

# Compressible quantum liquids: Field theory vs. holography

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[sachdev.physics.harvard.edu](http://sachdev.physics.harvard.edu)





**Liza Huijse**



**Max Metlitski**



**Brian Swingle**

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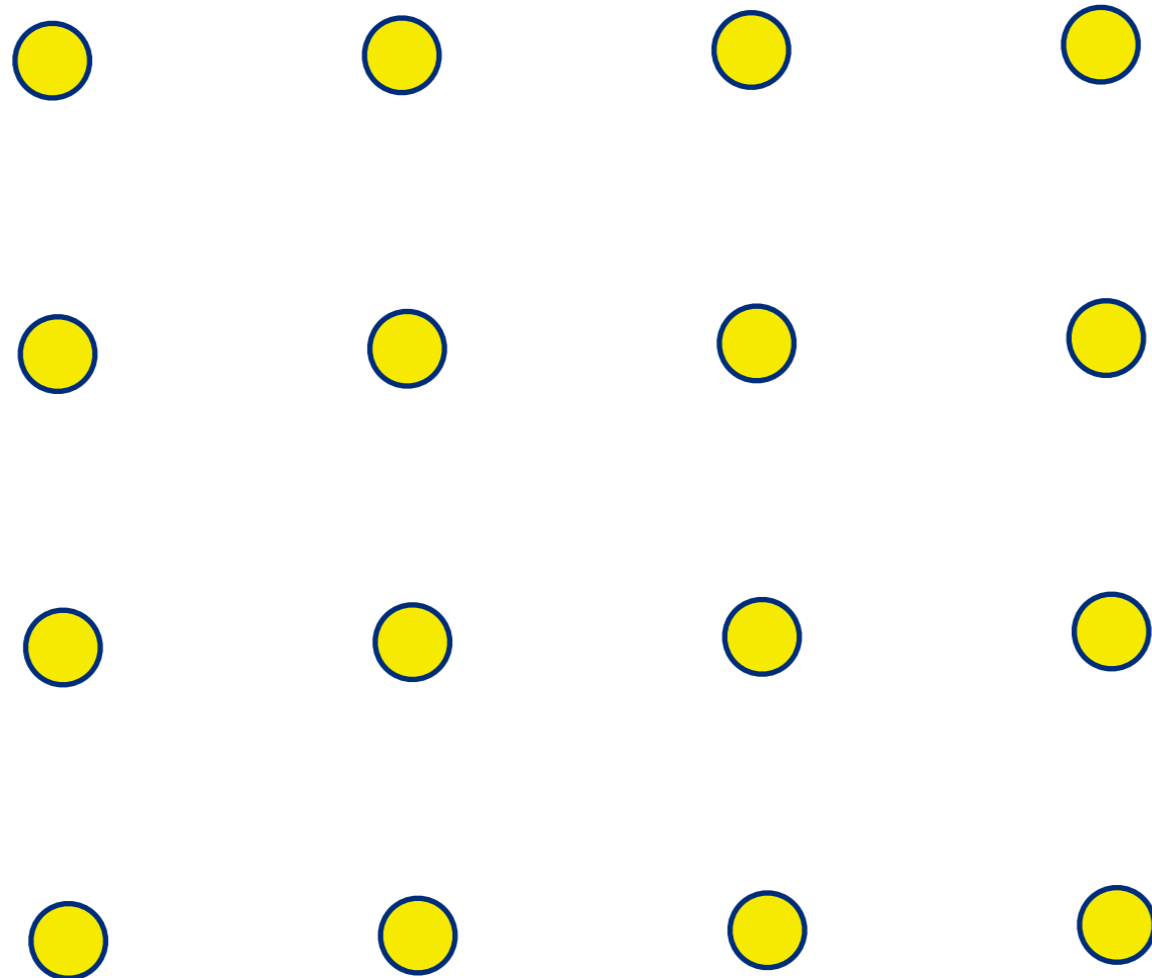
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- Compressible systems must be gapless.
- “Relativistic” quantum critical systems are compressible in  $d = 1$ , but not for  $d > 1$ .

# Compressible quantum matter

One compressible state is the **solid** (or “Wigner crystal” or “stripe”).

This state breaks translational symmetry.



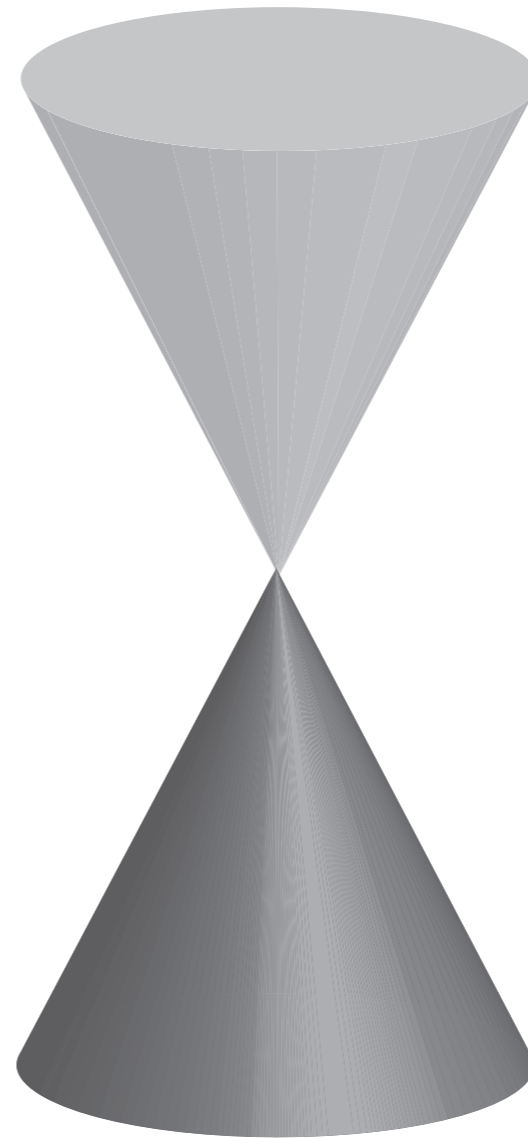
# Compressible quantum matter

Another familiar compressible state is  
the **superfluid**.

This state breaks the global  $U(1)$   
symmetry associated with  $Q$

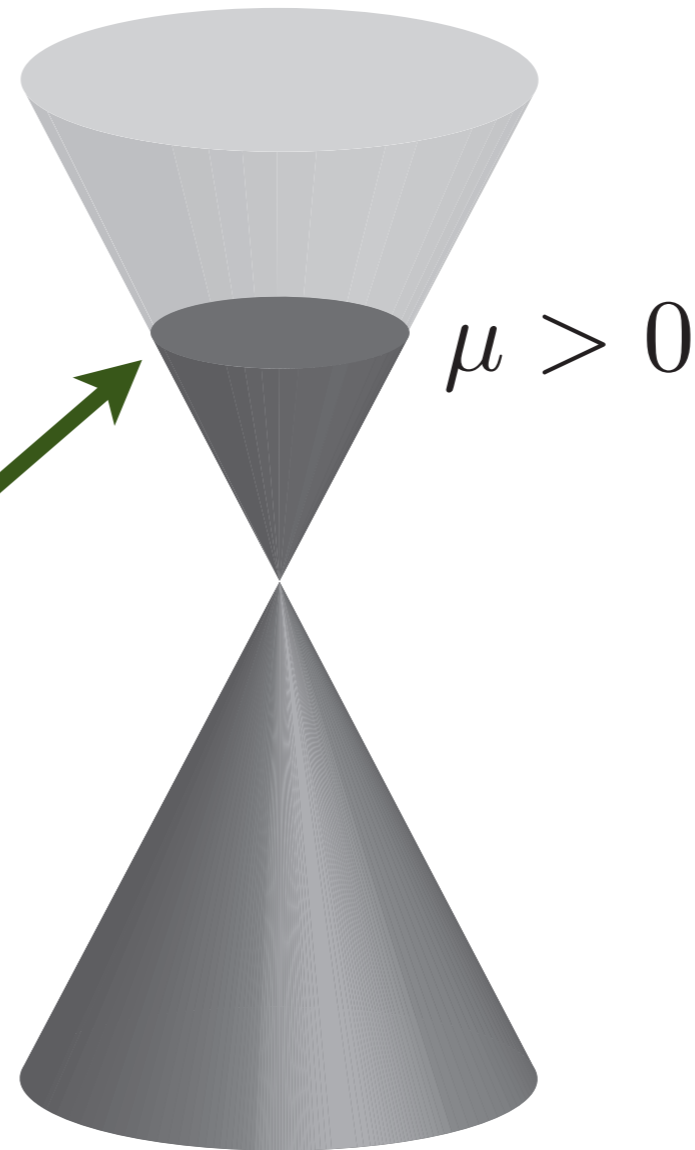


Condensate of  
fermion pairs

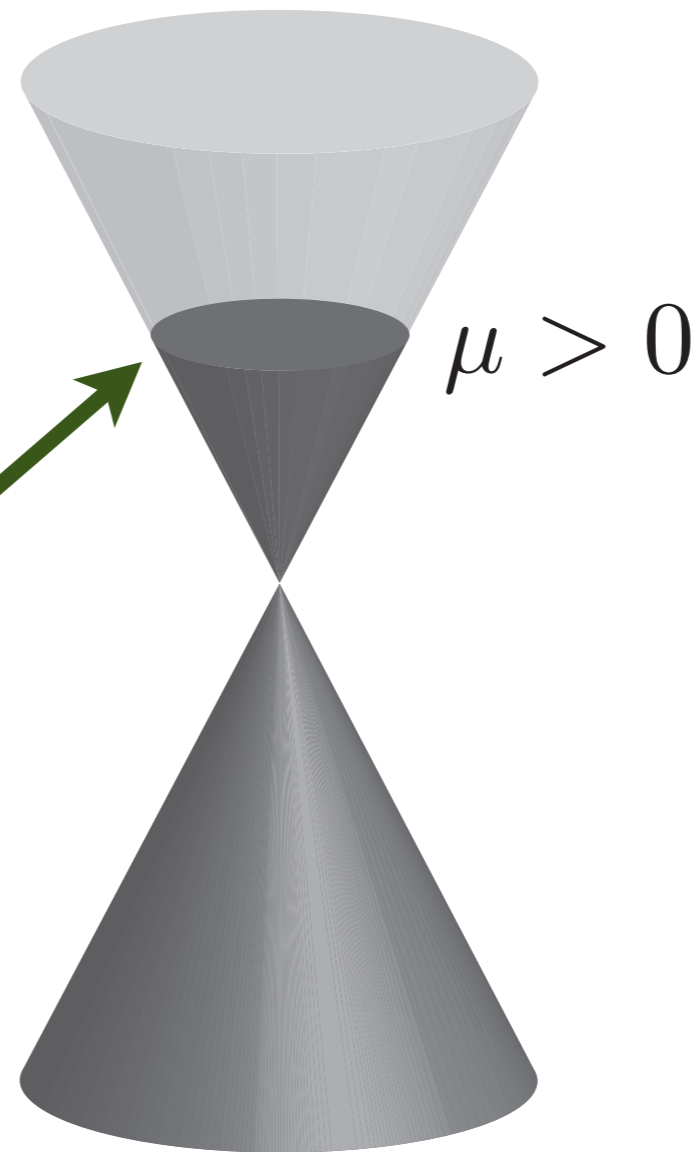


**Graphene**

The only other familiar compressible phase is a **Fermi Liquid** with a **Fermi surface**

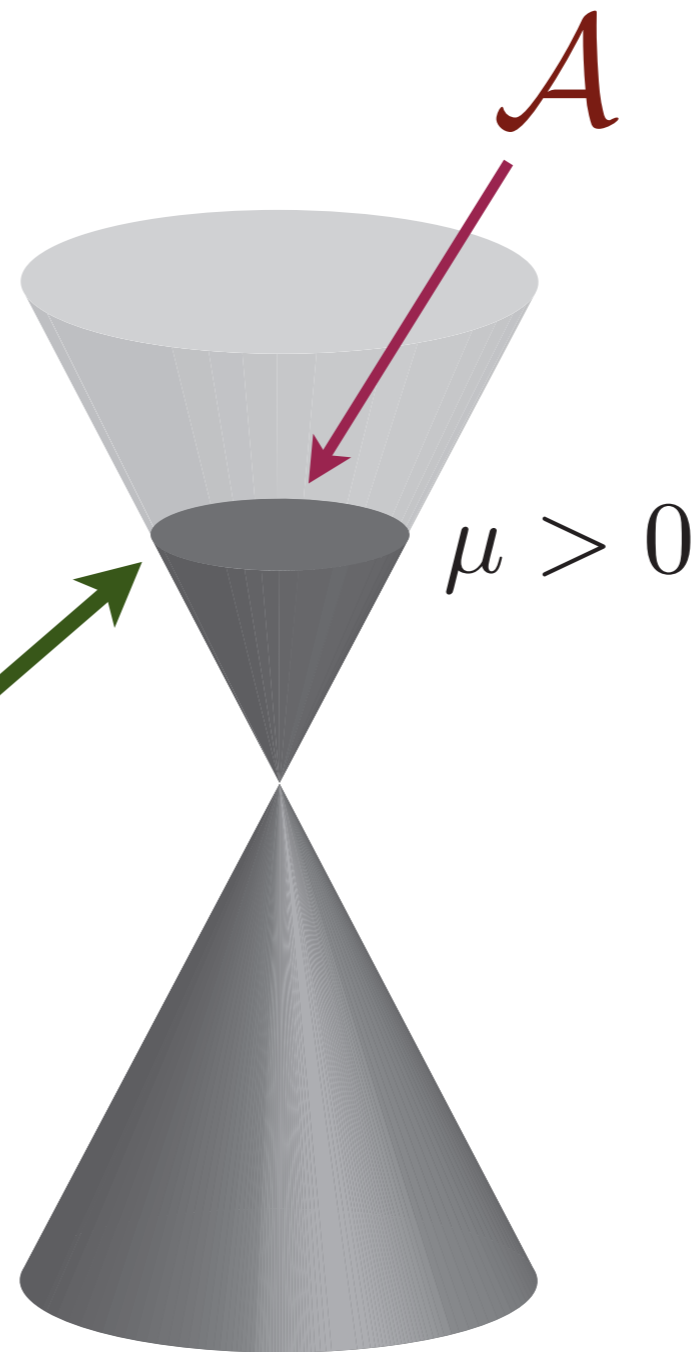


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- The *only* low energy excitations are long-lived quasiparticles near the Fermi surface.

The only other familiar compressible phase is a **Fermi Liquid** with a **Fermi surface**



- **Luttinger relation:** The total “volume (area)”  $\mathcal{A}$  enclosed by the Fermi surface is equal to  $\langle Q \rangle$ .

# Exotic phases of compressible quantum matter

I. Field theory

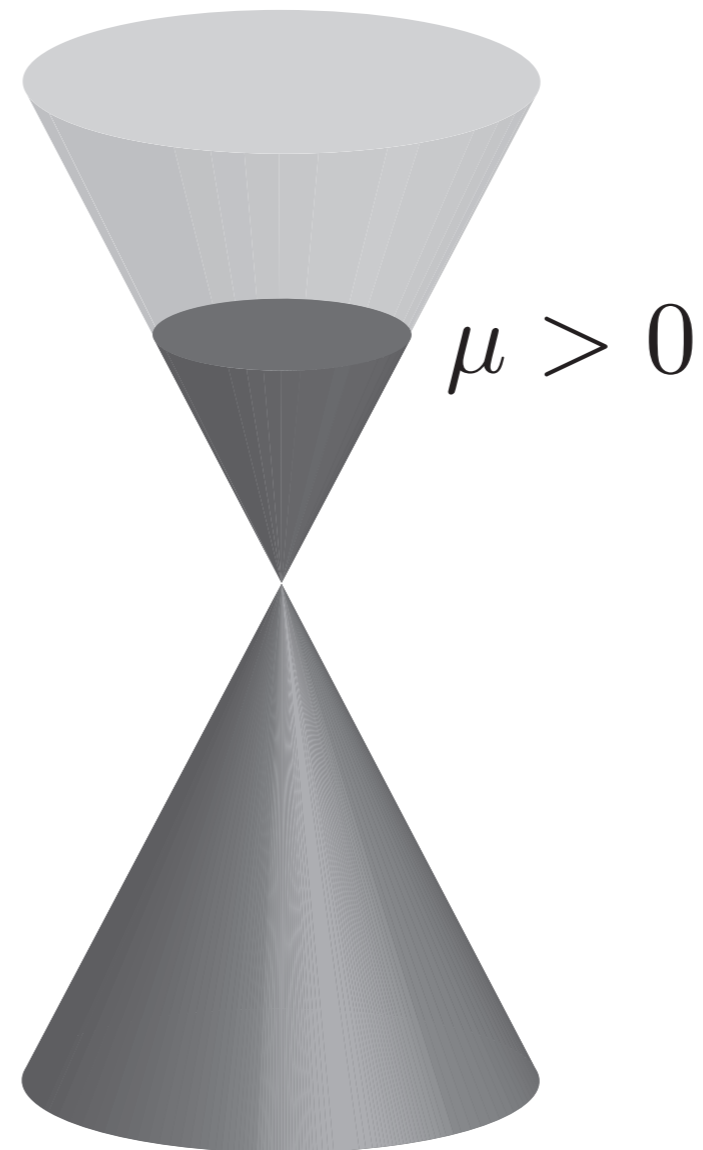
II. Holography

# Exotic phases of compressible quantum matter

I. Field theory

II. Holography

# The Fermi Liquid (FL)



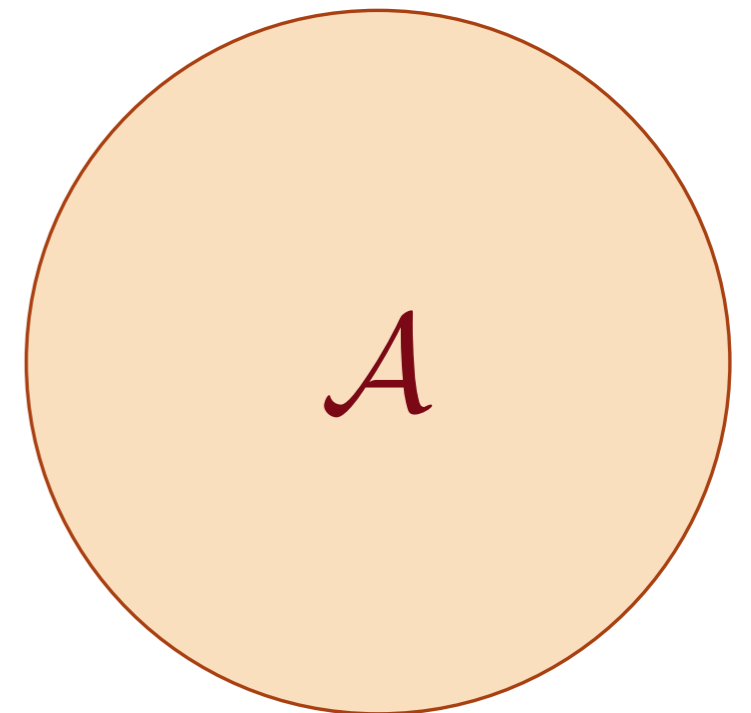
# The Fermi Liquid (FL)

Most common example: electrons with short-range interactions (or screened long-range interactions), which are adiabatically connected to the non-interacting limit. The electron Green's function  $G_f$  has a pole which crosses zero energy at  $k = k_F$ , and the Fermi surface has the same area as the non-interacting case.

$$\mathcal{L} = f_\sigma^\dagger \left( \partial_\tau - \frac{\nabla^2}{2m} - \mu \right) f_\sigma + 4 \text{ Fermi terms}$$

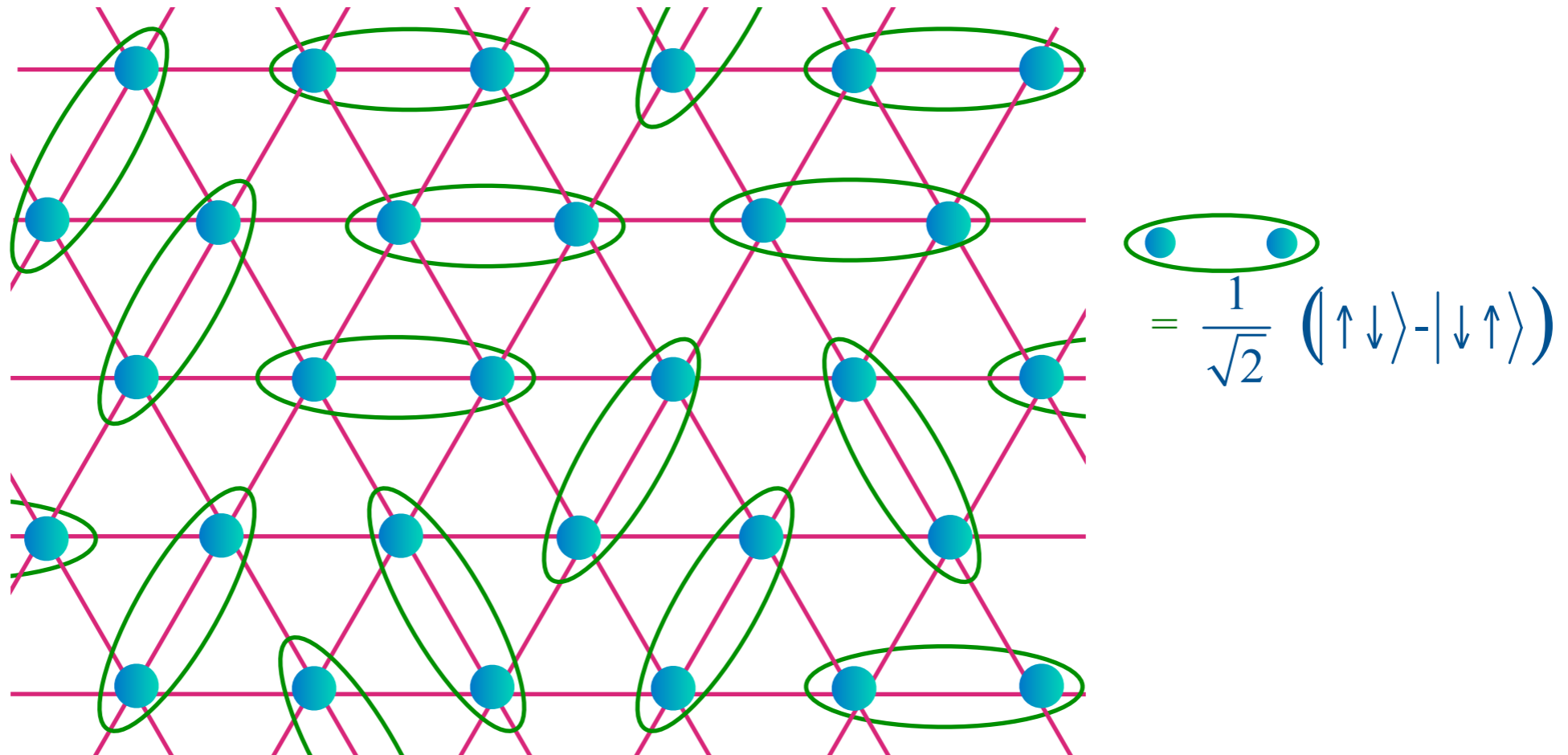
$$A = \langle f_\sigma^\dagger f_\sigma \rangle = \langle Q_\sigma \rangle$$

$$G_f = \frac{1}{\omega - v_F(k - k_F) + i\omega^2}$$



# The Non-Fermi Liquid (NFL)

- Model of a spin liquid (“Bose metal”): couple fermions to a dynamical gauge field  $A_\mu$ .



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# The Non-Fermi Liquid (NFL)

- Model of a spin liquid (“Bose metal”): couple fermions to a dynamical gauge field  $A_\mu$ .
- Longitudinal gauge fluctuations are screened by the fermions. But transverse gauge fluctuations remain unscreened, and are Landau-damped by excitations near the Fermi surface. The theory of a Fermi surface coupled to transverse gauge fluctuations is *strongly coupled in two spatial dimensions*.

S.-S. Lee, Phys. Rev. B **80**, 165102 (2009)

M. A. Metlitski and S. Sachdev, Phys. Rev. B **82**, 075127 (2010)

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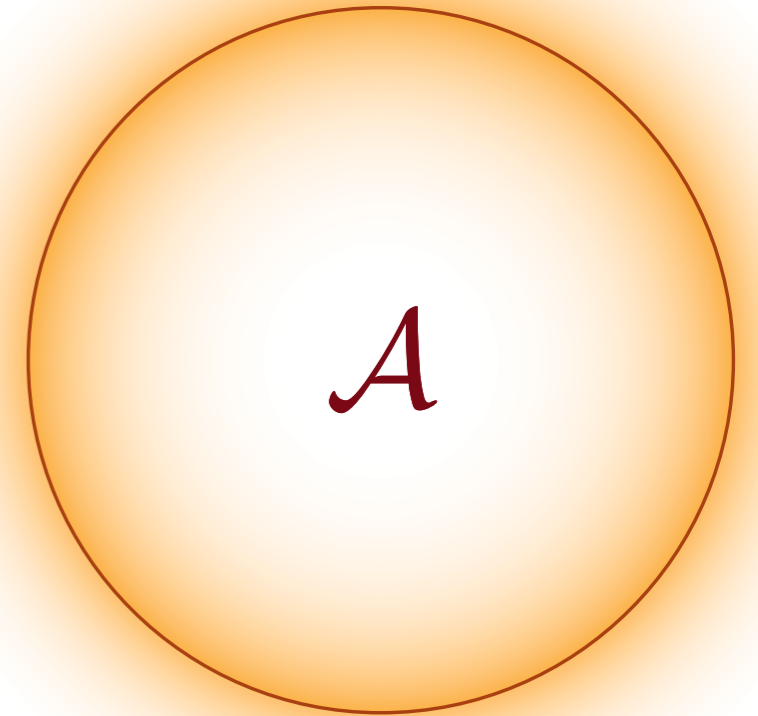
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- The overdamped transverse gauge modes lead to “non-Fermi liquid” broadening of the fermion pole near the Fermi surface.

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# The Non-Fermi Liquid (NFL)

- *The location of the Fermi surfaces is well defined, and the Luttinger relation applies as before.*

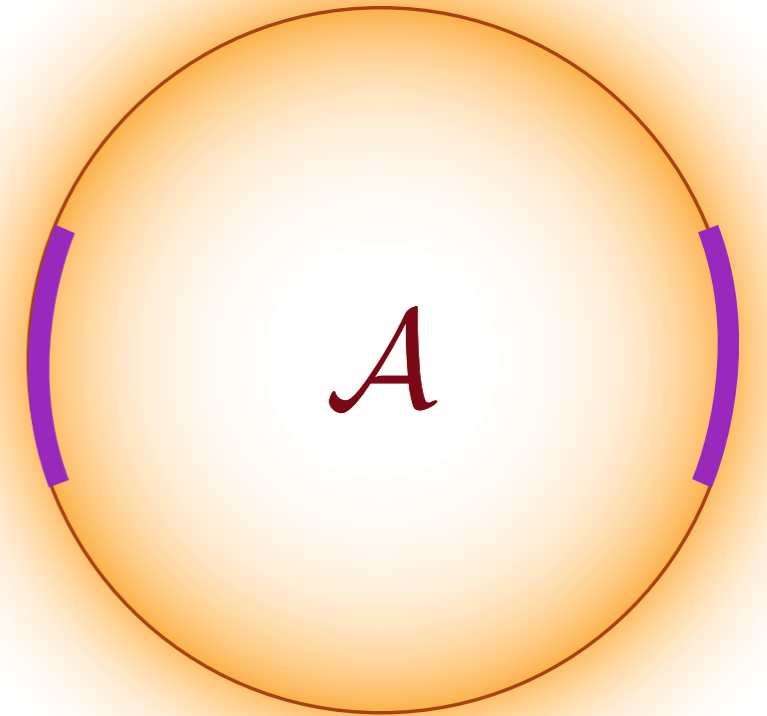


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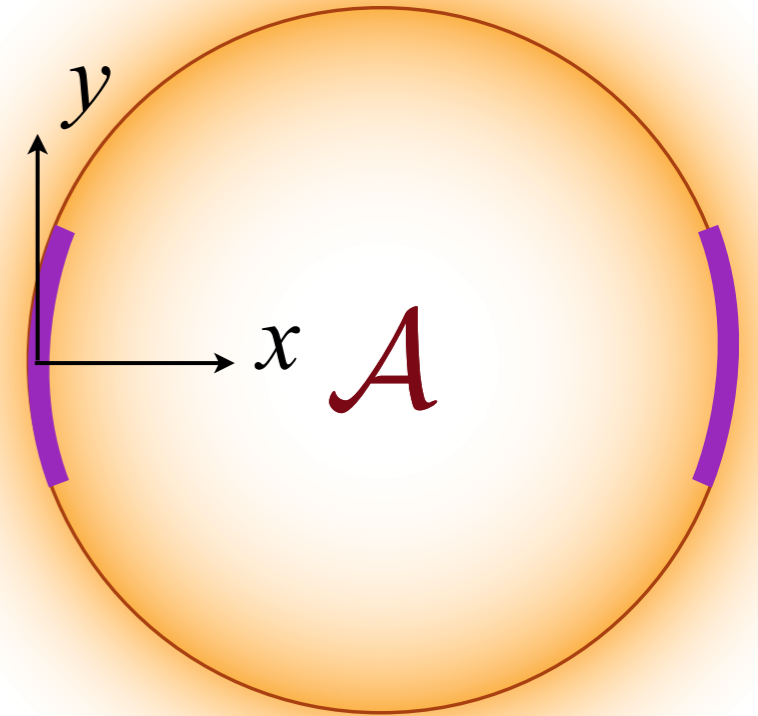
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- Fluctuations near the Fermi surface are described by a strongly-coupled two-patch theory. Ward identities allow consistent matching of the patches, and patches along different directions decouple in the low energy limit.
- The singularity of the Green's function upon approaching the Fermi surface is described by the scaling form

$$G_f^{-1} = q^{1-\eta} F(\omega/q^z)$$

where  $q_x = k_x - k_F$ ,  $q_y = k_y$ , and  $q = q_x + q_y^2/(2k_F)$ , and  $\eta$  and  $z$  are anomalous exponents. To three-loop order, we find  $\eta \neq 0$  and  $z = 3/2$ .

$$\text{One-loop order: } G_f^{-1} \sim v_F q + i\omega^{2/3}$$



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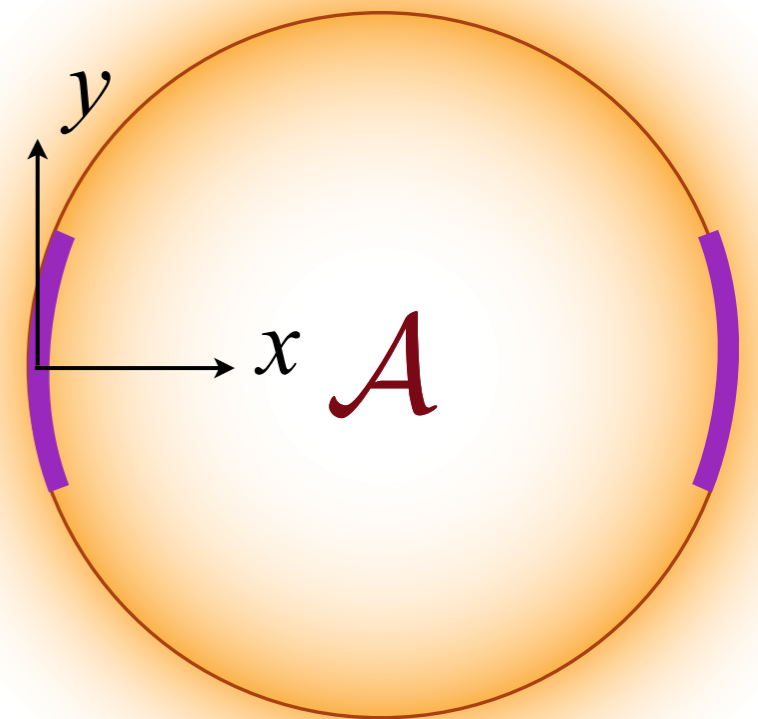
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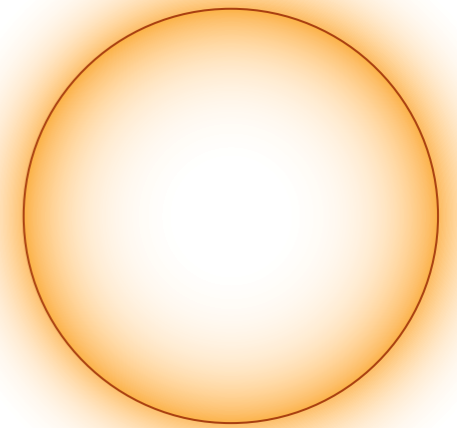


$$A = \langle f_\sigma^\dagger f_\sigma \rangle = \langle Q_\sigma \rangle$$

## Key question:

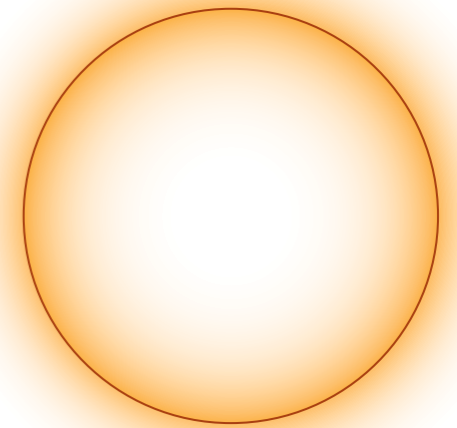
How do we detect the  
“hidden Fermi surfaces”  
of fermions with gauge charges  
in the non-Fermi liquid phases ?

These are not directly visible in the  
gauge-invariant fermion correlations  
computable via holography



One promising answer:

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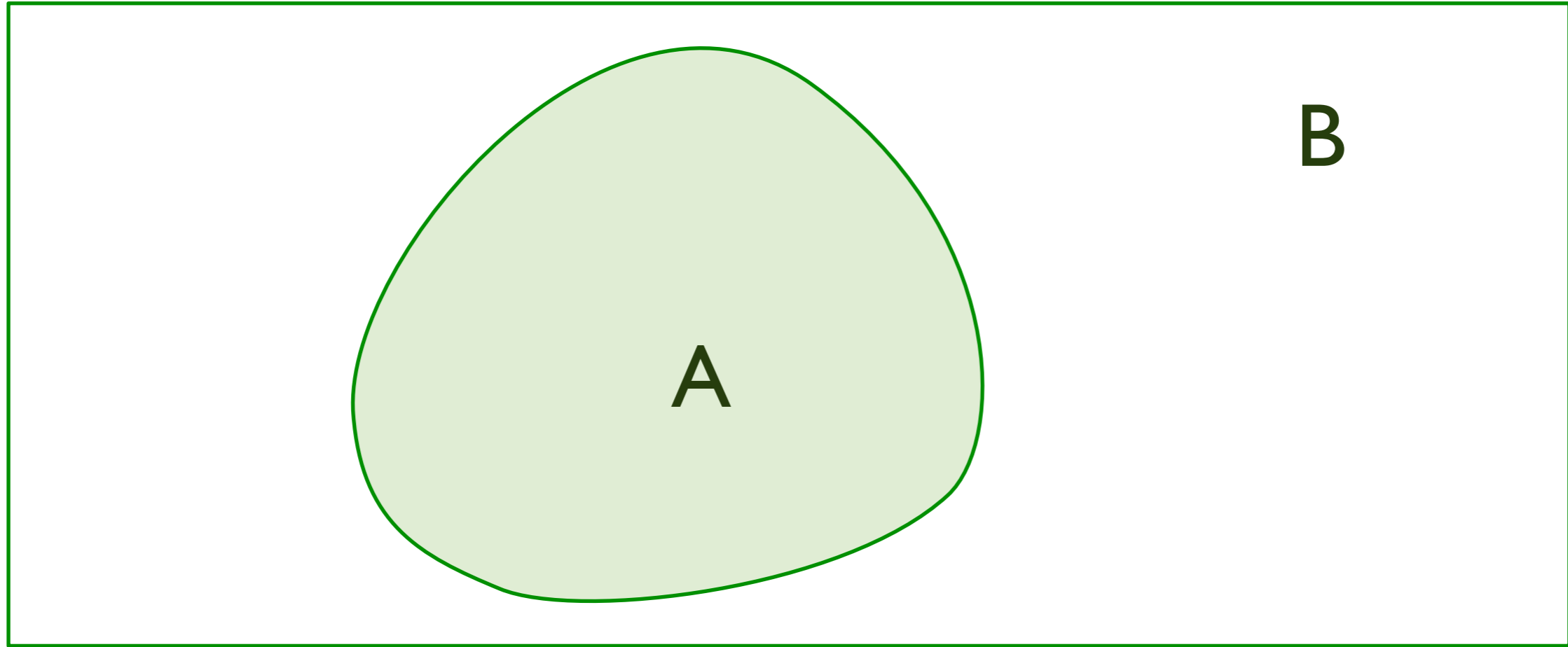


Compute  
entanglement entropy

N. Ogawa, T. Takayanagi, and T. Ugajin, arXiv:1111.1023

L. Huijse, B. Swingle, and S. Sachdev arXiv:1112.0573

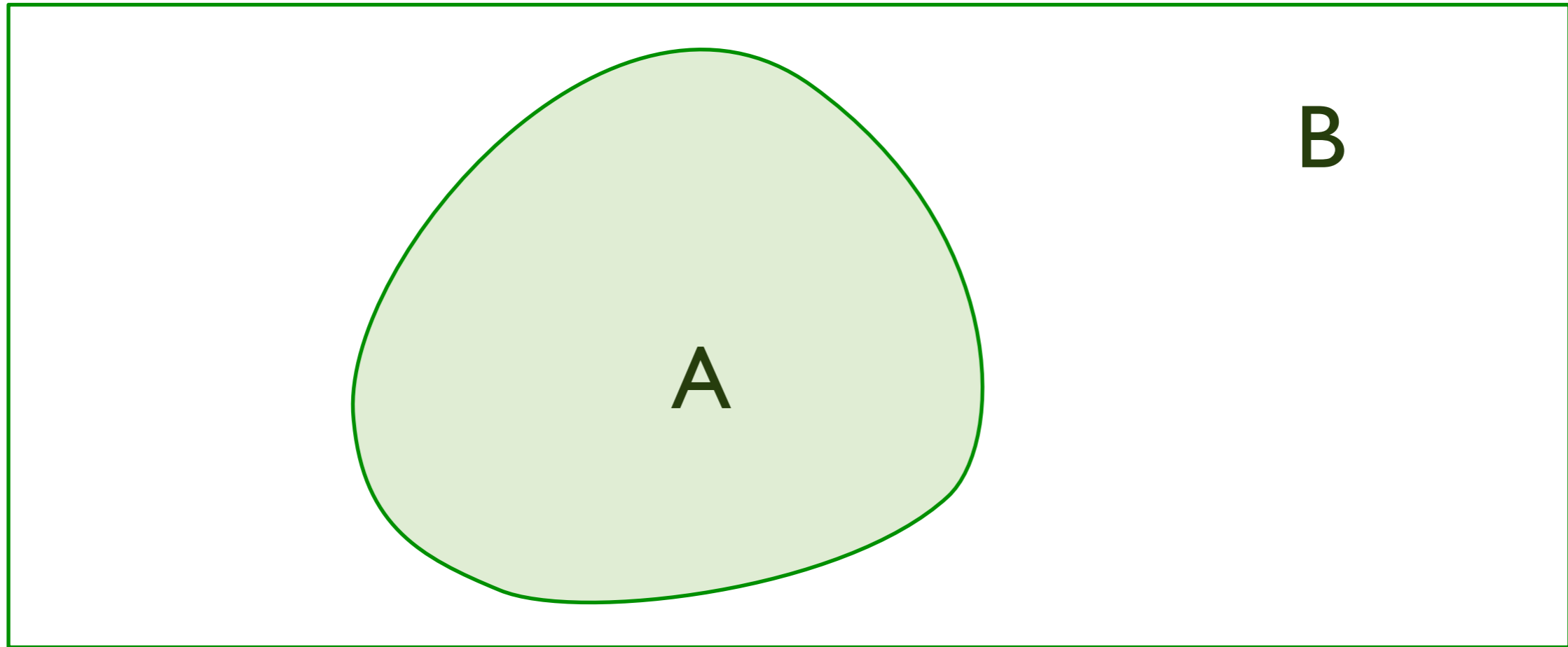
## Entanglement entropy of Fermi surfaces



$\rho_A = \text{Tr}_B \rho =$  density matrix of region  $A$

**Entanglement entropy**  $S_{EE} = -\text{Tr}(\rho_A \ln \rho_A)$

## Entanglement entropy of Fermi surfaces



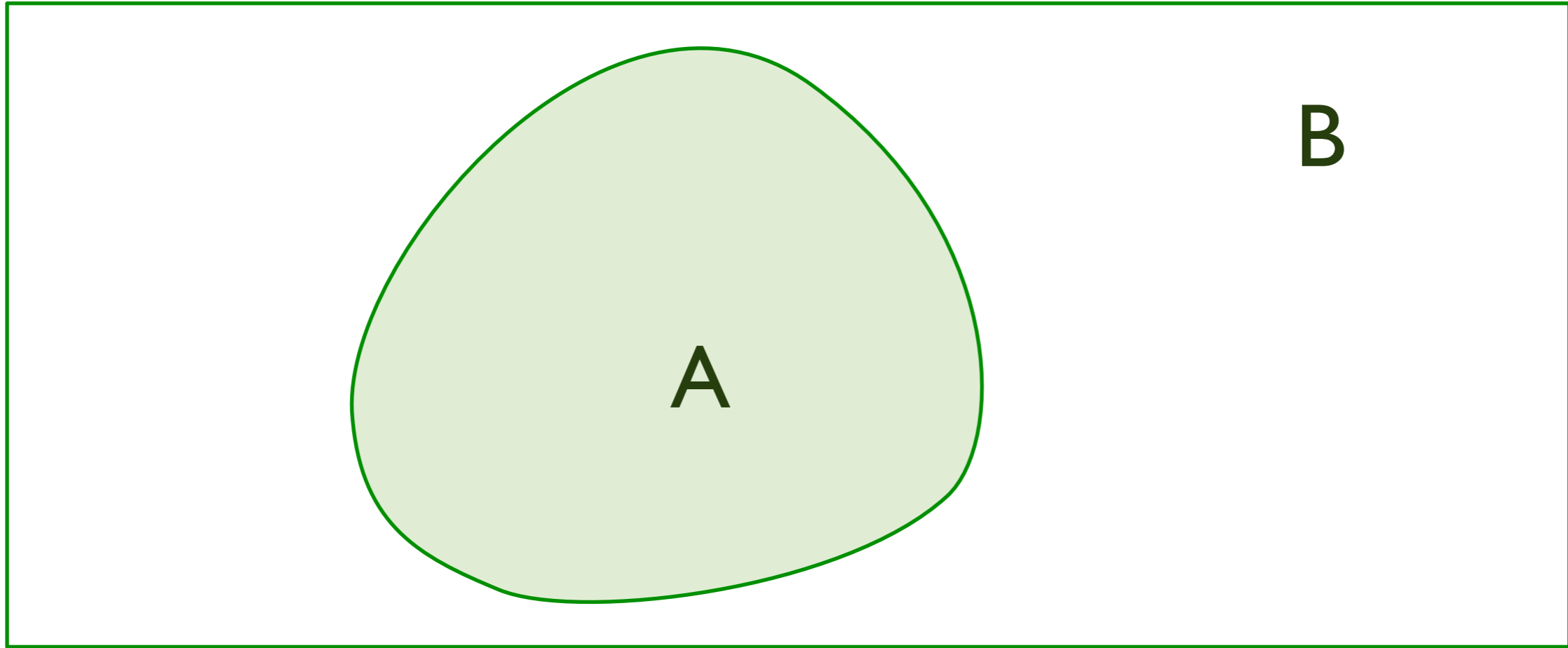
Logarithmic violation of “area law”:  $S_{EE} = \frac{1}{12} (k_F P) \ln(k_F P)$

for a circular Fermi surface with Fermi momentum  $k_F$ , where  $P$  is the perimeter of region A with an arbitrary smooth shape.

D. Gioev and I. Klich, *Physical Review Letters* **96**, 100503 (2006)

B. Swingle, *Physical Review Letters* **105**, 050502 (2010)

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Non-Fermi liquids have, at most, the “1/12” prefactor modified.

Y. Zhang, T. Grover, and A. Vishwanath, *Physical Review Letters* **107**, 067202 (2011)

# Exotic phases of compressible quantum matter

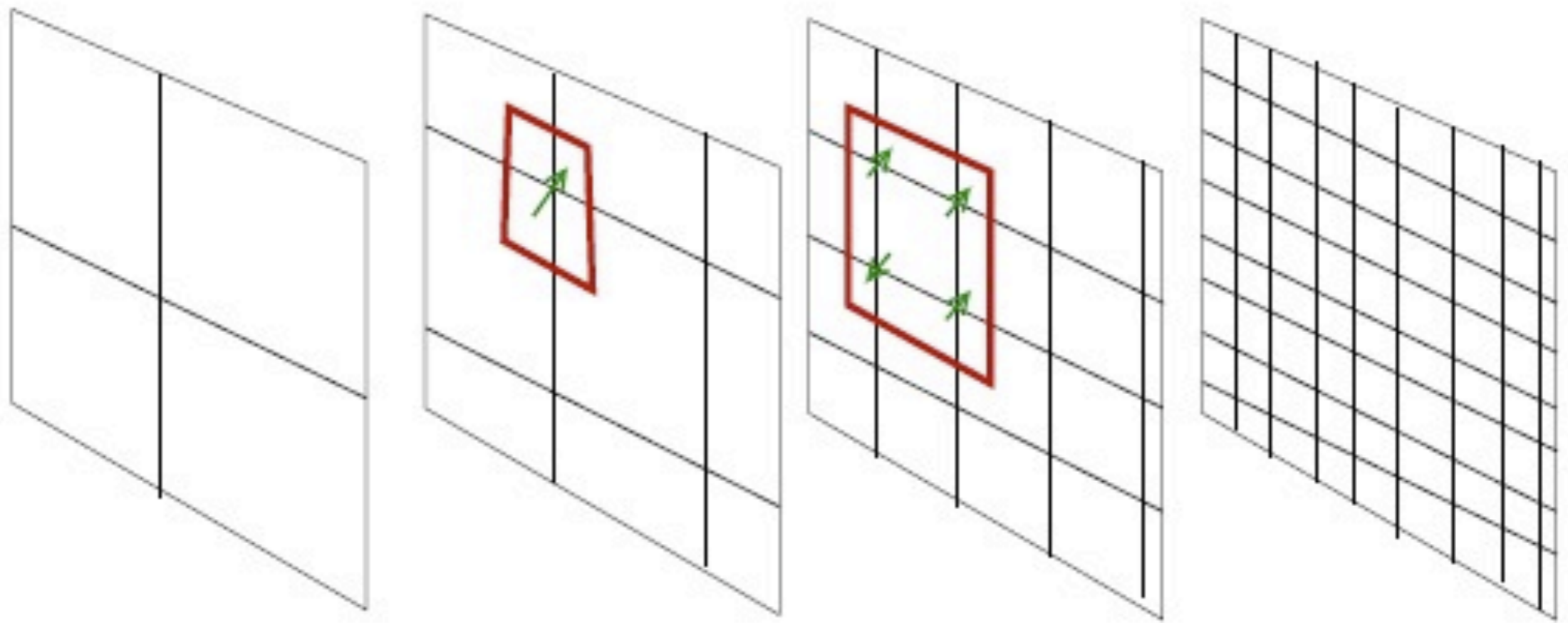
I. Field theory

II. Holography

# Exotic phases of compressible quantum matter

I. Field theory

II. Holography



$r$  ←

For a relativistic CFT in  $d$  spatial dimensions, the metric in the holographic space is uniquely fixed by demanding the following scale transformation ( $i = 1 \dots d$ )

$$x_i \rightarrow \zeta x_i \quad , \quad t \rightarrow \zeta t \quad , \quad ds \rightarrow ds$$

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$$x_i \rightarrow \zeta x_i \quad , \quad t \rightarrow \zeta t \quad , \quad ds \rightarrow ds$$

This gives the unique metric

$$ds^2 = \frac{1}{r^2} (-dt^2 + dr^2 + dx_i^2)$$

Reparametrization invariance in  $r$  has been used to the prefactor of  $dx_i^2$  equal to  $1/r^2$ . This fixes  $r \rightarrow \zeta r$  under the scale transformation. This is the metric of the space  $\text{AdS}_{d+2}$ .

In general, such scaling arguments lead to the most general metric

$$ds^2 = \frac{1}{r^2} \left( -\frac{dt^2}{r^{2d(z-1)/(d-\theta)}} + r^{2\theta/(d-\theta)} dr^2 + dx_i^2 \right)$$

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This metric transforms under rescaling as

$$\begin{aligned} x_i &\rightarrow \zeta x_i \\ t &\rightarrow \zeta^z t \\ ds &\rightarrow \zeta^{\theta/d} ds. \end{aligned}$$

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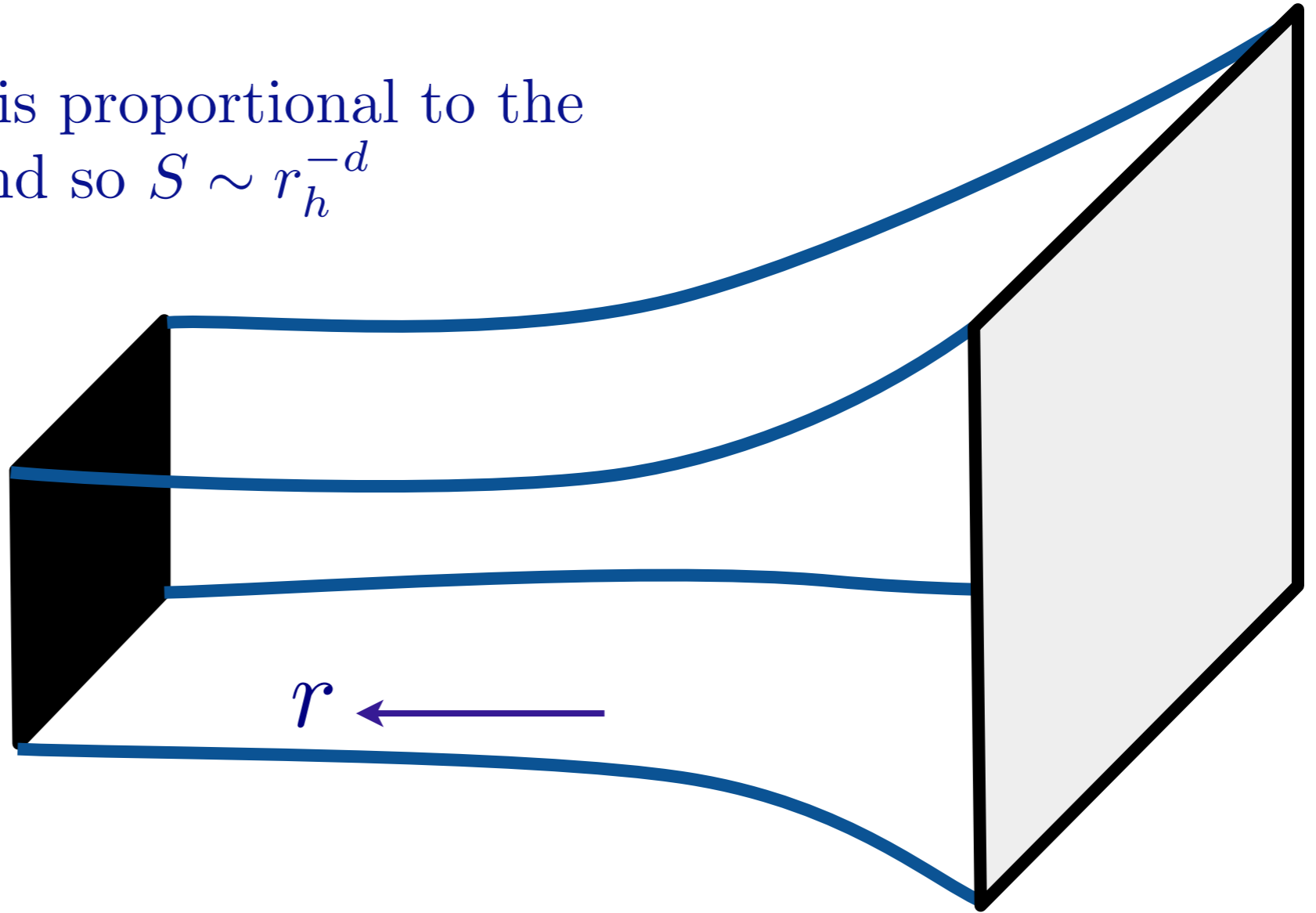
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**What is  $\theta$  ?** ( $\theta = 0$  for “relativistic” quantum critical points).

At  $T > 0$ , there is a “black-brane” at  $r = r_h$ .

The Beckenstein-Hawking entropy of the black-brane is the thermal entropy of the quantum system  $r = 0$ .

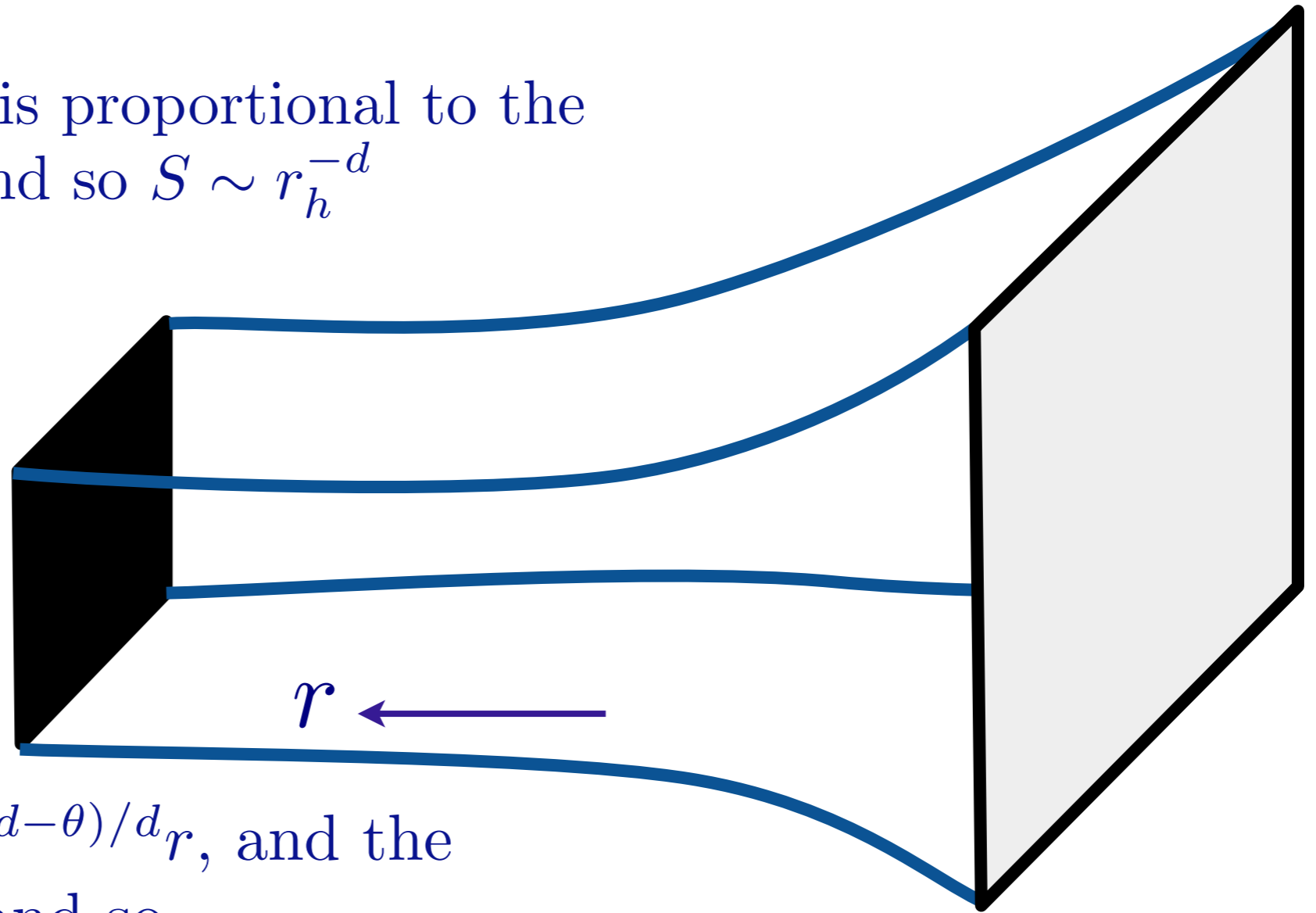
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Under rescaling  $r \rightarrow \zeta^{(d-\theta)/d} r$ , and the temperature  $T \sim t^{-1}$ , and so

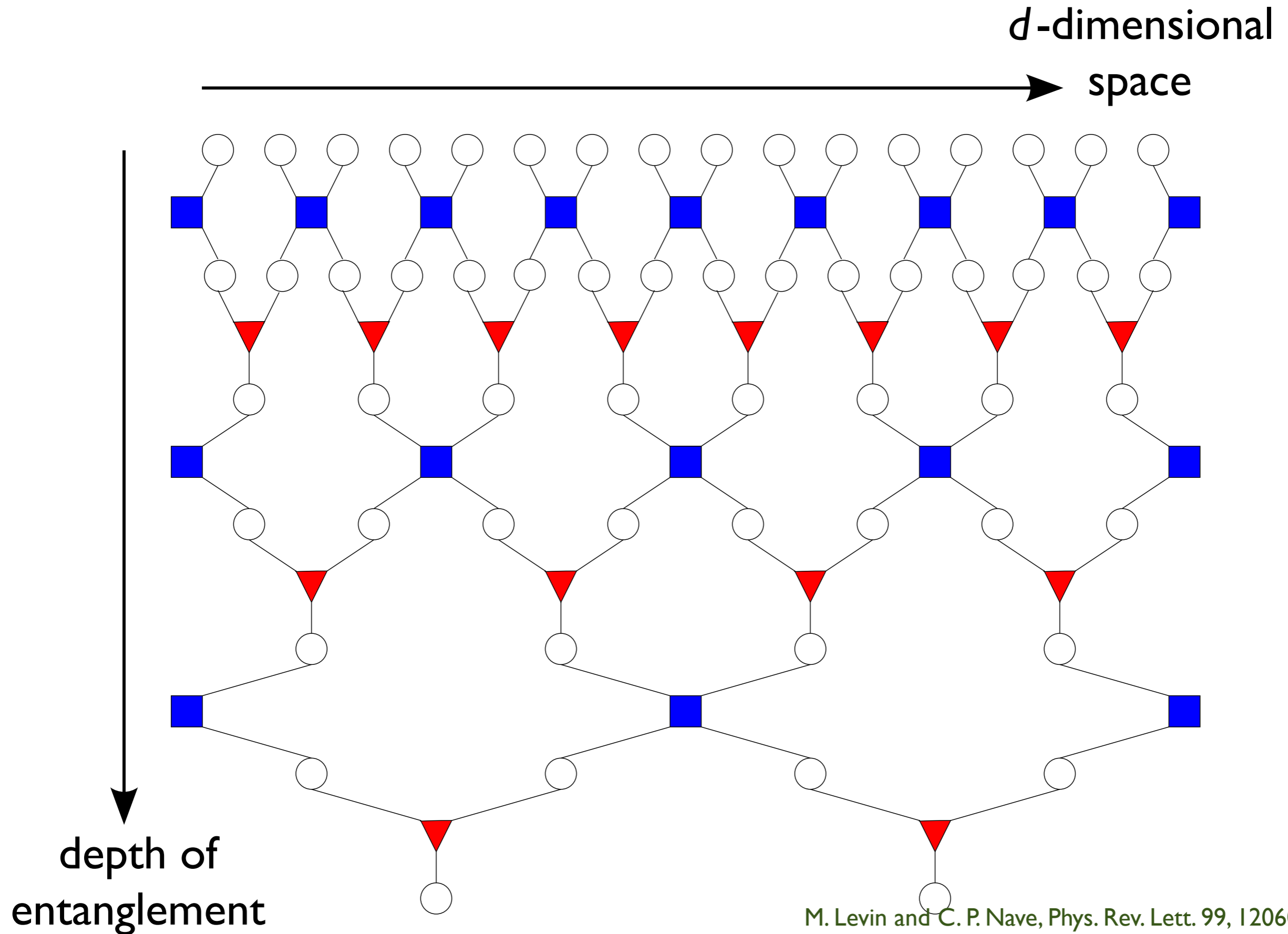
$$S \sim T^{(d-\theta)/z}$$

So  $\theta$  is the “violation of hyperscaling” exponent.

A non-Fermi liquid has gapless fermionic excitations on the Fermi surface, which disperse in the single transverse direction with dynamic critical exponent  $z$ . So we expect compressible quantum states to have an effective dimension  $d - \theta$  with

$$\theta = d - 1$$

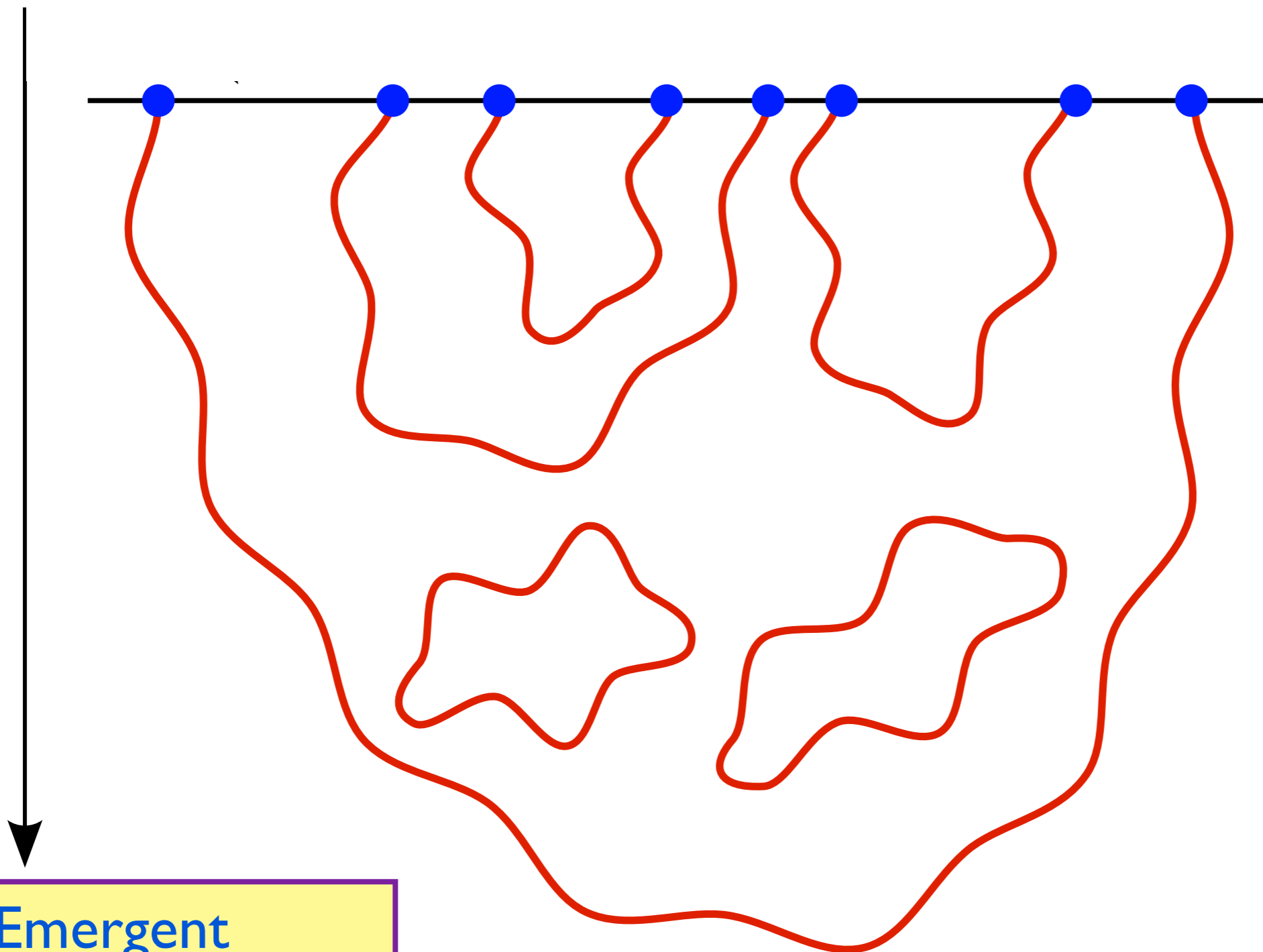
# Tensor network representation of entanglement



M. Levin and C. P. Nave, Phys. Rev. Lett. 99, 120601 (2007)  
F. Verstraete, M. M. Wolf, D. Perez-Garcia, and J. I. Cirac, Phys. Rev. Lett. 96, 220601 (2006)

String theory near  
a d-brane

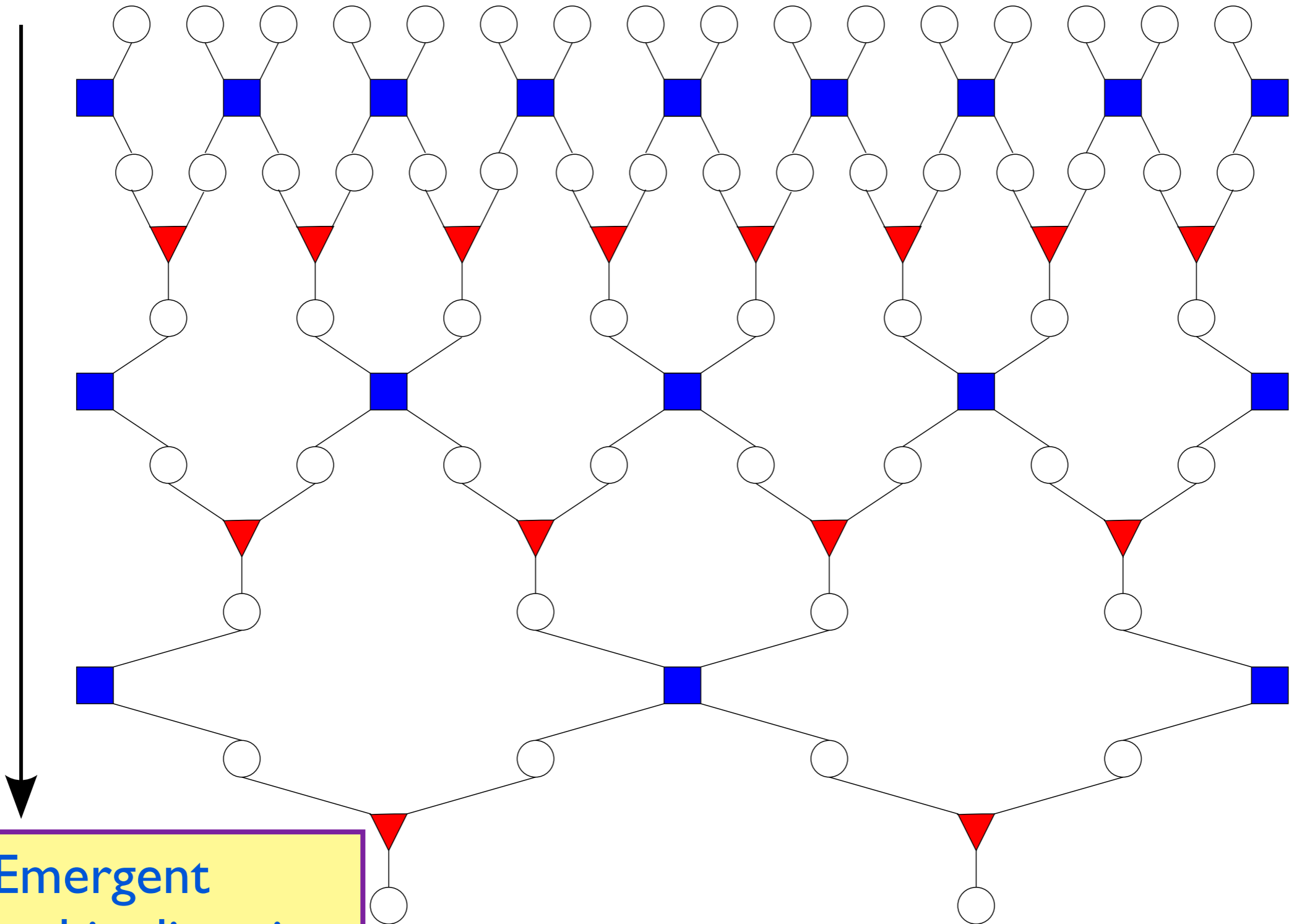
$d$ -dimensional  
space



Emergent  
holographic direction

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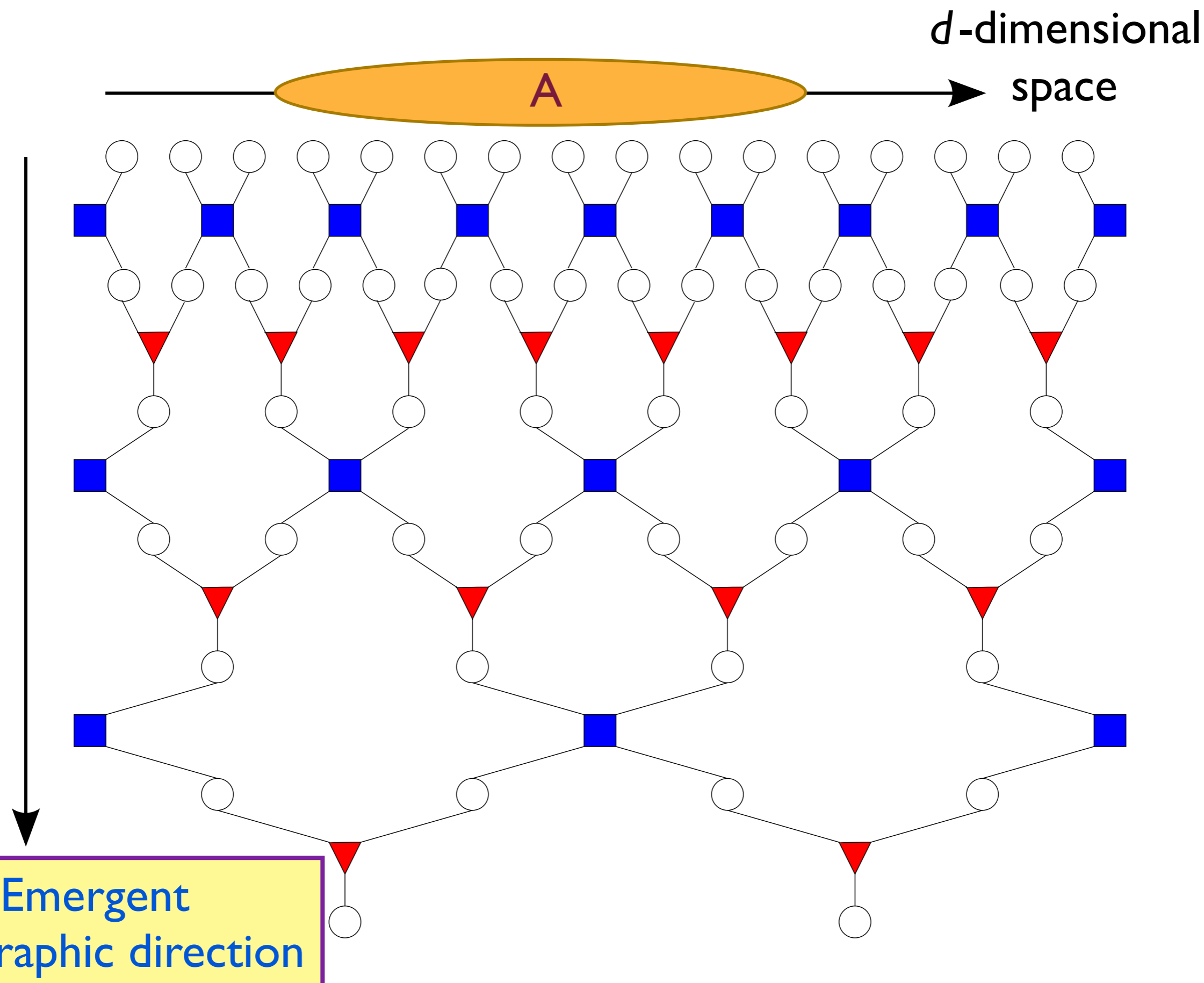
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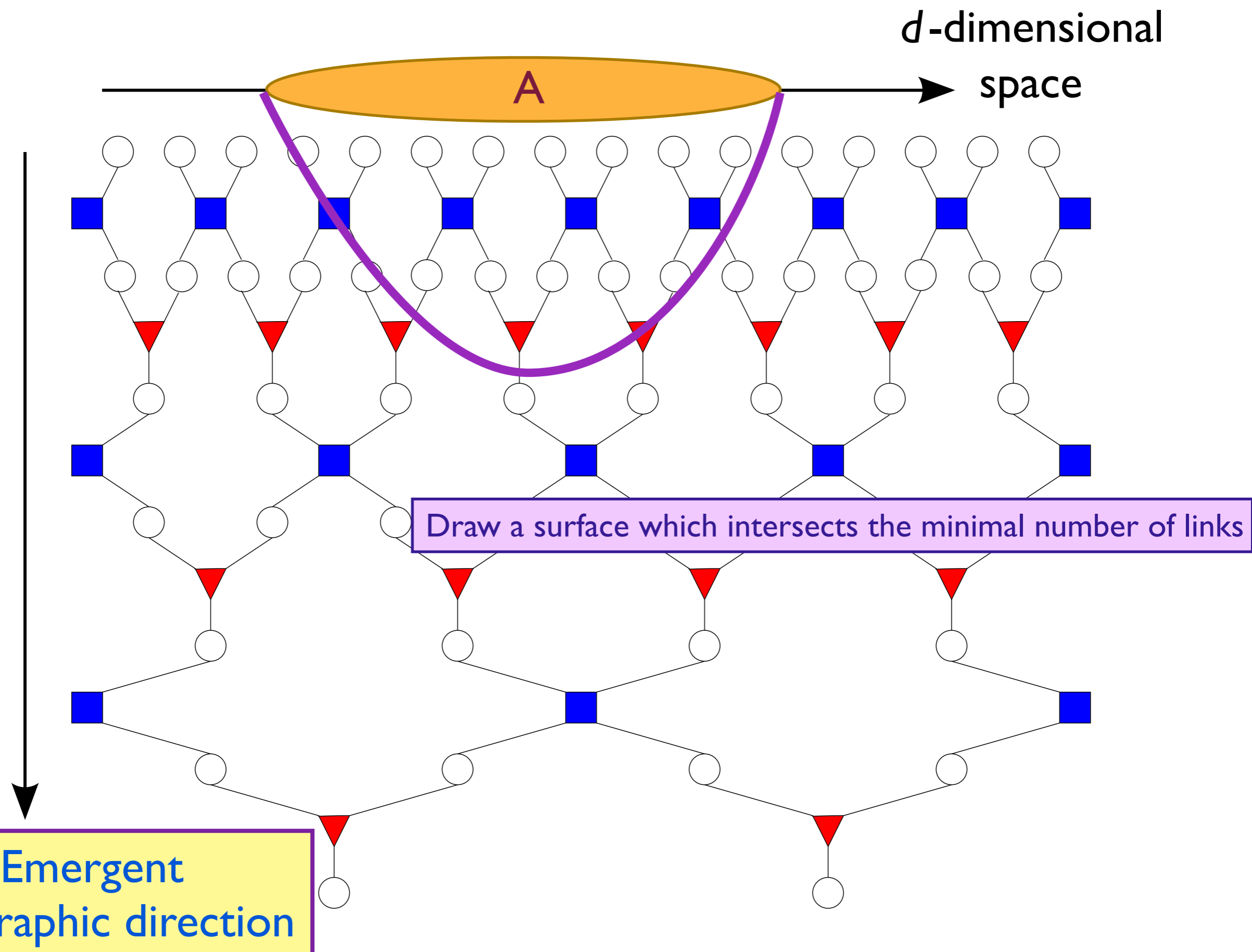
Emergent  
holographic direction

Brian Swingle, arXiv:0905.1317

# Entanglement entropy



# Entanglement entropy



## Entanglement entropy

The entanglement entropy of a region  $A$  on the boundary equals the minimal area of a surface in the higher-dimensional space whose boundary co-incides with that of  $A$ .

This can be seen in both the string and tensor-network pictures

S. Ryu and T. Takayanagi, Phys. Rev. Lett. 96, 18160 (2006).  
Brian Swingle, arXiv:0905.1317

## Entanglement entropy of the metric

$$ds^2 = \frac{1}{r^2} \left( -\frac{dt^2}{r^{2d(z-1)/(d-\theta)}} + r^{2\theta/(d-\theta)} dr^2 + dx_i^2 \right)$$

The area law is obeyed for

$$\theta \leq d - 1$$

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For  $\theta = d - 1$ , the value expected for compressible quantum states, the entanglement entropy has log-violation of the area law

$$S_E = \Xi \mathcal{Q}^{(d-1)/d} \Sigma \ln \left( \mathcal{Q}^{(d-1)/d} \Sigma \right).$$

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- $\Xi$  is a dimensionless constant which is *independent* of  $Q$  and of any property of the entangling region.

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- $\Sigma$  is the  $(d - 1)$ -dimensional surface area of entangling region (in  $d = 2$ ,  $\Sigma = P$  is the perimeter). Note  $S_E$  is otherwise independent of the shape of the entangling region, unlike other gapless systems. This is a characteristic property of a Fermi surface

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$$S_E = \Xi Q^{(d-1)/d} \Sigma \ln \left( Q^{(d-1)/d} \Sigma \right).$$

- $Q$  is the total conserved charge. The metric has a complicated dependence on  $Q$ , but  $S_E$  is just proportional to  $Q^{(d-1)/d}$ . Many UV details are irrelevant, and  $S_E$  flows to the universal  $Q$  dependence in the IR. By Luttinger's relation  $Q \sim k_F^d$ , and so the prefactor is the area of the Fermi surface, as expected from field theory.

## Inequalities

$$ds^2 = \frac{1}{r^2} \left( -\frac{dt^2}{r^{2d(z-1)/(d-\theta)}} + r^{2\theta/(d-\theta)} dr^2 + dx_i^2 \right)$$

The area law of entanglement entropy is obeyed for

$$\theta \leq d - 1.$$

The “null energy condition” of the gravity theory yields

$$z \geq 1 + \frac{\theta}{d}.$$

Remarkably, for  $d = 2$ ,  $\theta = d - 1$  and  $z = 1 + \theta/d$ , we obtain  $z = 3/2$ , the same value associated with the field theory.

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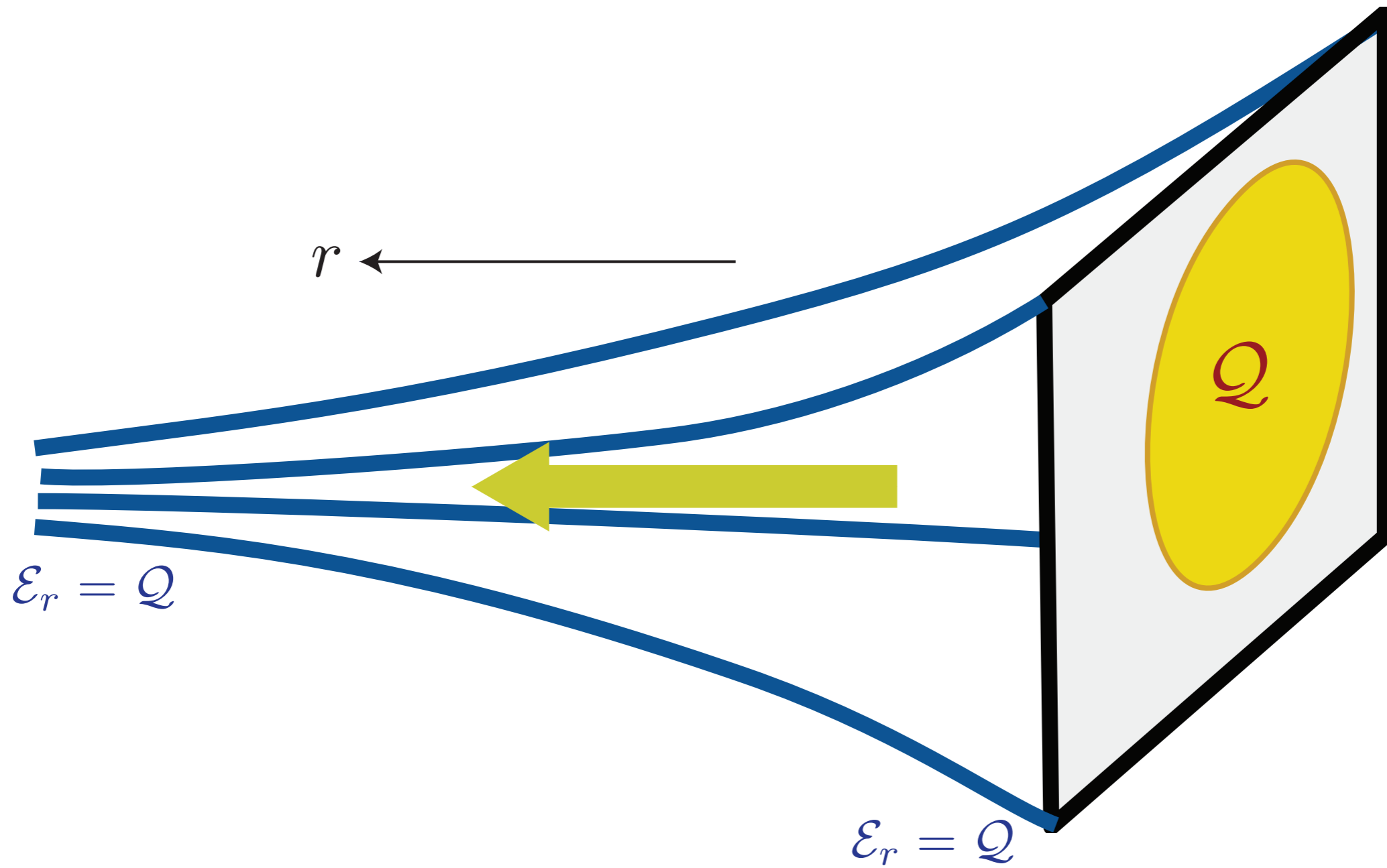
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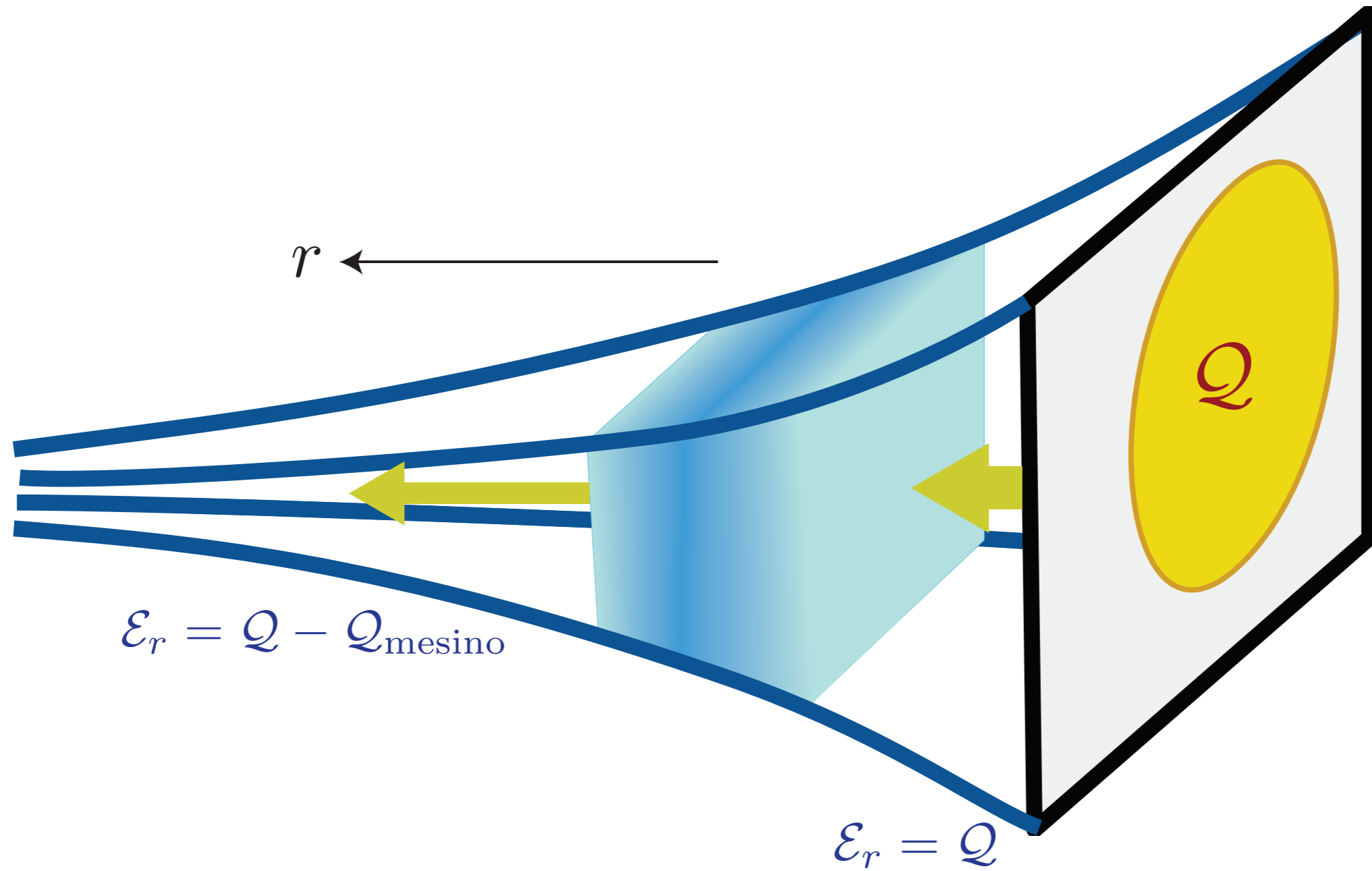
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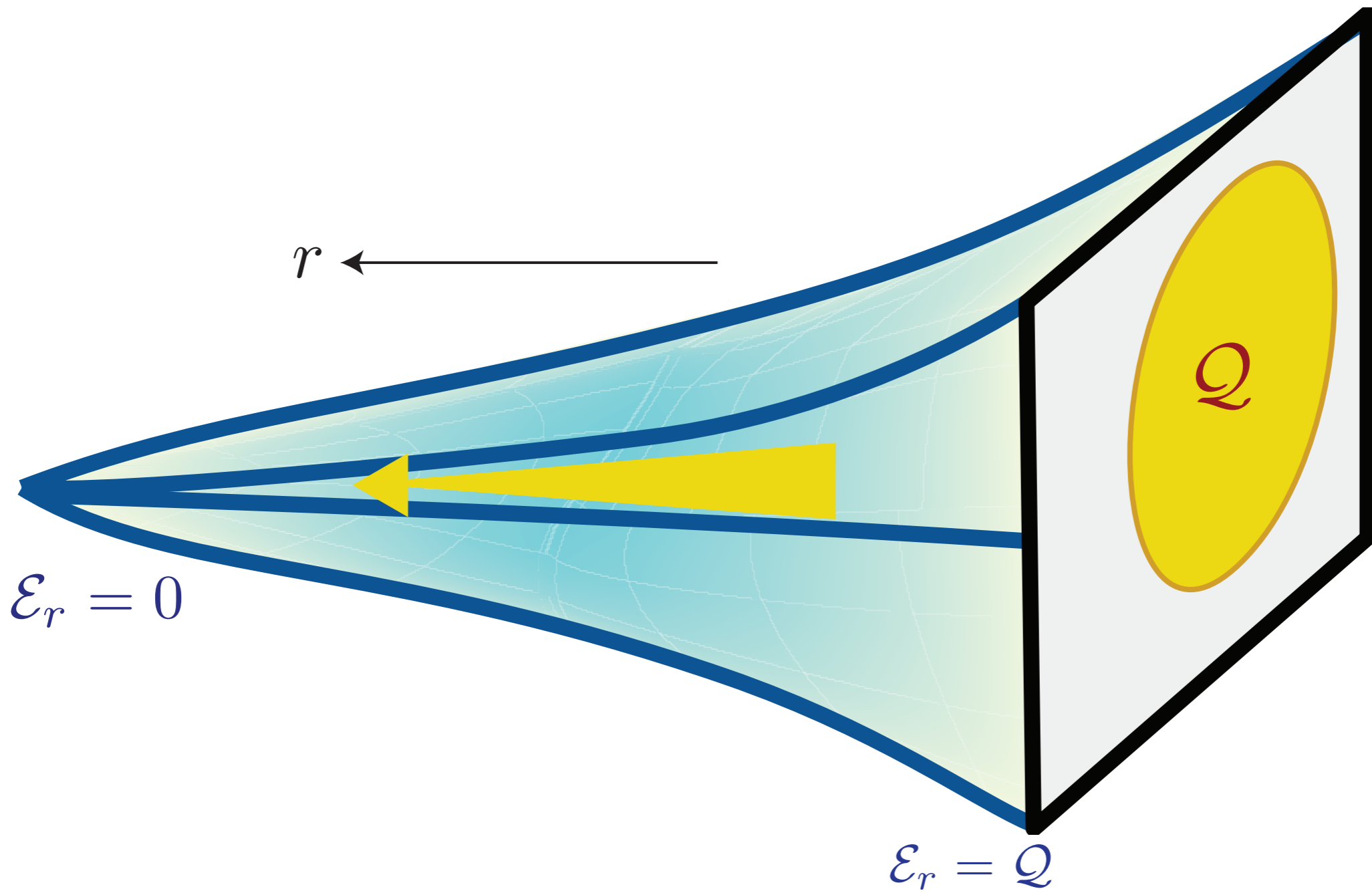
# Holographic theory of a non-Fermi liquid (NFL)



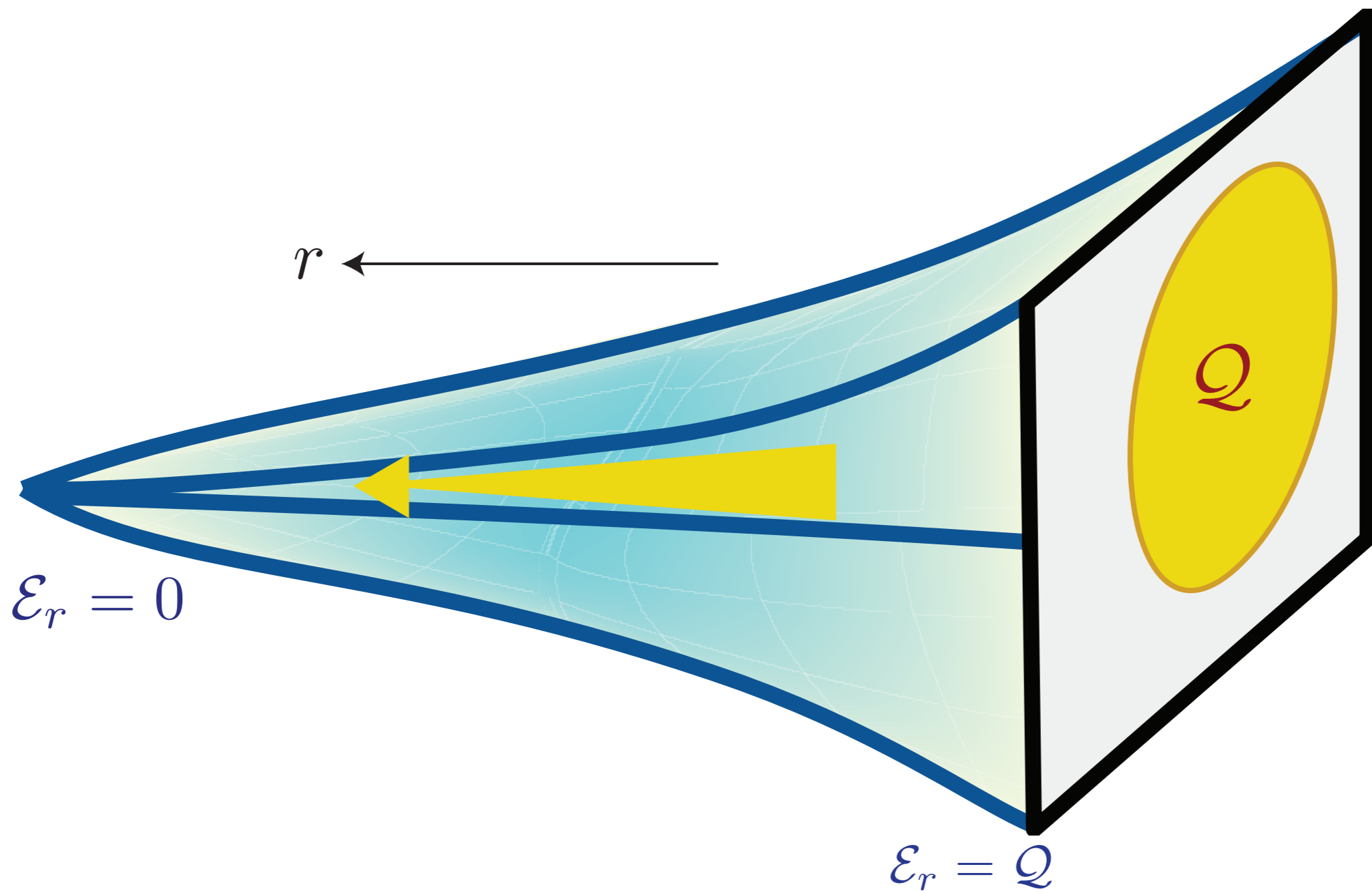
# Holographic theory of a fractionalized-Fermi liquid (FL\*)



# Holographic theory of a Fermi liquid (FL)



# Holographic theory of a Fermi liquid (FL)



Gauss Law in the bulk

$\Leftrightarrow$  Luttinger theorem on the boundary

# Theory of a non-Fermi liquid (NFL)

Field theory

Holography

A gauge-dependent Fermi surface of overdamped gapless fermions.

Fermi surface is hidden.

# Theory of a non-Fermi liquid (NFL)

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