

1. Review of Fermi liquid theory

Topological argument for the Luttinger theorem

2. Fractionalized Fermi liquid

A Fermi liquid co-existing with topological order for the pseudogap metal

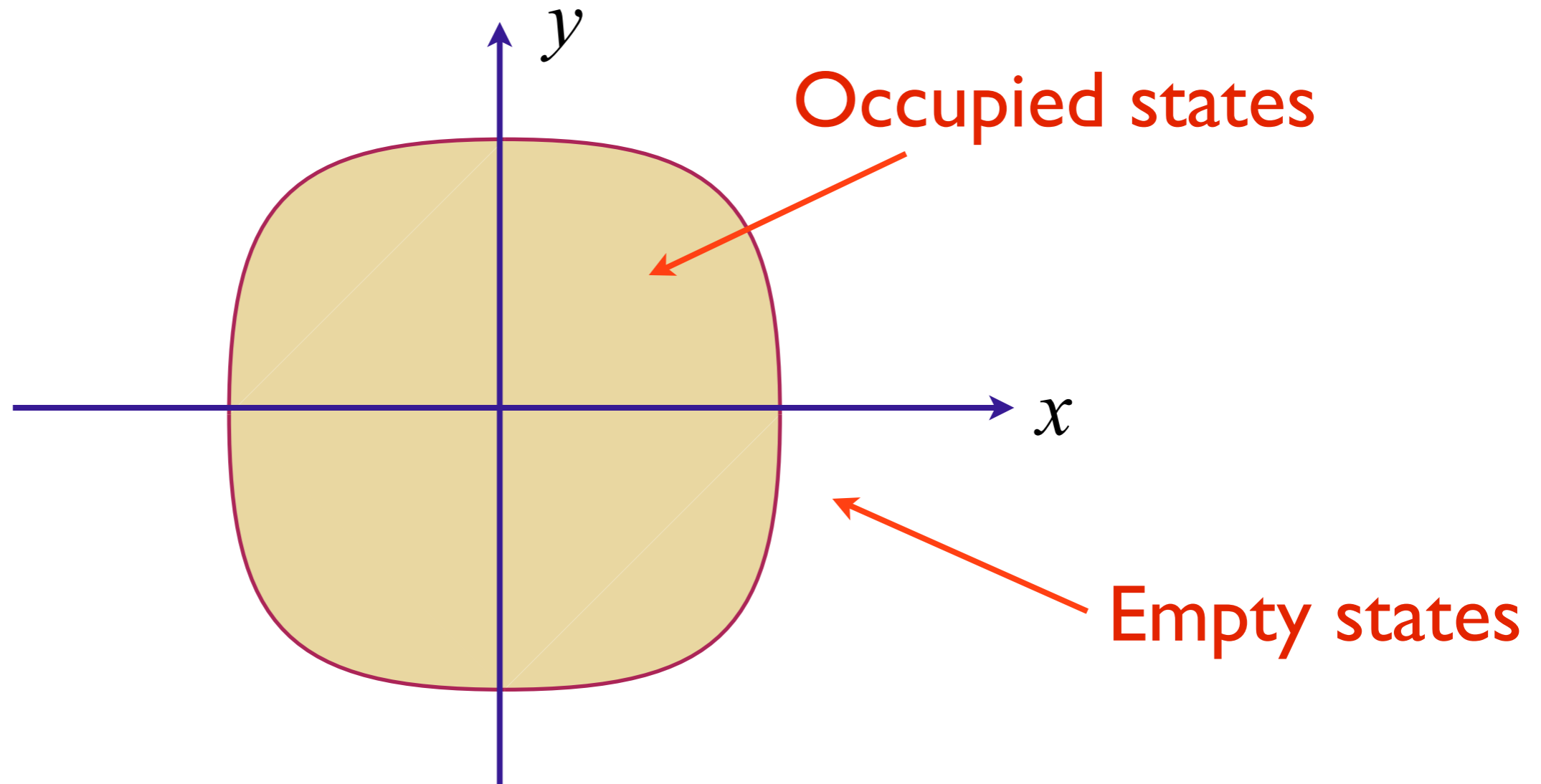
3. Quantum matter without quasiparticles

(A) A mean-field model of a non-Fermi liquid, and charged black holes

(B) Field theory of a non-Fermi liquid (Ising-nematic quantum critical point)

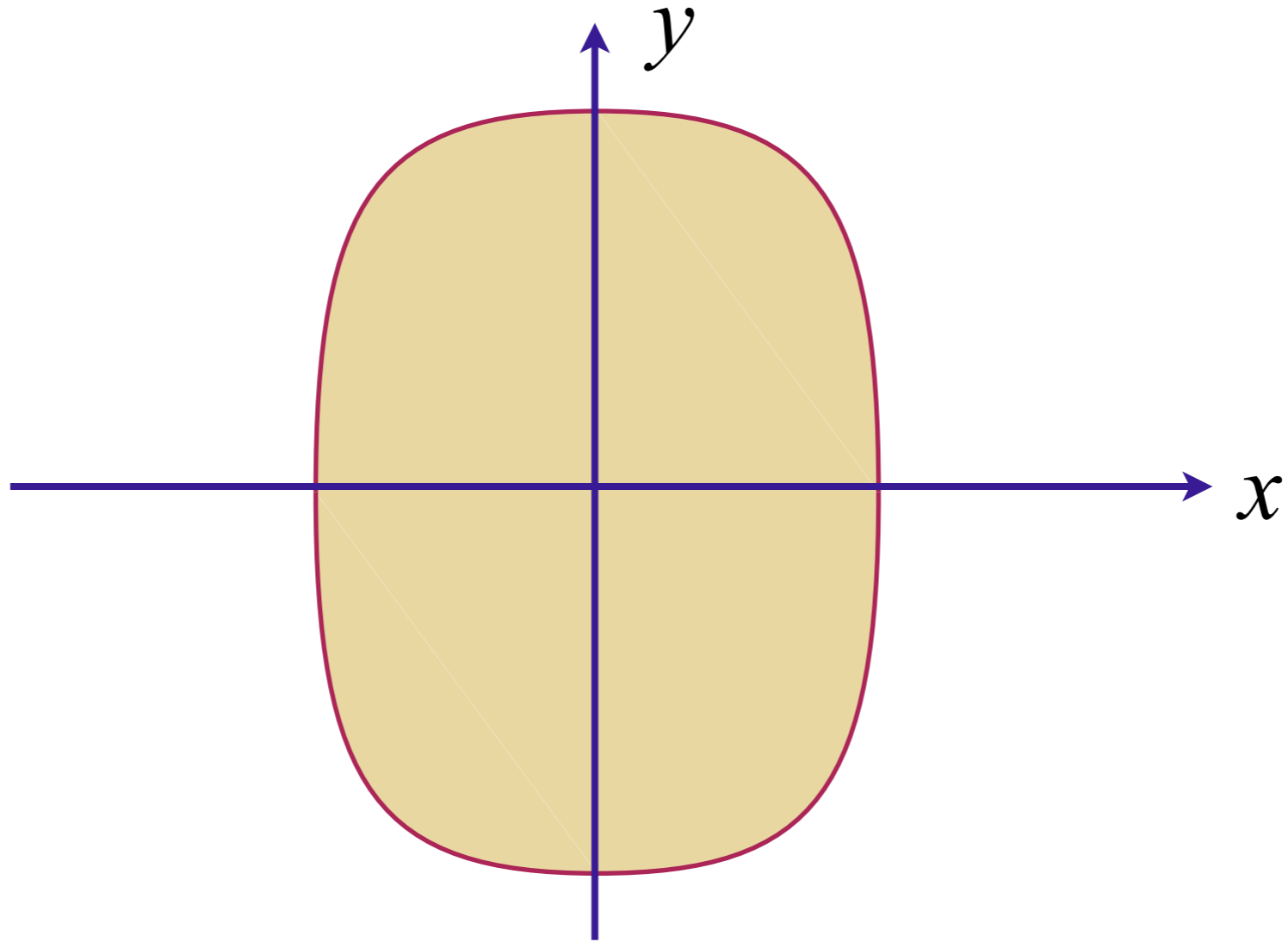
(C) Theory of transport in strange metals: application to the (less) strange metal in graphene

Quantum criticality of Ising-nematic ordering in a metal



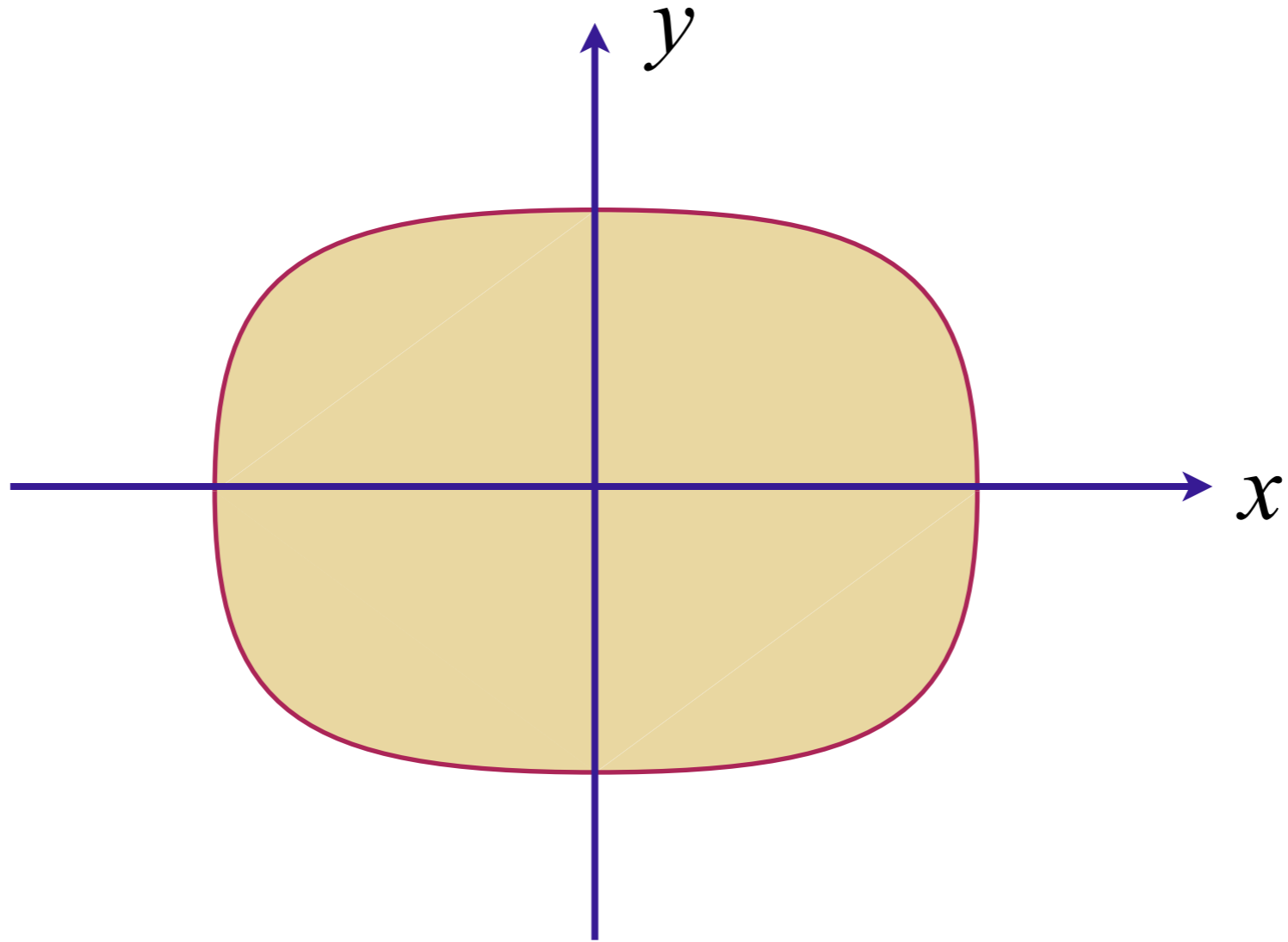
A metal with a Fermi surface
with full square lattice symmetry

Quantum criticality of Ising-nematic ordering in a metal



Spontaneous elongation along y direction:

Quantum criticality of Ising-nematic ordering in a metal



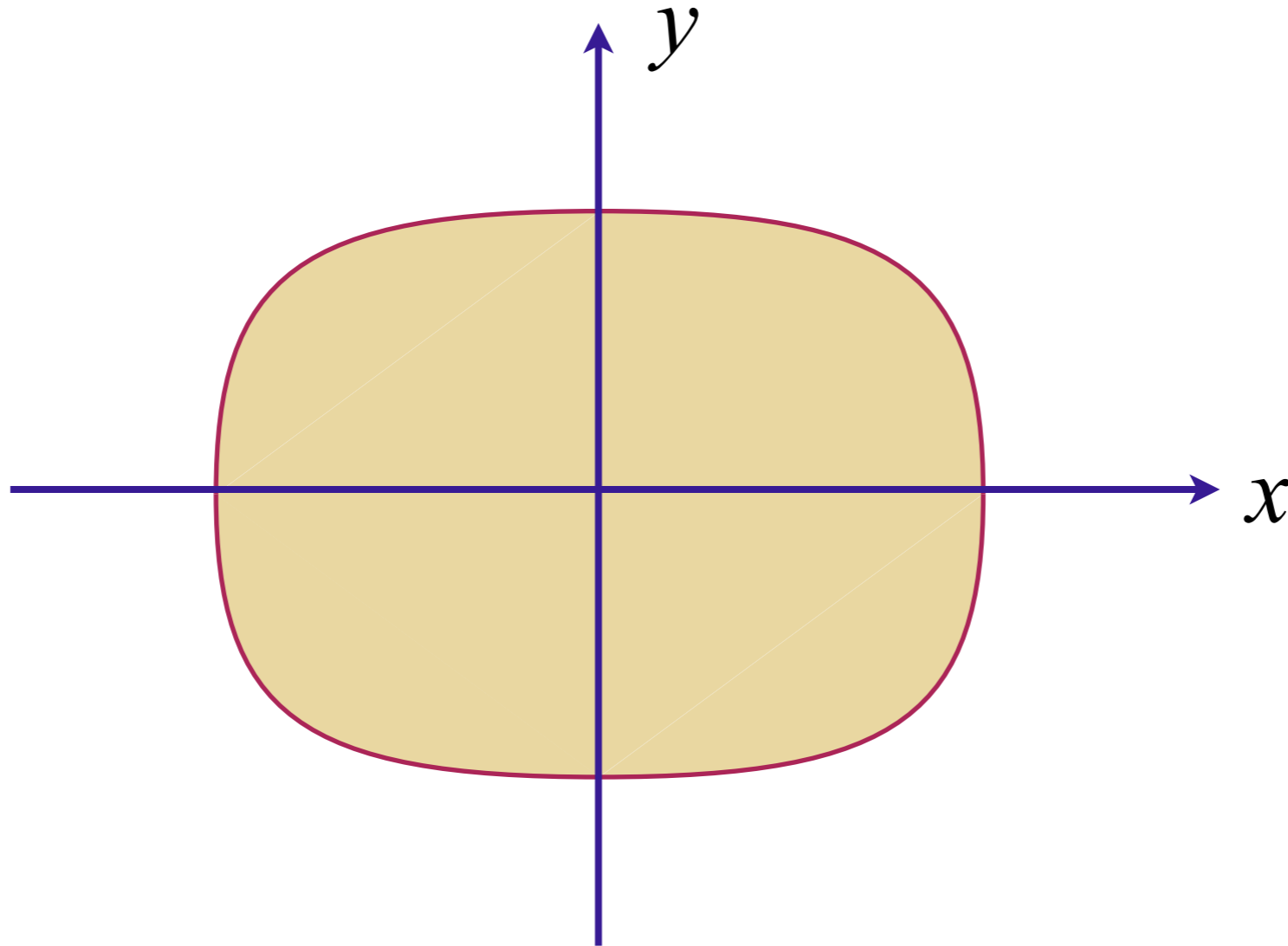
Spontaneous elongation along x direction:

Ising-nematic order parameter

$$\phi \sim \int d^2 k (\cos k_x - \cos k_y) c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma}$$

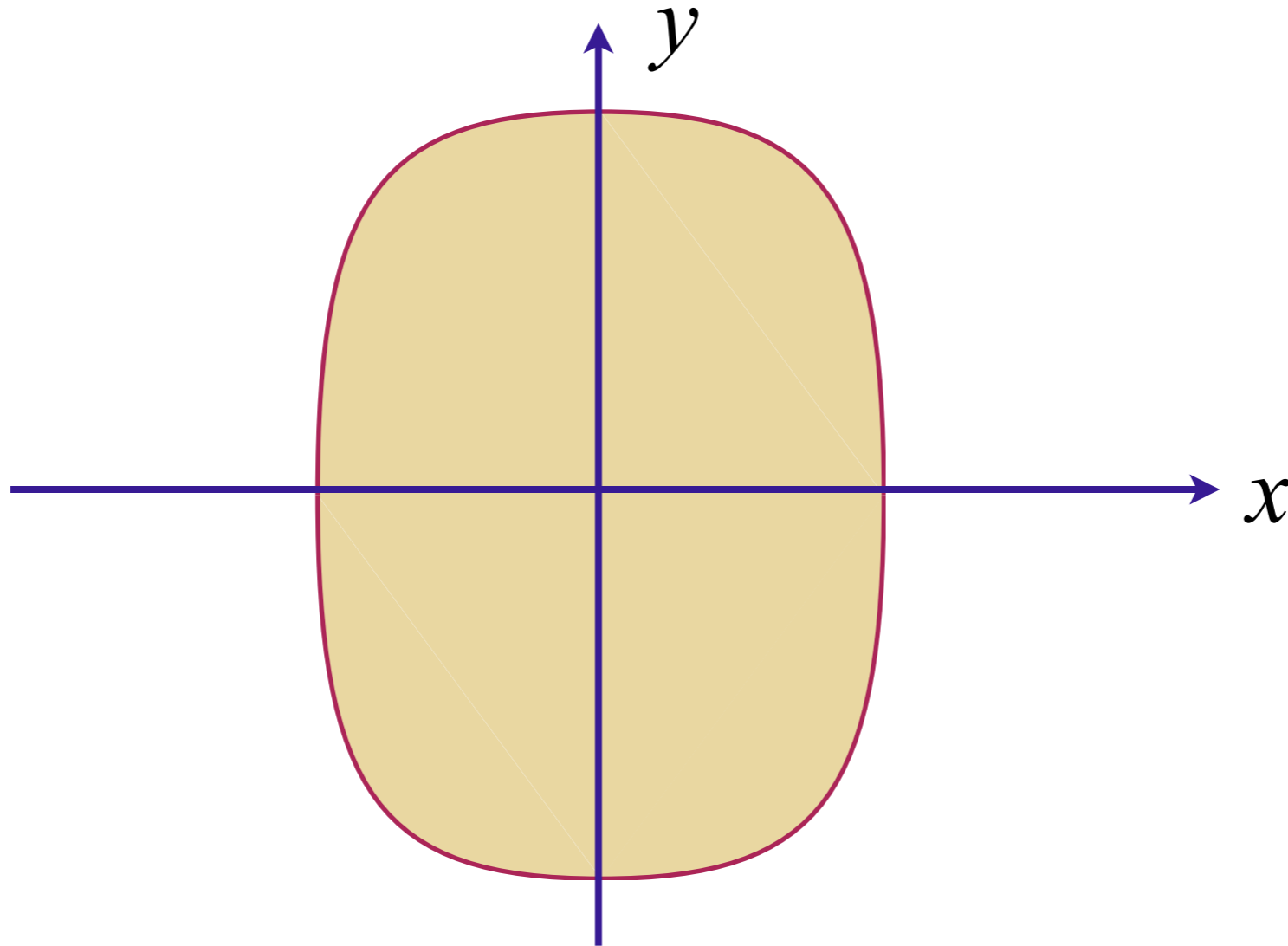
Measures spontaneous breaking of square lattice point-group symmetry of underlying Hamiltonian

Quantum criticality of Ising-nematic ordering in a metal



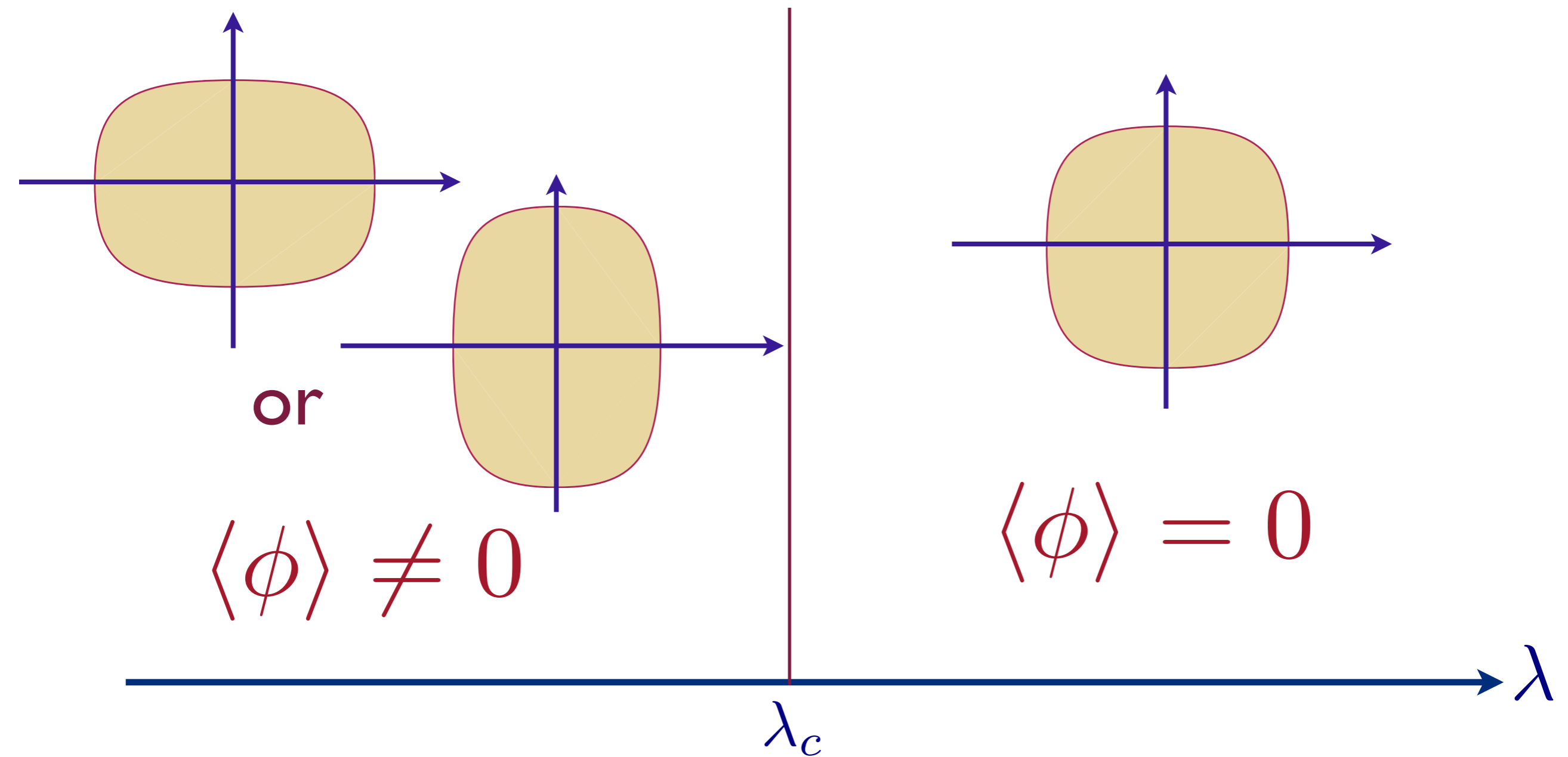
Spontaneous elongation along x direction:
Ising order parameter $\phi > 0$.

Quantum criticality of Ising-nematic ordering in a metal



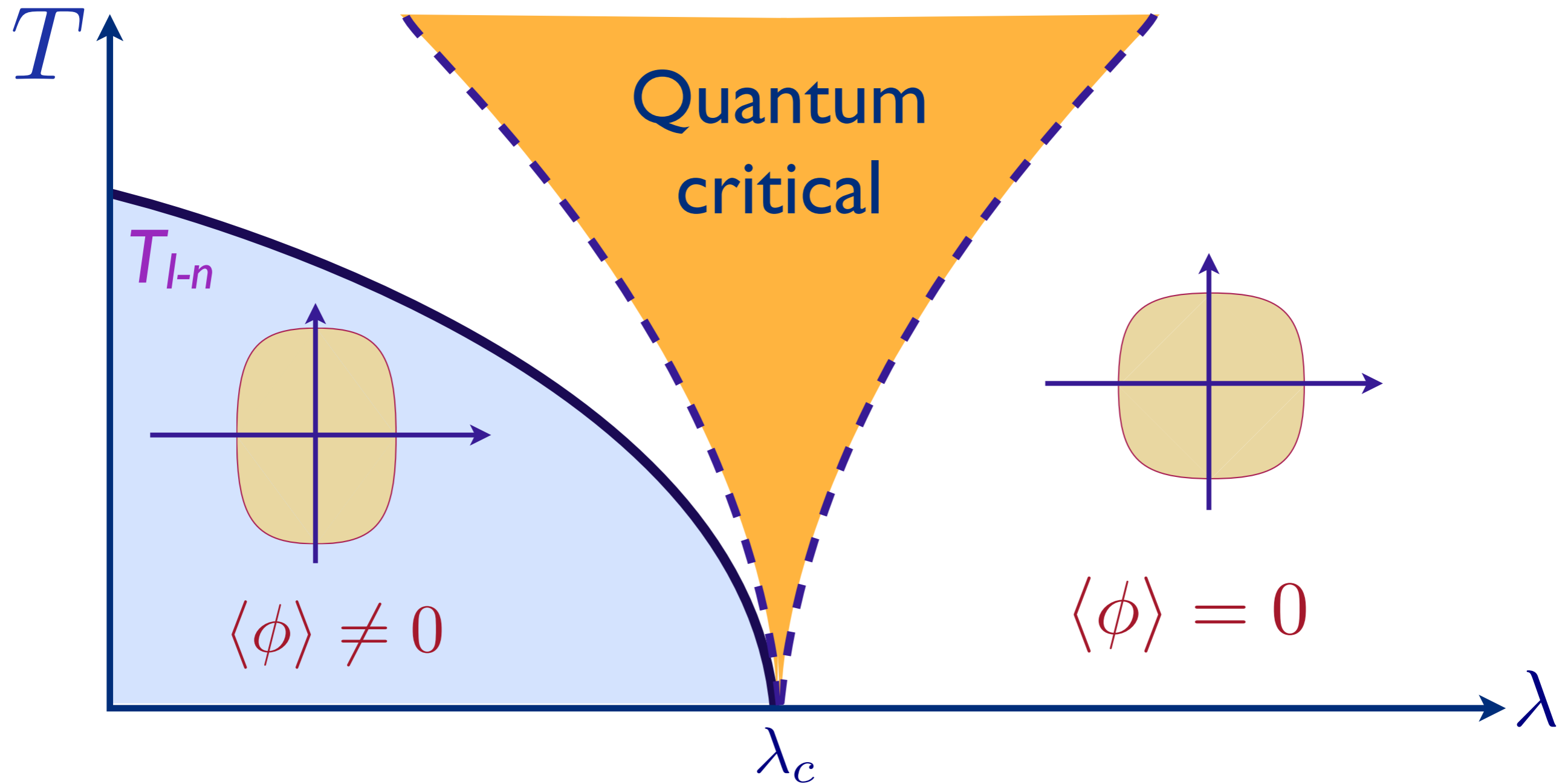
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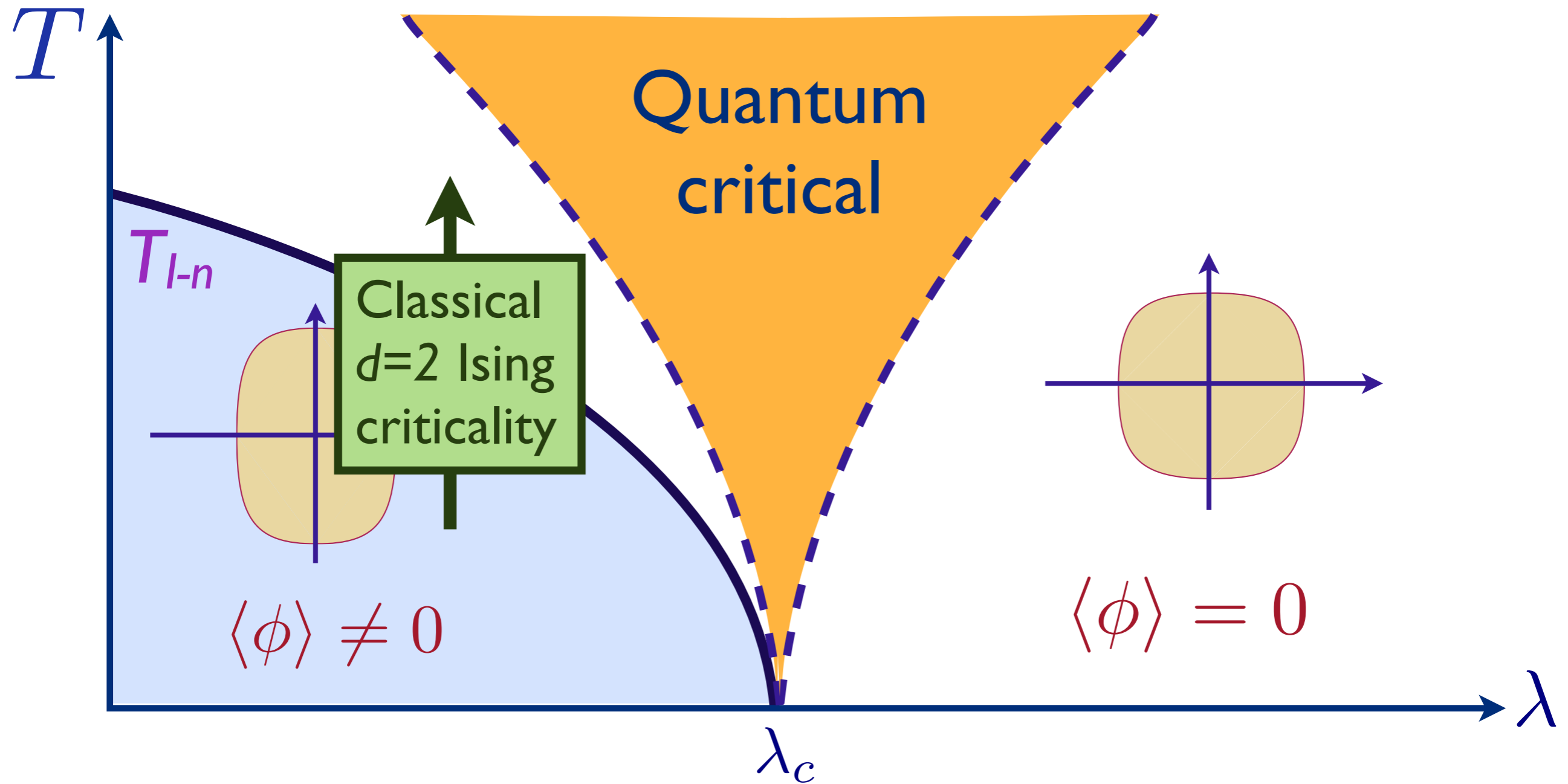
Pomeranchuk instability as a function of coupling λ

Quantum criticality of Ising-nematic ordering in a metal



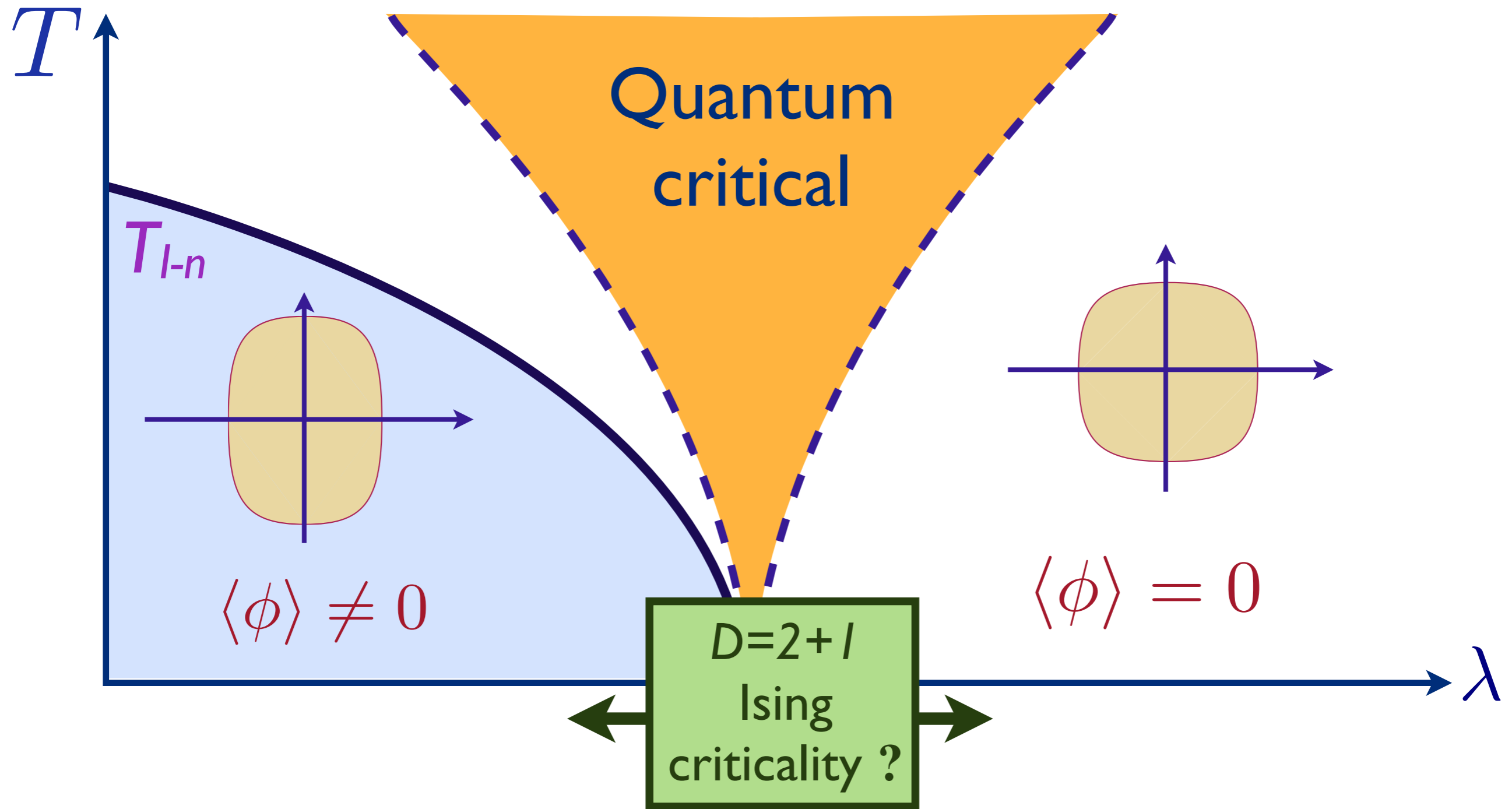
Phase diagram as a function of T and λ

Quantum criticality of Ising-nematic ordering in a metal



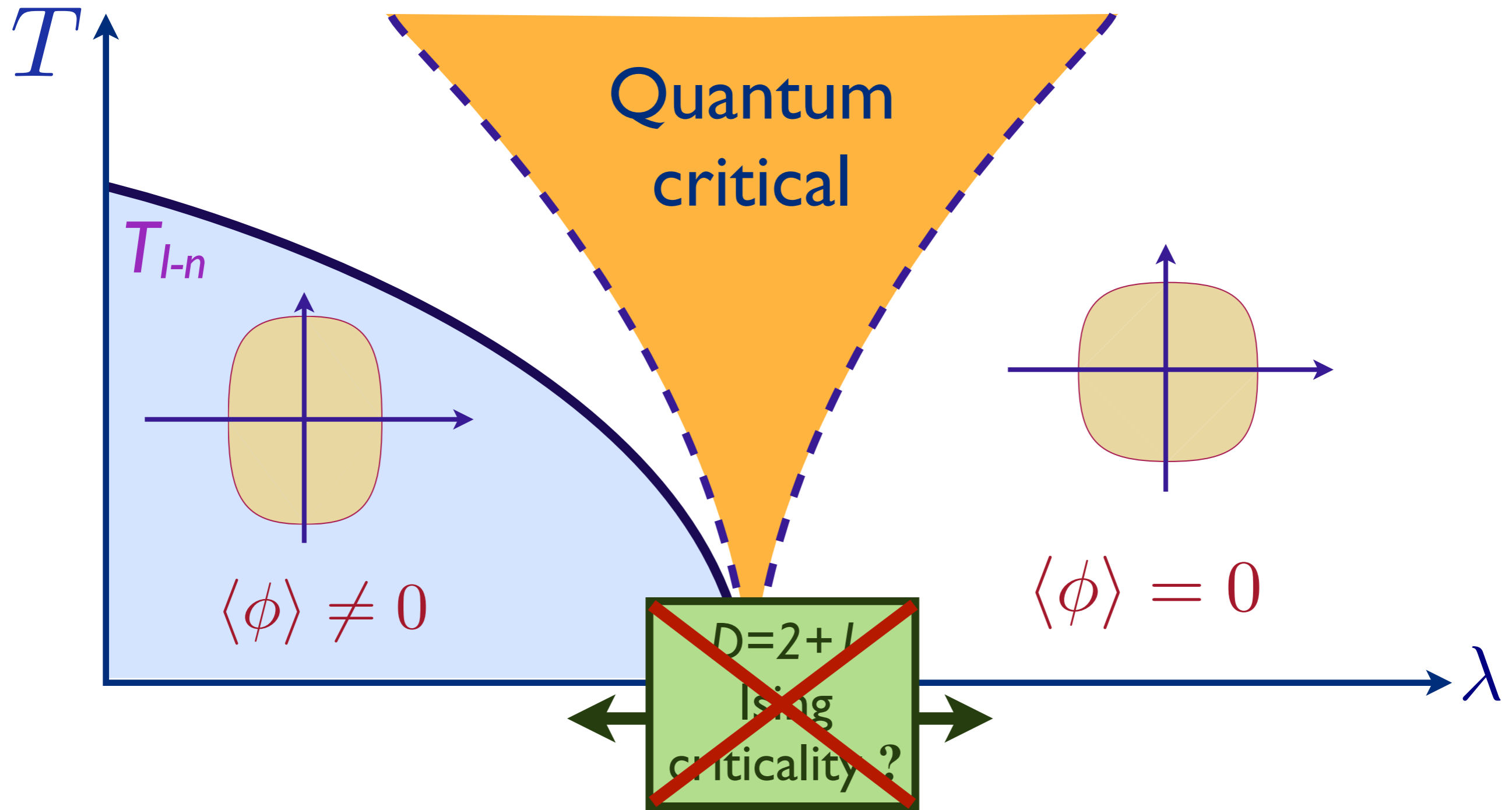
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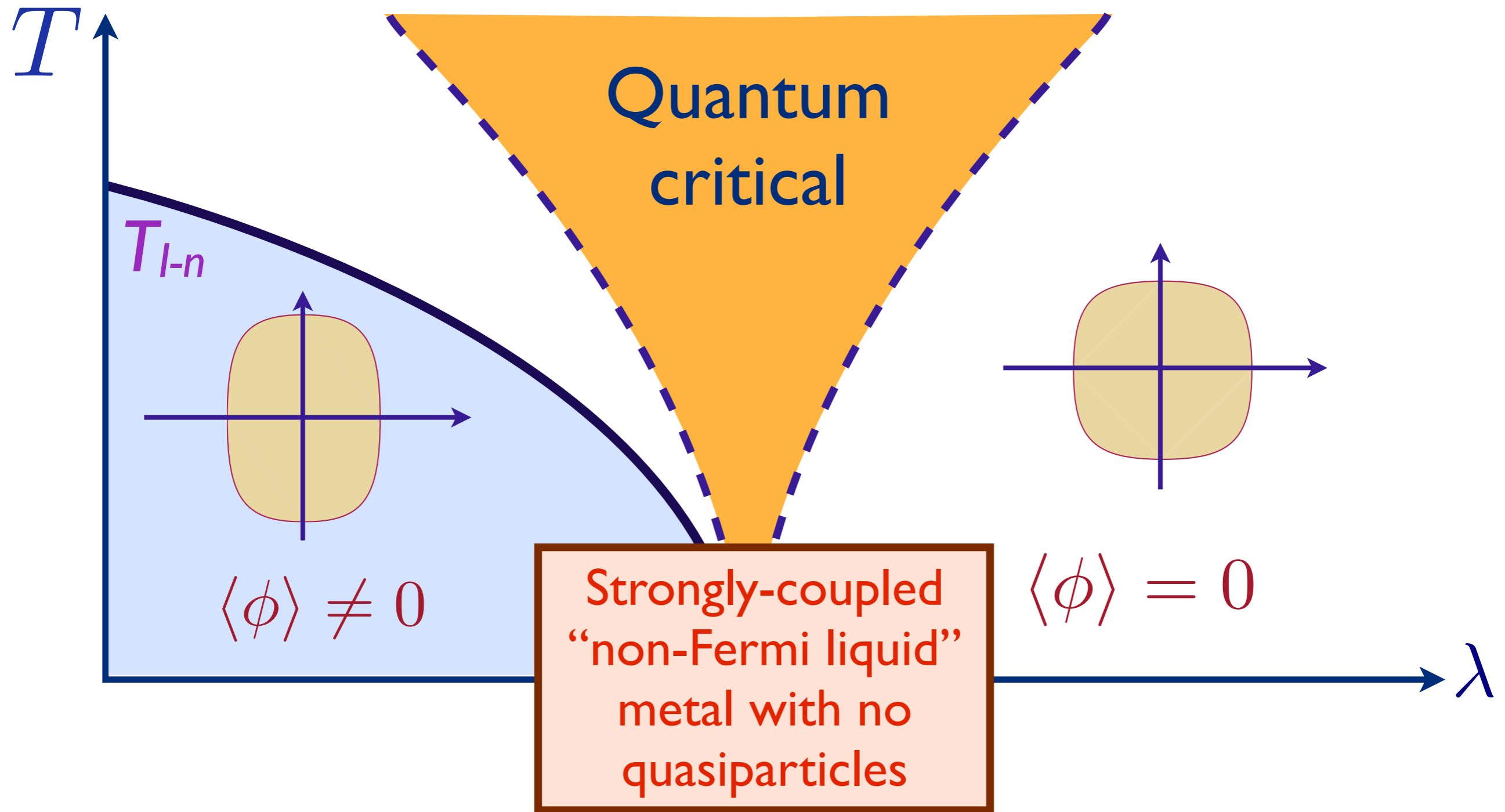
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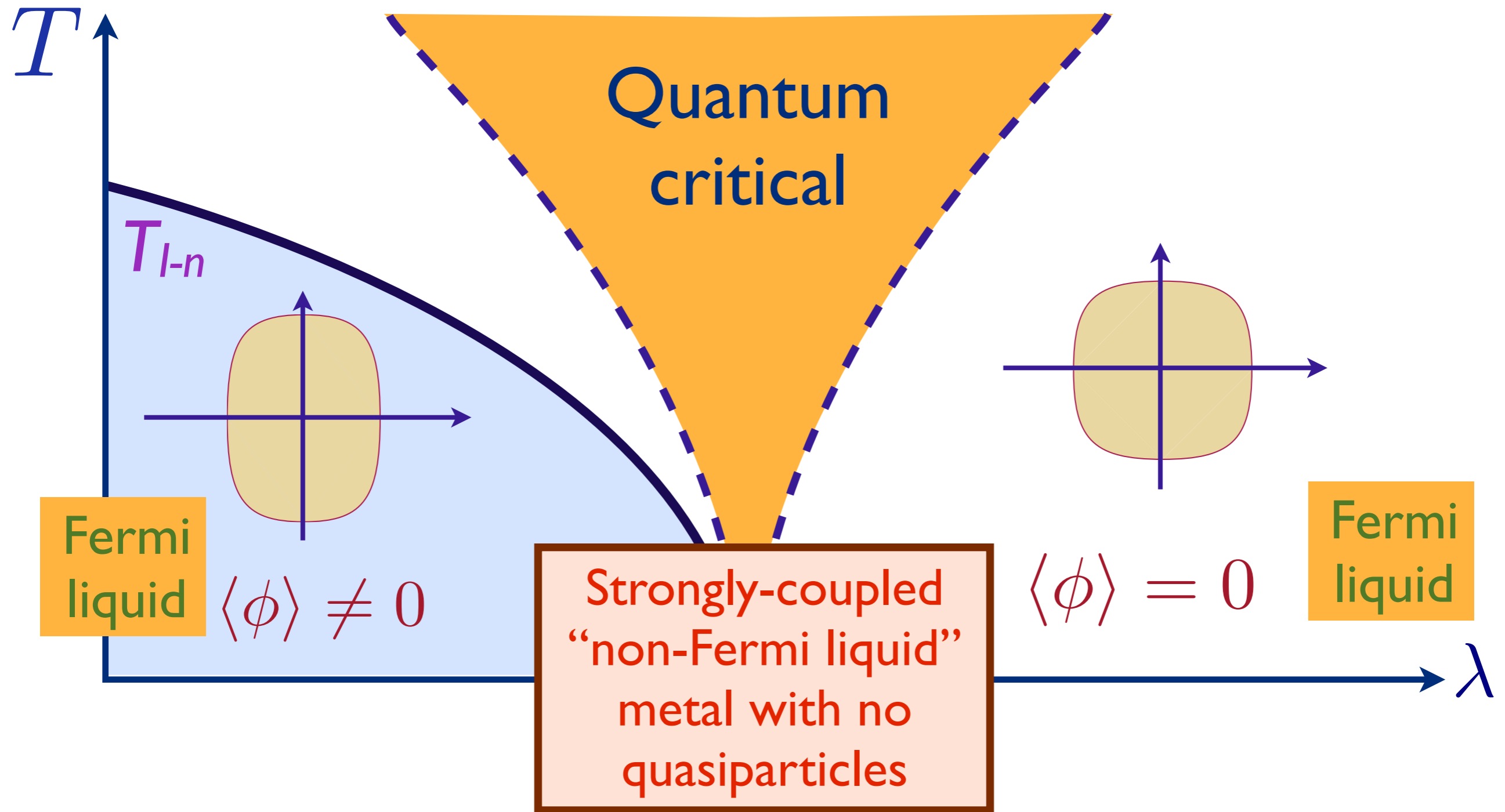
Phase diagram as a function of T and λ

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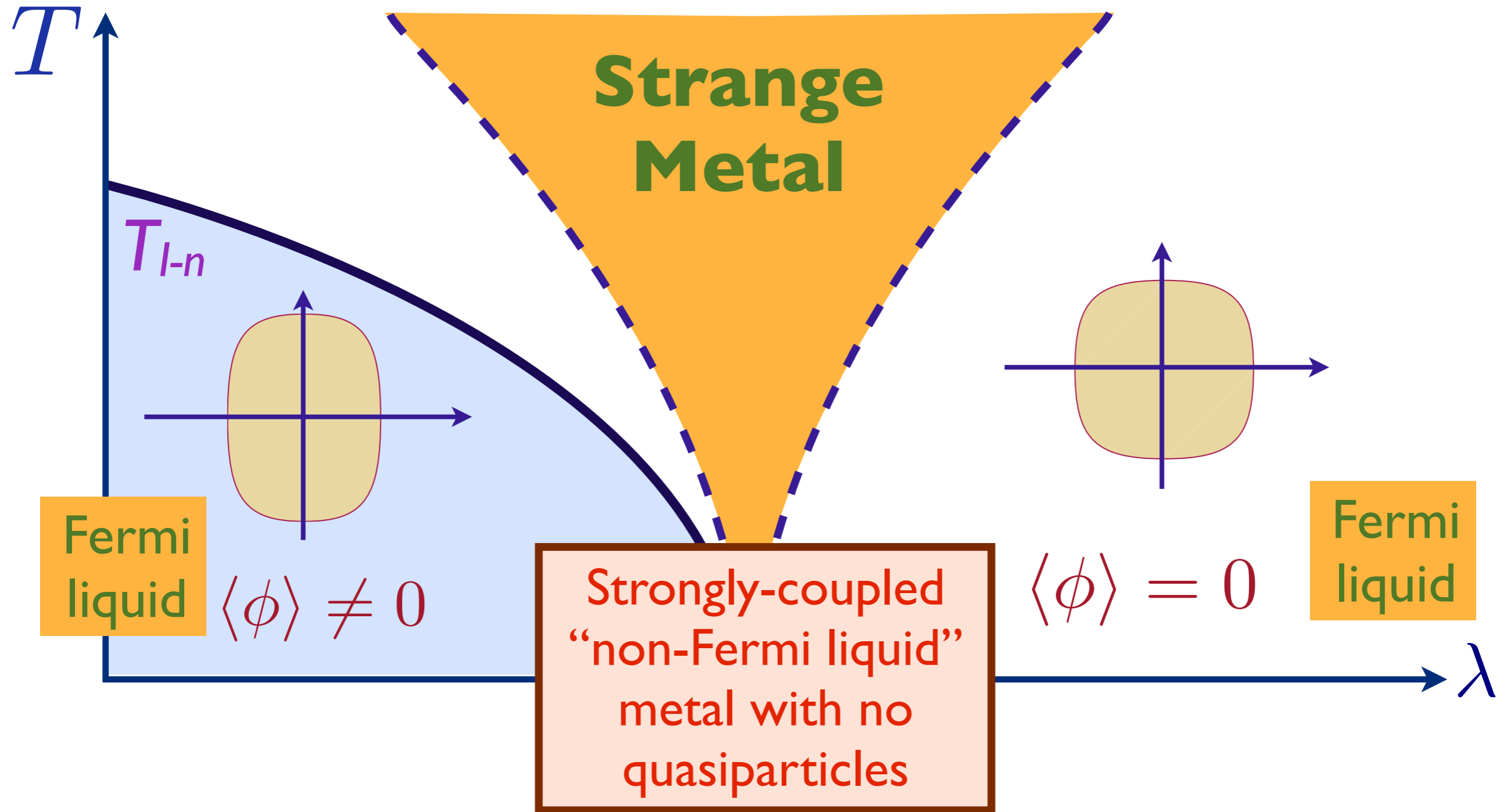
Phase diagram as a function of T and λ

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Phase diagram as a function of T and λ

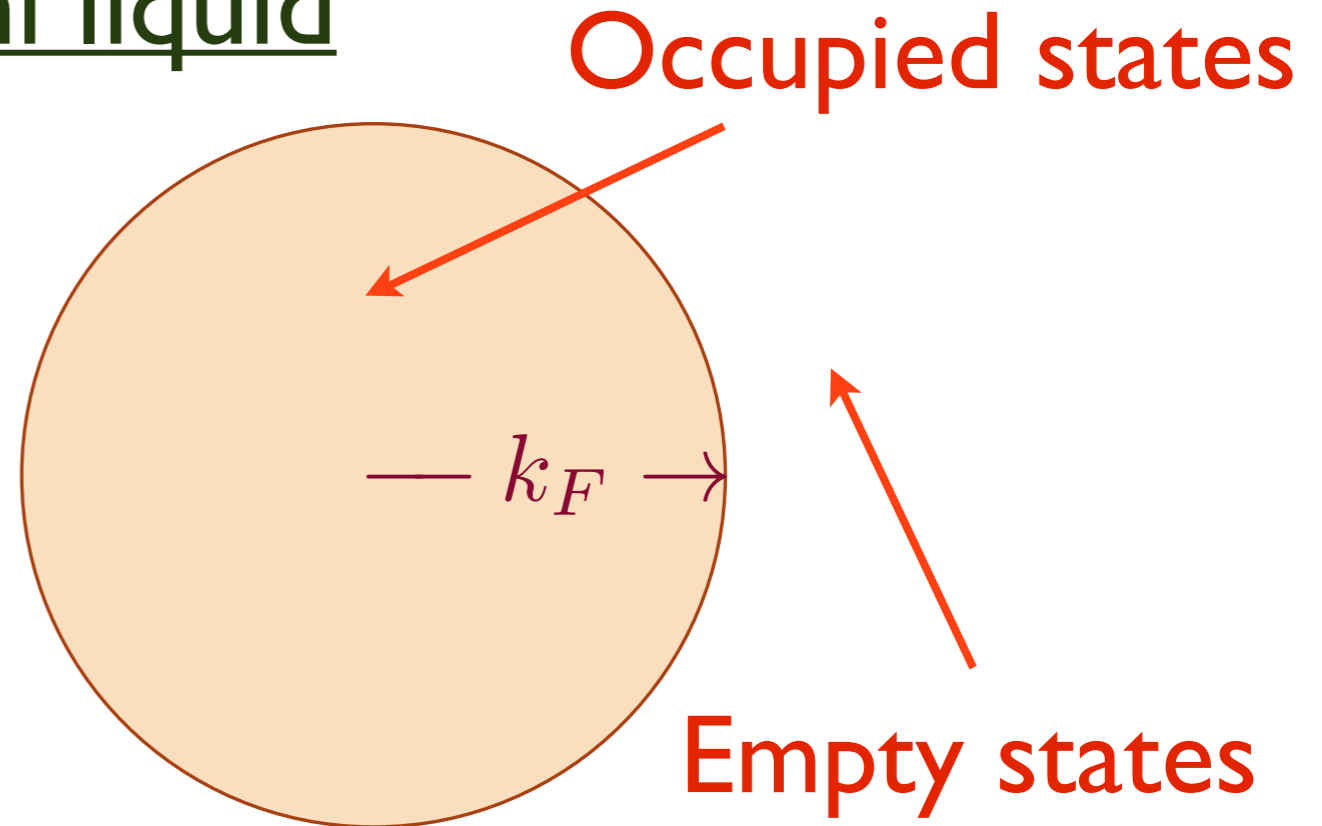
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Phase diagram as a function of T and λ

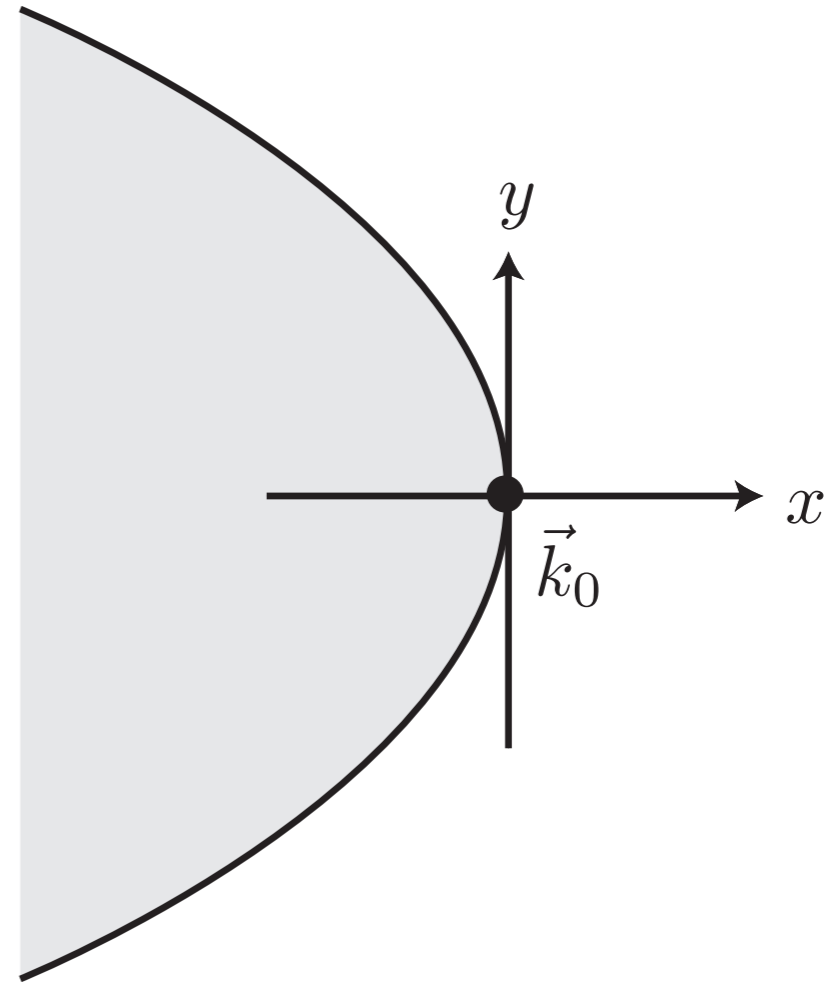
The Fermi liquid

$$\mathcal{L} = f_{\alpha}^{\dagger} \left(\partial_{\tau} - \frac{\nabla^2}{2m} - \mu \right) f_{\alpha} + u f_{\alpha}^{\dagger} f_{\beta}^{\dagger} f_{\beta} f_{\alpha}$$



The Fermi liquid: RG

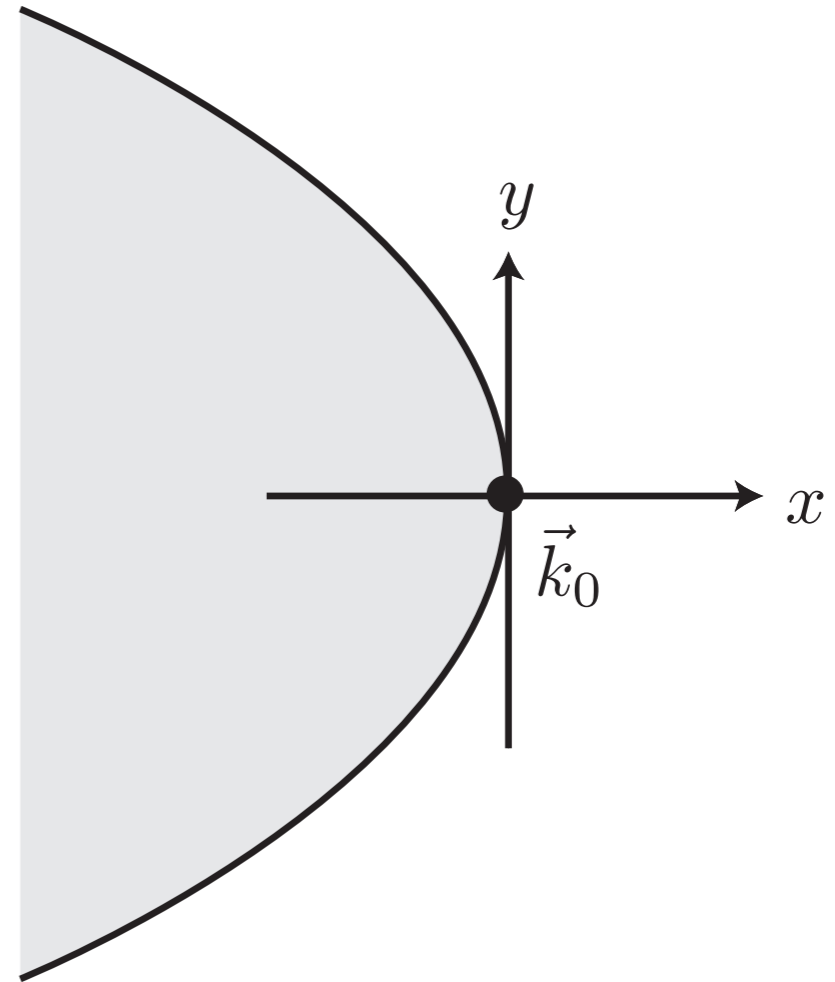
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- Expand fermion kinetic energy at wavevectors about \vec{k}_0 , by writing $f_{\alpha}(\vec{k}_0 + \vec{q}) = \psi_{\alpha}(\vec{q})$

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- Expand fermion kinetic energy at wavevectors about \vec{k}_0 , by writing $f_\alpha(\vec{k}_0 + \vec{q}) = \psi_\alpha(\vec{q})$

$$\mathcal{L}[\psi_\alpha] = \psi_\alpha^\dagger \left(\partial_\tau - i\partial_x - \partial_y^2 \right) \psi_\alpha + u \psi_\alpha^\dagger \psi_\beta^\dagger \psi_\beta \psi_\alpha$$

The Fermi liquid: RG

$$\mathcal{S}[\psi_\alpha] = \int d^{d-1}y dx d\tau \left[\psi_\alpha^\dagger (\partial_\tau - i\partial_x - \partial_y^2) \psi_\alpha + u \psi_\alpha^\dagger \psi_\beta^\dagger \psi_\beta \psi_\alpha \right]$$

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The kinetic energy is invariant under the rescaling $x \rightarrow x/s$, $y \rightarrow y/s^{1/2}$, and $\tau \rightarrow \tau/s^z$, provided $z = 1$ and

$$\psi \rightarrow \psi s^{(d+1)/4}.$$

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Then we find $u \rightarrow u s^{(1-d)/2}$, and so we have the RG flow

$$\frac{du}{d\ell} = \frac{(1-d)}{2} u$$

Interactions are *irrelevant* in $d = 2$!

The Fermi liquid: RG

$$\mathcal{S}[\psi_\alpha] = \int d^{d-1}y dx d\tau \left[\psi_\alpha^\dagger (\partial_\tau - i\partial_x - \partial_y^2) \psi_\alpha + u \psi_\alpha^\dagger \psi_\beta^\dagger \psi_\beta \psi_\alpha \right]$$

The fermion Green's function to order u^2 has the form (upto logs)

$$G(\vec{q}, \omega) = \frac{\mathcal{A}}{\omega - q_x - q_y^2 + ic\omega^2}$$

So the quasiparticle pole is sharp.

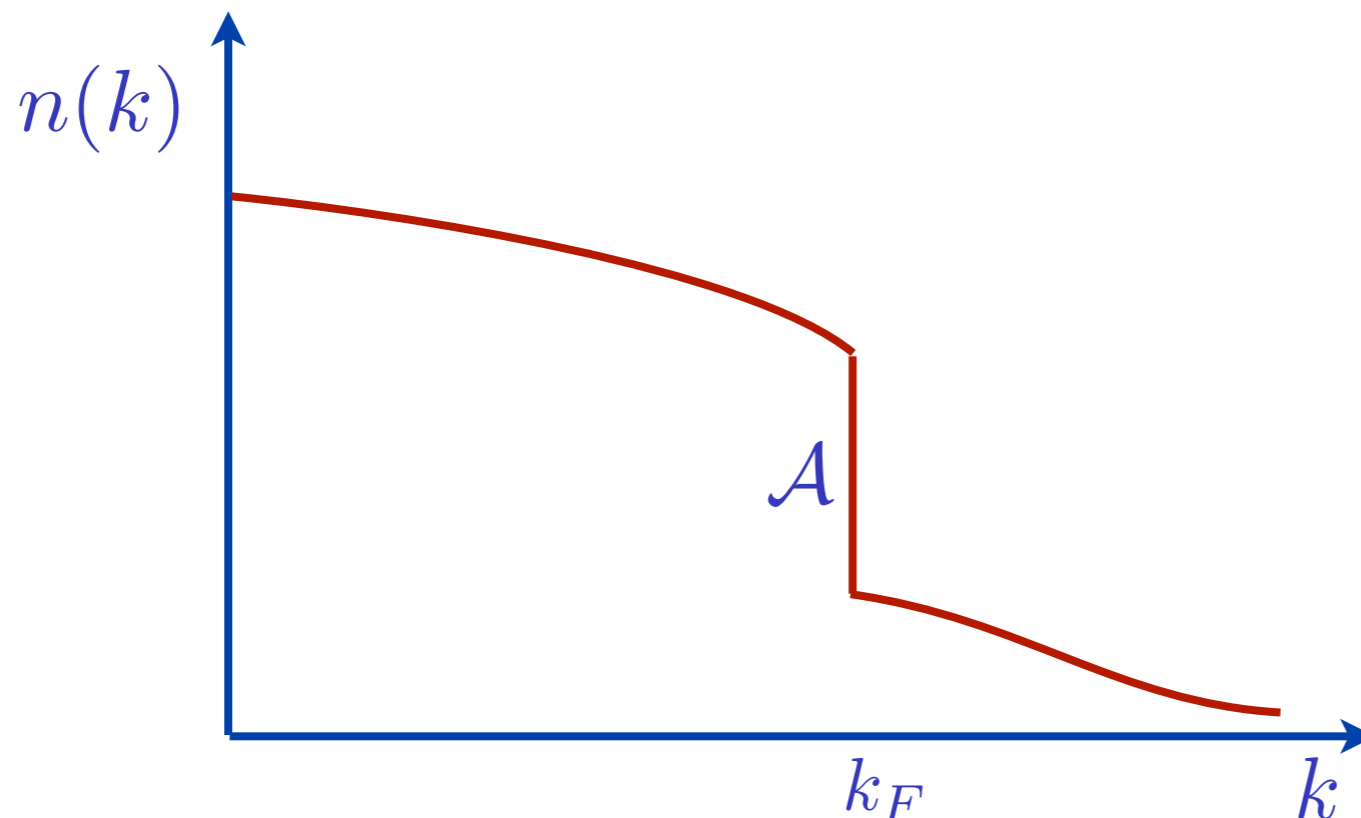
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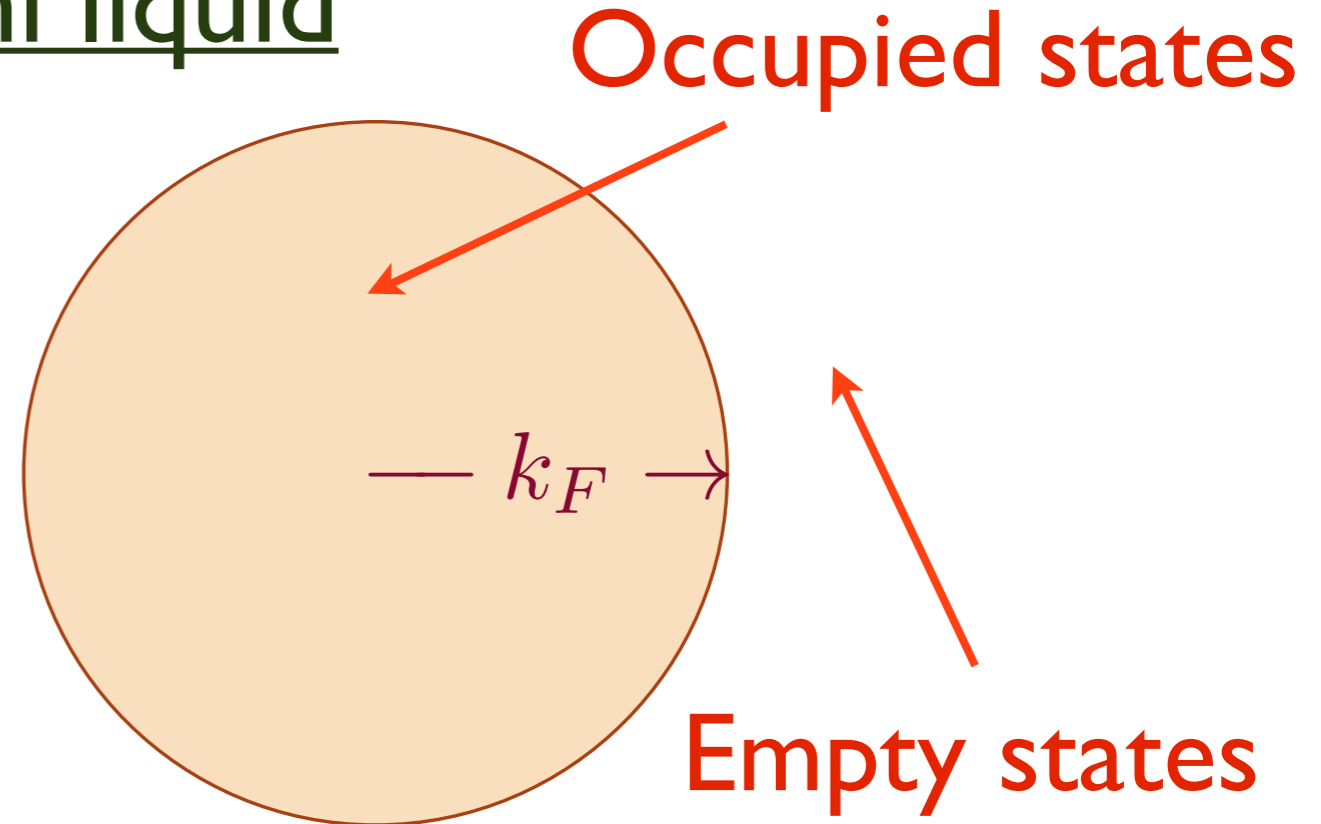
So the quasiparticle pole is sharp. And fermion momentum distribution function $n(\vec{k}) = \langle f_\alpha^\dagger(\vec{k}) f_\alpha(\vec{k}) \rangle$ had the following form:



The Fermi liquid

$$\mathcal{L} = f^\dagger \left(\partial_\tau - \frac{\nabla^2}{2m} - \mu \right) f$$

+ 4 Fermi terms

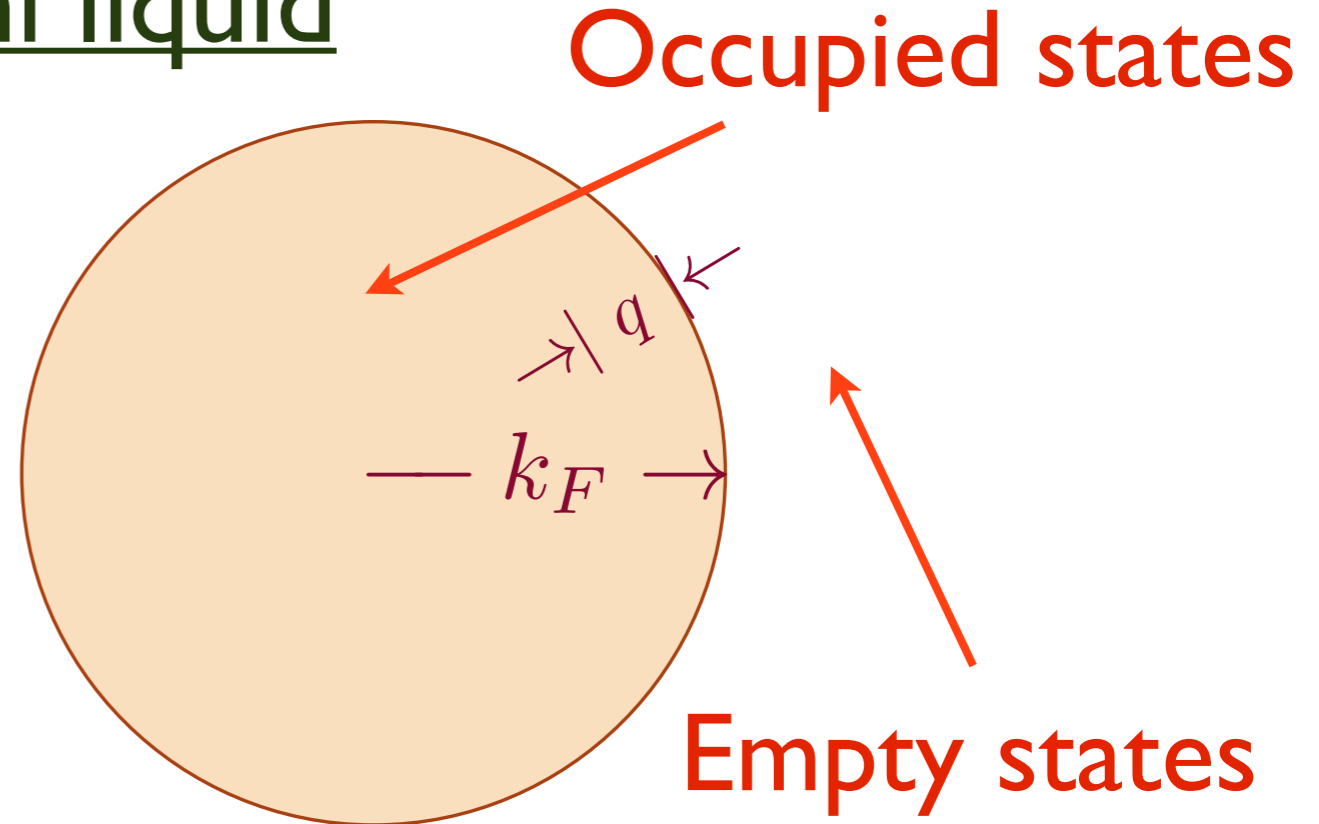


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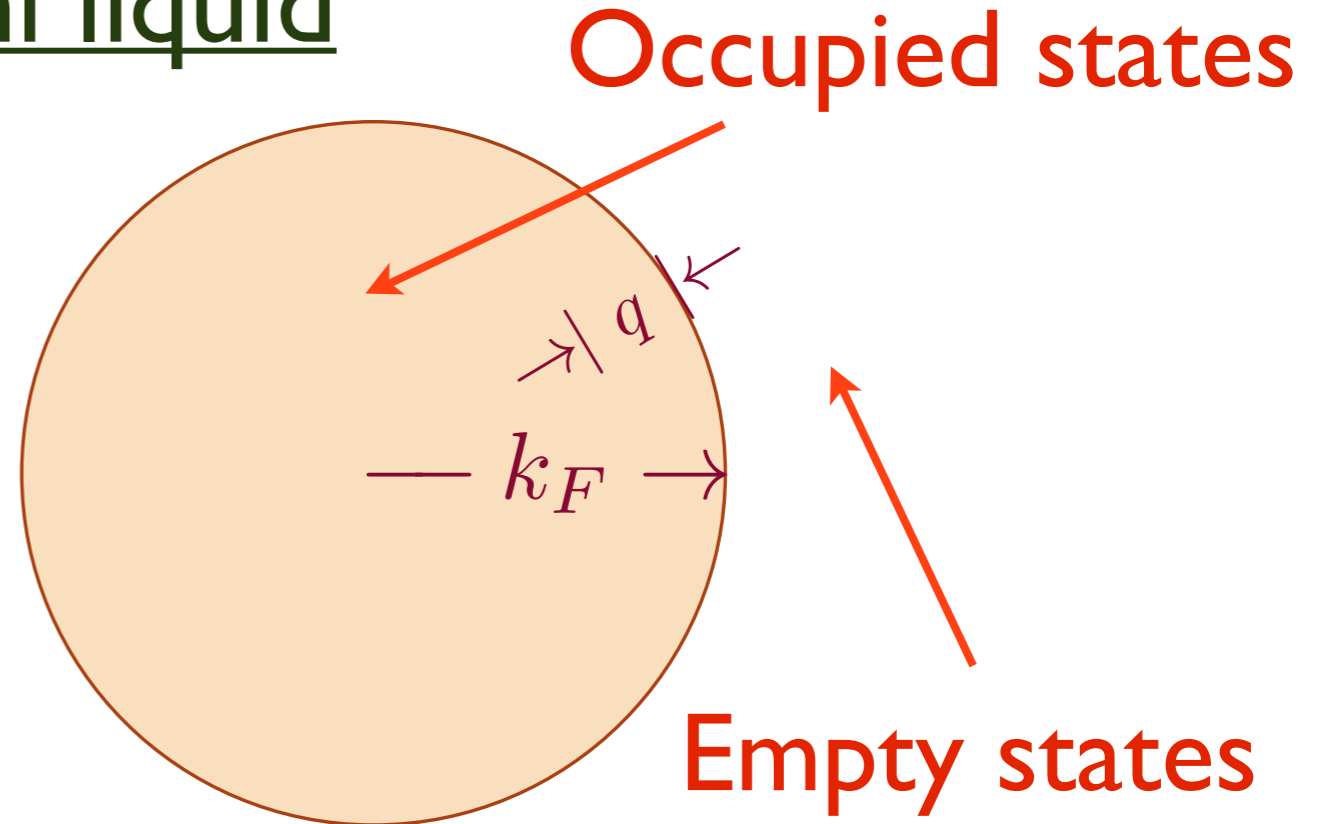


- Fermi wavevector obeys the Luttinger relation $k_F^d \sim \mathcal{Q}$, the fermion density
- Sharp particle and hole of excitations near the Fermi surface with energy $\omega \sim |q|^z$, with dynamic exponent $z = 1$.

The Fermi liquid

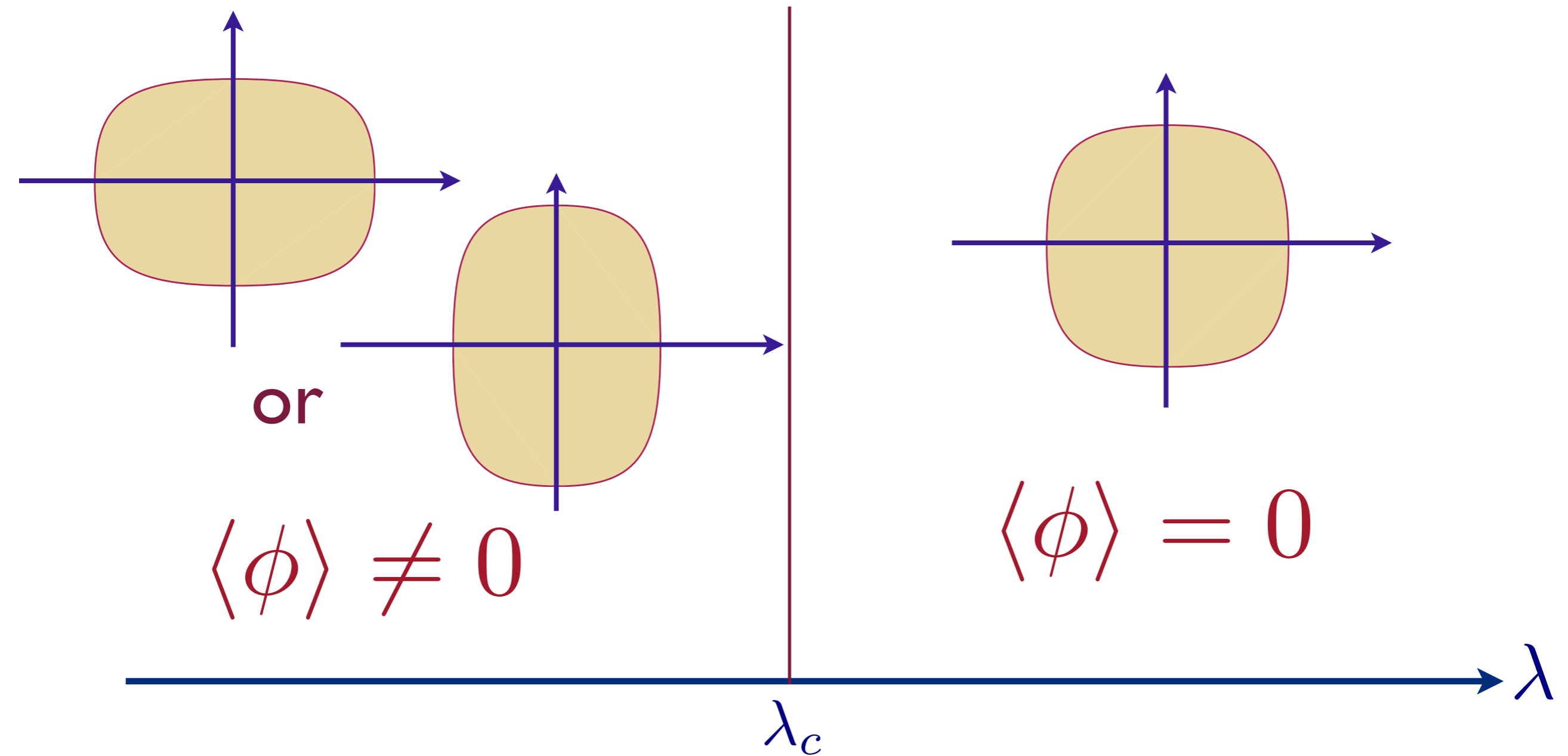
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- Fermi wavevector obeys the Luttinger relation $k_F^d \sim \mathcal{Q}$, the fermion density
- Sharp particle and hole of excitations near the Fermi surface with energy $\omega \sim |q|^z$, with dynamic exponent $z = 1$.
- The phase space density of fermions is effectively one-dimensional, so the entropy density $S \sim T$. It is useful to write this as $S \sim T^{(d-\theta)/z}$, with violation of hyperscaling exponent $\theta = d - 1$.

Quantum criticality of Ising-nematic ordering in a metal



Pomeranchuk instability as a function of coupling λ

Quantum criticality of Ising-nematic ordering in a metal

Effective action for Ising order parameter

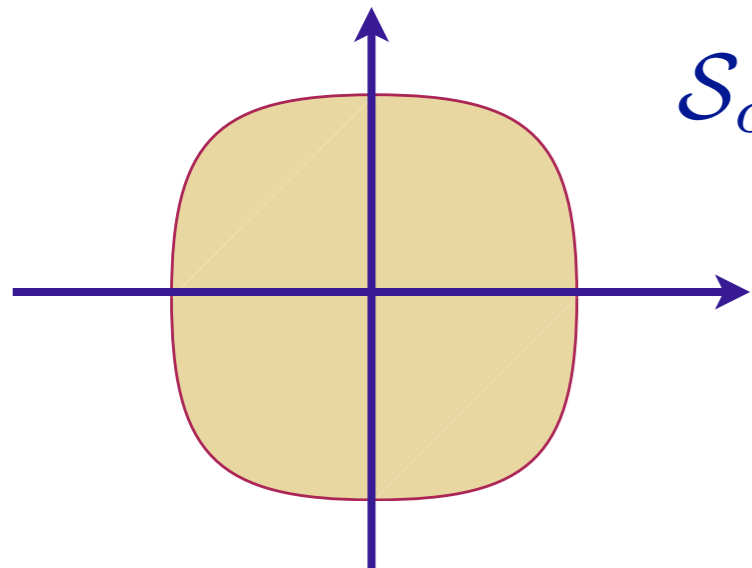
$$\mathcal{S}_\phi = \int d^2r d\tau [(\partial_\tau \phi)^2 + c^2 (\nabla \phi)^2 + (\lambda - \lambda_c) \phi^2 + u \phi^4]$$

Quantum criticality of Ising-nematic ordering in a metal

Effective action for Ising order parameter

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Effective action for electrons:

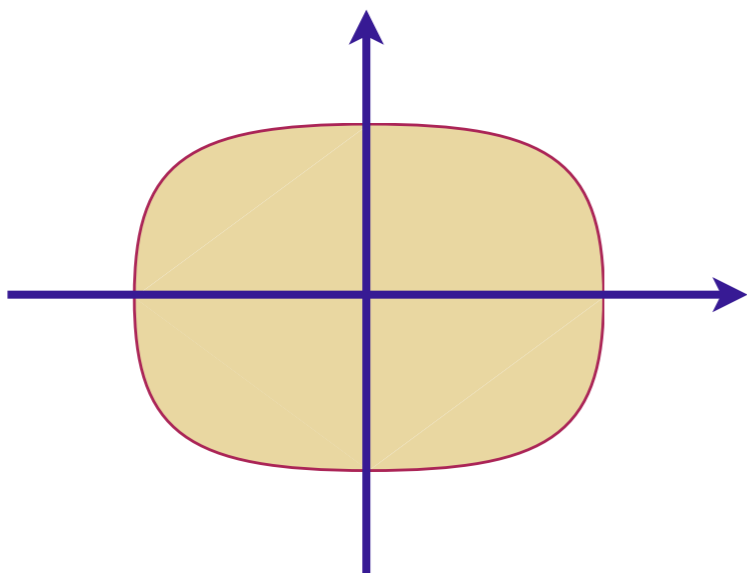

$$\begin{aligned} \mathcal{S}_c &= \int d\tau \sum_{\alpha=1}^{N_f} \left[\sum_i c_{i\alpha}^\dagger \partial_\tau c_{i\alpha} - \sum_{i<j} t_{ij} c_{i\alpha}^\dagger c_{j\alpha} \right] \\ &\equiv \sum_{\alpha=1}^{N_f} \sum_{\mathbf{k}} \int d\tau c_{\mathbf{k}\alpha}^\dagger (\partial_\tau + \varepsilon_{\mathbf{k}}) c_{\mathbf{k}\alpha} \end{aligned}$$

Quantum criticality of Ising-nematic ordering in a metal

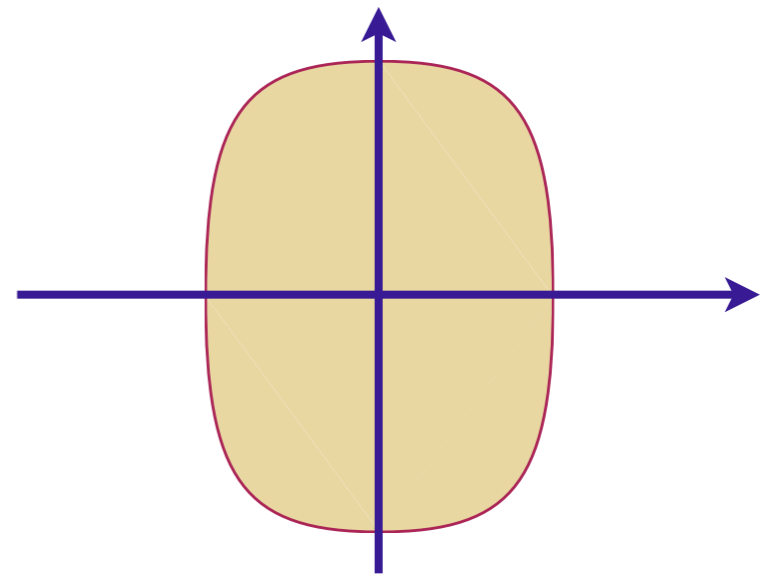
Coupling between Ising order and electrons

$$\mathcal{S}_{\phi c} = -g \int d\tau \sum_{\alpha=1}^{N_f} \sum_{\mathbf{k}, \mathbf{q}} \phi_{\mathbf{q}} (\cos k_x - \cos k_y) c_{\mathbf{k}+\mathbf{q}/2, \alpha}^\dagger c_{\mathbf{k}-\mathbf{q}/2, \alpha}$$

for spatially dependent ϕ



$$\langle \phi \rangle > 0$$



$$\langle \phi \rangle < 0$$

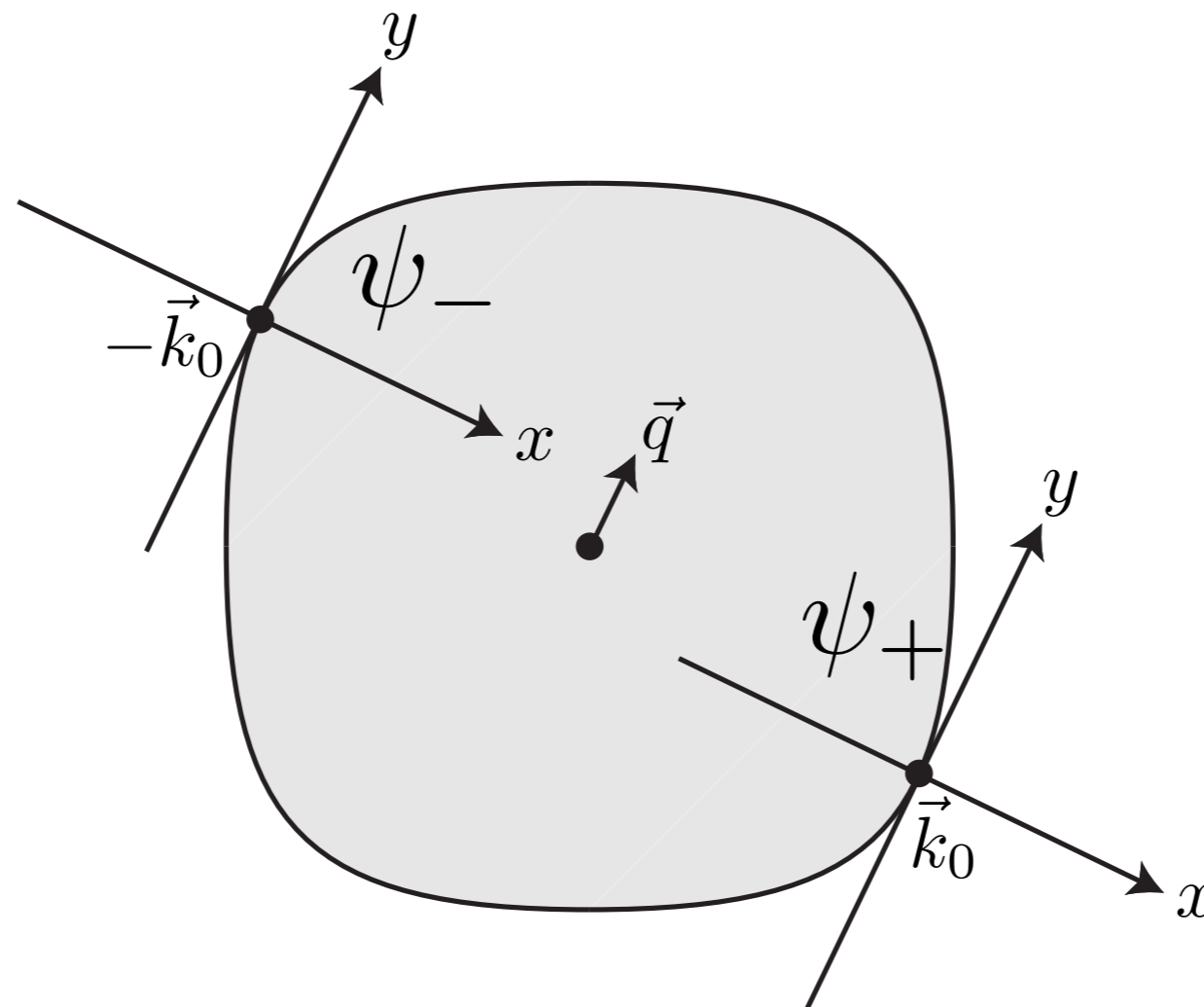
Quantum criticality of Ising-nematic ordering in a metal

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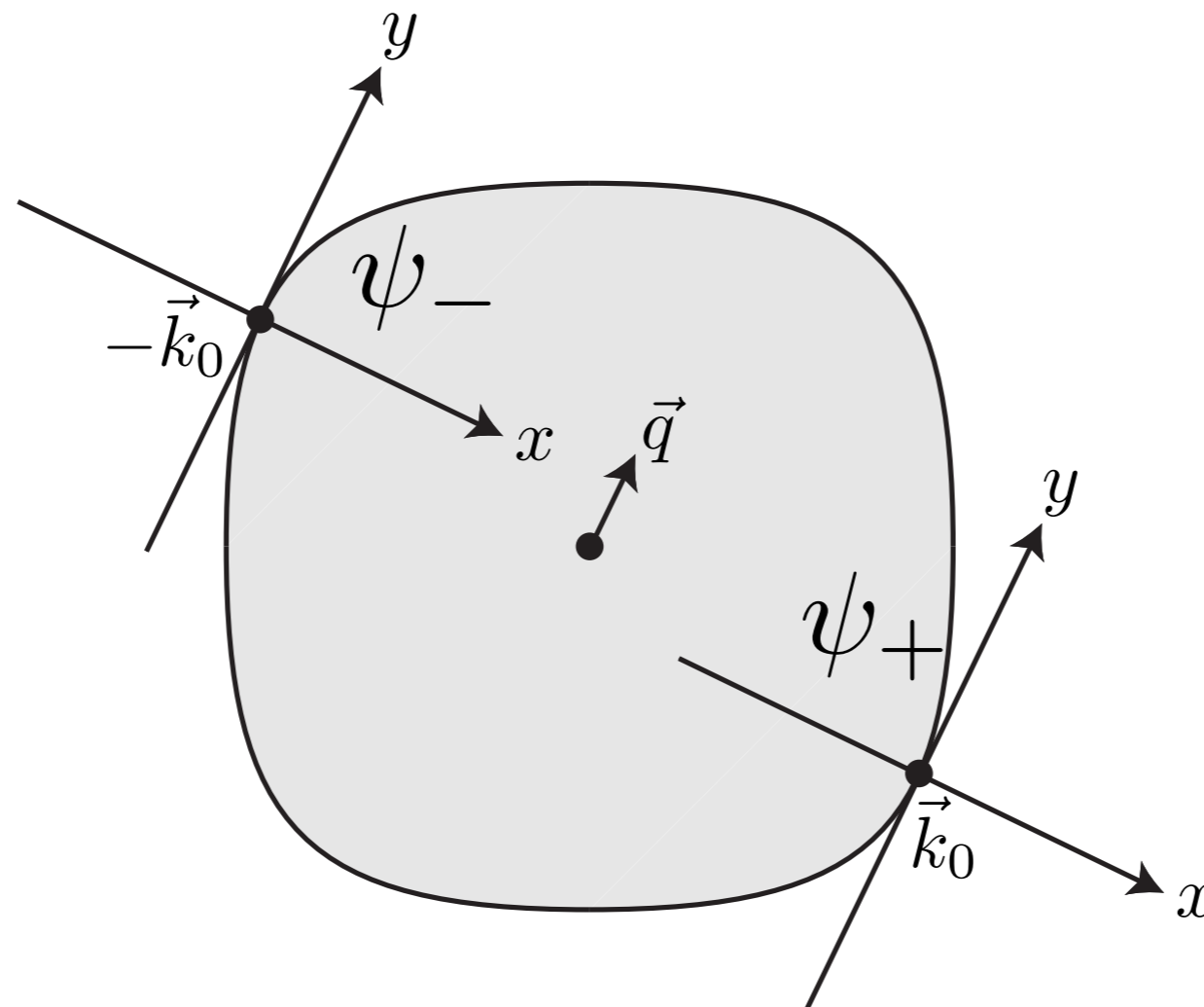
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Quantum criticality of Ising-nematic ordering in a metal



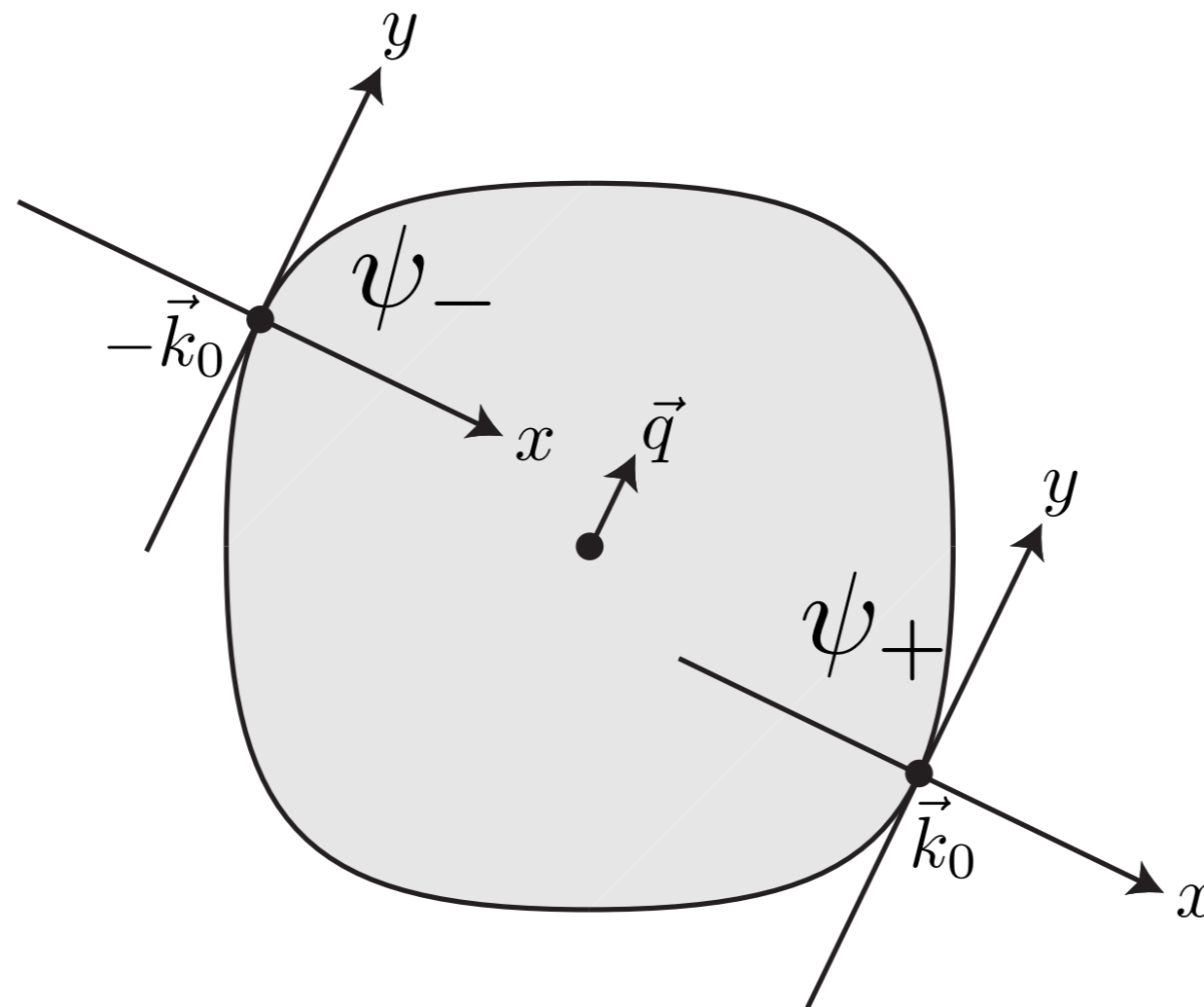
- ϕ fluctuation at wavevector \vec{q} couples most efficiently to fermions near $\pm\vec{k}_0$.

Quantum criticality of Ising-nematic ordering in a metal



- ϕ fluctuation at wavevector \vec{q} couples most efficiently to fermions near $\pm\vec{k}_0$.
- Expand fermion kinetic energy at wavevectors about $\pm\vec{k}_0$ and boson (ϕ) kinetic energy about $\vec{q} = 0$.

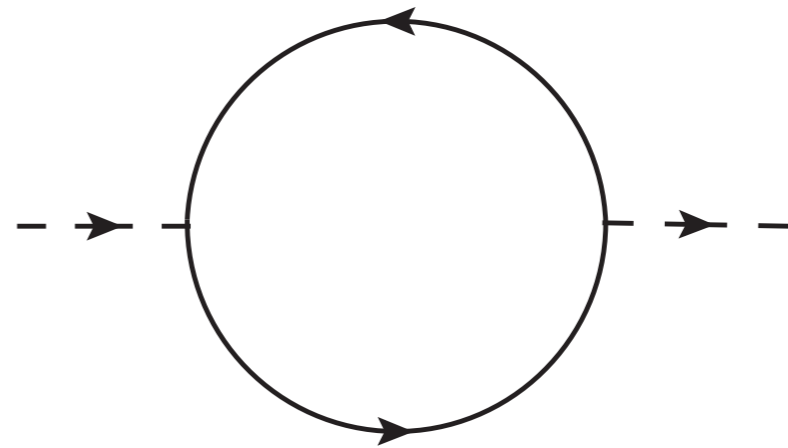
Quantum criticality of Ising-nematic ordering in a metal



$$\begin{aligned} \mathcal{L}[\psi_{\pm}, \phi] = & \psi_+^\dagger (\partial_\tau - i\partial_x - \partial_y^2) \psi_+ + \psi_-^\dagger (\partial_\tau + i\partial_x - \partial_y^2) \psi_- \\ & - \phi \left(\psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- \right) + \frac{1}{2g^2} (\partial_y \phi)^2 \end{aligned}$$

Quantum criticality of Ising-nematic ordering in a metal

$$\mathcal{L} = \psi_+^\dagger (\partial_\tau - i\partial_x - \partial_y^2) \psi_+ + \psi_-^\dagger (\partial_\tau + i\partial_x - \partial_y^2) \psi_- - \phi \left(\psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- \right) + \frac{1}{2g^2} (\partial_y \phi)^2$$



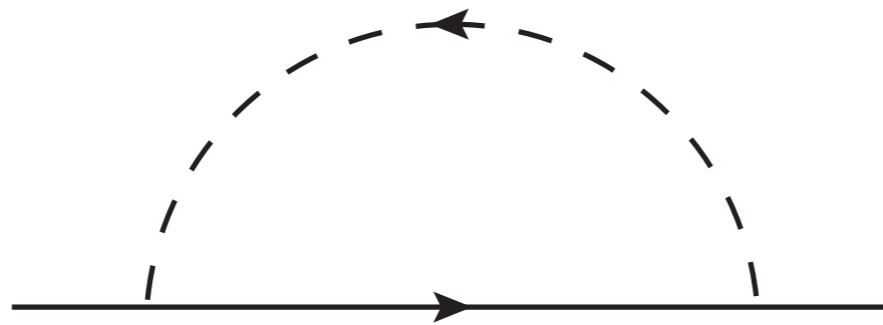
One loop ϕ self-energy with N_f fermion flavors:

$$\begin{aligned} \Sigma_\phi(\vec{q}, \omega) &= N_f \int \frac{d^2 k}{4\pi^2} \frac{d\Omega}{2\pi} \frac{1}{[-i(\Omega + \omega) + k_x + q_x + (k_y + q_y)^2] [-i\Omega - k_x + k_y^2]} \\ &= \frac{N_f}{4\pi} \frac{|\omega|}{|q_y|} \end{aligned}$$

Landau-damping

Quantum criticality of Ising-nematic ordering in a metal

$$\begin{aligned}\mathcal{L} = & \psi_+^\dagger (\partial_\tau - i\partial_x - \partial_y^2) \psi_+ + \psi_-^\dagger (\partial_\tau + i\partial_x - \partial_y^2) \psi_- \\ & - \phi \left(\psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- \right) + \frac{1}{2g^2} (\partial_y \phi)^2\end{aligned}$$

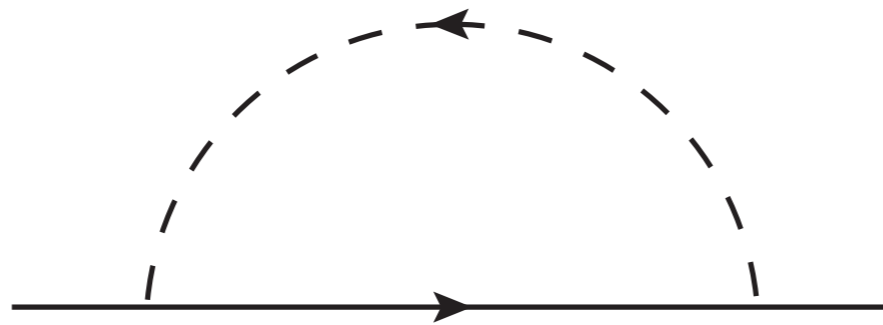


Electron self-energy at order $1/N_f$:

$$\begin{aligned}\Sigma(\vec{k}, \Omega) &= -\frac{1}{N_f} \int \frac{d^2q}{4\pi^2} \frac{d\omega}{2\pi} \frac{1}{[-i(\omega + \Omega) + k_x + q_x + (k_y + q_y)^2] \left[\frac{q_y^2}{g^2} + \frac{|\omega|}{|q_y|} \right]} \\ &= -i \frac{2}{\sqrt{3}N_f} \left(\frac{g^2}{4\pi} \right)^{2/3} \text{sgn}(\Omega) |\Omega|^{2/3}\end{aligned}$$

Quantum criticality of Ising-nematic ordering in a metal

$$\mathcal{L} = \psi_+^\dagger (\partial_\tau - i\partial_x - \partial_y^2) \psi_+ + \psi_-^\dagger (\partial_\tau + i\partial_x - \partial_y^2) \psi_- - \phi (\psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_-) + \frac{1}{2g^2} (\partial_y \phi)^2$$



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Quantum criticality of Ising-nematic ordering in a metal

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Schematic form of ϕ and fermion Green's functions in d dimensions

$$D(\vec{q}, \omega) = \frac{1/N_f}{q_\perp^2 + \frac{|\omega|}{|q_\perp|}}, \quad G_f(\vec{q}, \omega) = \frac{1}{q_x + q_\perp^2 - i \text{sgn}(\omega) |\omega|^{d/3} / N_f}$$

In the boson case, $q_\perp^2 \sim \omega^{1/z_b}$ with $z_b = 3/2$.

In the fermion case, $q_x \sim q_\perp^2 \sim \omega^{1/z_f}$ with $z_f = 3/d$.

Note $z_f < z_b$ for $d > 2 \Rightarrow$ Fermions have *higher* energy than bosons, and perturbation theory in g is OK.

Strongly-coupled theory in $d = 2$.

Quantum criticality of Ising-nematic ordering in a metal

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Schematic form of ϕ and fermion Green's functions in $d = 2$

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In *both* cases $q_x \sim q_y^2 \sim \omega^{1/z}$, with $z = 3/2$. Note that the bare term $\sim \omega$ in G_f^{-1} is irrelevant.

Strongly-coupled theory without quasiparticles.

Quantum criticality of Ising-nematic ordering in a metal

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Simple scaling argument for $z = 3/2$.

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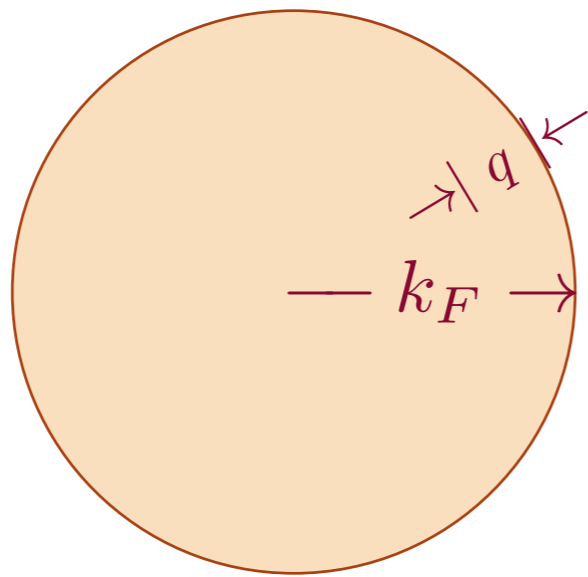
Simple scaling argument for $z = 3/2$.

Under the rescaling $x \rightarrow x/s$, $y \rightarrow y/s^{1/2}$, and $\tau \rightarrow \tau/s^z$, we find invariance provided

$$\begin{aligned} \phi &\rightarrow \phi s \\ \psi &\rightarrow \psi s^{(2z+1)/4} \\ g &\rightarrow g s^{(3-2z)/4} \end{aligned}$$

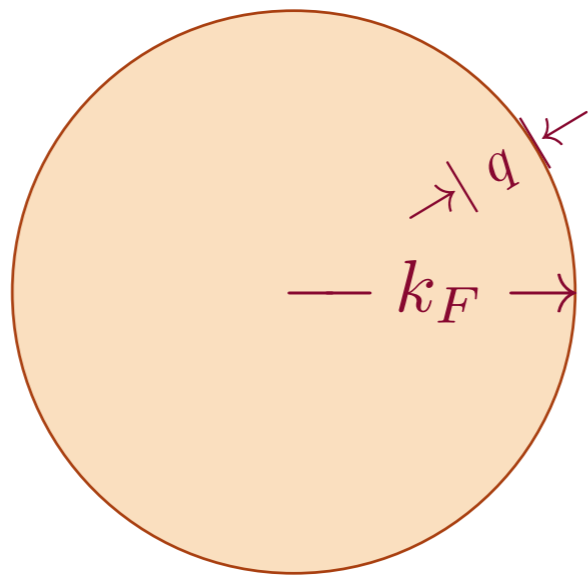
So the action is invariant provided $z = 3/2$.

FL Fermi liquid



- $k_F^d \sim Q$, the fermion density
- Sharp fermionic excitations near Fermi surface with $\omega \sim |q|^z$, and $z = 1$.
- Entropy density $S \sim T^{(d-\theta)/z}$ with violation of hyperscaling exponent $\theta = d - 1$.

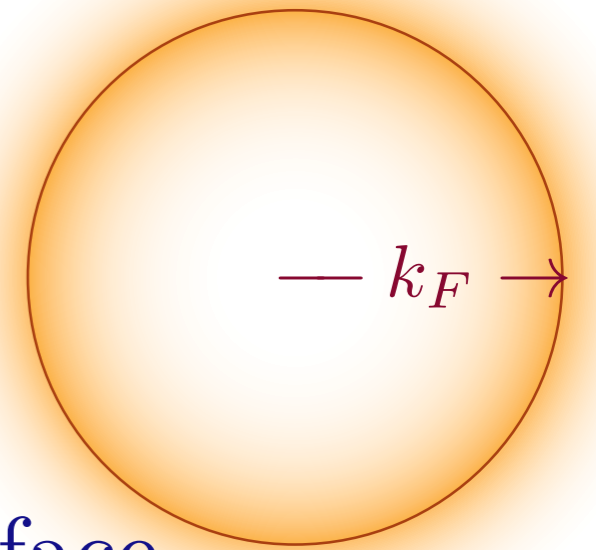
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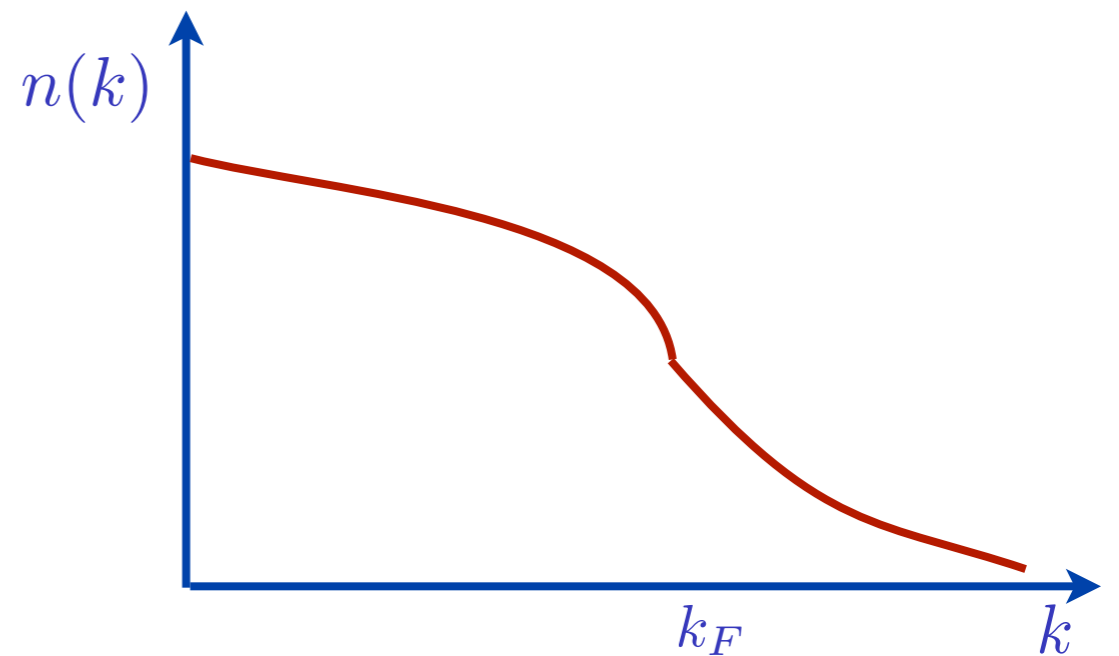
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- Sharp fermionic excitations near Fermi surface with $\omega \sim |q|^z$, and $z = 1$.
- Entropy density $S \sim T^{(d-\theta)/z}$ with violation of hyperscaling exponent $\theta = d - 1$.

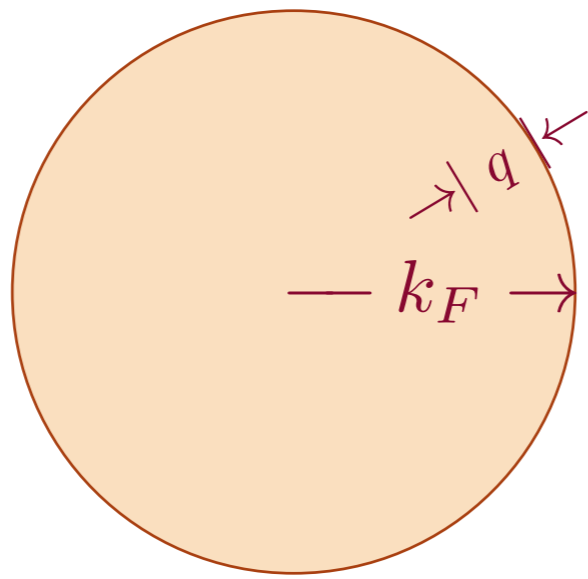
NFL Nematic QCP



- Fermi surface with $k_F^d \sim Q$.



FL Fermi liquid

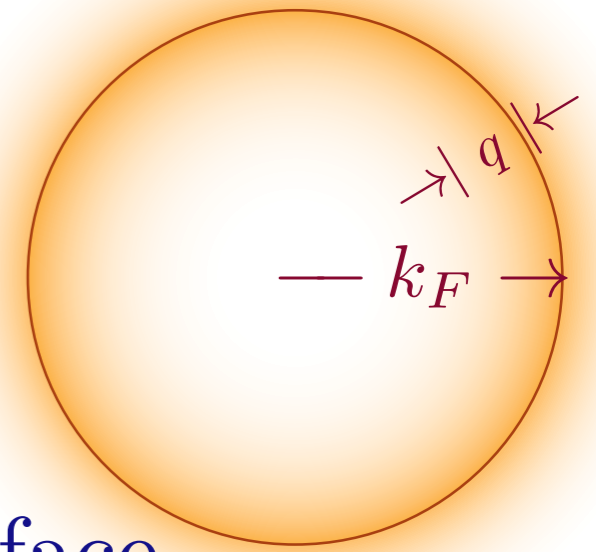


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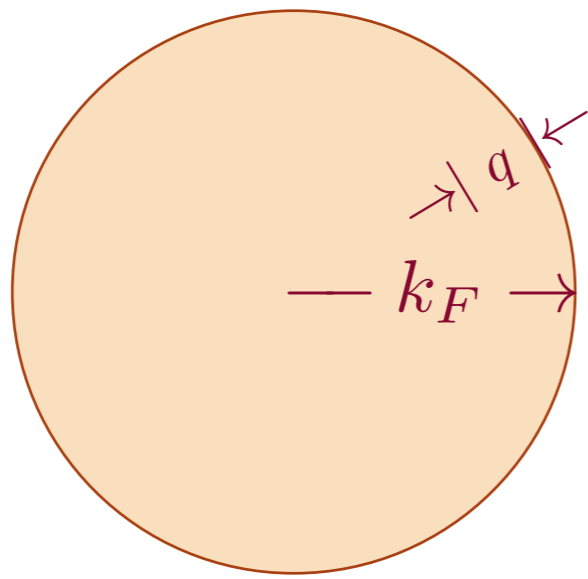
NFL Nematic QCP



- Fermi surface with $k_F^d \sim Q$.

- Diffuse fermionic excitations with $z = 3/2$ to three loops.

FL Fermi liquid

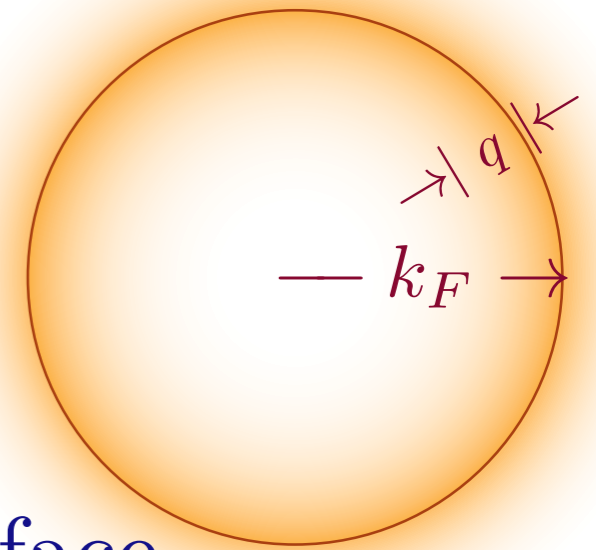


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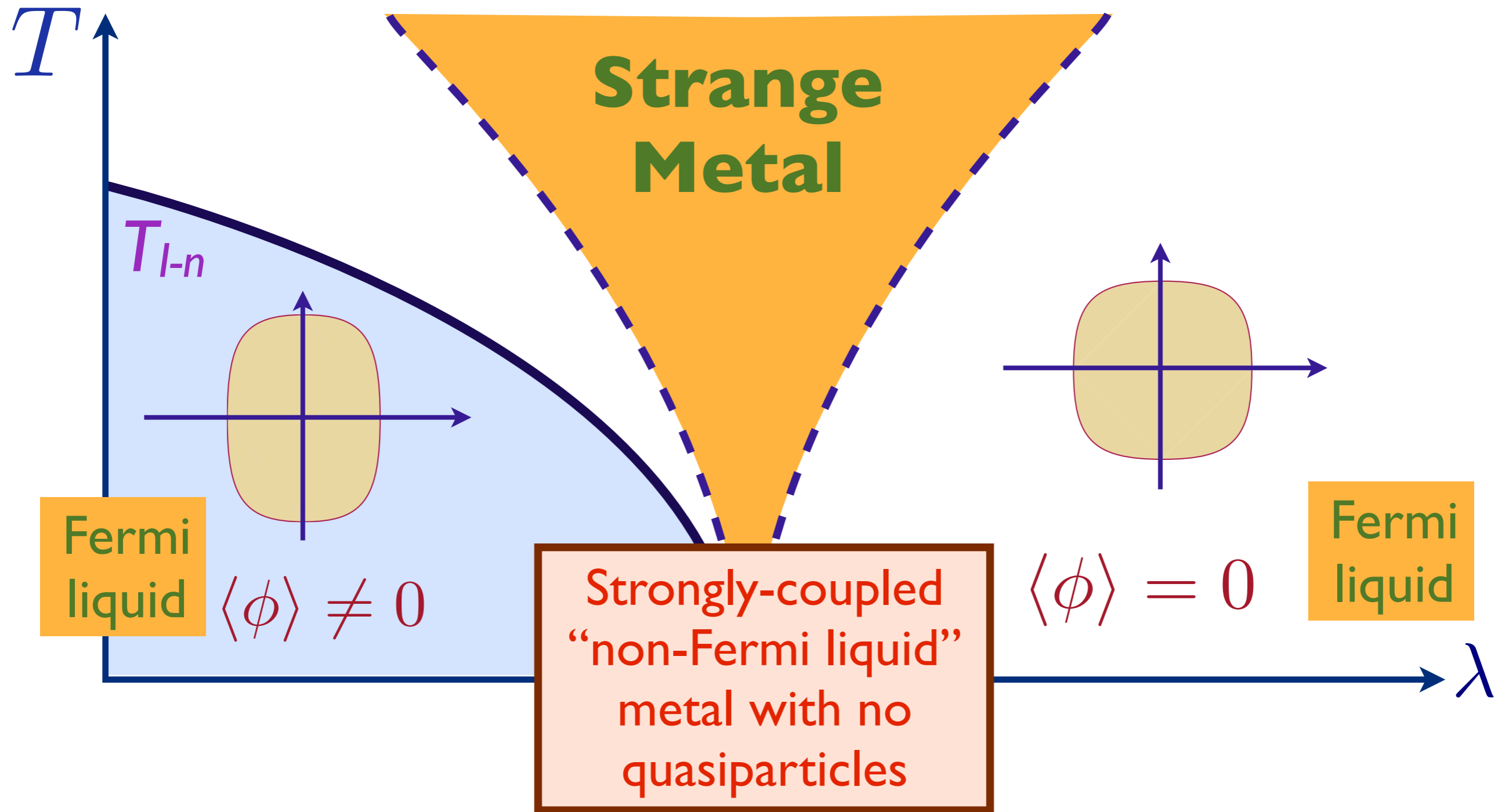


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Quantum criticality of Ising-nematic ordering in a metal



Phase diagram as a function of T and λ