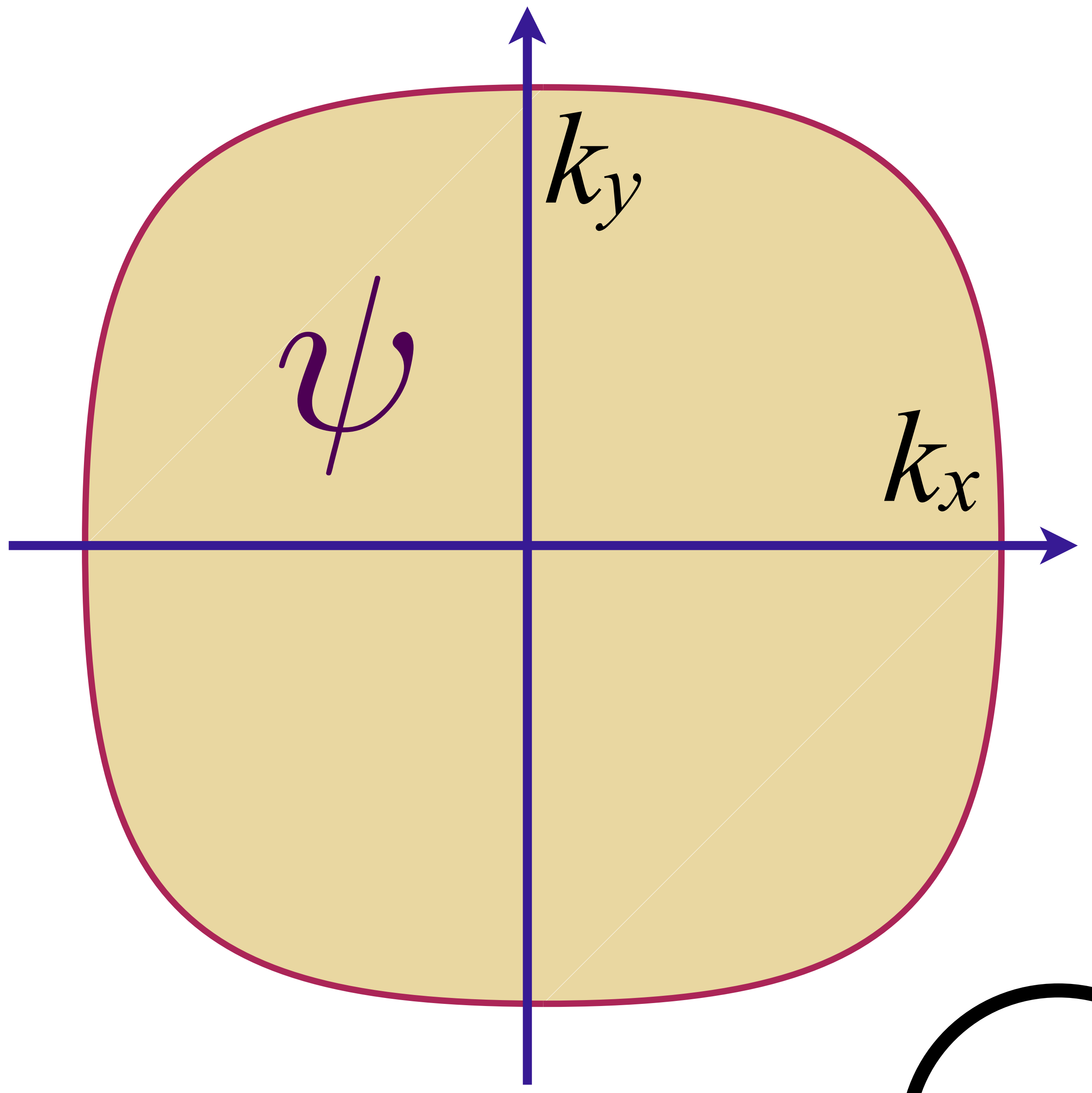
$$\Sigma(i\omega) \sim -iv^2 \mathrm{sgn}(\omega) - i \left( \frac{g^2}{v^2} + g'^2 \right) \omega \ln(1/|\omega|); \quad \frac{1}{\tau(\omega)} \sim \left( \frac{g^2}{v^2} + g'^2 \right) |\omega|$$

Marginal Fermi liquid self energy and $T\ln(1/T)$ specific heat

# Fermi surface coupled to a critical boson with disorder

$$\mathcal{L}_\psi = \psi^\dagger_{\boldsymbol{k}}\left(\frac{\partial}{\partial\tau}+\varepsilon(\boldsymbol{k})\right)\psi_{\boldsymbol{k}}$$

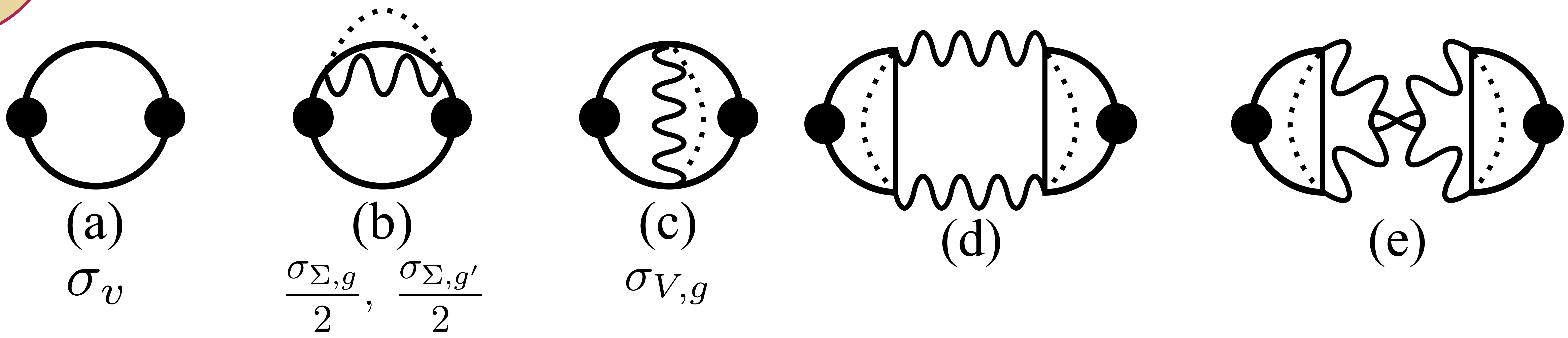
a critical boson $\phi$
*e.g.* Ising-nematic order



$$[\phi(\boldsymbol{r})]^2 + [g+g'(\boldsymbol{r})]\,\psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})\,\phi(\boldsymbol{r})$$

$$+v(\boldsymbol{r})\psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})$$

**Conductivity:**



(a) $\sigma_v$

(b) $\frac{\sigma_{\Sigma,g}}{2}, \frac{\sigma_{\Sigma,g'}}{2}$

(c) $\sigma_{V,g}$

(d)

(e)

+ all ladders and bubbles…..

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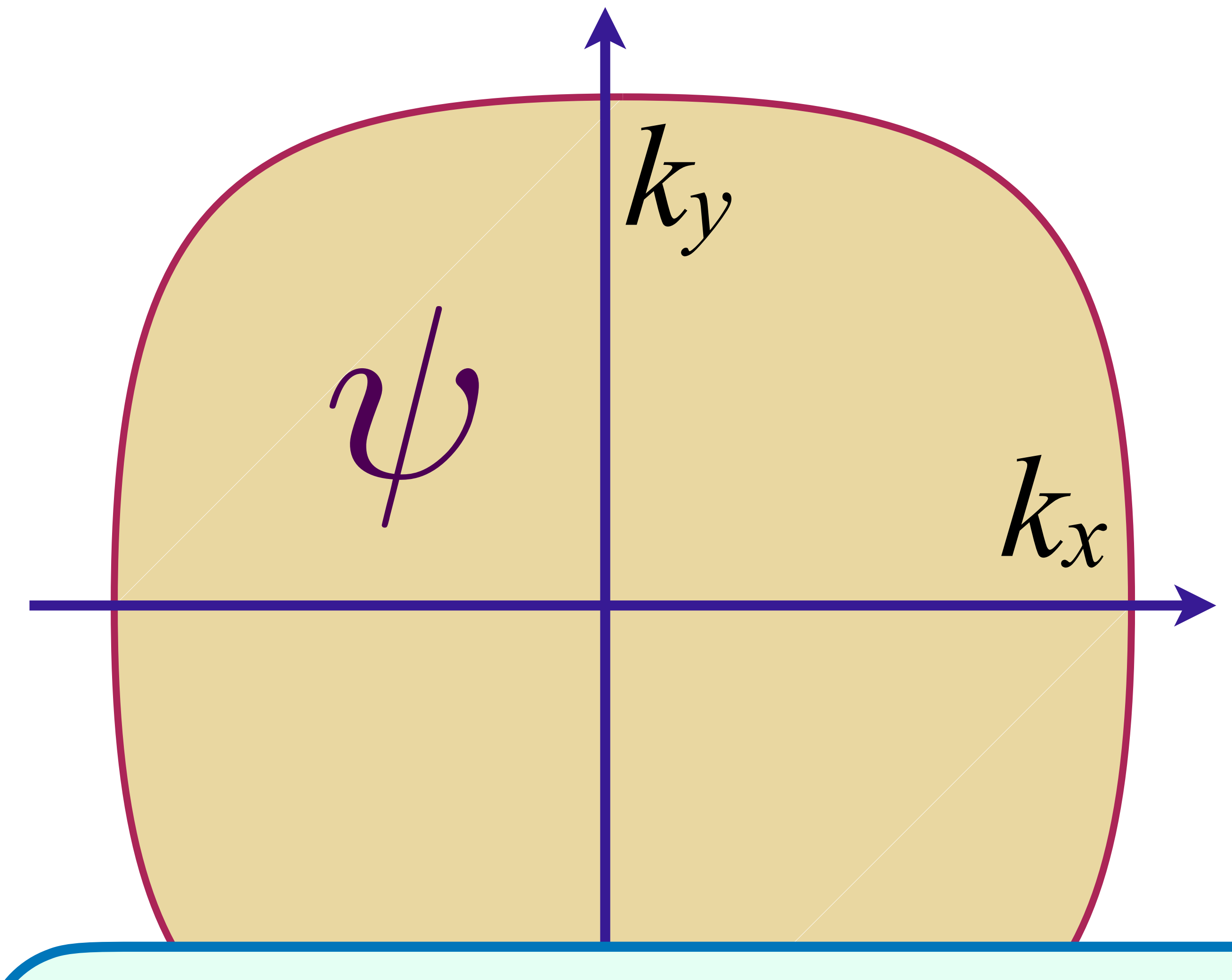

$$[\phi(\boldsymbol{r})]^2 + [g + g'(\boldsymbol{r})]\, \psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})\, \phi(\boldsymbol{r})$$

$$+ v(\boldsymbol{r})\psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})$$

Conductivity: $\sigma(\omega) \sim [1/\tau_{\rm trans}(\omega) - i\omega\, m^*_{\rm trans}(\omega)/m]^{-1}$

$$\frac{1}{\tau_{\rm trans}(\omega)} \sim v^2 + g'^2|\omega| \quad ; \quad \frac{m^*_{\rm trans}(\omega)}{m} \sim \frac{2g'^2}{\pi} \ln(\Lambda/\omega)$$

Residual resistivity is determined by $v^2$; Linear-in-$T$ resistivity determined by $g'^2$; Transport insensitive to $g$; Marginal Fermi liquid self energy and $T\ln(1/T)$ specific heat.

Fermi surface coupled to a critical boson:

No spatial disorder

*A non-Fermi liquid but NOT a strange metal*

---

Fermi surface coupled to a critical boson:

Potential disorder $v$

*A marginal Fermi liquid but NOT a strange metal*

---

Fermi surface coupled to a critical boson:

Interaction disorder $g'$

*A marginal Fermi liquid AND a strange metal*

# Summary

- SYK: a solvable toy model without particle-like excitations, exhibiting thermalization and many-body chaos in a time of order $\hbar/(k_B T)$, independent of microscopic energy scales.

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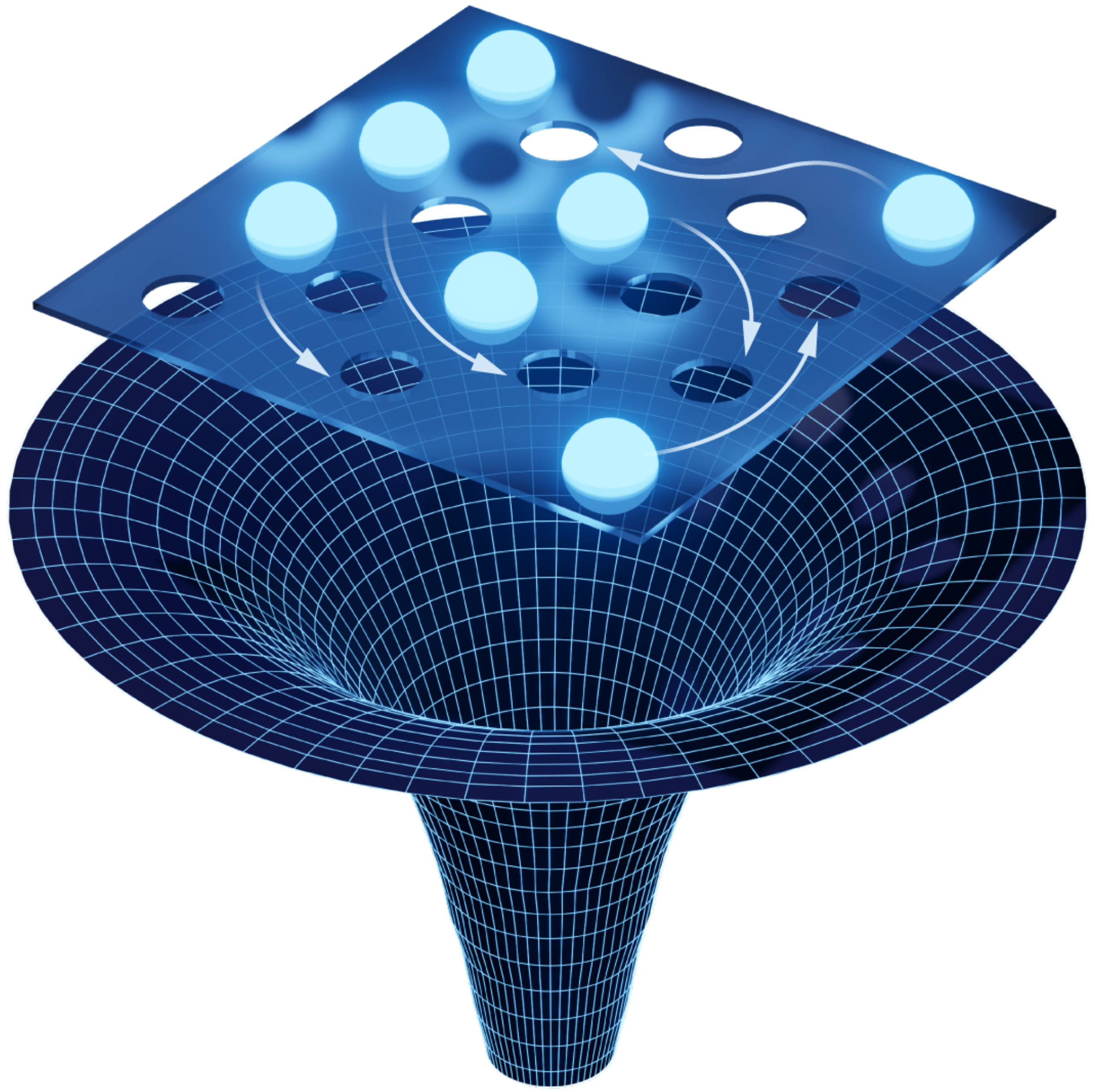
- Toy SYK model captures the correct universal low energy quantum theory of charged black holes, and provides a Hamiltonian realization of black hole microstates.



# Summary

- SYK: a solvable toy model without particle-like excitations, exhibiting thermalization and many-body chaos in a time of order $\hbar/(k_B T)$, independent of microscopic energy scales.

- Toy SYK model captures the correct universal low energy quantum theory of charged black holes, and provides a Hamiltonian realization of black hole microstates.

- Linear-$T$ resistivity, $T\ln(1/T)$ specific heat, $\sim 1/\omega$ optical conductivity, and marginal Fermi liquid electron spectrum *all* arise from a SYK-like model with spatially random interactions in a two-dimensional quantum-critical metal.

Aavishkar Patel, Haoyu Guo, Ilya Esterlis, S.S. arXiv: 2203.04990