Quantum matter and gauge-gravity duality

Isaac Newton Institute for Mathematical Sciences
Workshop on
"Condensed Matter, Black Holes and Holography"
April 16-20, 2012

Subir Sachdev

Talk online at sachdev.physics.harvard.edu

PHYSICS
VE RI TAS
HARVARD

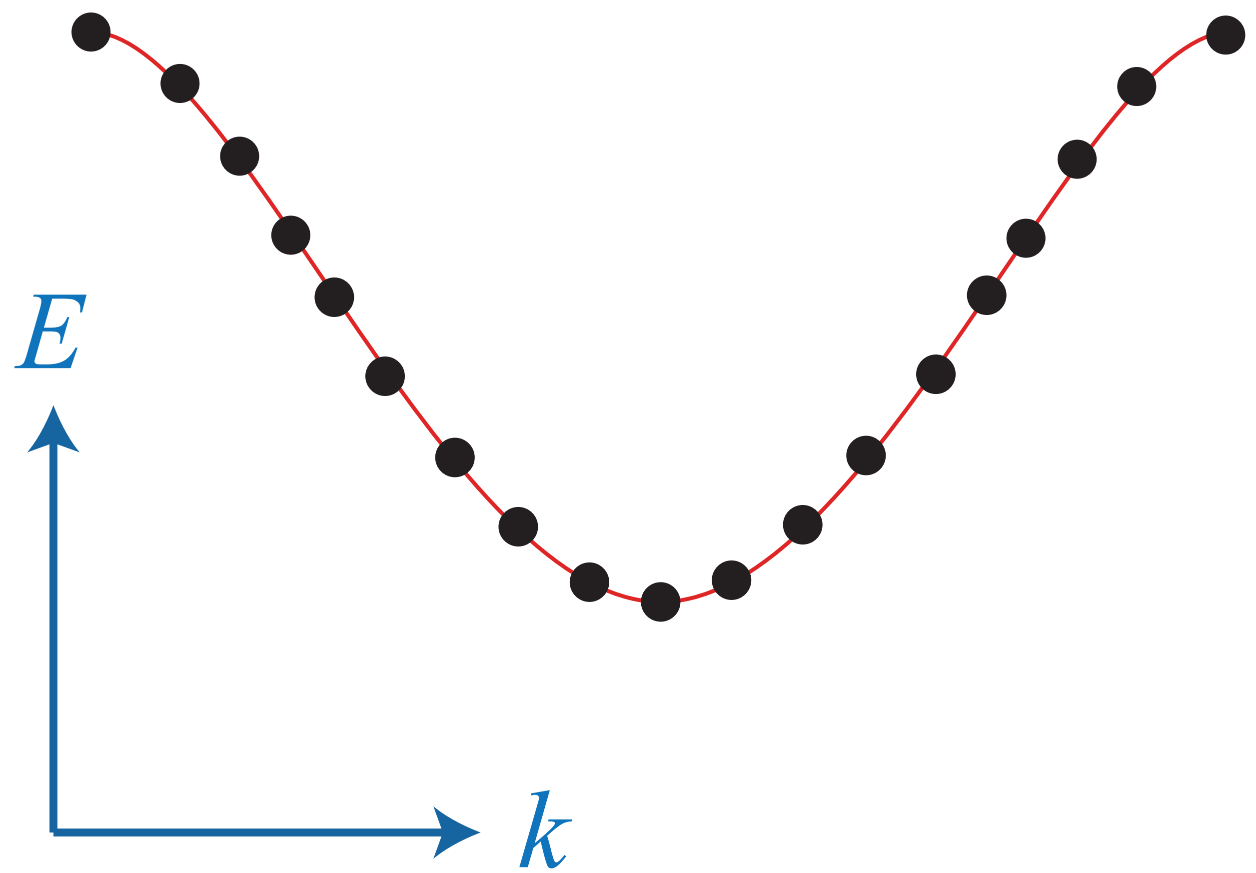
Lecture 1

Sommerfeld-Bloch theory of metals, insulators, and superconductors:

many-electron quantum states are adiabatically connected to independent electron states

Sommerfeld-Bloch theory of
metals, insulators, and superconductors:
many-electron quantum states are adiabatically connected to independent electron states

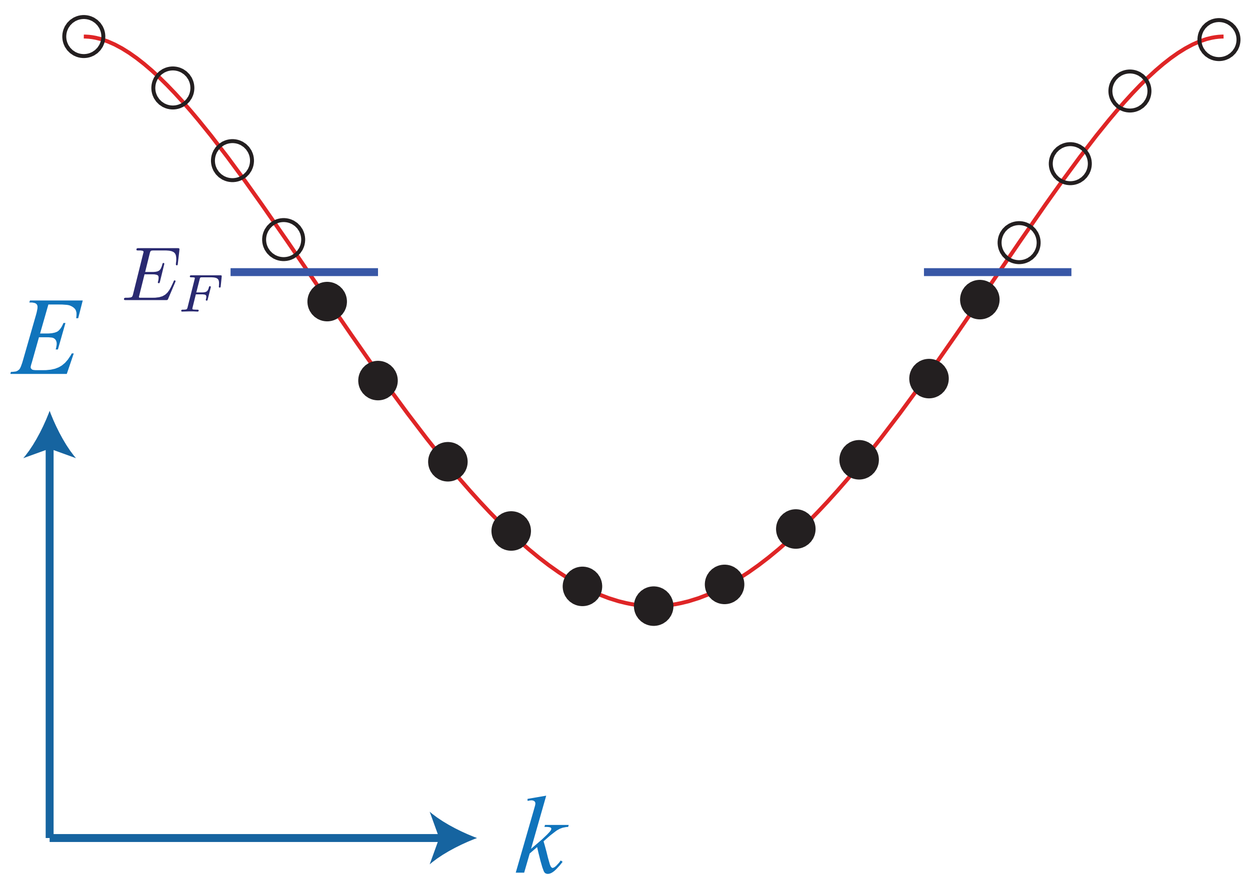
Band insulators



An even number of electrons per unit cell

Sommerfeld-Bloch theory of metals, insulators, and superconductors: many-electron quantum states are adiabatically connected to independent electron states

Metals

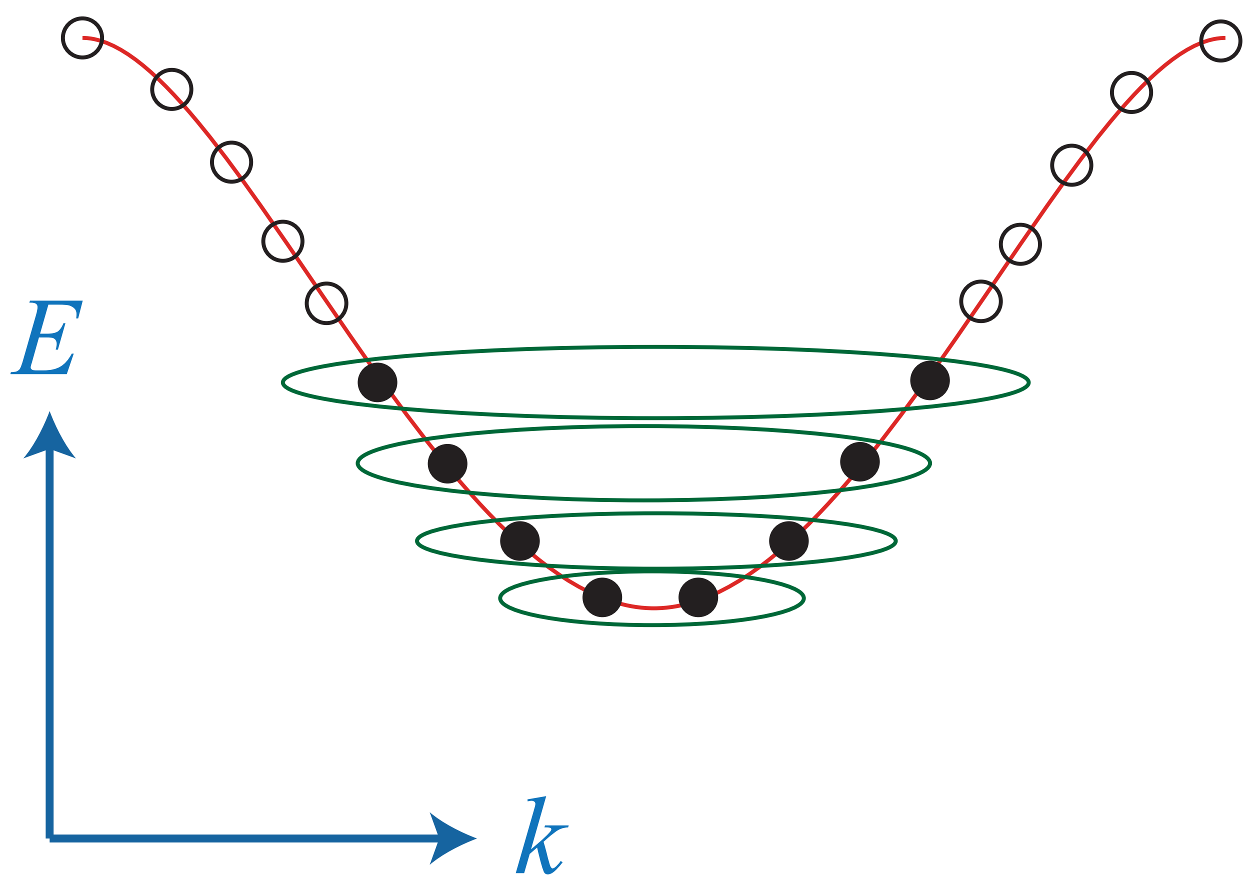


An odd number of electrons per unit cell

Sommerfeld-Bloch theory of
metals, insulators, and superconductors:

many-electron quantum states are adiabatically
connected to independent electron states

Superconductors



Sommerfeld-Bloch theory of
metals, insulators, and superconductors:
many-electron quantum states are adiabatically
connected to independent electron states

Modern phases of quantum matter

Not adiabatically connected to independent electron states:

many-particle quantum entanglement

Classify zero temperature ground states
of infinite quantum systems in *d* spatial dimensions
(on a lattice or in the continuum)
with translational invariance.

Classify zero temperature ground states
of infinite quantum systems in *d* spatial dimensions
(on a lattice or in the continuum)
with translational invariance.

Useful classification is provided by
nature of excitations with vanishing energy:

1. Gapped systems without zero energy excitations

Classify zero temperature ground states of infinite quantum systems in *d* spatial dimensions (on a lattice or in the continuum) with translational invariance.

Useful classification is provided by nature of excitations with vanishing energy:

1. Gapped systems without zero energy excitations

2. "Relativistic" systems with zero energy excitations at isolated points in momentum space

Classify zero temperature ground states of infinite quantum systems in d spatial dimensions (on a lattice or in the continuum) with translational invariance.

Useful classification is provided by nature of excitations with vanishing energy:

1. Gapped systems without zero energy excitations

2. "Relativistic" systems with zero energy excitations at isolated points in momentum space

3. "Compressible" systems with zero energy excitations on d-1 dimensional surfaces in momentum space.

Gapped quantum matter

Insulators, quantum Hall states

Compressible quantum matter

Metals, superconductors, strange metals

Conformal quantum matter

Graphene, antiferromagnets, ultracold atoms

Gapped quantum matter

Insulators, quantum Hall states

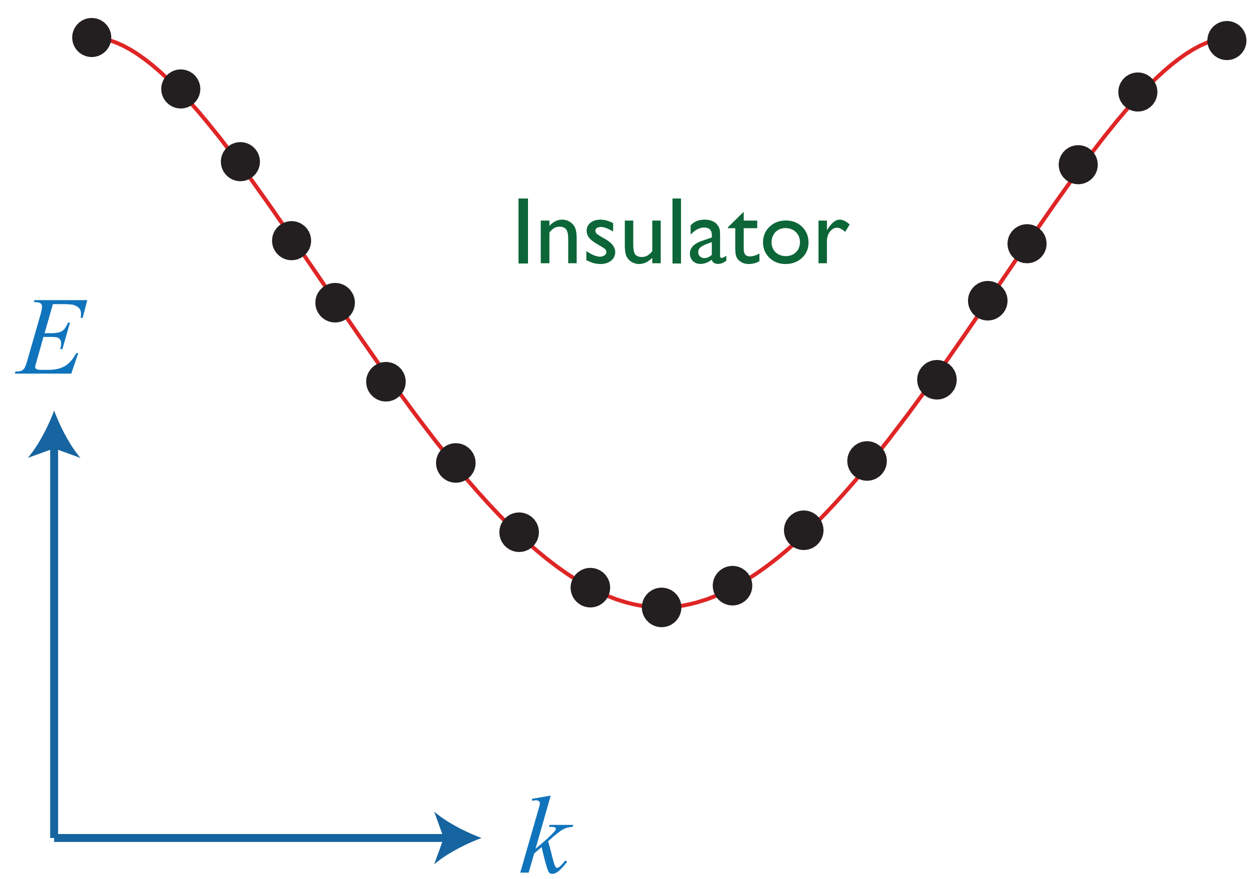
Compressible quantum matter

Metals, superconductors, strange metals

Conformal quantum matter

Graphene, antiferromagnets, ultracold atoms

Band insulators



An even number of electrons per unit cell

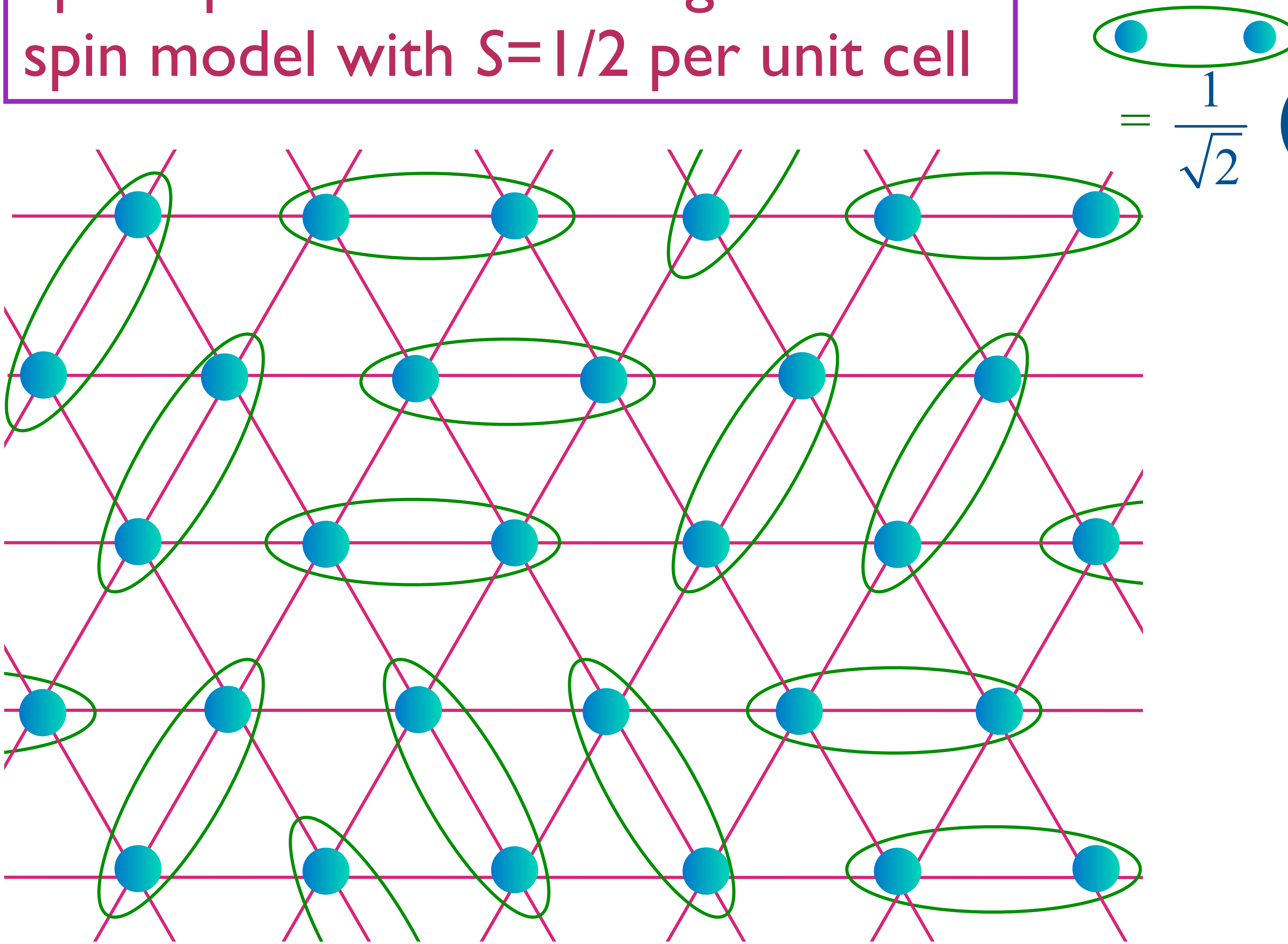
Mott insulator

Emergent excitations

An odd number of electrons per unit cell
but electrons are localized by Coulomb repulsion

Mott insulator: Triangular lattice antiferromagnet

Spin liquid obtained in a generalized spin model with $S=1/2$ per unit cell

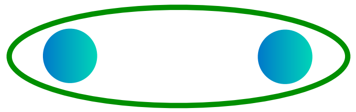


$$= \frac{1}{\sqrt{2}} \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right)$$

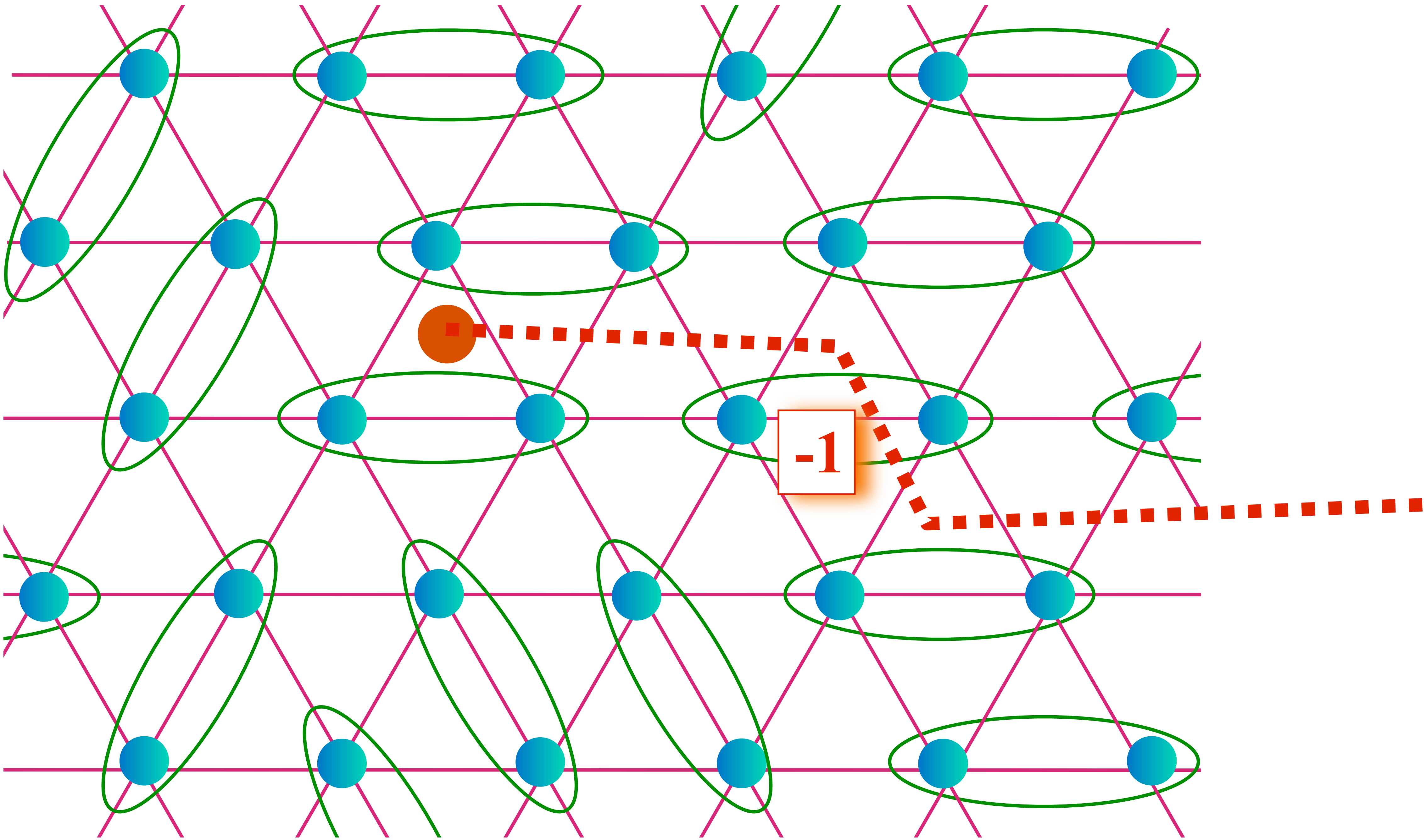
P. Fazekas and P. W. Anderson, *Philos. Mag.* **30**, 23 (1974).

Excitations of the Z_2 Spin liquid

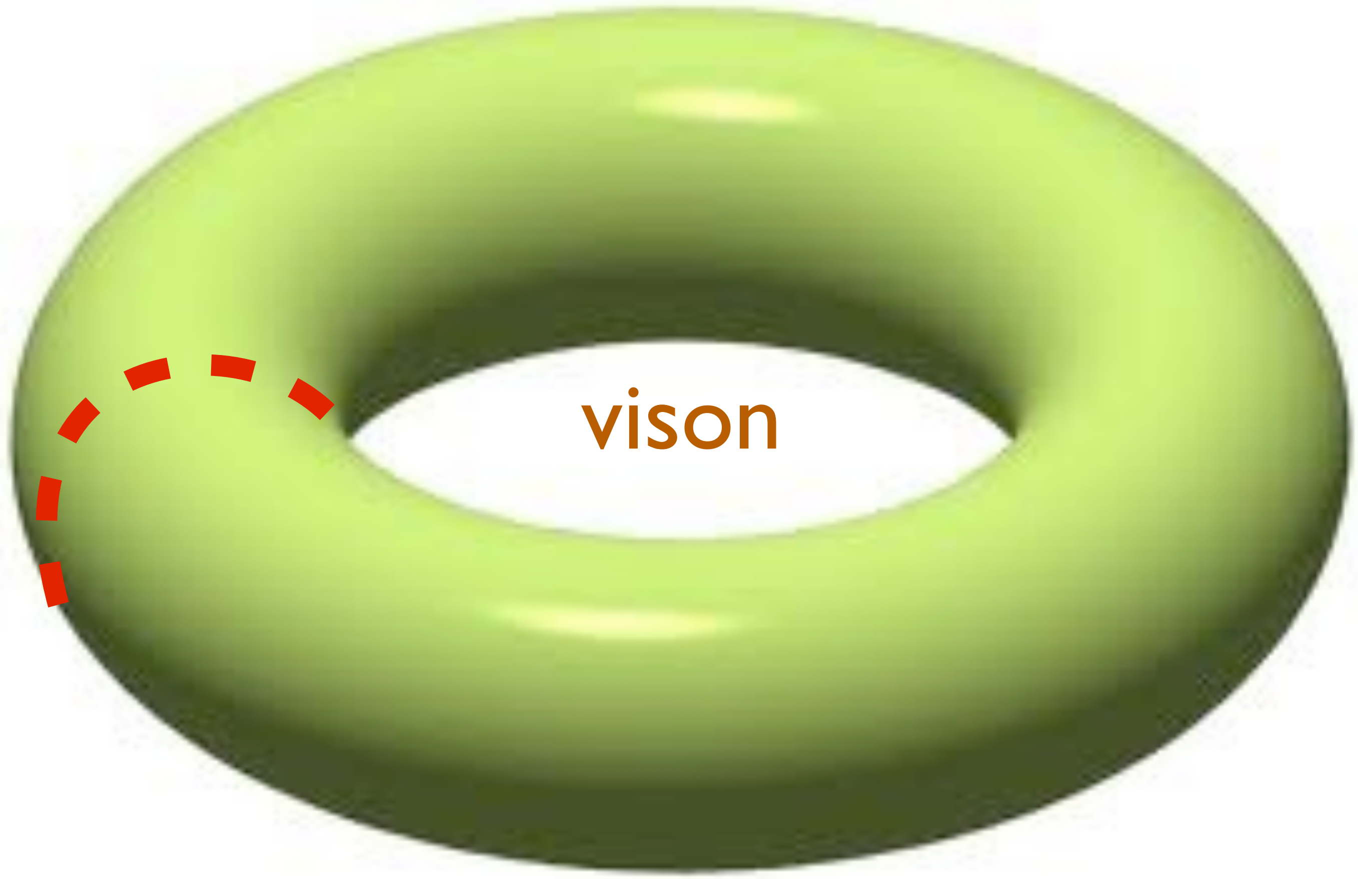
A vison



$$= \frac{1}{\sqrt{2}} \left(\left| \uparrow \downarrow \right\rangle - \left| \downarrow \uparrow \right\rangle \right)$$



Topological order in the Z_2 spin liquid ground state



4-fold degeneracy on the torus

Topological order in the Z_2 spin liquid ground state

These properties of the ground state can be described by effective theories:

- deconfined phase of a Z_2 gauge theory

N. Read and S. Sachdev, *Phys. Rev. Lett.* **66**, 1773 (1991)
T. Senthil and M.P.A. Fisher, *Phys. Rev. B* **63**, 134521 (2001)

- topological doubled Chern-Simons gauge theory

J. Maldacena, G. Moore, and N. Seiberg, JHEP 0110:005 (2001).
M. Freedman, C. Nayak, K. Shtengel, K. Walker, and Z. Wang, *Annals of Physics* **310**, 428 (2004).

Quantum Hall states

Similar topological properties,
but no time-reversal symmetry:

- ground state degeneracy on a torus
- universal entanglement entropy
- gapless edge states on spaces with boundaries
(can also happen for some spin liquids)
- topological Chern-Simons gauge theories

Gapped quantum matter

Insulators, quantum Hall states

Compressible quantum matter

Metals, superconductors, strange metals

Conformal quantum matter

Graphene, antiferromagnets, ultracold atoms

Gapped quantum matter

Insulators, quantum Hall states

Compressible quantum matter

Metals, superconductors, strange metals

Conformal quantum matter

Graphene, antiferromagnets, ultracold atoms

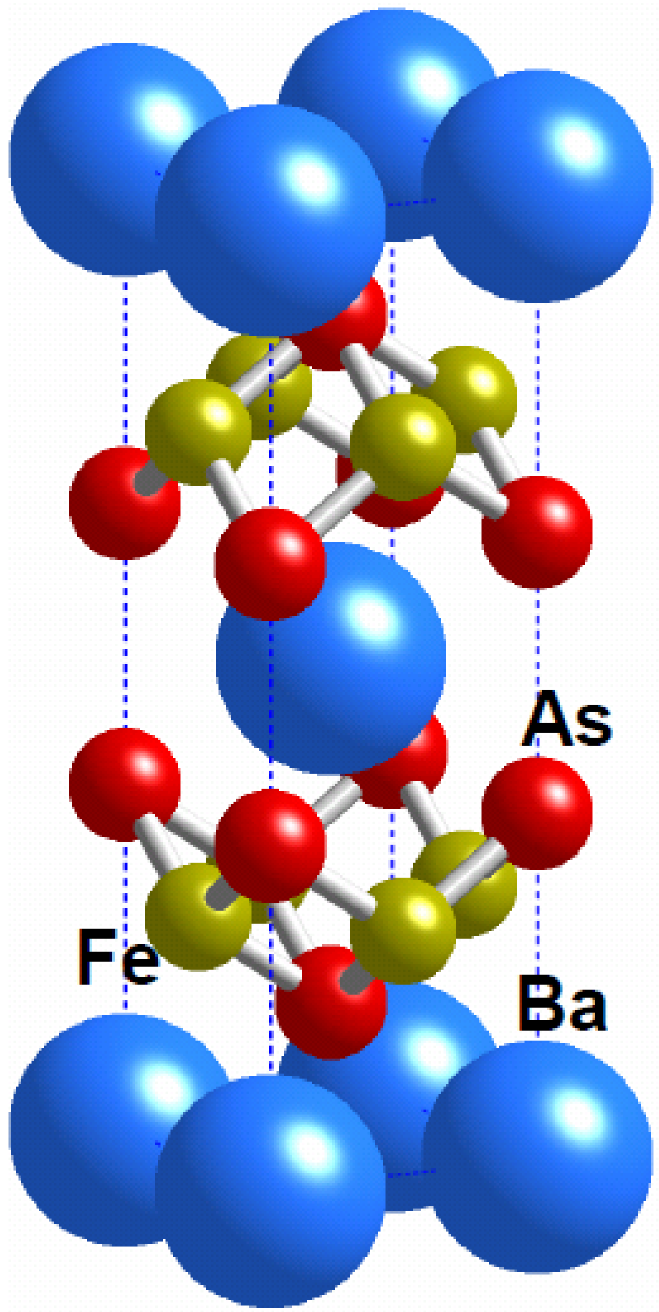
Liza Huijse

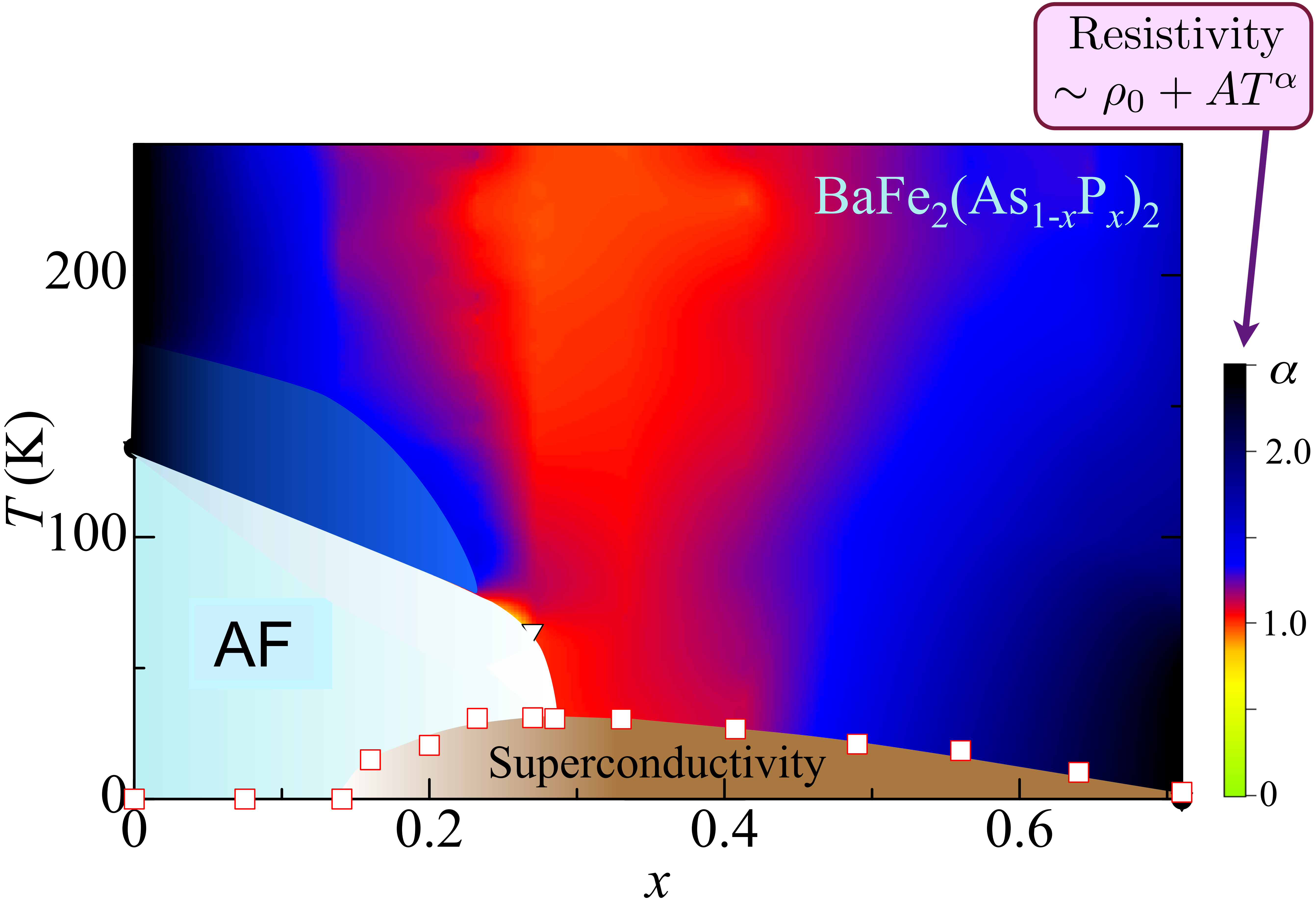
Max Metlitski

Brian Swingle

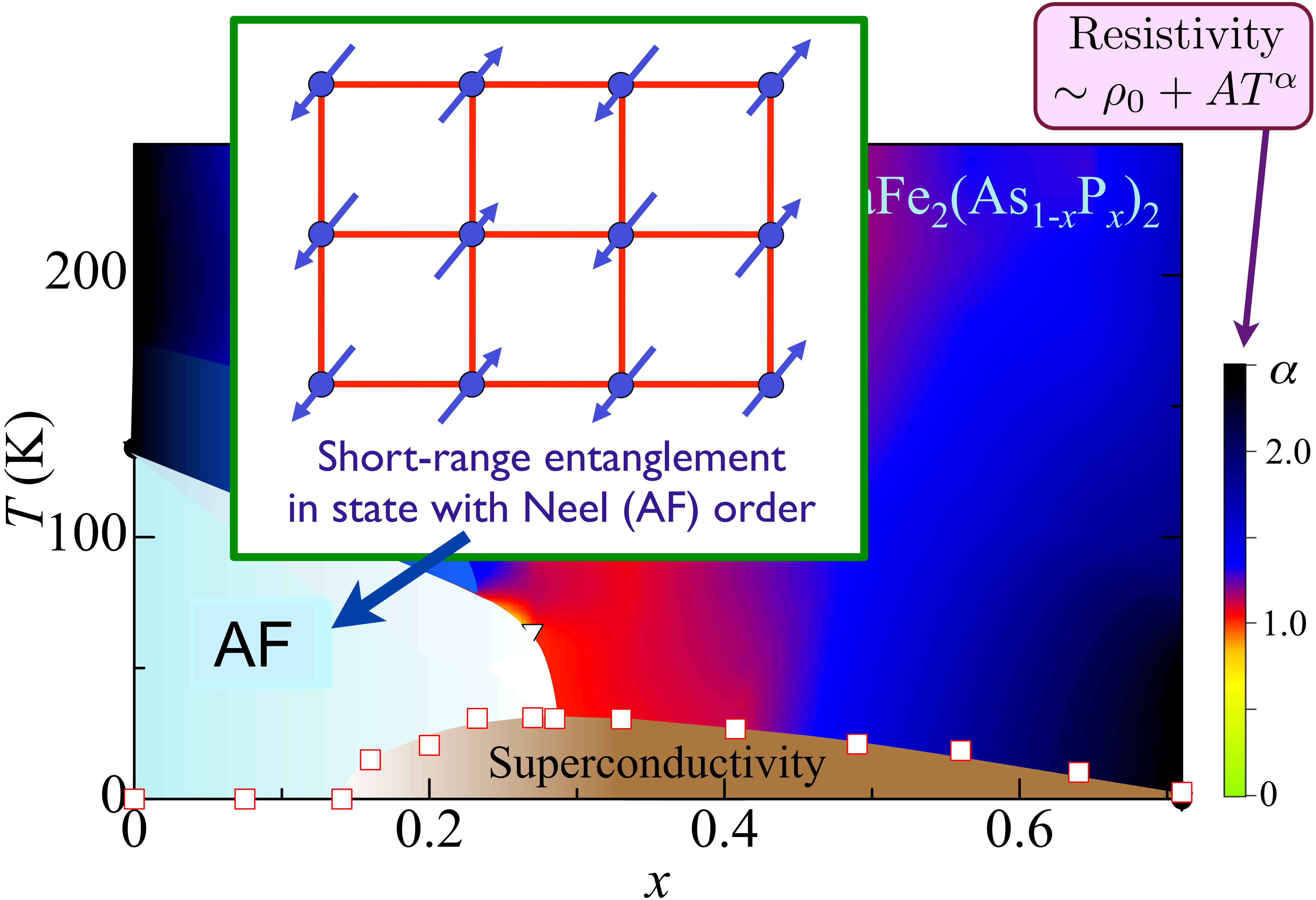
Iron pnictides:

a new class of high temperature superconductors

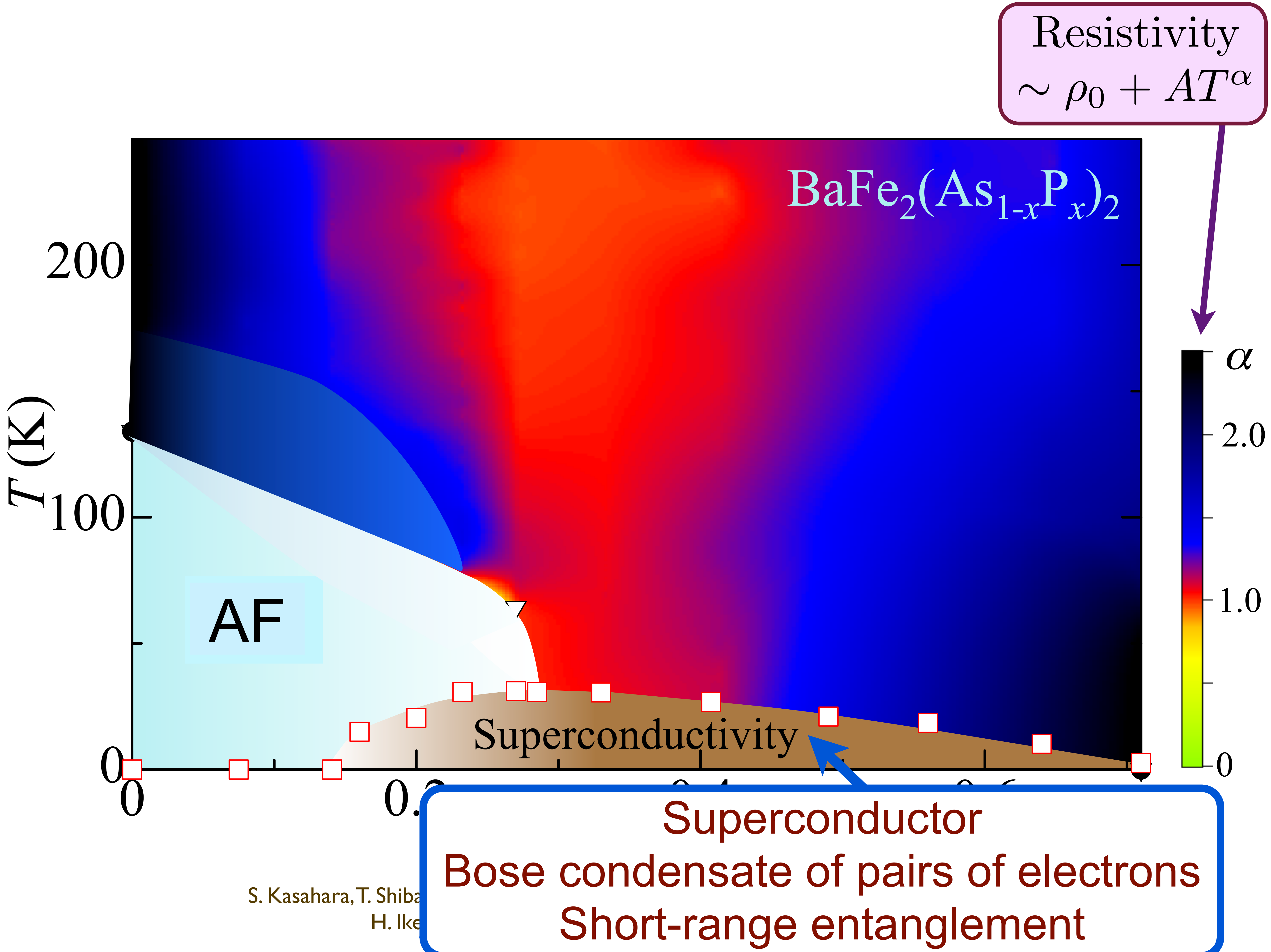


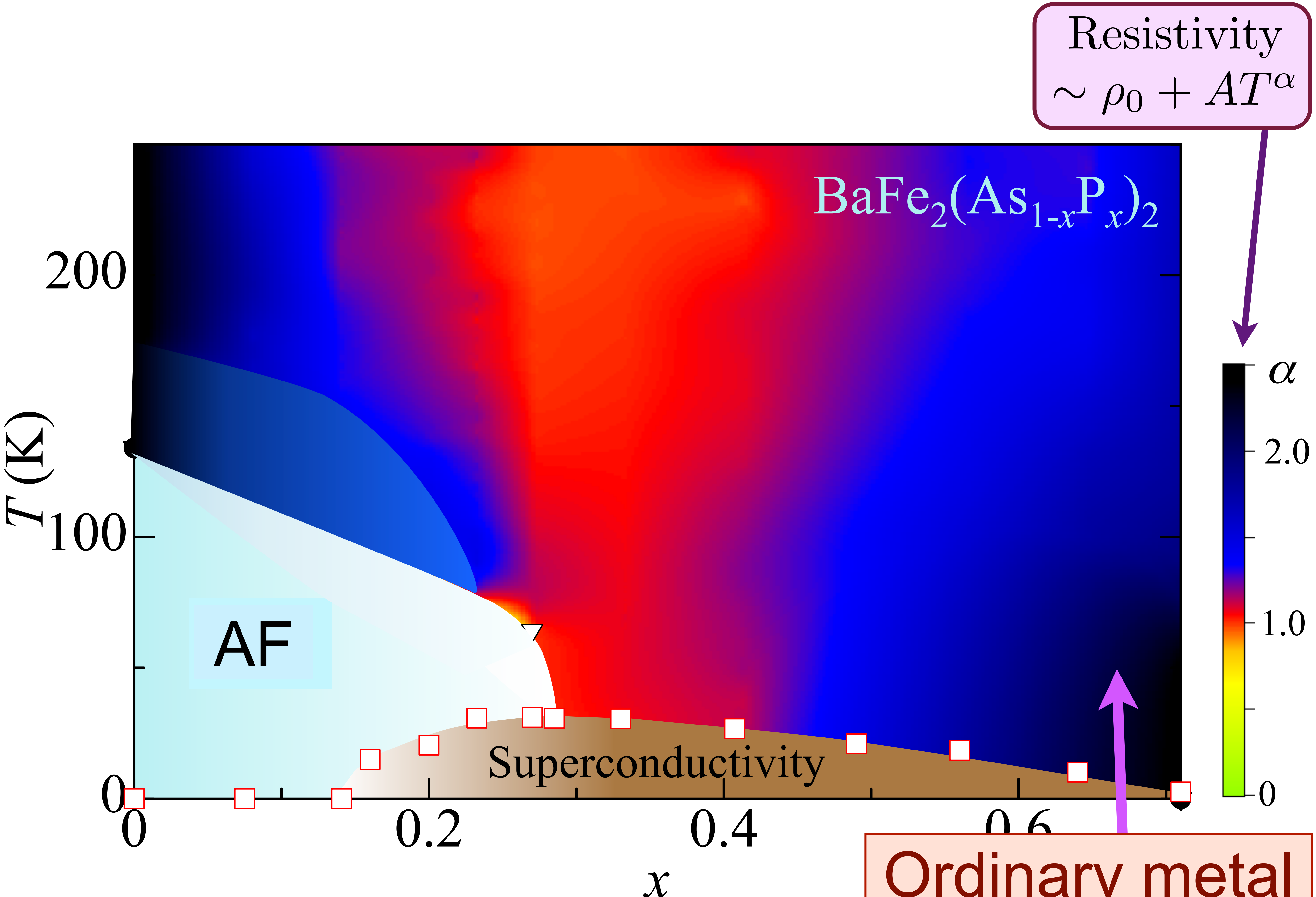


S. Kasahara, T. Shibauchi, K. Hashimoto, K. Ikada, S. Tonegawa, R. Okazaki, H. Shishido, H. Ikeda, H. Takeya, K. Hirata, T. Terashima, and Y. Matsuda, *Physical Review B* **81**, 184519 (2010)



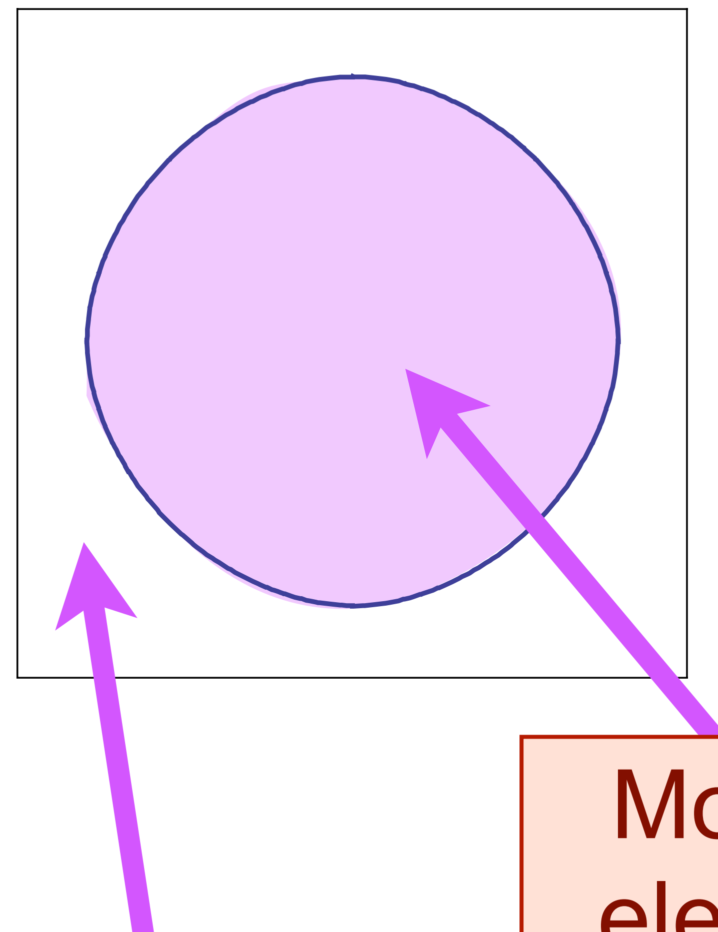
S. Kasahara, T. Shibauchi, K. Hashimoto, K. Ikada, S. Tonegawa, R. Okazaki, H. Shishido, H. Ikeda, H. Takeya, K. Hirata, T. Terashima, and Y. Matsuda, *Physical Review B* **81**, 184519 (2010)


Resistivity
$\sim \rho_0 + AT^\alpha$
$BaFe_2(As_{1-x}P_x)_2$
α
2.0
1.0
0
200
100
0
T (K)
AF
Superconductivity
Superconductor
Bose condensate of pairs of electrons
Short-range entanglement

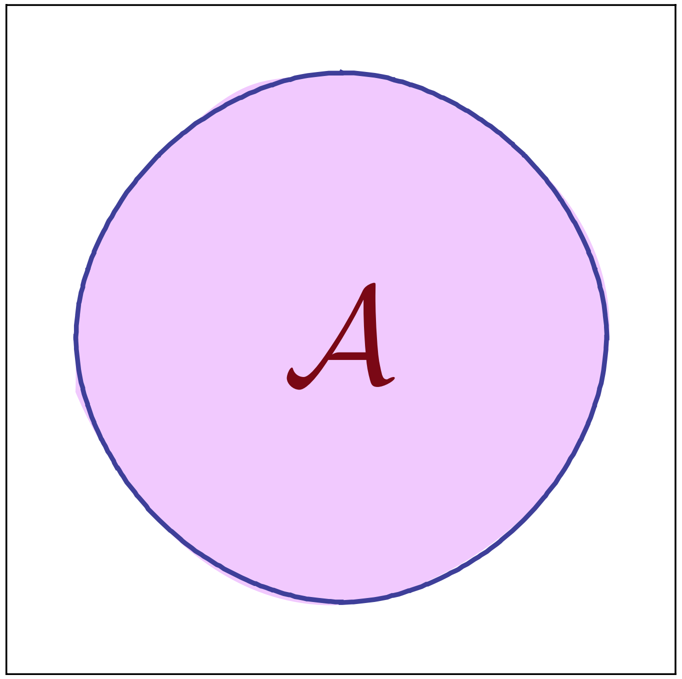


S. Kasahara, T. Shibauchi, K. Hashimoto, K. Ikada, S. Tonegawa, R. O
H. Ikeda, H. Takeya, K. Hirata, T. Terashima, and Y. Mat
Physical Review B **81**, 184519 (2010)

Sommerfeld-Bloch-Landau theory of ordinary metals

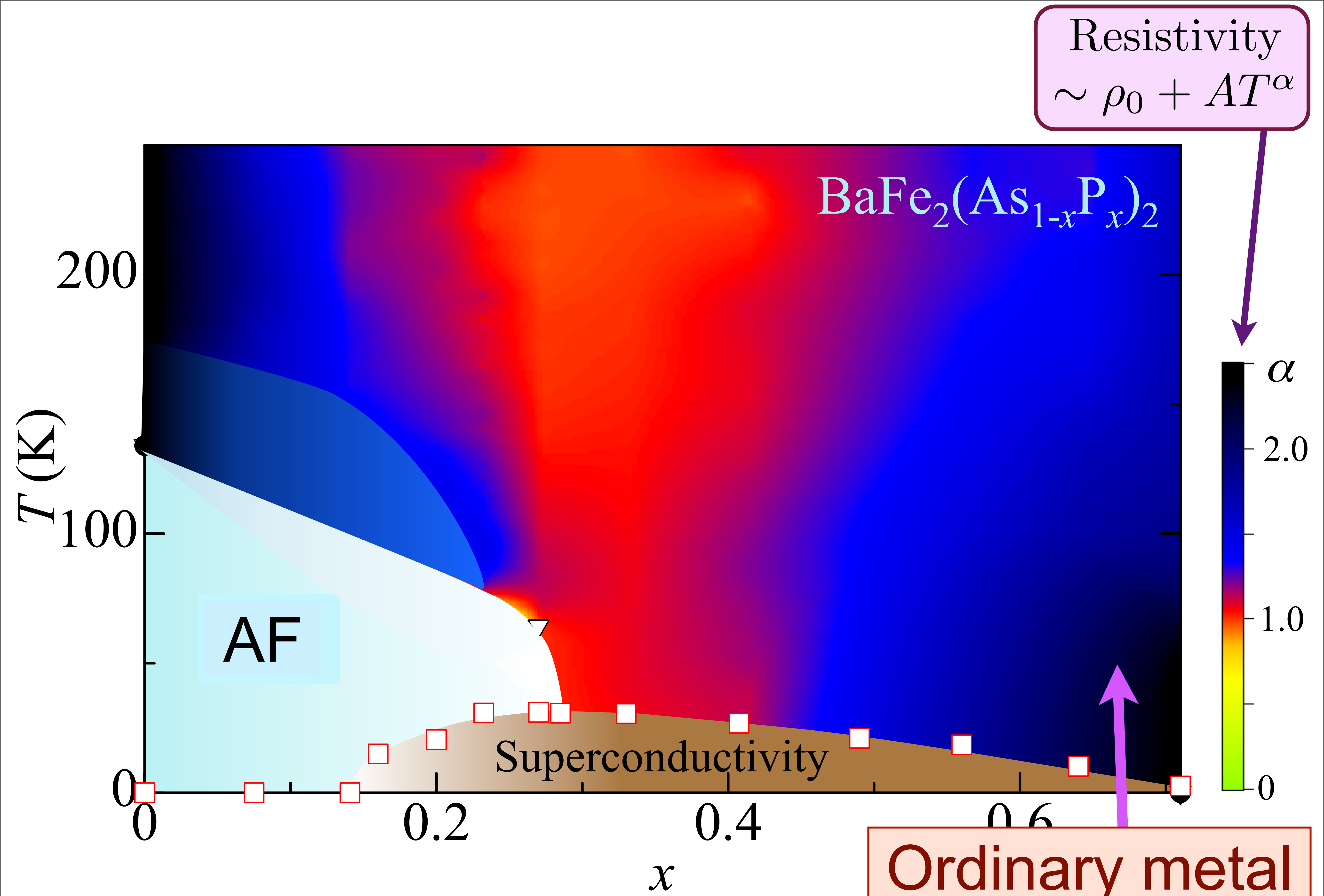


Sommerfeld-Bloch-Landau theory of ordinary metals

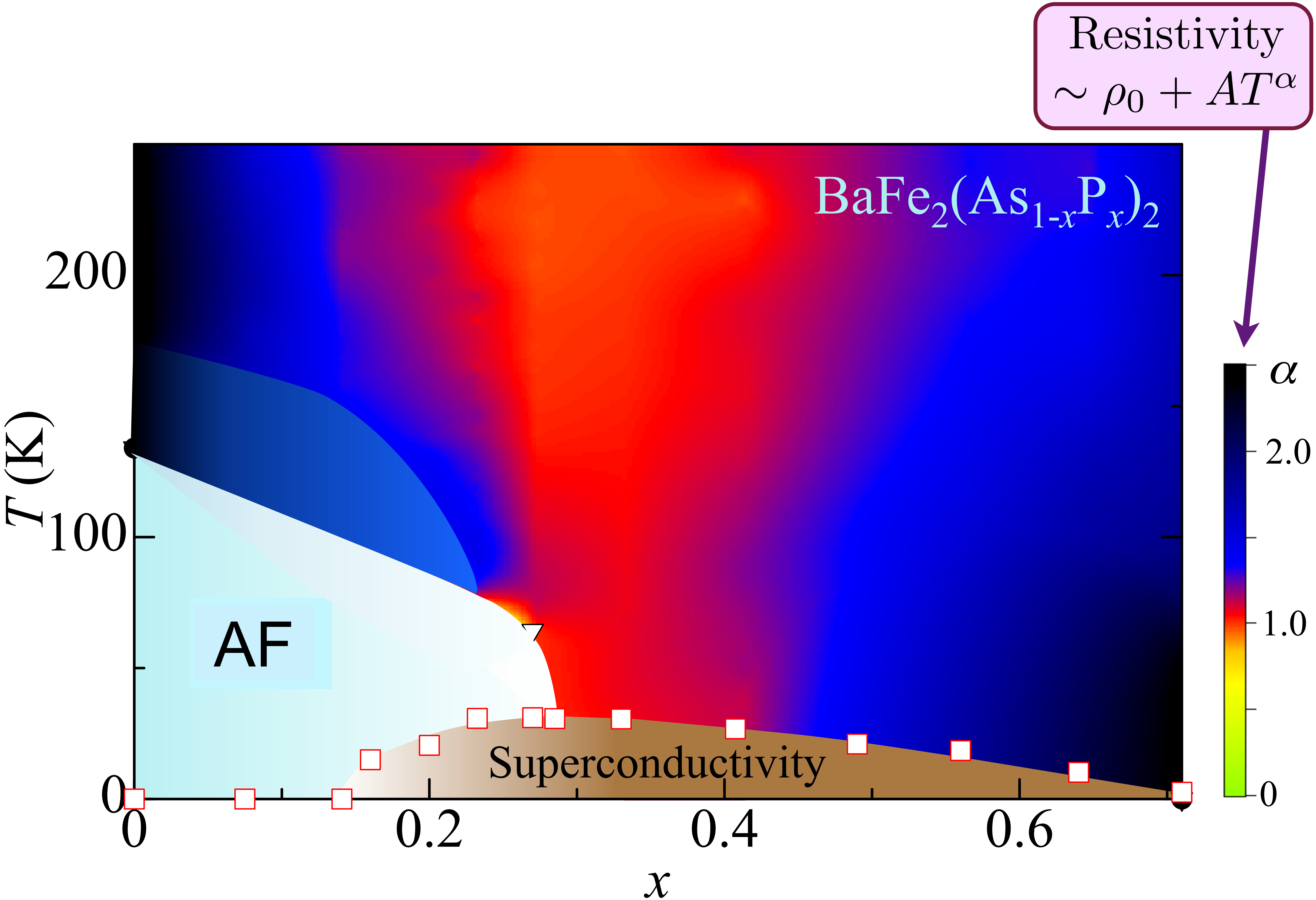


Key feature of the theory:
the **Fermi surface**

- Area enclosed by the Fermi surface $\mathcal{A} = \mathcal{Q}$, the electron density
- Excitations near the Fermi surface are responsible for the familiar properties of ordinary metals, such as resistivity $\sim T^2$.

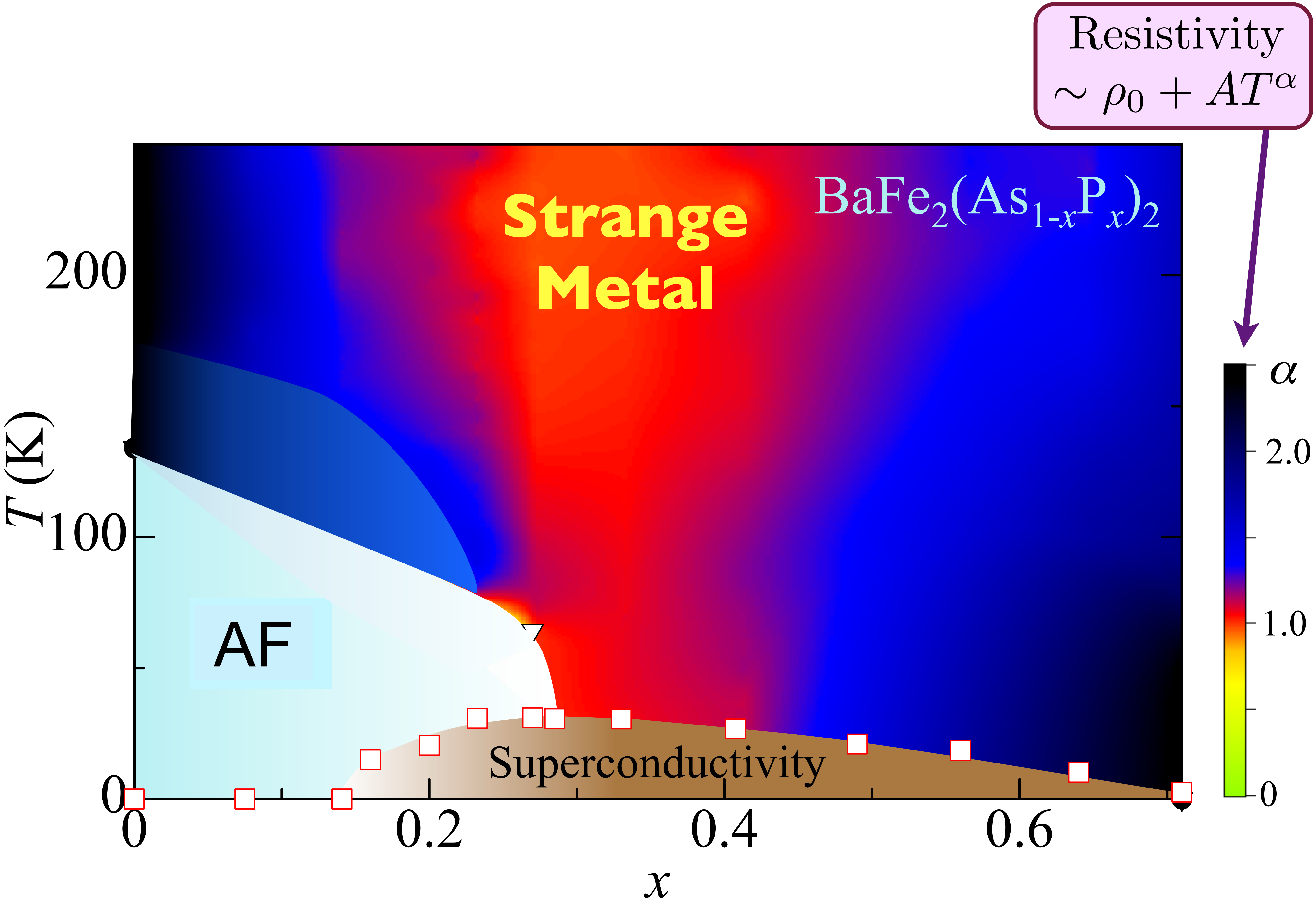


S. Kasahara, T. Shibauchi, K. Hashimoto, K. Ikada, S. Tonegawa, R. O
H. Ikeda, H. Takeya, K. Hirata, T. Terashima, and Y. Mat
Physical Review B **81**, 184519 (2010)



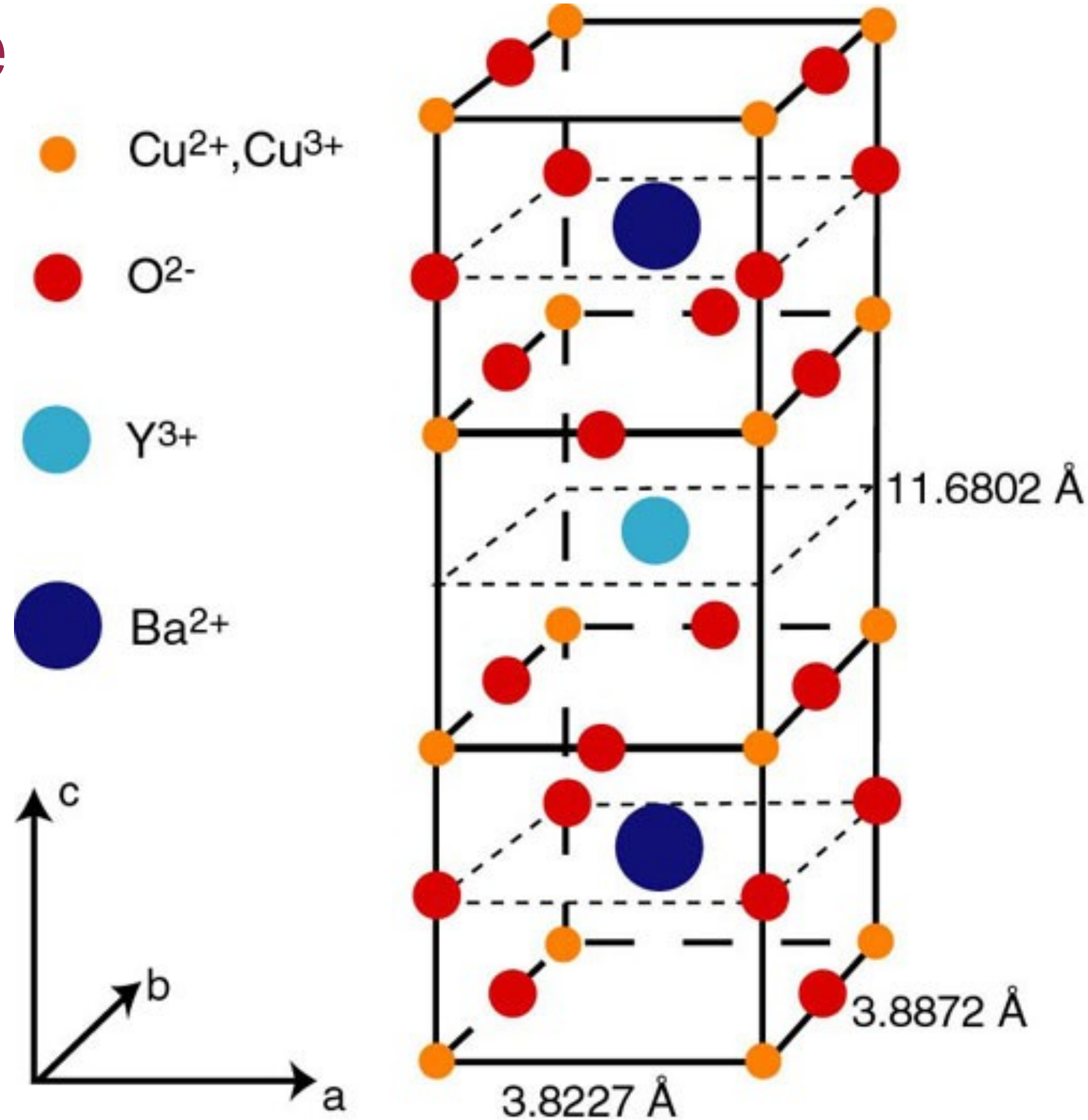
S. Kasahara, T. Shibauchi, K. Hashimoto, K. Ikada, S. Tonegawa, R. Okazaki, H. Shishido, H. Ikeda, H. Takeya, K. Hirata, T. Terashima, and Y. Matsuda, *Physical Review B* **81**, 184519 (2010)

Saturday, April 21, 2012



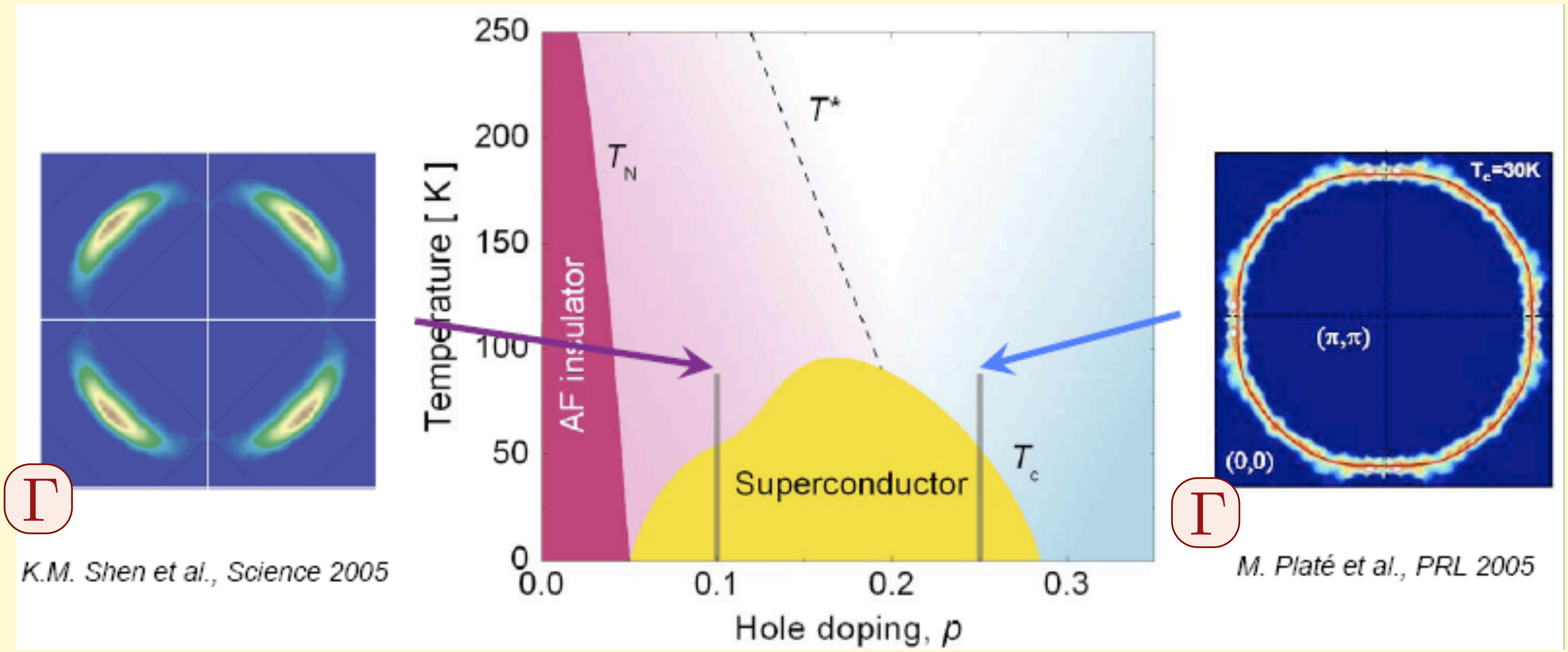
S. Kasahara, T. Shibauchi, K. Hashimoto, K. Ikada, S. Tonegawa, R. Okazaki, H. Shishido, H. Ikeda, H. Takeya, K. Hirata, T. Terashima, and Y. Matsuda, *Physical Review B* **81**, 184519 (2010)

High temperature superconductors



$$\mathrm{YBa_2Cu_3O}_{6+x}$$

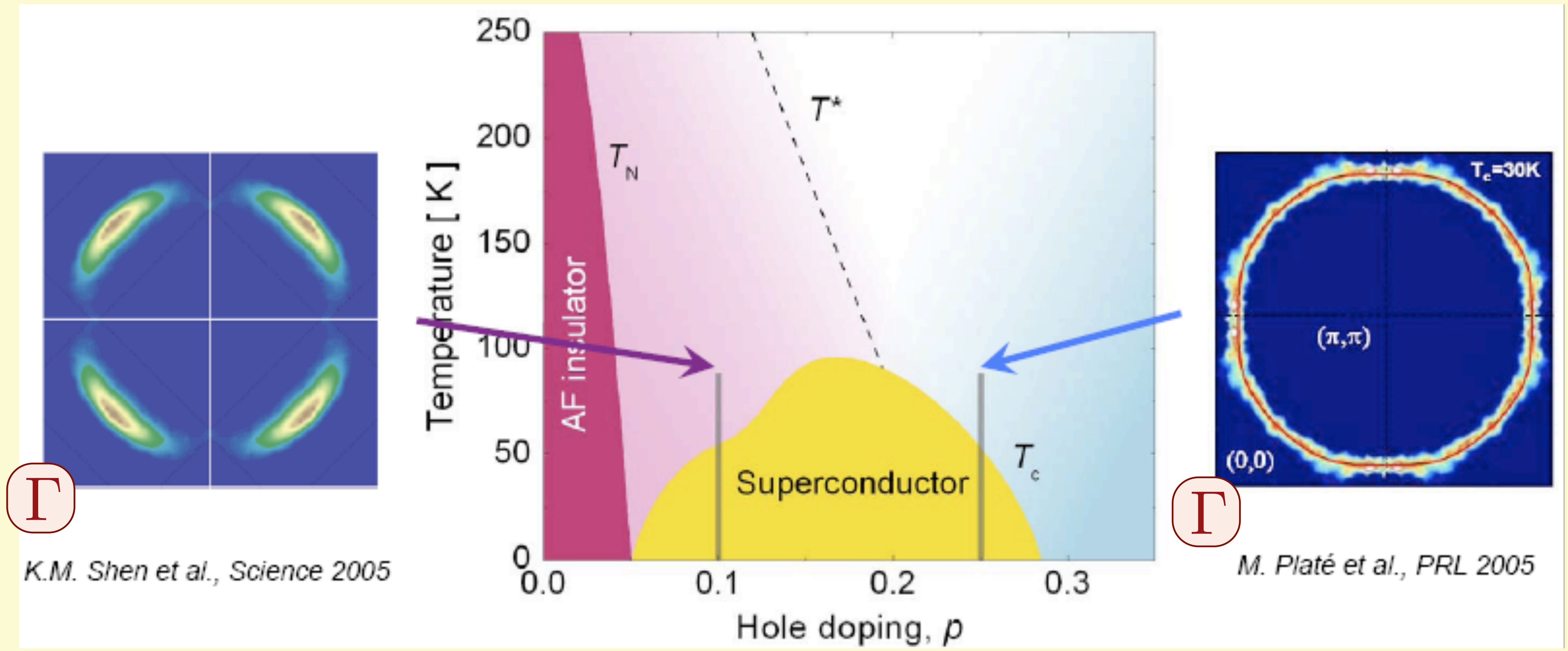
Evolution of the (ARPES) Fermi surface on the cuprate phase diagram



Smaller hole Fermi-pockets

Large hole Fermi surface

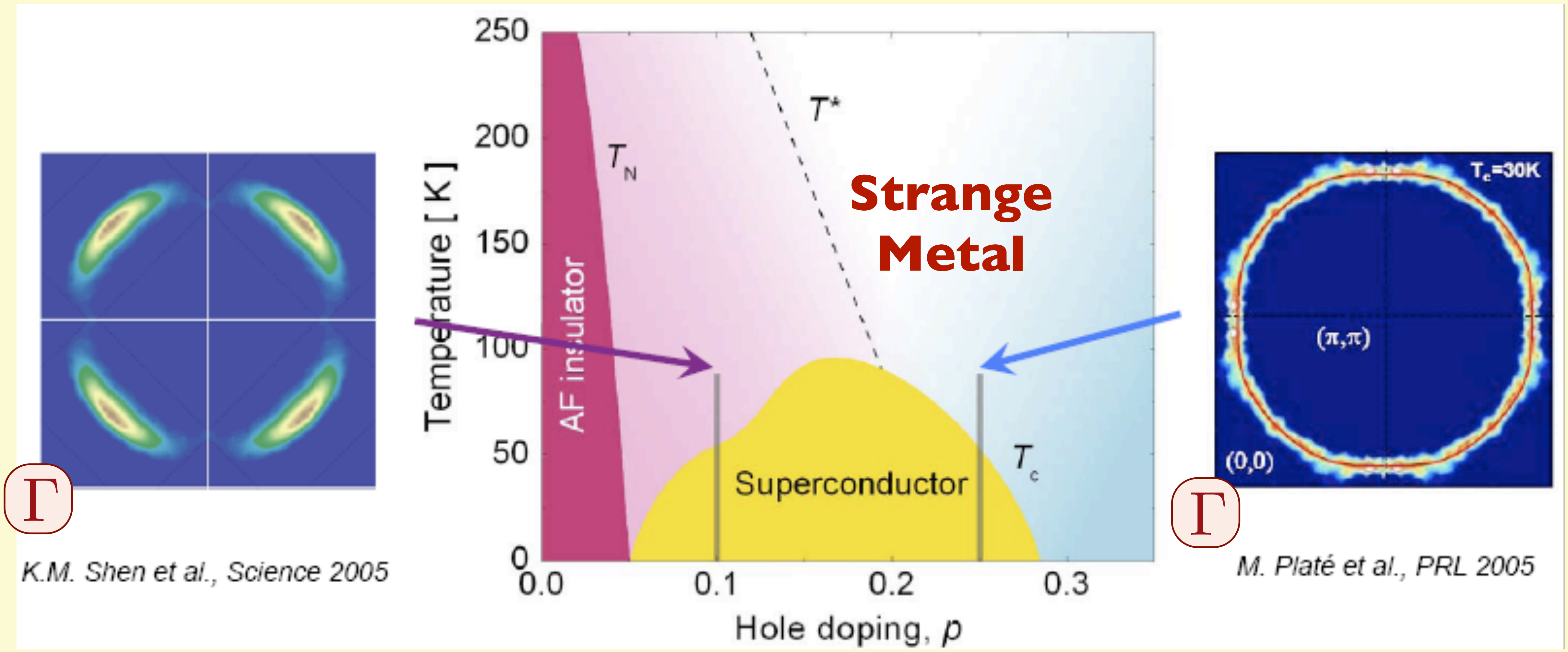
Evolution of the (ARPES) Fermi surface on the cuprate phase diagram



Smaller hole Fermi-pockets

Large hole Fermi surface

Evolution of the (ARPES) Fermi surface on the cuprate phase diagram



Smaller hole Fermi-pockets

Large hole Fermi surface

Compressible quantum matter

- Consider an infinite, continuum, translationally-invariant quantum system with a globally conserved U(1) charge $\mathcal{Q}$ (the "electron density") in spatial dimension $d > 1$.

Compressible quantum matter

- Consider an infinite, continuum, translationally-invariant quantum system with a globally conserved U(1) charge $\mathcal{Q}$ (the "electron density") in spatial dimension $d > 1$.

- Describe *zero temperature* phases where $d\langle \mathcal{Q} \rangle / d\mu \neq 0$, where μ (the "chemical potential") which changes the Hamiltonian, H, to $H - \mu\mathcal{Q}$.

Compressible quantum matter

The only compressible phase of traditional condensed matter physics which does not break the translational or $\mathrm{U}(1)$ symmetries is the Landau Fermi liquid

Compressible quantum matter

Compressible quantum matter

First challenge to string theory:

Classify and understand non-Fermi liquid phases of compressible quantum matter, *i.e.* strange metals

Strange metals

A. Field theory

B. Holography

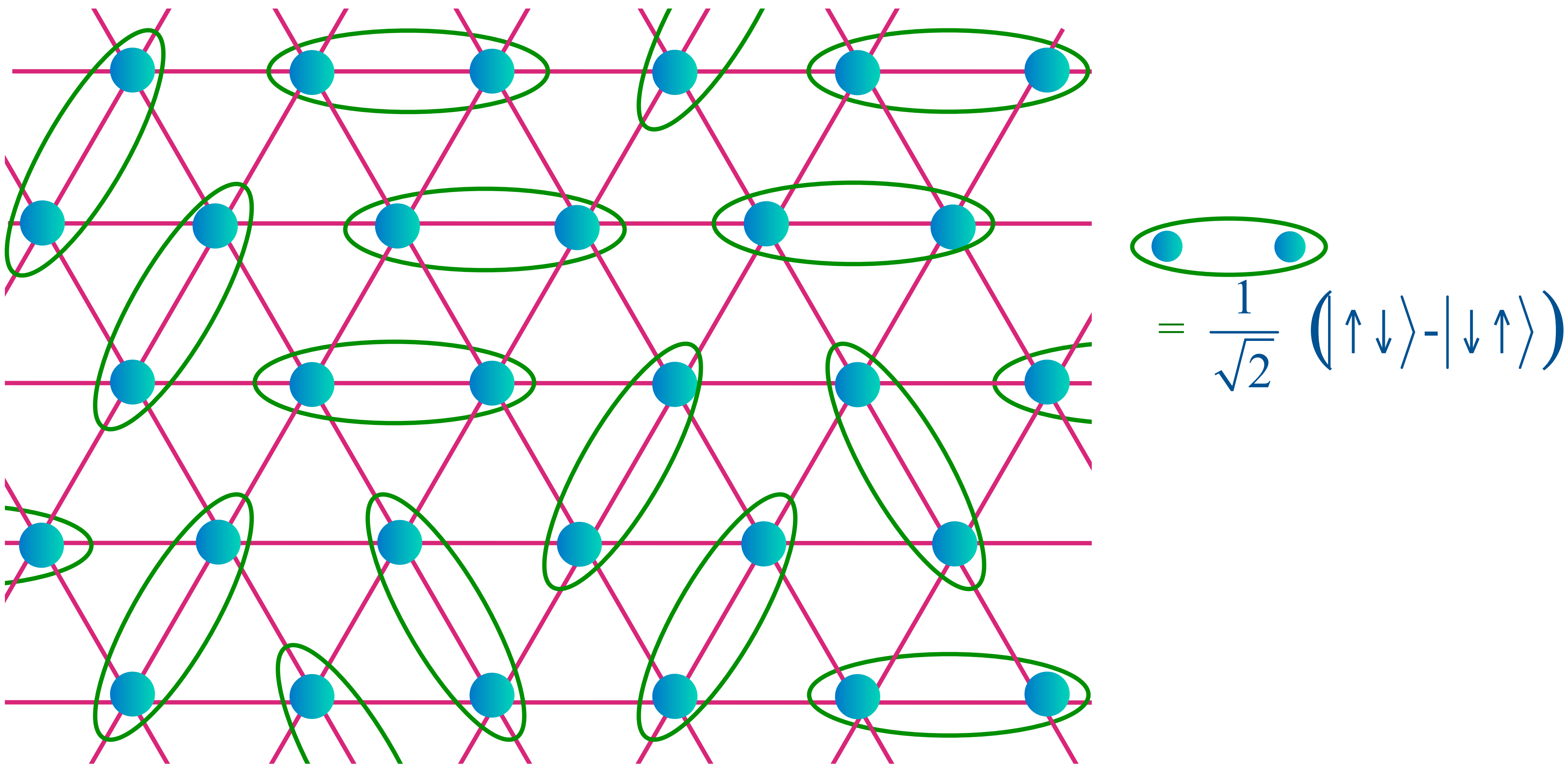
Strange metals

$\boxed{\textit{A. Field theory}}$

B. Holography

The Non-Fermi Liquid (NFL)

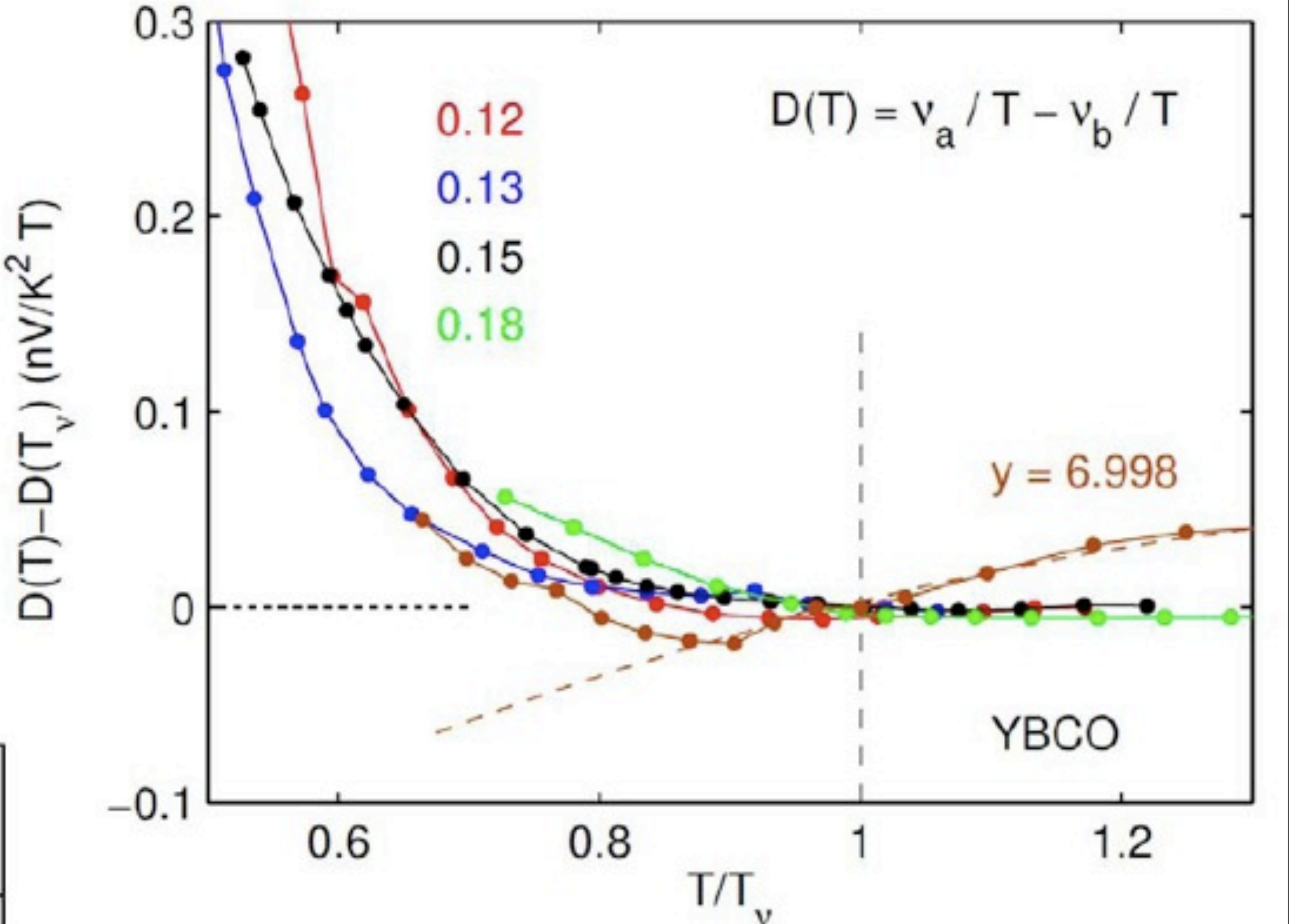
- Model of a spin liquid ("Bose metal"): couple fermions to a dynamical gauge field A_μ.

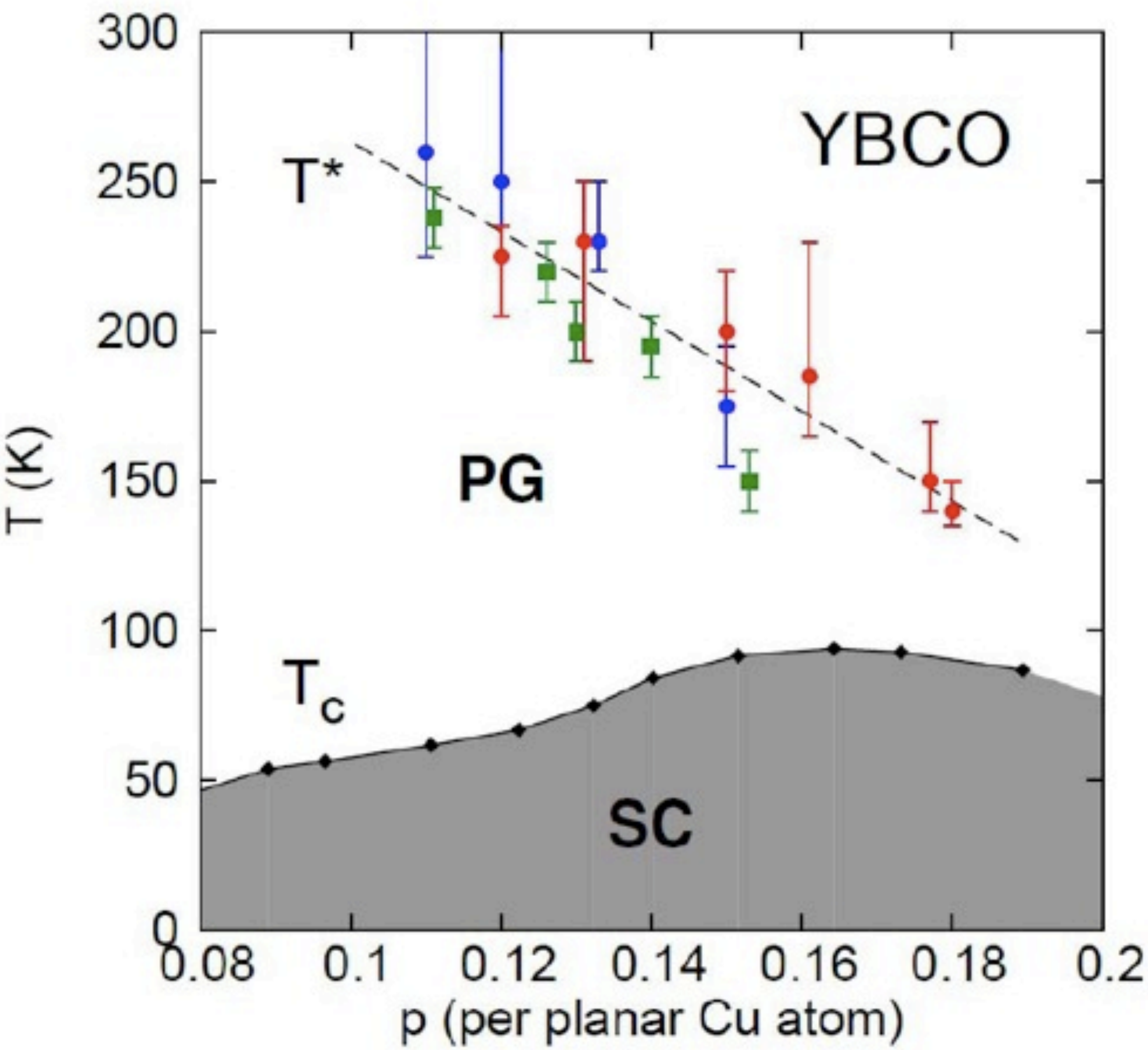


$$\mathcal{L} = f_\sigma^\dagger \left(\partial_\tau - iA_\tau - \frac{(\nabla - i\mathbf{A})^2}{2m} - \mu \right) f_\sigma$$

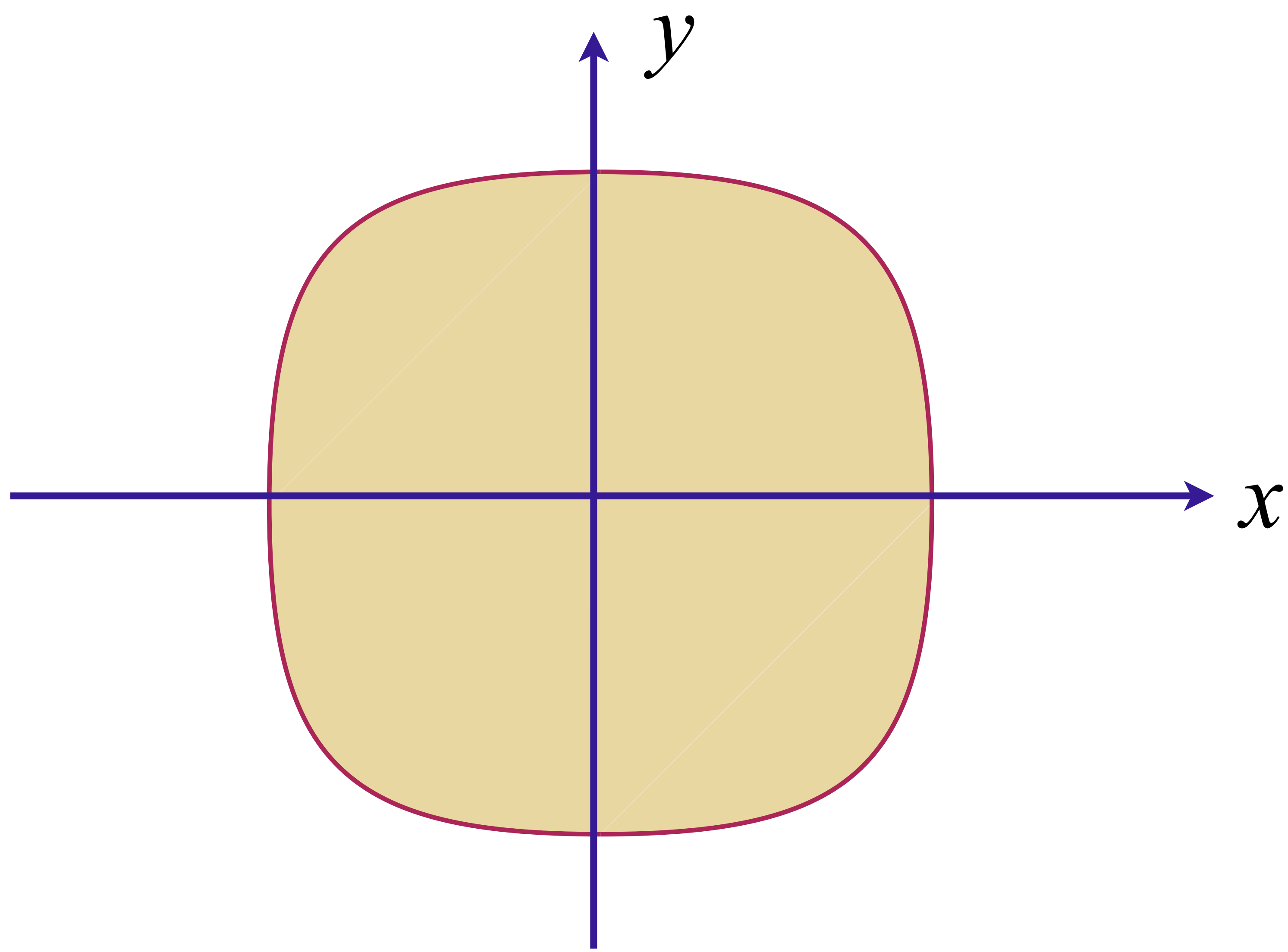
Broken rotational symmetry in the pseudogap phase of a high-Tc superconductor

R. Daou, J. Chang, David LeBoeuf, Olivier Cyr-Choiniere, Francis Laliberte, Nicolas Doiron-Leyraud, B. J. Ramshaw, Ruixing Liang, D.A. Bonn, W. N. Hardy, and Louis Taillefer
Nature, **463**, 519 (2010).



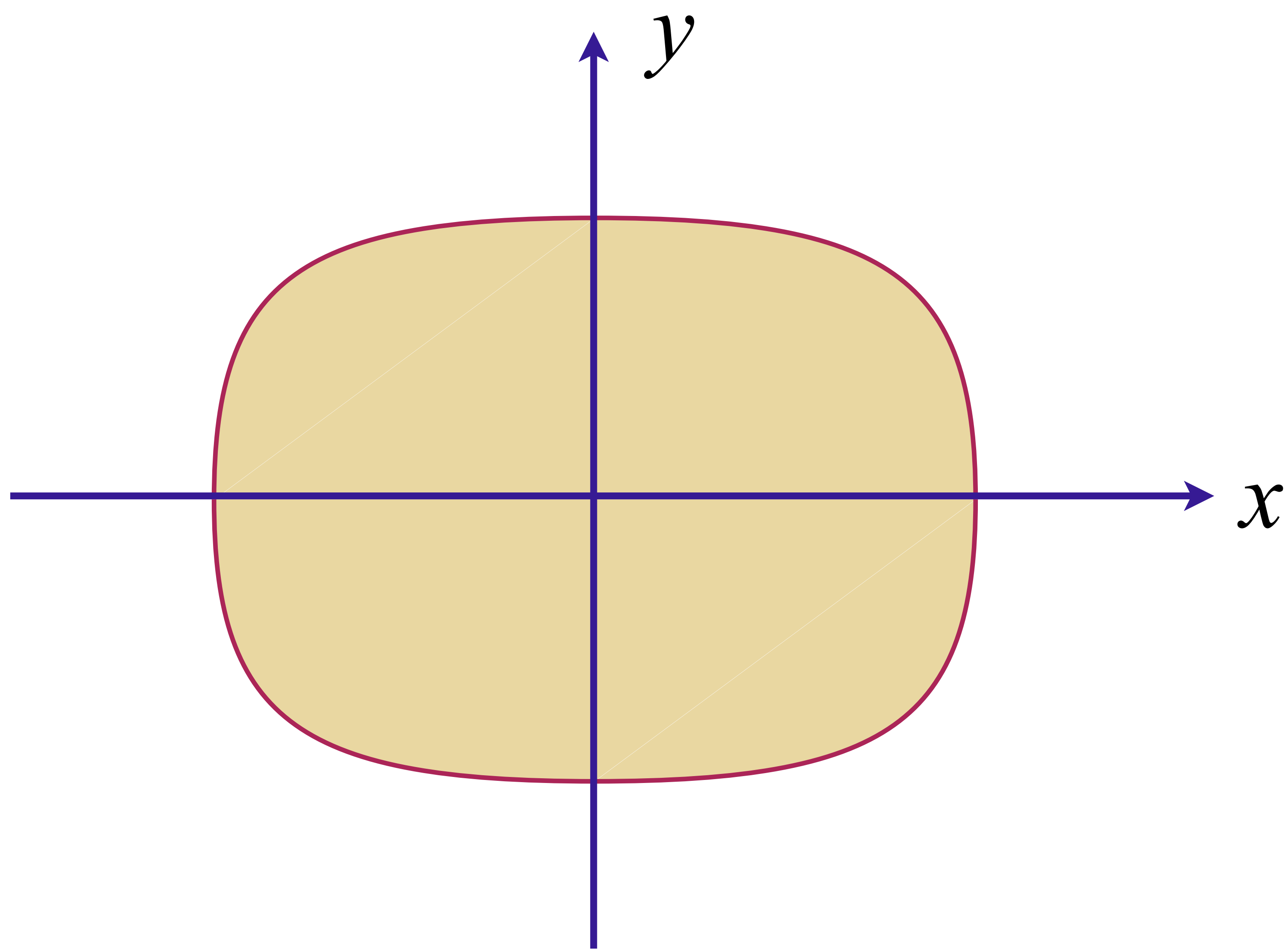


Quantum criticality of Ising-nematic ordering



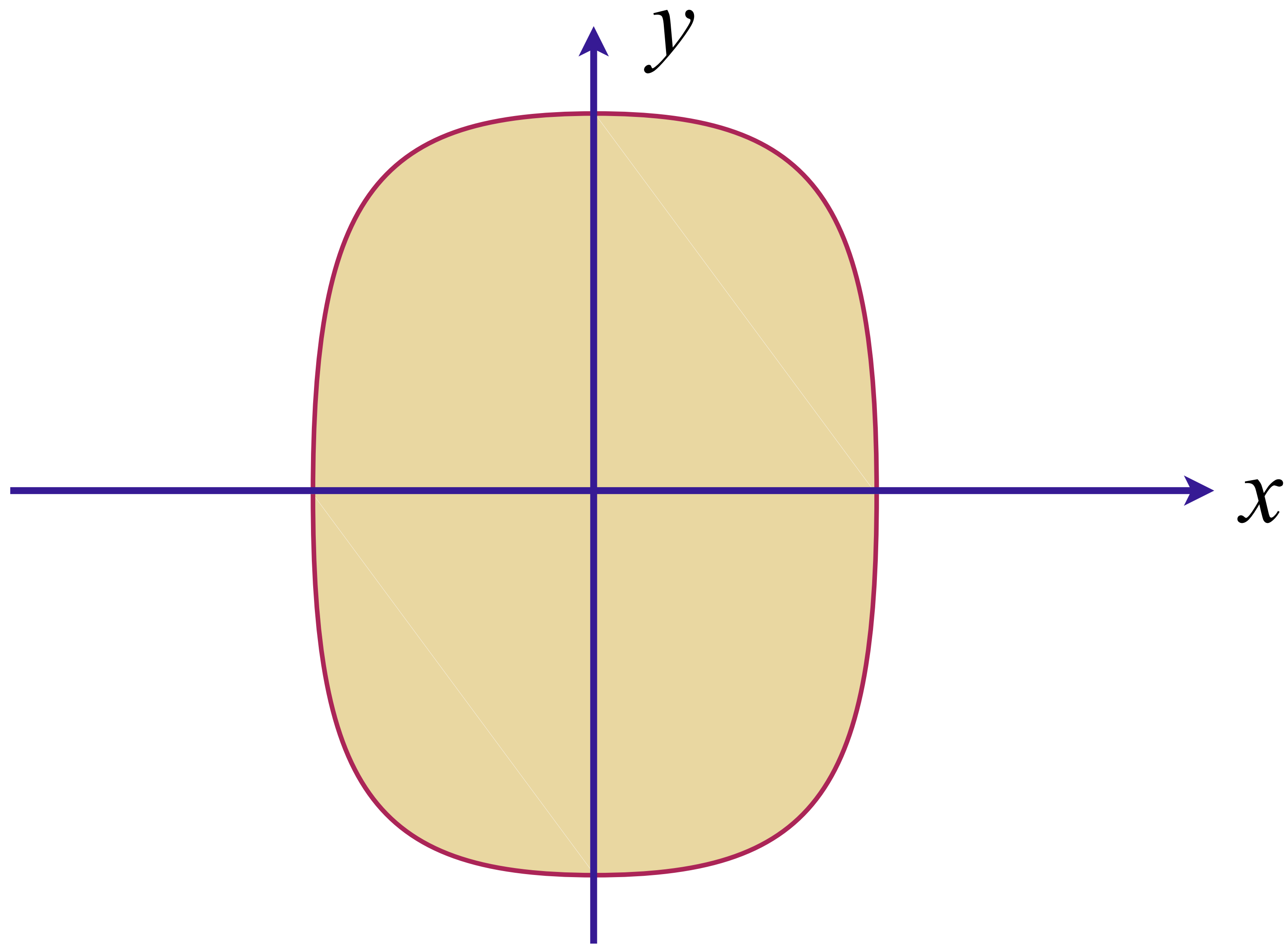
Fermi surface with full square lattice symmetry

Quantum criticality of Ising-nematic ordering



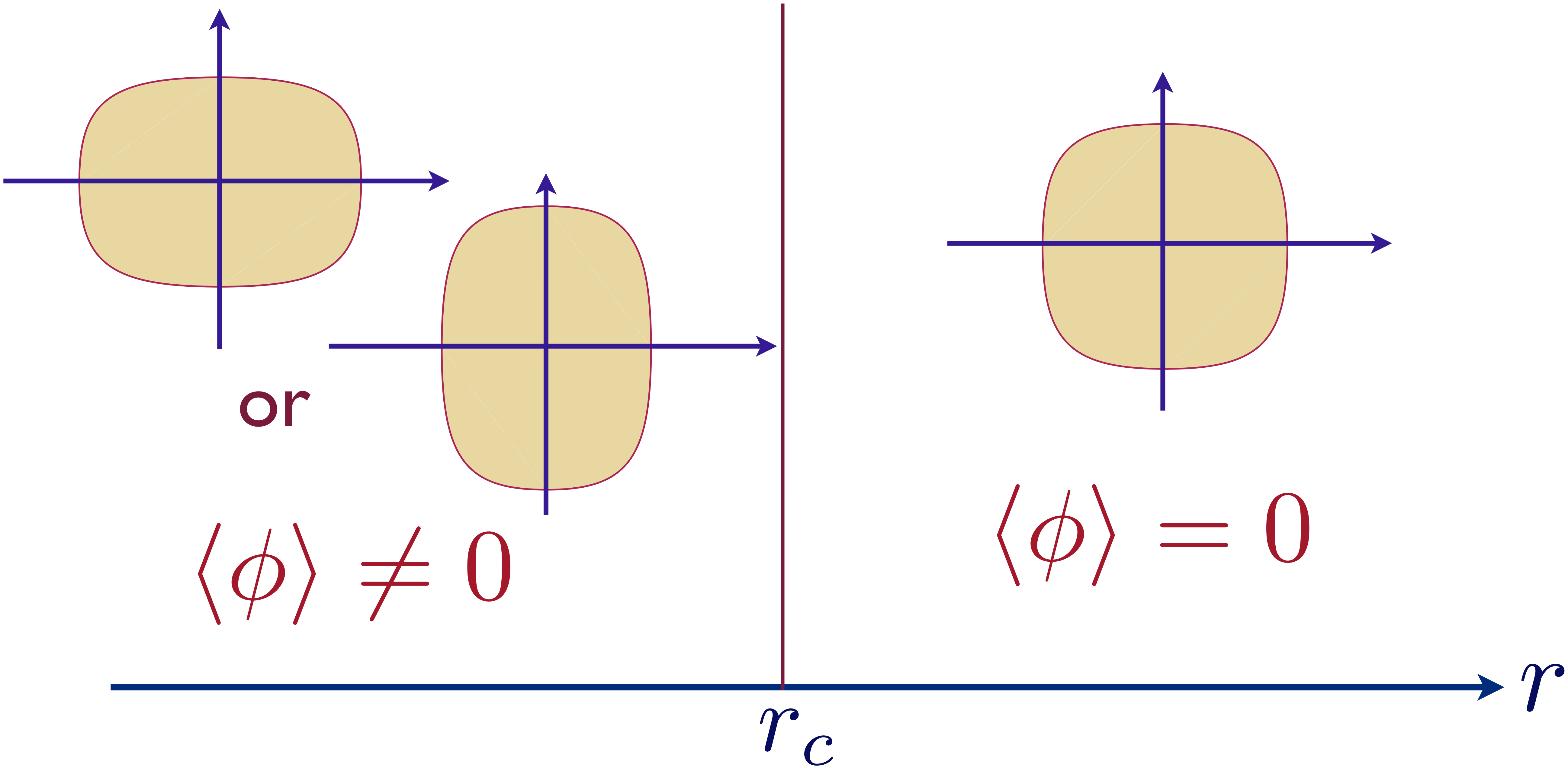
Spontaneous elongation along x direction:
Ising order parameter $\phi > 0$.

Quantum criticality of Ising-nematic ordering



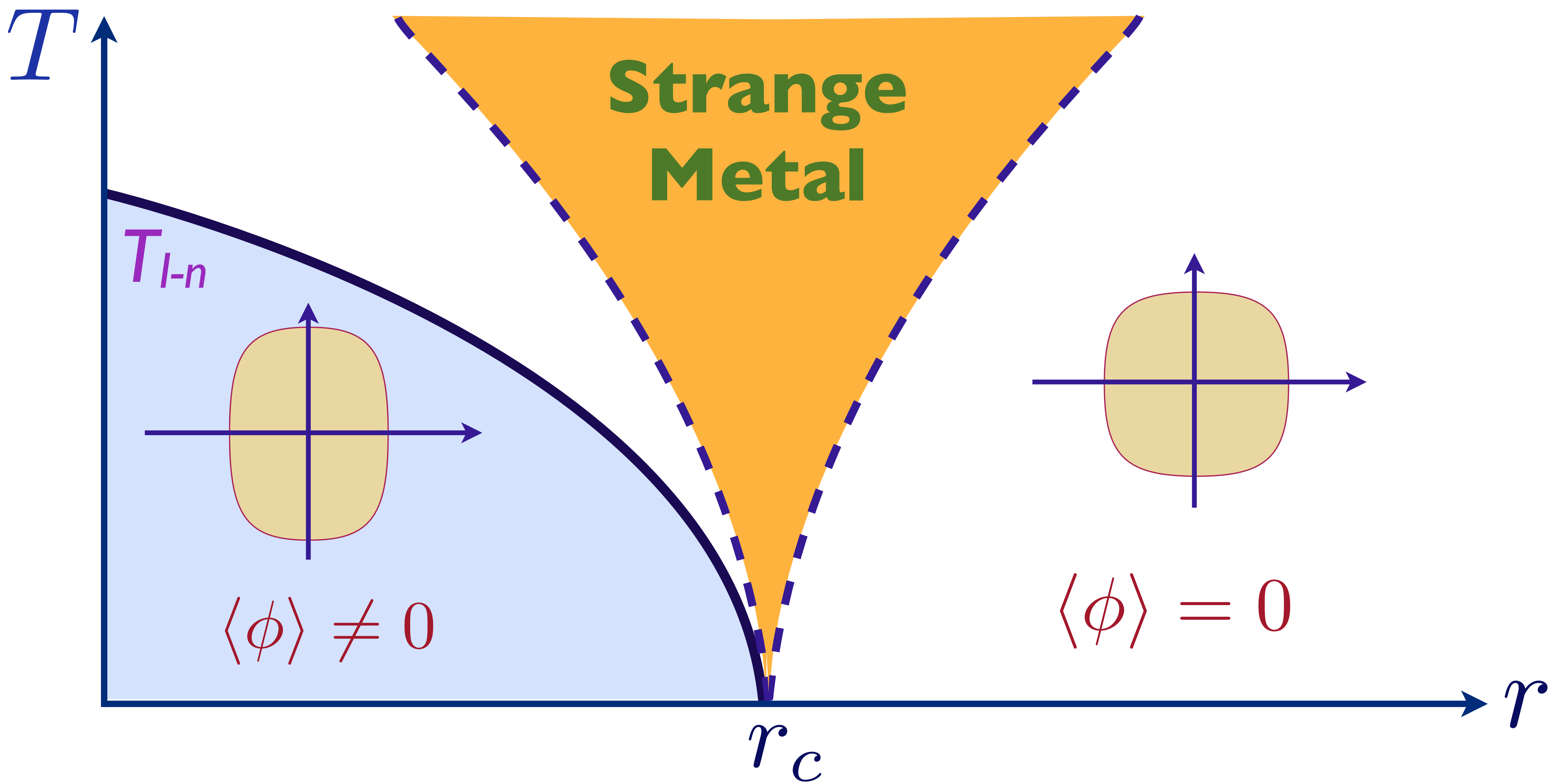
Spontaneous elongation along y direction:
Ising order parameter $\phi < 0$.

Quantum criticality of Ising-nematic ordering



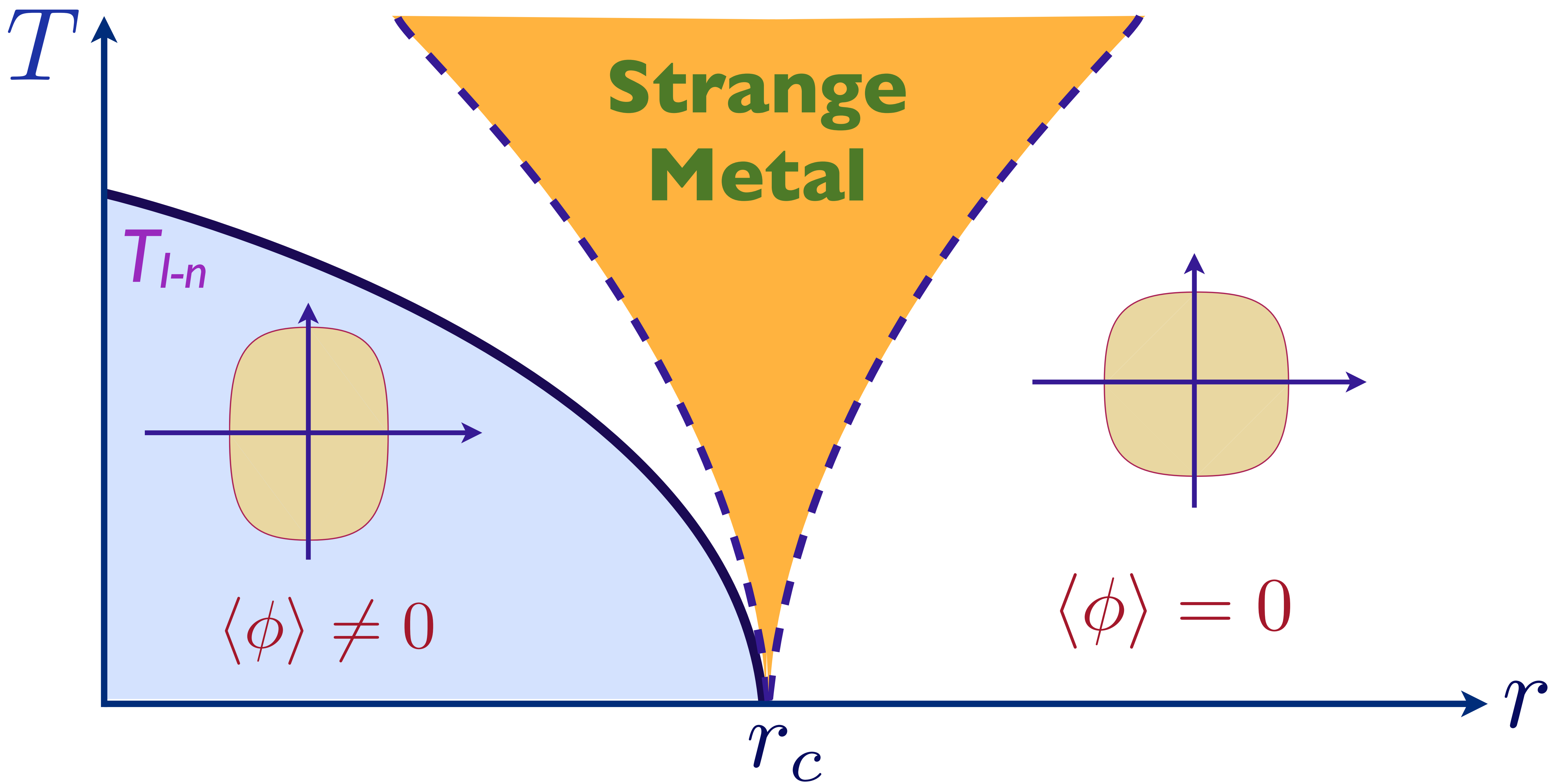
Pomeranchuk instability as a function of coupling r

Quantum criticality of Ising-nematic ordering



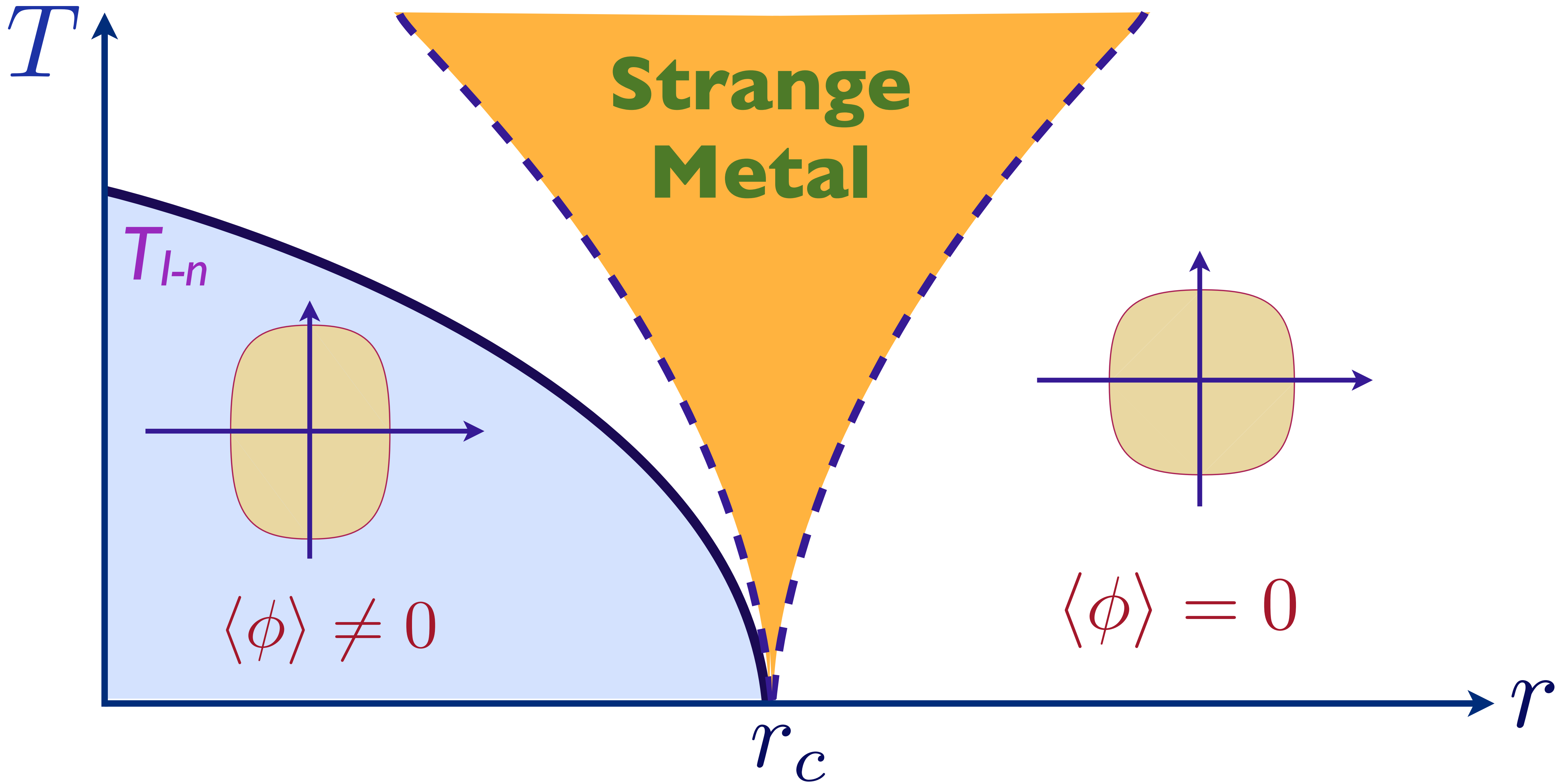
Phase diagram as a function of T and r

Quantum criticality of Ising-nematic ordering



Phase diagram as a function of T and r

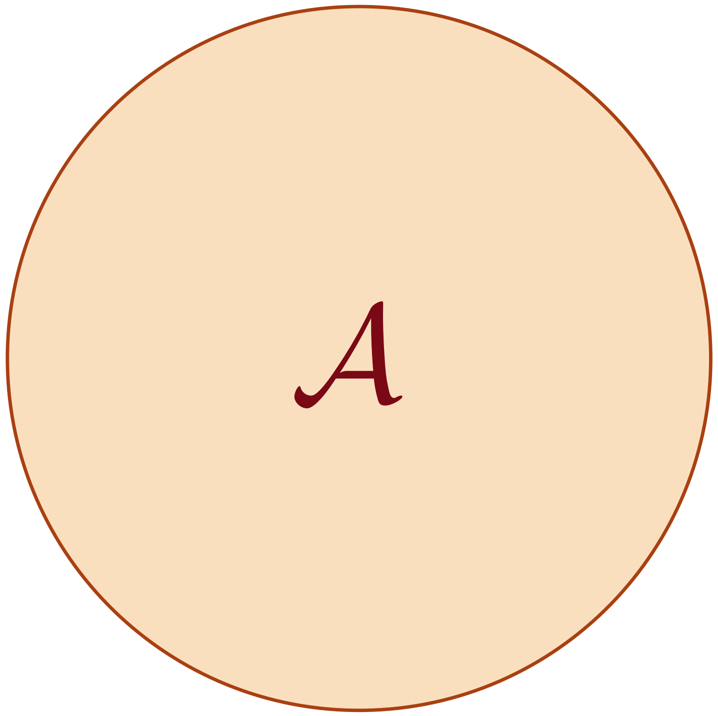
Quantum criticality of Ising-nematic ordering



Phase diagram as a function of T and r

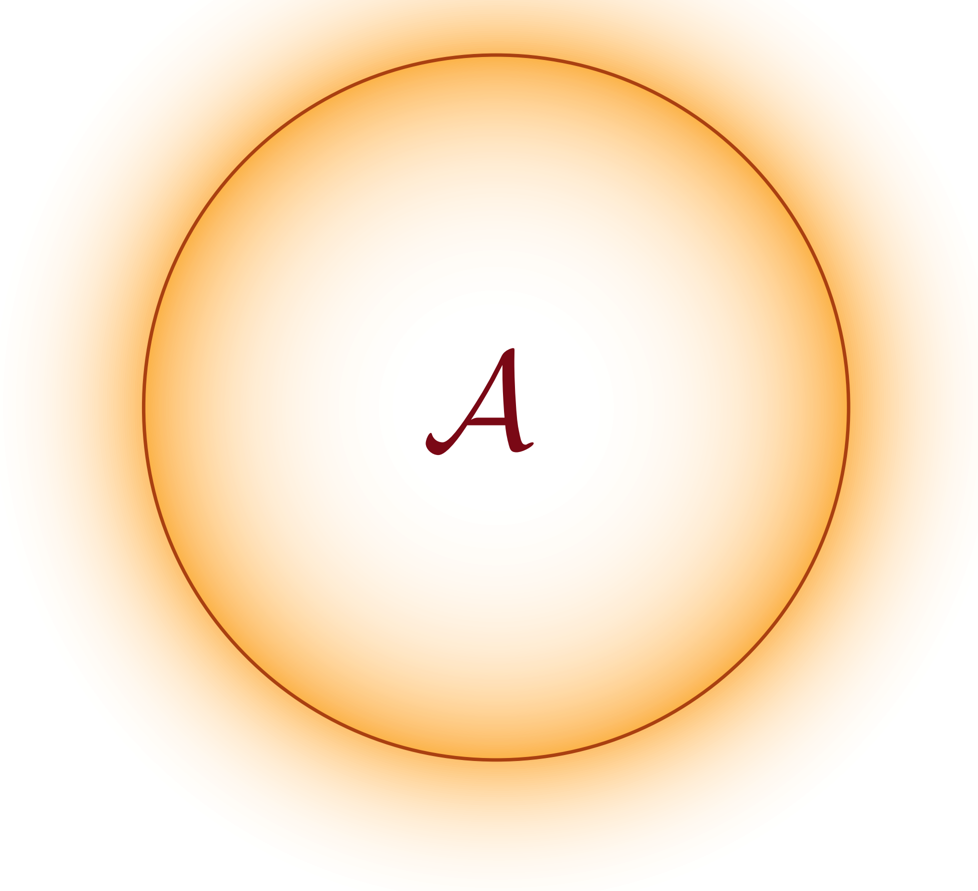
Low energy theory of this strange metal is essentially identical to that of a Fermi surface coupled to a gauge field

Fermi surface of an ordinary metal



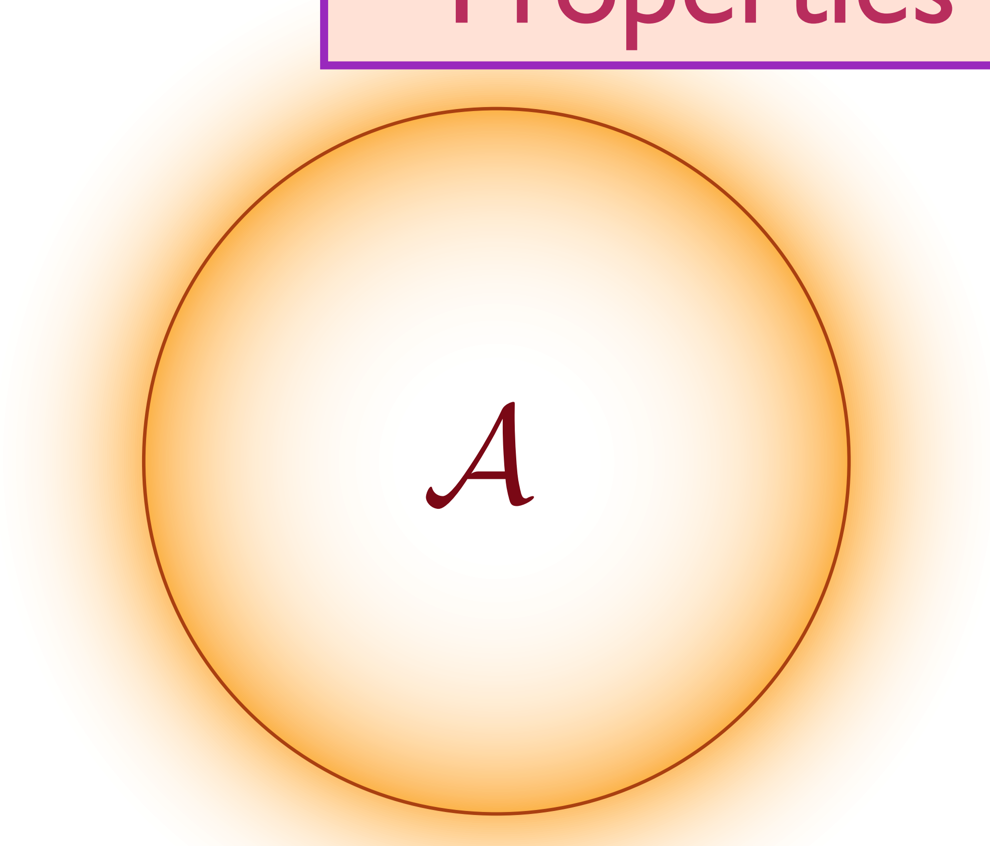
$$\mathcal{L} = f_\sigma^\dagger \left(\partial_\tau - \frac{\nabla^2}{2m} - \mu \right) f_\sigma$$

Fermions coupled to a gauge field



$$\mathcal{L} = f_\sigma^\dagger \left(\partial_\tau - i A_\tau - \frac{(\nabla - i\mathbf{A})^2}{2m} - \mu \right) f_\sigma$$

Properties of this strange metal

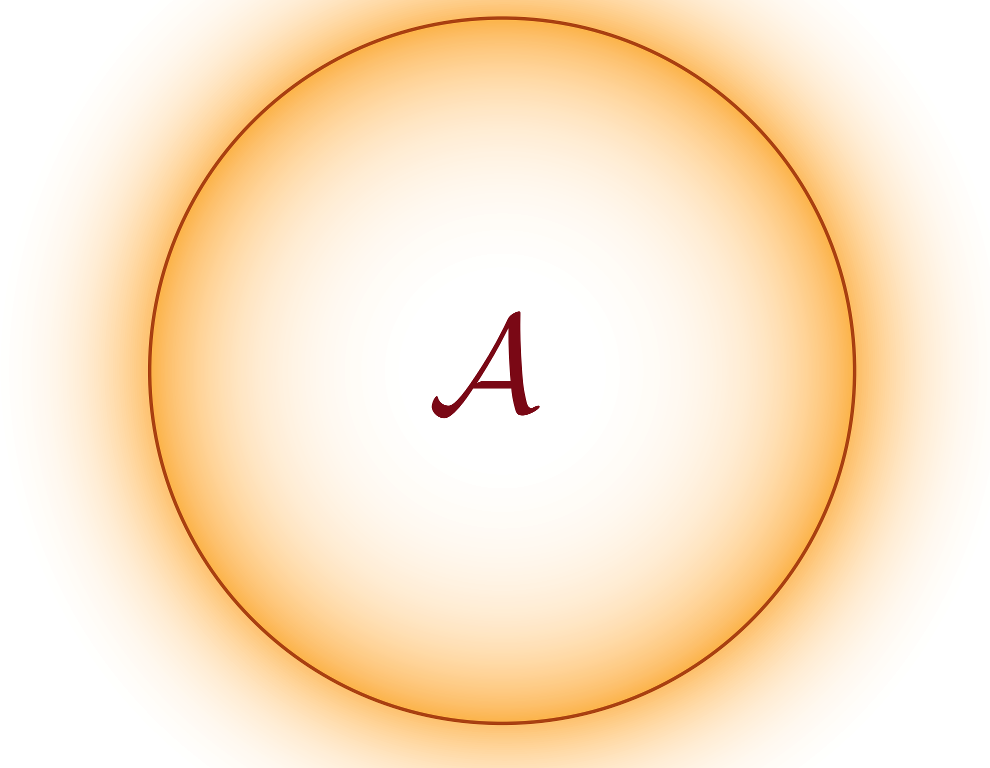


$$\mathcal{L} = f_\sigma^\dagger \left(\partial_\tau - i A_\tau - \frac{(\nabla - i\mathbf{A})^2}{2m} - \mu \right) f_\sigma$$

- There is a sharp Fermi surface defined by the (gauge-dependent) fermion Green's function: $G_f^{-1}(|\mathbf{k}| = k_F, \omega = 0) = 0$. This Green's function is not measurable, and so the Fermi surface is "*hidden*".

S.-S. Lee, Phys. Rev. B **80**, 165102 (2009)
M. A. Metlitski and S. Sachdev, Phys. Rev. B **82**, 075127 (2010)
D. F. Mross, J. McGreevy, H. Liu, and T. Senthil, Phys. Rev. B **82**, 045121 (2010)

Properties of this strange metal



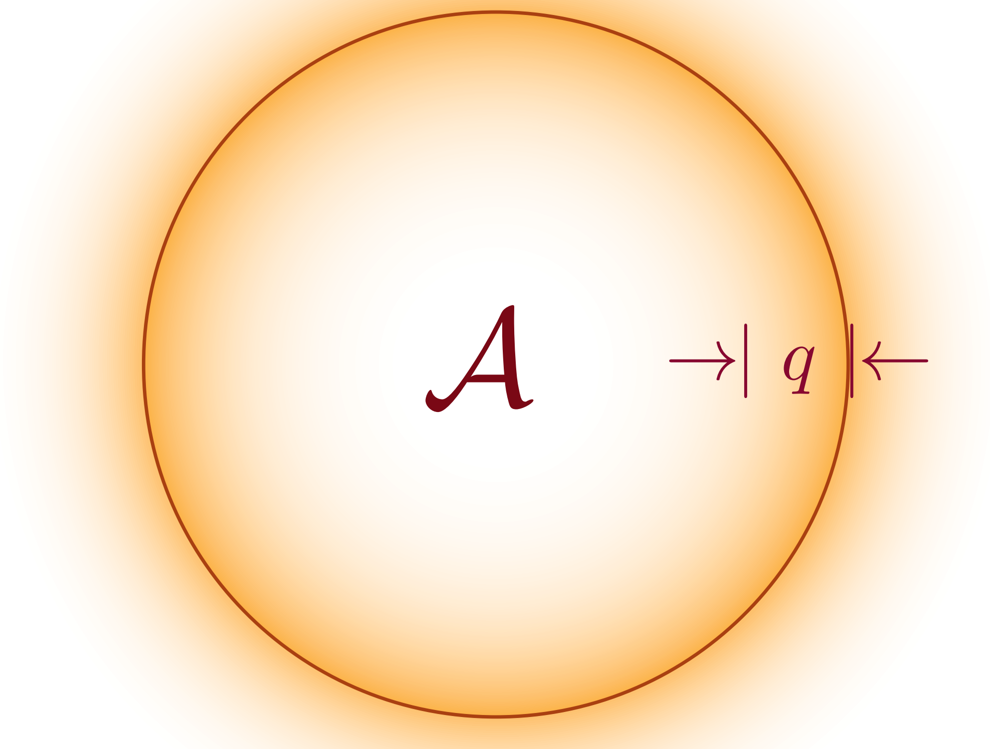
$$\mathcal{L} = f_\sigma^\dagger \left(\partial_\tau - i A_\tau - \frac{(\nabla - i\mathbf{A})^2}{2m} - \mu \right) f_\sigma$$

- There is a sharp Fermi surface defined by the (gauge-dependent) fermion Green's function: $G_f^{-1}(|\mathbf{k}| = k_F, \omega = 0) = 0$. This Green's function is not measurable, and so the Fermi surface is "*hidden*".

- Area enclosed by the Fermi surface $\mathcal{A} = \mathcal{Q}$, the fermion density

S.-S. Lee, Phys. Rev. B **80**, 165102 (2009)
M. A. Metlitski and S. Sachdev, Phys. Rev. B **82**, 075127 (2010)
D. F. Mross, J. McGreevy, H. Liu, and T. Senthil, Phys. Rev. B **82**, 045121 (2010)

Properties of this strange metal



$$\mathcal{L} = f_\sigma^\dagger \left(\partial_\tau - iA_\tau - \frac{(\nabla - i\mathbf{A})^2}{2m} - \mu \right) f_\sigma$$

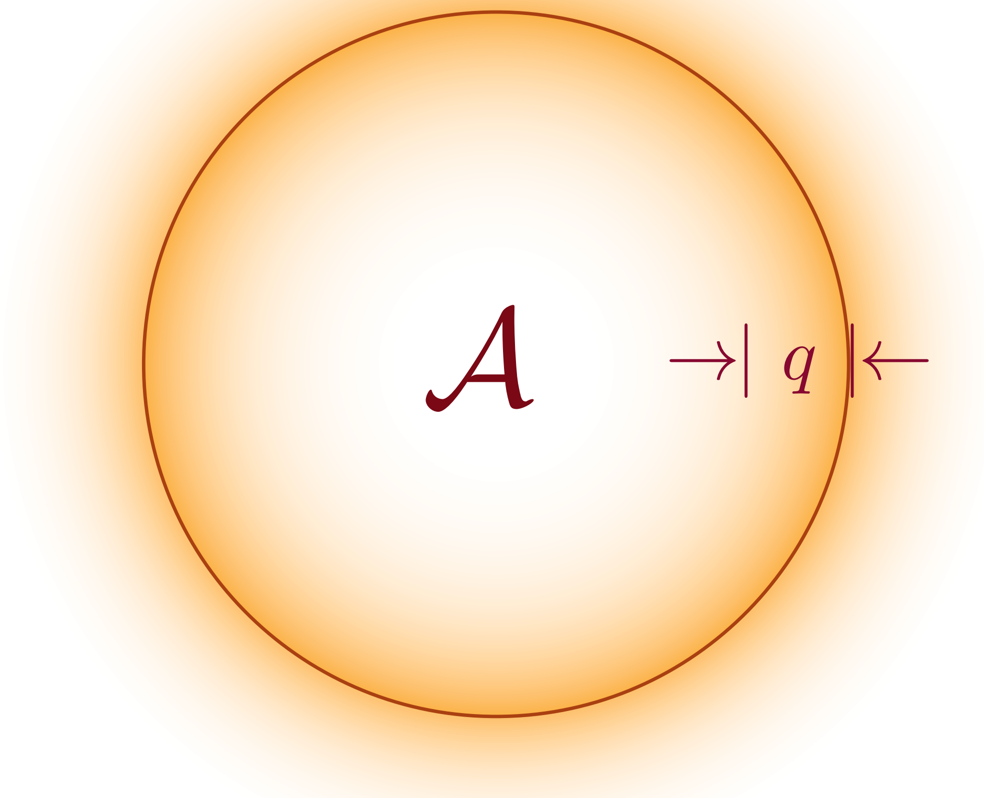
- There is a sharp Fermi surface defined by the (gauge-dependent) fermion Green's function: $G_f^{-1}(|\mathbf{k}| = k_F, \omega = 0) = 0$. This Green's function is not measurable, and so the Fermi surface is "*hidden*".

- Area enclosed by the Fermi surface $\mathcal{A} = \mathcal{Q}$, the fermion density

- Critical continuum of excitations near the Fermi surface with energy $\omega \sim |q|^z$, where $q = |\mathbf{k}| - k_F$ is the distance from the Fermi surface and z is the dynamic critical exponent.

S.-S. Lee, Phys. Rev. B **80**, 165102 (2009)
M. A. Metlitski and S. Sachdev, Phys. Rev. B **82**, 075127 (2010)
D. F. Mross, J. McGreevy, H. Liu, and T. Senthil, Phys. Rev. B **82**, 045121 (2010)

Properties of this strange metal

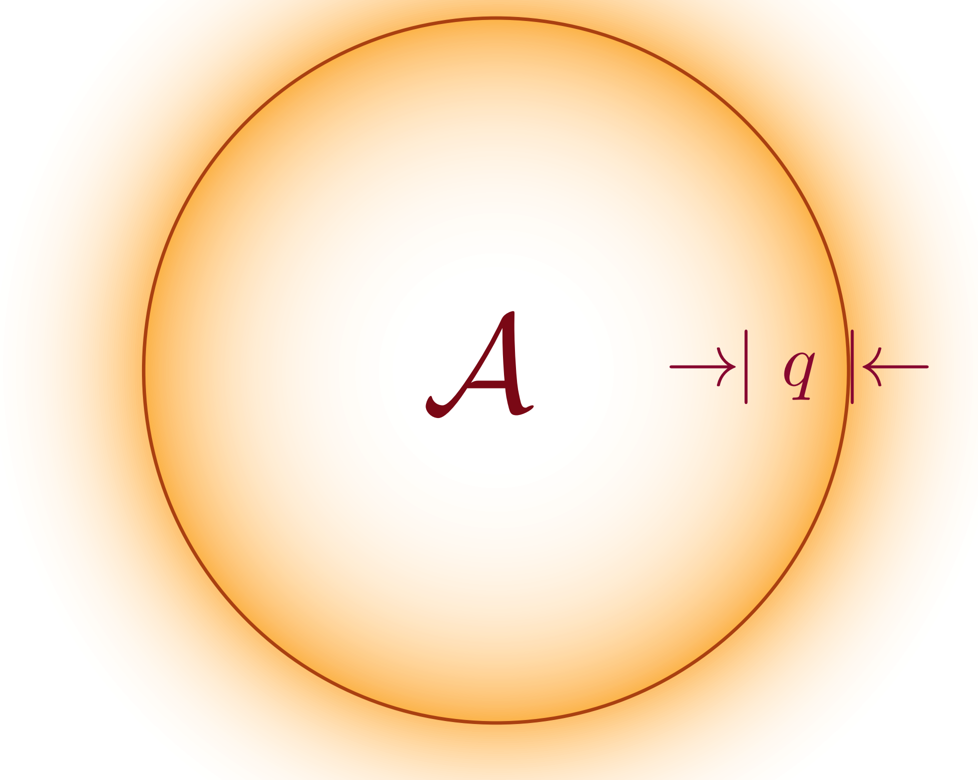


$$\mathcal{L} = f_\sigma^\dagger \left(\partial_\tau - iA_\tau - \frac{(\nabla - i\mathbf{A})^2}{2m} - \mu \right) f_\sigma$$

- Gauge-dependent Green's function $G_f^{-1} = q^{1-\eta} F(\omega/q^z)$. Three-loop computation shows $\eta \neq 0$ and $\boxed{z = 3/2}$.

S.-S. Lee, Phys. Rev. B **80**, 165102 (2009)
M. A. Metlitski and S. Sachdev, Phys. Rev. B **82**, 075127 (2010)
D. F. Mross, J. McGreevy, H. Liu, and T. Senthil, Phys. Rev. B **82**, 045121 (2010)

Properties of this strange metal



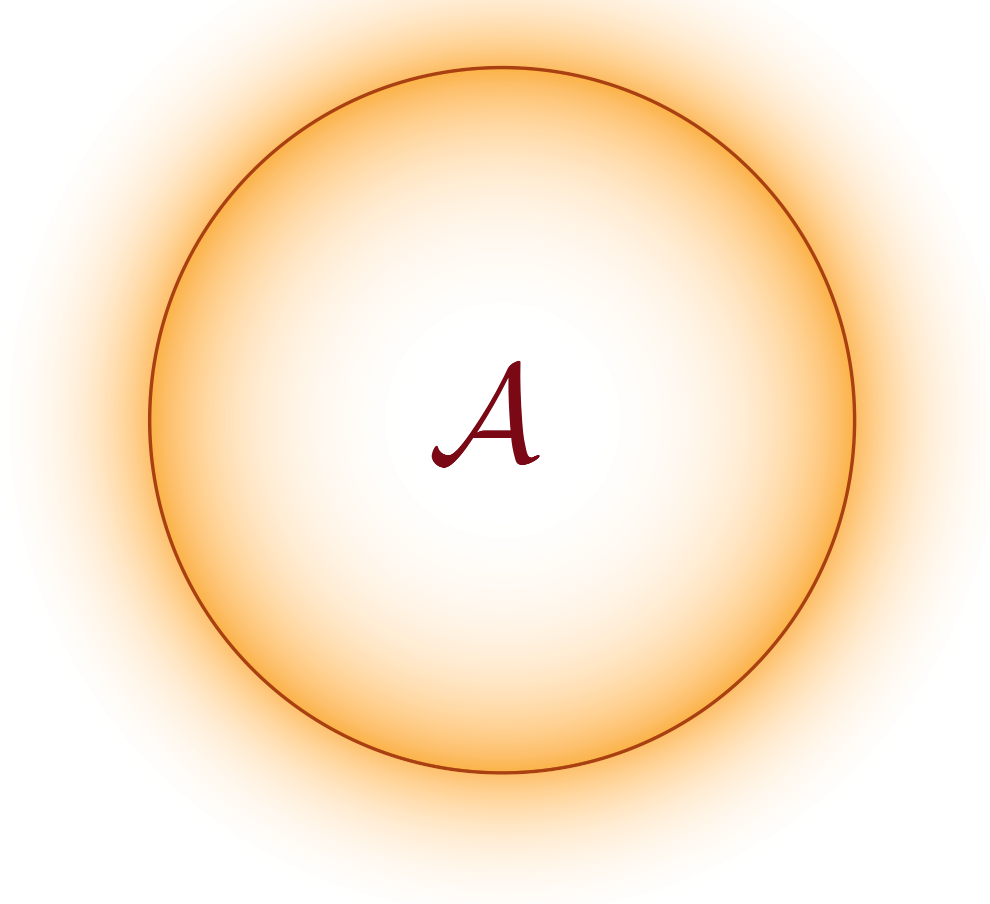
$$\mathcal{L} = f_\sigma^\dagger \left(\partial_\tau - i A_\tau - \frac{(\nabla - i\mathbf{A})^2}{2m} - \mu \right) f_\sigma$$

- Gauge-dependent Green's function $G_f^{-1} = q^{1-\eta} F(\omega/q^z)$. Three-loop computation shows $\eta \neq 0$ and $\boxed{z = 3/2}$.

- The phase space density of fermions is effectively one-dimensional, so the entropy density $S \sim T^{d_{\rm eff}/z}$ with $d_{\rm eff} = 1$.

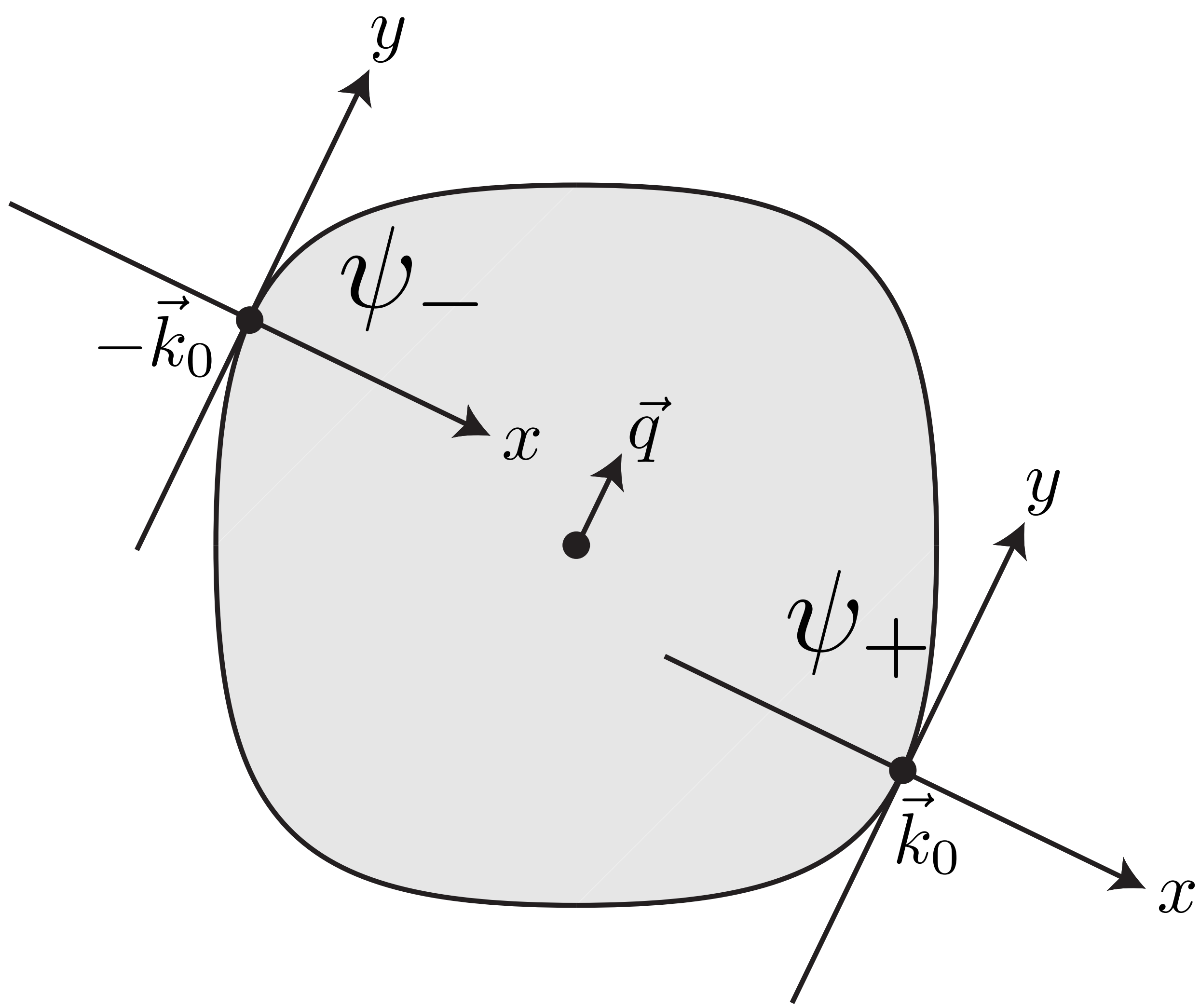
S.-S. Lee, Phys. Rev. B **80**, 165102 (2009)
M. A. Metlitski and S. Sachdev, Phys. Rev. B **82**, 075127 (2010)
D. F. Mross, J. McGreevy, H. Liu, and T. Senthil, Phys. Rev. B **82**, 045121 (2010)

Fermions coupled to a gauge field



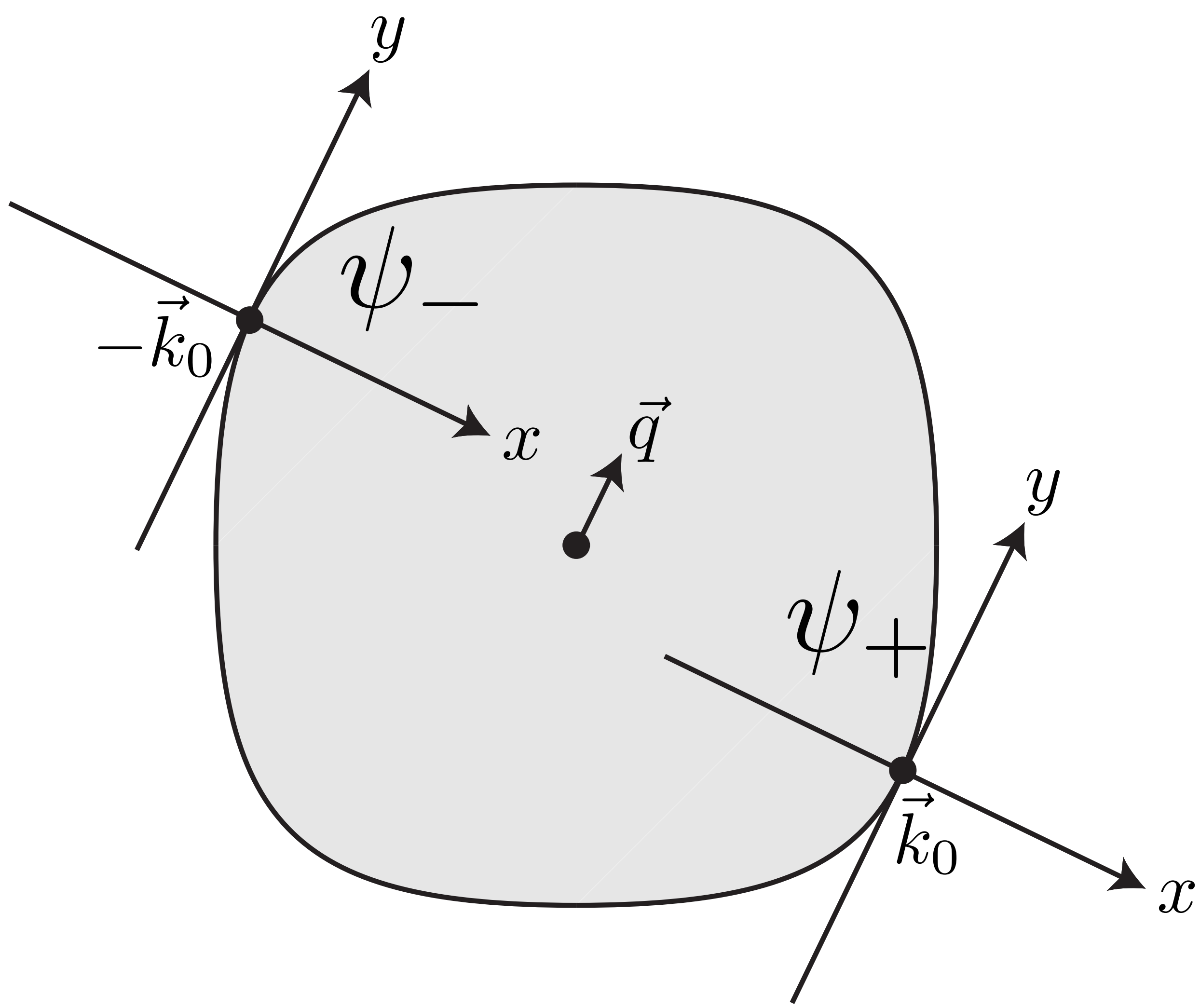
$$\mathcal{L} = f_\sigma^\dagger \left(\partial_\tau - i A_\tau - \frac{(\nabla - i\mathbf{A})^2}{2m} - \mu \right) f_\sigma$$

Field theory of this strange metal



- Gauge fluctuation at wavevector $\vec{q}$ couples most efficiently to fermions near $\pm\vec{k}_0$.

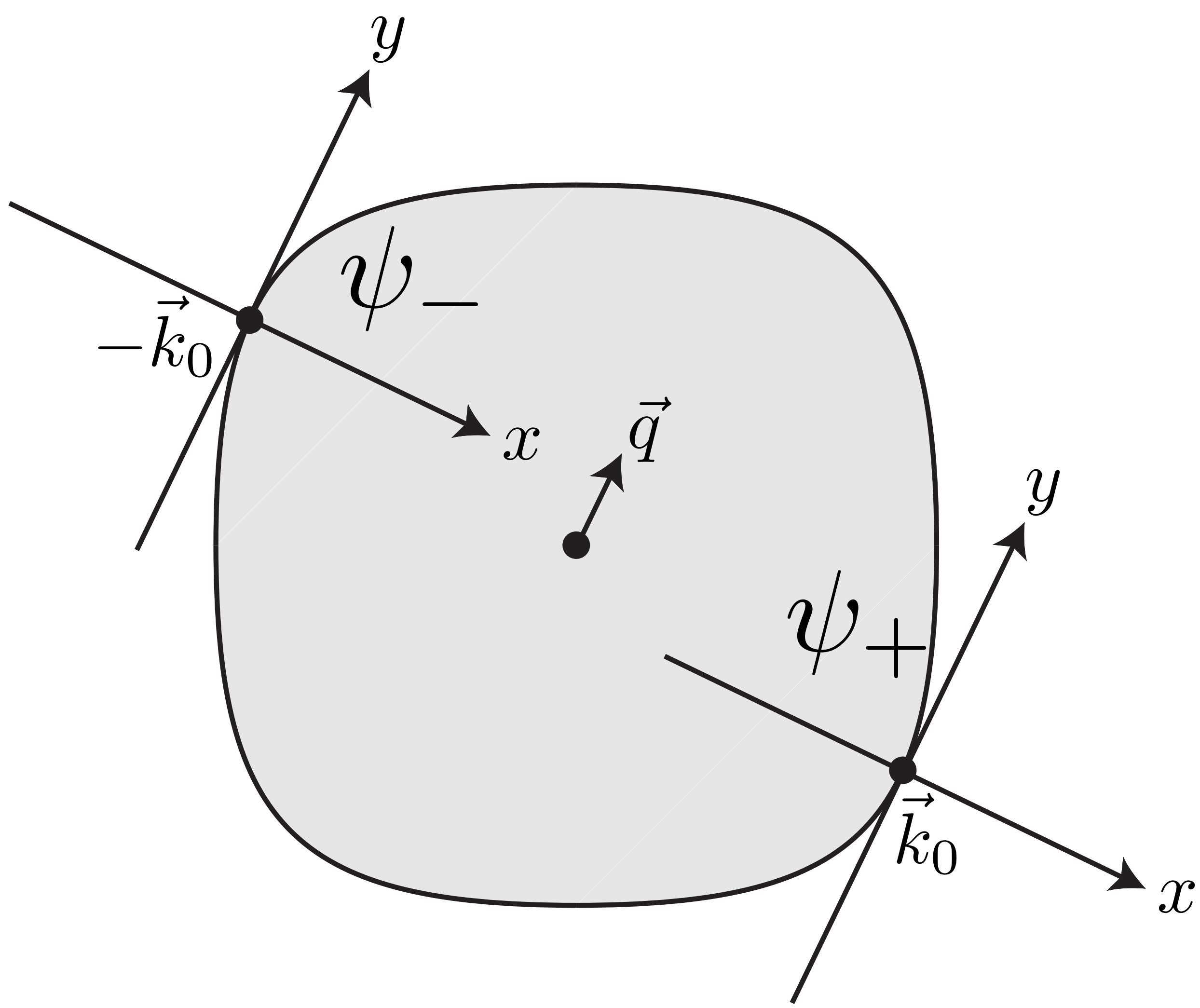
Field theory of this strange metal



- Gauge fluctuation at wavevector $\vec{q}$ couples most efficiently to fermions near $\pm\vec{k}_0$.

- Expand fermion kinetic energy at wavevectors about $\vec{k}_0$.

Field theory of this strange metal

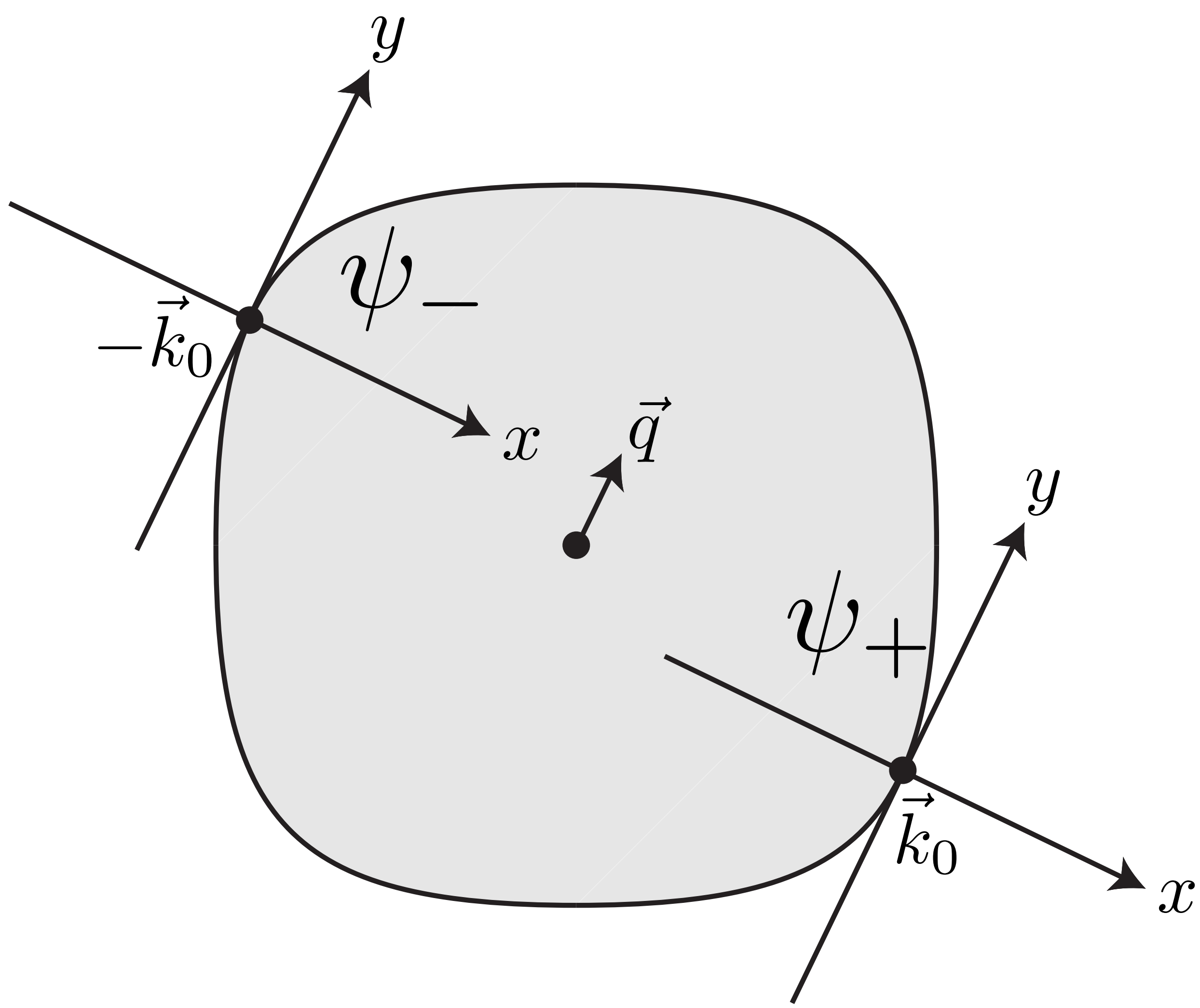


- Gauge fluctuation at wavevector $\vec{q}$ couples most efficiently to fermions near $\pm\vec{k}_0$.

- Expand fermion kinetic energy at wavevectors about $\vec{k}_0$.

- In Landau gauge, only need the component of the gauge field, a, orthogonal to $\vec{q}$.

Field theory of this strange metal



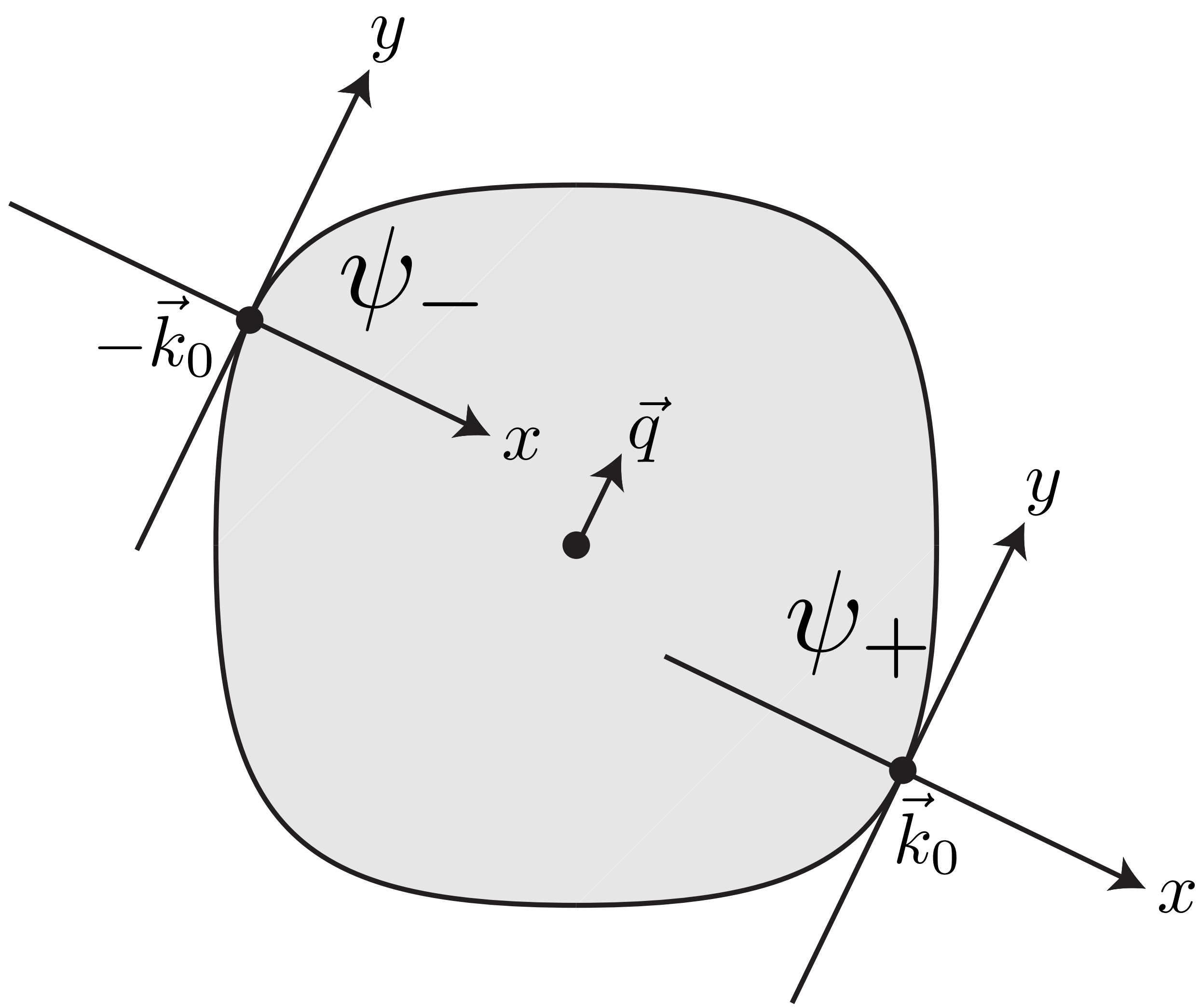
$$\mathcal{L}[\psi_\pm, a] =$$

$$\psi_+^\dagger \left(\partial_\tau - i\partial_x - \partial_y^2\right) \psi_+ + \psi_-^\dagger \left(\partial_\tau + i\partial_x - \partial_y^2\right) \psi_-$$

$$-a \left(\psi_+^\dagger \psi_+ - \psi_-^\dagger \psi_-\right) + \frac{1}{2g^2} \left(\partial_y a\right)^2$$

M. A. Metlitski and S. Sachdev, Phys. Rev. B **82**, 075127 (2010)

Field theory of the Ising-nematic critical point



$$\mathcal{L}[\psi_\pm, \phi] =$$

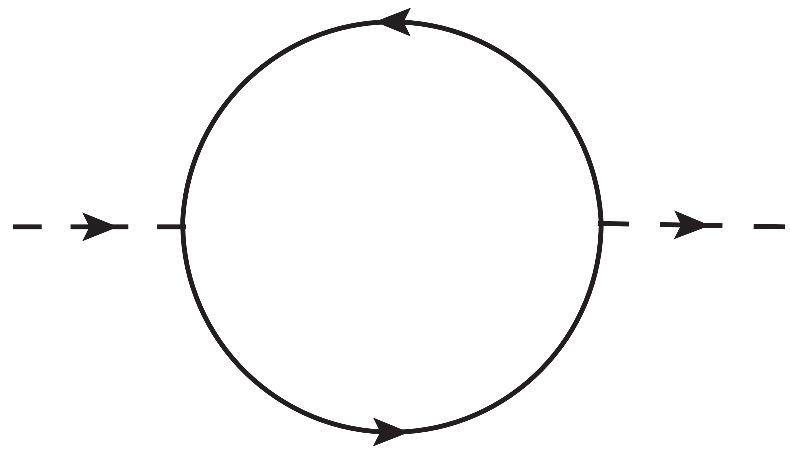
$$\psi_+^\dagger \left(\partial_\tau - i\partial_x - \partial_y^2\right) \psi_+ + \psi_-^\dagger \left(\partial_\tau + i\partial_x - \partial_y^2\right) \psi_-$$

$$-\phi \left(\psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_-\right) + \frac{1}{2g^2} \left(\partial_y \phi\right)^2$$

M. A. Metlitski and S. Sachdev, Phys. Rev. B **82**, 075127 (2010)

Field theory of this strange metal

$$\mathcal{L} = \psi_+^\dagger \left(\partial_\tau - i\partial_x - \partial_y^2\right)\psi_+ + \psi_-^\dagger \left(\partial_\tau + i\partial_x - \partial_y^2\right)\psi_- - a\left(\psi_+^\dagger\psi_+ - \psi_-^\dagger\psi_-\right) + \frac{1}{2g^2}\left(\partial_y a\right)^2$$



One loop photon self-energy with N_f fermion flavors:

$$D(\vec{q},\omega) = N_f \int \frac{d^2k}{4\pi^2}\frac{d\Omega}{2\pi}\frac{1}{\left[-i(\Omega+\omega)+k_x+q_x+(k_y+q_y)^2\right]\left[-i\Omega-k_x+k_y^2\right]}$$
$$= \frac{N_f}{4\pi}\frac{|\omega|}{|q_y|}$$

Landau-damping of photon

Lecture 2

Strange metals

A. Field theory

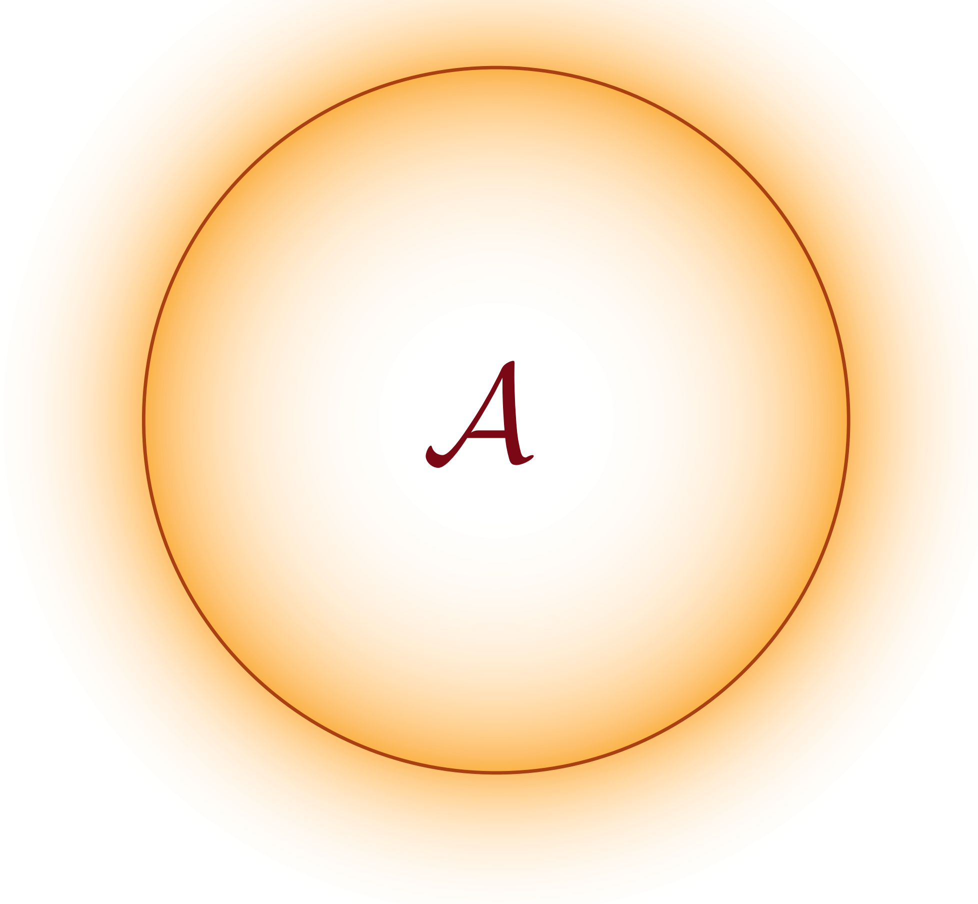
B. Holography

Strange metals

A. Field theory

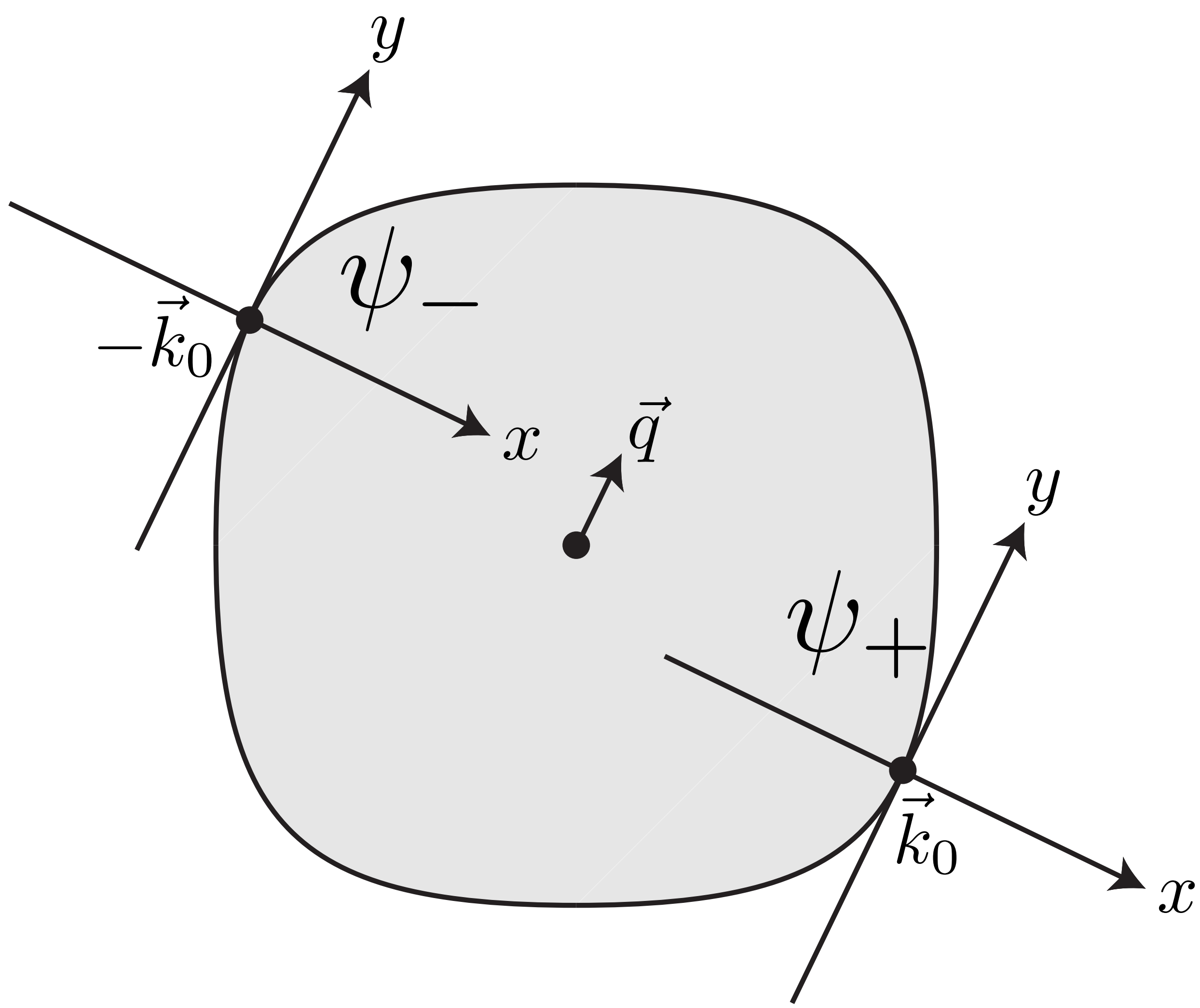
B. Holography

Fermions coupled to a gauge field



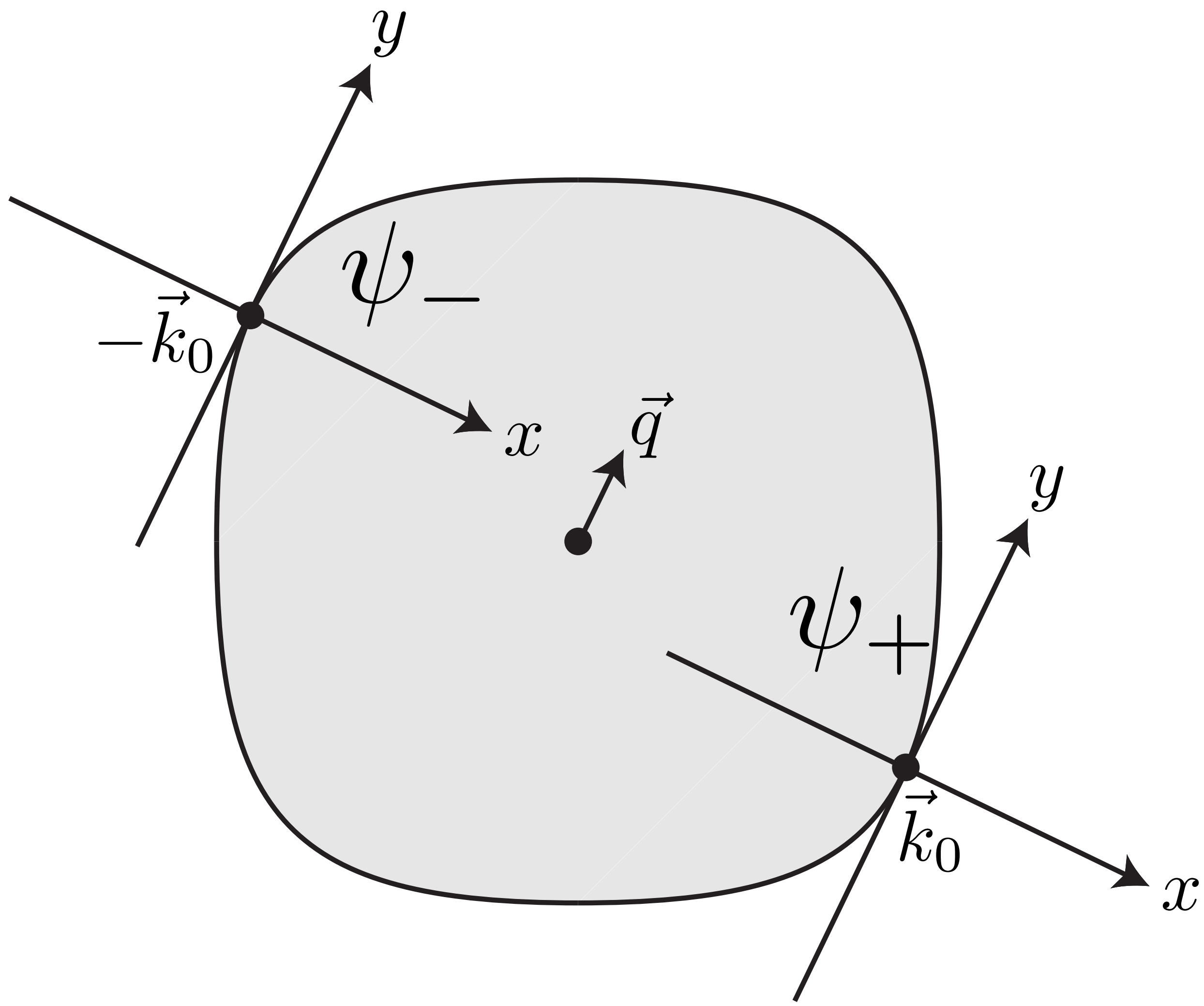
$$\mathcal{L} = f_\sigma^\dagger \left(\partial_\tau - i A_\tau - \frac{(\nabla - i\mathbf{A})^2}{2m} - \mu \right) f_\sigma$$

Field theory of this strange metal



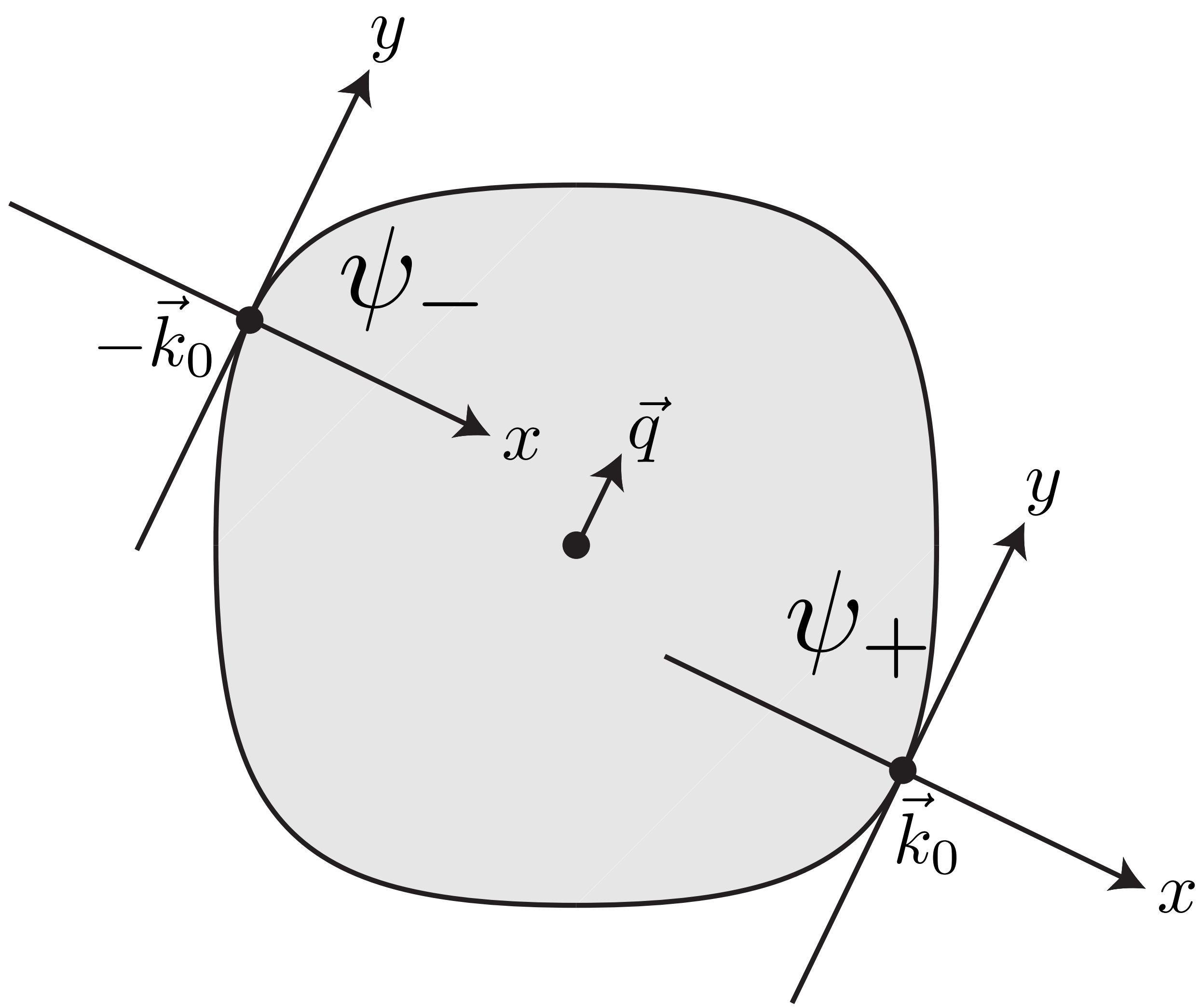
- Gauge fluctuation at wavevector $\vec{q}$ couples most efficiently to fermions near $\pm\vec{k}_0$.

Field theory of this strange metal



- Gauge fluctuation at wavevector $\vec{q}$ couples most efficiently to fermions near $\pm\vec{k}_0$.
- Expand fermion kinetic energy at wavevectors about $\vec{k}_0$.

Field theory of this strange metal

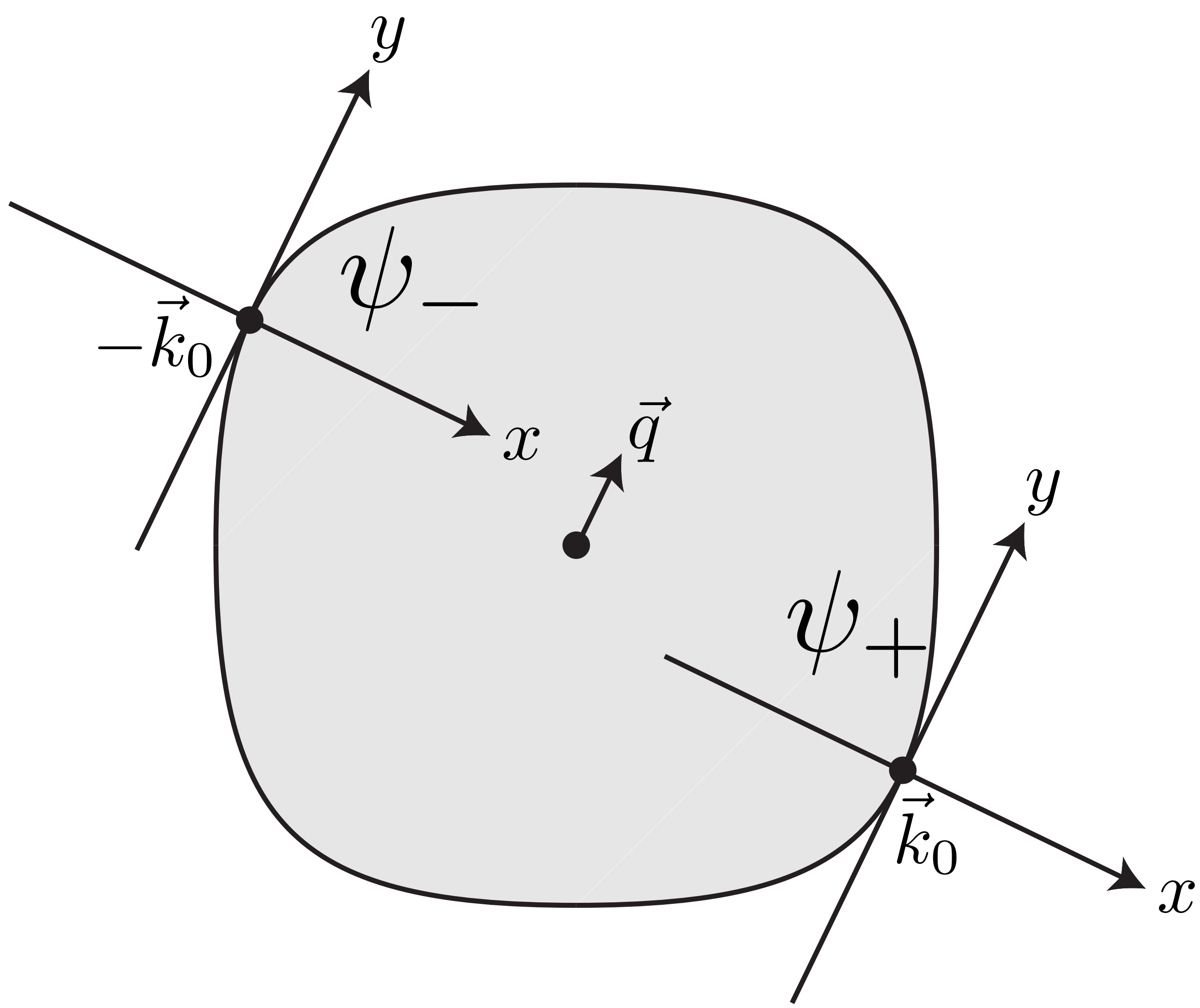


- Gauge fluctuation at wavevector $\vec{q}$ couples most efficiently to fermions near $\pm\vec{k}_0$.

- Expand fermion kinetic energy at wavevectors about $\vec{k}_0$.

- In Landau gauge, only need the component of the gauge field, a, orthogonal to $\vec{q}$.

Field theory of this strange metal



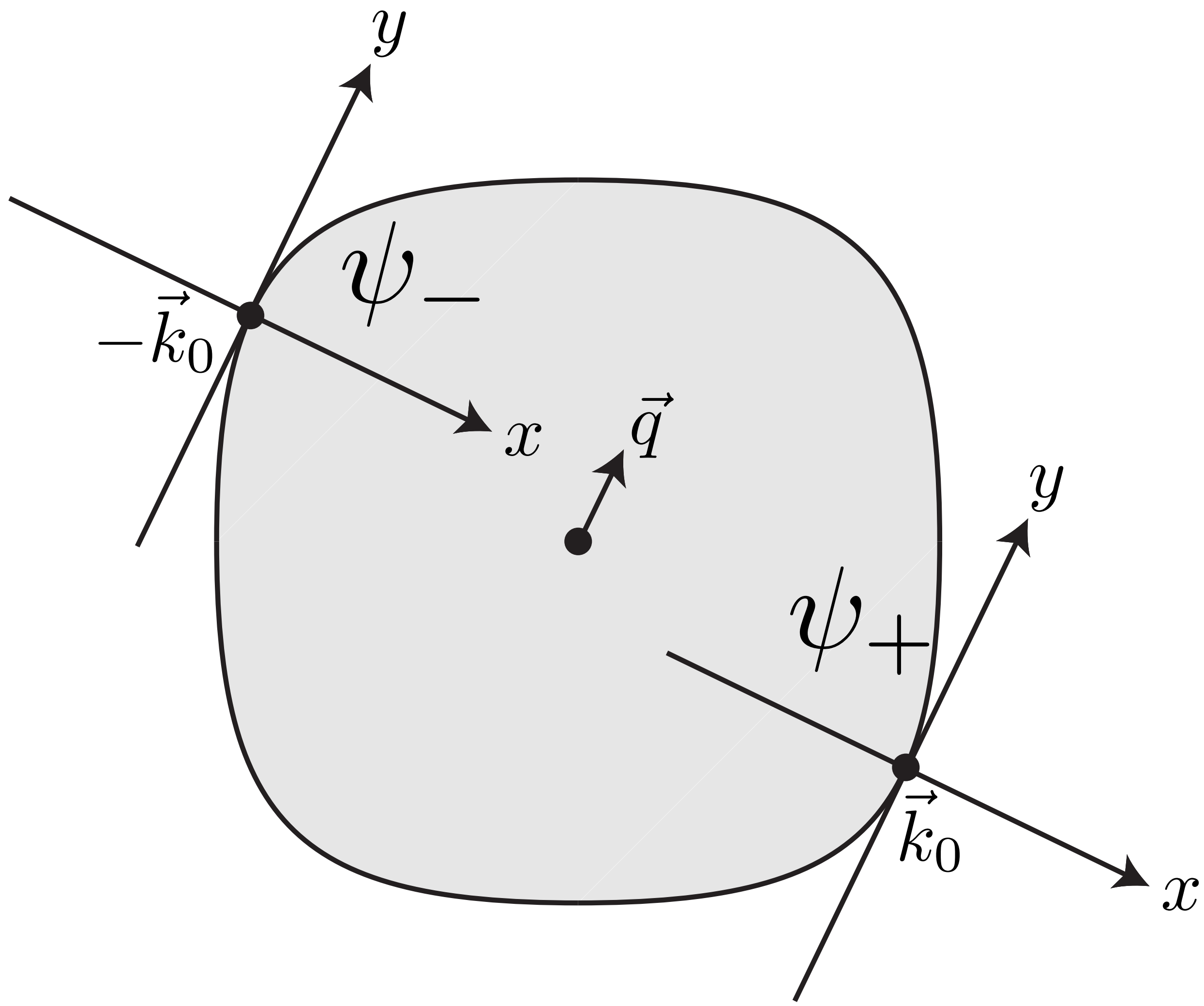
$$\mathcal{L}[\psi_\pm, a] =$$

$$\psi_+^\dagger \left(\partial_\tau - i\partial_x - \partial_y^2\right) \psi_+ + \psi_-^\dagger \left(\partial_\tau + i\partial_x - \partial_y^2\right) \psi_-$$

$$-a \left(\psi_+^\dagger \psi_+ - \psi_-^\dagger \psi_-\right) + \frac{1}{2g^2} \left(\partial_y a\right)^2$$

M. A. Metlitski and S. Sachdev, Phys. Rev. B **82**, 075127 (2010)

Field theory of the Ising-nematic critical point



$$\mathcal{L}[\psi_\pm, \phi] =$$

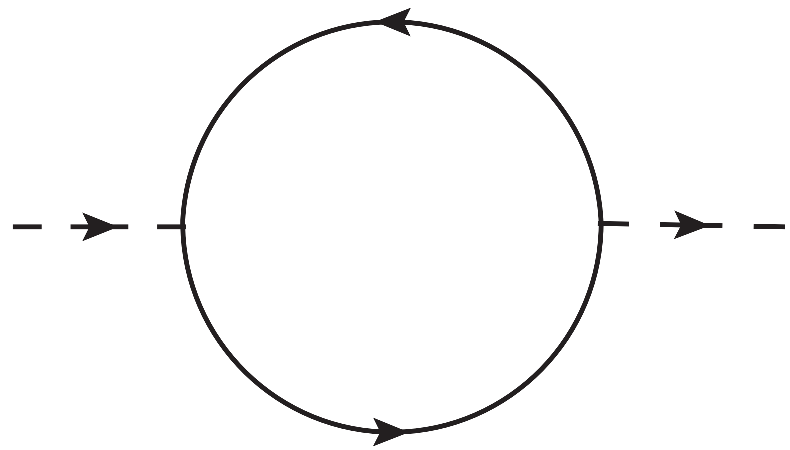
$$\psi_+^\dagger \left(\partial_\tau - i\partial_x - \partial_y^2\right) \psi_+ + \psi_-^\dagger \left(\partial_\tau + i\partial_x - \partial_y^2\right) \psi_-$$

$$-\phi \left(\psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_-\right) + \frac{1}{2g^2} \left(\partial_y \phi\right)^2$$

M. A. Metlitski and S. Sachdev, Phys. Rev. B **82**, 075127 (2010)

Field theory of this strange metal

$$
\begin{aligned}
\mathcal{L} \;=\;& \psi_+^\dagger \left(\partial_\tau - i\partial_x - \partial_y^2\right) \psi_+ + \psi_-^\dagger \left(\partial_\tau + i\partial_x - \partial_y^2\right) \psi_- \\
& - a \left(\psi_+^\dagger \psi_+ - \psi_-^\dagger \psi_-\right) + \frac{1}{2g^2} \left(\partial_y a\right)^2
\end{aligned}
$$



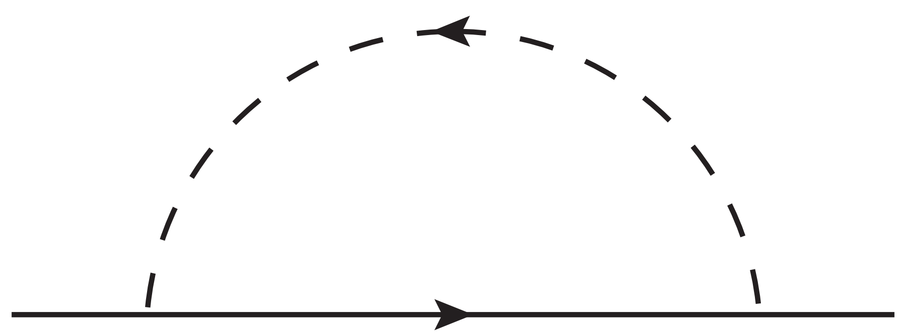
One loop photon self-energy with N_f fermion flavors:

$$
\begin{aligned}
D(\vec{q},\omega) &= N_f \int \frac{d^2k}{4\pi^2} \frac{d\Omega}{2\pi} \frac{1}{\left[-i(\Omega+\omega) + k_x + q_x + (k_y+q_y)^2\right]\left[-i\Omega - k_x + k_y^2\right]} \\
&= \frac{N_f}{4\pi} \frac{|\omega|}{|q_y|}
\end{aligned}
$$

Landau-damping of photon

Field theory of this strange metal

$$\mathcal{L} = \psi_+^\dagger \left(\partial_\tau - i\partial_x - \partial_y^2\right) \psi_+ + \psi_-^\dagger \left(\partial_\tau + i\partial_x - \partial_y^2\right) \psi_- - a\left(\psi_+^\dagger \psi_+ - \psi_-^\dagger \psi_-\right) + \frac{1}{2g^2}\left(\partial_y a\right)^2$$



Electron self-energy at order $1/N_f$:

$$\Sigma(\vec{k},\Omega) = -\frac{1}{N_f}\int \frac{d^2q}{4\pi^2}\frac{d\omega}{2\pi}\frac{1}{\left[-i(\omega+\Omega)+k_x+q_x+(k_y+q_y)^2\right]\left[\frac{q_y^2}{g^2}+\frac{|\omega|}{|q_y|}\right]}$$

$$= -i\frac{2}{\sqrt{3}N_f}\left(\frac{g^2}{4\pi}\right)^{2/3}\mathrm{sgn}(\Omega)|\Omega|^{2/3}$$

Field theory of this strange metal

$$\begin{aligned}\mathcal{L} &= \psi_+^\dagger \left(\partial_\tau - i\partial_x - \partial_y^2\right)\psi_+ + \psi_-^\dagger \left(\partial_\tau + i\partial_x - \partial_y^2\right)\psi_- \\ &\quad - a\left(\psi_+^\dagger \psi_+ - \psi_-^\dagger \psi_-\right) + \frac{1}{2g^2}\left(\partial_y a\right)^2\end{aligned}$$

Schematic form of photon and fermion Green's functions

$$D(\vec{q},\omega) = \frac{1/N_f}{q_y^2 + \dfrac{|\omega|}{|q_y|}} \quad , \quad G_f(\vec{q},\omega) = \frac{1}{q_x + q_y^2 - i\,\mathrm{sgn}(\omega)|\omega|^{2/3}/N_f}$$

In *both* cases $q_x \sim q_y^2 \sim \omega^{1/z}$, with $z = 3/2$. Note that the bare term $\sim \omega$ in G_f^{-1} is irrelevant.

Strongly-coupled theory without quasiparticles.

Field theory of this strange metal

$$\mathcal{L} = \psi_+^\dagger \left(\partial_\tau - i\partial_x - \partial_y^2\right)\psi_+ + \psi_-^\dagger \left(\partial_\tau + i\partial_x - \partial_y^2\right)\psi_- - a\left(\psi_+^\dagger\psi_+ - \psi_-^\dagger\psi_-\right) + \frac{1}{2g^2}\left(\partial_y a\right)^2$$

Simple scaling argument for $z = 3/2$.

Field theory of this strange metal

$$\mathcal{L}_{\rm scaling} = \psi_+^\dagger \left(-i\partial_x - \partial_y^2\right)\psi_+ + \psi_-^\dagger \left(+i\partial_x - \partial_y^2\right)\psi_- - \lambda\, a \left(\psi_+^\dagger \psi_+ - \psi_-^\dagger \psi_-\right) + \left(\partial_y a\right)^2$$

Simple scaling argument for $z = 3/2$.

Field theory of this strange metal

$$
\begin{aligned}
\mathcal{L}_{\rm scaling} \quad = \quad & \psi_+^\dagger \left(-i \partial_x - \partial_y^2 \right) \psi_+ + \psi_-^\dagger \left(+i \partial_x - \partial_y^2 \right) \psi_- \\
& - \lambda \, a \left(\psi_+^\dagger \psi_+ - \psi_-^\dagger \psi_- \right) + \left(\partial_y a \right)^2
\end{aligned}
$$

Simple scaling argument for $z = 3/2$.

Under the rescaling $x \to x/s$, $y \to y/s^{1/2}$, and $\tau \to \tau/s^z$, we find invariance provided

$$
\begin{aligned}
a & \to a \, s^{(2z+1)/4} \\
\psi & \to \psi \, s^{(2z+1)/4} \\
\lambda & \to \lambda \, s^{(3-2z)/4}
\end{aligned}
$$

So the action is invariant provided $z = 3/2$.

Field theory of this strange metal

$$\begin{aligned}\mathcal{L} &= \psi_+^\dagger\left(\partial_\tau - i\partial_x - \partial_y^2\right)\psi_+ + \psi_-^\dagger\left(\partial_\tau + i\partial_x - \partial_y^2\right)\psi_- \\ &\quad - a\left(\psi_+^\dagger\psi_+ - \psi_-^\dagger\psi_-\right) + \frac{1}{2g^2}\left(\partial_y a\right)^2\end{aligned}$$

Schematic form of photon and fermion Green's functions

$$D(\vec{q},\omega) = \frac{1/N_f}{q_y^2 + \dfrac{|\omega|}{|q_y|}} \quad , \quad G_f(\vec{q},\omega) = \frac{1}{q_x + q_y^2 - i\,\mathrm{sgn}(\omega)|\omega|^{2/3}/N_f}$$

In *both* cases $q_x \sim q_y^2 \sim \omega^{1/z}$, with $z = 3/2$. Note that the bare term $\sim \omega$ in G_f^{-1} is irrelevant.

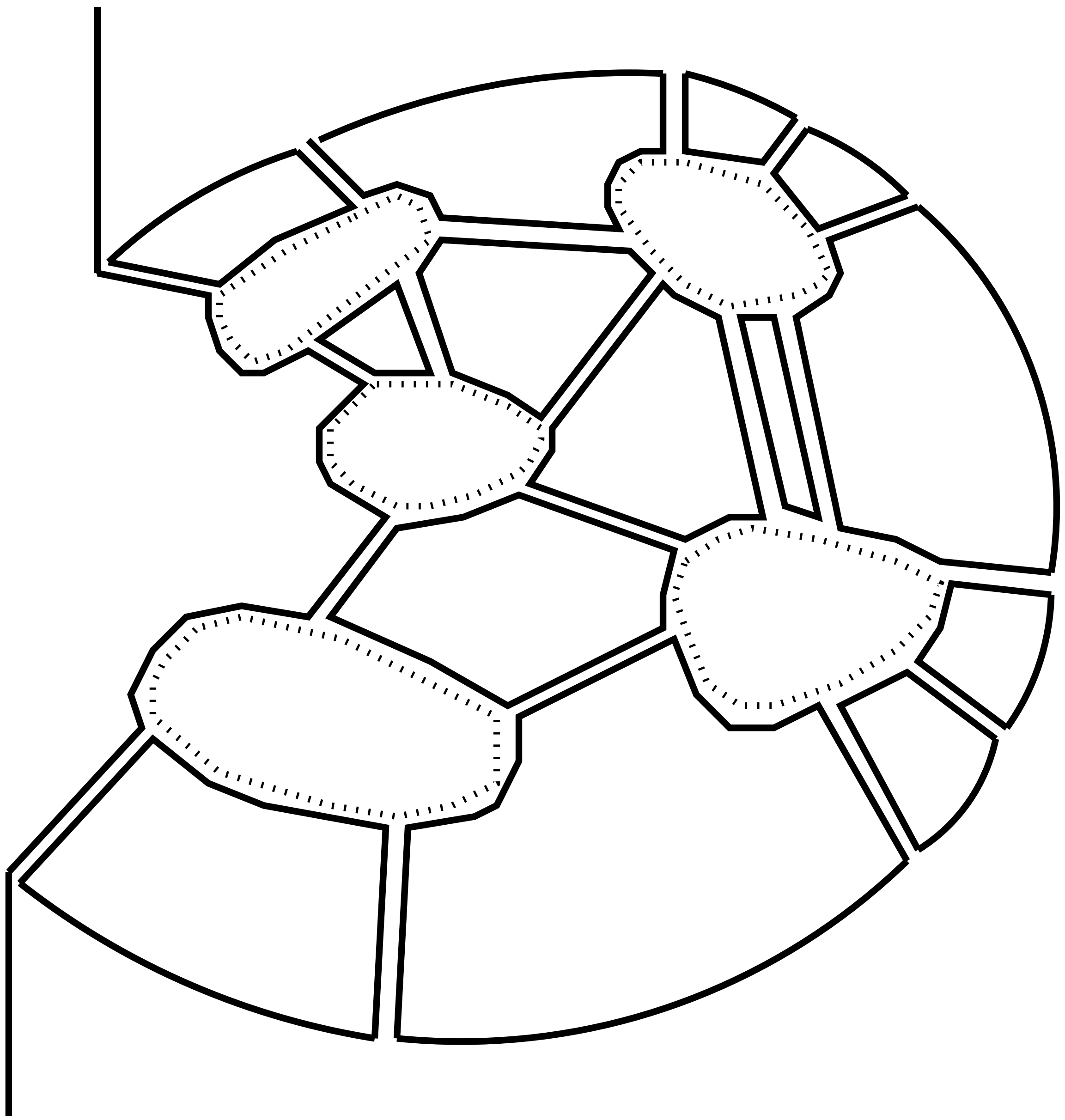
Strongly-coupled theory without quasiparticles.

Field theory of this strange metal

$$\mathcal{L} = \psi_+^\dagger \left(\partial_\tau - i\partial_x - \partial_y^2\right) \psi_+ + \psi_-^\dagger \left(\partial_\tau + i\partial_x - \partial_y^2\right) \psi_- - a\left(\psi_+^\dagger \psi_+ - \psi_-^\dagger \psi_-\right) + \frac{1}{2g^2} \left(\partial_y a\right)^2$$

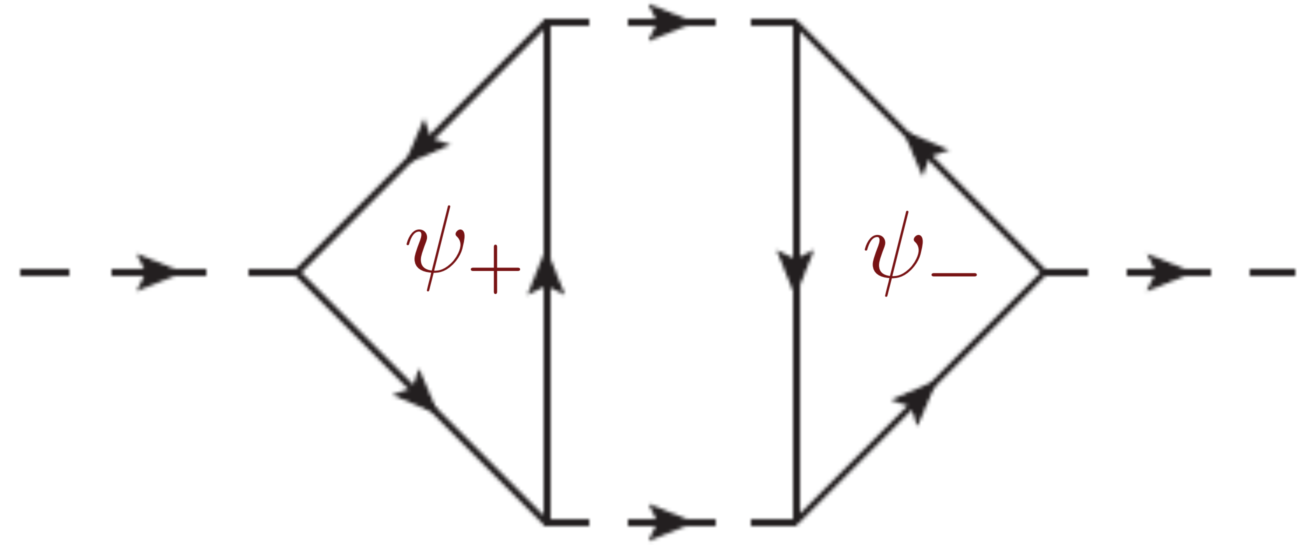
The $1/N_f$ expansion is *not* determined by counting fermion loops, because of infrared singularities created by the Fermi surface. The $|\omega|^{2/3}/N_f$ fermion self-energy leads to additional powers of N_f, and a breakdown in the loop expansion.

Computations in the $1/N$ expansion



All planar graphs of ψ_+ alone are as important as the leading term

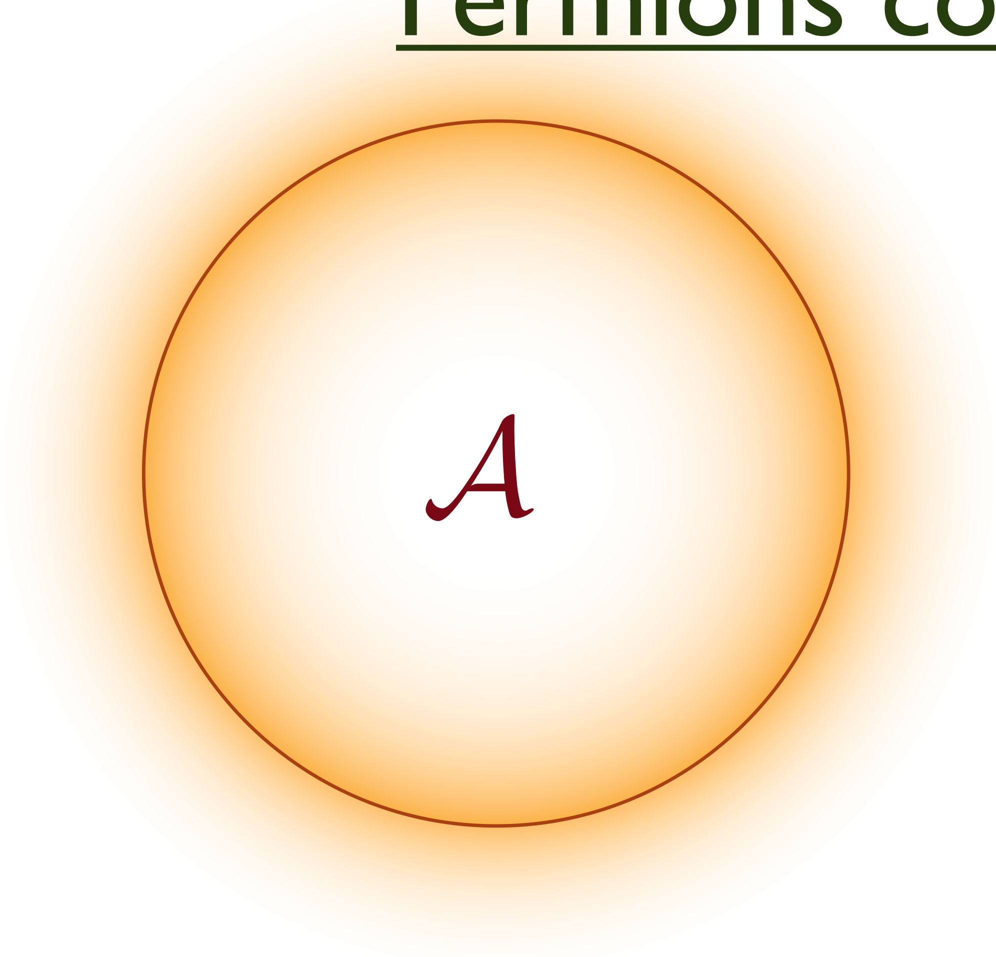
Sung-Sik Lee, *Physical Review* B **80**, 165102 (2009)



Graph mixing ψ_+ and ψ_- is $\mathcal{O}\left(N^{3/2}\right)$ (instead of $\mathcal{O}\left(N\right)$), violating genus expansion

M. A. Metlitski and S. Sachdev, Phys. Rev. B **82**, 075127 (2010)

Fermions coupled to a gauge field



$$\mathcal{L} = f_\sigma^\dagger \left(\partial_\tau - i A_\tau - \frac{(\nabla - i\mathbf{A})^2}{2m} - \mu \right) f_\sigma$$

Fermions and bosons coupled to a gauge field

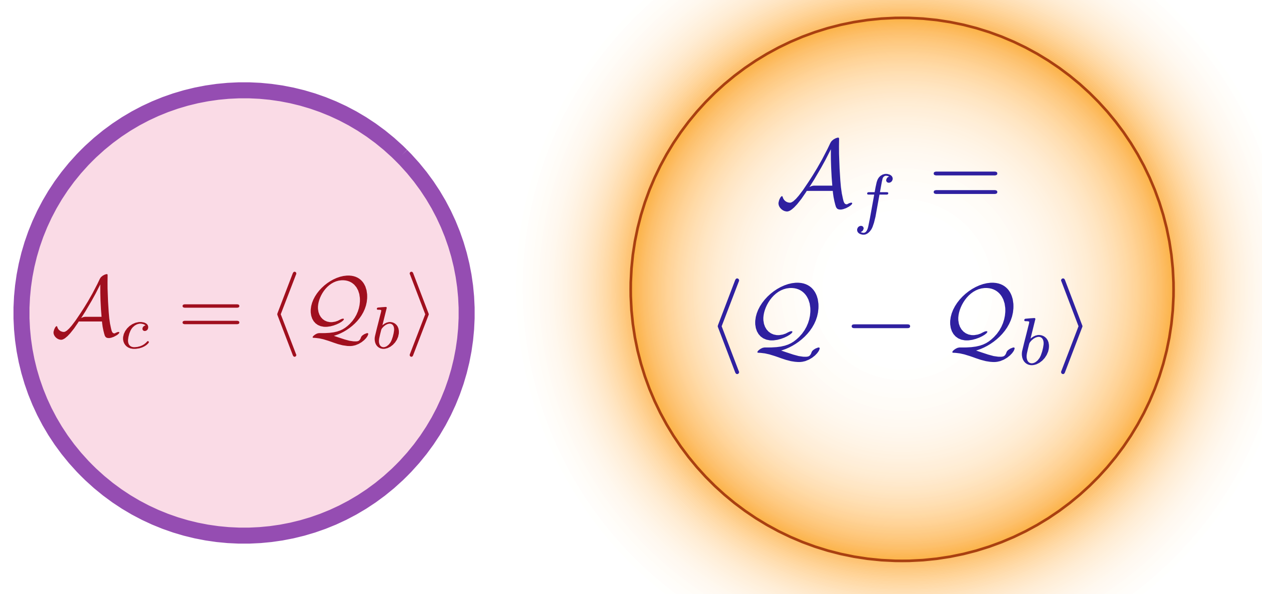
$$\mathcal{L} = f^\dagger \left(\partial_\tau - iA_\tau - \frac{(\nabla - i\mathbf{A})^2}{2m} - \mu \right) f$$
$$+ \; b^\dagger \left(\partial_\tau + iA_\tau - \frac{(\nabla + i\mathbf{A})^2}{2m_b} - \mu_b \right) b + s|b|^2 - g\, b^\dagger f^\dagger f b + \ldots$$

Fermions and bosons coupled to a gauge field

$$\begin{aligned}\mathcal{L} &= f^\dagger\left(\partial_\tau - iA_\tau - \frac{(\nabla - i\mathbf{A})^2}{2m} - \mu\right)f \\ &+ b^\dagger\left(\partial_\tau + iA_\tau - \frac{(\nabla + i\mathbf{A})^2}{2m_b} - \mu_b\right)b + s|b|^2 - g\, b^\dagger f^\dagger f b + \ldots\end{aligned}$$

Another strange metal: the fractionalized Fermi liquid (FL*)

Bosons can bind with fermions to form a gauge-neutral fermion $c \sim b\,f$. The FL* phase has 2 Fermi surfaces: the gauge-neutral Fermi surface of c, and the gauge-charged Fermi surface of f. They enclose a *combined* area equal to $\langle \mathcal{Q} \rangle$.



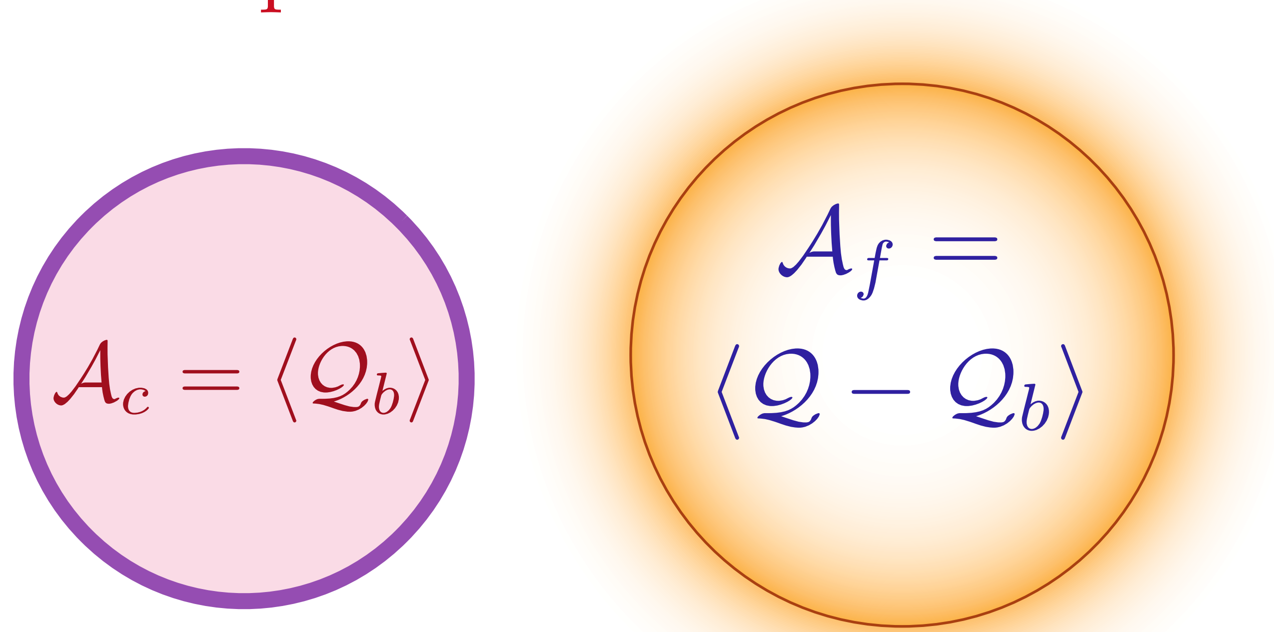
T. Senthil, M. Vojta, and S. Sachdev, *Physical Review* B **69**, 035111 (2004)

Fermions and bosons coupled to a gauge field

$$
\begin{aligned}
\mathcal{L} &= f^\dagger \left(\partial_\tau - i A_\tau - \frac{(\nabla - i\mathbf{A})^2}{2m} - \mu \right) f \\
&+ b^\dagger \left(\partial_\tau + i A_\tau - \frac{(\nabla + i\mathbf{A})^2}{2m_b} - \mu_b \right) b + s|b|^2 - g\, b^\dagger f^\dagger f b + \ldots
\end{aligned}
$$

Another strange metal: the fractionalized Fermi liquid (FL*)

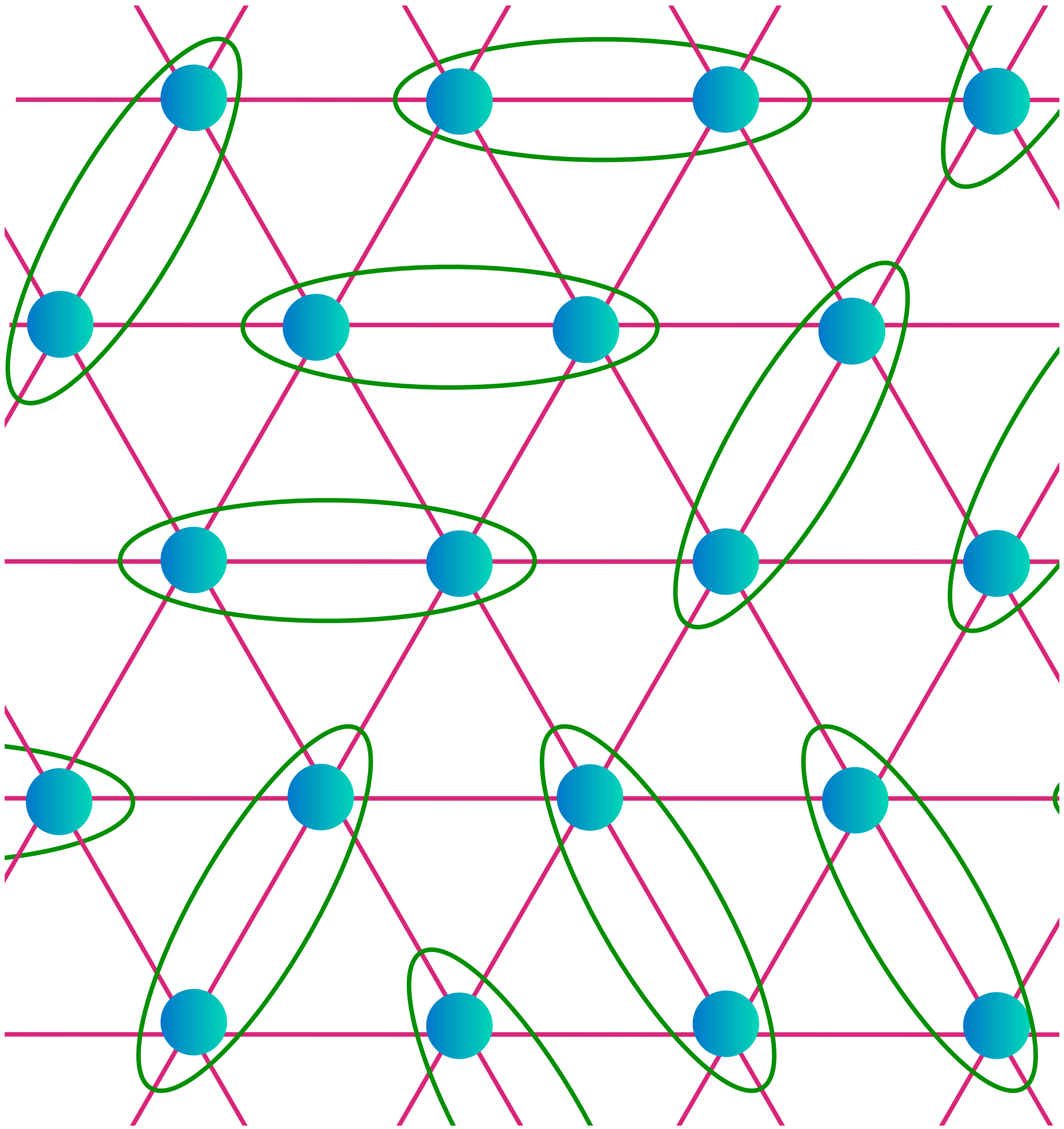
In holography:
the c Fermi surface is that of the "probe" fermion;
the f Fermi surface is "hidden" past the horizon.



S. Sachdev, *Physical Review Letters* **105**, 151602 (2010)

Kondo lattice model

Another strange metal: the fractionalized Fermi liquid (FL*)



Spin liquid of *f* electrons

Fermi surface of *c* conduction electrons

T. Senthil, M. Vojta, and S. Sachdev, *Physical Review* B **69**, 035111 (2004)

Strange metals

A. Field theory

B. Holography

Strange metals

A. Field theory

$\boxed{\textit{B. Holography}}$

Study the large N_c limit of a SU(N_c) Yang-Mills gauge field coupled to adjoint (matrix) fermions at a non-zero chemical potential

Study the large N_c limit of a SU(N_c) Yang-Mills gauge field coupled to adjoint (matrix) fermions at a non-zero chemical potential

Landau damping of photons and fermion quasiparticle decay exist already in the $N_c \to \infty$ limit. Schematic form of photon and fermion Green's functions

$$D(\vec{q},\omega) = \frac{1}{q_y^2 + \frac{|\omega|}{|q_y|}} \quad , \quad G_f(\vec{q},\omega) = \frac{1}{q_x + q_y^2 - i\text{sgn}(\omega)|\omega|^{2/3}}$$

Note the absence of a $1/N_c$ prefactor for the fermion self-energy. This implies there are no anomalous powers of N_c from infrared singularities in the large N_c limit. Naive counting of powers of N_c from the trace over color indices remains valid.

Study the large N_c limit of a SU(N_c) Yang-Mills gauge field coupled to adjoint (matrix) fermions at a non-zero chemical potential

- 'tHooft's argument for the large N_c limit of pure Yang-Mills theory applies unchanged. Feynman diagrams acquire a factor of $1/N_c^g$ where g is genus of the surface defined by the Feynman diagram.

Study the large N_c limit of a SU(N_c) Yang-Mills gauge field coupled to adjoint (matrix) fermions at a non-zero chemical potential

- 'tHooft's argument for the large N_c limit of pure Yang-Mills theory applies unchanged. Feynman diagrams acquire a factor of $1/N_c^g$ where g is genus of the surface defined by the Feynman diagram.

- The $N_c \to \infty$ limit requires summation of all "planar" graphs.

Study the large N_c limit of a SU(N_c) Yang-Mills gauge field coupled to adjoint (matrix) fermions at a non-zero chemical potential

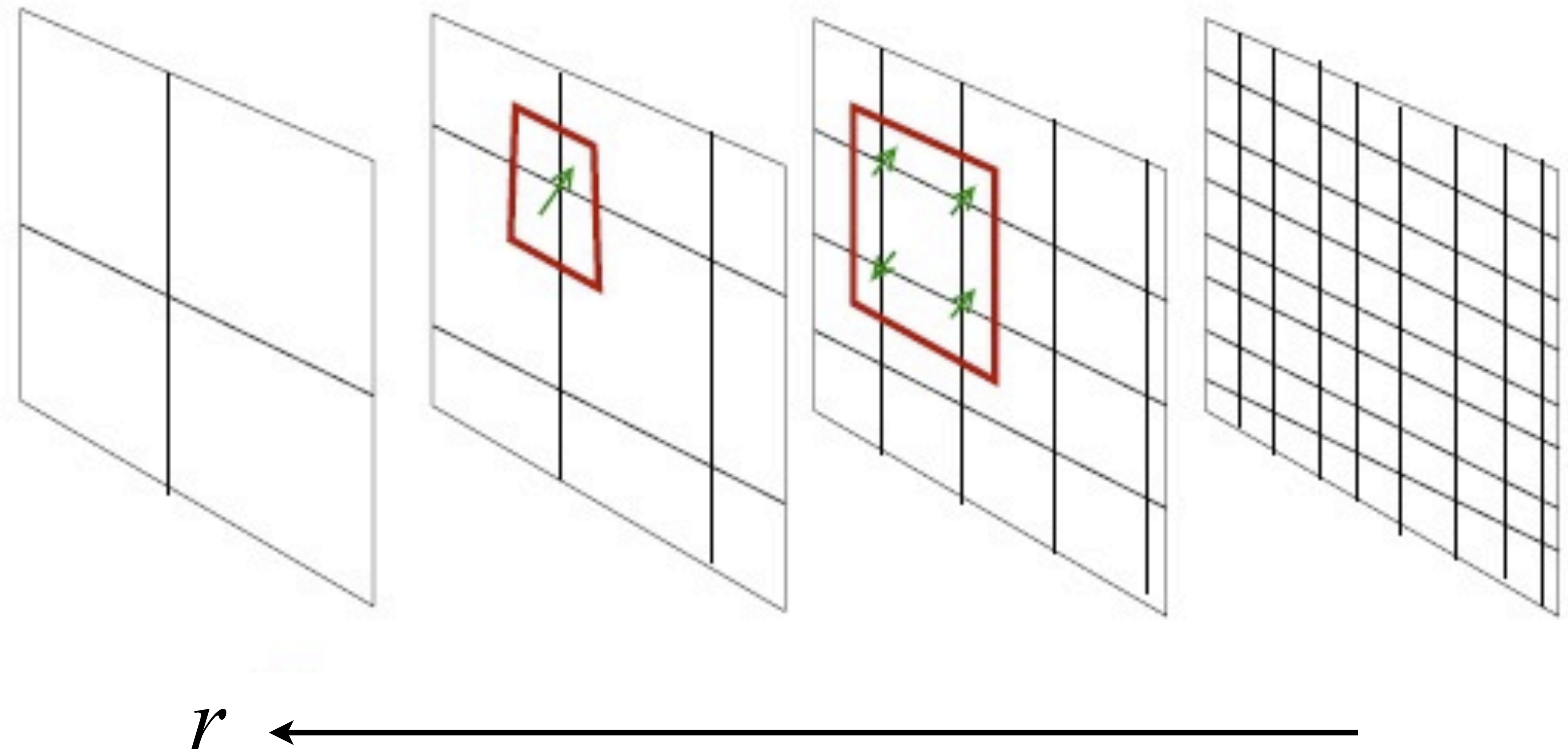
- 'tHooft's argument for the large N_c limit of pure Yang-Mills theory applies unchanged. Feynman diagrams acquire a factor of $1/N_c^g$ where g is genus of the surface defined by the Feynman diagram.

- The $N_c \to \infty$ limit requires summation of all "planar" graphs.

- The pairing instability to superconducting phases is subdominant in the $1/N_c$ expansion.

Study the large N_c limit of a SU(N_c) Yang-Mills gauge field coupled to adjoint (matrix) fermions at a non-zero chemical potential

- 'tHooft's argument for the large N_c limit of pure Yang-Mills theory applies unchanged. Feynman diagrams acquire a factor of $1/N_c^g$ where g is genus of the surface defined by the Feynman diagram.
- The $N_c \to \infty$ limit requires summation of all "planar" graphs.
- The pairing instability to superconducting phases is subdominant in the $1/N_c$ expansion.
- We will now present a conjectured gravity dual of this theory.



J. McGreevy, arXiv0909.0518

For a relativistic CFT in d spatial dimensions, the metric in the holographic space is uniquely fixed by demanding the following scale transformaion $(i = 1 \ldots d)$

$$x_i \to \zeta x_i \quad , \quad t \to \zeta t \quad , \quad ds \to ds$$

For a relativistic CFT in d spatial dimensions, the metric in the holographic space is uniquely fixed by demanding the following scale transformaion $(i = 1 \ldots d)$

$$x_i \to \zeta x_i \quad , \quad t \to \zeta t \quad , \quad ds \to ds$$

This gives the unique metric

$$ds^2 = \frac{1}{r^2}\left(-dt^2 + dr^2 + dx_i^2\right)$$

Reparametrization invariance in r has been used to the prefactor of dx_i^2 equal to $1/r^2$. This fixes $r \to \zeta r$ under the scale transformation. This is the metric of the space AdS_{d+2}.

Now consider the metric which transforms under rescaling as

$$\begin{aligned} x_i &\rightarrow \zeta\, x_i \\ t &\rightarrow \zeta^z\, t \\ ds &\rightarrow \zeta^{\theta/d}\, ds. \end{aligned}$$

This identifies z as the dynamic critical exponent ($z = 1$ for "relativistic" quantum critical points).

What is θ ? ($\theta = 0$ for "relativistic" quantum critical points).

L. Huijse, S. Sachdev, B. Swingle, Physical Review B **85**, 035121 (2012)

Now consider the metric which transforms under rescaling as

$$\begin{aligned} x_i &\rightarrow \zeta\, x_i \\ t &\rightarrow \zeta^z\, t \\ ds &\rightarrow \zeta^{\theta/d}\, ds. \end{aligned}$$

This identifies z as the dynamic critical exponent ($z = 1$ for "relativistic" quantum critical points).

What is θ ? ($\theta = 0$ for "relativistic" quantum critical points).

The most general choice of such a metric is

$$\boxed{ds^2 = \frac{1}{r^2}\left(-\frac{dt^2}{r^{2d(z-1)/(d-\theta)}} + r^{2\theta/(d-\theta)}dr^2 + dx_i^2\right)}$$

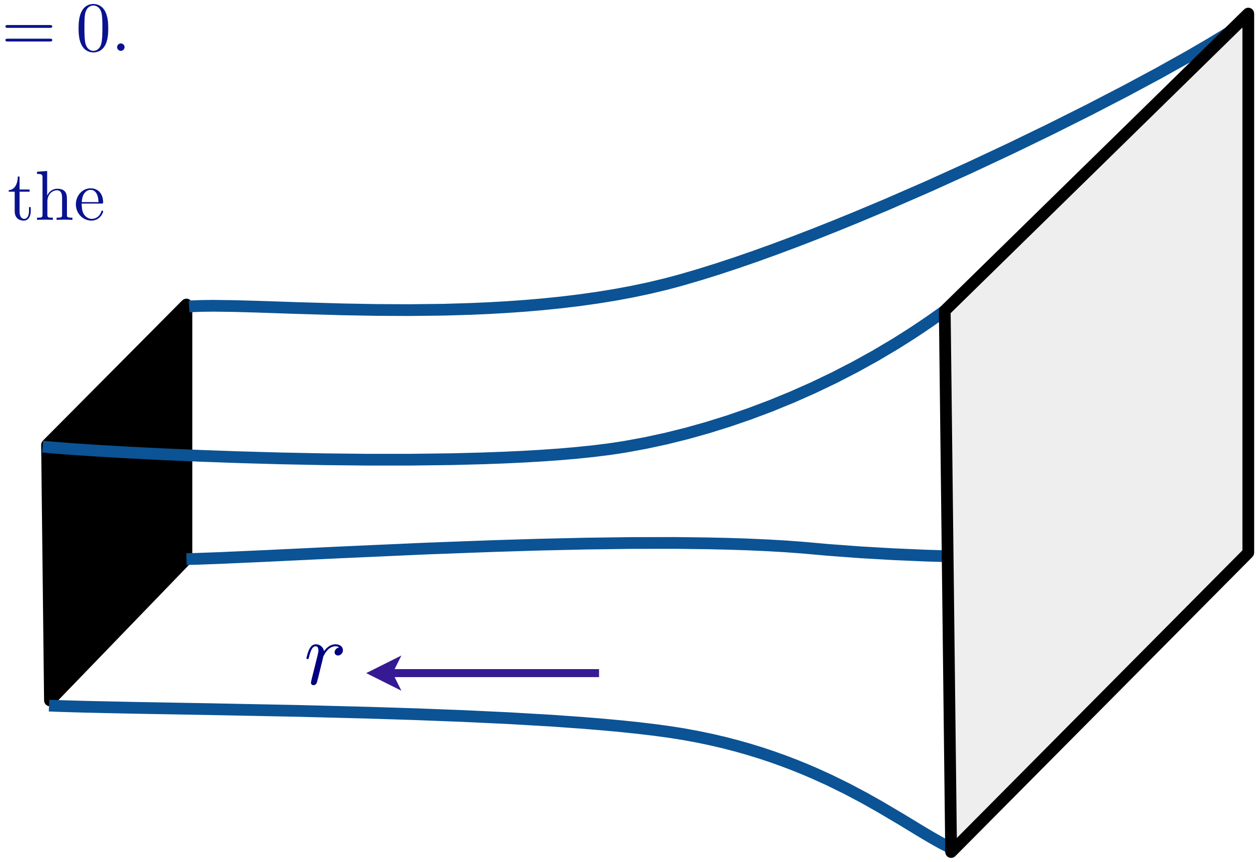
We have used reparametrization invariance in r to choose so that it scales as $r \rightarrow \zeta^{(d-\theta)/d} r$.

L. Huijse, S. Sachdev, B. Swingle, Physical Review B **85**, 035121 (2012)

At $T > 0$, there is a "black-brane" at $r = r_h$.

The Beckenstein-Hawking entropy of the black-brane is the thermal entropy of the quantum system $r = 0$.

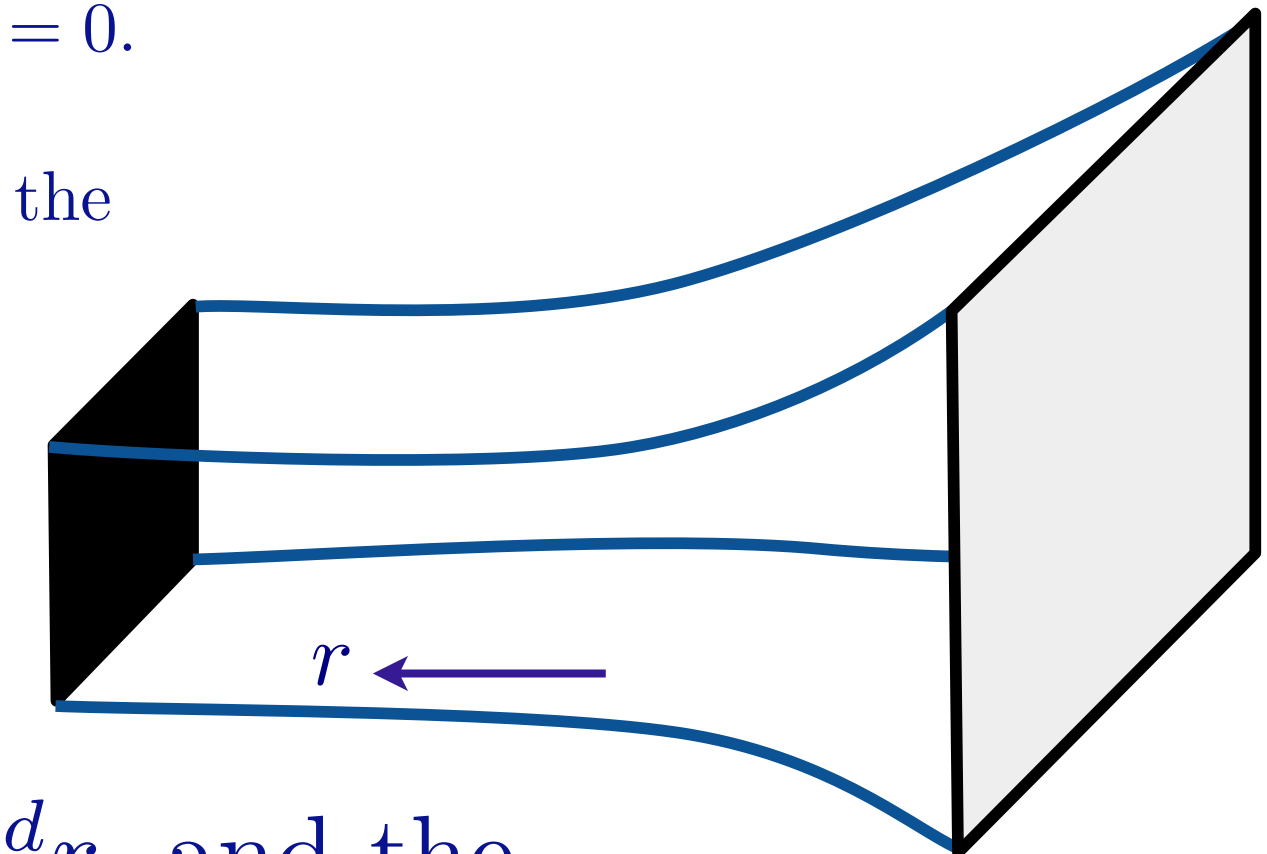
The entropy density, S, is proportional to the "area" of the horizon, and so $S \sim r_h^{-d}$



At $T > 0$, there is a "black-brane" at $r = r_h$.

The Beckenstein-Hawking entropy of the black-brane is the thermal entropy of the quantum system $r = 0$.

The entropy density, S, is proportional to the "area" of the horizon, and so $S \sim r_h^{-d}$



Under rescaling $r \to \zeta^{(d-\theta)/d} r$, and the temperature $T \sim t^{-1}$, and so

$$S \sim T^{(d-\theta)/z} = T^{d_{\rm eff}/z}$$

where $\theta = d - d_{\rm eff}$ measures "dimension deficit" in the phase space of low energy degrees of a freedom.

Holography of non-Fermi liquids

$$\boxed{ds^2 = \frac{1}{r^2}\left(-\frac{dt^2}{r^{2d(z-1)/(d-\theta)}} + r^{2\theta/(d-\theta)}dr^2 + dx_i^2\right)}$$

A non-Fermi liquid has gapless fermionic excitations on the Fermi surface, which disperse in the single transverse direction with dynamic critical exponent z, with entropy density $\sim T^{1/z}$. So we expect compressible quantum states to have $\boxed{d_{\text{eff}} = 1}$, or

$$\boxed{\theta = d - 1}$$

L. Huijse, S. Sachdev, B. Swingle, Physical Review B **85**, 035121 (2012)

Holography of non-Fermi liquids

$$ds^2 = \frac{1}{r^2}\left(-\frac{dt^2}{r^{2d(z-1)/(d-\theta)}} + r^{2\theta/(d-\theta)}dr^2 + dx_i^2\right)$$

$$\theta = d - 1$$

L. Huijse, S. Sachdev, B. Swingle, Physical Review B **85**, 035121 (2012)

Holography of non-Fermi liquids

$$\boxed{ds^2 = \frac{1}{r^2}\left(-\frac{dt^2}{r^{2d(z-1)/(d-\theta)}} + r^{2\theta/(d-\theta)}dr^2 + dx_i^2\right)}$$

$$\boxed{\theta = d - 1}$$

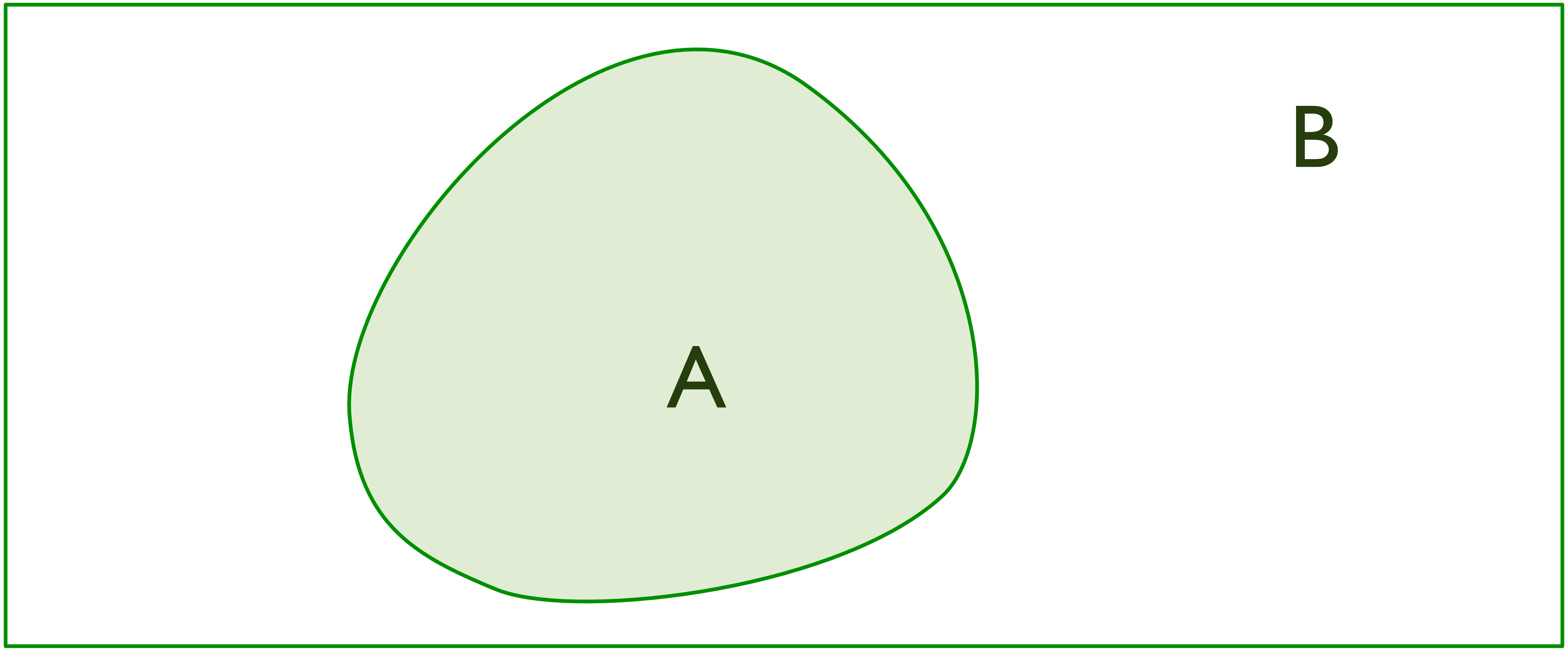
This metric is constrained by "null energy condition" of gravity. Using the metric and Einstein's equation we can obtain the stress-energy tensor $T_{\mu\nu}$. Then the constraint $T_{\mu\nu}N^\mu N^\nu \geq 0$ with N_μ a null vector yeids

$$\boxed{z \geq 1 + \frac{\theta}{d}.}$$

Remarkably, for $d = 2$, $\theta = d - 1$ and $z = 1 + \theta/d$, we obtain $\boxed{z \geq 3/2;}$ the lower bound is precisely the value of the field theory!

N. Ogawa, T. Takayanagi, and T. Ugajin, arXiv:1111.1023
L. Huijse, S. Sachdev, B. Swingle, Physical Review B **85**, 035121 (2012)

Entanglement entropy

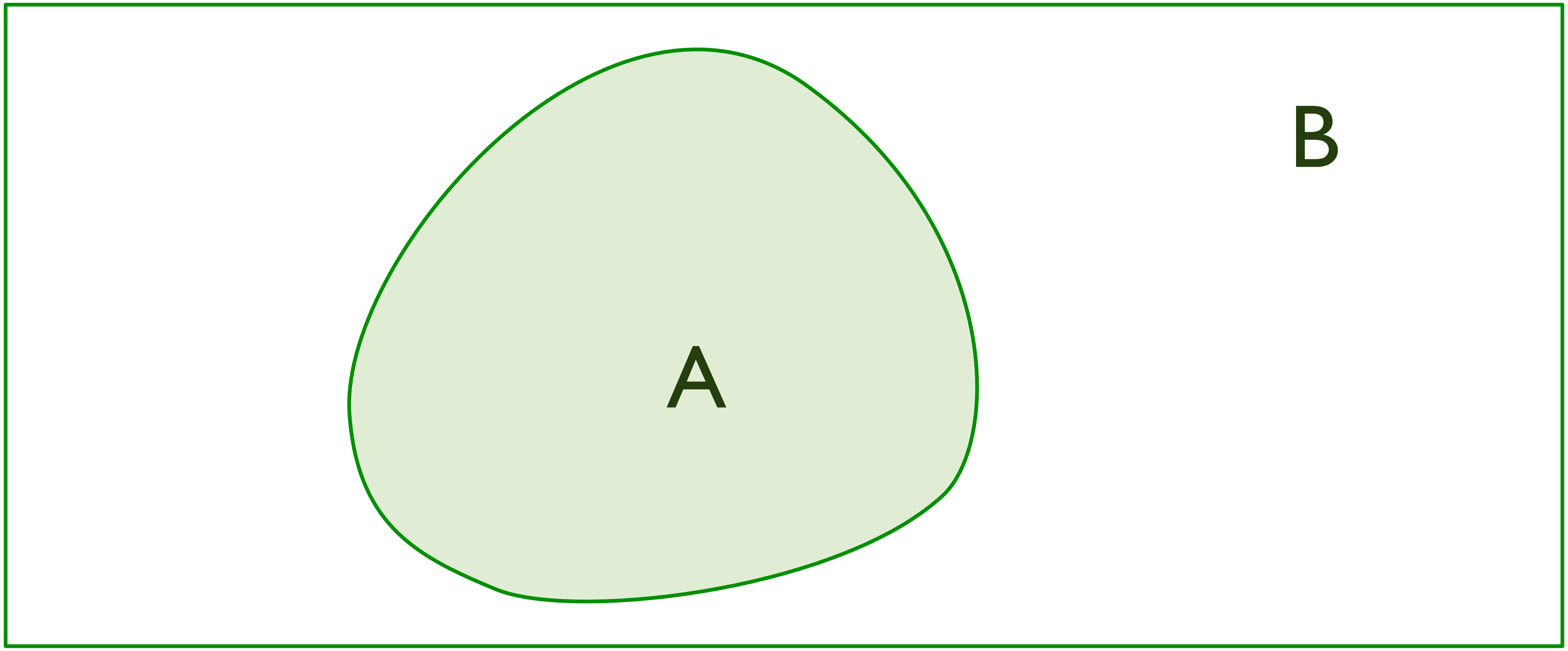


Measure strength of quantum entanglement of region A with region B.

$$\rho_A = \mathrm{Tr}_B \rho = \text{density matrix of region } A$$

Entanglement entropy $S_{EE} = -\mathrm{Tr}\left(\rho_A \ln \rho_A\right)$

Entanglement entropy of Fermi surfaces

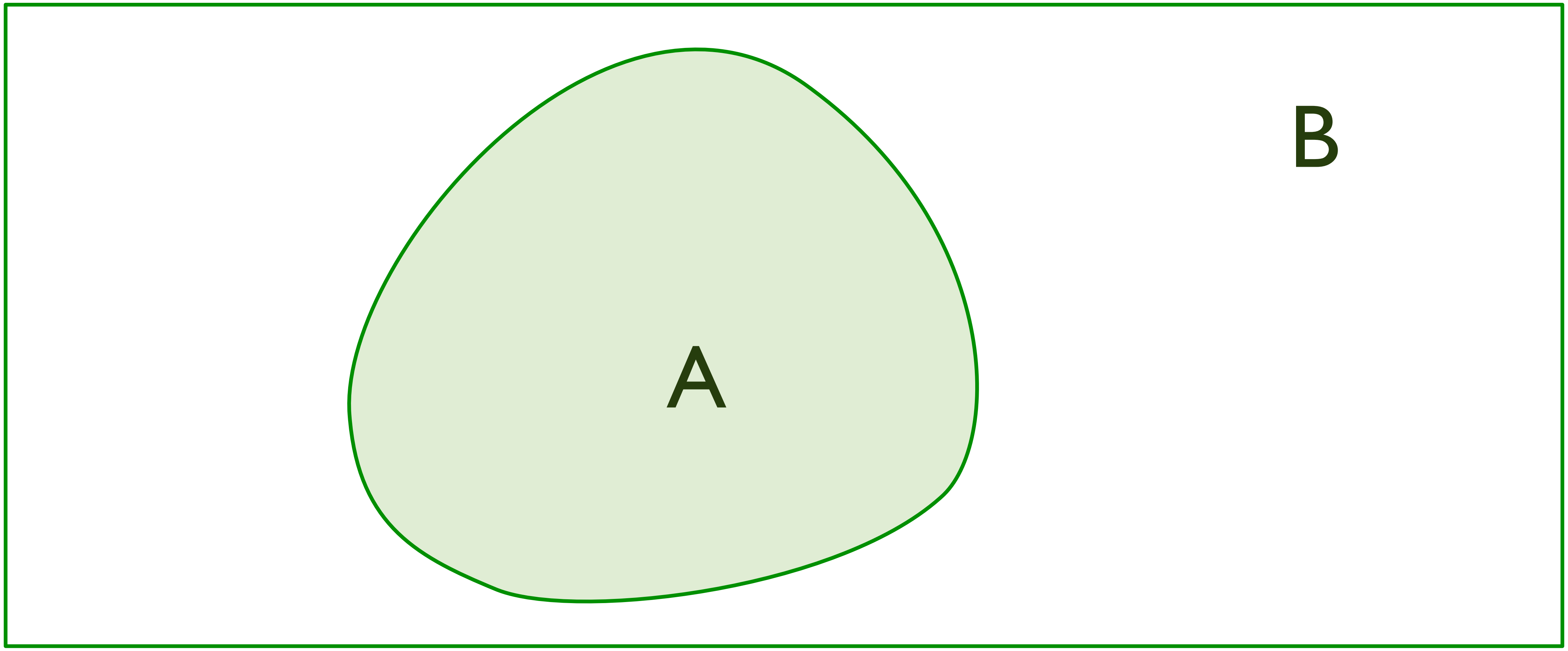


Logarithmic violation of "area law": $S_{EE} = \frac{1}{12}(k_F P)\ln(k_F P)$

for a circular Fermi surface with Fermi momentum k_F,
where P is the perimeter of region A with an arbitrary smooth shape.

D. Gioev and I. Klich, *Physical Review Letters* **96**, 100503 (2006)
B. Swingle, *Physical Review Letters* **105**, 050502 (2010)

Entanglement entropy of Fermi surfaces



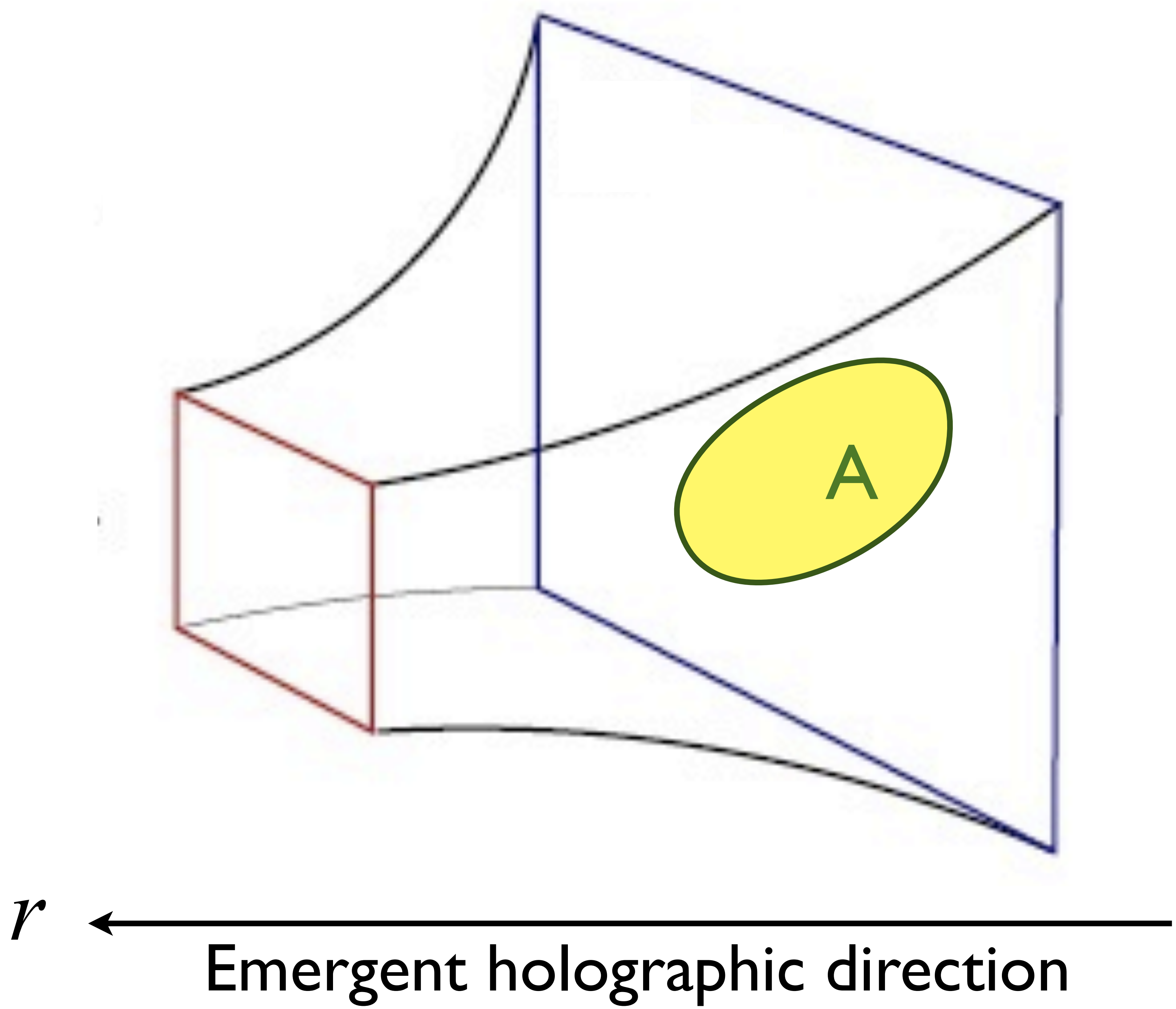
Logarithmic violation of "area law": $S_{EE} = \frac{1}{12}(k_F P)\ln(k_F P)$

for a circular Fermi surface with Fermi momentum k_F,
where P is the perimeter of region A with an arbitrary smooth shape.

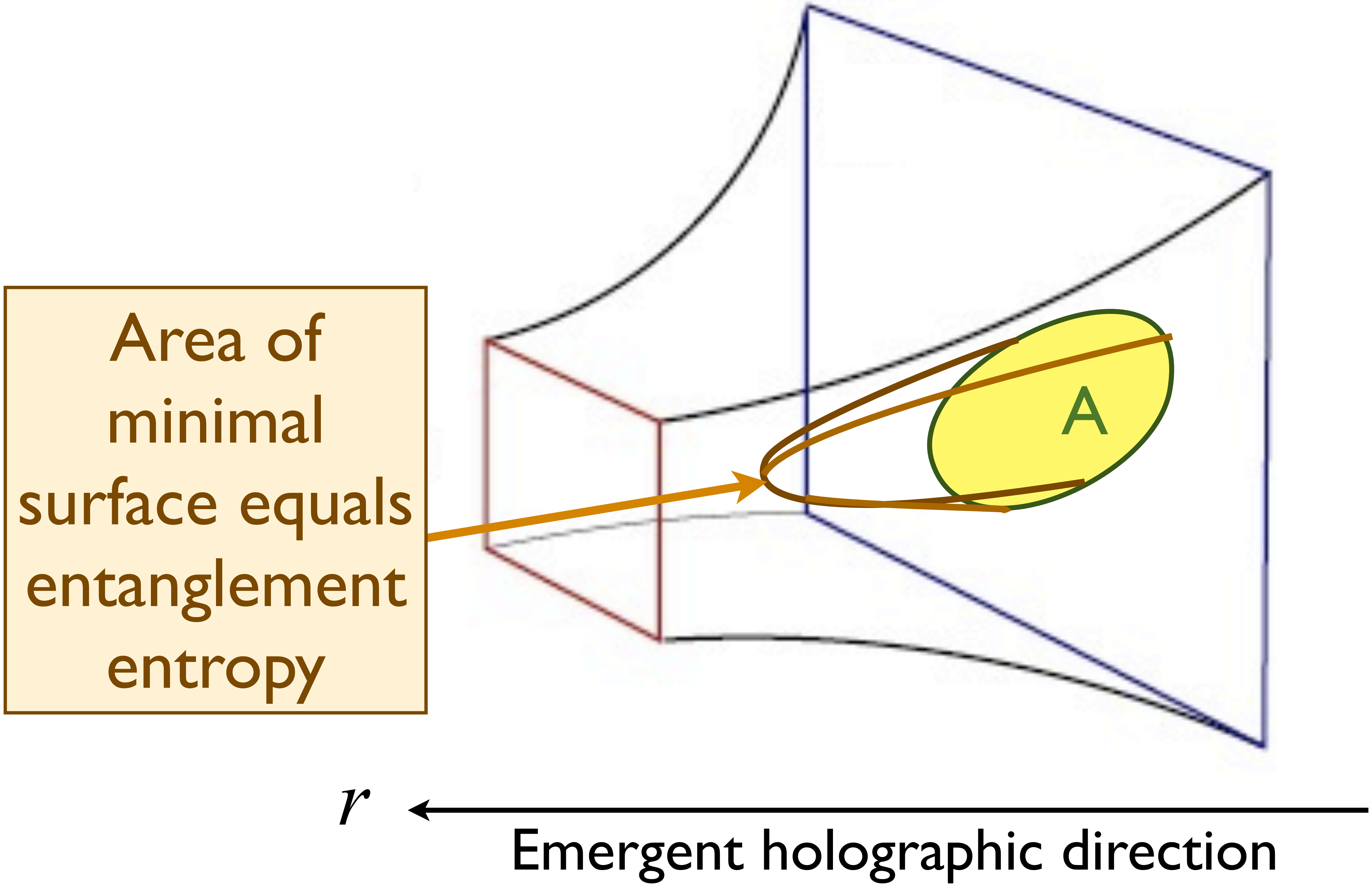
Non-Fermi liquids have, at most, the "1/12" prefactor modified.

Y. Zhang, T. Grover, and A. Vishwanath, *Physical Review Letters* **107**, 067202 (2011)

Holographic entanglement entropy



Holographic entanglement entropy



S. Ryu and T. Takayanagi, Phys. Rev. Lett. 96, 18160 (2006).

Holography of non-Fermi liquids

$$ds^2 = \frac{1}{r^2}\left(-\frac{dt^2}{r^{2d(z-1)/(d-\theta)}} + r^{2\theta/(d-\theta)}dr^2 + dx_i^2\right)$$

$$\theta = d - 1$$

L. Huijse, S. Sachdev, B. Swingle, Physical Review B **85**, 035121 (2012)

Holography of non-Fermi liquids

$$\boxed{ds^2 = \frac{1}{r^2}\left(-\frac{dt^2}{r^{2d(z-1)/(d-\theta)}} + r^{2\theta/(d-\theta)}dr^2 + dx_i^2\right)}$$

$$\boxed{\theta = d - 1}$$

- The entanglement entropy exhibits logarithmic violation of the area law only for this value of θ !!

N. Ogawa, T. Takayanagi, and T. Ugajin, arXiv:1111.1023

L. Huijse, S. Sachdev, B. Swingle, Physical Review B **85**, 035121 (2012)

Holography of non-Fermi liquids

$$\boxed{ds^2 = \frac{1}{r^2}\left(-\frac{dt^2}{r^{2d(z-1)/(d-\theta)}} + r^{2\theta/(d-\theta)}dr^2 + dx_i^2\right)}$$

$$\boxed{\theta = d - 1}$$

- The entanglement entropy exhibits logarithmic violation of the area law only for this value of θ !!

- The logarithmic violation is of the form $P \ln P$, where P is the perimeter of the entangling region. This form is *independent* of the shape of the entangling region, just as is expected for a (hidden) Fermi surface !!!

L. Huijse, S. Sachdev, B. Swingle, Physical Review B **85**, 035121 (2012)

Summary

Compressible quantum matter

- Consider an infinite, continuum, translationally-invariant quantum system with a globally conserved U(1) charge $\mathcal{Q}$ (the "electron density") in spatial dimension $d > 1$.

- Describe *zero temperature* phases where $d\langle\mathcal{Q}\rangle/d\mu \neq 0$, where μ (the "chemical potential") which changes the Hamiltonian, H, to $H - \mu\mathcal{Q}$.

Theory of a non-Fermi liquid (NFL)

Field theory	Holography
A gauge-dependent Fermi surface of overdamped gapless fermions.	Fermi surface is hidden.

Theory of a non-Fermi liquid (NFL)

Field theory	Holography
A gauge-dependent Fermi surface of overdamped gapless fermions.	Fermi surface is hidden.
Thermal entropy density $S \sim T^{1/z}$ in $d = 2$, where z is the dynamic critical exponent.	Thermal entropy density $S \sim T^{1/z}$ in all d for hyperscaling violation exponent $\theta = d - 1$, and z the dynamic critical exponent.

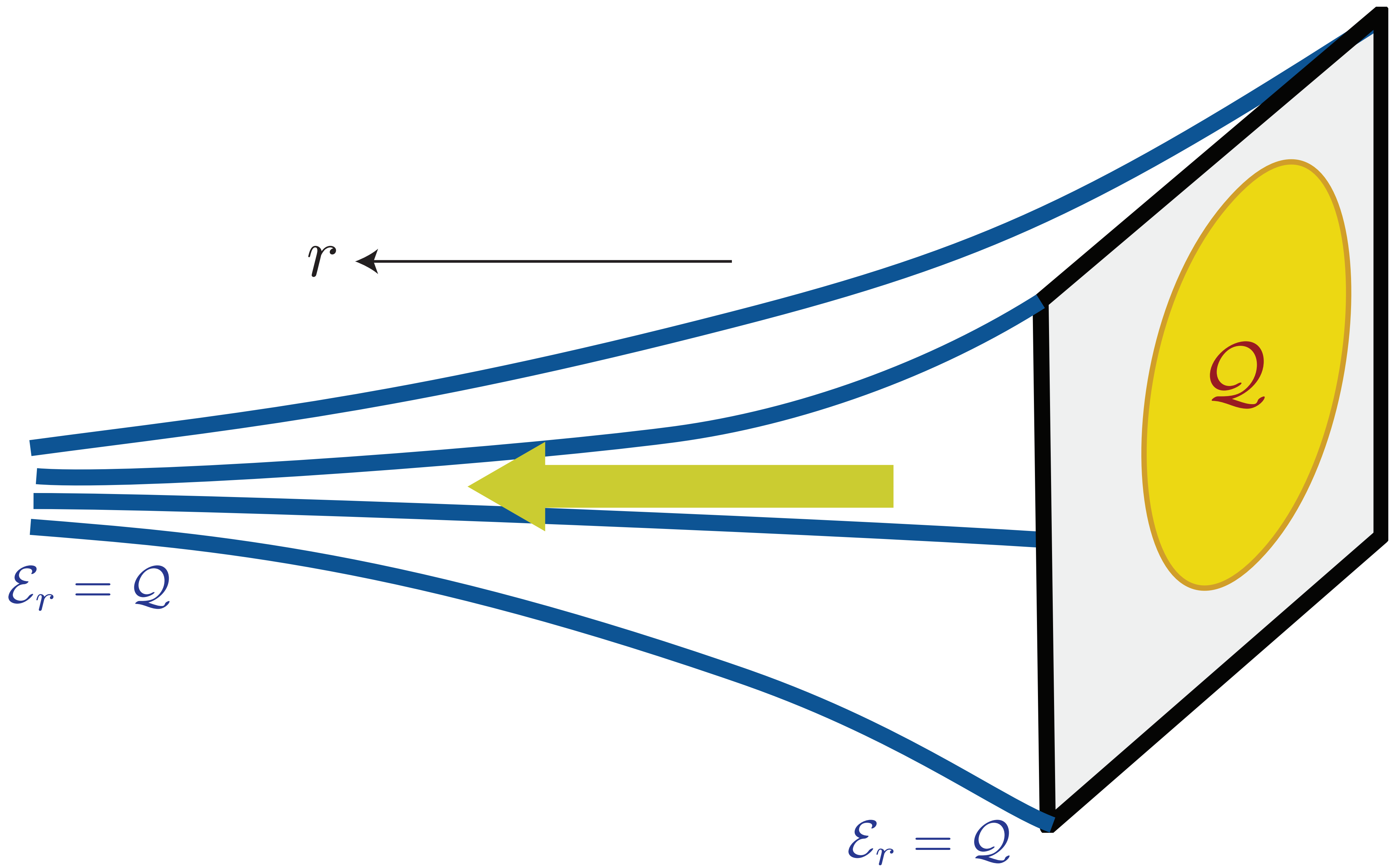
Theory of a non-Fermi liquid (NFL)

Field theory	Holography
A gauge-dependent Fermi surface of overdamped gapless fermions.	Fermi surface is hidden.
Thermal entropy density $S \sim T^{1/z}$ in $d = 2$, where z is the dynamic critical exponent.	Thermal entropy density $S \sim T^{1/z}$ in all d for hyperscaling violation exponent $\theta = d - 1$, and z the dynamic critical exponent.
Logarithmic violation of area law of entanglement entropy, with prefactor proportional to the boundary area of the entangling region, but independent of its shape.	Logarithmic violation of area law of entanglement entropy, with prefactor proportional to the boundary area of the entangling region, but independent of its shape.

Theory of a non-Fermi liquid (NFL)

Field theory	Holography
Three-loop analysis shows $z = 3/2$ in $d = 2$.	Existence of gravity dual implies $z \geq 1 + \theta/d$; leads to $z \geq 3/2$ for $\theta = d-1$ in $d = 2$.

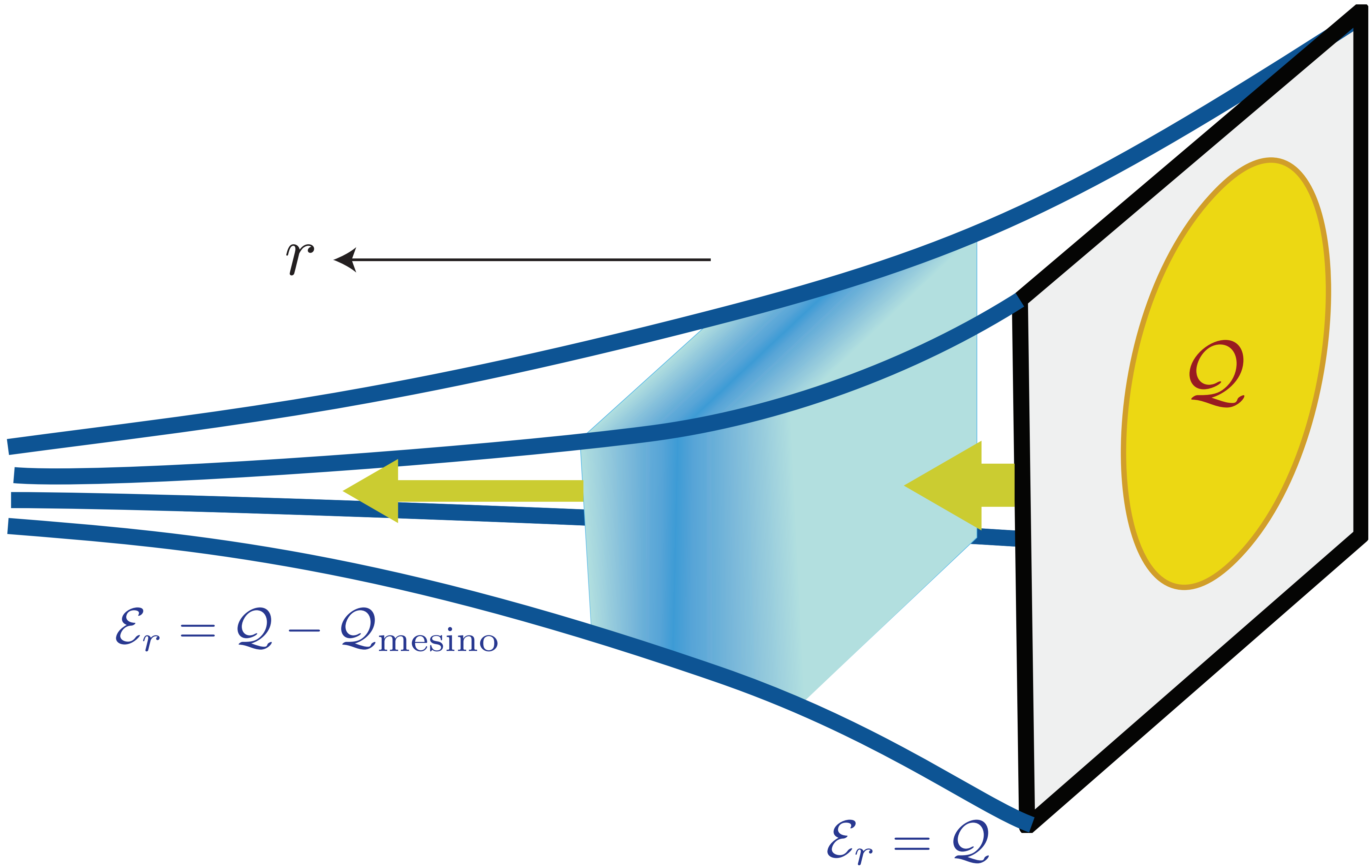
Holographic theory of a non-Fermi liquid (NFL)



Gauss Law in the bulk

$\Leftrightarrow$ Luttinger theorem on the boundary

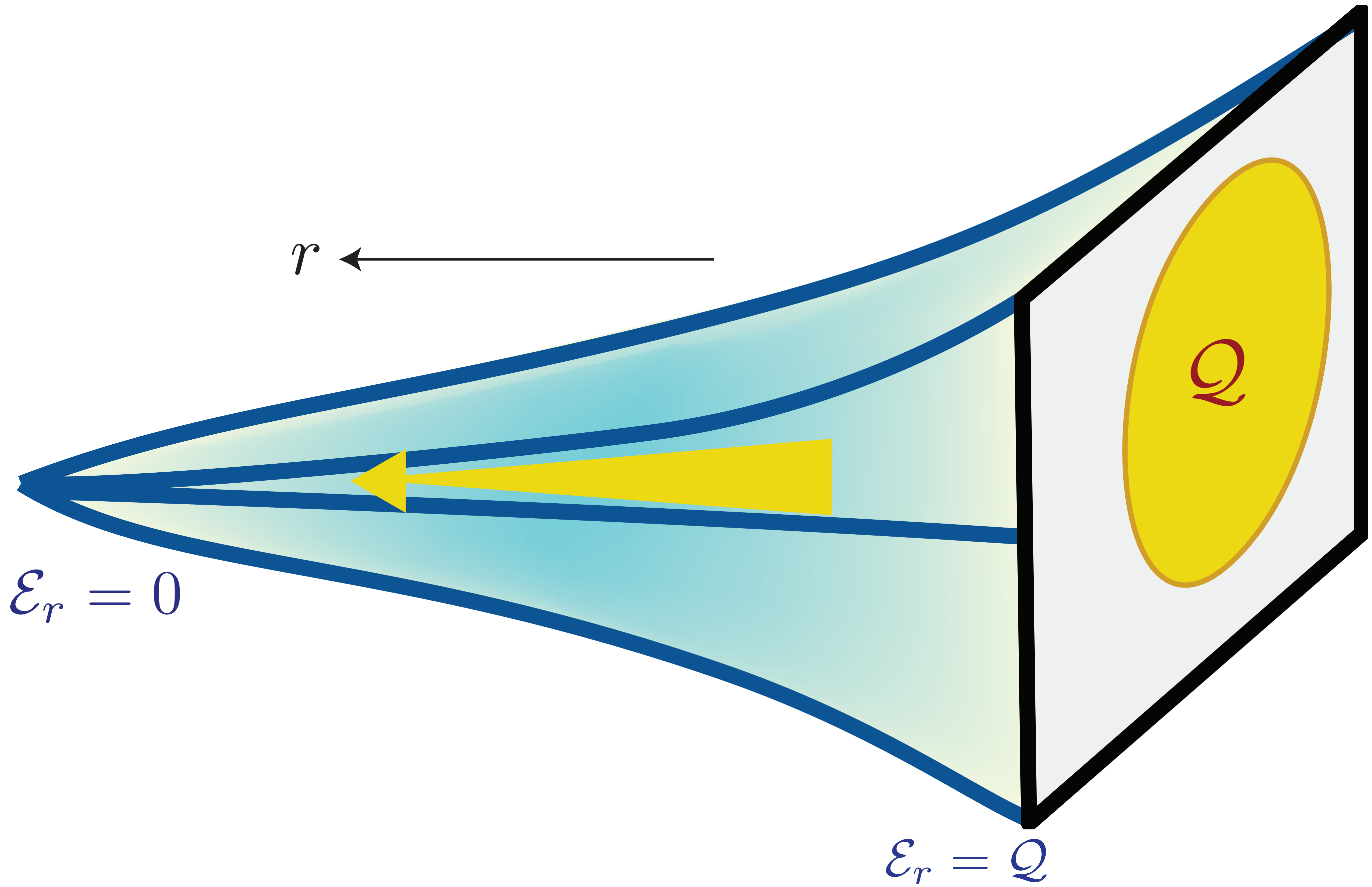
Holographic theory of a fractionalized-Fermi liquid (FL*)



Gauss Law in the bulk

$\Leftrightarrow$ Luttinger theorem on the boundary

Holographic theory of a Fermi liquid (FL)

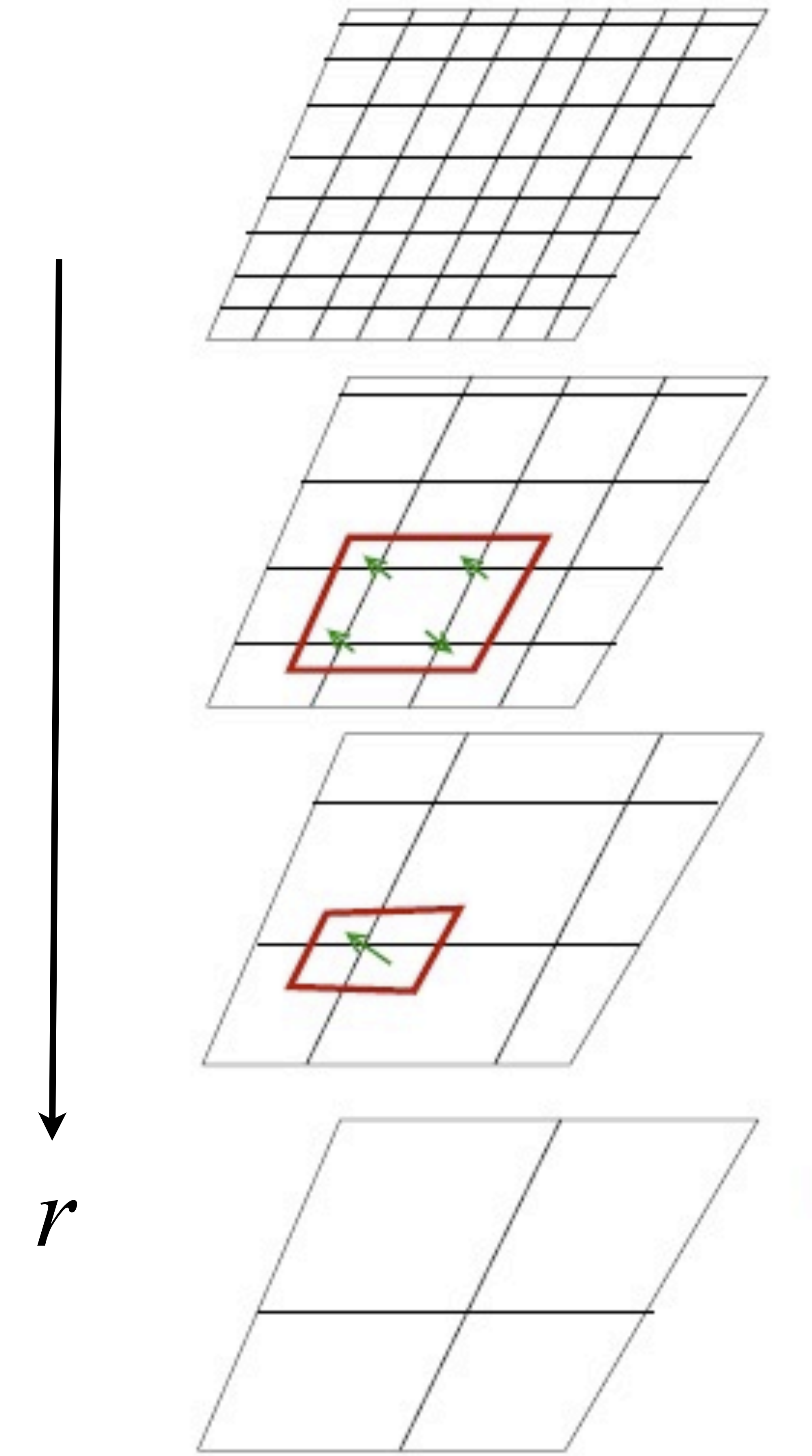


Gauss Law in the bulk

$\Leftrightarrow$ Luttinger theorem on the boundary

Lecture 3


ASCENSO PROHIBIDO
CLIMBING FORBIDDEN
NO SUBIR



J. McGreevy, arXiv0909.0518

Consider the metric which transforms under rescaling as

$$\begin{aligned} x_i &\rightarrow \zeta\, x_i \\ t &\rightarrow \zeta^z\, t \\ ds &\rightarrow \zeta^{\theta/d}\, ds. \end{aligned}$$

This identifies z as the dynamic critical exponent ($z = 1$ for "relativistic" quantum critical points).

θ is the violation of hyperscaling exponent.
The most general choice of such a metric is

$$\boxed{ds^2 = \frac{1}{r^2}\left(-\frac{dt^2}{r^{2d(z-1)/(d-\theta)}} + r^{2\theta/(d-\theta)}dr^2 + dx_i^2\right)}$$

We have used reparametrization invariance in r to choose so that it scales as $r \rightarrow \zeta^{(d-\theta)/d} r$.

L. Huijse, S. Sachdev, B. Swingle, Physical Review B **85**, 035121 (2012)

$$\boxed{ds^2 = \frac{1}{r^2}\left(-\frac{dt^2}{r^{2d(z-1)/(d-\theta)}} + r^{2\theta/(d-\theta)}dr^2 + dx_i^2\right)}$$

$$ds^2 = \frac{1}{r^2}\left(-\frac{dt^2}{r^{2d(z-1)/(d-\theta)}} + r^{2\theta/(d-\theta)}dr^2 + dx_i^2\right)$$

- The thermal entropy density scales as

$$S \sim T^{(d-\theta)/z}.$$

The third law of thermodynamics requires $\theta < d$.

$$\boxed{ds^2 = \frac{1}{r^2}\left(-\frac{dt^2}{r^{2d(z-1)/(d-\theta)}} + r^{2\theta/(d-\theta)}dr^2 + dx_i^2\right)}$$

- The thermal entropy density scales as

$$S \sim T^{(d-\theta)/z}.$$

  The third law of thermodynamics requires $\theta < d$.

- The entanglement entropy, S_E, of an entangling region with boundary surface 'area' Σ scales as

$$S_E \sim \begin{cases} \Sigma & , \quad \text{for } \theta < d-1 \\ \Sigma \ln \Sigma & , \quad \text{for } \theta = d-1 \\ \Sigma^{\theta/(d-1)} & , \quad \text{for } \theta > d-1 \end{cases}$$

  All local quantum field theories obey the "area law" (upto log violations) and so $\theta \le d-1$.

$$ds^2 = \frac{1}{r^2}\left(-\frac{dt^2}{r^{2d(z-1)/(d-\theta)}} + r^{2\theta/(d-\theta)}dr^2 + dx_i^2\right)$$

- The thermal entropy density scales as

$$S \sim T^{(d-\theta)/z}.$$

The third law of thermodynamics requires $\theta < d$.

- The entanglement entropy, S_E, of an entangling region with boundary surface 'area' Σ scales as

$$S_E \sim \begin{cases} \Sigma & , \quad \text{for } \theta < d-1 \\ \Sigma \ln \Sigma & , \quad \text{for } \theta = d-1 \\ \Sigma^{\theta/(d-1)} & , \quad \text{for } \theta > d-1 \end{cases}$$

All local quantum field theories obey the "area law" (upto log violations) and so $\theta \leq d-1$.

- The null energy condition implies $z \geq 1 + \dfrac{\theta}{d}$.

$$ds^2 = \frac{1}{r^2}\left(-\frac{dt^2}{r^{2d(z-1)/(d-\theta)}} + r^{2\theta/(d-\theta)}dr^2 + dx_i^2\right)$$

- The thermal entropy density scales as

$$S \sim T^{(d-\theta)/z}.$$

The third law of thermodynamics requires $\theta < d$.

- The entanglement entropy, S_E, of an entangling region with boundary surface 'area' Σ scales as

$$S_E \sim \begin{cases} \Sigma & , \quad \text{for } \theta < d-1 \\ \Sigma \ln \Sigma & , \quad \text{for } \theta = d-1 \\ \Sigma^{\theta/(d-1)} & , \quad \text{for } \theta > d-1 \end{cases}$$

All local quantum field theories obey the "area law" (upto log violations) and so $\theta \leq d-1$.

- The null energy condition implies $z \geq 1 + \dfrac{\theta}{d}$.

Strange metals

A. Field theory

$\boxed{\textit{B. Holography}}$

Holography of non-Fermi liquids

$$\boxed{ds^2 = \frac{1}{r^2}\left(-\frac{dt^2}{r^{2d(z-1)/(d-\theta)}} + r^{2\theta/(d-\theta)}dr^2 + dx_i^2\right)}$$

$$\boxed{\theta = d - 1}$$

- The value of θ is fixed by requiring that the thermal entropy density $S \sim T^{1/z}$ for general d.
  Conjecture: this metric then describes a compressible state with a *hidden* Fermi surface.

L. Huijse, S. Sachdev, B. Swingle, Physical Review B **85**, 035121 (2012)

Holography of non-Fermi liquids

$$\boxed{ds^2 = \frac{1}{r^2}\left(-\frac{dt^2}{r^{2d(z-1)/(d-\theta)}} + r^{2\theta/(d-\theta)}dr^2 + dx_i^2\right)}$$

$$\boxed{\theta = d - 1}$$

- The value of θ is fixed by requiring that the thermal entropy density $S \sim T^{1/z}$ for general d.
  Conjecture: this metric then describes a compressible state with a *hidden* Fermi surface.

- The null energy condition yields the inequality $z \geq 1 + \theta/d$. For $d = 2$ and $\theta = 1$ this yields $z \geq 3/2$. The field theory analysis gave $z = 3/2$ to three loops !

L. Huijse, S. Sachdev, B. Swingle, Physical Review B **85**, 035121 (2012)

Holography of non-Fermi liquids

$$ds^2 = \frac{1}{r^2}\left(-\frac{dt^2}{r^{2d(z-1)/(d-\theta)}} + r^{2\theta/(d-\theta)}dr^2 + dx_i^2\right)$$

$$\theta = d - 1$$

- The entanglement entropy exhibits logarithmic violation of the area law only for this value of θ !!

N. Ogawa, T. Takayanagi, and T. Ugajin, arXiv:1111.1023
L. Huijse, S. Sachdev, B. Swingle, Physical Review B **85**, 035121 (2012)

Holography of non-Fermi liquids

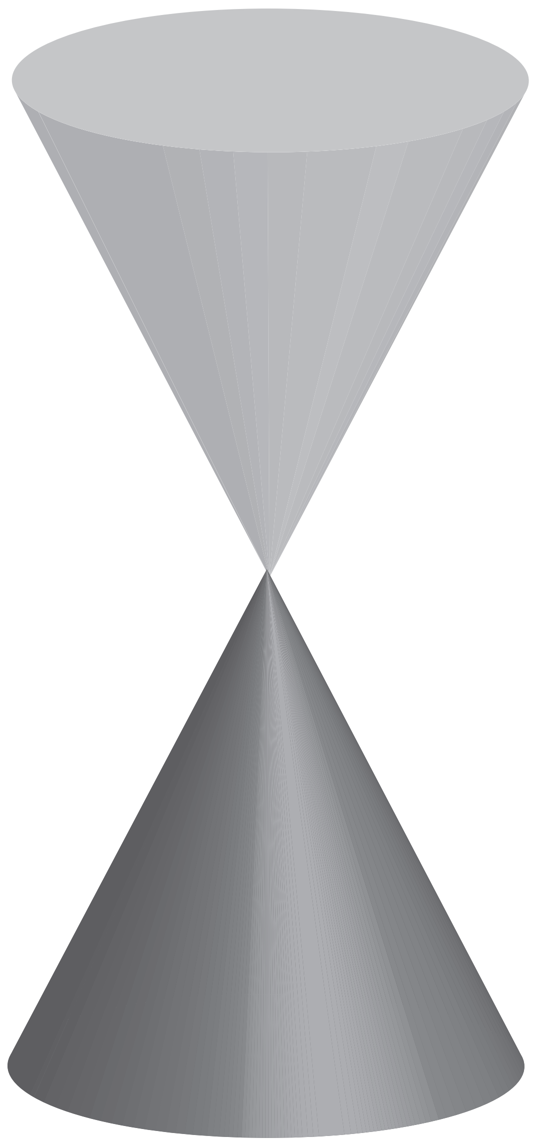
$$ds^2 = \frac{1}{r^2}\left(-\frac{dt^2}{r^{2d(z-1)/(d-\theta)}} + r^{2\theta/(d-\theta)}dr^2 + dx_i^2\right)$$

$$\theta = d - 1$$

- The entanglement entropy exhibits logarithmic violation of the area law only for this value of θ !!
- The logarithmic violation is of the form $P \ln P$, where P is the perimeter of the entangling region. This form is *independent* of the shape of the entangling region, just as is expected for a (hidden) Fermi surface !!!

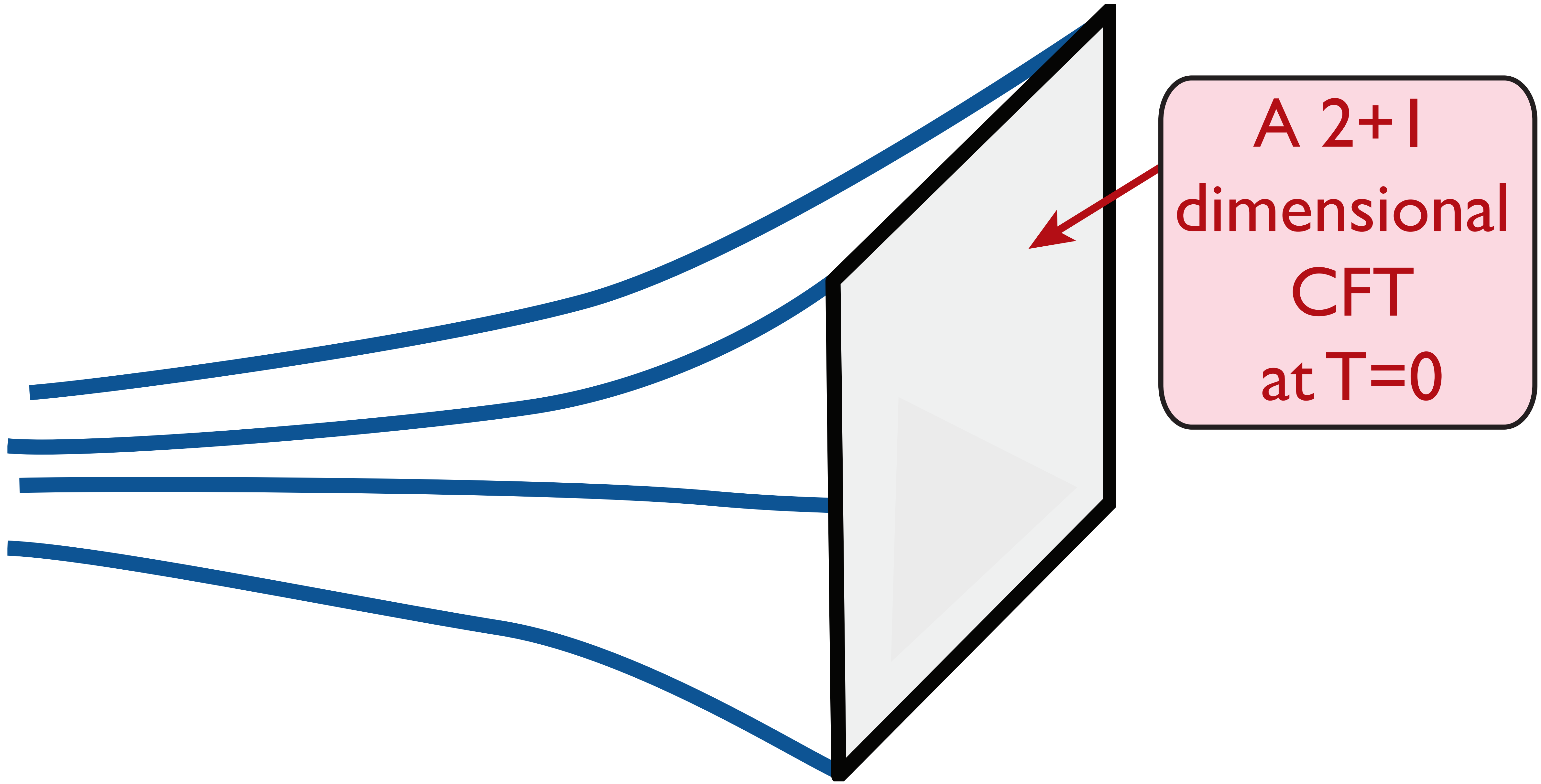
L. Huijse, S. Sachdev, B. Swingle, Physical Review B **85**, 035121 (2012)

Begin with a CFT



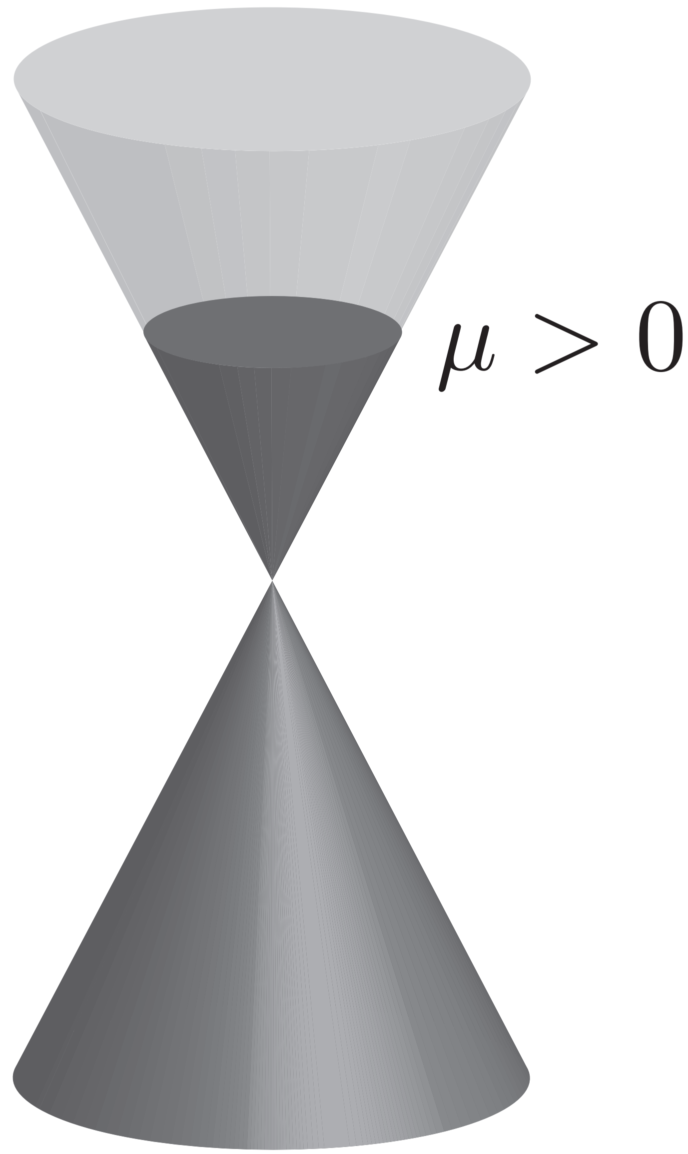
Dirac fermions + gauge field + ...

Holographic representation: AdS_4

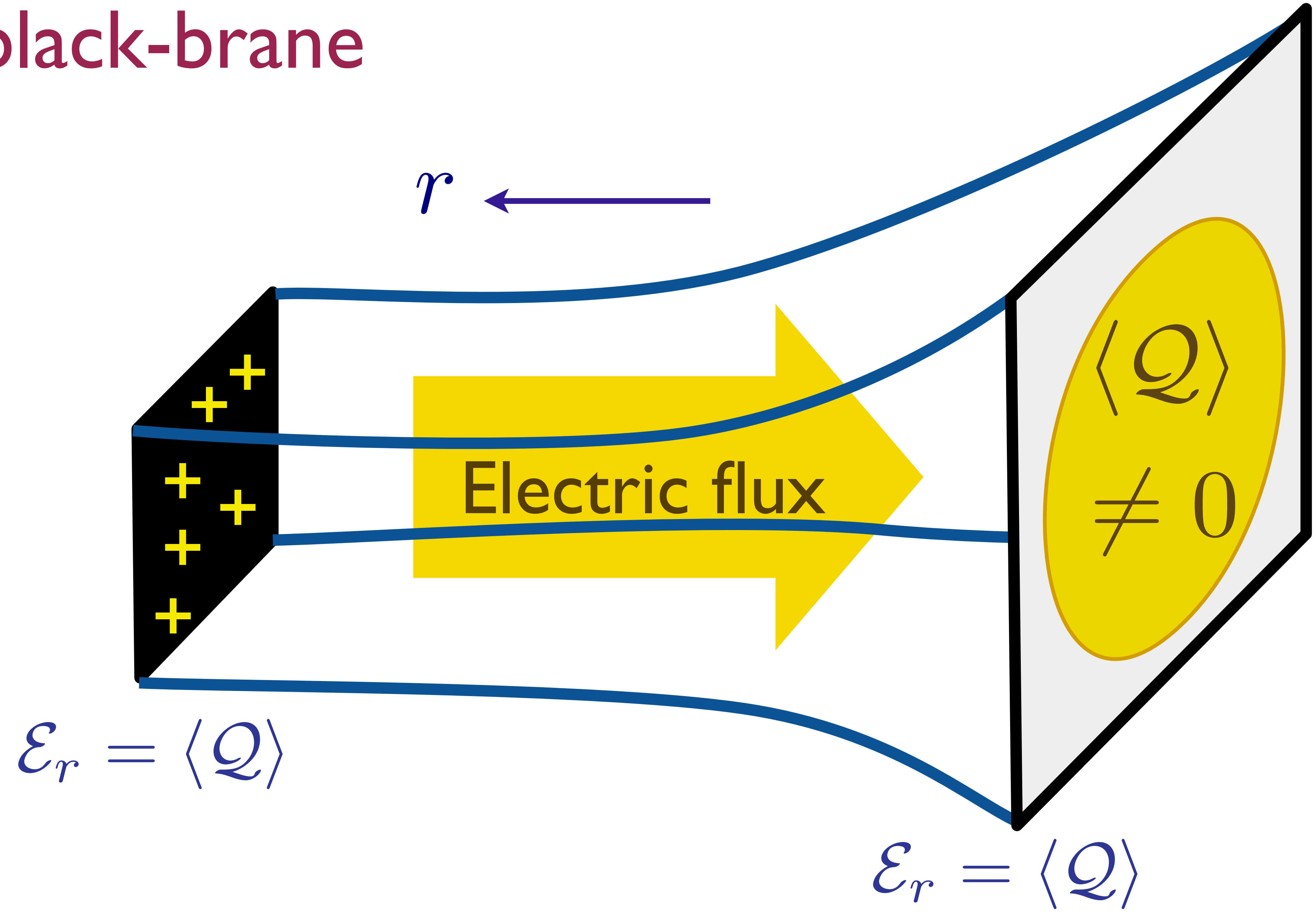


$$\mathcal{S} = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}\left(R + \frac{6}{L^2}\right)\right]$$

Apply a chemical potential to the "deconfined" CFT



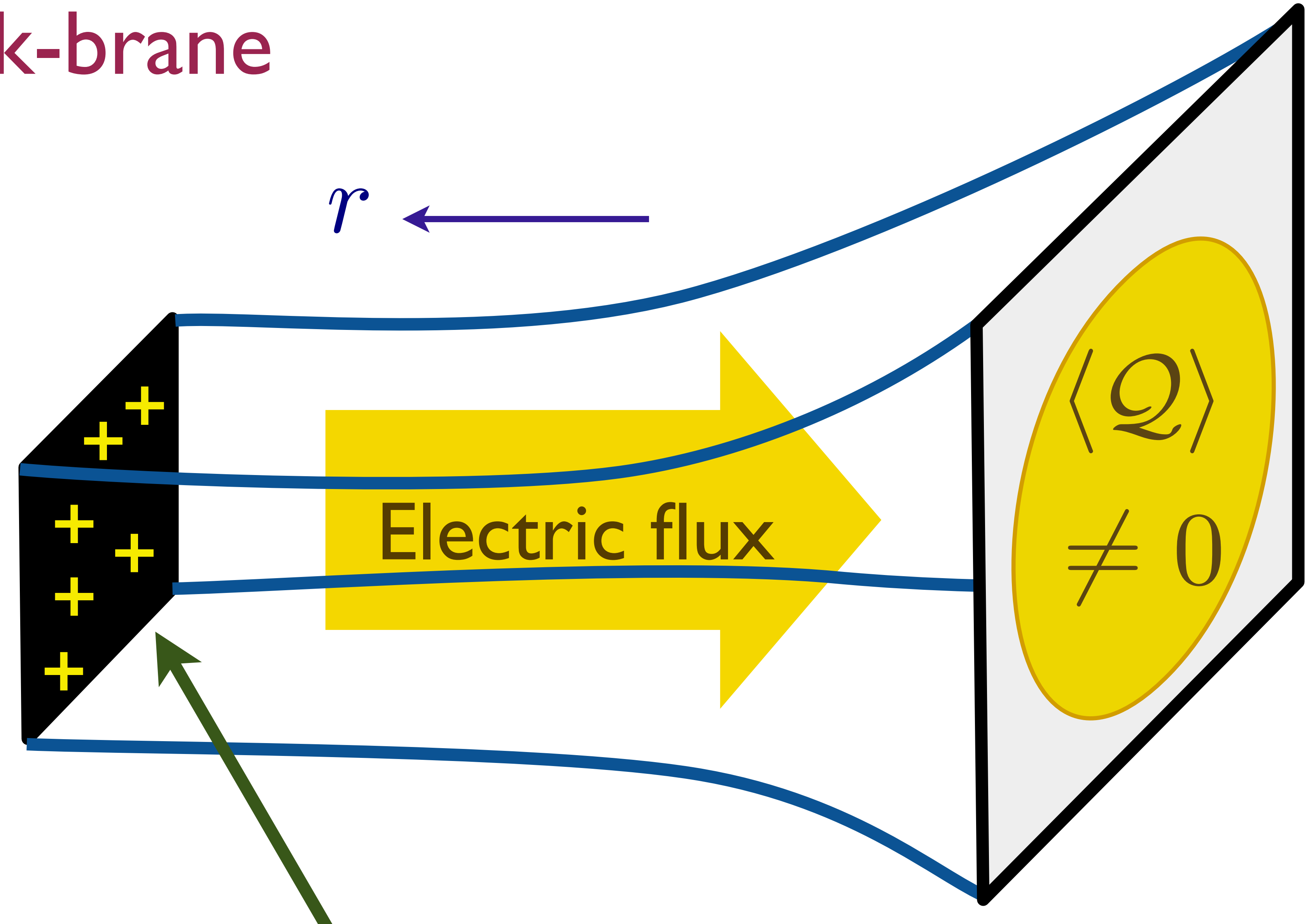
The Maxwell-Einstein theory of the applied chemical potential yields a AdS_4-Reissner-Nordtröm black-brane



$$\mathcal{S} = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa^2} \left(R + \frac{6}{L^2} \right) - \frac{1}{4e^2} F_{ab} F^{ab} \right]$$

S.A. Hartnoll, P. K. Kovtun, M. Müller, and S. Sachdev, Physical Review B **76**, 144502 (2007)

The Maxwell-Einstein theory of the applied chemical potential yields a AdS4-Reissner-Nordtröm black-brane



At $T = 0$, we obtain an extremal black-brane, with a near-horizon (IR) metric of $\mathrm{AdS}_2 \times R^2$

$$ds^2 = \frac{L^2}{6}\left(\frac{-dt^2 + dr^2}{r^2}\right) + dx^2 + dy^2$$

T. Faulkner, H. Liu, J. McGreevy, and D. Vegh, arXiv:0907.2694

Artifacts of AdS$_2$ X R^2

- Corresponds to $\theta \to d$ and $z \to \infty$. This implies non-zero entropy density at $T = 0$, and "volume" law for entanglement entropy.

- Green's function of a probe fermion (a *mesino*) can have a Fermi surface, but self energies are momentum independent, and the singular behavior is the same on and off the Fermi surface

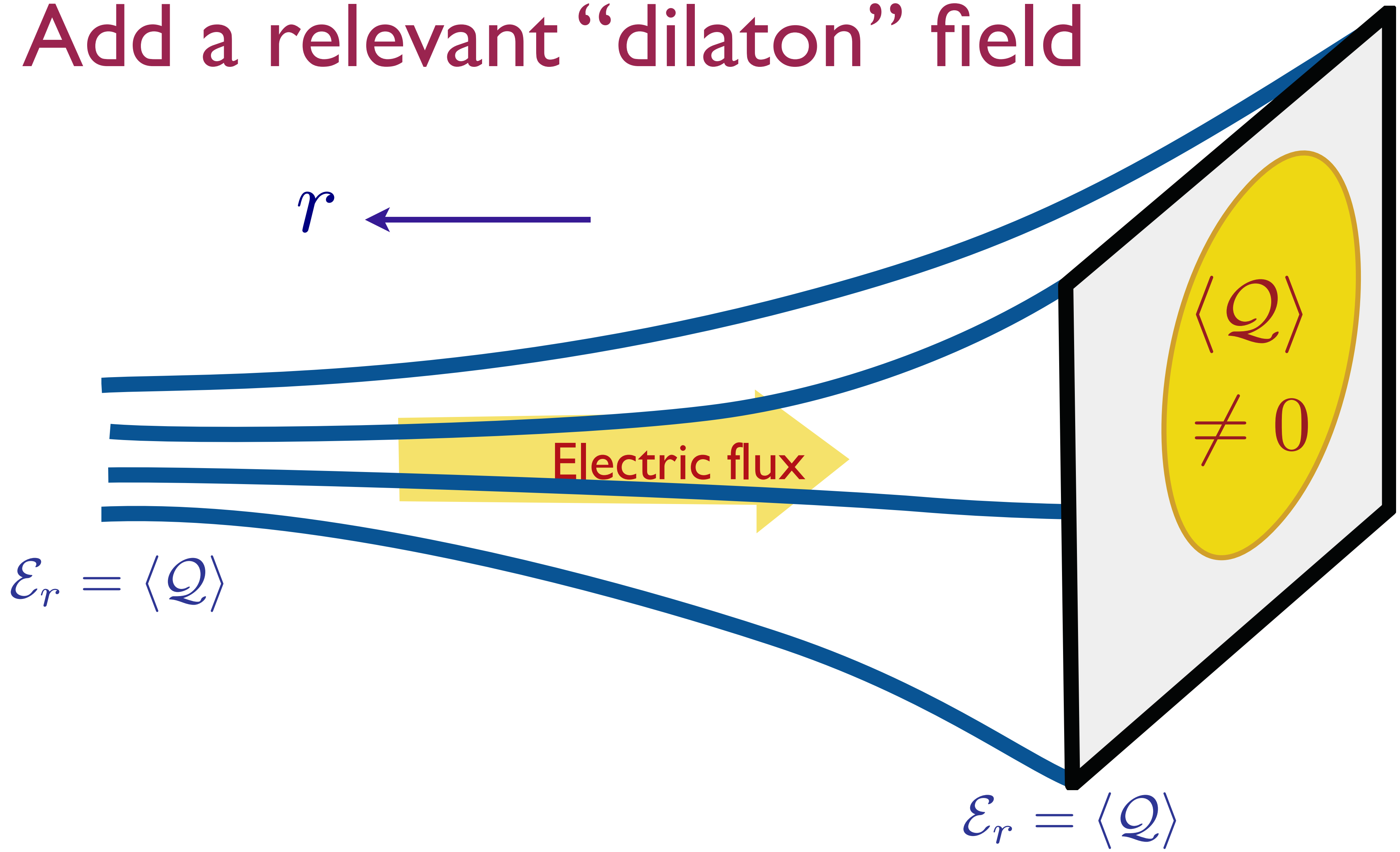
- Deficit of order $\sim N^2$ in the volume enclosed by the mesino Fermi surfaces: presumably associated with "hidden Fermi surfaces" of gauge-charged particles (the *quarks*).

T. Faulkner, H. Liu, J. McGreevy, and D. Vegh, arXiv:0907.2694

S. Sachdev, *Phys. Rev. Lett.* **105**, 151602 (2010).

Holographic theory of a non-Fermi liquid (NFL)

Add a relevant "dilaton" field



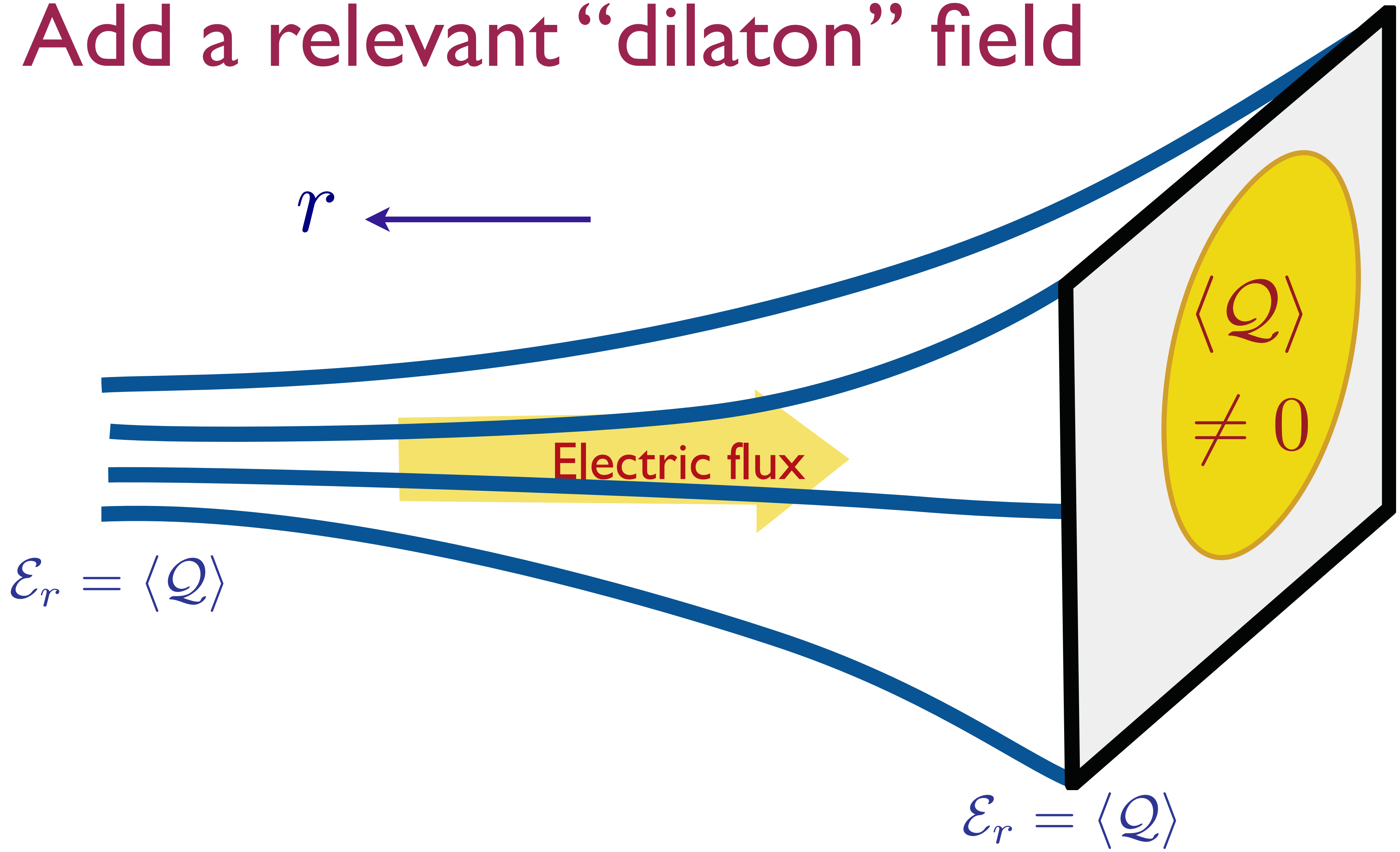
$$\mathcal{S} = \int d^{d+2}x \sqrt{-g} \left[\frac{1}{2\kappa^2} \left(R - 2(\nabla\Phi)^2 - \frac{V(\Phi)}{L^2} \right) - \frac{Z(\Phi)}{4e^2} F_{ab}F^{ab} \right]$$

with $Z(\Phi) = Z_0 e^{\alpha\Phi}$, $V(\Phi) = -V_0 e^{-\beta\Phi}$, as $\Phi \to \infty$.

C. Charmousis, B. Gouteraux, B. S. Kim, E. Kiritsis and R. Meyer, JHEP **1011**, 151 (2010).
S. S. Gubser and F. D. Rocha, Phys. Rev. D **81**, 046001 (2010).
N. Iizuka, N. Kundu, P. Narayan and S. P. Trivedi, arXiv:1105.1162 [hep-th].

Holographic theory of a non-Fermi liquid (NFL)

Add a relevant "dilaton" field



Leads to metric $ds^2 = L^2 \left(-f(r)dt^2 + g(r)dr^2 + \frac{dx^2 + dy^2}{r^2} \right)$

with $f(r) \sim r^{-\gamma}$, $g(r) \sim r^{\delta}$, $\Phi(r) \sim \ln(r)$ as $r \to \infty$.

C. Charmousis, B. Gouteraux, B. S. Kim, E. Kiritsis and R. Meyer, JHEP **1011**, 151 (2010).
S. S. Gubser and F. D. Rocha, Phys. Rev. D **81**, 046001 (2010).
N. Iizuka, N. Kundu, P. Narayan and S. P. Trivedi, arXiv:1105.1162 [hep-th].

Holographic theory of a non-Fermi liquid (NFL)

$$ds^2 = \frac{1}{r^2}\left(-\frac{dt^2}{r^{2d(z-1)/(d-\theta)}} + r^{2\theta/(d-\theta)}dr^2 + dx_i^2\right)$$

The $r \to \infty$ metric has the above form with

$$\begin{aligned}
\theta &= \frac{d^2\beta}{\alpha + (d-1)\beta} \\
z &= 1 + \frac{\theta}{d} + \frac{8(d(d-\theta)+\theta)^2}{d^2(d-\theta)\alpha^2}.
\end{aligned}$$

Note $z \geq 1 + \theta/d$.

Holographic theory of a non-Fermi liquid (NFL)

$$\boxed{ds^2 = \frac{1}{r^2}\left(-\frac{dt^2}{r^{2d(z-1)/(d-\theta)}} + r^{2\theta/(d-\theta)}dr^2 + dx_i^2\right)}$$

The solution also specifies the missing numerical prefactors in the metric. In general, these depend upon the details on the UV boundary condition as $r \to 0$. However, the coefficient of dx_i^2/r^2 turns out to be *independent* of the UV boundary conditions, and proportional to $\mathcal{Q}^{2\theta/(d(d-\theta))}$.

The square-root of this coefficient is the prefactor of the log divergence in the entanglement entropy for $\theta = d-1$.

Holography of non-Fermi liquids

$$\boxed{ds^2 = \frac{1}{r^2}\left(-\frac{dt^2}{r^{2d(z-1)/(d-\theta)}} + r^{2\theta/(d-\theta)}dr^2 + dx_i^2\right)}$$

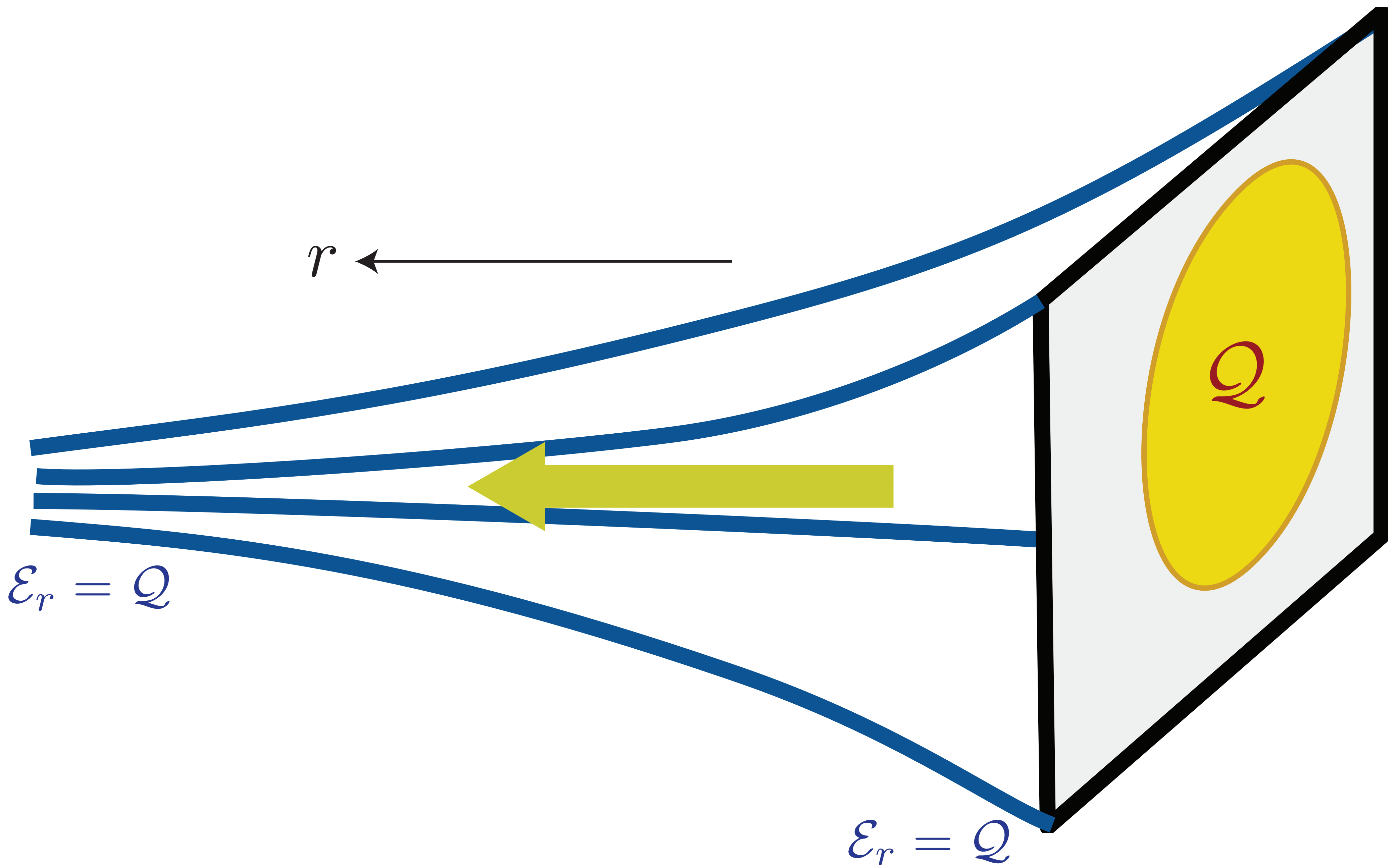
$$\boxed{\theta = d - 1}$$

- The entanglement entropy has log-violation of the area law
$$S_E = \Xi\, \mathcal{Q}^{(d-1)/d}\Sigma \ln\left(\mathcal{Q}^{(d-1)/d}\Sigma\right).$$

  where Σ is surface area of the entangling region, and Ξ is a dimensionless constant which is *independent* of $\mathcal{Q}$ and of any property of the entangling region.
  Note $\mathcal{Q}^{(d-1)/d} \sim k_F^{d-1}$ via the Luttinger relation, and then S_E is just as expected for a Fermi surface !!!!

L. Huijse, S. Sachdev, B. Swingle, Physical Review B **85**, 035121 (2012)

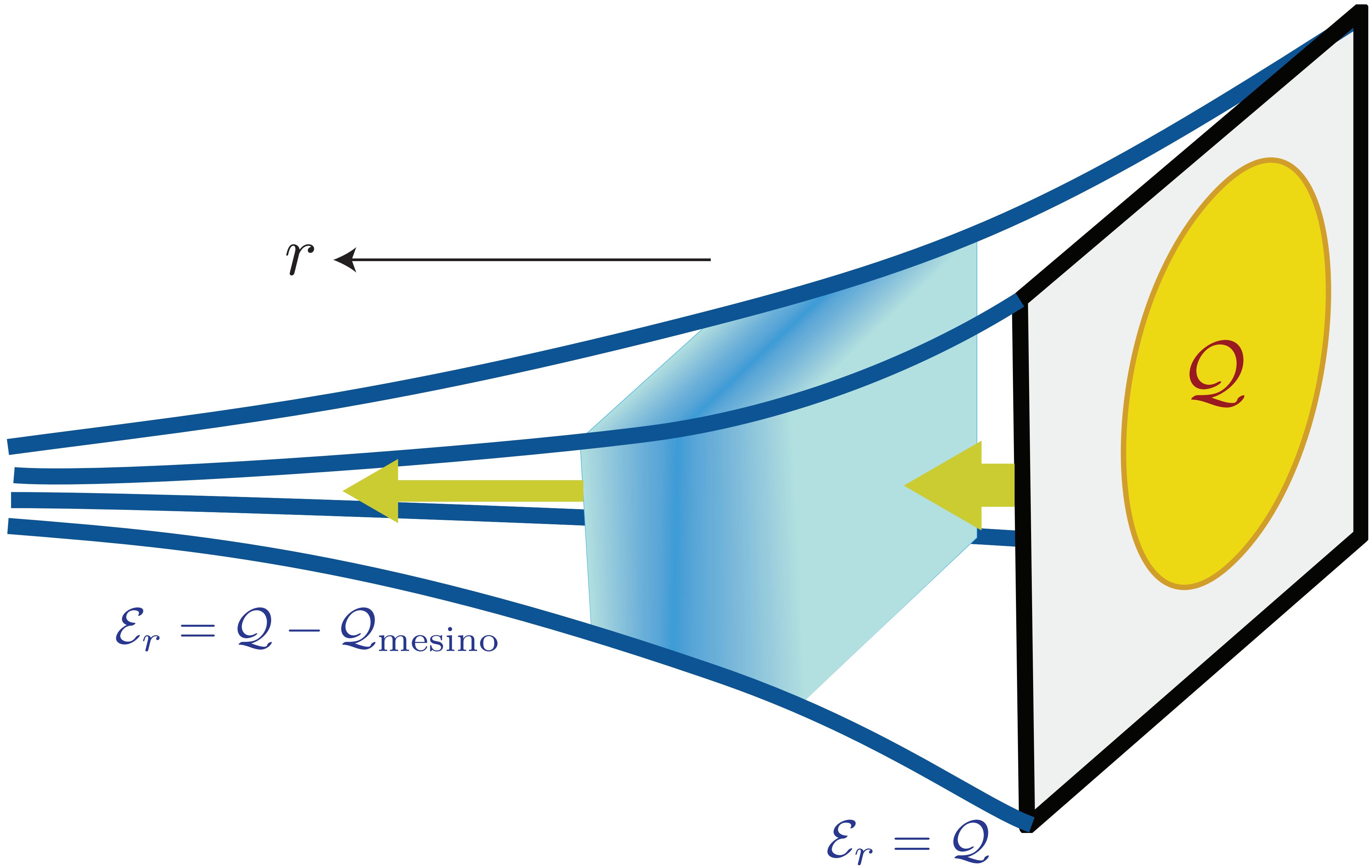
Holographic theory of a non-Fermi liquid (NFL)



Gauss Law in the bulk

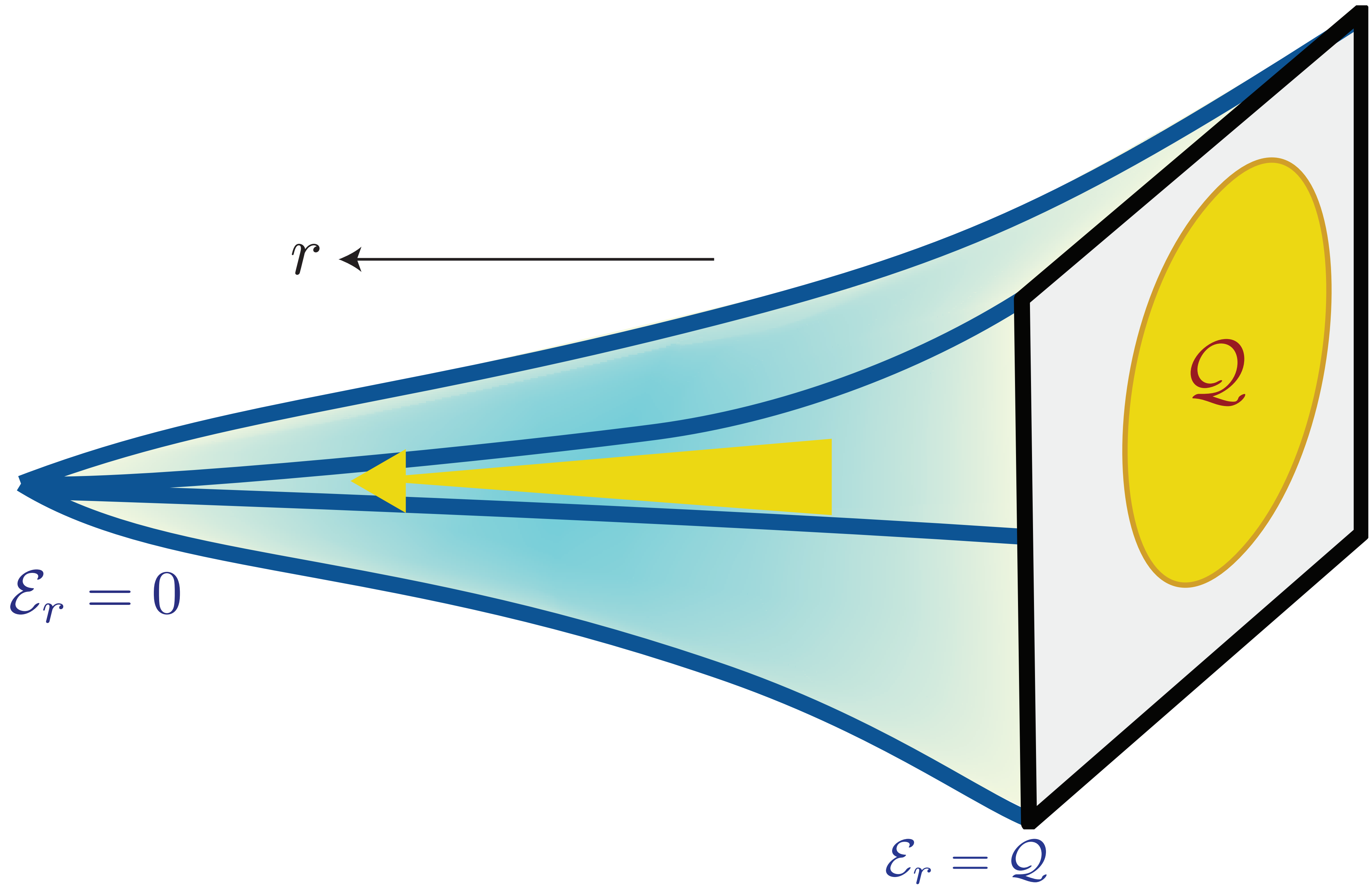
$\Leftrightarrow$ Luttinger theorem on the boundary

Holographic theory of a fractionalized-Fermi liquid (FL*)



- Now the entanglement entropy yields that the Fermi momentum of the hidden Fermi surface is given by $k_F^d \sim \mathcal{Q} - \mathcal{Q}_{\text{mesino}}$, just as expected by the extended Luttinger relation.

Holographic theory of a Fermi liquid (FL)



Gauss Law in the bulk

$\Leftrightarrow$ Luttinger theorem on the boundary

Gapped quantum matter

Insulators, quantum Hall states

Compressible quantum matter

Metals, superconductors, strange metals

Conformal quantum matter

Graphene, antiferromagnets, ultracold atoms

Gapped quantum matter

Insulators, quantum Hall states

Compressible quantum matter

Metals, superconductors, strange metals

Conformal quantum matter

Graphene, antiferromagnets, ultracold atoms

Conformal quantum matter

A. Field theory:graphene

B. Field theory: superfluid-insulator transition

C. Holography

Conformal quantum matter

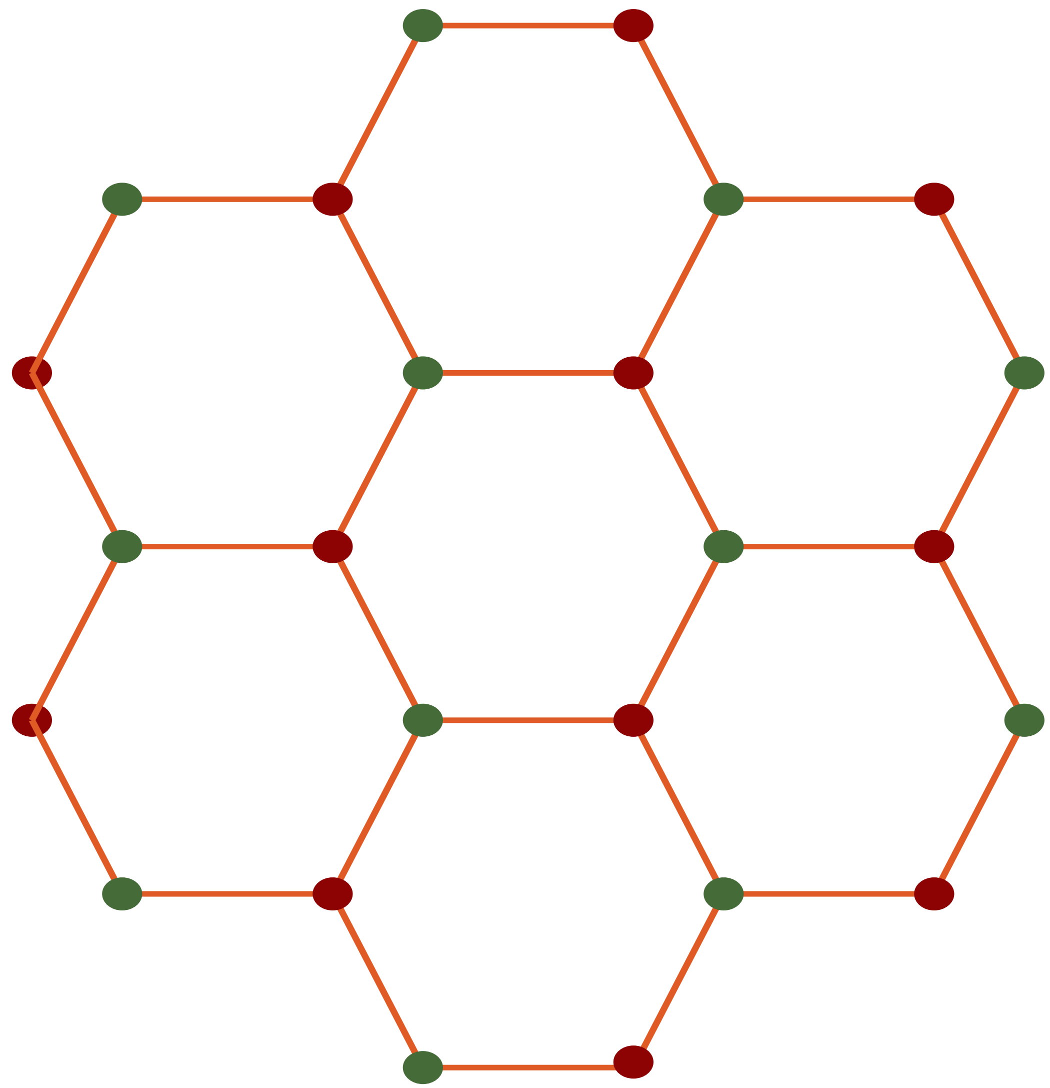
A. Field theory:graphene

B. Field theory: superfluid-insulator transition

C. Holography

Honeycomb lattice

(describes graphene after adding long-range Coulomb interactions)



$$H = -t \sum_{\langle ij \rangle} c_{i\alpha}^{\dagger} c_{j\alpha} + U \sum_{i} \left(n_{i\uparrow} - \frac{1}{2} \right) \left(n_{i\downarrow} - \frac{1}{2} \right)$$

The Hubbard Model

$$H = -\sum_{i,j} t_{ij} c^\dagger_{i\alpha} c_{j\alpha} + U \sum_i \left(n_{i\uparrow} - \frac{1}{2}\right)\left(n_{i\downarrow} - \frac{1}{2}\right) - \mu \sum_i c^\dagger_{i\alpha} c_{i\alpha}$$

In the limit of large U, and at a density of one particle per site, this maps onto the Heisenberg antiferromagnet

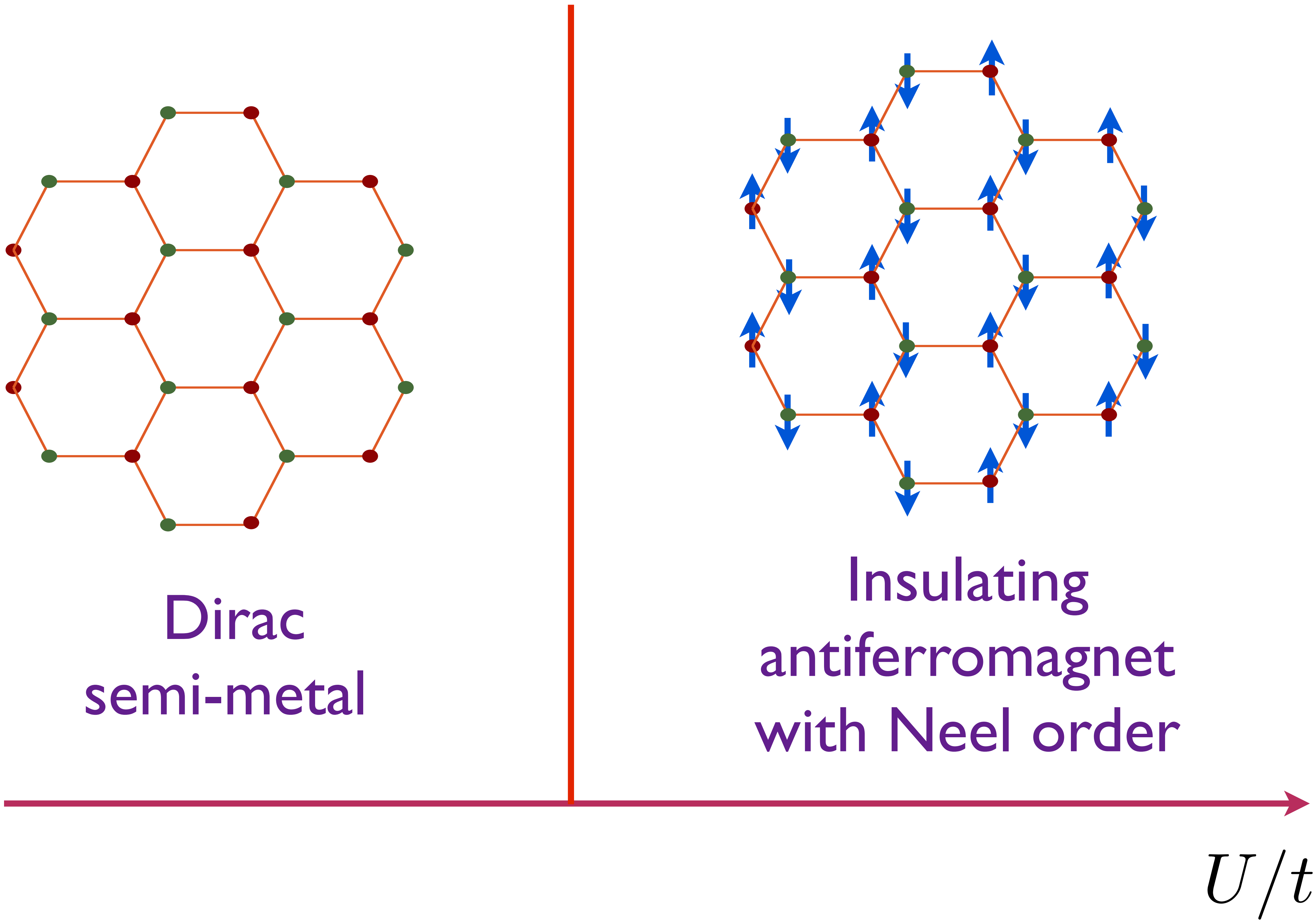
$$H_{AF} = \sum_{i<j} J_{ij} S_i^a S_j^a$$

where $a = x, y, z$,

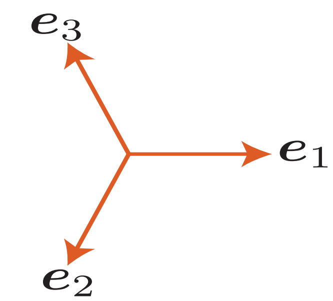
$$S_i^a = \frac{1}{2} c^{a\dagger}_{i\alpha} \sigma^a_{\alpha\beta} c_{i\beta},$$

with σ^a the Pauli matrices and

$$J_{ij} = \frac{4t_{ij}^2}{U}$$


Dirac
semi-metal
Insulating
antiferromagnet
with Neel order
U/t

Honeycomb lattice at half filling.



We define the unit length vectors

$$\mathbf{e}_1 = (1, 0) \quad , \quad \mathbf{e}_2 = (-1/2, \sqrt{3}/2) \quad , \quad \mathbf{e}_3 = (-1/2, -\sqrt{3}/2). \tag{1}$$

Note that $\mathbf{e}_i \cdot \mathbf{e}_j = -1/2$ for $i \neq j$, and $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 = 0$.
We take the origin of co-ordinates of the honeycomb lattice at the center of an *empty hexagon*. The A sublattice sites closest to the origin are at $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, while the B sublattice sites closest to the origin are at $-\mathbf{e}_1$, $-\mathbf{e}_2$, and $-\mathbf{e}_3$.
The reciprocal lattice is generated by the wavevectors

$$\mathbf{G}_1 = \frac{4\pi}{3}\mathbf{e}_1 \quad , \quad \mathbf{G}_2 = \frac{4\pi}{3}\mathbf{e}_2 \quad , \quad \mathbf{G}_3 = \frac{4\pi}{3}\mathbf{e}_3 \tag{2}$$

The first Brillouin zone is a hexagon whose vertices are given by

$$\mathbf{Q}_1 = \frac{1}{3}(\mathbf{G}_2 - \mathbf{G}_3) \quad , \quad \mathbf{Q}_2 = \frac{1}{3}(\mathbf{G}_3 - \mathbf{G}_1) \quad , \quad \mathbf{Q}_3 = \frac{1}{3}(\mathbf{G}_1 - \mathbf{G}_2), \tag{3}$$

and $-\mathbf{Q}_1$, $-\mathbf{Q}_2$, and $-\mathbf{Q}_3$.

We define the Fourier transform of the fermions by

$$c_A(\mathbf{k}) = \sum_{\mathbf{r}} c_A(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} \tag{4}$$

and similarly for c_B.
The hopping Hamiltonian is

$$H_0 = -t \sum_{\langle ij \rangle} \left(c^\dagger_{Ai\alpha} c_{Bj\alpha} + c^\dagger_{Bj\alpha} c_{Ai\alpha} \right) \tag{5}$$

where α is a spin index. If we introduce Pauli matrices τ^a in sublattice space ($a = x, y, z$), this Hamiltonian can be written as

$$\begin{aligned} H_0 &= \int \frac{d^2k}{4\pi^2} c^\dagger(\mathbf{k}) \Big[-t \Big(\cos(\mathbf{k}\cdot\mathbf{e}_1) + \cos(\mathbf{k}\cdot\mathbf{e}_2) + \cos(\mathbf{k}\cdot\mathbf{e}_3) \Big) \tau^x \\ &\quad + t \Big(\sin(\mathbf{k}\cdot\mathbf{e}_1) + \sin(\mathbf{k}\cdot\mathbf{e}_2) + \sin(\mathbf{k}\cdot\mathbf{e}_3) \Big) \tau^y \Big] c(\mathbf{k}) \end{aligned} \tag{6}$$

The low energy excitations of this Hamiltonian are near $\mathbf{k} \approx \pm \mathbf{Q}_1$.

In terms of the fields near $\mathbf{Q}_1$ and $-\mathbf{Q}_1$, we define

$$\begin{aligned}
\Psi_{A1\alpha}(\mathbf{k}) &= c_{A\alpha}(\mathbf{Q}_1 + \mathbf{k}) \\
\Psi_{A2\alpha}(\mathbf{k}) &= c_{A\alpha}(-\mathbf{Q}_1 + \mathbf{k}) \\
\Psi_{B1\alpha}(\mathbf{k}) &= c_{B\alpha}(\mathbf{Q}_1 + \mathbf{k}) \\
\Psi_{B2\alpha}(\mathbf{k}) &= c_{B\alpha}(-\mathbf{Q}_1 + \mathbf{k}) \qquad (7)
\end{aligned}$$

We consider Ψ to be a 8 component vector, and introduce Pauli matrices ρ^a which act in the $1, 2$ valley space. Then the Hamiltonian is

$$H_0 = \int \frac{d^2k}{4\pi^2} \Psi^\dagger(\mathbf{k}) \Big(v\tau^y k_x + v\tau^x \rho^z k_y \Big) \Psi(\mathbf{k}), \qquad (8)$$

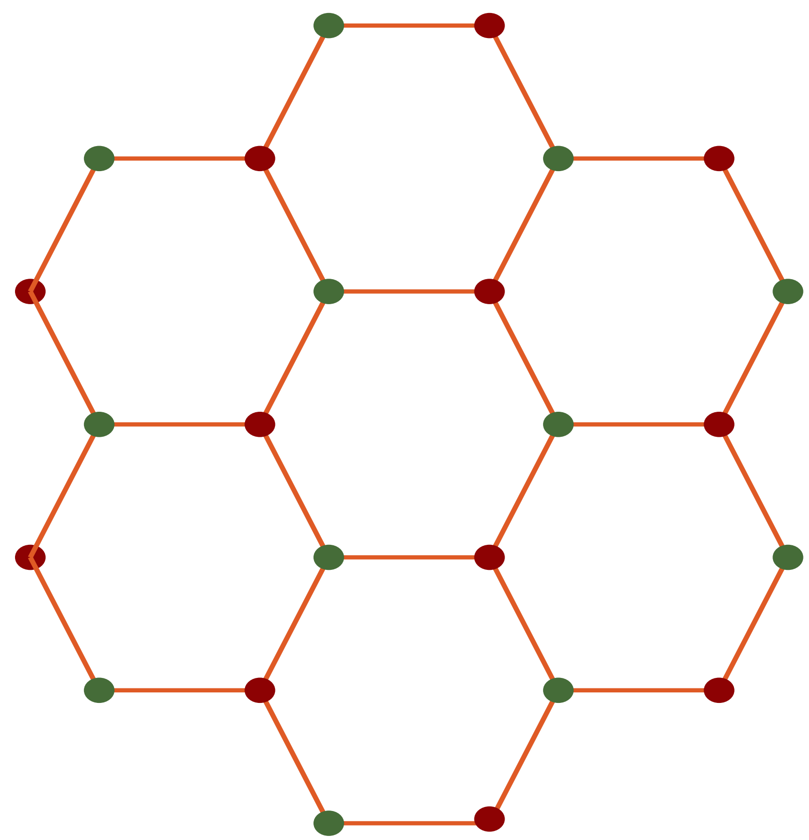
where $v = 3t/2$; below we set $v = 1$. Now define $\overline{\Psi} = \Psi^\dagger \rho^z \tau^z$. Then we can write the imaginary time Lagrangian as

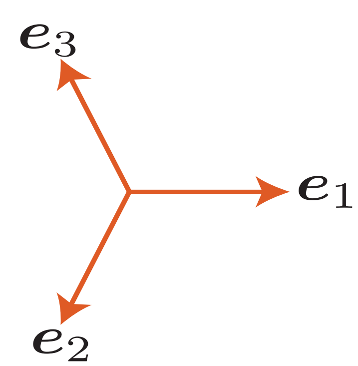
$$\mathcal{L}_0 = -i\overline{\Psi} \left(\omega\gamma_0 + k_x\gamma_1 + k_y\gamma_2 \right) \Psi \qquad (9)$$

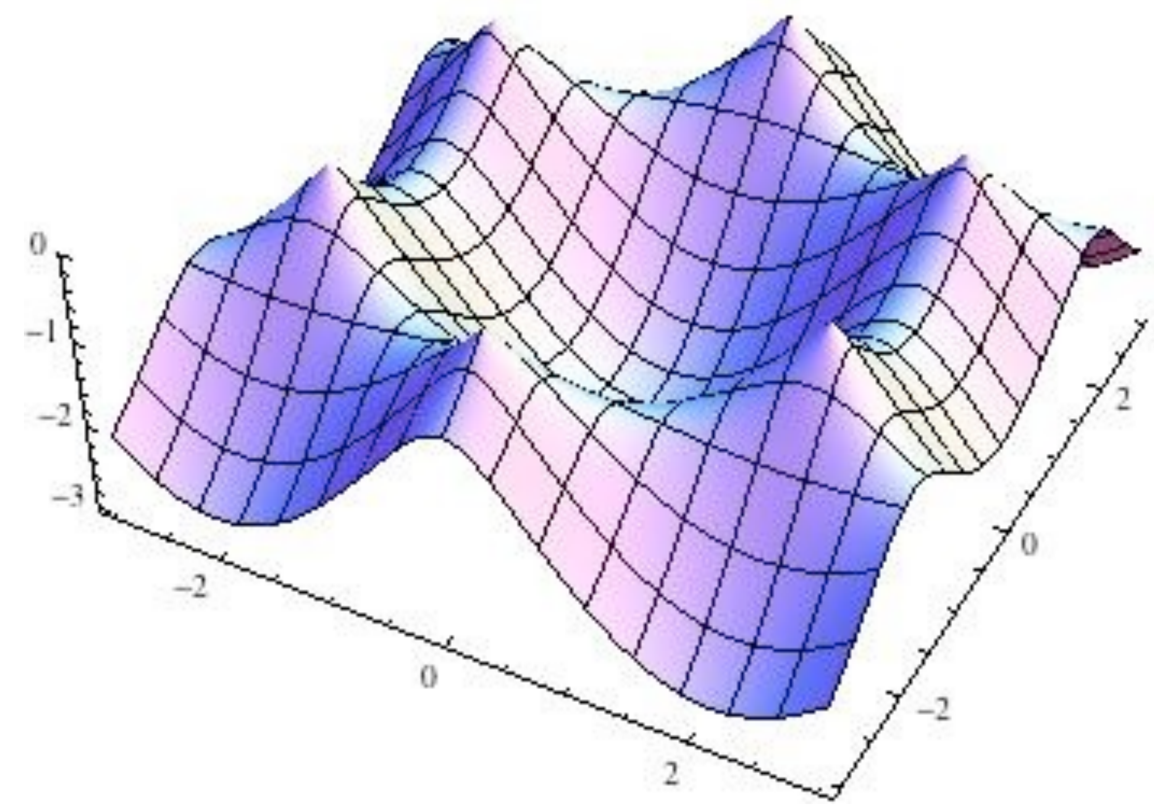
where

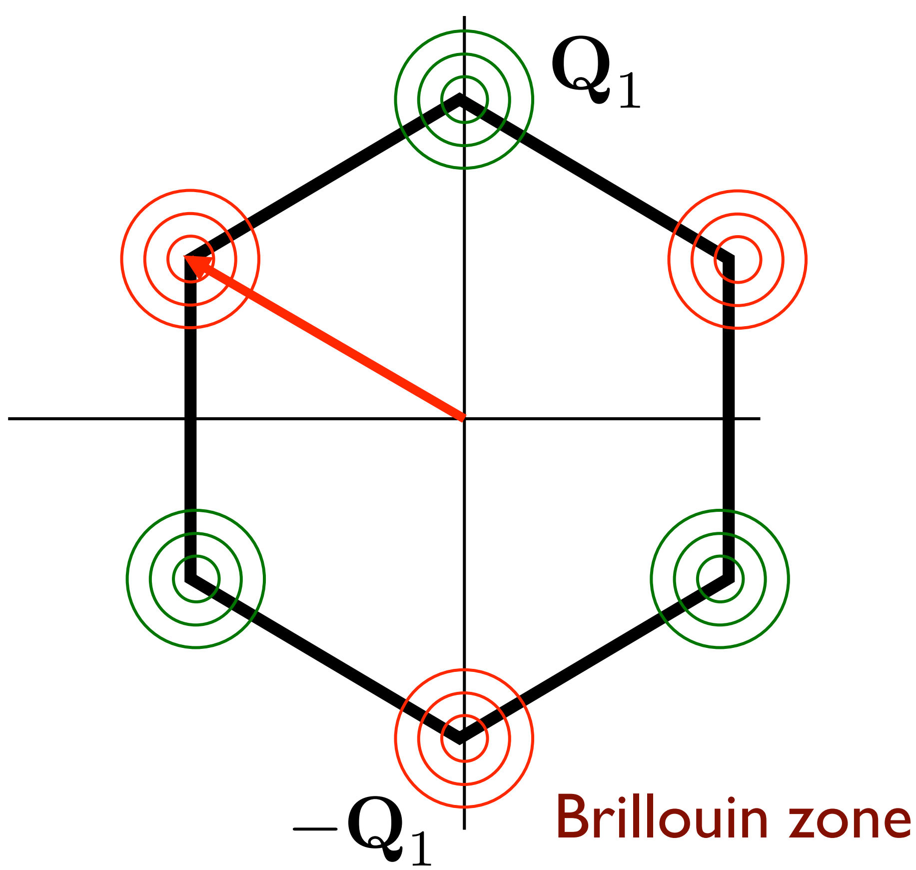
$$\gamma_0 = -\rho^z\tau^z \quad \gamma_1 = \rho^z\tau^x \quad \gamma_2 = -\tau^y \qquad (10)$$

Graphene









Semi-metal with massless Dirac fermions

Exercise: Observe that $\mathcal{L}_0$ is invariant under the scaling transformation $x' = xe^{-\ell}$ and $\tau' = \tau e^{-\ell}$. Write the Hubbard interaction U in terms of the Dirac fermions, and show that it has the tree-level scaling transformation $U' = Ue^{-\ell}$. So argue that all short-range interactions are *irrelevant* in the Dirac semi-metal phase.

Exercise: Observe that $\mathcal{L}_0$ is invariant under the scaling transformation $x' = xe^{-\ell}$ and $\tau' = \tau e^{-\ell}$. Write the Hubbard interaction U in terms of the Dirac fermions, and show that it has the tree-level scaling transformation $U' = Ue^{-\ell}$. So argue that all short-range interactions are *irrelevant* in the Dirac semi-metal phase.

Antiferromagnetism

We use the operator equation (valid on each site i):

$$U\left(n_\uparrow - \frac{1}{2}\right)\left(n_\downarrow - \frac{1}{2}\right) = -\frac{2U}{3}S^{a2} + \frac{U}{4} \tag{11}$$

Then we decouple the interaction via

$$\exp\left(\frac{2U}{3}\sum_i \int d\tau S_i^{a2}\right) = \int \mathcal{D}J_i^a(\tau)\exp\left(-\sum_i \int d\tau \left[\frac{3}{8U}J_i^{a2} - J_i^a S_i^a\right]\right) \tag{12}$$

We now integrate out the fermions, and look for the saddle point of the resulting effective action for J_i^a. At the saddle-point we find that the lowest energy is achieved when the vector has opposite orientations on the A and B sublattices. Anticipating this, we look for a continuum limit in terms of a field φ^a where

$$J_A^a = \varphi^a \quad , \quad J_B^a = -\varphi^a \tag{13}$$

The coupling between the field φ^a and the Ψ fermions is given by

$$\begin{aligned} \sum_i J_i^a c_{i\alpha}^\dagger \sigma_{\alpha\beta}^a c_{i\beta} &= \varphi^a \left(c_{A\alpha}^\dagger \sigma_{\alpha\beta}^a c_{A\beta} - c_{B\alpha}^\dagger \sigma_{\alpha\beta}^a c_{B\beta} \right) \\ &= \varphi^a \Psi^\dagger \tau^z \sigma^a \Psi = -\varphi^a \overline{\Psi} \rho^z \sigma^a \Psi \end{aligned} \tag{14}$$

From this we motivate the low energy theory

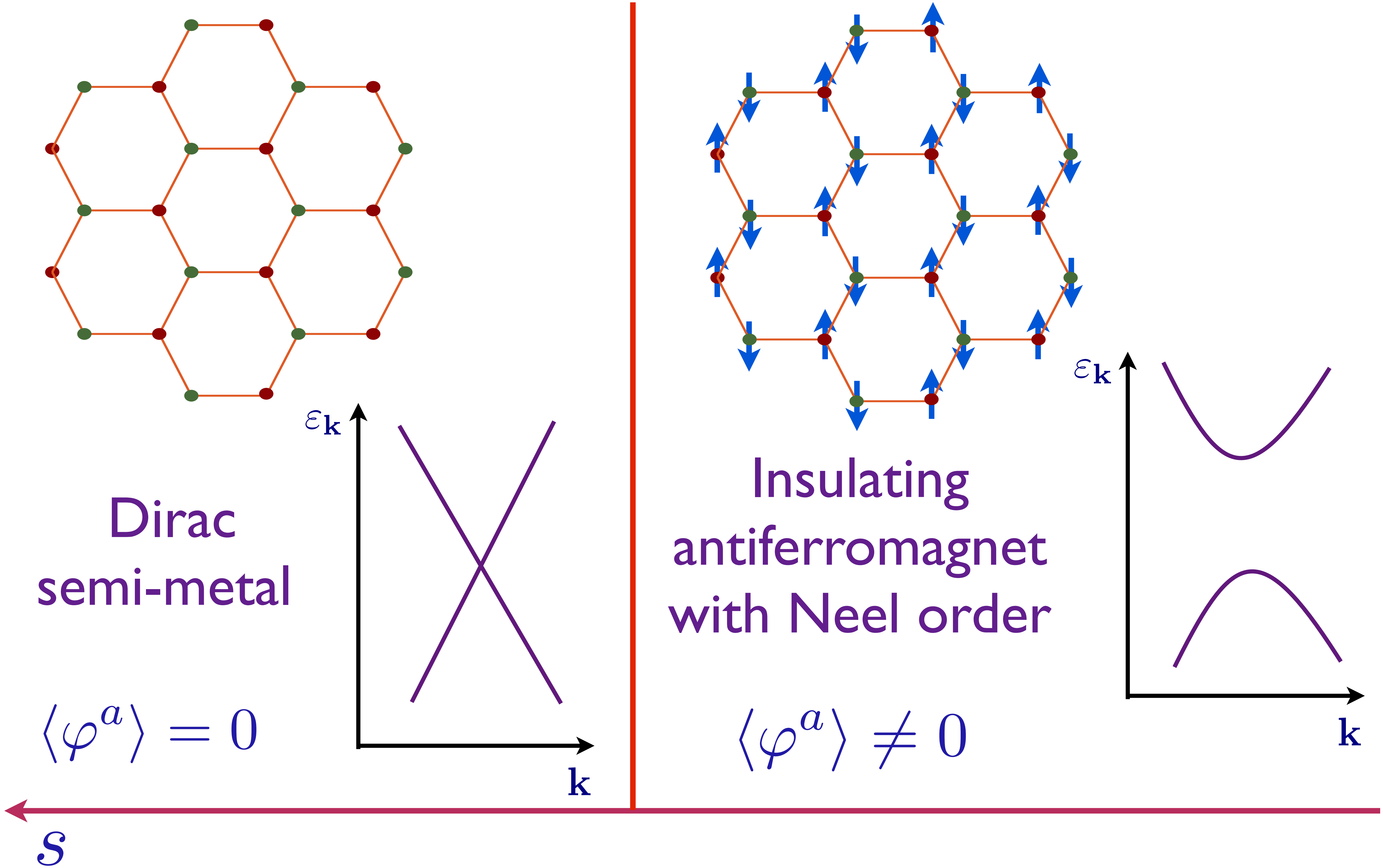
$$\mathcal{L} = \overline{\Psi} \gamma_\mu \partial_\mu \Psi + \frac{1}{2} \left[(\partial_\mu \varphi^a)^2 + s \varphi^{a2} \right] + \frac{u}{24} \left(\varphi^{a2} \right)^2 - \lambda \varphi^a \overline{\Psi} \rho^z \sigma^a \Psi \tag{15}$$

Note that the matrix $\rho^z \sigma^a$ commutes with all the γ_μ; hence $\rho^z \sigma^a$ is a matrix in "flavor" space. This is the Gross-Neveu model, and it describes the quantum phase transition from the Dirac semi-metal to an insulating Néel state In mean-field theory, the

Dirac semi-metal is obtained for $s > 0$ with $\langle \varphi^a \rangle = 0$. The Néel state obtains for $s < 0$, and we have $\varphi^a = N_0 \delta_{az}$ (say), and so the dispersion of the electrons is

$$\omega_k = \pm \sqrt{k^2 + \lambda^2 N_0^2} \tag{16}$$

near the points $\pm \mathbf{Q}_1$. These form the conduction and valence bands of the insulator.


Dirac
semi-metal
$\langle \varphi^a \rangle = 0$
$\varepsilon_{\mathbf{k}}$
$\mathbf{k}$
Insulating
antiferromagnet
with Neel order
$\langle \varphi^a \rangle \neq 0$
$\varepsilon_{\mathbf{k}}$
$\mathbf{k}$
s

From this we motivate the low energy theory

$$\mathcal{L} = \overline{\Psi}\gamma_\mu\partial_\mu\Psi + \frac{1}{2}\left[(\partial_\mu\varphi^a)^2 + s\varphi^{a2}\right] + \frac{u}{24}\left(\varphi^{a2}\right)^2 - \lambda\varphi^a\overline{\Psi}\rho^z\sigma^a\Psi \tag{15}$$

Note that the matrix $\rho^z\sigma^a$ commutes with all the γ_μ; hence $\rho^z\sigma^a$ is a matrix in "flavor" space. This is the Gross-Neveu model, and it describes the quantum phase transition from the Dirac semi-metal to an insulating Néel state

From this we motivate the low energy theory

$$\mathcal{L} = \overline{\Psi}\gamma_\mu\partial_\mu\Psi + \frac{1}{2}\left[(\partial_\mu\varphi^a)^2 + s\varphi^{a2}\right] + \frac{u}{24}\left(\varphi^{a2}\right)^2 - \lambda\varphi^a\overline{\Psi}\rho^z\sigma^a\Psi \tag{15}$$

Note that the matrix $\rho^z\sigma^a$ commutes with all the γ_μ; hence $\rho^z\sigma^a$ is a matrix in "flavor" space. This is the Gross-Neveu model, and it describes the quantum phase transition from the Dirac semi-metal to an insulating Néel state

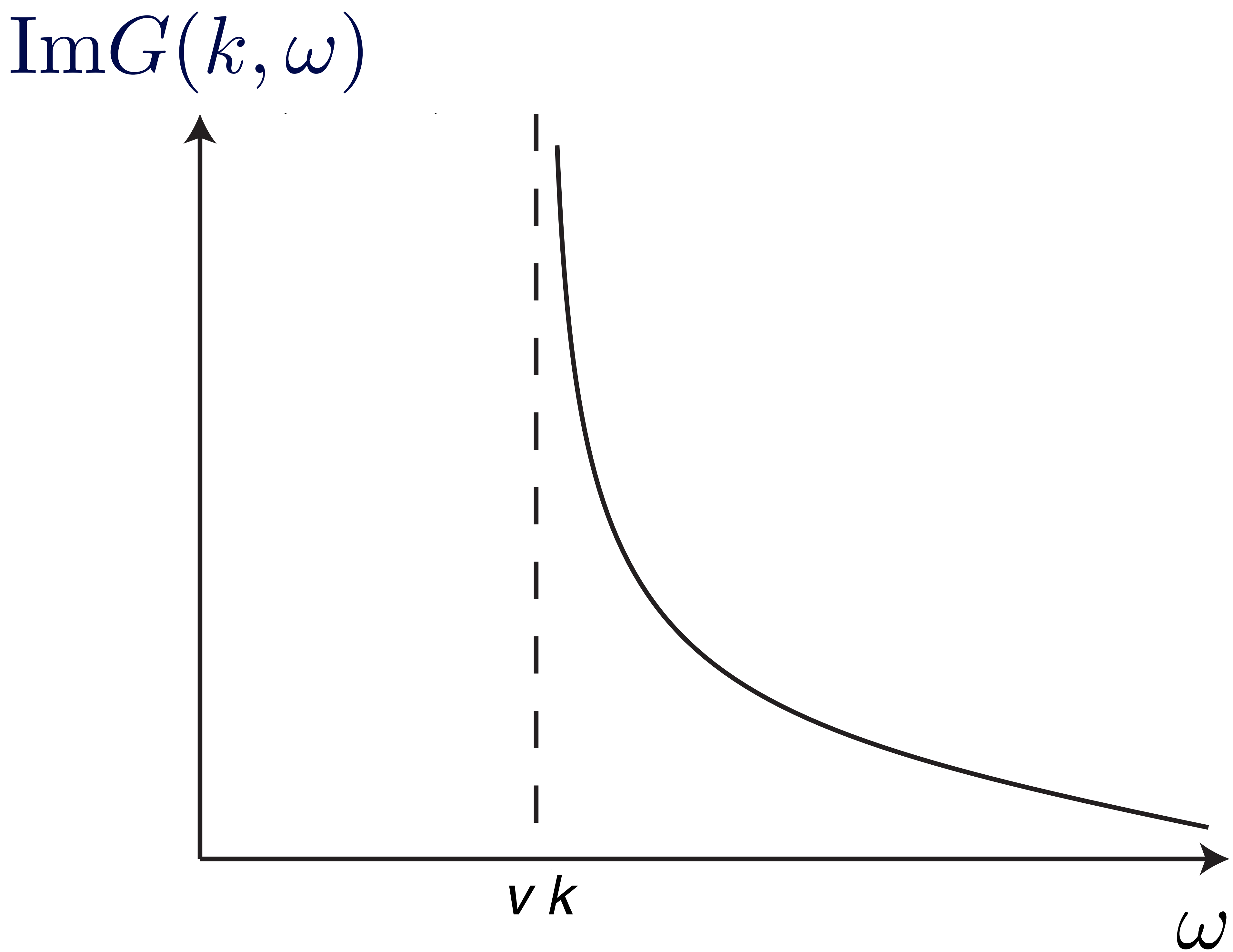
Exercise: Perform a tree-level RG transformation on $\mathcal{L}$. The quadratic gradient terms are invariant under $\Psi' = \Psi e^{\ell}$ and $\varphi' = \varphi e^{\ell/2}$. Show that this leads to $s' = se^{2\ell}$. Thus s is a relevant perturbation which drives the system into either the semi-metal or antiferromagnetic insulator. The quantum critical point is reached by tuning s to its critical value ($= 0$ at tree level). Show that the couplings u and λ are both relevant perturbations at this critical point. Thus, while interactions are irrelevant in the Dirac semi-metal (and in the insulator), they are strongly relevant at the quantum-critical point.

An analysis of this quantum critical point requires a RG analysis which goes beyond tree-level. Such an analysis can be controlled in an expansion in $1/N$ (where N is the number of fermion flavors) or $(3-d)$ (where d is the spatial dimensionality. Such analyses show that the couplings u and λ reach a RG fixed point which describes a conformal field theory (CFT).

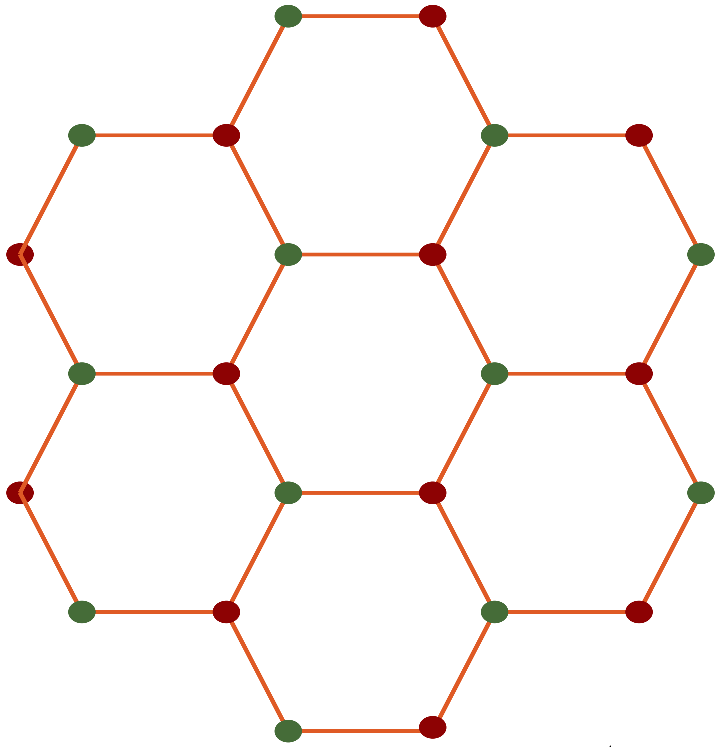
An important result of such an analysis is the following structure in the electron Green's function:

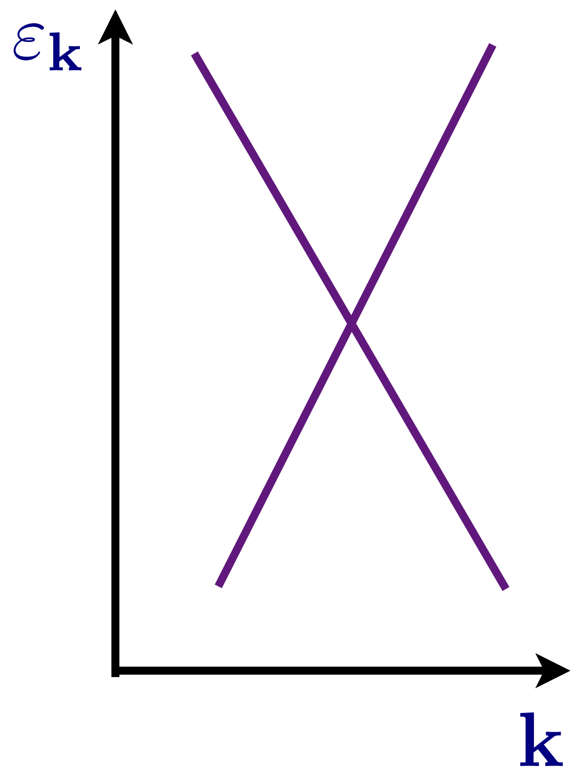
$$G(k,\omega) = \left\langle \Psi(k,\omega); \Psi^\dagger(k,\omega) \right\rangle \sim \frac{i\omega + v k_x \tau^y + v k_y \tau^x \rho^z}{(\omega^2 + v^2 k_x^2 + v^2 k_y^2)^{1-\eta/2}} \quad (17)$$

where $\eta > 0$ is the *anomalous dimension* of the fermion. Note that this leads to a fermion spectral density which has no quasiparticle pole: thus the quantum critical point has no well-defined quasiparticle excitations.



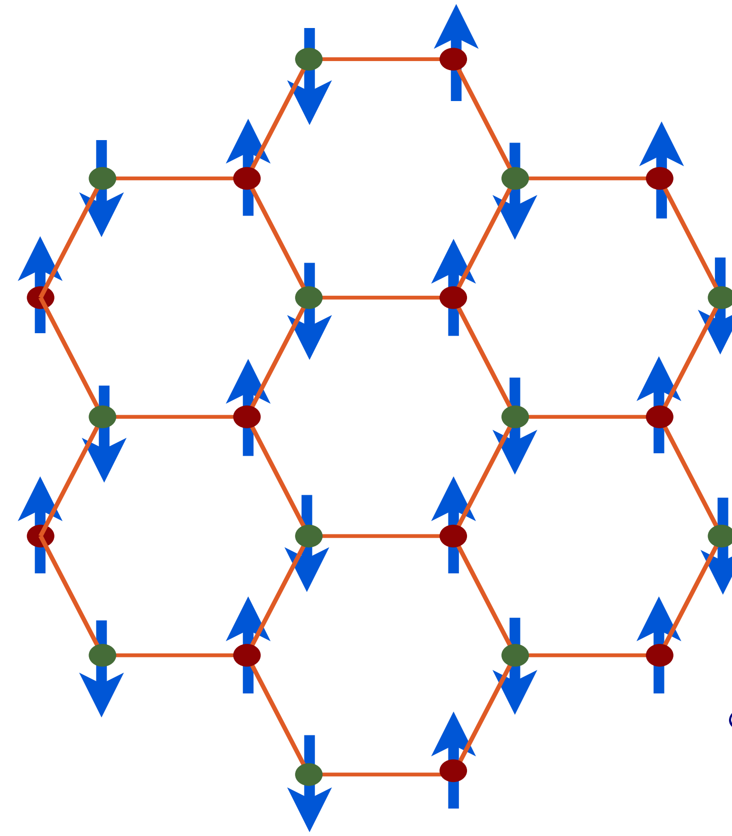
$\mathrm{Im} G(k, \omega)$
$v\,k$
ω

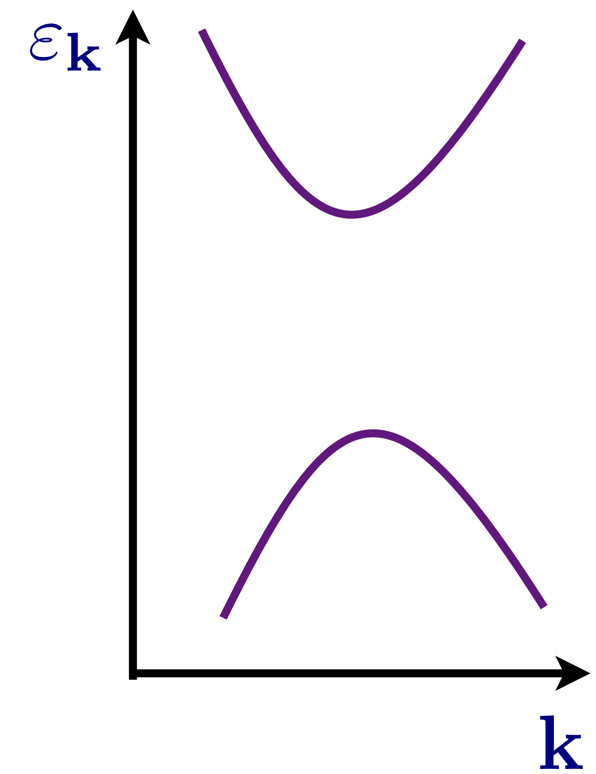




Dirac
semi-metal

$$\langle \varphi^a \rangle = 0$$



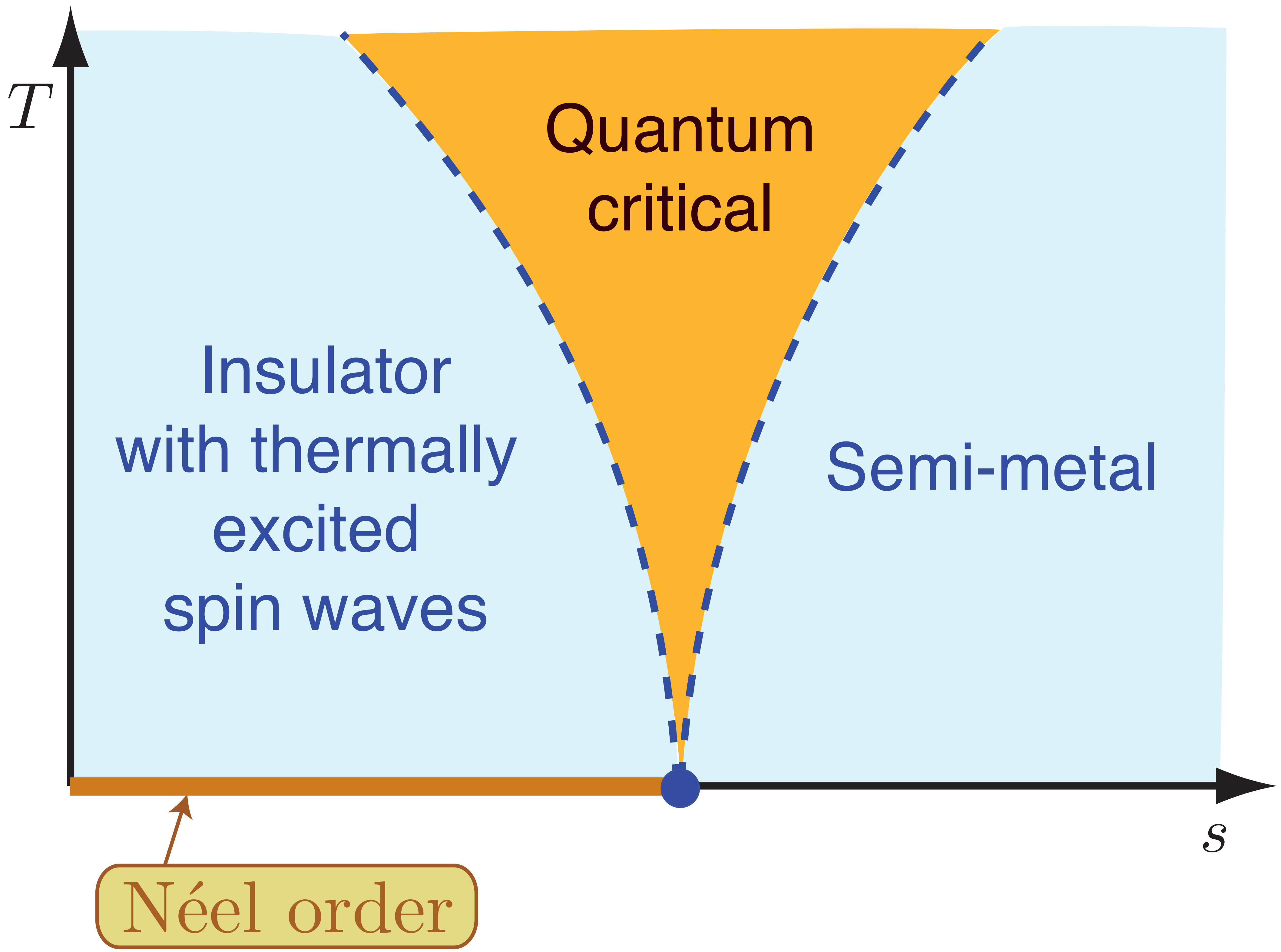


Insulating
antiferromagnet
with Neel order

$$\langle \varphi^a \rangle \neq 0$$

s

Quantum phase transition described by a strongly-coupled conformal field theory without well-defined quasiparticles


T
Quantum critical
Insulator with thermally excited spin waves
Semi-metal
s
Néel order

Electrical transport

The conserved electrical current is

$$J_\mu = -i\overline{\Psi}\gamma_\mu \Psi. \tag{1}$$

Let us compute its two-point correlator, $K_{\mu\nu}(k)$ at a spacetime momentum k_μ at $T=0$. At leading order, this is given by a one fermion loop diagram which evaluates to

$$\begin{aligned} K_{\mu\nu}(k) &= \int \frac{d^3p}{8\pi^3} \frac{\mathrm{Tr}\left[\gamma_\mu(i\gamma_\lambda p_\lambda + m\rho^z\sigma^z)\gamma_\nu(i\gamma_\delta(k_\delta + p_\delta) + m\rho^z\sigma^z)\right]}{(p^2+m^2)((p+k)^2+m^2)} \\ &= -\frac{2}{\pi}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\int_0^1 dx \frac{k^2 x(1-x)}{\sqrt{m^2+k^2x(1-x)}}, \end{aligned} \tag{2}$$

where the mass $m=0$ in the semi-metal and at the quantum critical point, while $m = |\lambda N_0|$ in the insulator. Note that the current correlation is purely transverse, and this follows from the requirement of current conservation

$$k_\mu K_{\mu\nu} = 0. \tag{3}$$

Of particular interest to us is the K_{00} component, after analytic continuation to Minkowski space where the spacetime momentum k_μ is replaced by (ω, k). The conductivity is obtained from this correlator via the Kubo formula

$$\sigma(\omega) = \lim_{k \to 0} \frac{-i\omega}{k^2} K_{00}(\omega, k). \tag{4}$$

In the insulator, where $m > 0$, analysis of the integrand in Eq. (2) shows that that the spectral weight of the density correlator has a gap of $2m$ at $k = 0$, and the conductivity in Eq. (4) vanishes. These properties are as expected in any insulator.
In the metal, and at the critical point, where $m = 0$, the fermionic spectrum is gapless, and so is that of the charge correlator. The density correlator in Eq. (2) and the conductivity in Eq. (4) evaluate to the simple universal results

$$\begin{aligned} K_{00}(\omega, k) &= \frac{1}{4} \frac{k^2}{\sqrt{k^2 - \omega^2}} \\ \sigma(\omega) &= 1/4. \end{aligned} \tag{5}$$

Going beyond one-loop, we find *no change* in these results in the semi-metal to all orders in perturbation theory. At the quantum critical point, there are no anomalous dimensions for the conserved current, but the amplitude does change yielding

$$
\begin{aligned}
K_{00}(\omega, k) &= \mathcal{K} \frac{k^2}{\sqrt{k^2 - \omega^2}} \\
\sigma(\omega) &= \mathcal{K},
\end{aligned}
\tag{6}
$$

where $\mathcal{K}$ is a universal number dependent only upon the universality class of the quantum critical point. The value of the $\mathcal{K}$ for the Gross-Neveu model is not known exactly, but can be estimated by computations in the $(3-d)$ or $1/N$ expansions.

Lecture 4

Conformal quantum matter

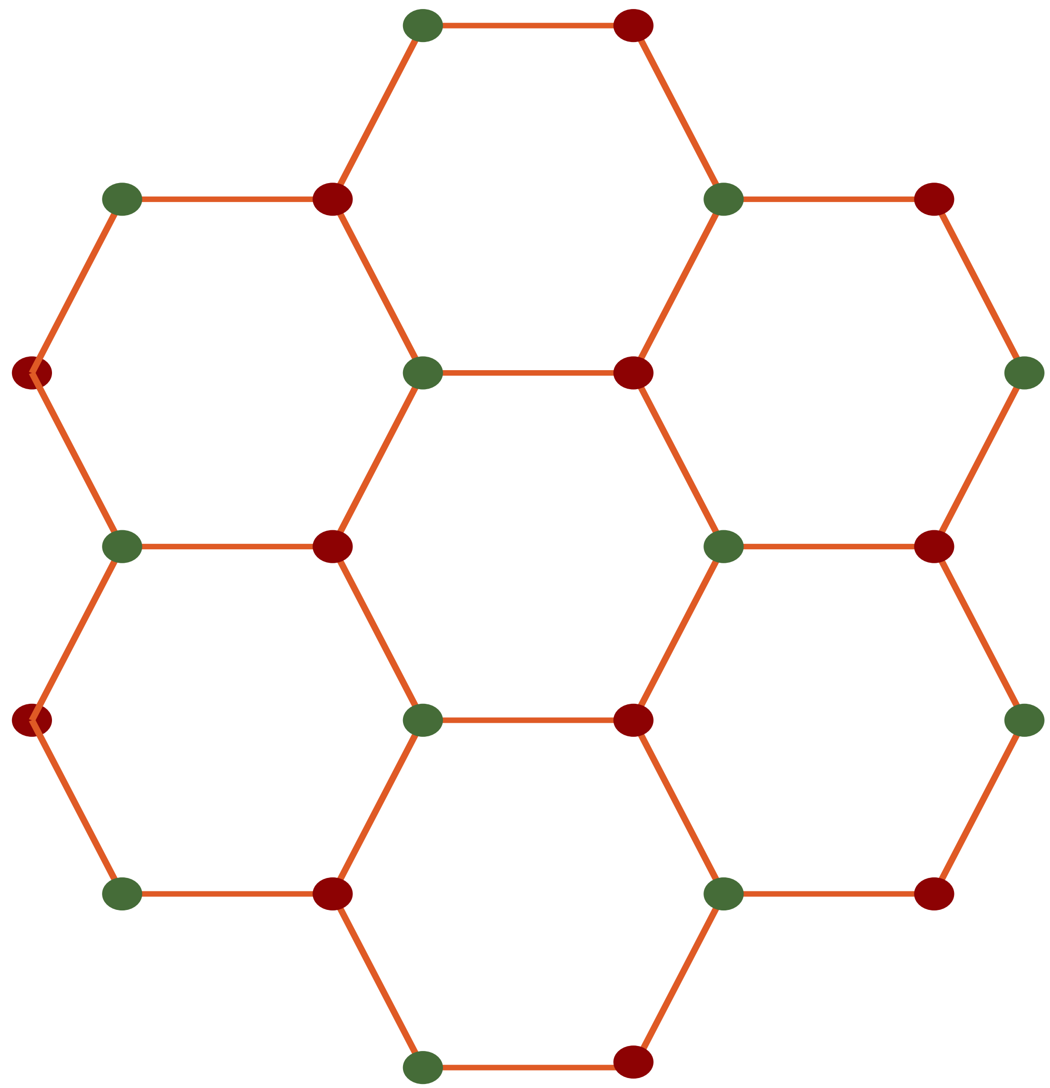
A. Field theory:graphene

B. Field theory: superfluid-insulator transition

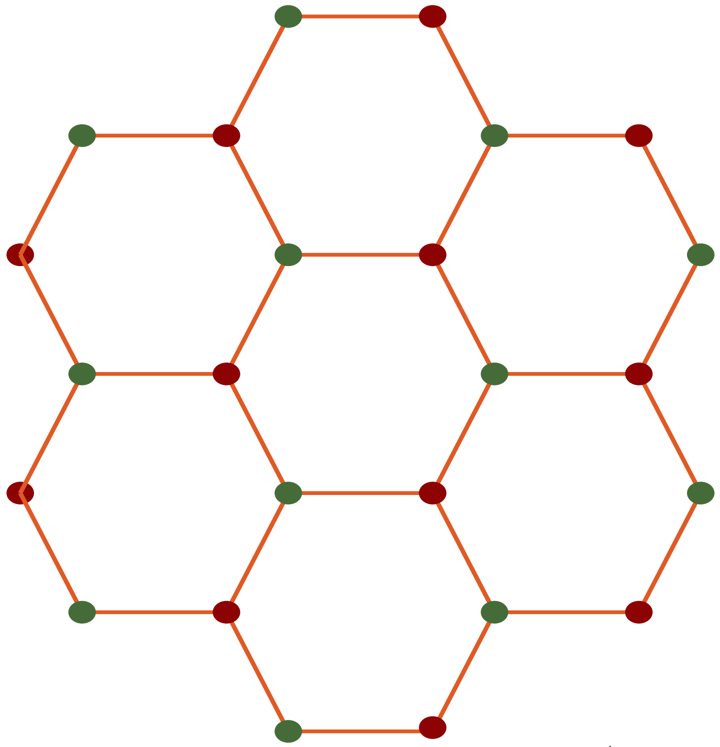
C. Holography

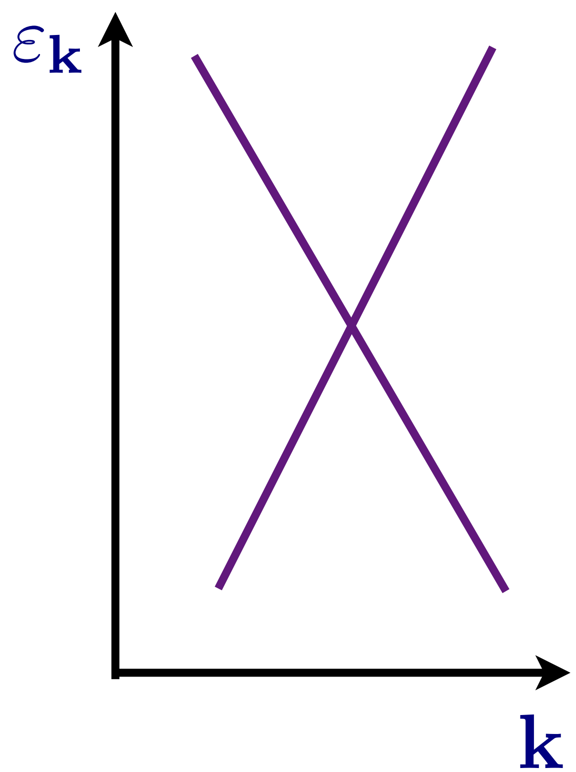
Honeycomb lattice

(describes graphene after adding long-range Coulomb interactions)



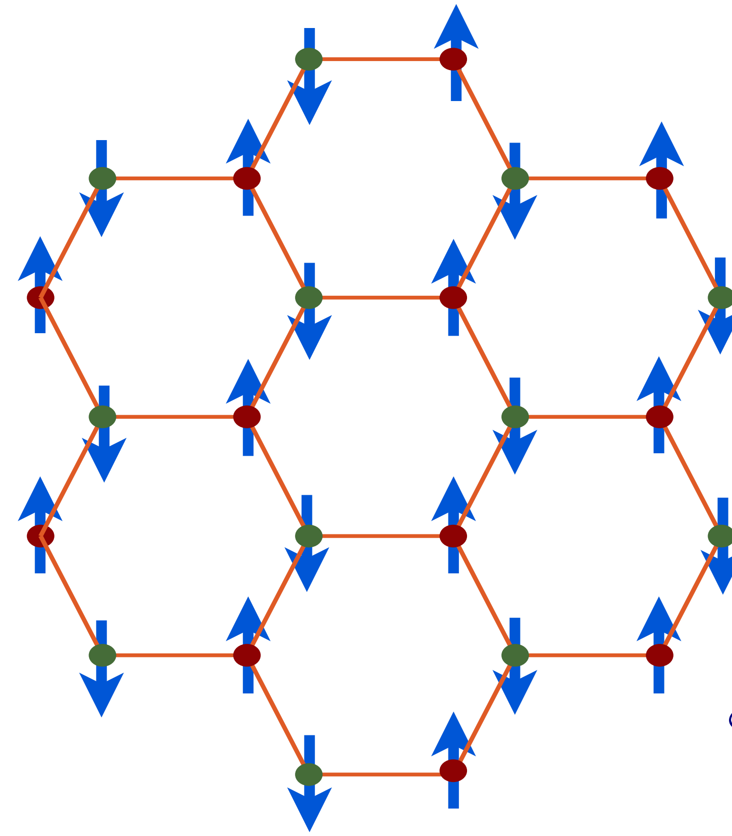
$$H = -t \sum_{\langle ij \rangle} c_{i\alpha}^{\dagger} c_{j\alpha} + U \sum_{i} \left(n_{i\uparrow} - \frac{1}{2} \right) \left(n_{i\downarrow} - \frac{1}{2} \right)$$



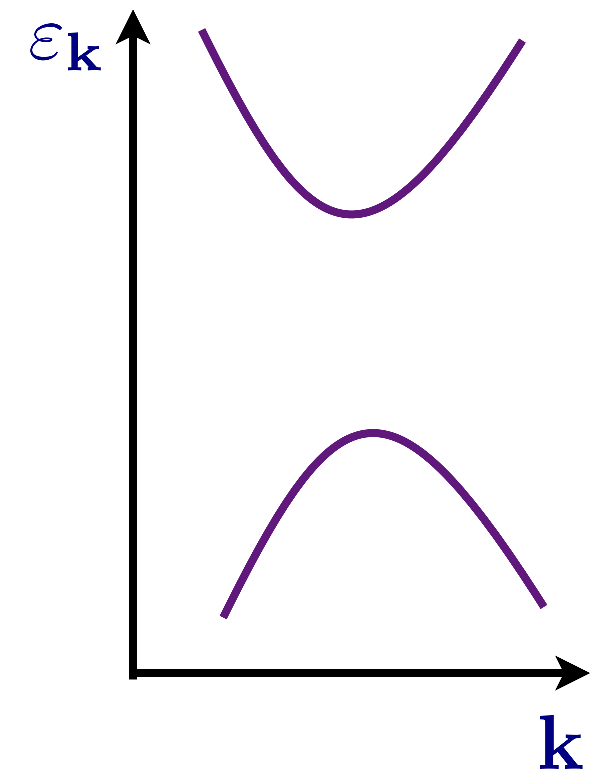


Dirac
semi-metal

$$\langle \varphi^a \rangle = 0$$



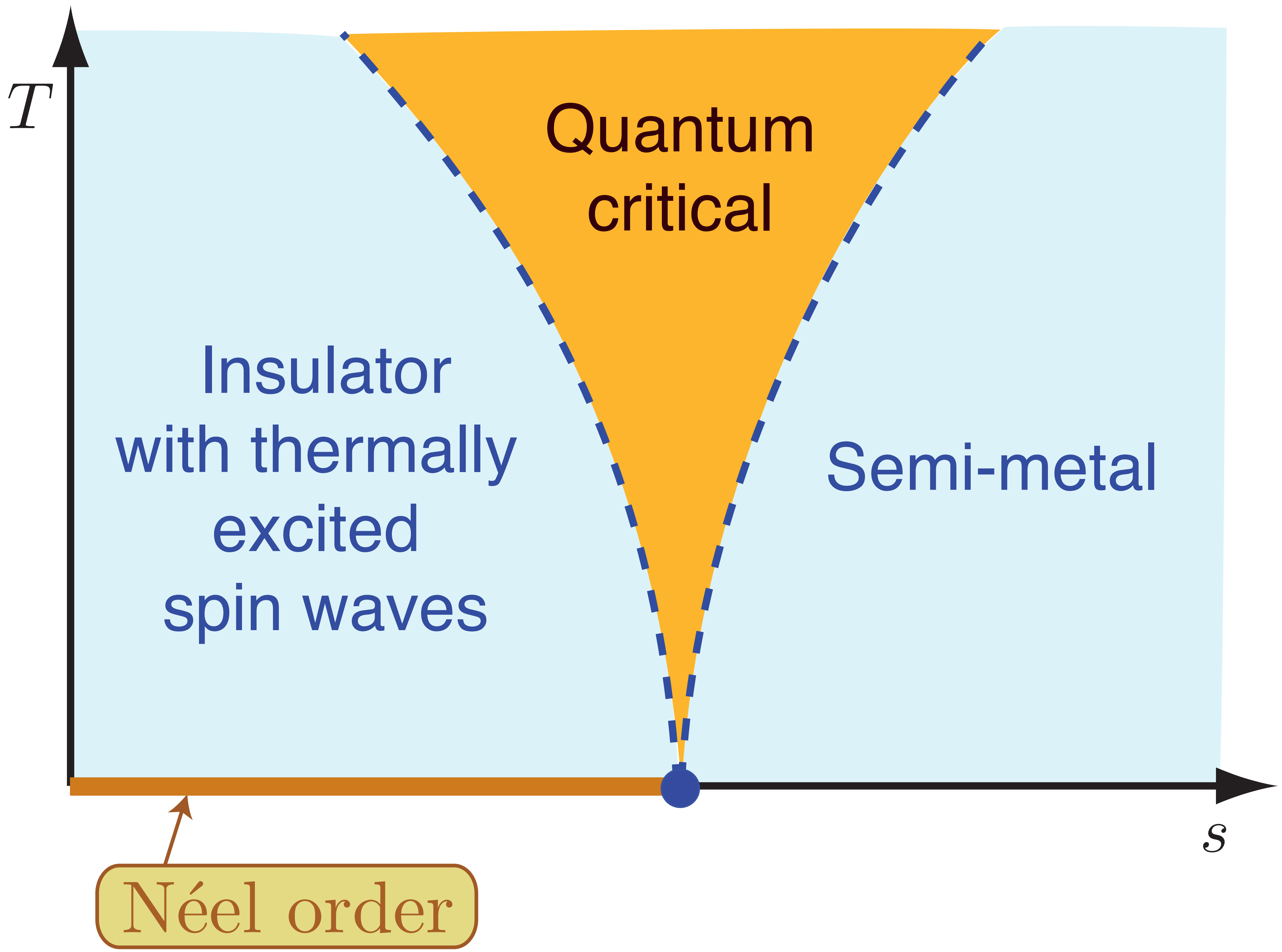
Insulating
antiferromagnet
with Neel order



$$\langle \varphi^a \rangle \neq 0$$

s

Quantum phase transition described by a strongly-coupled conformal field theory without well-defined quasiparticles



T
Quantum
critical
Insulator
with thermally
excited
spin waves
Semi-metal
s
Néel order

Electrical transport

The conserved electrical current is

$$J_\mu = -i\overline{\Psi}\gamma_\mu \Psi. \tag{1}$$

Let us compute its two-point correlator, $K_{\mu\nu}(k)$ at a spacetime momentum k_μ at $T = 0$. At leading order, this is given by a one fermion loop diagram which evaluates to

$$\begin{aligned} K_{\mu\nu}(k) &= \int \frac{d^3 p}{8\pi^3} \frac{\mathrm{Tr}\left[\gamma_\mu (i\gamma_\lambda p_\lambda + m\rho^z\sigma^z)\gamma_\nu (i\gamma_\delta(k_\delta + p_\delta) + m\rho^z\sigma^z)\right]}{(p^2+m^2)((p+k)^2+m^2)} \\ &= -\frac{2}{\pi}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\int_0^1 dx \frac{k^2 x(1-x)}{\sqrt{m^2 + k^2 x(1-x)}}, \end{aligned} \tag{2}$$

where the mass $m = 0$ in the semi-metal and at the quantum critical point, while $m = |\lambda N_0|$ in the insulator. Note that the current correlation is purely transverse, and this follows from the requirement of current conservation

$$k_\mu K_{\mu\nu} = 0. \tag{3}$$

Of particular interest to us is the K_{00} component, after analytic continuation to Minkowski space where the spacetime momentum k_μ is replaced by (ω, k). The conductivity is obtained from this correlator via the Kubo formula

$$\sigma(\omega) = \lim_{k \to 0} \frac{-i\omega}{k^2} K_{00}(\omega, k). \tag{4}$$

In the insulator, where $m > 0$, analysis of the integrand in Eq. (2) shows that that the spectral weight of the density correlator has a gap of $2m$ at $k = 0$, and the conductivity in Eq. (4) vanishes. These properties are as expected in any insulator.
In the metal, and at the critical point, where $m = 0$, the fermionic spectrum is gapless, and so is that of the charge correlator. The density correlator in Eq. (2) and the conductivity in Eq. (4) evaluate to the simple universal results

$$\begin{aligned} K_{00}(\omega, k) &= \frac{1}{4} \frac{k^2}{\sqrt{k^2 - \omega^2}} \\ \sigma(\omega) &= 1/4. \end{aligned} \tag{5}$$

Going beyond one-loop, we find *no change* in these results in the semi-metal to all orders in perturbation theory. At the quantum critical point, there are no anomalous dimensions for the conserved current, but the amplitude does change yielding

$$
\begin{aligned}
K_{00}(\omega, k) &= \mathcal{K} \frac{k^2}{\sqrt{k^2 - \omega^2}} \\
\sigma(\omega) &= \mathcal{K},
\end{aligned}
\tag{6}
$$

where $\mathcal{K}$ is a universal number dependent only upon the universality class of the quantum critical point. The value of the $\mathcal{K}$ for the Gross-Neveu model is not known exactly, but can be estimated by computations in the $(3-d)$ or $1/N$ expansions.

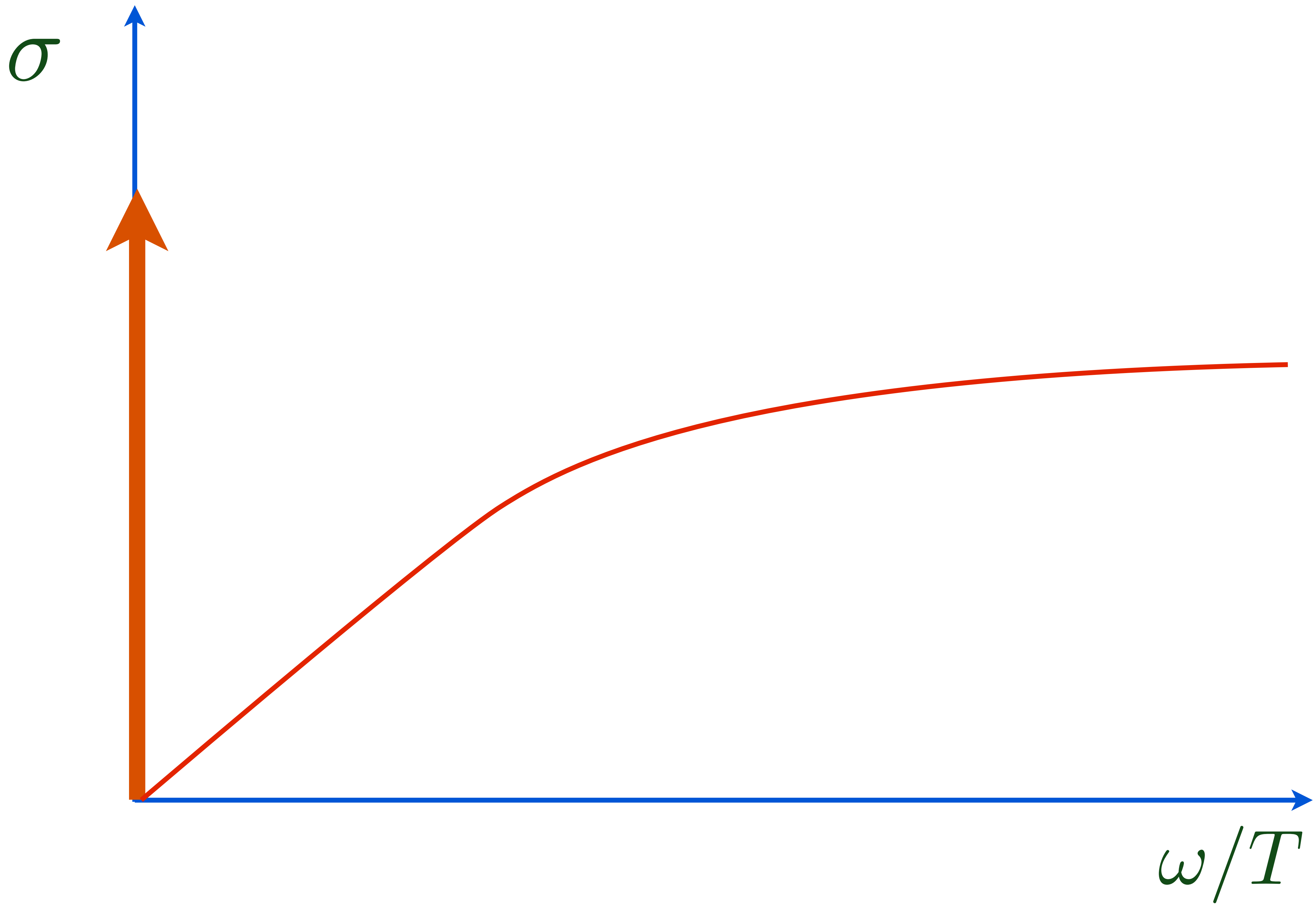
Non-zero temperatures

At the quantum-critical point at one-loop order, we can set $m = 0$, and then repeat the computation in Eq. (2) at $T > 0$. This only requires replacing the integral over the loop frequency by a summation over the Matsubara frequencies, which are quantized by odd multiples of πT. Such a computation, via Eq. (4) leads to the conductivity

$$\mathrm{Re}[\sigma(\omega)] = (2T \ln 2)\, \delta(\omega) + \frac{1}{4} \tanh\left(\frac{|\omega|}{4T}\right); \qquad (7)$$

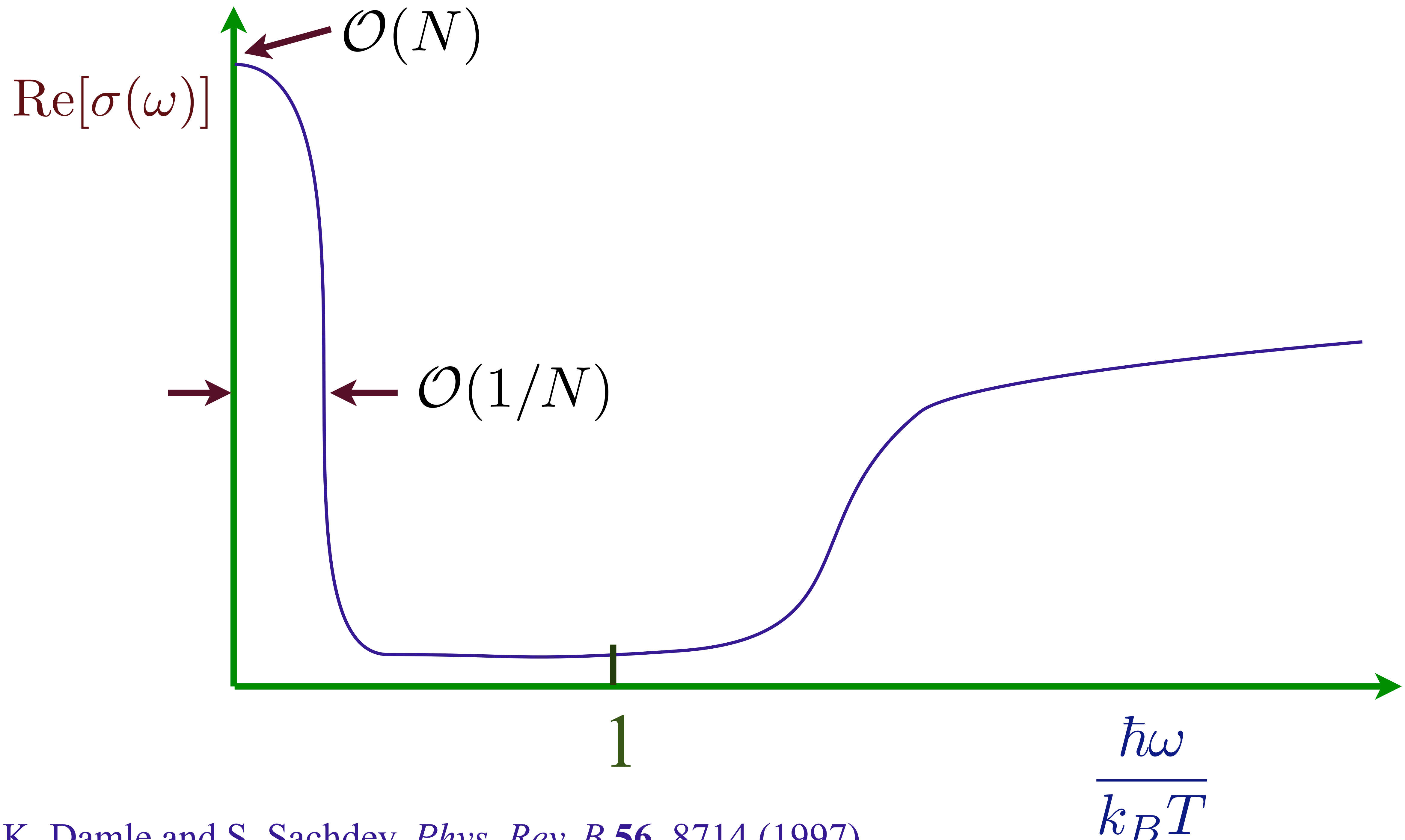
the imaginary part of $\sigma(\omega)$ is the Hilbert transform of $\mathrm{Re}[\sigma(\omega)] - 1/4$. Note that this reduces to Eq. (5) in the limit $\omega \gg T$. However, the most important new feature of Eq. (7) arises for $\omega \ll T$, where we find a delta function at zero frequency in the real part. Thus the d.c. conductivity is infinite at this order, arising from the collisionless transport of thermally excited carriers.

Electrical transport in a free-field theory for *T* > *0*



Vector large N expansion for CFT3

$$\sigma = \frac{Q^2}{h}\Sigma\left(\frac{\hbar\omega}{k_B T}\right); \quad \Sigma \to \text{a universal function}$$



K. Damle and S. Sachdev, *Phys. Rev. B* **56**, 8714 (1997).

Conformal quantum matter

A. Field theory:graphene

B. Field theory: superfluid-insulator transition

C. Holography

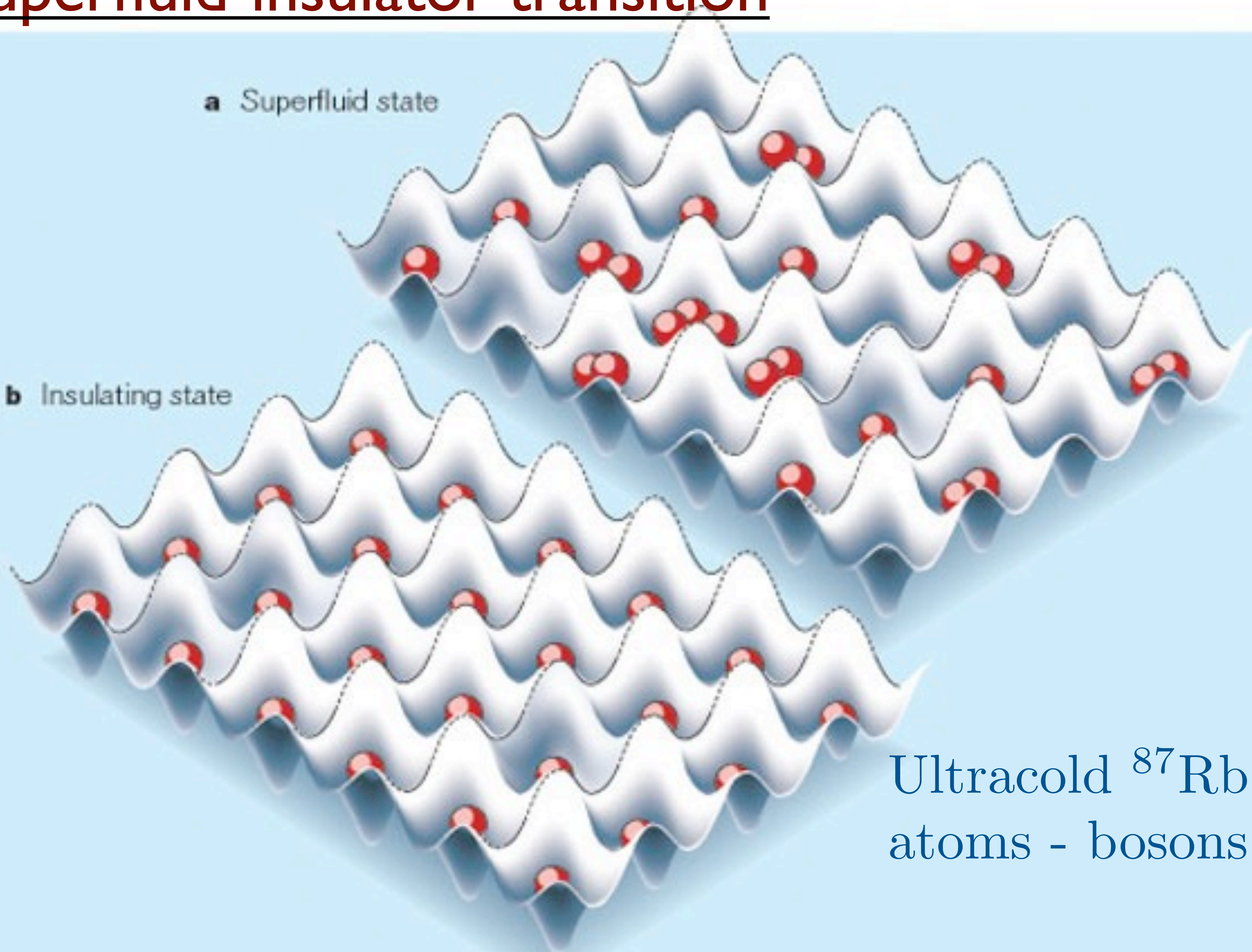
Conformal quantum matter

A. Field theory: graphene

$\boxed{\textit{B. Field theory: superfluid-insulator transition}}$

C. Holography

Superfluid-insulator transition



Ultracold ^{87}Rb atoms - bosons

M. Greiner, O. Mandel, T. Esslinger, T. W. Hänsch, and I. Bloch, *Nature* **415**, 39 (2002).

The Superfluid-Insulator transition

Boson Hubbard model

Degrees of freedom: Bosons, $b_j^\dagger$, hopping between the sites, j, of a lattice, with short-range repulsive interactions.

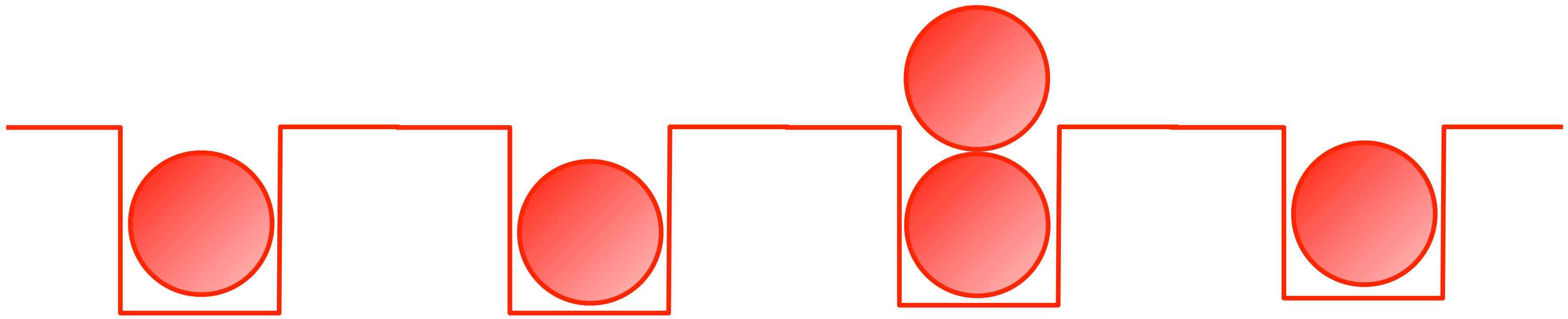
$$H = -t \sum_{\langle ij \rangle} b_i^\dagger b_j - \mu \sum_j n_j + \frac{U}{2} \sum_j n_j (n_j - 1) + \cdots$$

$$n_j \equiv b_j^\dagger b_j$$

$$[b_j, b_k^\dagger] = \delta_{jk}$$

M.P. A. Fisher, P.B. Weichmann, G. Grinstein, and D.S. Fisher, *Phys. Rev. B* **40**, 546 (1989).

Excitations of the insulator:



Particles ~ $\psi^\dagger$

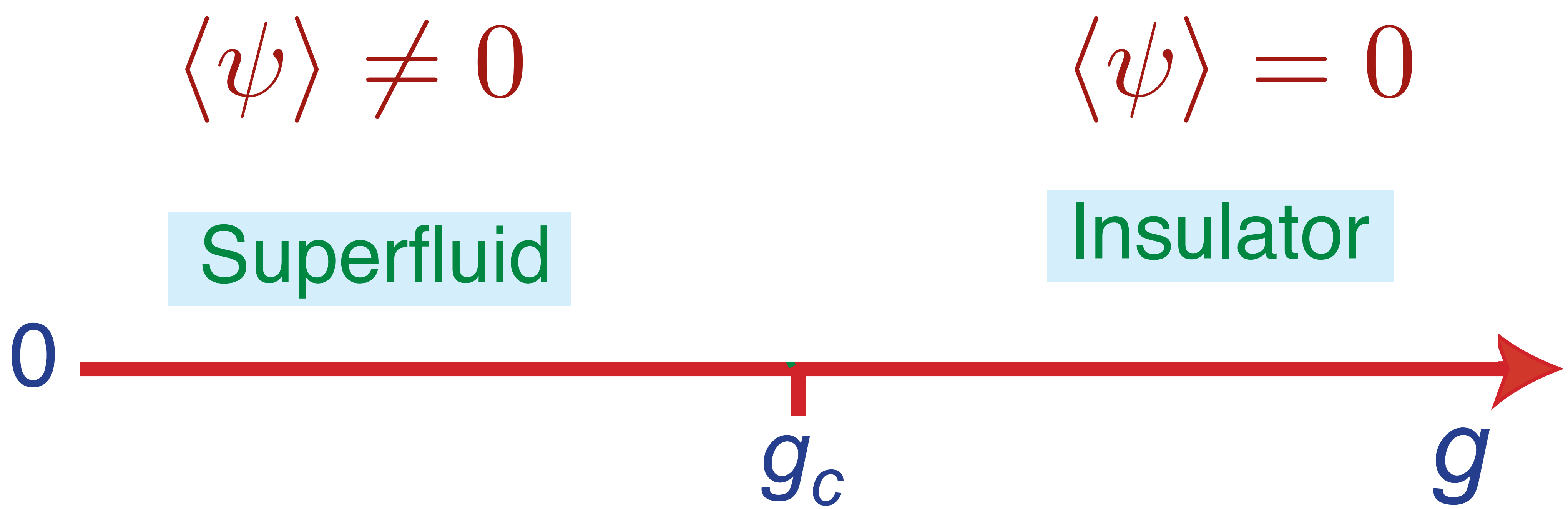
Holes ~ ψ

Density of particles = density of holes $\Rightarrow$

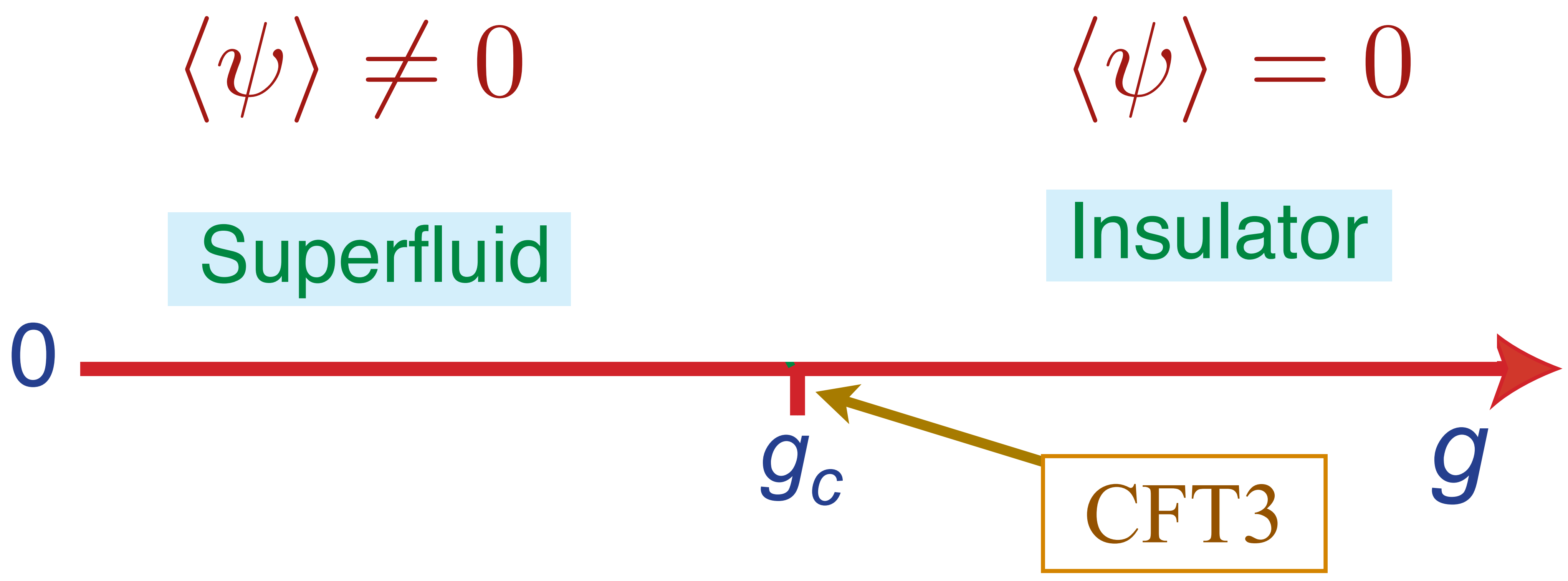
"relativistic" field theory for ψ:

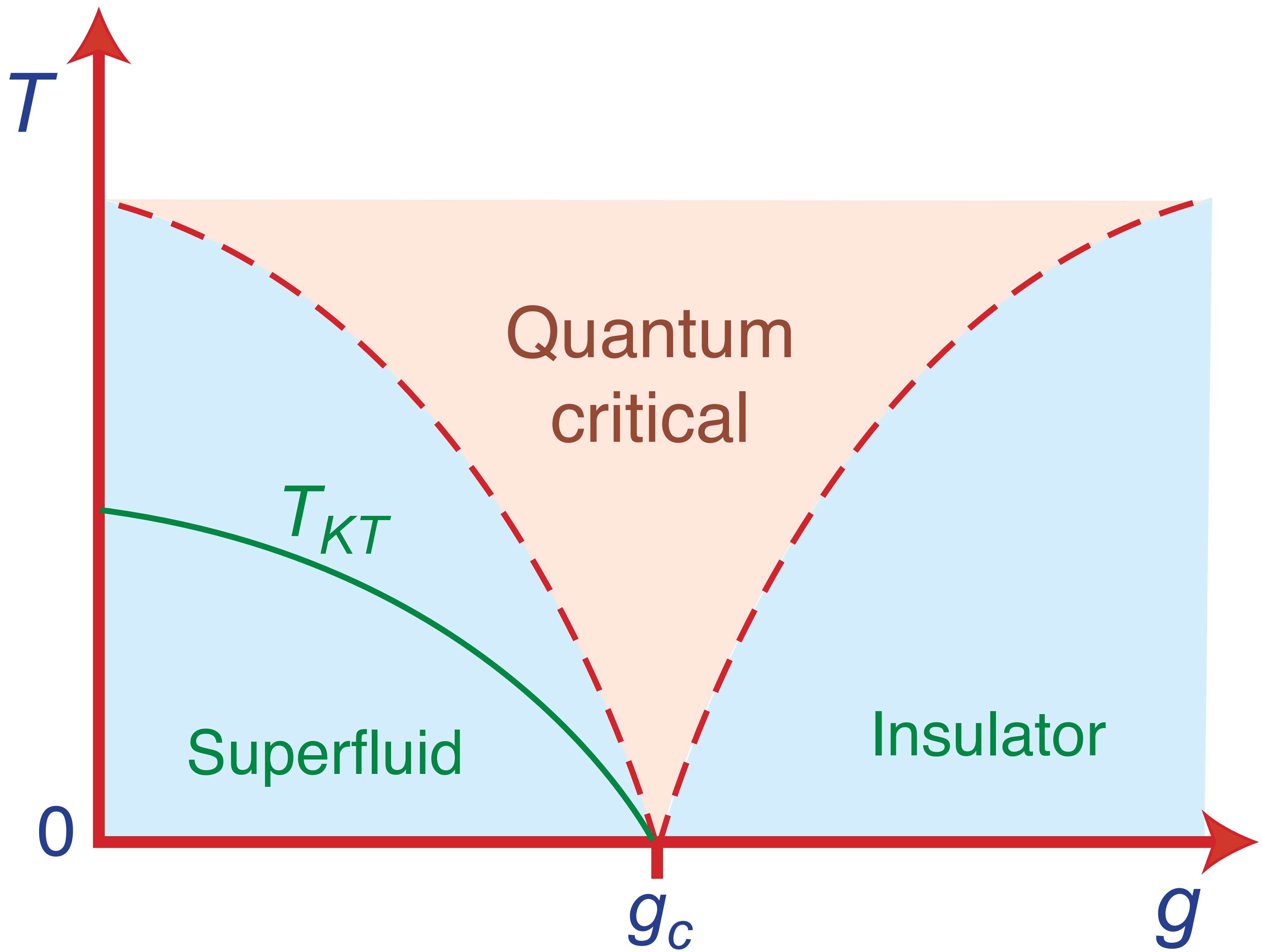
$$\mathcal{S} = \int d^2 r d\tau \left[|\partial_\tau \psi|^2 + v^2 |\vec{\nabla}\psi|^2 + (g - g_c)|\psi|^2 + \frac{u}{2}|\psi|^4 \right]$$

M.P. A. Fisher, P.B. Weichmann, G. Grinstein, and D.S. Fisher, *Phys. Rev. B* **40**, 546 (1989).

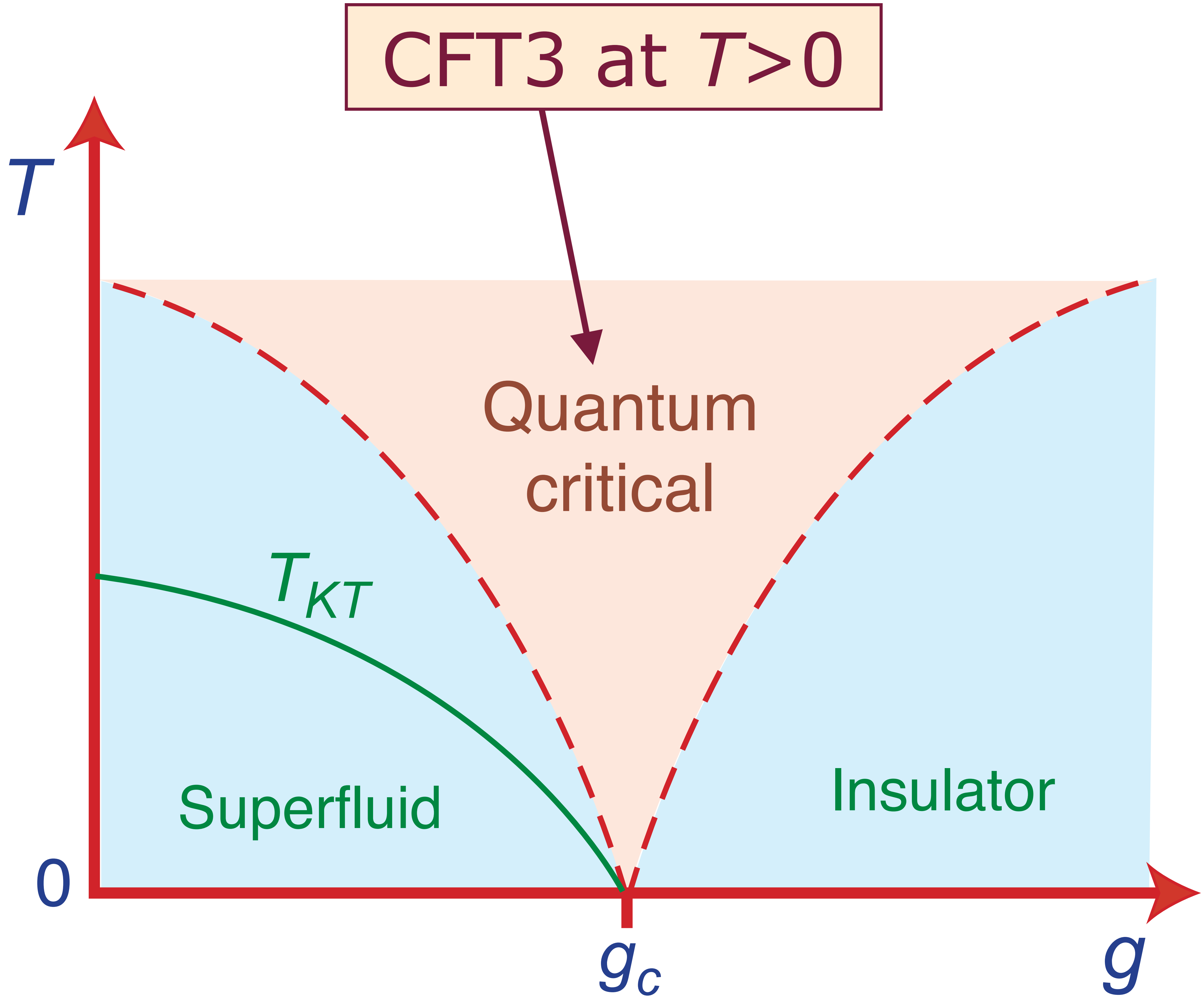


$\langle \psi \rangle \neq 0$
$\langle \psi \rangle = 0$
Superfluid
Insulator
0
g_c
g


$\langle \psi \rangle \neq 0$
$\langle \psi \rangle = 0$
Superfluid
Insulator
0
g_c
g
CFT3



T
Quantum
critical
T_{KT}
Superfluid
Insulator
0
g_c
g


CFT3 at $T>0$
T
Quantum
critical
T_{KT}
Superfluid
Insulator
0
g_c
g

Quantum critical transport

Quantum *"nearly perfect fluid"*
with shortest possible
equilibration time, $\tau_{\rm eq}$

$$\boxed{\tau_{\rm eq} = \mathcal{C}\frac{\hbar}{k_B T}}$$

where $\mathcal{C}$ is a *universal* constant

S. Sachdev, *Quantum Phase Transitions,* Cambridge (1999).

Quantum critical transport

Transport co-oefficients not determined by collision rate, but by universal constants of nature

Conductivity

$$\boxed{\sigma = \frac{Q^2}{h} \times [\text{Universal constant } \mathcal{O}(1)\ ]}$$

(Q is the "charge" of one boson)

M.P.A. Fisher, G. Grinstein, and S.M. Girvin, *Phys. Rev. Lett*. **64**, 587 (1990)
K. Damle and S. Sachdev, *Phys. Rev. B* **56**, 8714 (1997).

Quantum critical transport

Transport co-oefficients not determined by collision rate, but by universal constants of nature

Momentum transport

$$\frac{\eta}{s} \equiv \frac{\text{viscosity}}{\text{entropy density}}$$

$$= \frac{\hbar}{k_B} \times [\text{Universal constant } \mathcal{O}(1)\ ]$$

P. Kovtun, D. T. Son, and A. Starinets, *Phys. Rev. Lett.* **94,** 11601 (2005)

Quantum critical transport

Describe charge transport using Boltzmann theory of interacting bosons:

$$\frac{dv}{dt} + \frac{v}{\tau_c} = F.$$

This gives a frequency (ω) dependent conductivity

$$\sigma(\omega) = \frac{\sigma_0}{1 - i\,\omega\,\tau_c}$$

where $\tau_c \sim \hbar/(k_B T)$ is the time between boson collisions.

Quantum critical transport

Describe charge transport using Boltzmann theory of interacting bosons:

$$\frac{dv}{dt} + \frac{v}{\tau_c} = F.$$

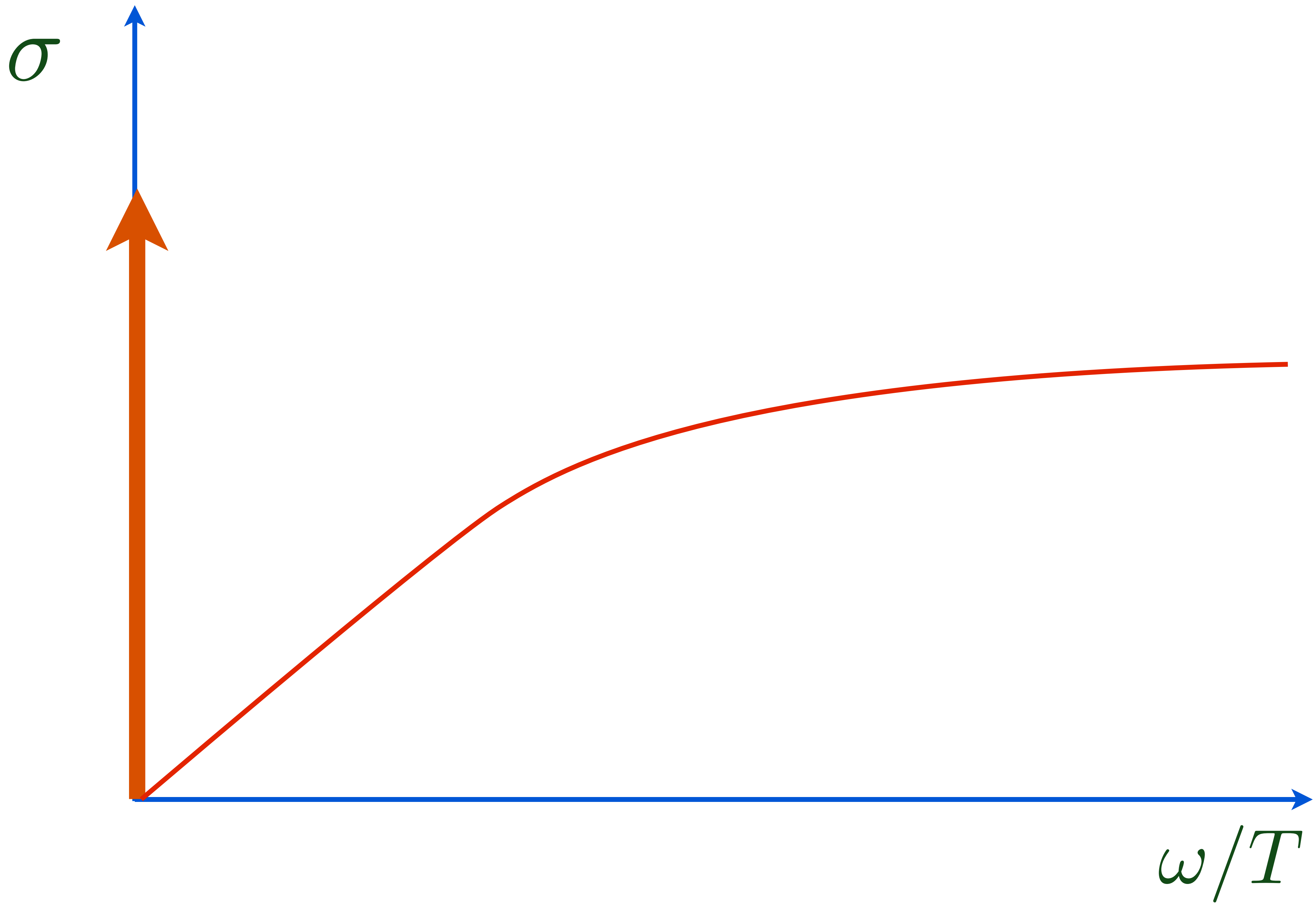
This gives a frequency (ω) dependent conductivity

$$\sigma(\omega) = \frac{\sigma_0}{1 - i\,\omega\,\tau_c}$$

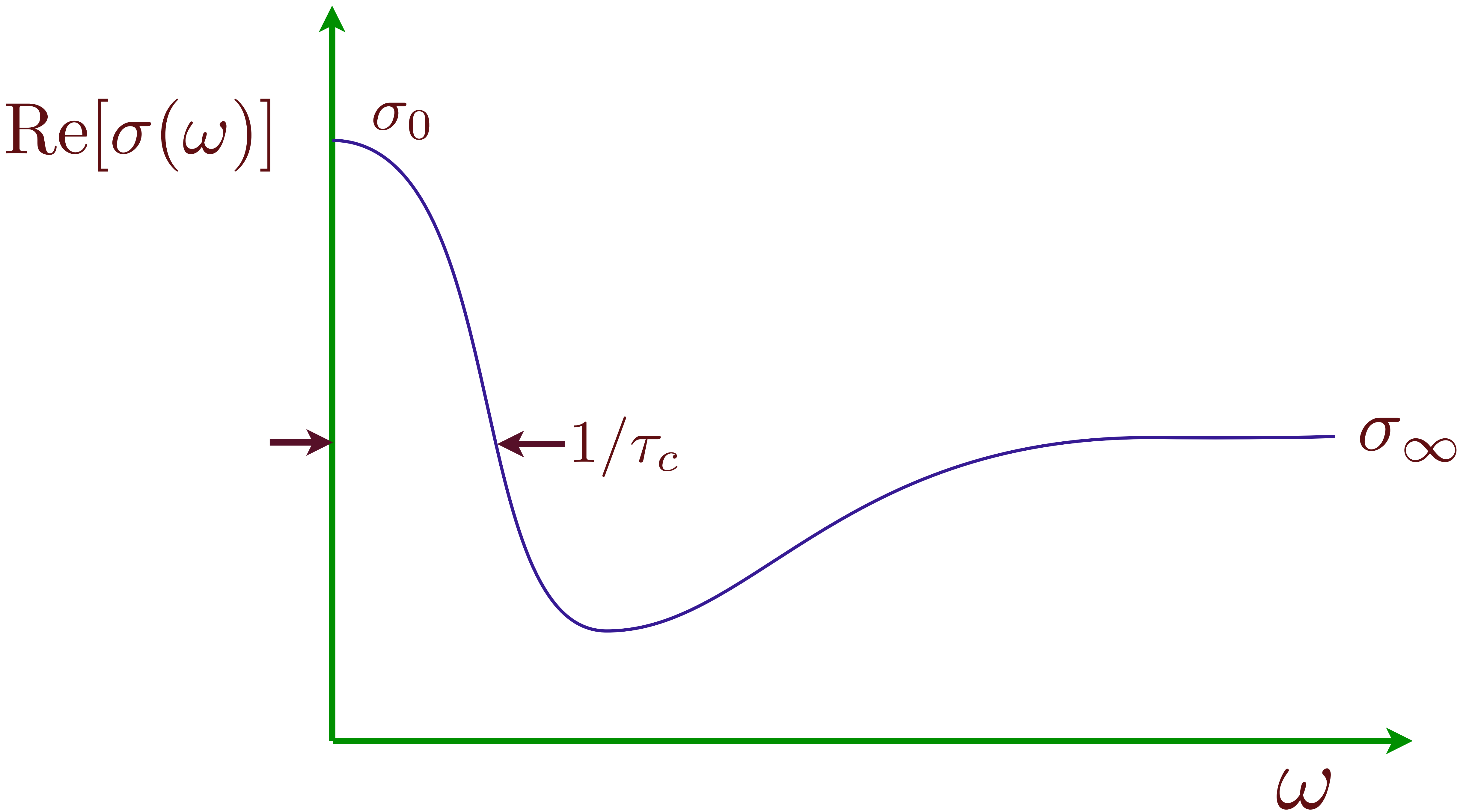
where $\tau_c \sim \hbar/(k_B T)$ is the time between boson collisions.

Also, we have $\sigma(\omega \to \infty) = \sigma_\infty$, associated with the density of states for particle-hole creation (the "optical conductivity") in the CFT3.

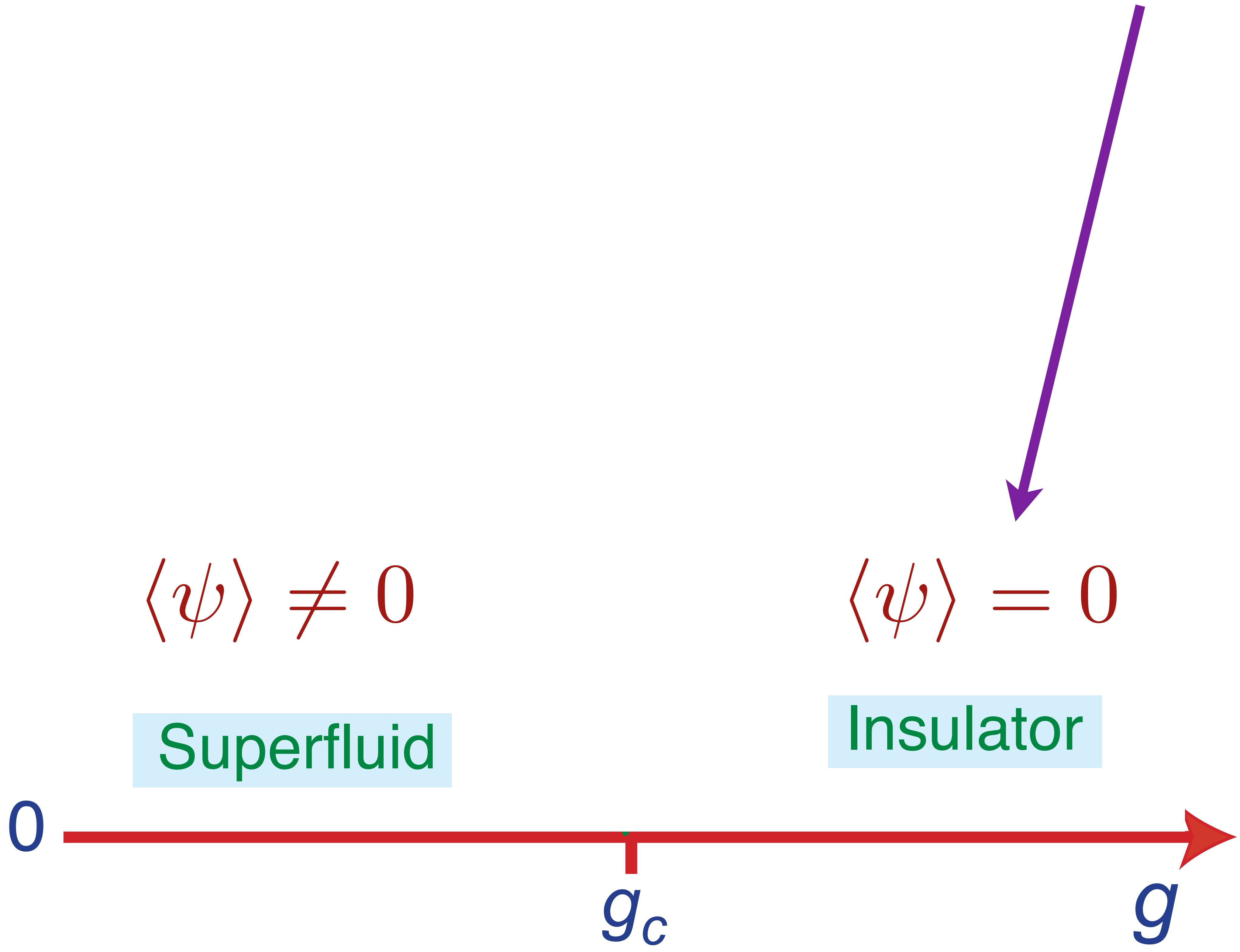
Electrical transport in a free-field theory for *T* > *0*



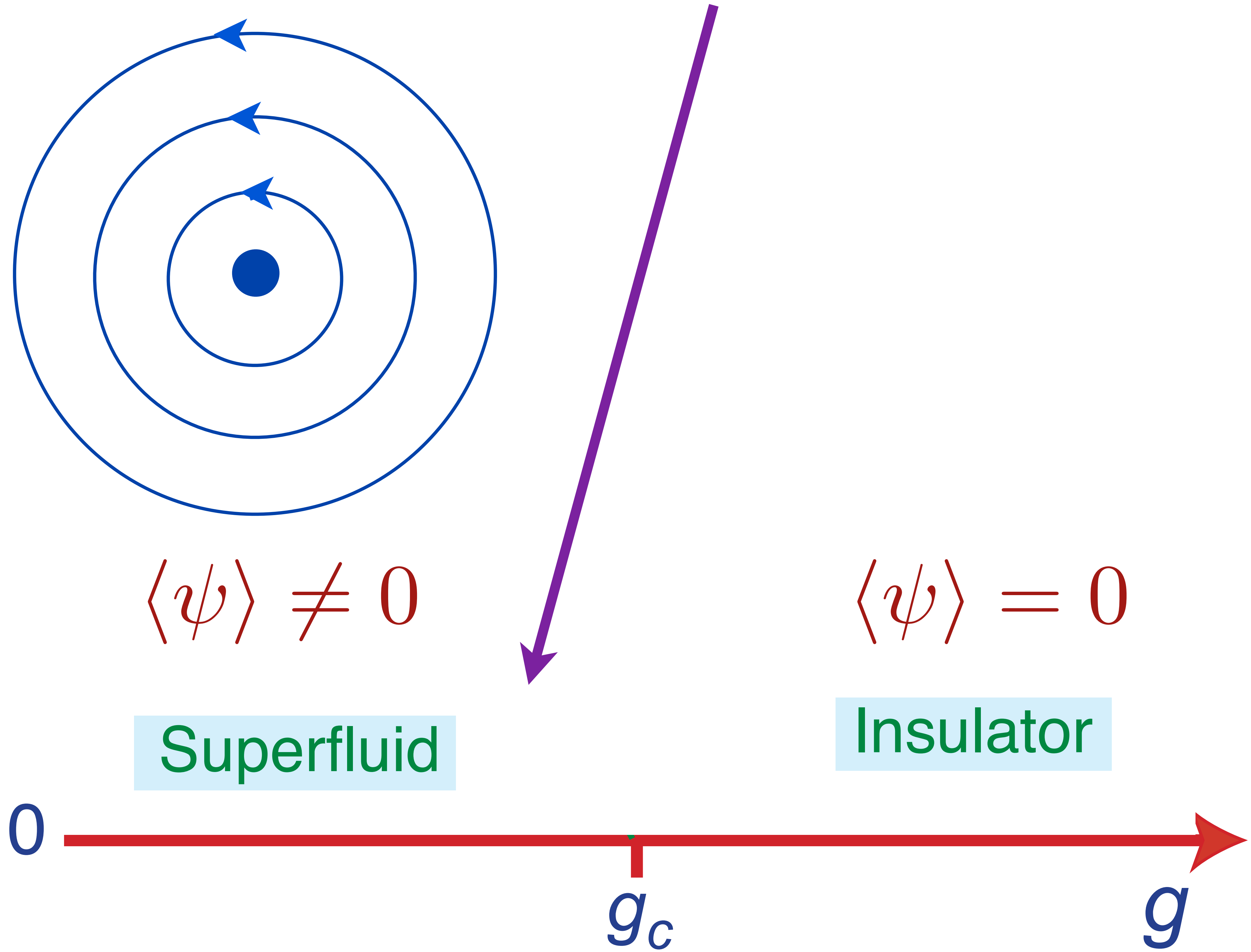
Boltzmann theory of bosons



So far, we have described the quantum critical point using the boson particle and hole excitations of the insulator.



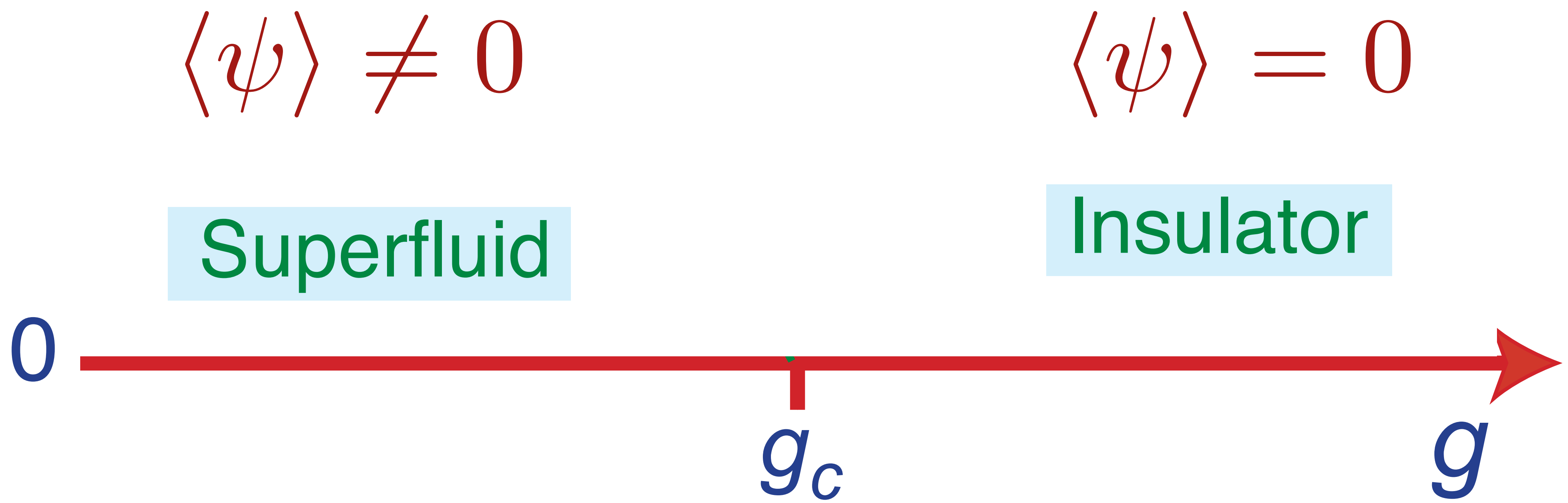
However, we could equally well describe the conductivity using the excitations of the superfluid, which are *vortices*.



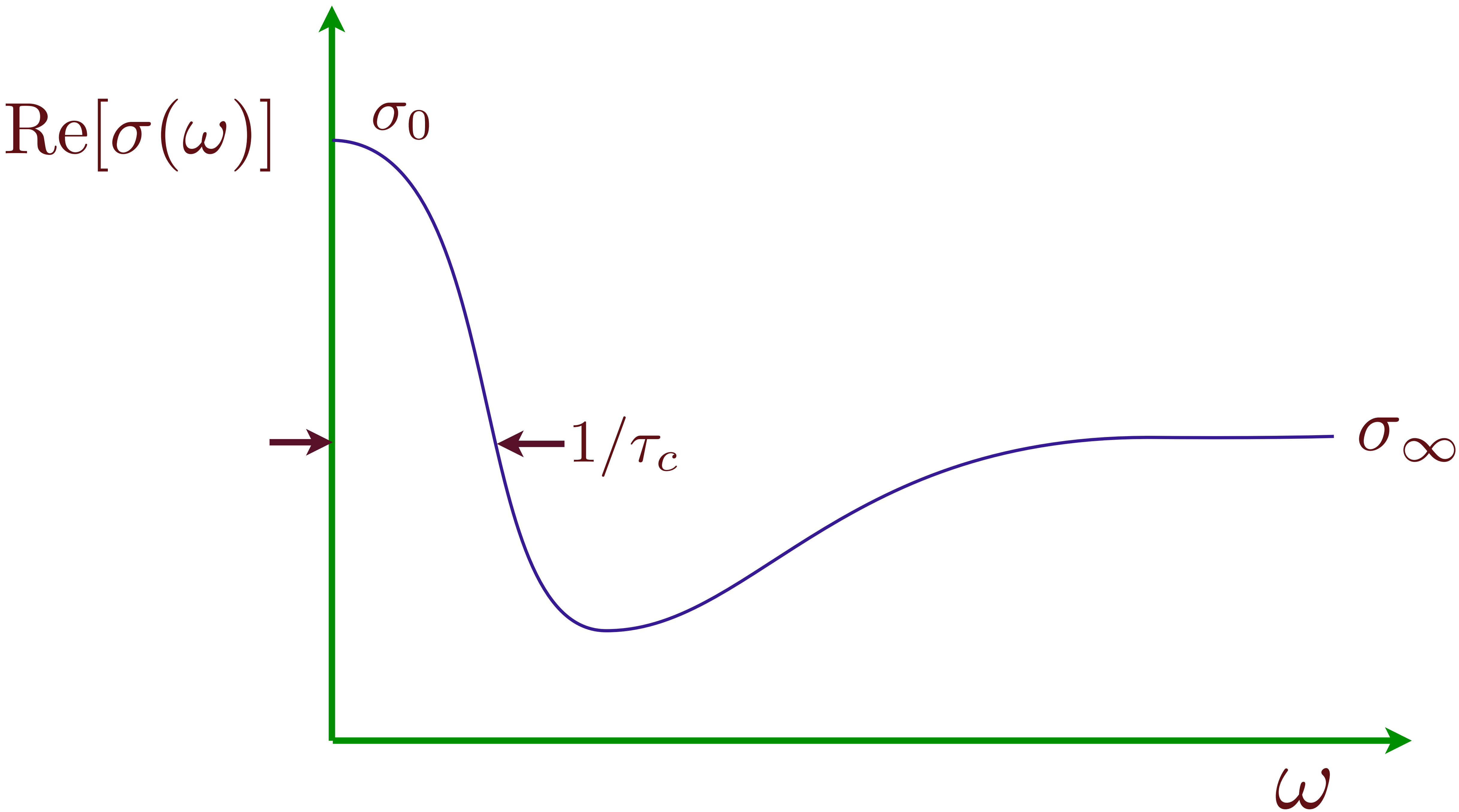
However, we could equally well describe the conductivity using the excitations of the superfluid, which are *vortices*.

These are quantum particles (in 2+1 dimensions) which described by a (mirror/e.m.) “dual” CFT3 with an emergent U(1) gauge field. Their $T > 0$ dynamics can also be described by a Boltzmann equation:

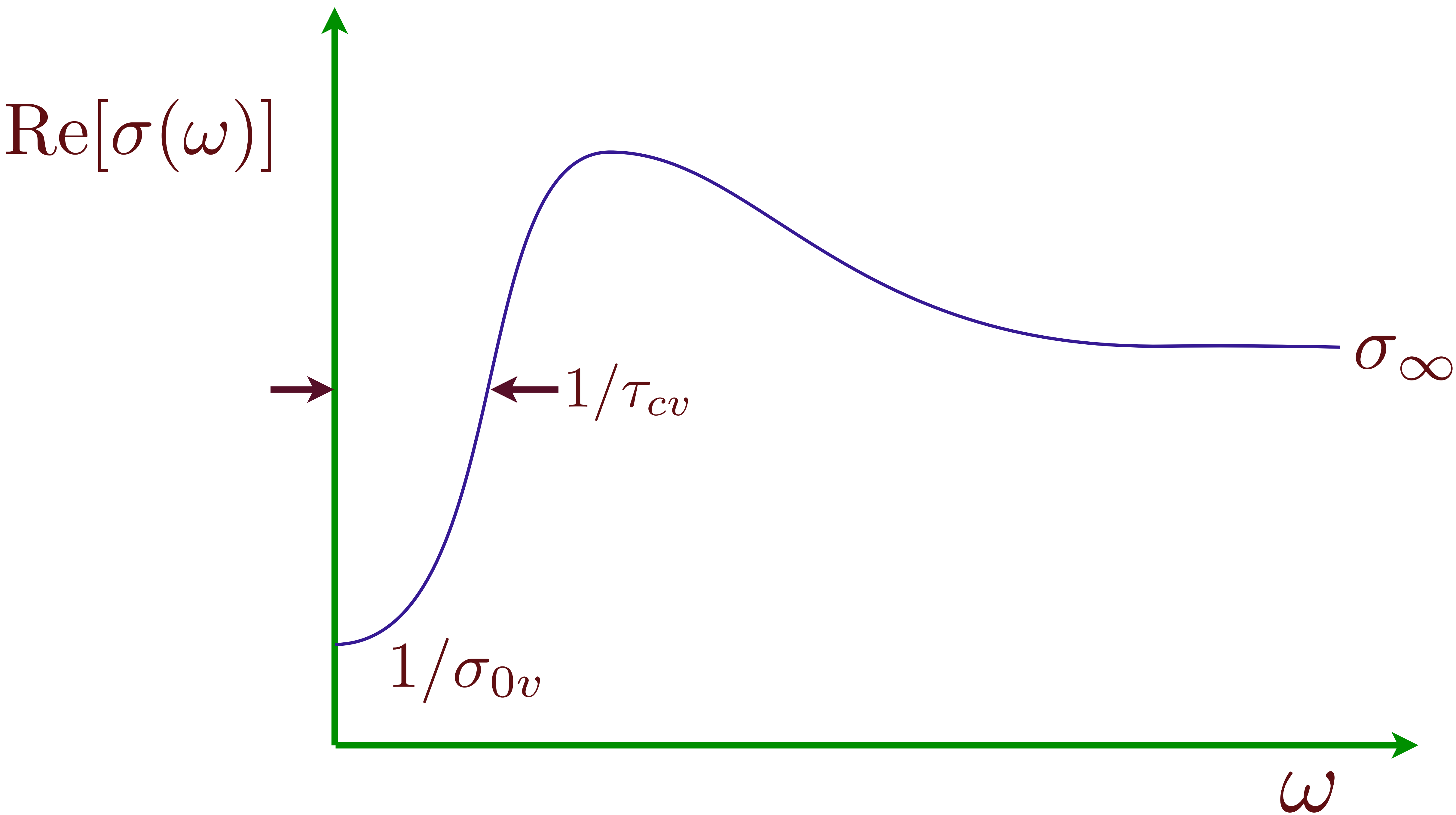
$$\text{Conductivity} = \text{Resistivity of vortices}$$



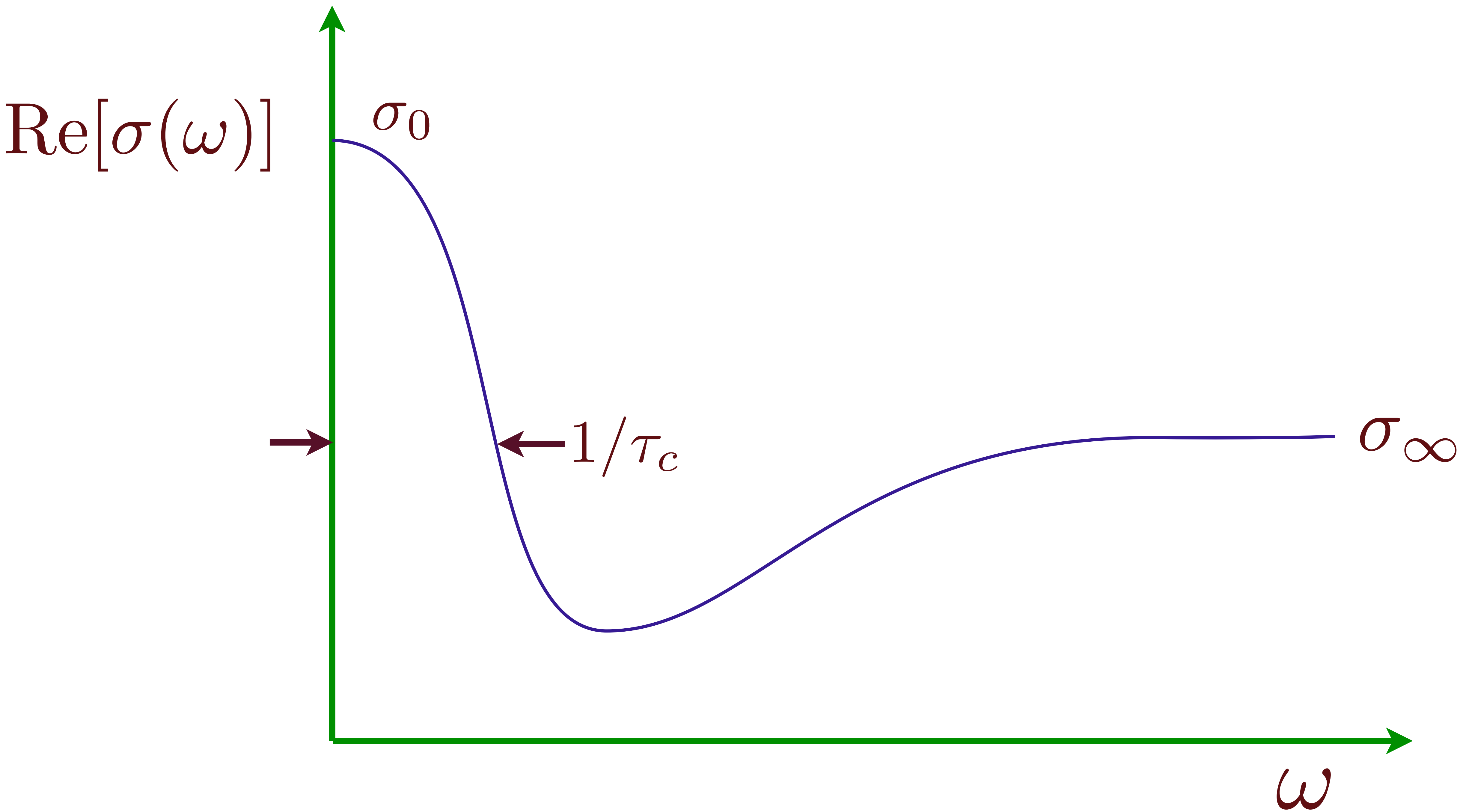
Boltzmann theory of bosons



Boltzmann theory of vortices



Boltzmann theory of bosons



Conformal quantum matter

A. Field theory:graphene

B. Field theory: superfluid-
insulator transition

C. Holography

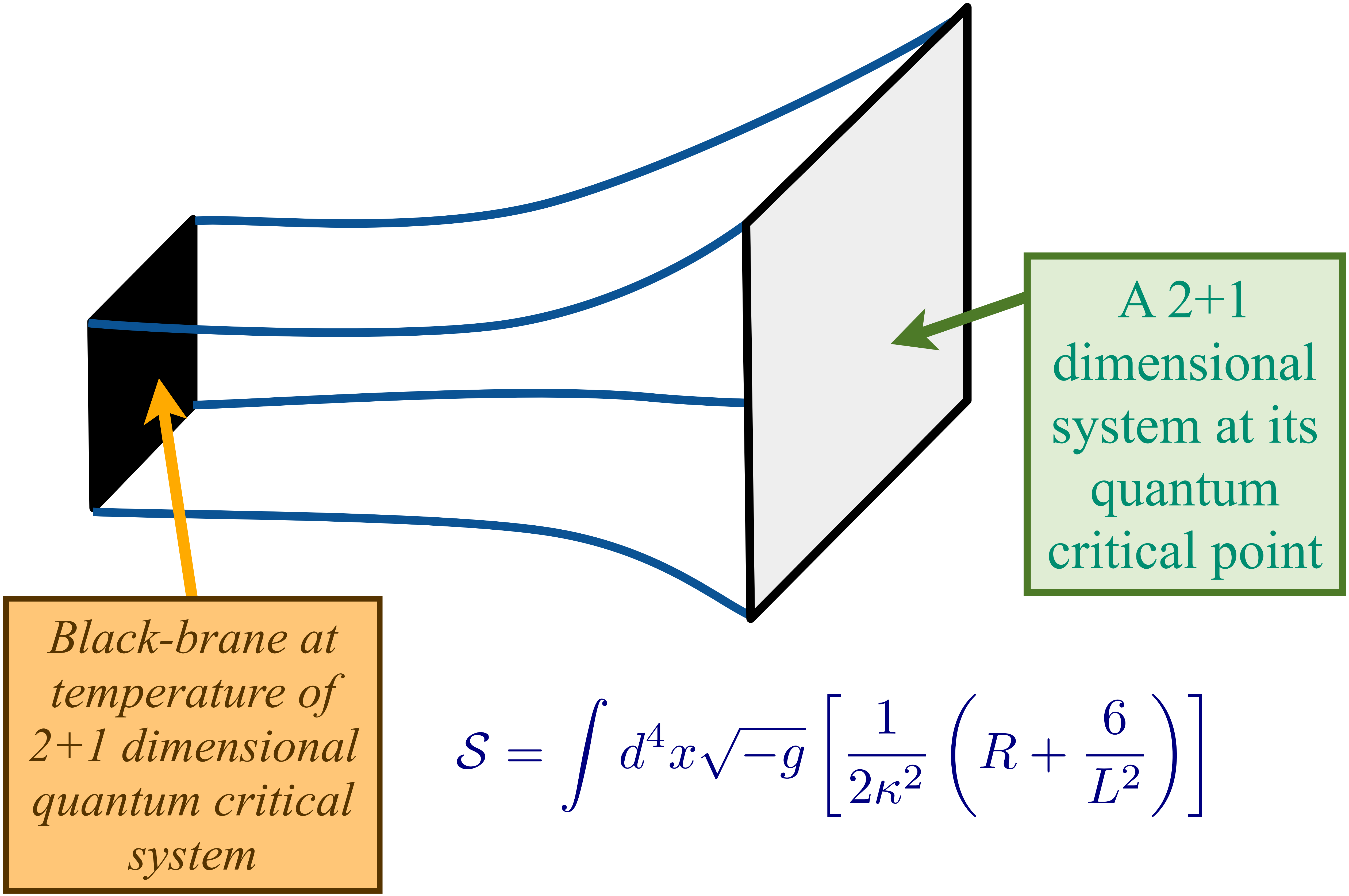
Conformal quantum matter

A. Field theory:graphene

*B. Field theory: superfluid-
insulator transition*

C. Holography

AdS4-Schwarzschild black-brane



$$\mathcal{S} = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa^2} \left(R + \frac{6}{L^2} \right) \right]$$

AdS$_4$ theory of "nearly perfect fluids"

To leading order in a gradient expansion, charge transport in an infinite set of strongly-interacting CFT3s can be described by Einstein-Maxwell gravity/electrodynamics on AdS$_4$-Schwarzschild

$$\mathcal{S}_{EM} = \int d^4x \sqrt{-g} \left[-\frac{1}{4e^2} F_{ab} F^{ab} \right] .$$

C. P. Herzog, P. K. Kovtun, S. Sachdev, and D. T. Son, *Phys. Rev.* D **75**, 085020 (2007).

AdS4 theory of "nearly perfect fluids"

To leading order in a gradient expansion, charge transport in an infinite set of strongly-interacting CFT3s can be described by Einstein-Maxwell gravity/electrodynamics on AdS$_4$-Schwarzschild

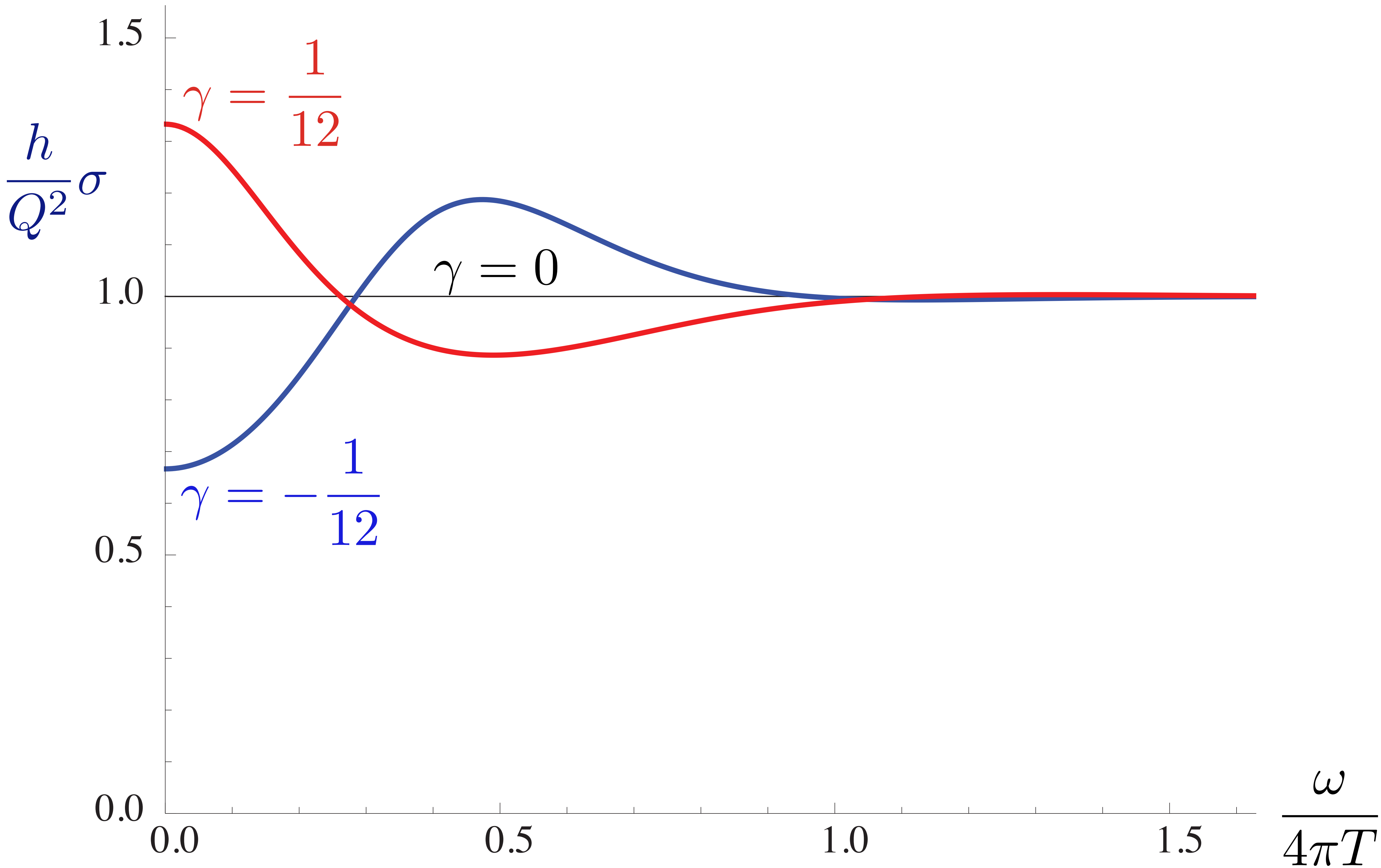
We include all possible 4-derivative terms: after suitable field redefinitions, the required theory has only *one* dimensionless constant γ (L is the radius of AdS$_4$):

$$\mathcal{S}_{EM} = \int d^4x \sqrt{-g} \left[-\frac{1}{4e^2} F_{ab} F^{ab} + \frac{\gamma L^2}{e^2} C_{abcd} F^{ab} F^{cd} \right],$$

where C_{abcd} is the Weyl curvature tensor.
Stability and causality constraints restrict $|\gamma| < 1/12$.

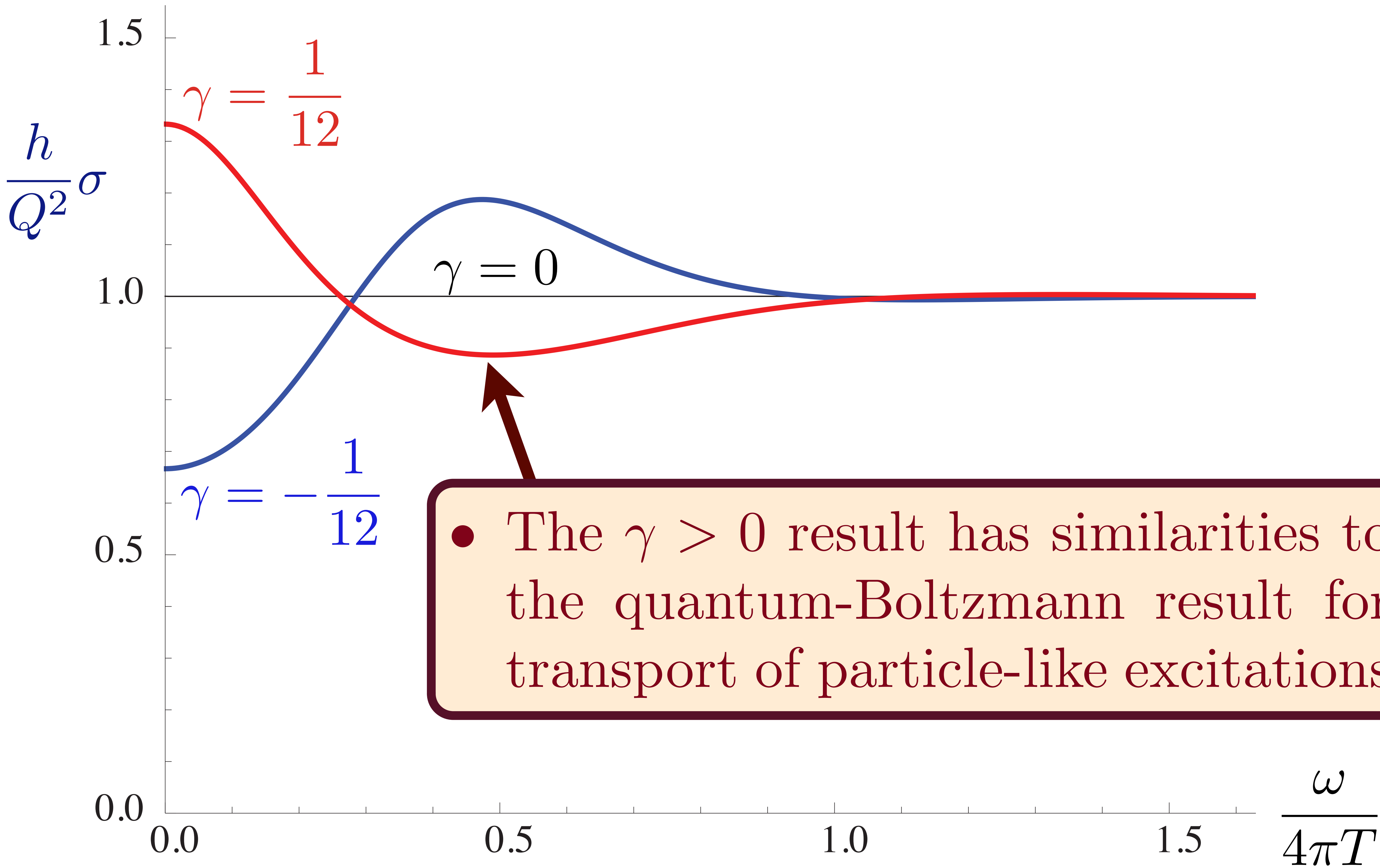
R. C. Myers, S. Sachdev, and A. Singh, *Physical Review D* **83**, 066017 (2011)

AdS$_4$ theory of strongly interacting "perfect fluids"



R. C. Myers, S. Sachdev, and A. Singh, *Physical Review D* **83**, 066017 (2011)

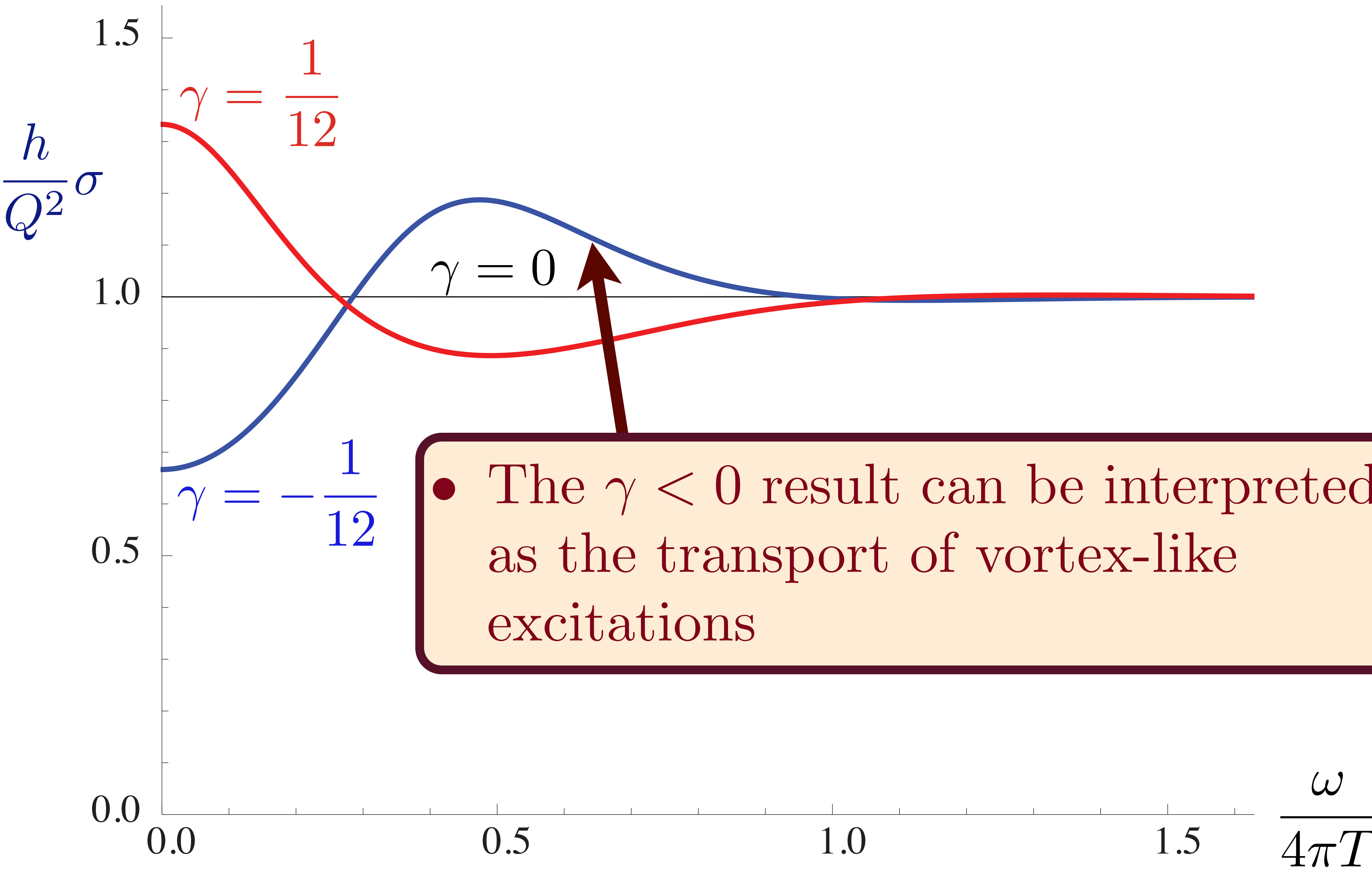
AdS4 theory of strongly interacting "perfect fluids"



- The $\gamma > 0$ result has similarities to the quantum-Boltzmann result for transport of particle-like excitations

R. C. Myers, S. Sachdev, and A. Singh, *Physical Review D* **83**, 066017 (2011)

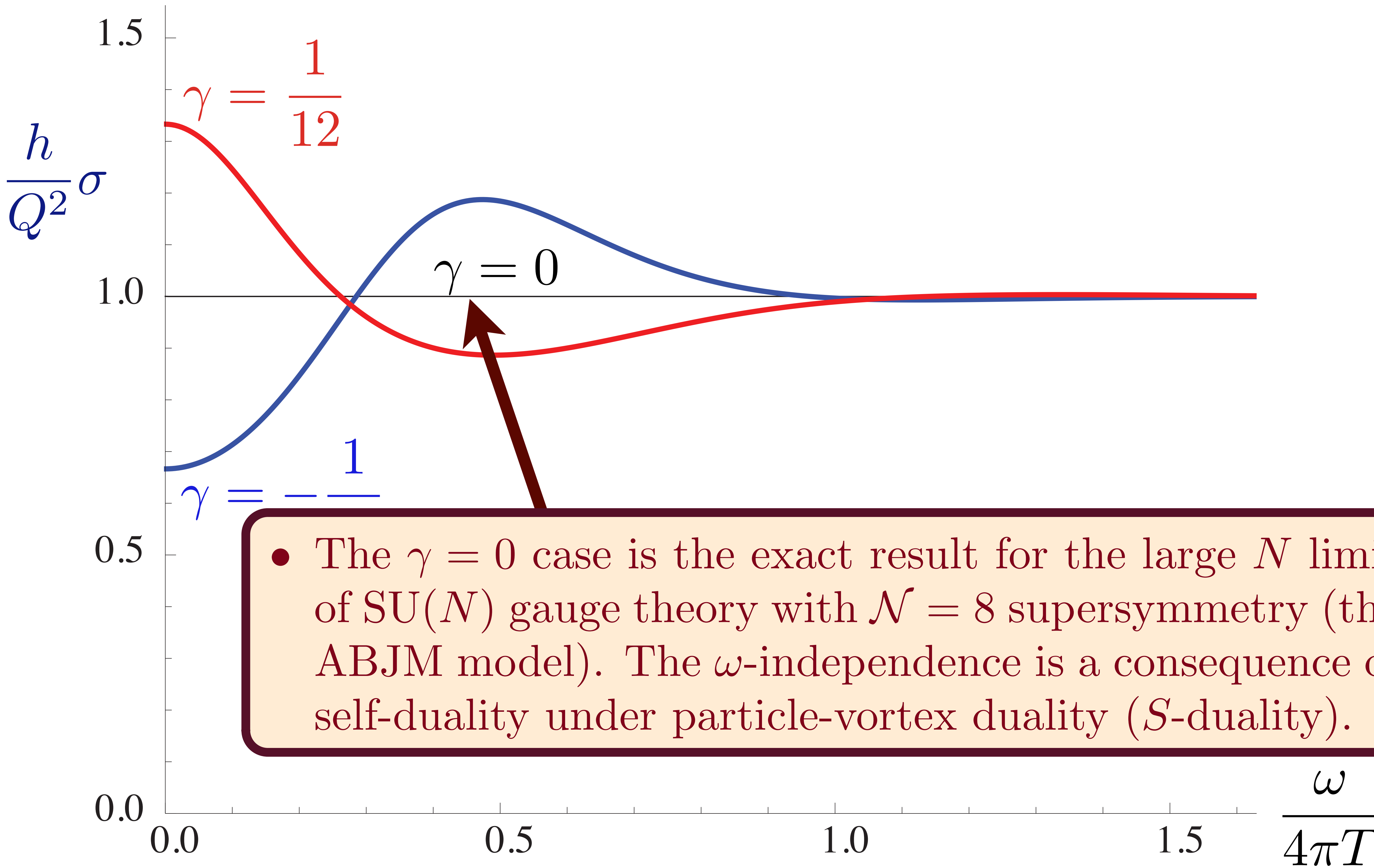
AdS_4 theory of strongly interacting “perfect fluids”



- The $\gamma < 0$ result can be interpreted as the transport of vortex-like excitations

R. C. Myers, S. Sachdev, and A. Singh, *Physical Review D* **83**, 066017 (2011)

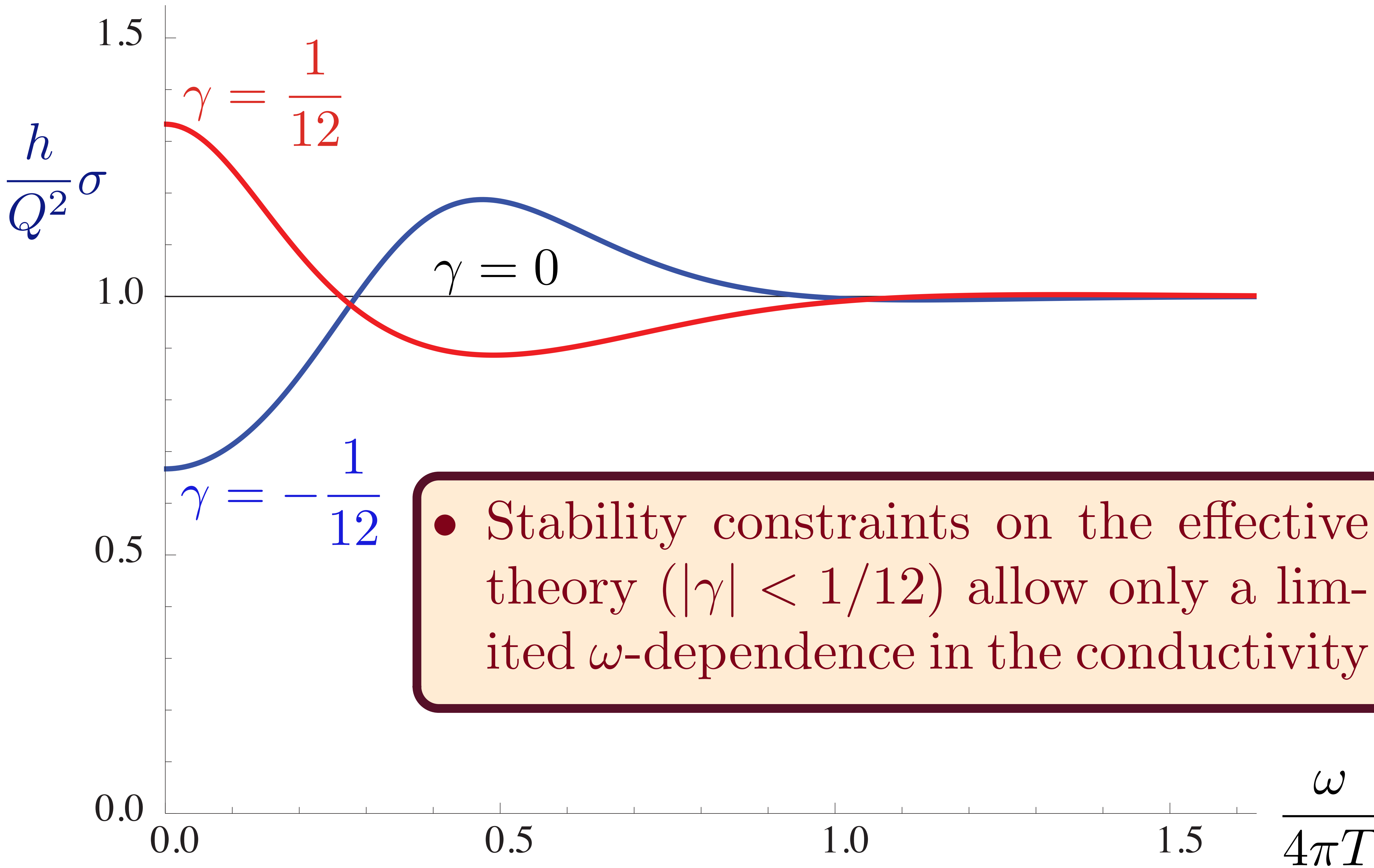
AdS$_4$ theory of strongly interacting "perfect fluids"



- The $\gamma = 0$ case is the exact result for the large N limit of SU(N) gauge theory with $\mathcal{N} = 8$ supersymmetry (the ABJM model). The ω-independence is a consequence of self-duality under particle-vortex duality (S-duality).

R. C. Myers, S. Sachdev, and A. Singh, *Physical Review D* **83**, 066017 (2011)

AdS$_4$ theory of strongly interacting "perfect fluids"



- Stability constraints on the effective theory ($|\gamma| < 1/12$) allow only a limited ω-dependence in the conductivity

R. C. Myers, S. Sachdev, and A. Singh, *Physical Review D* **83**, 066017 (2011)

AdS$_4$ theory of "nearly perfect fluids"

Theory for transport of conserved quantities in CFT3s:

$$\mathcal{S}_{EM} = \int d^4x\sqrt{-g}\left[-\frac{1}{4e^2}F_{ab}F^{ab} + \frac{\gamma L^2}{e^2}C_{abcd}F^{ab}F^{cd}\right],$$

where C_{abcd} is the Weyl curvature tensor.

General approach:

- Theory has 2 free dimensionless parameters: e^2 and γ. We match these to correlators of the CFT3 of interest at $\omega \gg T$: e^2 is determines the current correlator $\langle J_\mu J_\nu \rangle$, while γ determines the 3-point function $\langle T_{\mu\nu} J_\rho J_\sigma \rangle$, where $T_{\mu\nu}$ is the stress-energy tensor.

R. C. Myers, S. Sachdev, and A. Singh, *Physical Review D* **83**, 066017 (2011)

AdS$_4$ theory of "nearly perfect fluids"

Theory for transport of conserved quantities in CFT3s:

$$\mathcal{S}_{EM} = \int d^4x \sqrt{-g} \left[-\frac{1}{4e^2} F_{ab} F^{ab} + \frac{\gamma L^2}{e^2} C_{abcd} F^{ab} F^{cd} \right] ,$$

where C_{abcd} is the Weyl curvature tensor.

General approach:

- Theory has 2 free dimensionless parameters: e^2 and γ. We match these to correlators of the CFT3 of interest at $\omega \gg T$: e^2 is determines the current correlator $\langle J_\mu J_\nu \rangle$, while γ determines the 3-point function $\langle T_{\mu\nu} J_\rho J_\sigma \rangle$, where $T_{\mu\nu}$ is the stress-energy tensor.

- We determine these $\omega \gg T$ correlators of the CFT3 by other methods (*e.g.* vector large N expansion), and so obtain values of e^2 and γ.

R. C. Myers, S. Sachdev, and A. Singh, *Physical Review D* **83**, 066017 (2011)

AdS$_4$ theory of "nearly perfect fluids"

Theory for transport of conserved quantities in CFT3s:

$$\mathcal{S}_{EM} = \int d^4x\sqrt{-g}\left[-\frac{1}{4e^2}F_{ab}F^{ab} + \frac{\gamma L^2}{e^2}C_{abcd}F^{ab}F^{cd}\right],$$

where C_{abcd} is the Weyl curvature tensor.

General approach:

- Theory has 2 free dimensionless parameters: e^2 and γ. We match these to correlators of the CFT3 of interest at $\omega \gg T$: e^2 is determines the current correlator $\langle J_\mu J_\nu \rangle$, while γ determines the 3-point function $\langle T_{\mu\nu} J_\rho J_\sigma \rangle$, where $T_{\mu\nu}$ is the stress-energy tensor.

- We determine these $\omega \gg T$ correlators of the CFT3 by other methods (*e.g.* vector large N expansion), and so obtain values of e^2 and γ.

- We use $\mathcal{S}_{EM}$ to extrapolate to transport properties for $\omega \ll T$. This step is traditionally carried out by descendants of the Boltzmann equation.

R. C. Myers, S. Sachdev, and A. Singh, *Physical Review D* **83**, 066017 (2011)

Conclusions

Quantum criticality and conformal field theories

- New insights and solvable models for diffusion and transport of strongly interacting systems near quantum critical points

Conclusions

Quantum criticality and conformal field theories

- New insights and solvable models for diffusion and transport of strongly interacting systems near quantum critical points

- The description is far removed from, and complementary to, that of the quantum Boltzmann equation which builds on the quasiparticle/vortex picture.

Conclusions

Quantum criticality and conformal field theories

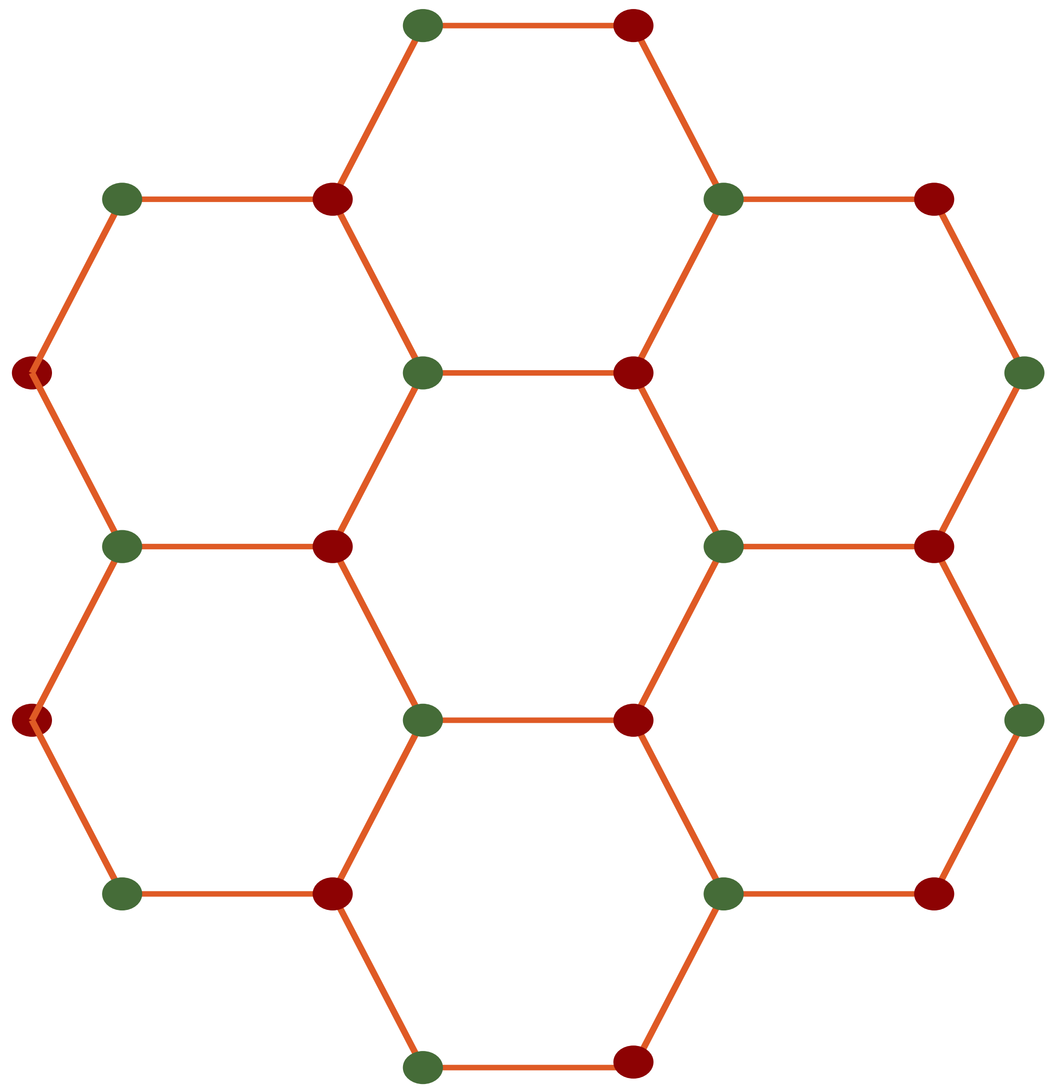
- New insights and solvable models for diffusion and transport of strongly interacting systems near quantum critical points

- The description is far removed from, and complementary to, that of the quantum Boltzmann equation which builds on the quasiparticle/vortex picture.

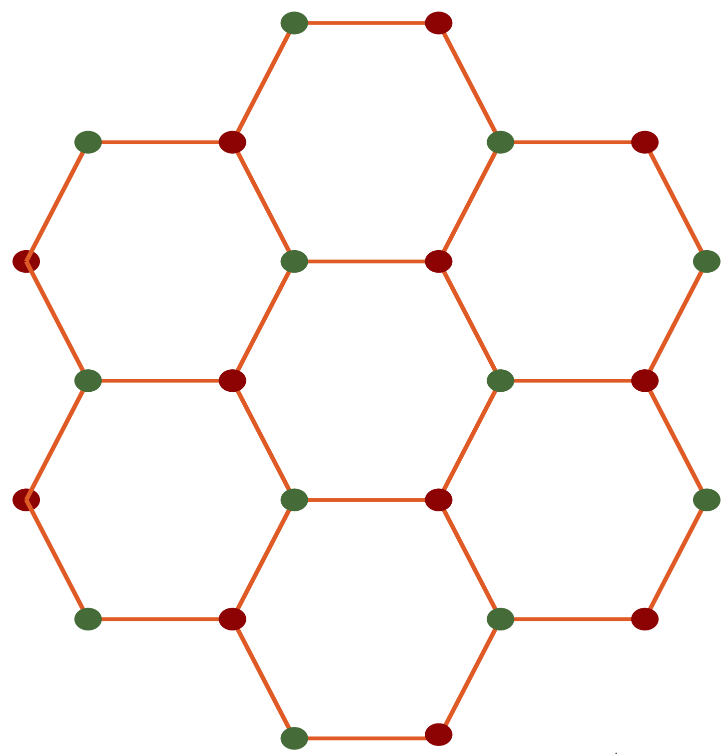
- Prospects for experimental tests of frequency-dependent, non-linear, and non-equilibrium transport

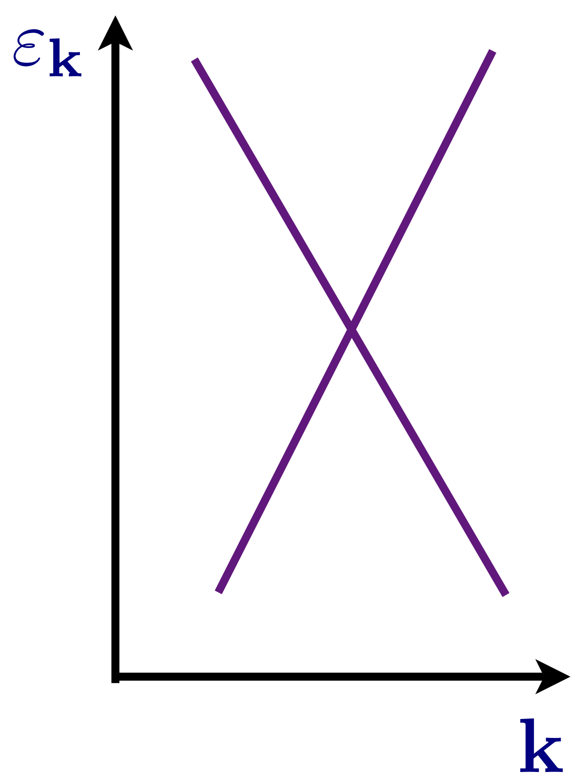
Honeycomb lattice

(describes graphene after adding long-range Coulomb interactions)



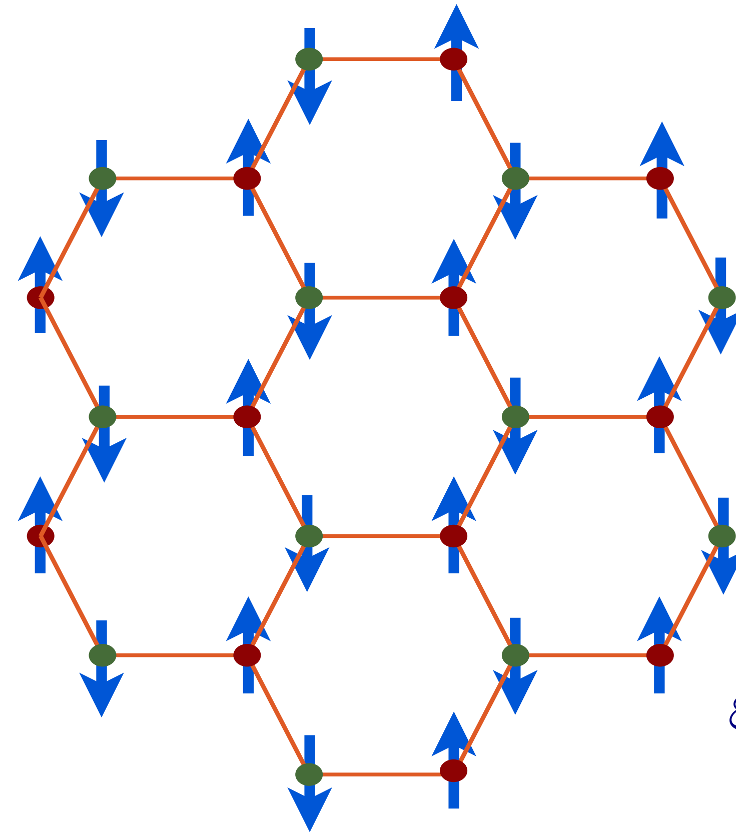
$$H = -t \sum_{\langle ij \rangle} c_{i\alpha}^{\dagger} c_{j\alpha} + U \sum_{i} \left(n_{i\uparrow} - \frac{1}{2} \right) \left(n_{i\downarrow} - \frac{1}{2} \right)$$



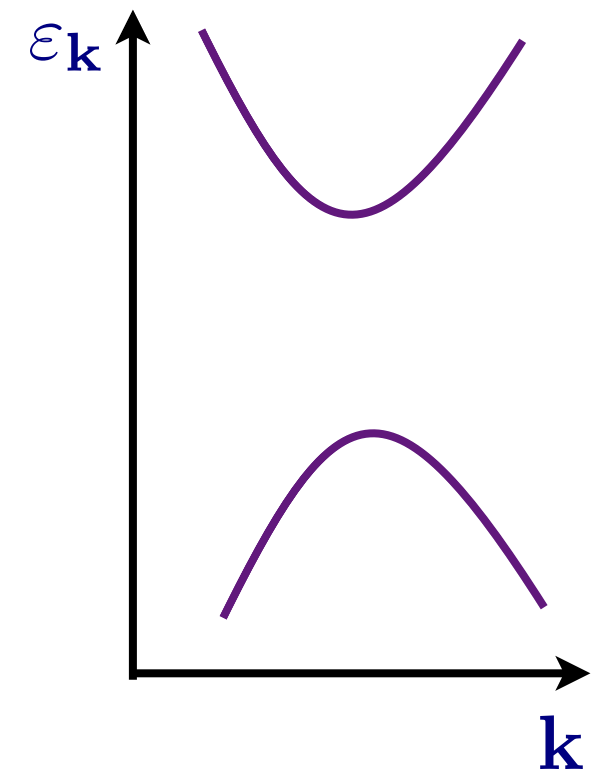


Dirac
semi-metal

$$\langle \varphi^a \rangle = 0$$



Insulating
antiferromagnet
with Neel order

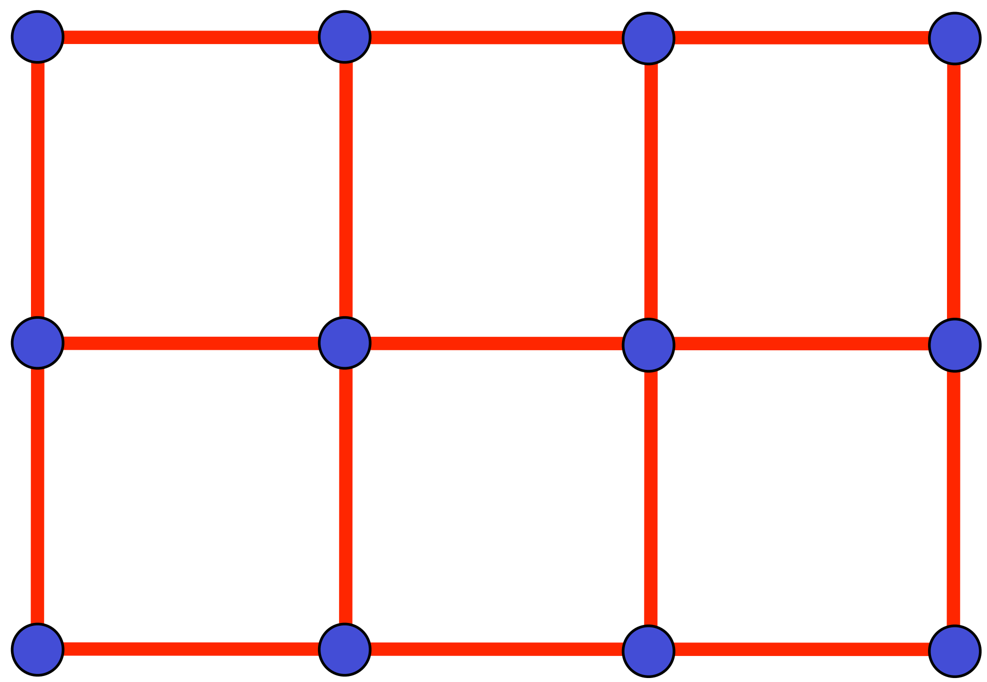


$$\langle \varphi^a \rangle \neq 0$$

s

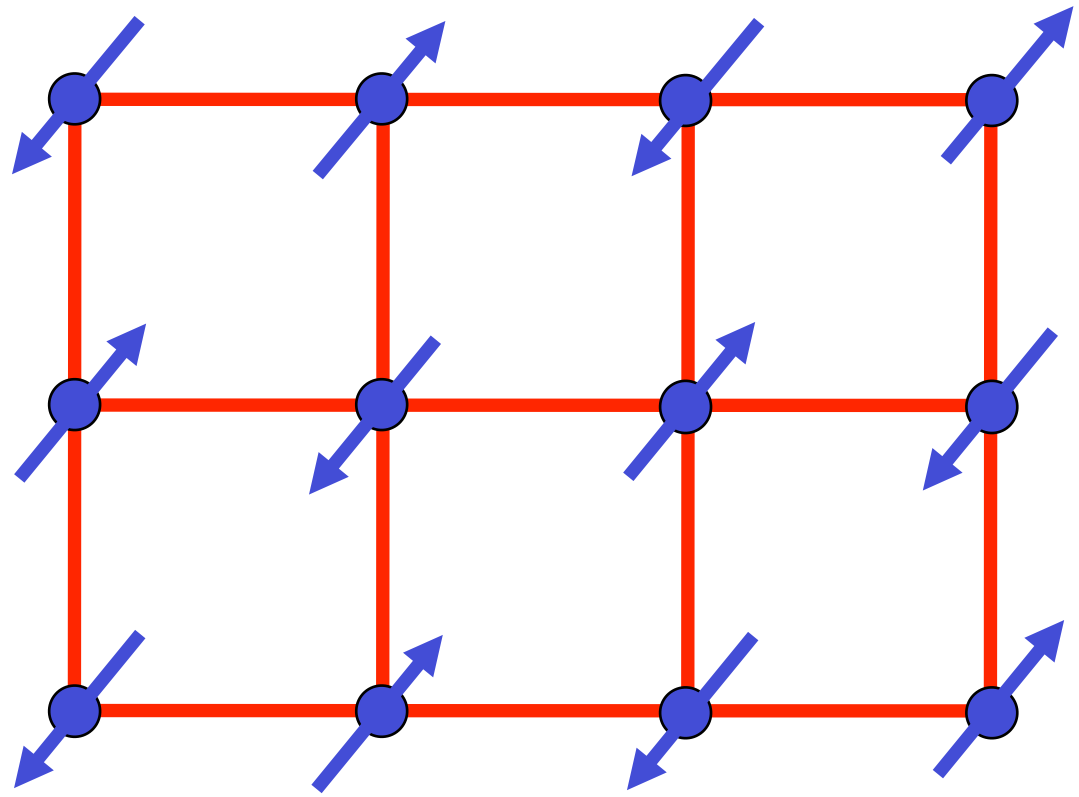
Quantum phase transition described by a strongly-coupled conformal field theory without well-defined quasiparticles

Square lattice



$$H = -t \sum_{\langle ij \rangle} c_{i\alpha}^{\dagger} c_{j\alpha} + U \sum_i \left(n_{i\uparrow} - \frac{1}{2} \right) \left(n_{i\downarrow} - \frac{1}{2} \right)$$

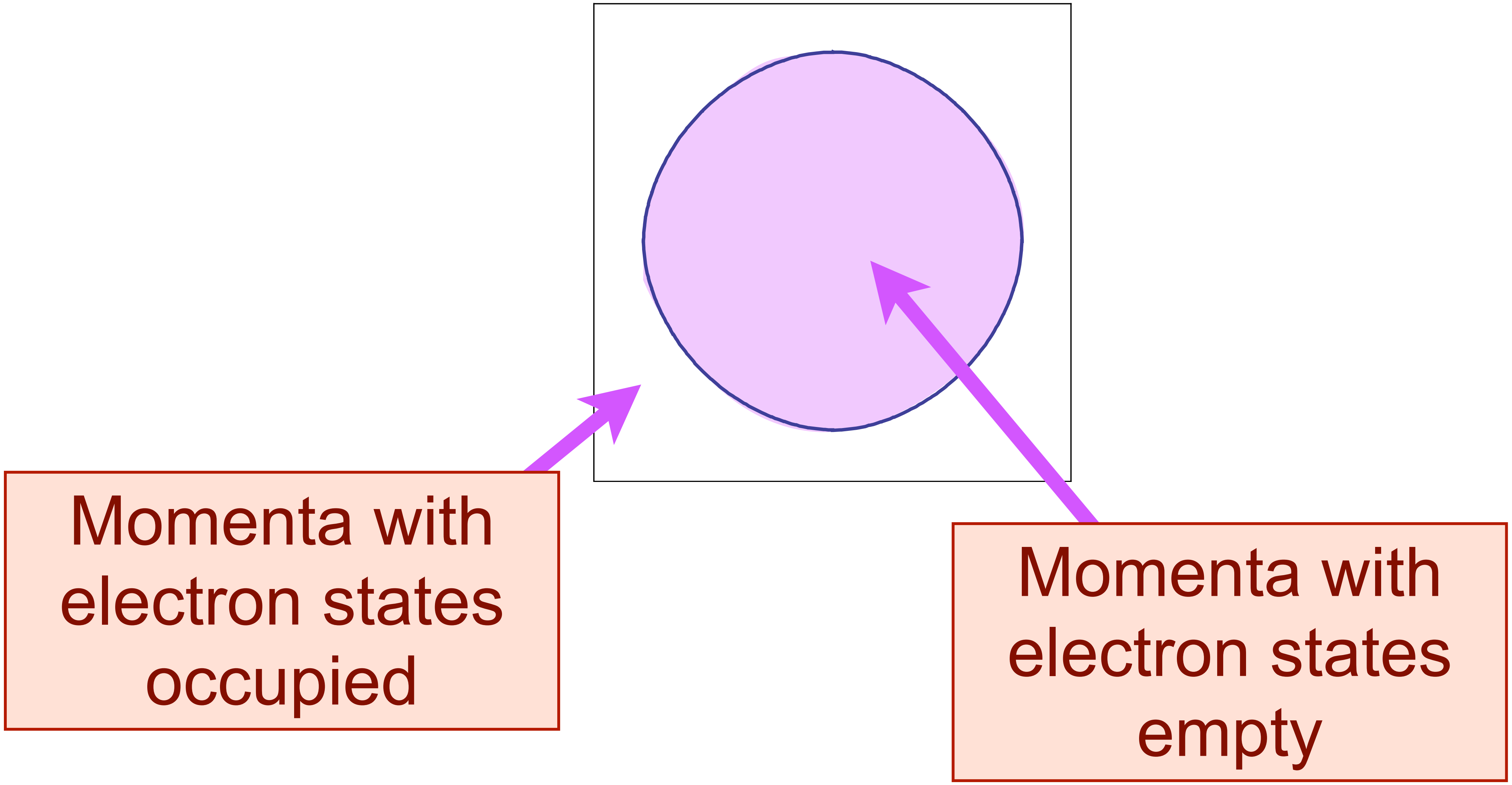
Square lattice



Large *U/t*

$$H = -t \sum_{\langle ij \rangle} c_{i\alpha}^{\dagger} c_{j\alpha} + U \sum_i \left(n_{i\uparrow} - \frac{1}{2} \right) \left(n_{i\downarrow} - \frac{1}{2} \right)$$

Square lattice



Small *U/t*: Fermi liquid metal

$$H = -t \sum_{\langle ij \rangle} c_{i\alpha}^{\dagger} c_{j\alpha} + U \sum_{i} \left(n_{i\uparrow} - \frac{1}{2} \right) \left(n_{i\downarrow} - \frac{1}{2} \right)$$

Boson-fermion theory for both phases

$$\mathcal{S} = \int d^2 r d\tau \left[\mathcal{L}_c + \mathcal{L}_\varphi + \mathcal{L}_{c\varphi} \right]$$

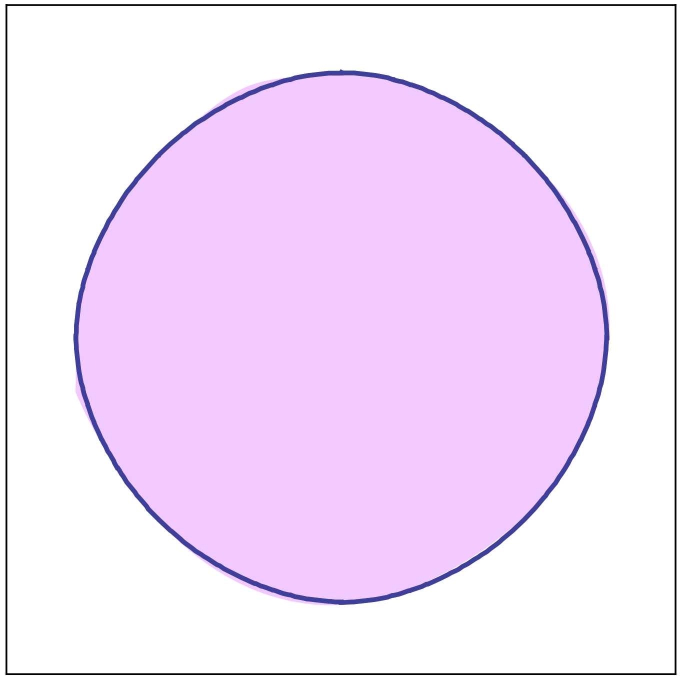
$$\mathcal{L}_c = c_a^\dagger \varepsilon(-i\boldsymbol{\nabla}) c_a$$

$$\mathcal{L}_\varphi = \frac{1}{2}(\boldsymbol{\nabla}\varphi_\alpha)^2 + \frac{r}{2}\varphi_\alpha^2 + \frac{u}{4}\left(\varphi_\alpha^2\right)^2$$

$$\mathcal{L}_{c\varphi} = \lambda\, \varphi_\alpha\, e^{i\mathbf{K}\cdot\mathbf{r}}\, c_a^\dagger\, \sigma_{ab}^\alpha\, c_b .$$

Decouple the U interaction by a boson field φ_α representing the antiferromagnetic order, just as for the honeycomb lattice; $\lambda^2 \sim U$, What is the influence of a non-zero antiferromagnetic order, $\langle \varphi_\alpha \rangle \neq 0$, on the Fermi surface of the metal ?

Metal with "large" Fermi surface

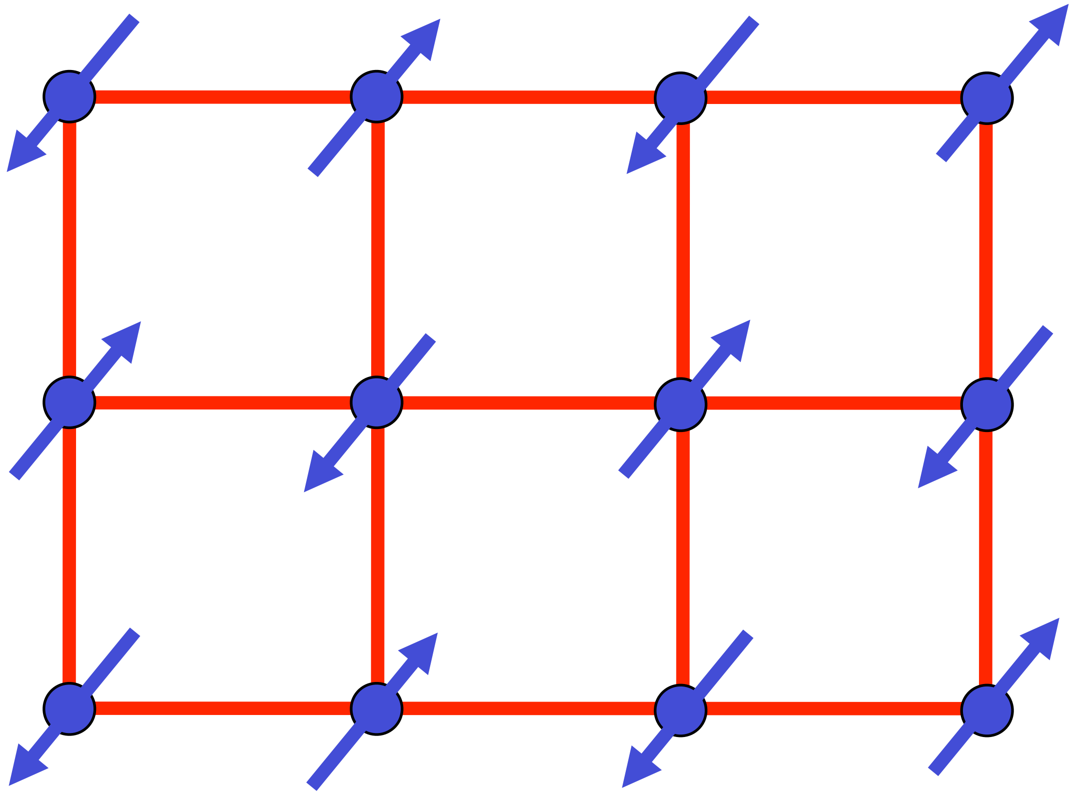


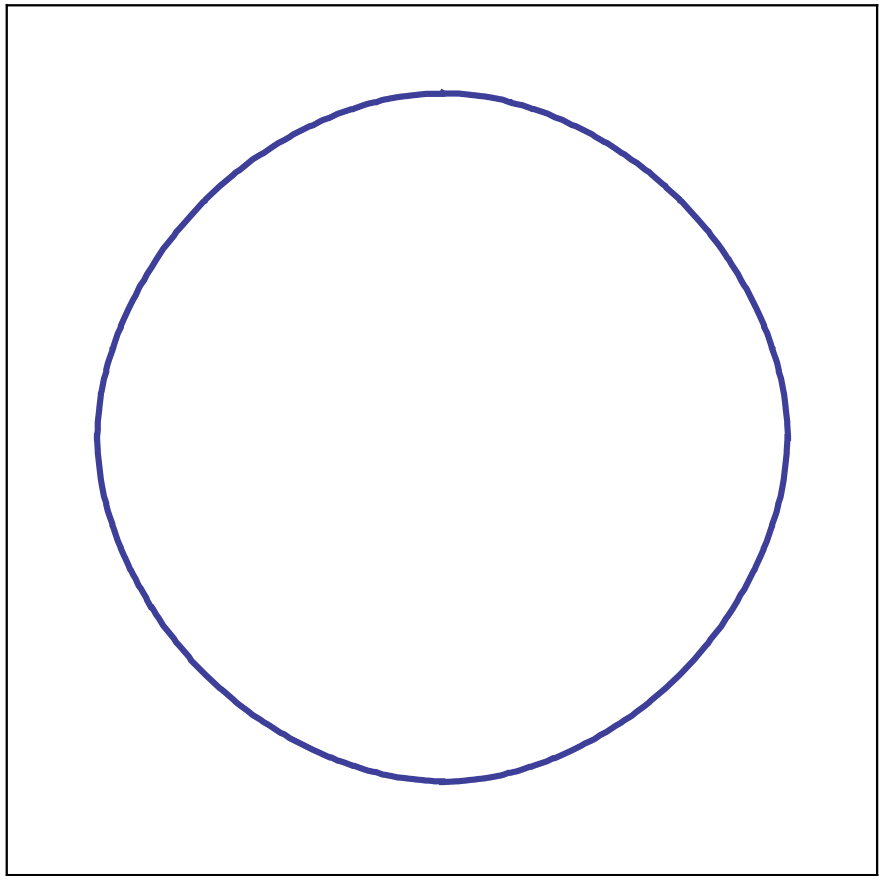
+

The electron spin polarization obeys

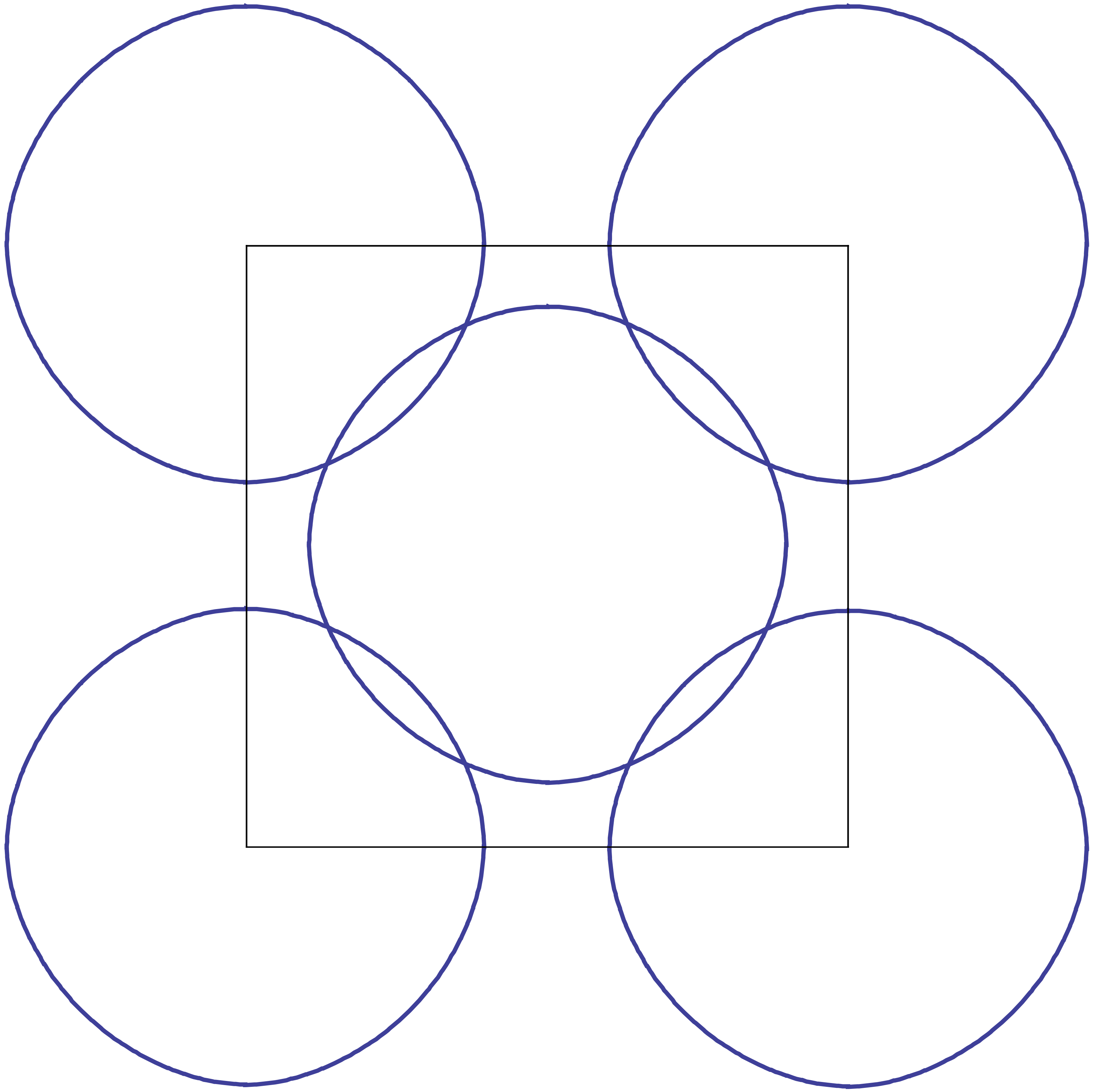
$$\left\langle \vec{S}(\mathbf{r},\tau) \right\rangle = \vec{\varphi}(\mathbf{r},\tau) e^{i\mathbf{K}\cdot\mathbf{r}}$$

where $\mathbf{K}$ is the ordering wavevector.

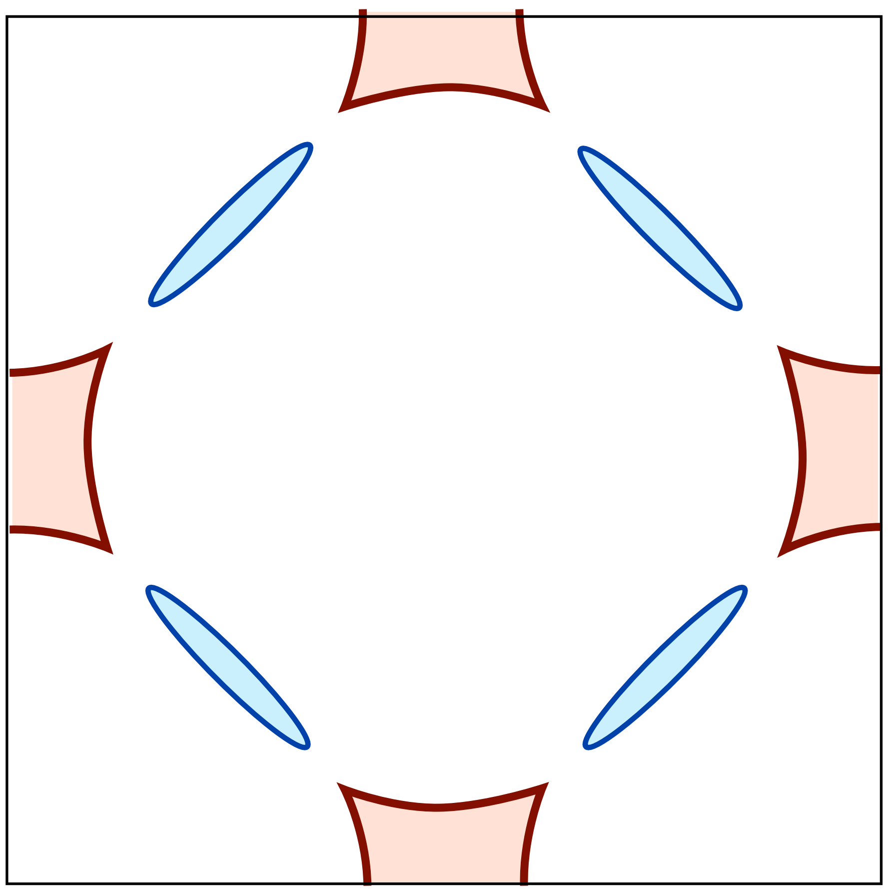




Metal with “large” Fermi surface

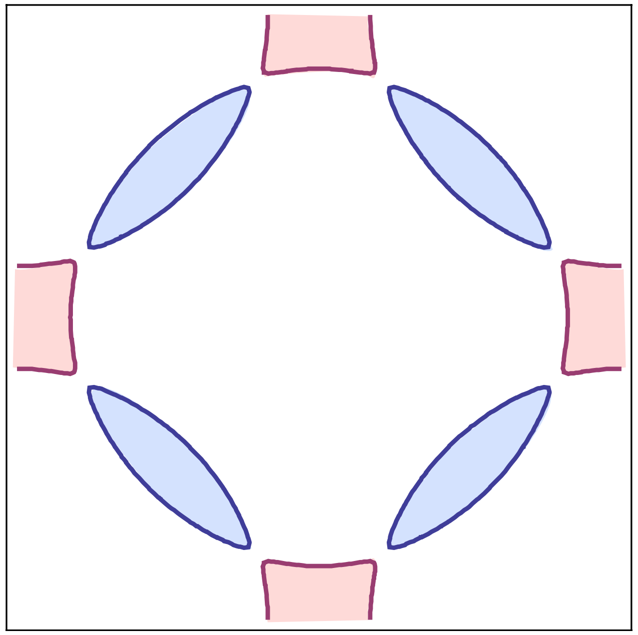


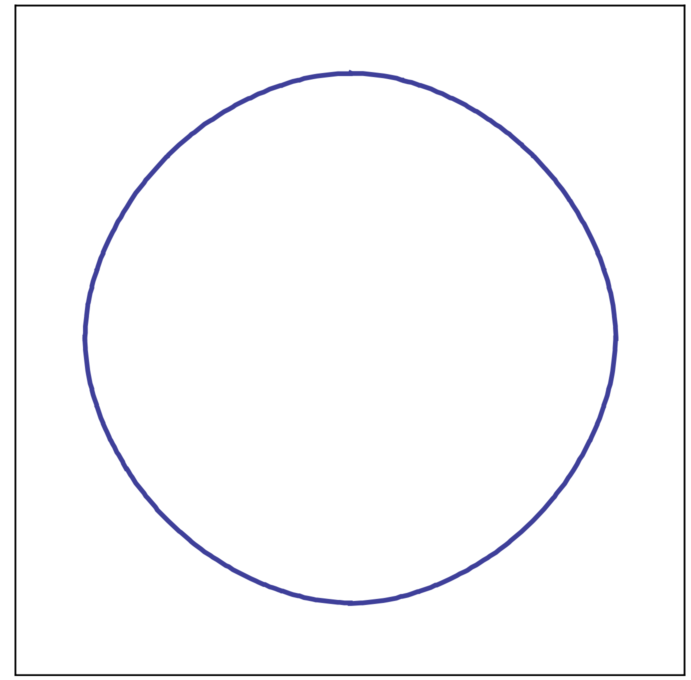
Fermi surfaces translated by $\mathbf{K} = (\pi, \pi)$.



Fermi surface reconstruction
into electron and hole pockets in
antiferromagnetic phase with $\langle \vec{\varphi} \rangle \neq 0$

Quantum phase transition with Fermi surface reconstruction





$\langle \vec{\varphi} \rangle \neq 0$

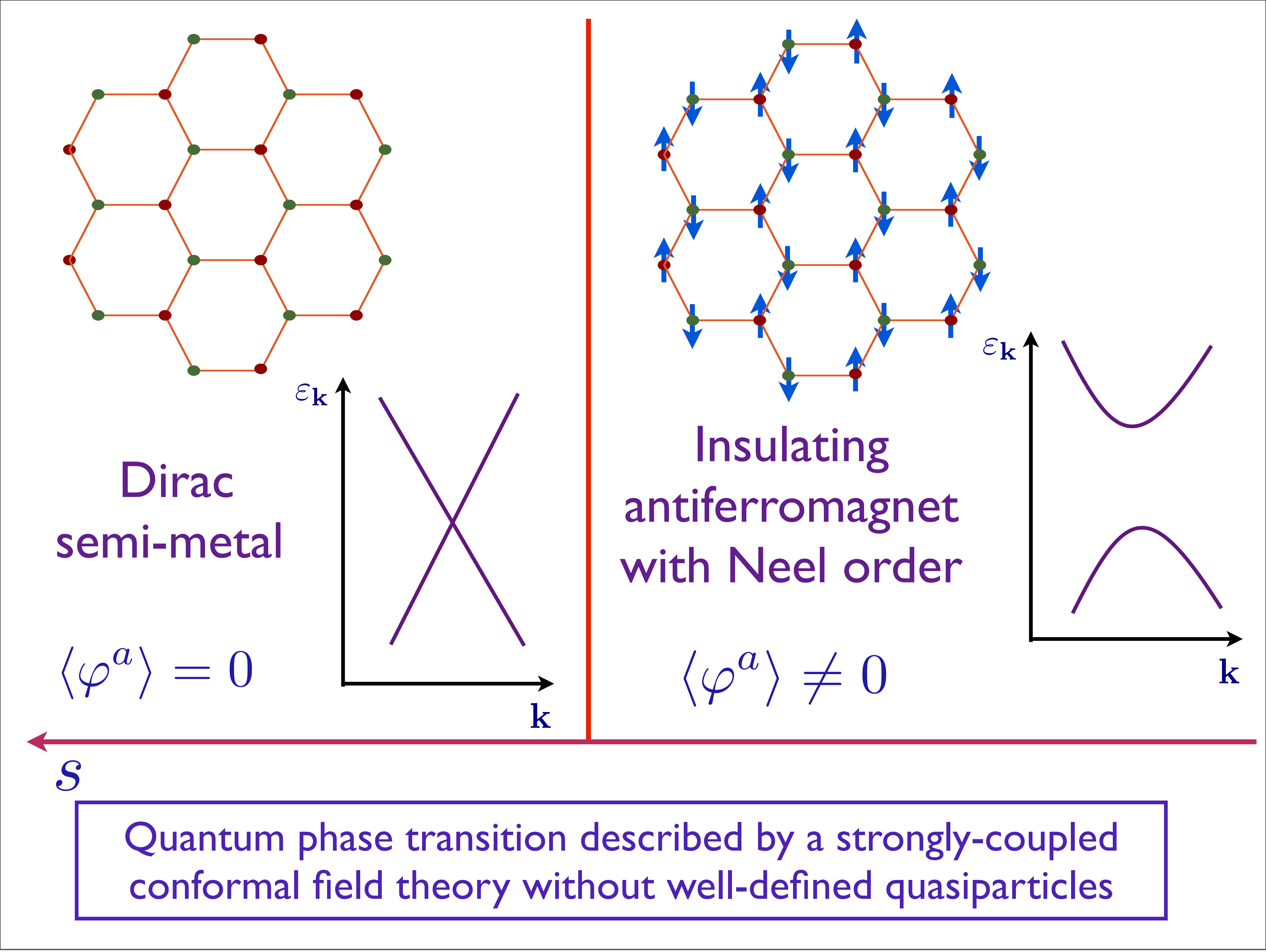
$\langle \vec{\varphi} \rangle = 0$

Metal with electron and hole pockets

Metal with "large" Fermi surface

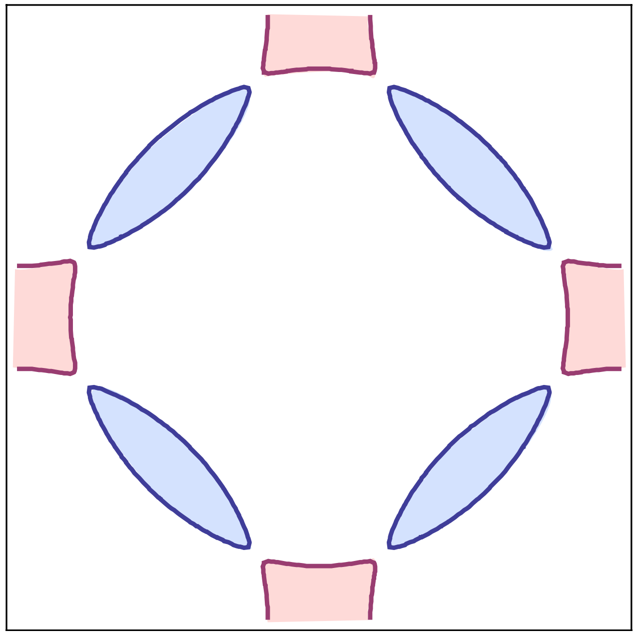
Increasing interaction

Unlike the honeycomb lattice, onset of antiferromagnetic order does not lead to an insulator, but to another metal with "pocket" Fermi surfaces.



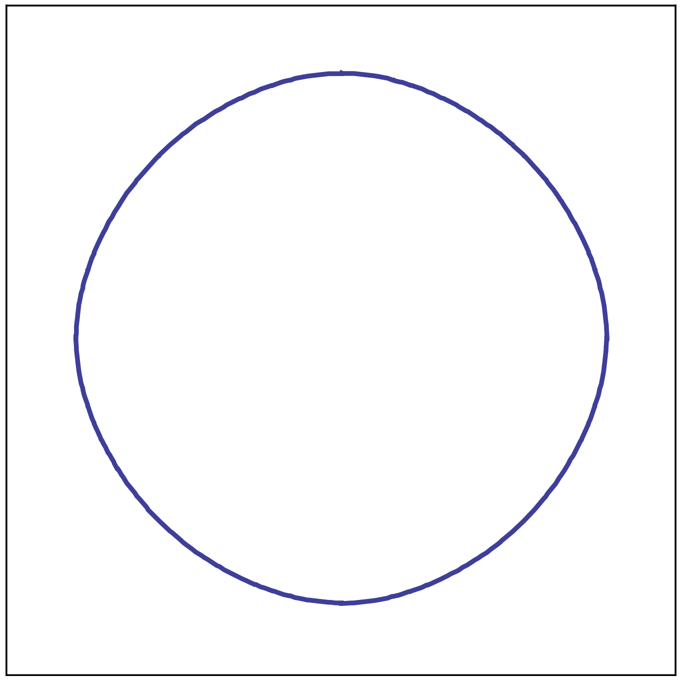
$\varepsilon_{\mathbf{k}}$
Dirac
semi-metal
$\langle \varphi^a \rangle = 0$
$\mathbf{k}$
$\varepsilon_{\mathbf{k}}$
Insulating
antiferromagnet
with Neel order
$\langle \varphi^a \rangle \neq 0$
$\mathbf{k}$
s
Quantum phase transition described by a strongly-coupled
conformal field theory without well-defined quasiparticles

Quantum phase transition with Fermi surface reconstruction



$\langle \vec{\varphi} \rangle \neq 0$

Metal with electron and hole pockets



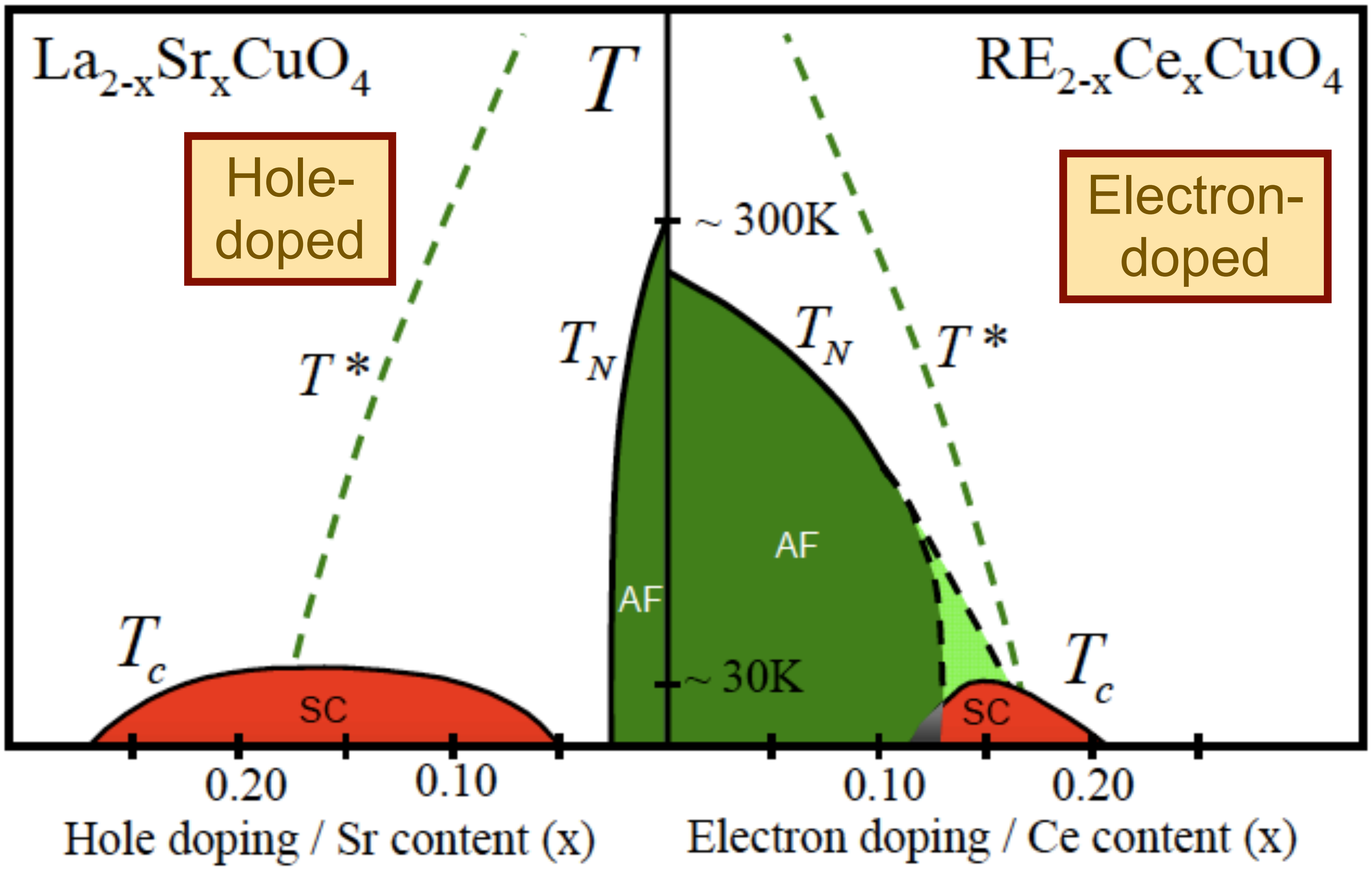
$\langle \vec{\varphi} \rangle = 0$

Metal with "large" Fermi surface

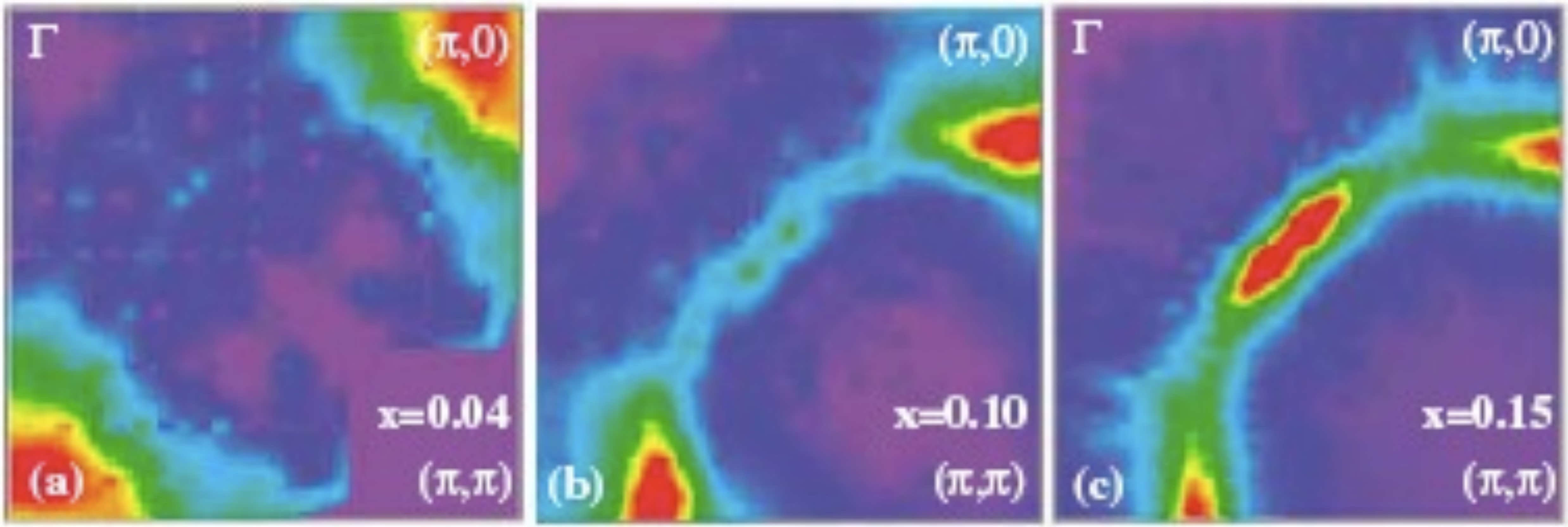
Increasing interaction

Unlike the honeycomb lattice, onset of antiferromagnetic order does not lead to an insulator, but to another metal with "pocket" Fermi surfaces.

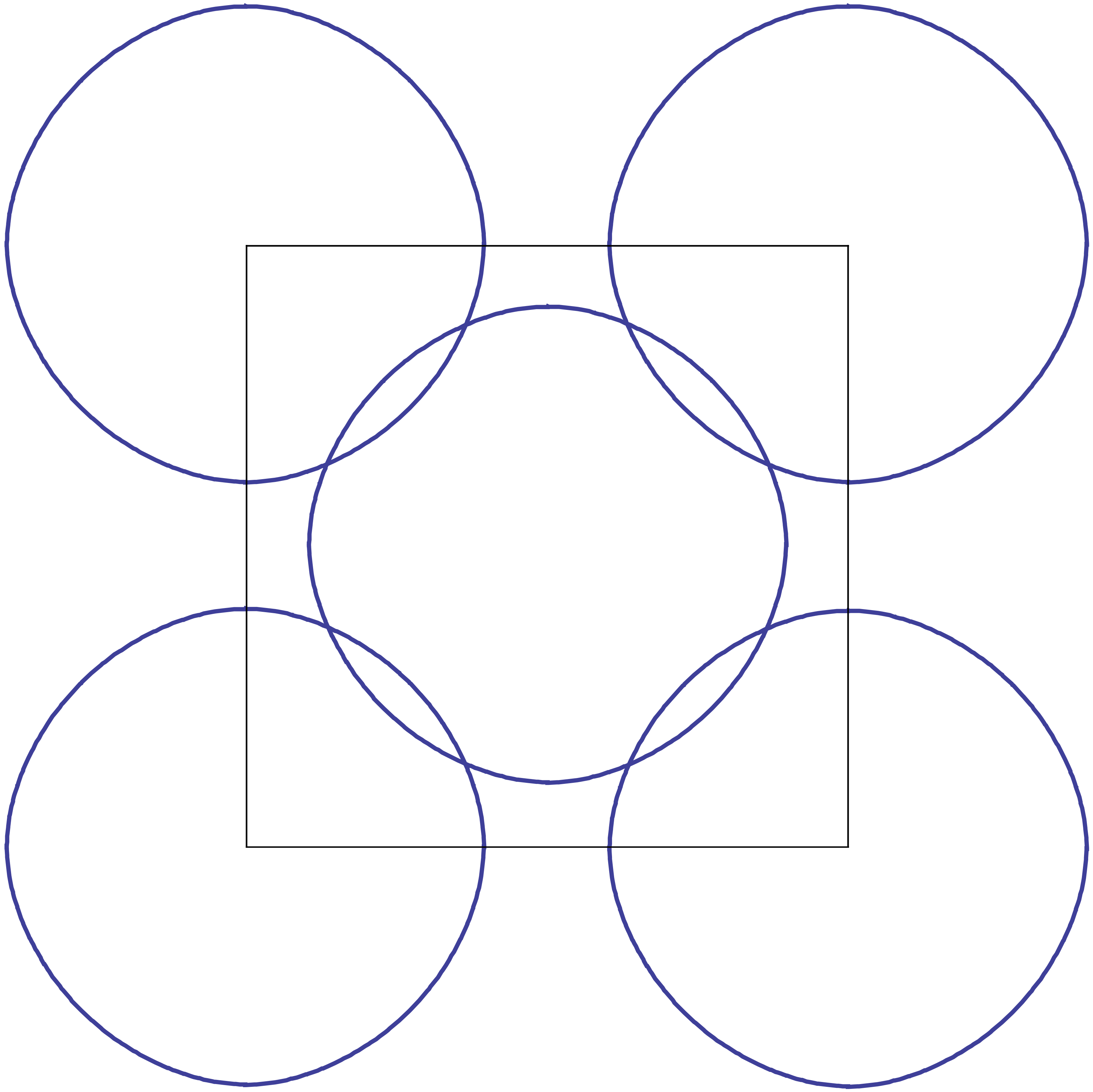
Hole and electron-doped cuprate superconductors



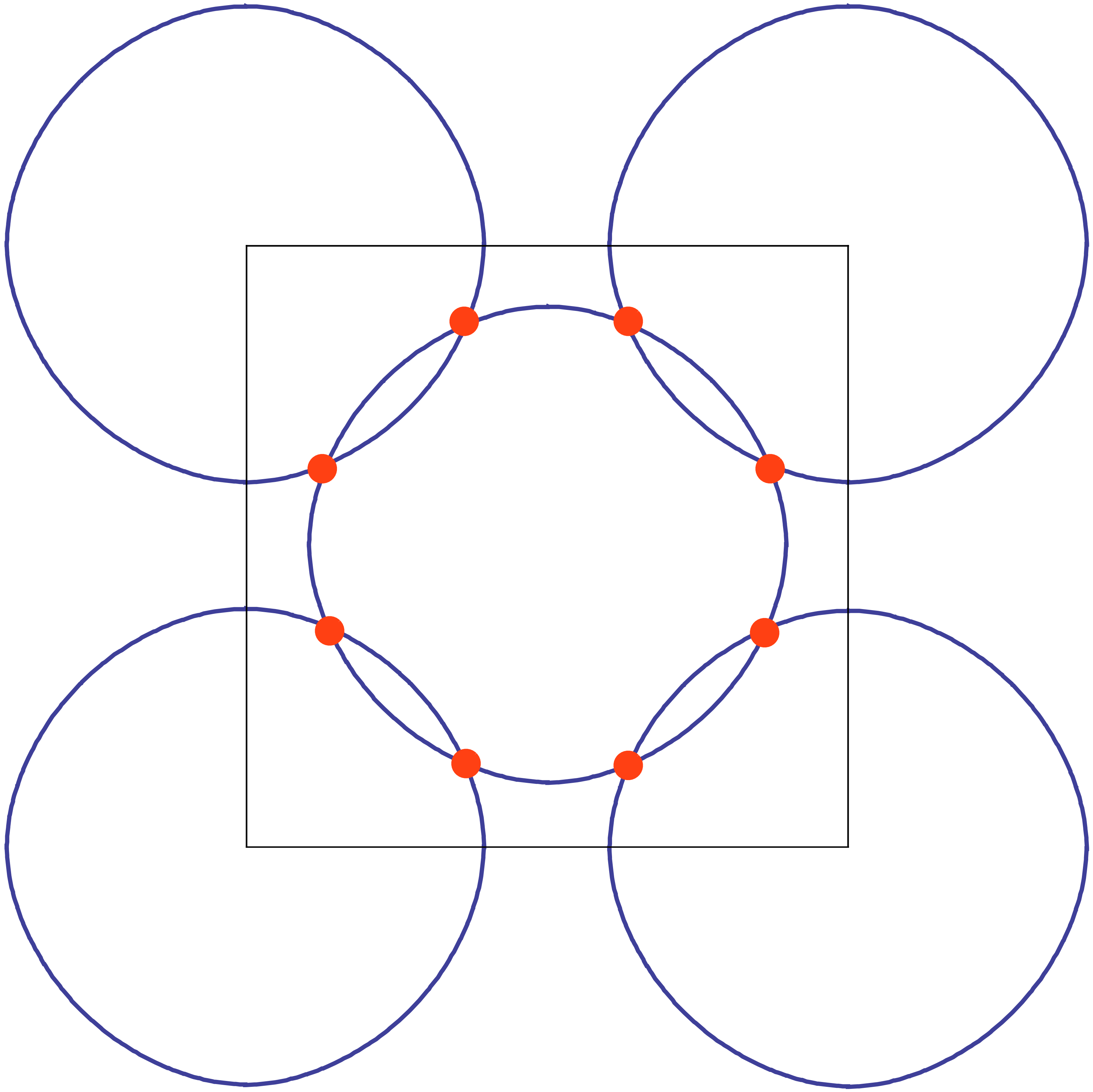
Photoemission in $Nd_{2-x}Ce_xCuO_4$



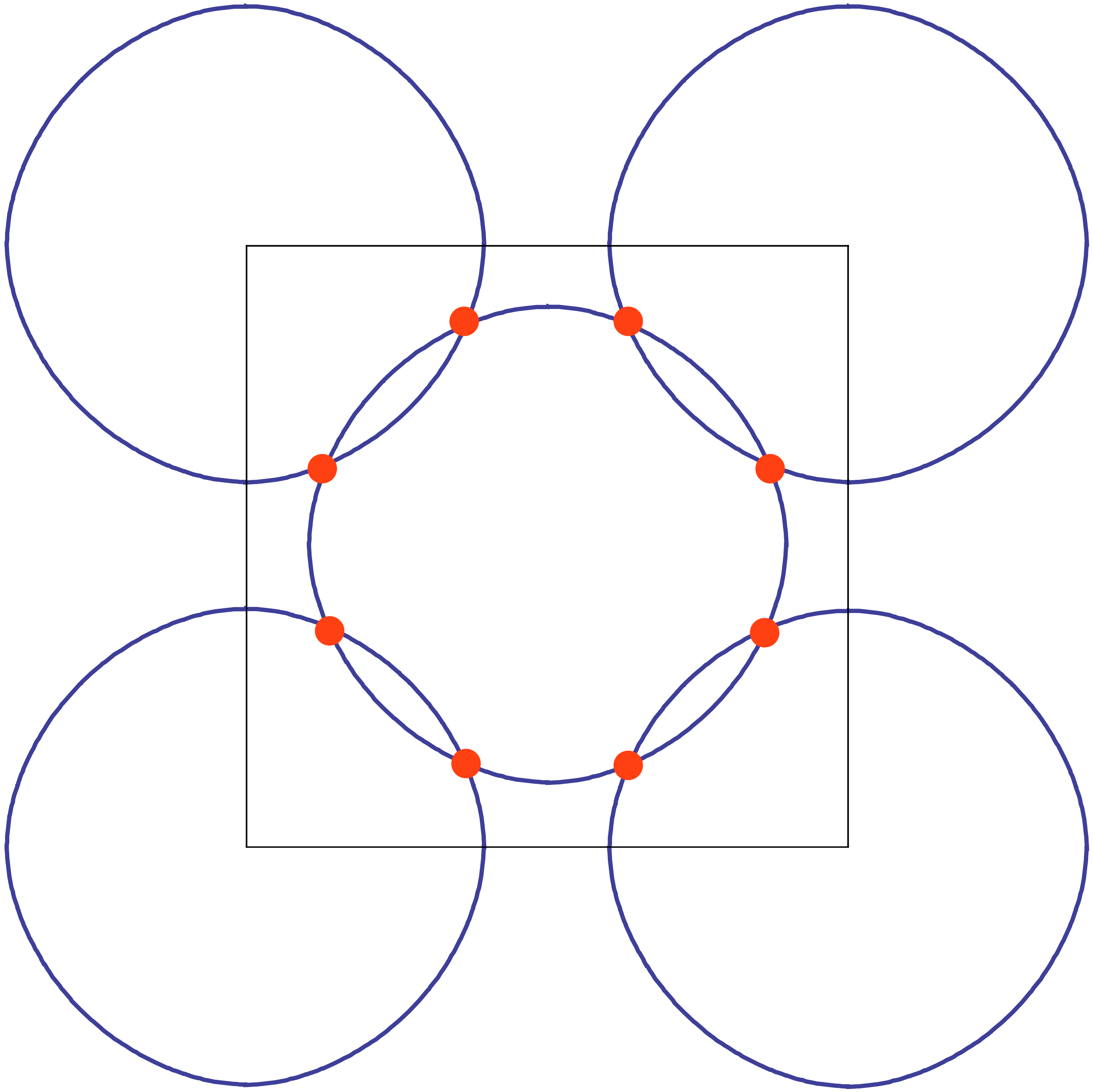
N. P. Armitage *et al.*, Phys. Rev. Lett. **88**, 257001 (2002).



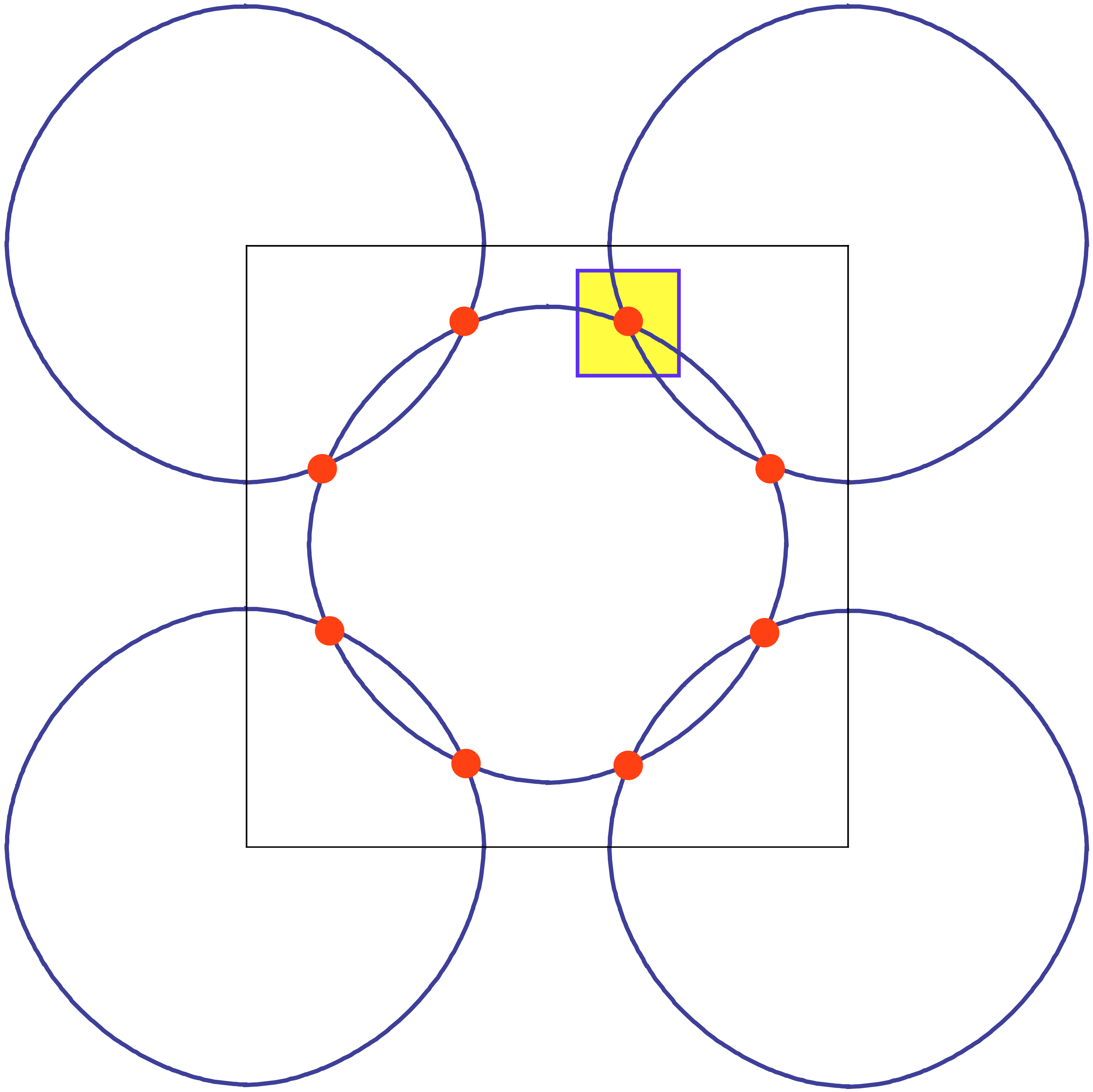
Fermi surfaces translated by $\mathbf{K} = (\pi, \pi)$.



“Hot” spots

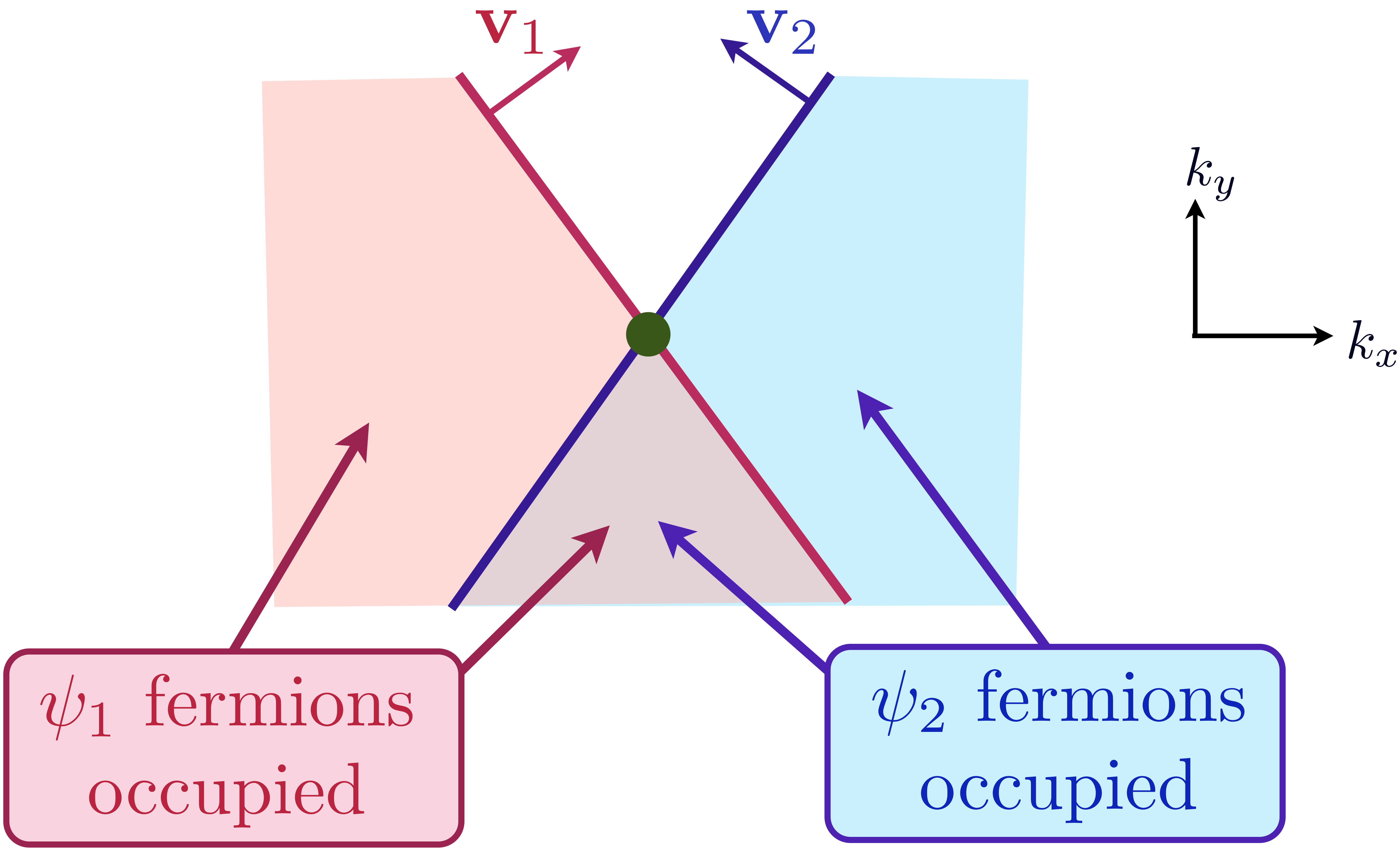


Low energy theory for critical point near hot spots



Low energy theory for critical point near hot spots

Theory has fermions $\psi_{1,2}$ (with Fermi velocities $\mathbf{v}_{1,2}$) and boson order parameter $\vec{\varphi}$, interacting with coupling λ



Low energy theory for critical point near hot spots

$$\mathcal{S} = \int d^2 r d\tau \left[\mathcal{L}_\psi + \mathcal{L}_\varphi + \mathcal{L}_{\psi\varphi} \right]$$

$$\mathcal{L}_\psi = \psi_{1a}^\dagger \left(\frac{\partial}{\partial \tau} - i \mathbf{v}_1 \cdot \boldsymbol{\nabla} \right) \psi_{1a}$$

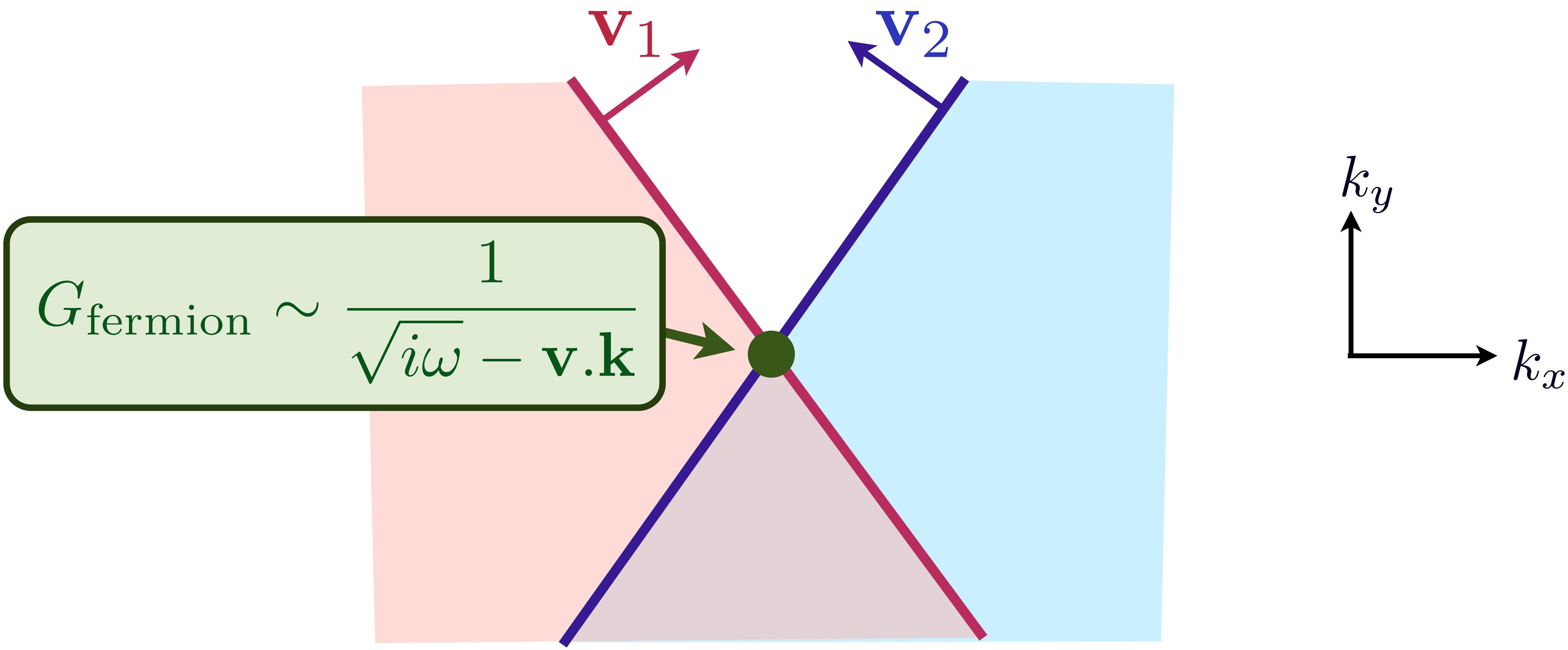
$$+ \psi_{2a}^\dagger \left(\frac{\partial}{\partial \tau} - i \mathbf{v}_2 \cdot \boldsymbol{\nabla} \right) \psi_{2a}$$

$$\mathcal{L}_\varphi = \frac{1}{2} (\boldsymbol{\nabla} \varphi_\alpha)^2 + \frac{r}{2} \varphi_\alpha^2 + \frac{u}{4} \left(\varphi_\alpha^2 \right)^2$$

$$\mathcal{L}_{\psi\varphi} = \lambda \varphi_\alpha \sigma_{ab}^\alpha \left(\psi_{1a}^\dagger \psi_{2b} + \psi_{2a}^\dagger \psi_{1b} \right).$$

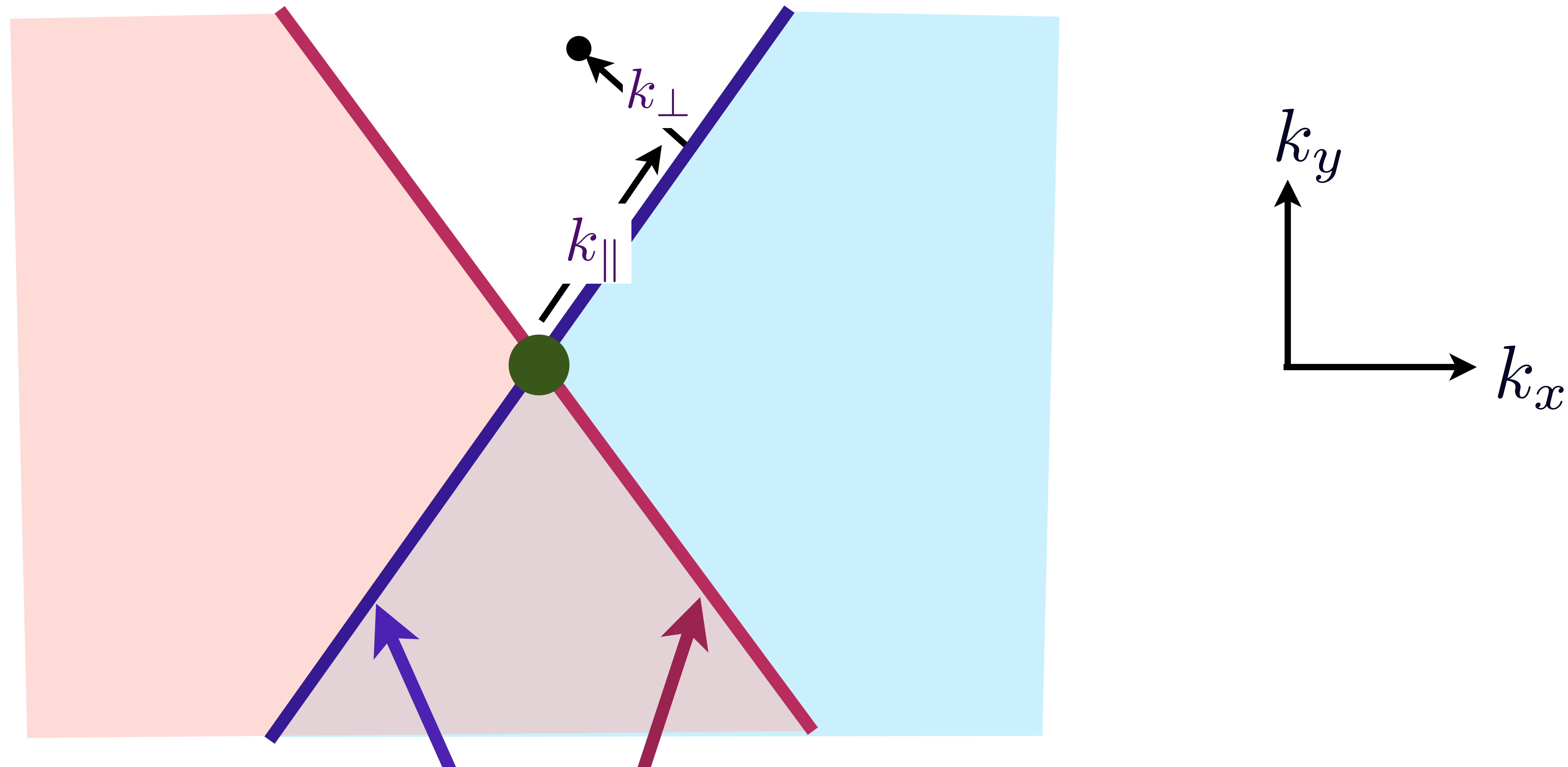
“Yukawa” coupling between fermions and antiferromagnetic order:
$\lambda^2 \sim U$, the Hubbard repulsion

Critical point theory is strongly coupled in $d = 2$



A. J. Millis, *Phys. Rev. B* **45**, 13047 (1992)
Ar. Abanov and A.V. Chubukov, *Phys. Rev. Lett.* **93**, 255702 (2004)

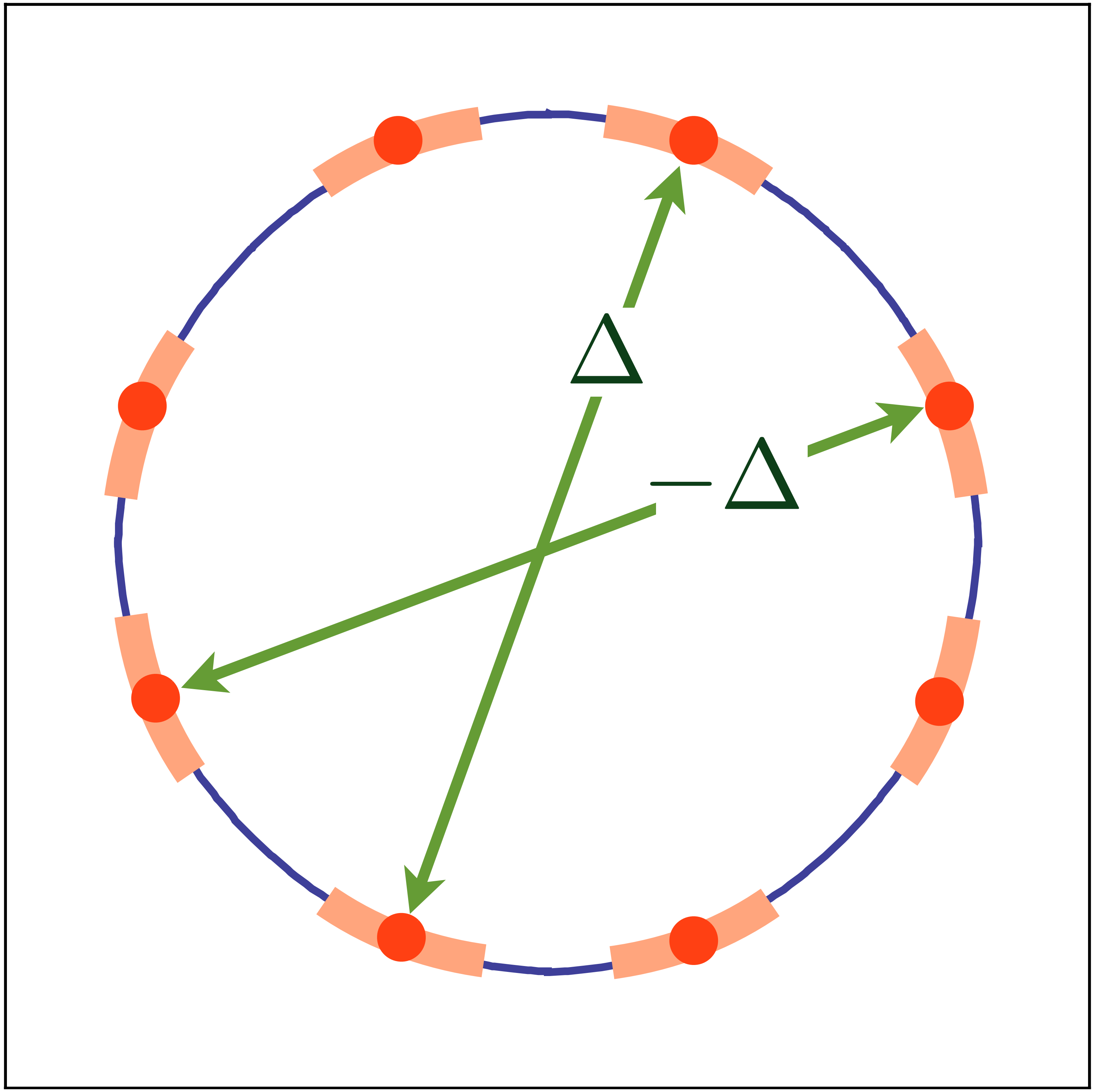
Critical point theory is strongly coupled in $d = 2$



$$G_{\text{fermion}} = \frac{Z(k_\parallel)}{\omega - v_F(k_\parallel) k_\perp}, \quad Z(k_\parallel) \sim v_F(k_\parallel) \sim k_\parallel$$

M. A. Metlitski and S. Sachdev, *Phys. Rev. B* **85**, 075127 (2010)

$$\left\langle c_{\mathbf{k}\alpha}^{\dagger} c_{-\mathbf{k}\beta}^{\dagger} \right\rangle = \varepsilon_{\alpha\beta} \Delta (\cos k_x - \cos k_y)$$

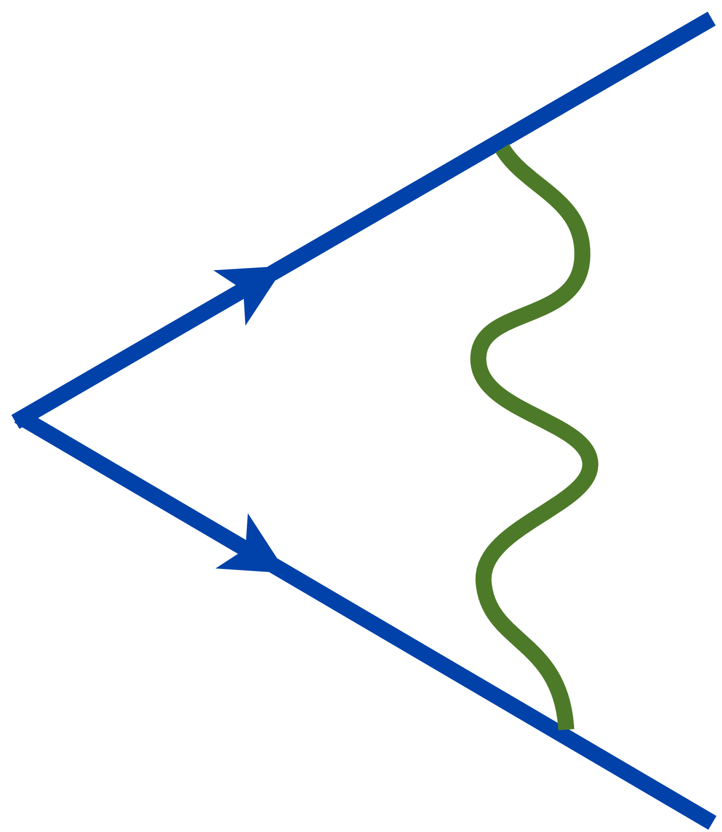


Unconventional pairing at *and near* hot spots

Enhancement of pairing susceptibility by interactions

Antiferromagnetic critical point

$$1 + \frac{\sin\theta}{2\pi}\log^2\left(\frac{E_F}{\omega}\right)$$



M. A. Metlitski and S. Sachdev, *Phys. Rev. B* **85**, 075127 (2010)

Enhancement of pairing susceptibility by interactions

Antiferromagnetic critical point

$$1 + \frac{\sin\theta}{2\pi} \log^2 \left(\frac{E_F}{\omega} \right)$$

M. A. Metlitski and S. Sachdev, *Phys. Rev. B* **85**, 075127 (2010)

Enhancement of pairing susceptibility by interactions

Antiferromagnetic critical point

$$1 + \frac{\sin\theta}{2\pi} \log^{\boxed{2}} \left(\frac{E_F}{\omega} \right)$$

M. A. Metlitski and S. Sachdev, *Phys. Rev. B* **85**, 075127 (2010)

Enhancement of pairing susceptibility by interactions

Antiferromagnetic critical point

$$1 + \frac{\sin\theta}{2\pi}\log^2\left(\frac{E_F}{\omega}\right)$$

Fermi energy

M.A. Metlitski and S. Sachdev, *Phys. Rev. B* **85**, 075127 (2010)

Enhancement of pairing susceptibility by interactions

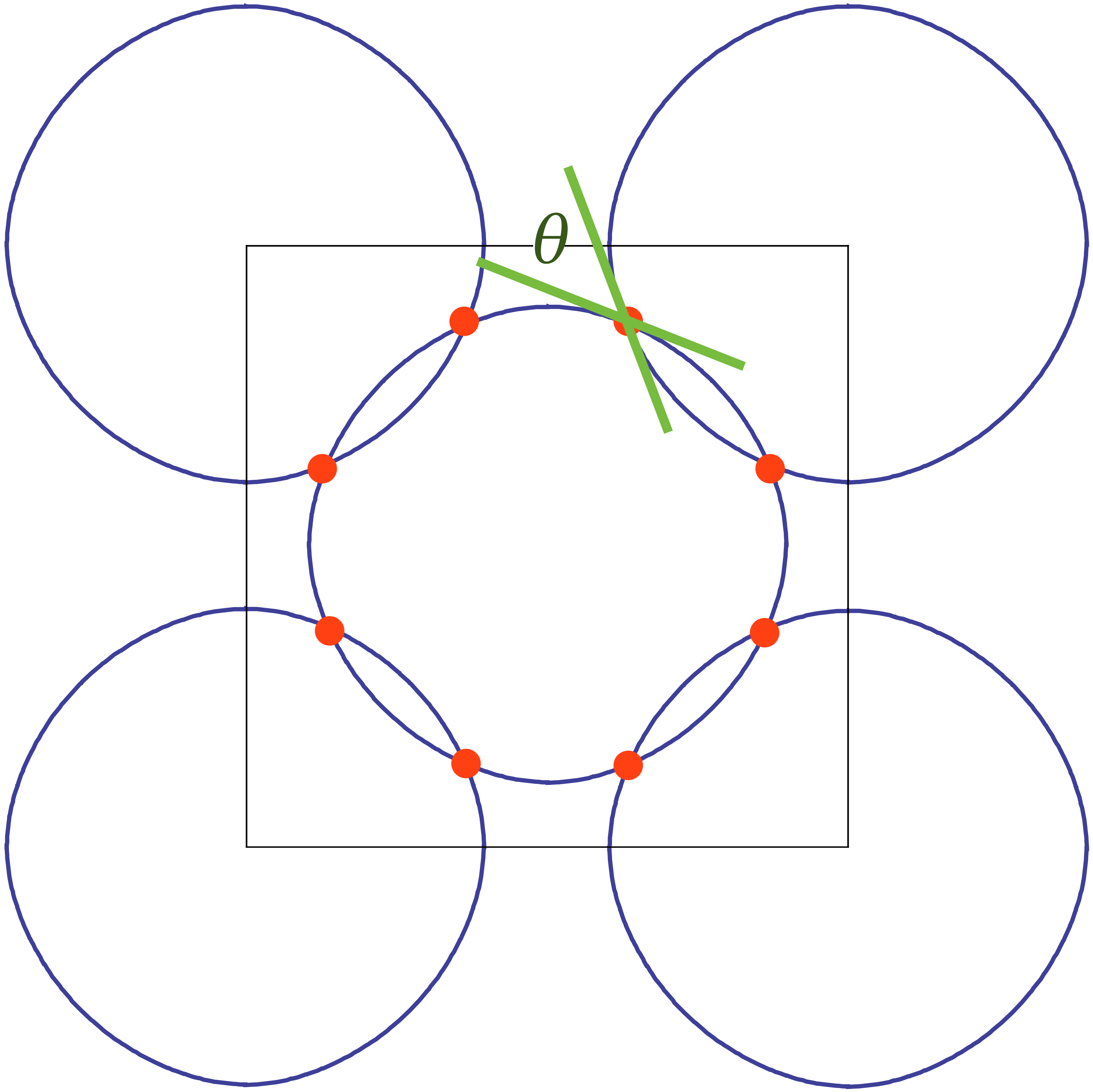
Antiferromagnetic critical point

$$1 + \frac{\sin\theta}{2\pi} \log^2 \left(\frac{E_F}{\omega} \right)$$

Fermi energy

θ is the angle between Fermi lines.
Independent of interaction strength U in 2 dimensions.

(see also Ar. Abanov, A.V. Chubukov, and A. M. Finkel'stein, *Europhys. Lett.* **54**, 488 (2001))
M.A. Metlitski and S. Sachdev, *Phys. Rev. B* **85**, 075127 (2010)


θ

Enhancement of pairing susceptibility by interactions

Antiferromagnetic critical point

$$1 + \frac{\sin\theta}{2\pi} \log^2\left(\frac{E_F}{\omega}\right)$$

- $\log^2$ singularity arises from Fermi lines; singularity *at* hot spots is weaker.
- Interference between BCS and quantum-critical logs.
- Momentum dependence of self-energy is crucial.
- Not suppressed by $1/N$ factor in $1/N$ expansion.

M.A. Metlitski and S. Sachdev, *Phys. Rev. B* **85**, 075127 (2010)