Large N theories of non-Fermi liquids

Gravity and Emergent Gauge Fields
in Condensed and Synthetic Matter
Mainz Institute for Theoretical Physics,
Johannes Gutenberg University
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Talk online: sachdev.physics.harvard.edu

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What is a non-Fermi liquid ?

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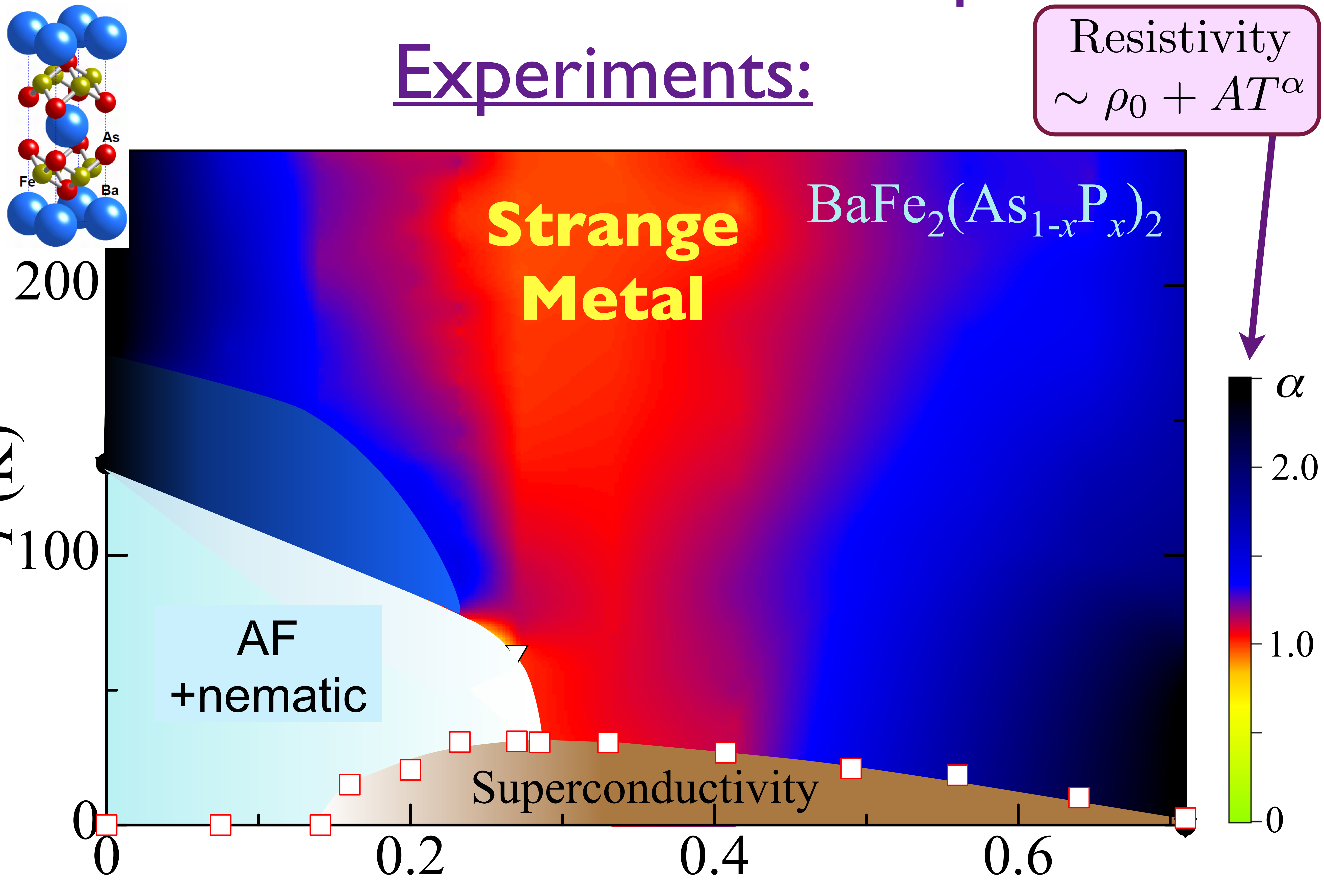
Experiments:

Resistivity
$\sim \rho_0 + AT^\alpha$



S. Kasahara et al. PRB **81**, 184519 (2010)

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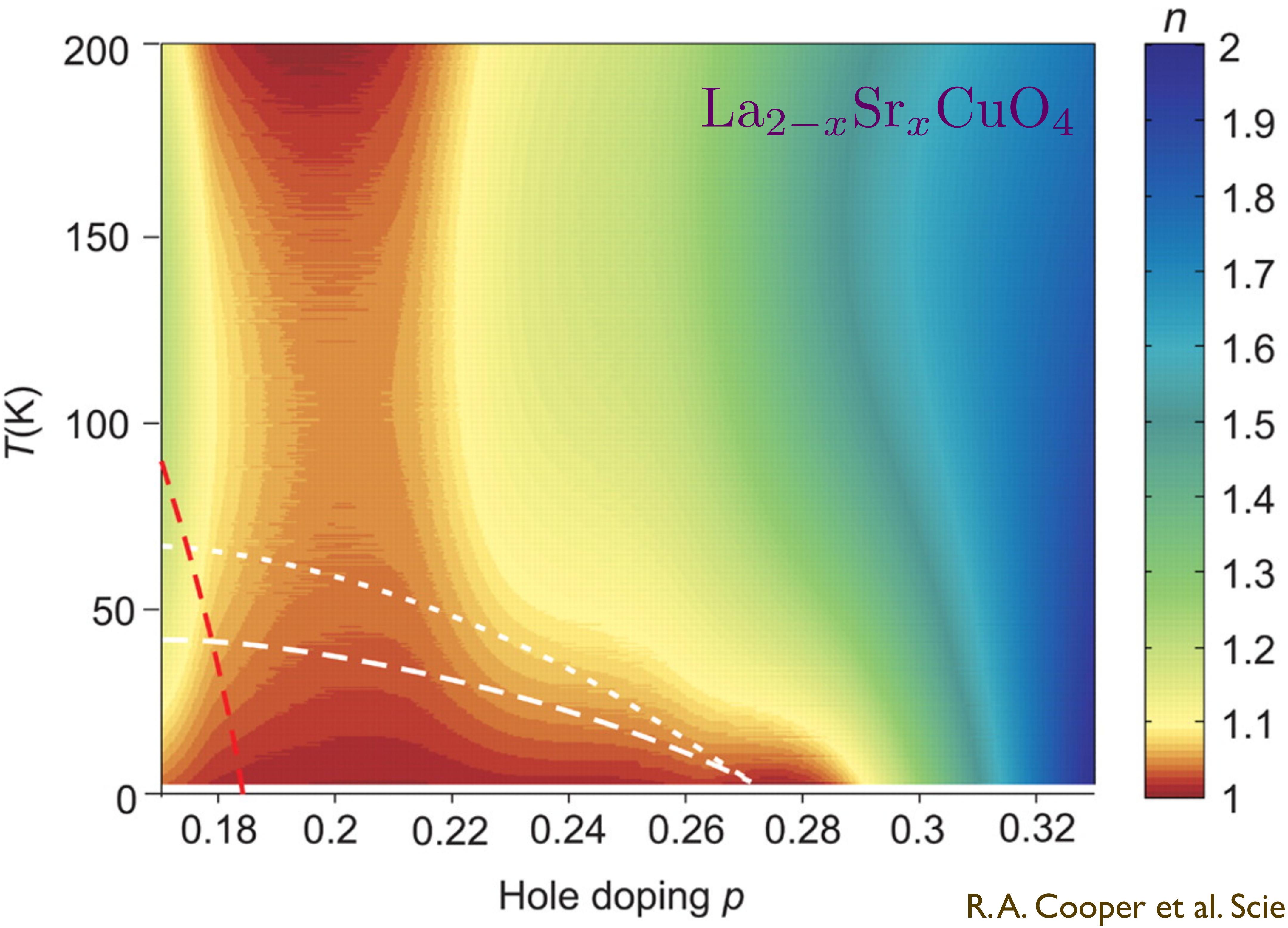

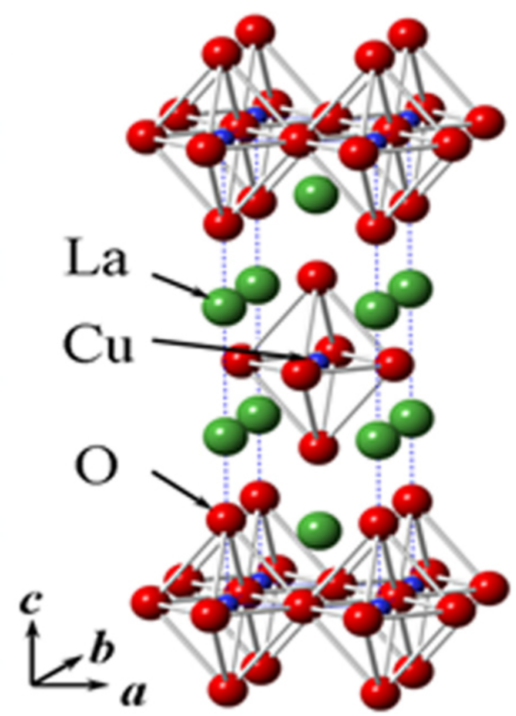

R.A. Cooper et al. Science **323**, 603 (2009)

What is a non-Fermi liquid ?

- **A compressible state of quantum matter:**
  There is a global U(1) charge $\mathcal{Q}$ with commutes with Hamiltonian, and upon applying a chemical potential change $\delta\mu$

$$\begin{aligned} \mathcal{H} &\Rightarrow \mathcal{H} - \delta\mu\,\mathcal{Q} \\ \frac{\delta\langle\mathcal{Q}\rangle}{\delta\mu} &\neq 0 \quad \text{as } T \to 0 \text{ in the thermodynamic limit} \end{aligned}$$

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  Dirac/Weyl fermions, CFTs in spatial dimension $d > 1$ are *not* compressible.

- **No quasiparticle excitations:**
  In the presence of quasiparticles, the low-lying many-body energies E can be identified as a set $\{n_\alpha\}$ of quasiparticles with energy ε_α

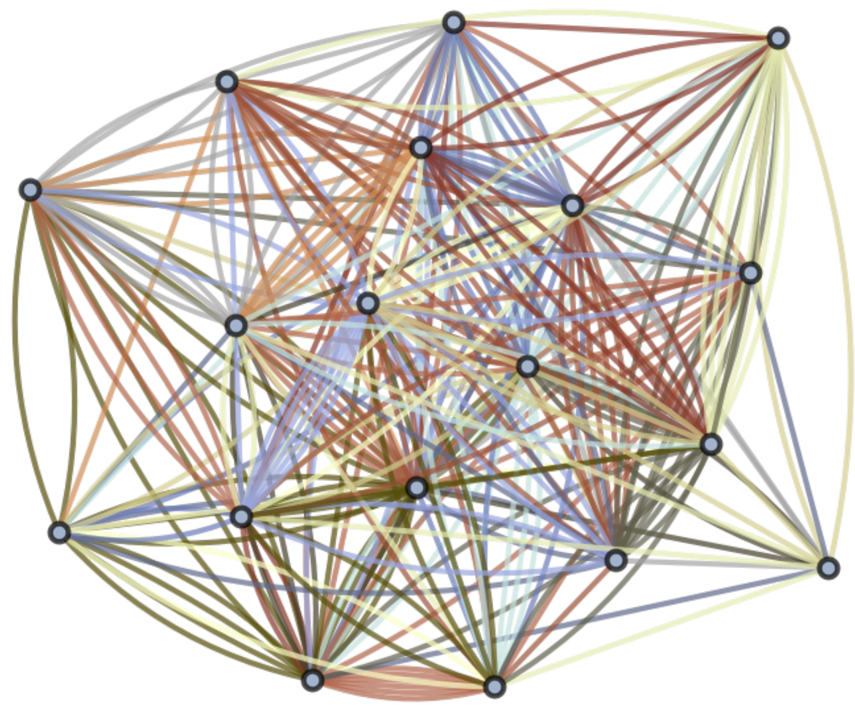
$$E = \sum_\alpha n_\alpha \varepsilon_\alpha + \sum_{\alpha,\beta} F_{\alpha\beta} n_\alpha n_\beta + \ldots$$

  Luttinger liquids, systems with an energy gap (TQFTs), do have quasiparticles.

Examples of non-Fermi liquids

Examples of non-Fermi liquids

1. Complex SYK model



Examples of non-Fermi liquids

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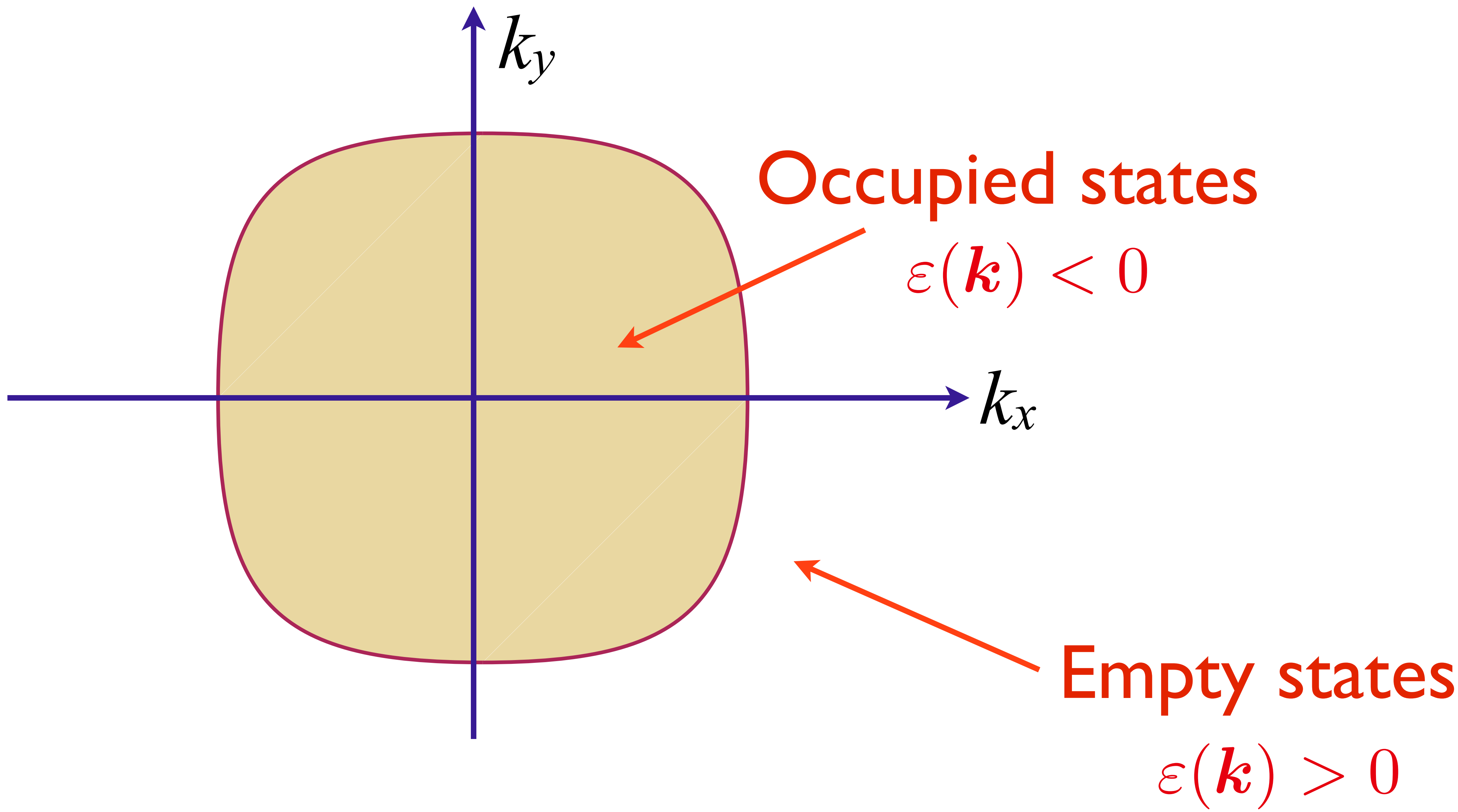

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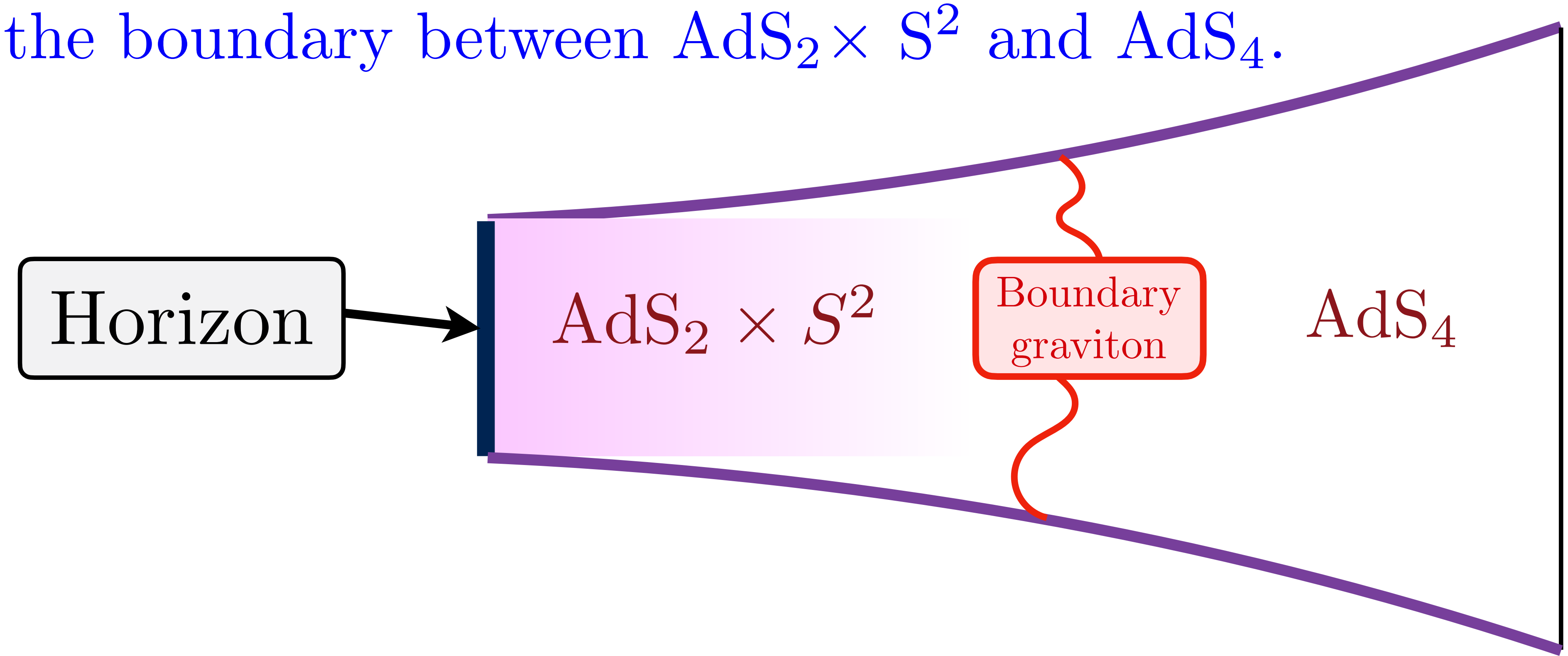

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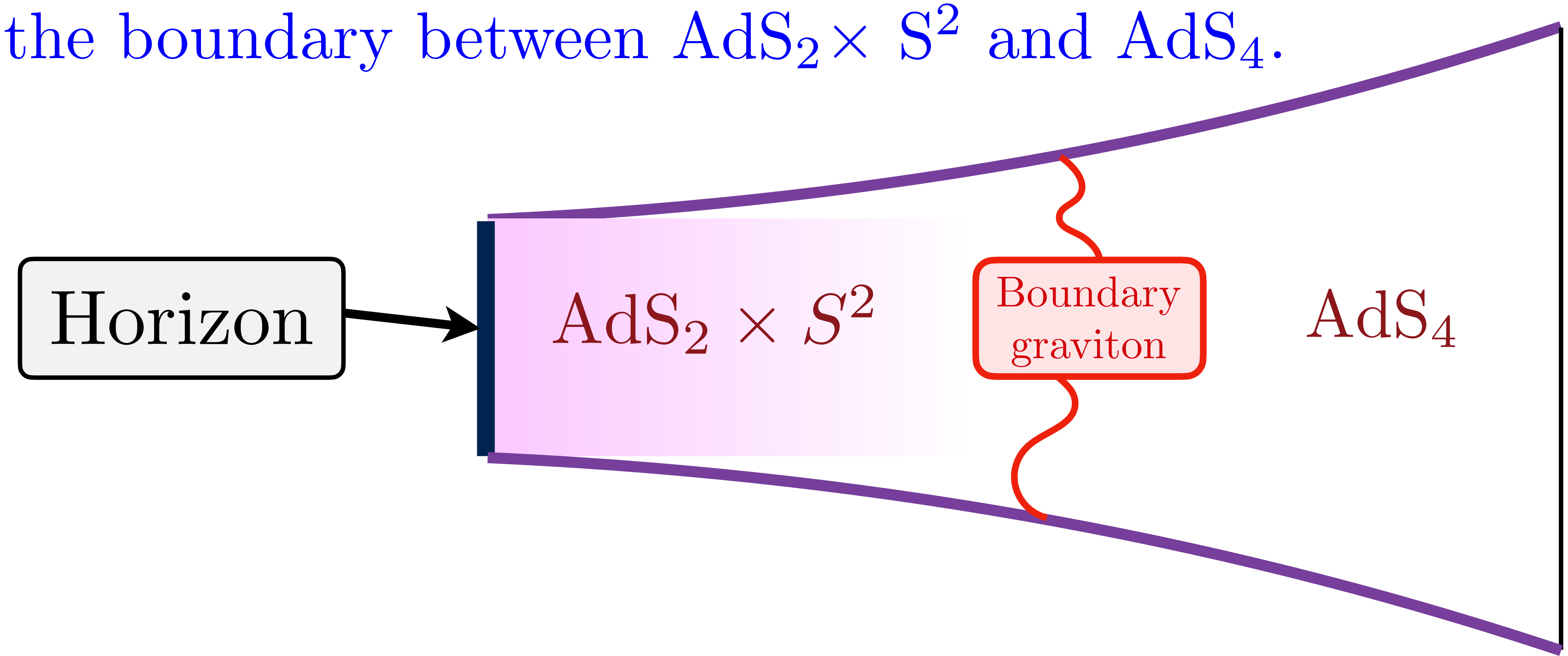
Large N limit of $\mathcal{N} = 8$ SYM_3 in a chemical potential on a sphere.



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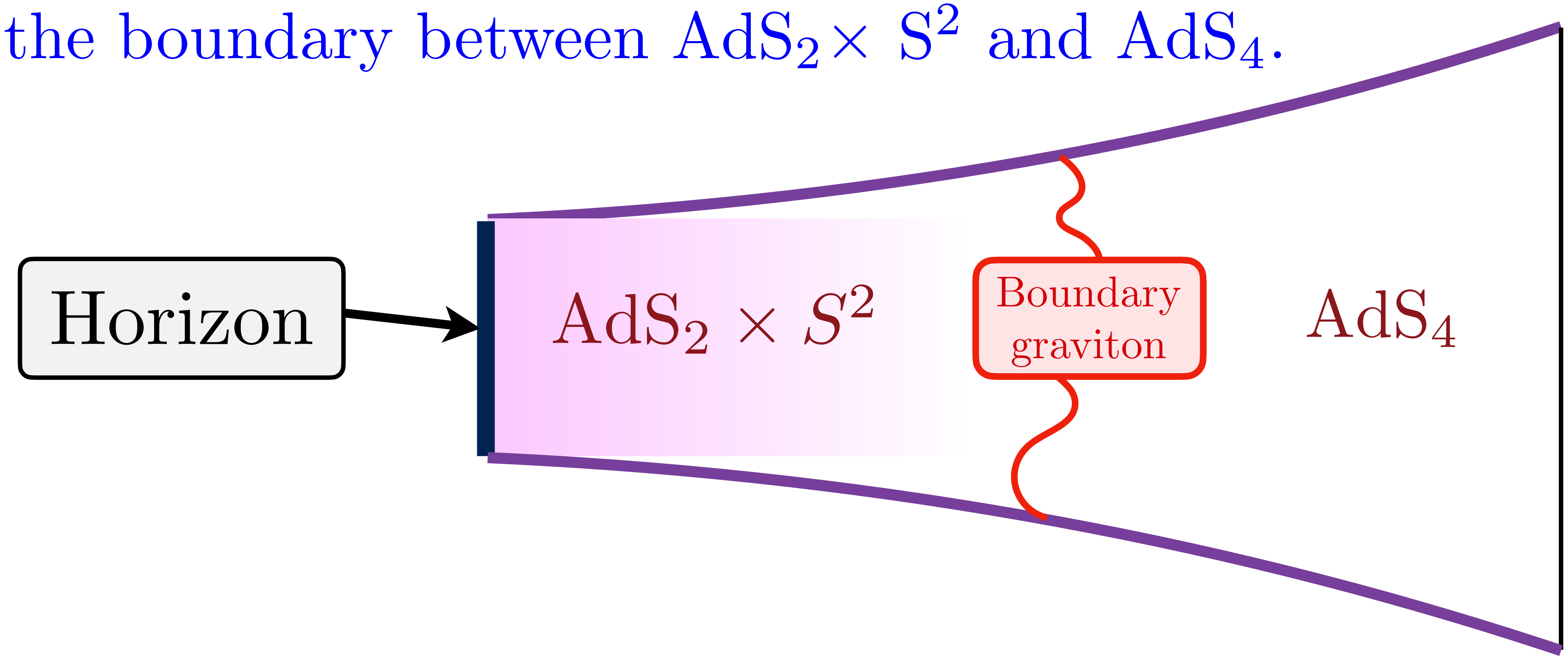
Holographic dual at low T



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4. Infinite, 2+1-dimensional charged black branes in 4 spacetime dimensions, at a temperature $T \ll \mathcal{Q}^{1/2}$, where $\mathcal{Q}$ is the charge density. Low T behavior is controlled by near-horizon $\mathrm{AdS}_2 \times \mathrm{R}^2$. Probe fermionic matter fields have a critical Fermi surface.

Large N limit of $\mathcal{N} = 8$ SYM_3 in a chemical potential on a sphere.

Faulkner, Iqbal, Liu, McGreevy; Cubrovic, Zaanen, Schalm (2009)

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   Large N limit of $\mathcal{N} = 8$ SYM_3 in a chemical potential in infinite flat space.

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Will present large N theory. Distinct from holographic theory 4.

Ilya Esterlis, Haoyu Guo, Aavishkar Patel, S.S. arXiv: 2103.08615

1. The complex SYK model

2. Critical Fermi surfaces: review

3. Critical Fermi surfaces: large *N* theory

The complex SYK model

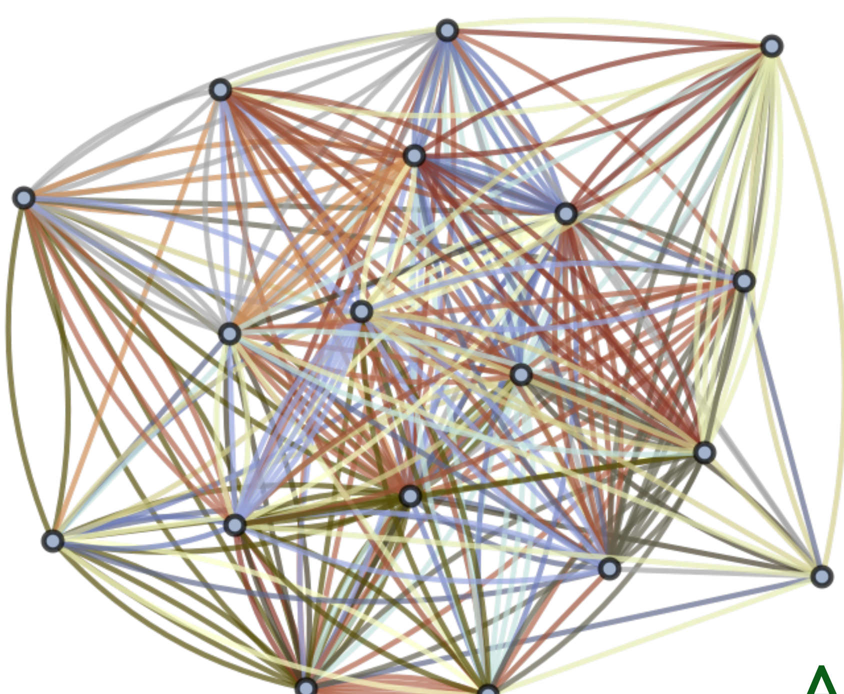
(See also: the "2-Body Random Ensemble" in nuclear physics; did not obtain the large *N* limit;
T.A. Brody, J. Flores, J.B. French, P.A. Mello, A. Pandey, and S.S.M. Wong, Rev. Mod. Phys. **53**, 385 (1981))

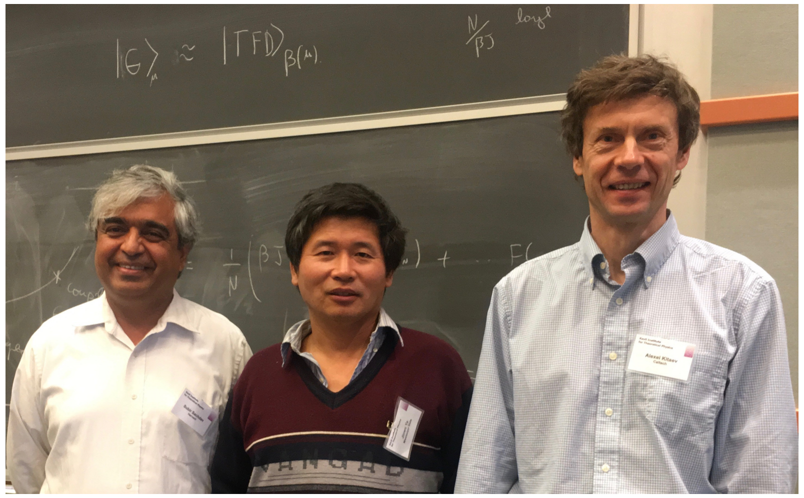
$$H = \frac{1}{(2N)^{3/2}} \sum_{\alpha,\beta,\gamma,\delta=1}^{N} U_{\alpha\beta;\gamma\delta}\, c_\alpha^\dagger c_\beta^\dagger c_\gamma c_\delta - \mu \sum_\alpha c_\alpha^\dagger c_\alpha$$

$$c_\alpha c_\beta + c_\beta c_\alpha = 0 \quad , \quad c_\alpha c_\beta^\dagger + c_\beta^\dagger c_\alpha = \delta_{\alpha\beta}$$

$$\mathcal{Q} = \frac{1}{N} \sum_\alpha c_\alpha^\dagger c_\alpha$$

$U_{\alpha\beta;\gamma\delta}$ are independent random variables with $\overline{U_{\alpha\beta;\gamma\delta}} = 0$ and $\overline{|U_{\alpha\beta;\gamma\delta}|^2} = U^2$
$N \to \infty$ yields critical strange metal.





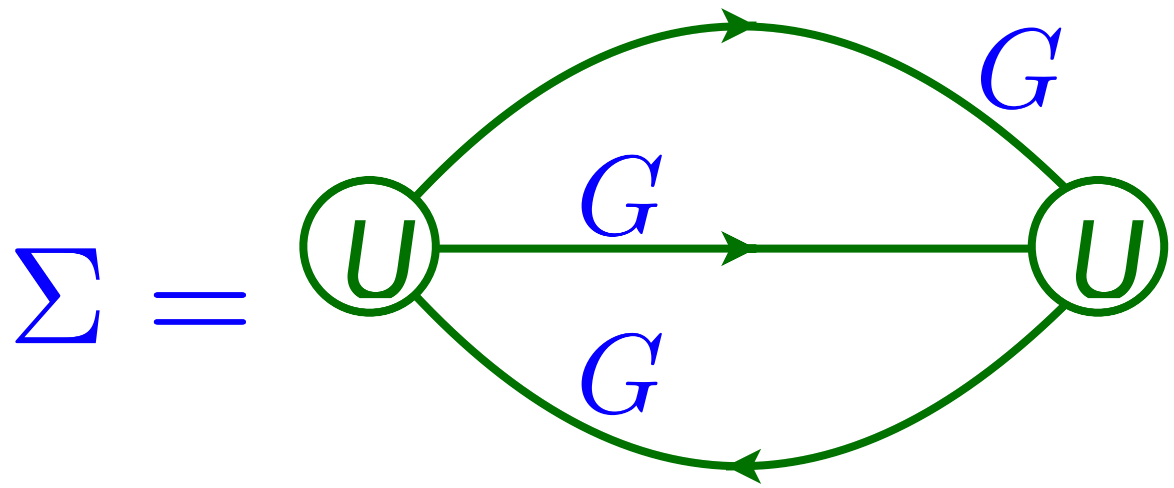
S. Sachdev and J. Ye, PRL **70**, 3339 (1993)

A. Kitaev, unpublished; S. Sachdev, PRX **5**, 041025 (2015)

The complex SYK model

Feynman graph expansion in $U_{\alpha\beta;\gamma\delta}$, and graph-by-graph average, yields exact equations in the large N limit:

$$\boxed{\begin{gathered} G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = -U^2 G^2(\tau) G(-\tau) \\ G(\tau = 0^-) = \mathcal{Q}. \end{gathered}}$$



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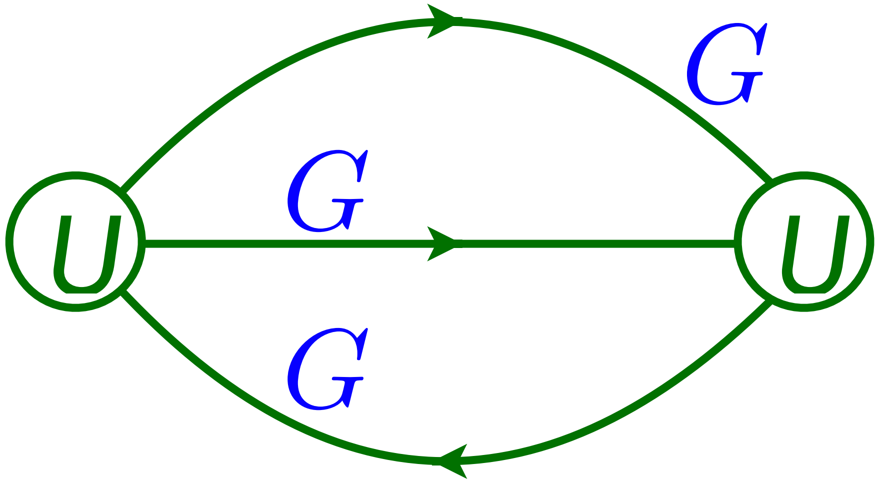
$\Sigma =$ 

- Solution for $1/|\tau|, T \ll U$ has a conformal structure:

$$G(\tau) = -A \frac{e^{-2\pi\mathcal{E}T\tau}}{\sqrt{1+e^{-4\pi\mathcal{E}}}} \left(\frac{T}{\sin(\pi T\tau)}\right)^{1/2} ,$$

where the 'particle-hole asymmetry' is determined by $\mathcal{E}$ which is a universal known function of $\mathcal{Q}$

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A. Georges and O. Parcollet PRB **59**, 5341 (1999)

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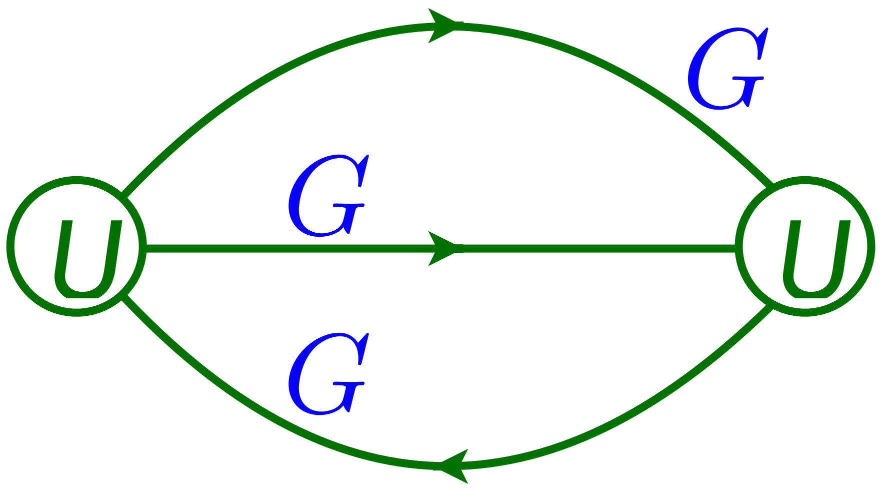
$\Sigma =$ 

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$$\Sigma(\omega) = U^{1/2} T^{1/2} \Phi\left(\frac{\omega}{T}\right),$$

$\Phi(z)$: universal scaling function.
Typical decay time of excitations $\sim T$, and independent of U: absence of quasiparticles.

G-Σ path integral

After introducing replicas $a = 1 \ldots n$, and integrating out the disorder, the partition function can be written as

$$
\begin{aligned}
Z = \int \mathcal{D}c_{ia}(\tau) \exp \Bigg[& - \sum_{ia} \int_0^\beta d\tau \, c_{ia}^\dagger \left(\frac{\partial}{\partial \tau} - \mu \right) c_{ia} \\
& - \frac{U^2}{4N^3} \sum_{ab} \int_0^\beta d\tau d\tau' \left| \sum_i c_{ia}^\dagger(\tau) c_{ib}(\tau') \right|^4 \Bigg] .
\end{aligned}
$$

For simplicity, we neglect the replica indices, and introduce the identity

$$
\begin{aligned}
1 = \int \mathcal{D}G(\tau_1, \tau_2) \mathcal{D}\Sigma(\tau_1, \tau_2) \exp \Bigg[& -N \int_0^\beta d\tau_1 d\tau_2 \Sigma(\tau_1, \tau_2) \Bigg(G(\tau_2, \tau_1) \\
& + \frac{1}{N} \sum_i c_i(\tau_2) c_i^\dagger(\tau_1) \Bigg) \Bigg] .
\end{aligned}
$$

G-Σ path integral

Then the partition function can be written as a path integral with an action S analogous to a Luttinger-Ward functional

$$
\begin{aligned}
Z &= \int \mathcal{D}G(\tau_1, \tau_2)\mathcal{D}\Sigma(\tau_1, \tau_2) \exp(-NS) \\
S &= \ln\det\left[\delta(\tau_1 - \tau_2)(\partial_{\tau_1} + \mu) - \Sigma(\tau_1, \tau_2)\right] \\
&\quad + \int d\tau_1 d\tau_2 \left[\Sigma(\tau_1, \tau_2)G(\tau_2, \tau_1) + (U^2/2)G^2(\tau_2, \tau_1)G^2(\tau_1, \tau_2)\right]
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$$Z = \int \mathcal{D}G(\tau_1,\tau_2)\mathcal{D}\Sigma(\tau_1,\tau_2)\exp(-NS)$$

$$S = \ln\det\left[\cancel{\delta(\tau_1-\tau_2)(\partial_{\tau_1}+\mu)} - \Sigma(\tau_1,\tau_2)\right]$$

$$+ \int d\tau_1 d\tau_2 \left[\Sigma(\tau_1,\tau_2)G(\tau_2,\tau_1) + (U^2/2)G^2(\tau_2,\tau_1)G^2(\tau_1,\tau_2)\right]$$

At frequencies $\ll U$, the time derivative in the determinant is less important, and without it the path integral is invariant under the reparametrization and gauge transformations

A. Georges and O. Parcollet
PRB **59**, 5341 (1999)
A. Kitaev, 2015
S. Sachdev, PRX **5,** 041025 (2015)

$$\tau = f(\sigma)$$

$$G(\tau_1,\tau_2) = [f'(\sigma_1)f'(\sigma_2)]^{-1/4}\,\frac{g(\sigma_1)}{g(\sigma_2)}\,\widetilde{G}(\sigma_1,\sigma_2)$$

$$\Sigma(\tau_1,\tau_2) = [f'(\sigma_1)f'(\sigma_2)]^{-3/4}\,\frac{g(\sigma_1)}{g(\sigma_2)}\,\widetilde{\Sigma}(\sigma_1,\sigma_2)$$

where $f(\sigma)$ and $g(\sigma)$ are arbitrary functions.

G-Σ path integral

Reparametrization and phase zero modes

We can write the path integral for the SYK model as

$$\mathcal{Z} = \int \mathcal{D}G(\tau_1, \tau_2)\mathcal{D}\Sigma(\tau_1, \tau_2) e^{-NS[G,\Sigma]}$$

for a known action $S[G, \Sigma]$. We find the saddle point, G_s, Σ_s, and only focus on the "Nambu-Goldstone" modes associated with breaking reparameterization and U(1) gauge symmetries by writing

$$G(\tau_1, \tau_2) = [f'(\tau_1)f'(\tau_2)]^{1/4} G_s(f(\tau_1) - f(\tau_2)) e^{i\phi(\tau_1) - i\phi(\tau_2)}$$

(and similarly for Σ). Then the path integral is approximated by

$$\mathcal{Z} = \int \mathcal{D}f(\tau)\mathcal{D}\phi(\tau) e^{-E_0/T + Ns_0 - NS_{\text{eff}}[f,\phi]},$$

where $E_0 \propto N$ is the ground state energy.

J. Maldacena and D. Stanford, arXiv:1604.07818;
R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, arXiv.1612.00849;
S. Sachdev, PRX **5,** 041025 (2015); J. Maldacena, D. Stanford, and Zhenbin Yang, arXiv:1606.01857;
K. Jensen, arXiv:1605.06098; J. Engelsoy, T.G. Mertens, and H. Verlinde, arXiv:1606.03438

G-Σ path integral

Symmetry arguments, and explicit computations, show that the effective action is

$$\boxed{S_{\rm eff}[f,\phi] = \frac{NK}{2}\int_0^{1/T} d\tau(\partial_\tau\phi + i(2\pi\mathcal{E}T)\partial_\tau f)^2 - \frac{N\gamma}{4\pi^2}\int_0^{1/T} d\tau\, \{\tan(\pi T f(\tau)),\tau\},}$$

where $f(\tau)$ is a monotonic map from $[0,1/T]$ to $[0,1/T]$, the couplings K, γ, and $\mathcal{E}$ can be related to thermodynamic derivatives and we have used the Schwarzian:

$$\{g,\tau\} \equiv \frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2 .$$

Specifically, an argument constraining the effective at $T=0$ is

$$S_{\rm eff}\left[f(\tau) = \frac{a\tau+b}{c\tau+d}, \phi(\tau)=0\right] = 0,$$

and this is origin of the Schwarzian.

J. Maldacena and D. Stanford, arXiv:1604.07818;
R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, PRB **95**, 155131 (2017);
A. Gaikwad, L.K. Joshi, G. Mandal, and S.R. Wadia, arXiv:1802.07746

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The fluctuations of ϕ lead to a breakdown of scaling at a time $t \sim N/U$.

Bagrets, Altland, Kamenev (2017)

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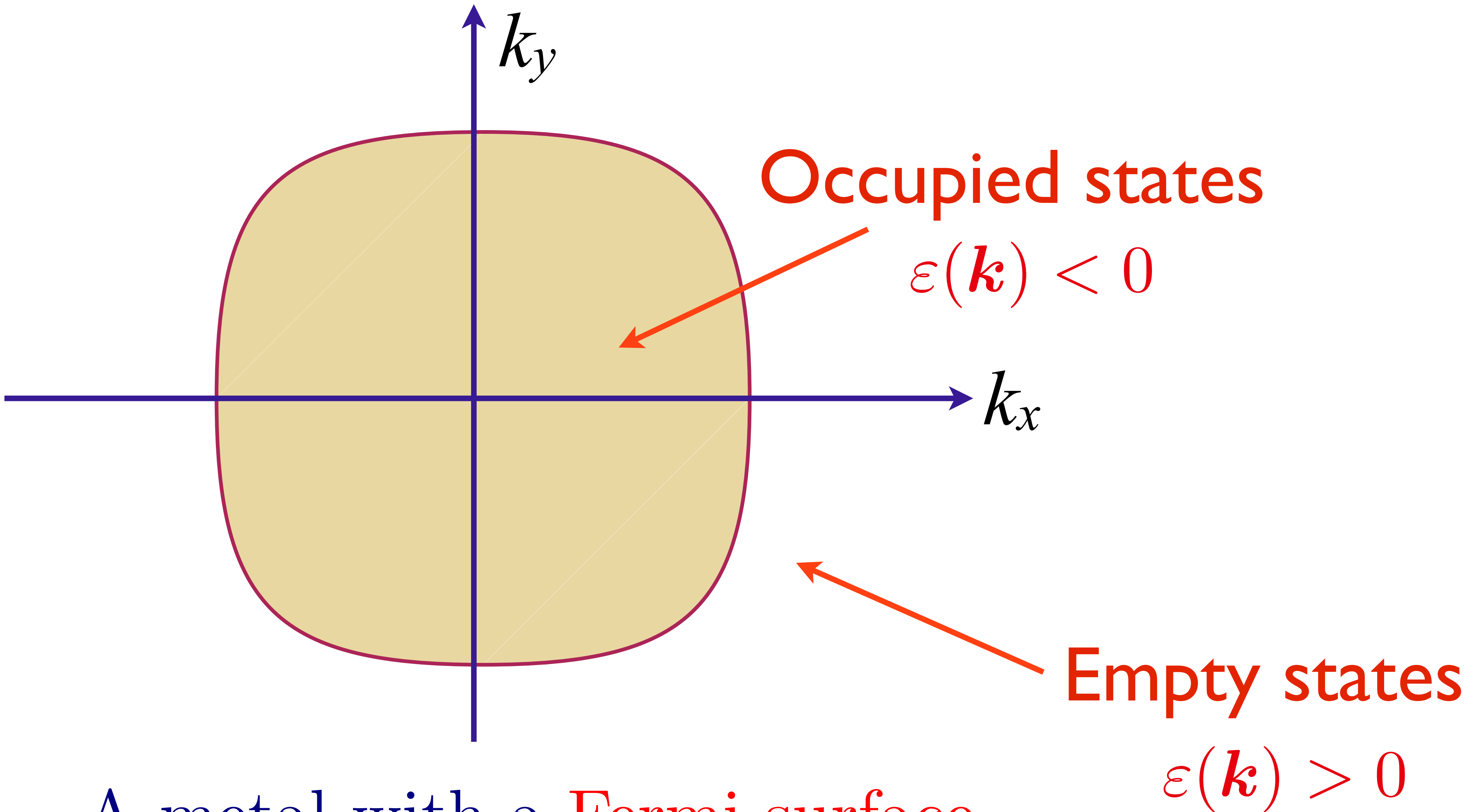
2. Critical Fermi surfaces: review

3. Critical Fermi surfaces: large N theory

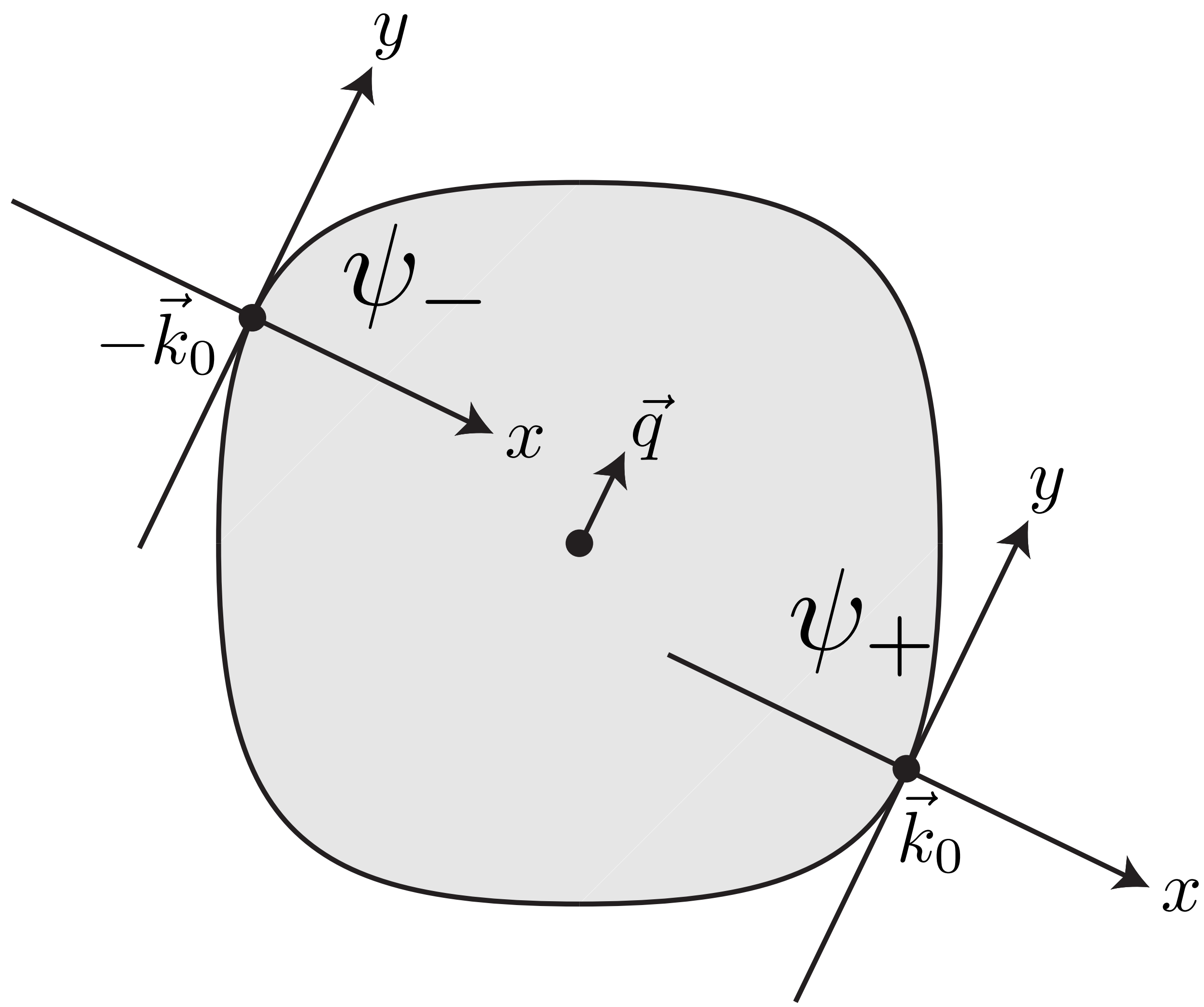
Fermi surface coupled to a gauge field



A metal with a Fermi surface
minimally coupled to a gauge field $\boldsymbol{A}$

$$\mathcal{L} = c_{\boldsymbol{k}}^{\dagger} \left(\frac{\partial}{\partial \tau} + \varepsilon(-i\boldsymbol{\nabla} - g\boldsymbol{A}) - \mu \right) c_{\boldsymbol{k}} + \frac{1}{2}(\boldsymbol{\nabla} \times \boldsymbol{A})^2$$

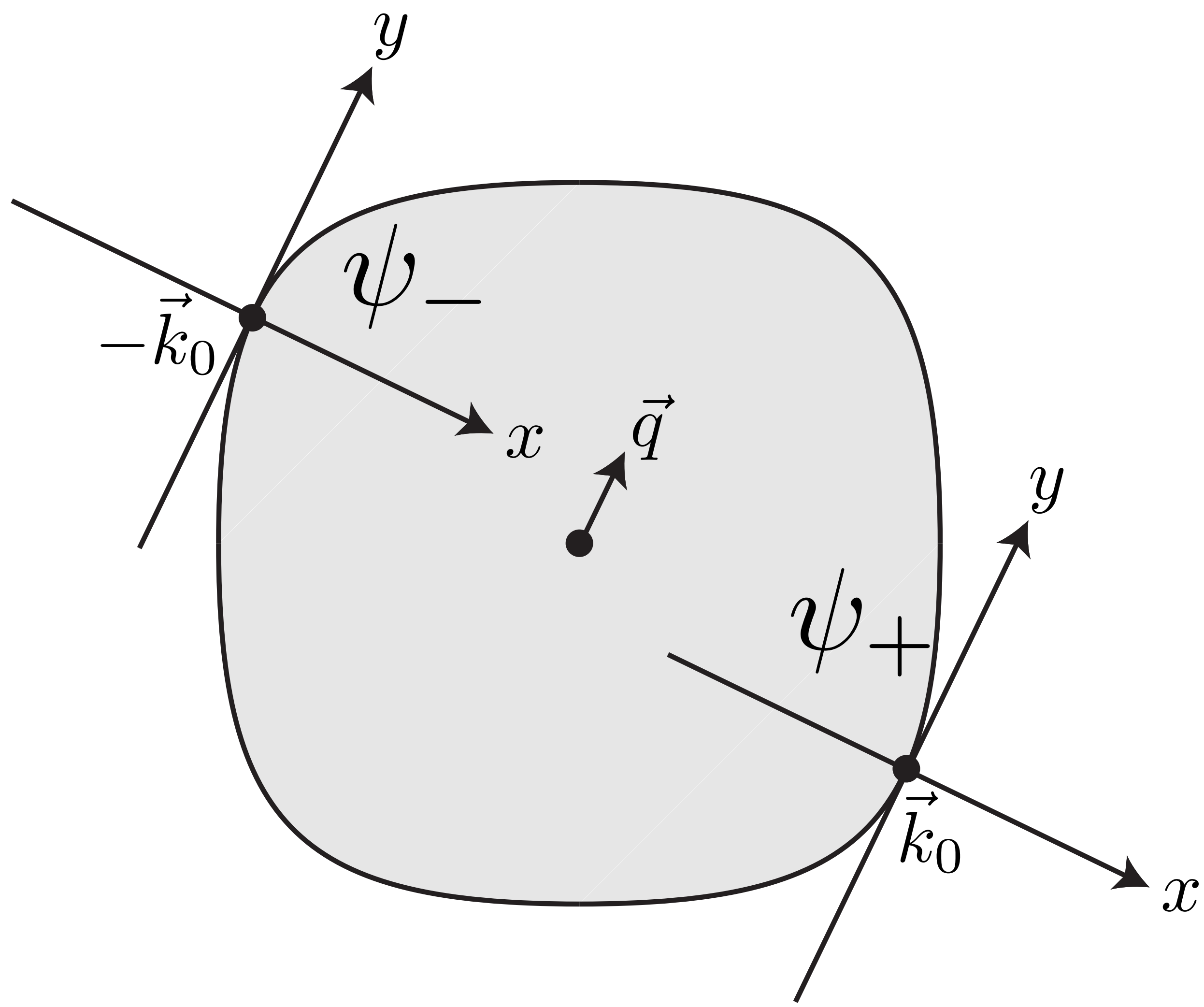
Fermi surface coupled to a gauge field



- Gauge fluctuation at wavevector $\boldsymbol{q}$ couples most efficiently to fermions near $\pm\boldsymbol{k}_0$.

- Expand fermion kinetic energy at wavevectors about $\pm\boldsymbol{k}_0$. In Landau gauge $\boldsymbol{A} = (a, 0)$.

Fermi surface coupled to a gauge field



$$\mathcal{L}[\psi_\pm, a] =$$

$$\psi_+^\dagger \left(\partial_\tau - i\partial_x - \partial_y^2\right)\psi_+ + \psi_-^\dagger \left(\partial_\tau + i\partial_x - \partial_y^2\right)\psi_-$$

$$-g\, a \left(\psi_+^\dagger \psi_+ - \psi_-^\dagger \psi_-\right) + \frac{1}{2}\left(\partial_y a\right)^2$$

M. A. Metlitski and S. Sachdev,
Phys. Rev. B **82**, 075127 (2010)

Fermi surface coupled to a gauge field

$$\mathcal{L} = \psi_+^\dagger \left(\partial_\tau - i\partial_x - \partial_y^2\right)\psi_+ + \psi_-^\dagger \left(\partial_\tau + i\partial_x - \partial_y^2\right)\psi_- - g\,a\left(\psi_+^\dagger\psi_+ - \psi_-^\dagger\psi_-\right) + \frac{1}{2}\left(\partial_y a\right)^2$$

Simple scaling argument for $z = 3/2$.

Under the rescaling $x \to x/s$, $y \to y/s^{1/2}$, and $\tau \to \tau/s^z$, we find invariance provided

$$\begin{aligned} a &\to a\, s^{(2z+1)/4} \\ \psi &\to \psi\, s^{(2z+1)/4} \\ g &\to g\, s^{(3-2z)/4} \end{aligned}$$

So the action is invariant provided $z = 3/2$.

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\mathcal{L} &= \psi_+^\dagger \left(\cancel{\partial_\tau} - i\partial_x - \partial_y^2\right)\psi_+ + \psi_-^\dagger \left(\cancel{\partial_\tau} + i\partial_x - \partial_y^2\right)\psi_- \\
&\quad - g\, a \left(\psi_+^\dagger \psi_+ - \psi_-^\dagger \psi_-\right) + \frac{1}{2}\left(\partial_y a\right)^2
\end{aligned}
$$

Because the bare time derivatives are irrelevant, the critical theory has an emergent time reparameterization symmetry:

$$
\begin{aligned}
\tau &\to f(\tau) \\
d\tau &\to f'(\tau)\, d\tau \\
x &\to [f'(\tau)]^{1/z}\, x \\
y &\to [f'(\tau)]^{1/(2z)}\, y \\
a &\to [f'(\tau)]^{-(2z+1)/(4z)}\, a \\
\psi &\to [f'(\tau)]^{-(2z+1)/(4z)}\, \psi
\end{aligned}
$$

Fermi surface coupled to a gauge field

$$\mathcal{L} = \psi_+^\dagger \left(\cancel{\partial_\tau} - i\partial_x - \partial_y^2\right)\psi_+ + \psi_-^\dagger\left(\cancel{\partial_\tau} + i\partial_x - \partial_y^2\right)\psi_- - g\, a\left(\psi_+^\dagger\psi_+ - \psi_-^\dagger\psi_-\right) + \frac{1}{2}\left(\partial_y a\right)^2$$

Various uncontrolled computations show

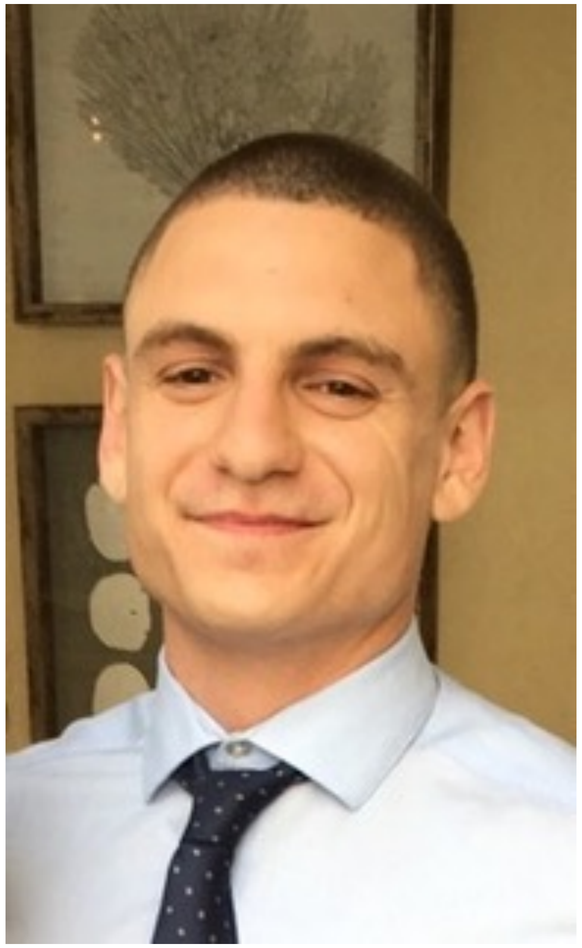
$$G(\boldsymbol{k}, i\omega) = \frac{1}{i\omega - v_F(|\boldsymbol{k}| - k_F) + i\#\, g^{4/3}\mathrm{sgn}(\omega)|\omega|^{2/3}}$$

Needed: systematic theory, which can also address possible pairing and density-wave instabilities, and the role of time-reparameterizations

1. The complex SYK model

2. Critical Fermi surfaces: review

3. Critical Fermi surfaces: large N theory



Ilya Esterlis

Aavishkar Patel

Haoyu Guo

Ilya Esterlis, Haoyu Guo, Aavishkar Patel, S.S. arXiv: 2103.08615

Large N theory of a critical Fermi surface

N flavors of fermions $\psi_{\pm\alpha}$,
M flavors of a boson a_α, and
a "Yukawa coupling" $g_{\alpha\beta\gamma}$ which is a random function in flavor space. Note: there is *no spatial randomness*. Take the large N limit with M/N fixed.

$$\begin{aligned}
\mathcal{L} &= \psi^\dagger_{+\alpha}\left(\partial_\tau - i\partial_x - \partial_y^2\right)\psi_{+\alpha} + \psi^\dagger_{-\alpha}\left(\partial_\tau + i\partial_x - \partial_y^2\right)\psi_{-\alpha} \\
&\quad - \frac{g_{\alpha\beta\gamma}}{N} a_\alpha \left(\eta_{+\alpha}\psi^\dagger_{+\beta}\psi_{+\gamma} + \eta_{-\alpha}\psi^\dagger_{-\beta}\psi_{-\gamma}\right) + \frac{1}{2}\left(\partial_y a_\alpha\right)^2
\end{aligned}$$

$\eta_{\pm\alpha} = \pm 1$ depending upon nature of a_α: gauge group, Higgs field, order parameter ....

$$\overline{g_{\alpha\beta\gamma}} = 0 \quad , \quad \overline{|g_{\alpha\beta\gamma}|^2} = g^2$$

Large N theory of a critical Fermi surface

We can now proceed just as in the SYK model: we obtain a theory for Green's functions which are bilocal in both space and time. Using the spacetime co-ordinate $X \equiv (\tau, x, y)$, we can write the averaged partition function

$$\overline{\mathcal{Z}}_{\psi\phi} = \int \mathcal{D}G(X_1, X_2)\mathcal{D}\Sigma(X_1, X_2)\mathcal{D}D(X_1, X_2) \times \mathcal{D}\Pi(X_1, X_2) \exp\left[-N I(G, \Sigma, D, \Pi)\right] .$$

The G-Σ-D-Π action is now

$$\begin{aligned} I(G, \Sigma, D, \Pi) &= \frac{g^2}{2}\mathrm{Tr}\left(G \cdot [GD]\right) - \mathrm{Tr}(G \cdot \Sigma) + \frac{1}{2}\mathrm{Tr}(D \cdot \Pi) \\ &- \ln\det\left[\left(\partial_{\tau_1} - i\partial_{x_1} - \partial_{y_1}^2\right)\delta(X_1 - X_2) + \Sigma(X_1, X_2)\right] \\ &+ \frac{1}{2}\ln\det\left[\left(-K\partial_{y_1}^2\right)\delta(X_1 - X_2) - \Pi(X_1, X_2)\right] . \end{aligned}$$

where we have introduced notation

$$\mathrm{Tr}\left(f \cdot g\right) \equiv \int dX_1 dX_2\, f(X_2, X_1) g(X_1, X_2) .$$

Large N theory of a critical Fermi surface

Saddle-point equations

$$G(\boldsymbol{k}, i\omega) = \frac{1}{i\omega - k_x - k_y^2 - \Sigma(\boldsymbol{k}, i\omega)}, \quad D(\boldsymbol{k}, i\omega) = \frac{1}{k_y^2 - \Pi(\boldsymbol{k}, i\omega)}$$

$$\Sigma(\boldsymbol{r}, \tau) = g^2 D(\boldsymbol{r}, \tau) G(\boldsymbol{r}, \tau), \quad \Pi(\boldsymbol{r}, \tau) = -g^2 G(-\boldsymbol{r}, -\tau) G(\boldsymbol{r}, \tau)$$

Large N theory of a critical Fermi surface

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Exact solution at low energies:

$$\Sigma(\boldsymbol{k}, \omega) = g^{4/3} T^{2/3} \Phi\left(\frac{\omega}{T}\right),$$

where $\Phi(z)$ is a universal scaling function, obtained by analytical continuation from imaginary Matsubara frequencies $\omega_n = (2n-1)\pi T$

$$\begin{aligned}
\Phi\left(\frac{i\omega_n}{T}\right) &= -i\,\mathrm{sgn}(\omega_n)\frac{2^{5/3}}{3\sqrt{3}} H_{1/3}\left(\frac{|\omega_n| - \pi T}{2\pi T}\right) \\
H_r(n \in \mathbb{Z}^+) &= \sum_{j=1}^{n} \frac{1}{j^r}
\end{aligned}$$

Large N theory of a critical Fermi surface

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Large N theory of a critical Fermi surface

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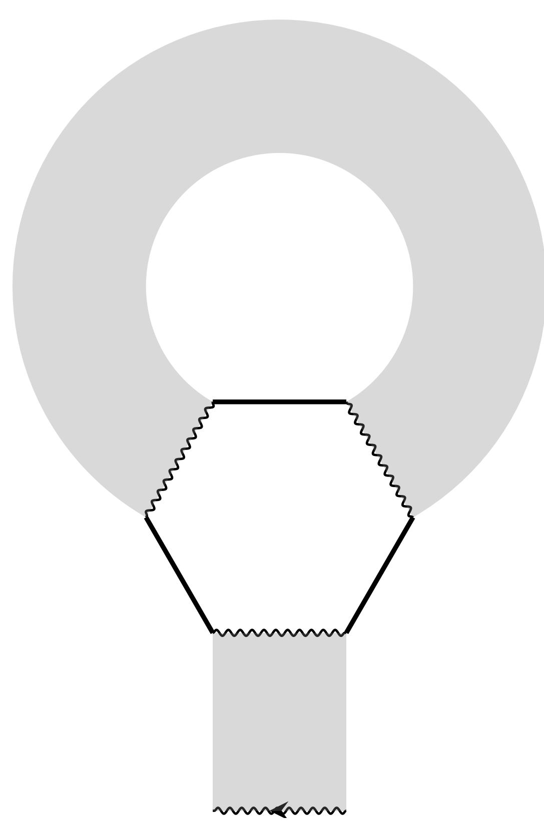
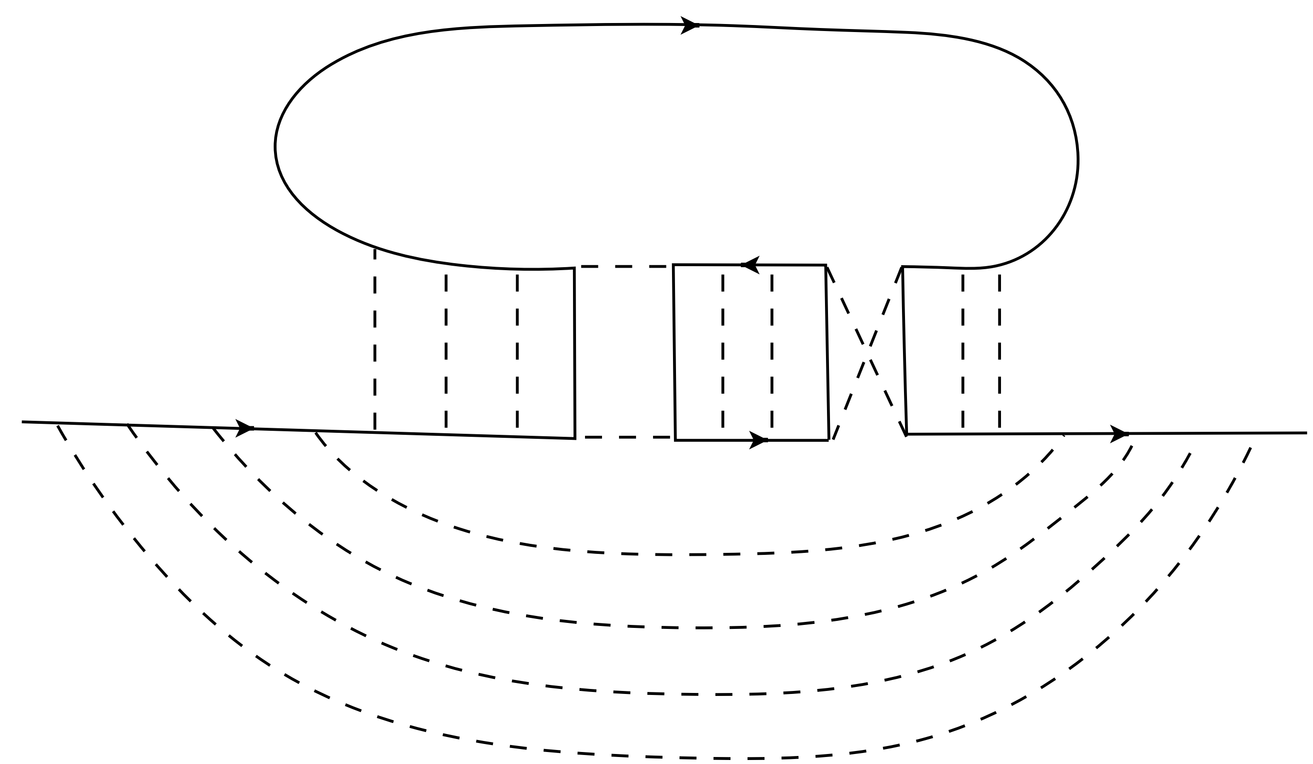
- Dynamic scaling exponent z, and fermion anomalous dimension can (in principle) have corrections at higher orders in $1/N$: these can be systematically computed in theory of fields bilocal in spacetime.





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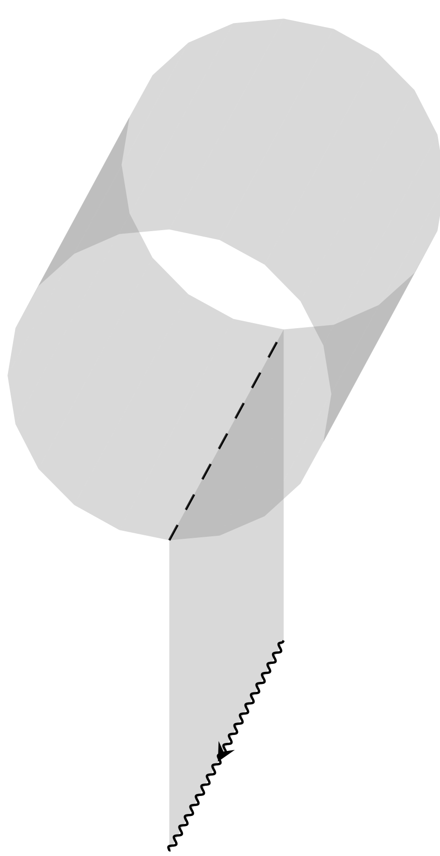

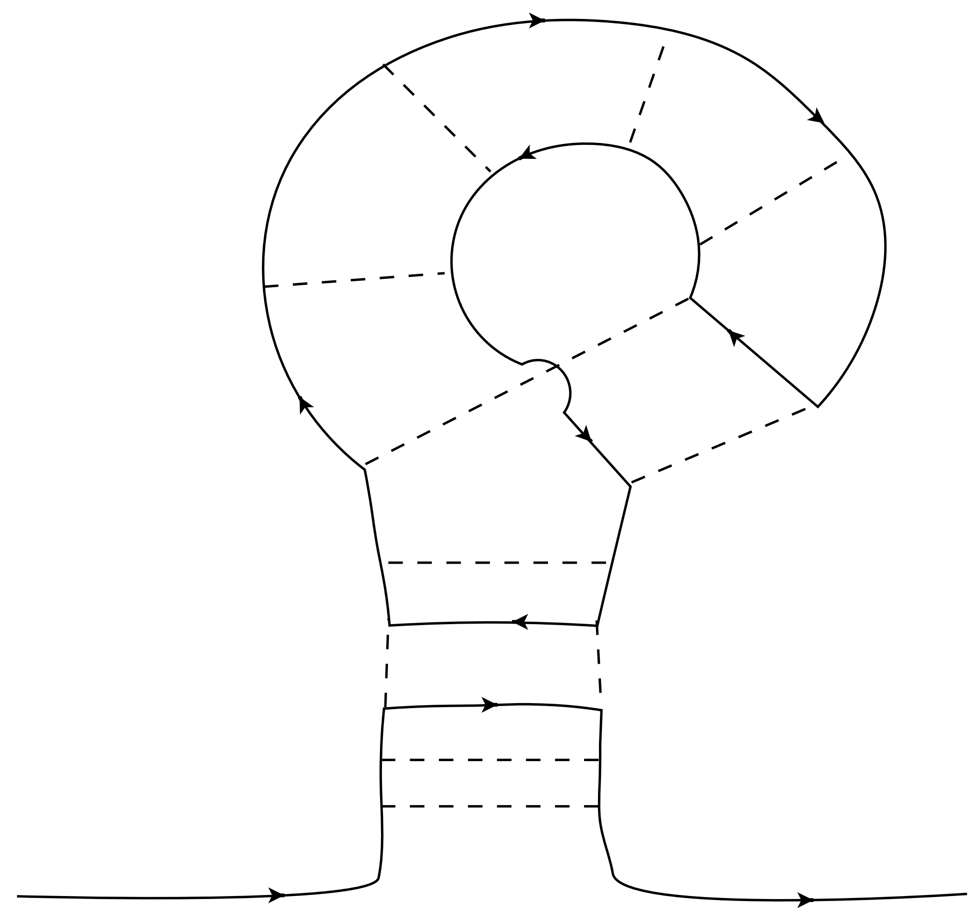

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- Can determine scaling dimensions of Cooper pair and $2k_F$ density wave operators in $N = \infty$ theory: these are universal functions of M/N, $\eta_{\pm\alpha} = \pm 1$ (and ratios of diamagnetic susceptibilities with multiple gauge fields). Complex scaling dimension implies an instability to superconductivity/density wave order.