

# Theories of non-Fermi liquids

Quantum Criticality and Topology in Itinerant Electron Systems  
Center for Nonlinear Studies, Los Alamos National Laboratory  
Albuquerque, August 15-19, 2016

Subir Sachdev

Talk online: [sachdev.physics.harvard.edu](http://sachdev.physics.harvard.edu)



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# What is a non-Fermi liquid ?

- A compressible phase at  $T = 0$ : the density  $\mathcal{Q}$  varies smoothly as a function of  $\mu$ . Global U(1) symmetry is unbroken.
- No quasiparticle excitations
- Shortest possible local-equilibration/de-phasing/transition-to-quantum-chaos with

$$\tau_{\varphi} \geq C \frac{\hbar}{k_B T}$$

S. Sachdev, *Quantum Phase Transitions*,  
Cambridge (1999)

$$\tau_L \geq \frac{1}{2\pi} \frac{\hbar}{k_B T}$$

J. Maldacena, S. H. Shenker and D. Stanford,  
arXiv:1503.01409

In Fermi liquids,  $\tau \sim 1/T^2$ ;  
in gapped systems,  $\tau \sim e^{\Delta/T}$ .

# Theories of non-Fermi liquids

- Sachdev-Ye-Kitaev (SYK) model
- Ising-nematic criticality in  $d=2$
- Spin density wave criticality in  $d=2$
- Graphene

# Theories of non-Fermi liquids

- Sachdev-Ye-Kitaev (SYK) model
- Ising-nematic criticality in  $d=2$
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- Graphene

# Infinite-range model with quasiparticles

$$H = \frac{1}{(N)^{1/2}} \sum_{i,j=1}^N t_{ij} c_i^\dagger c_j + \dots$$

$$c_i c_j + c_j c_i = 0 \quad , \quad c_i c_j^\dagger + c_j^\dagger c_i = \delta_{ij}$$

$$\frac{1}{N} \sum_i c_i^\dagger c_i = Q$$

$t_{ij}$  are independent random variables with  $\overline{t_{ij}} = 0$  and  $\overline{|t_{ij}|^2} = t^2$

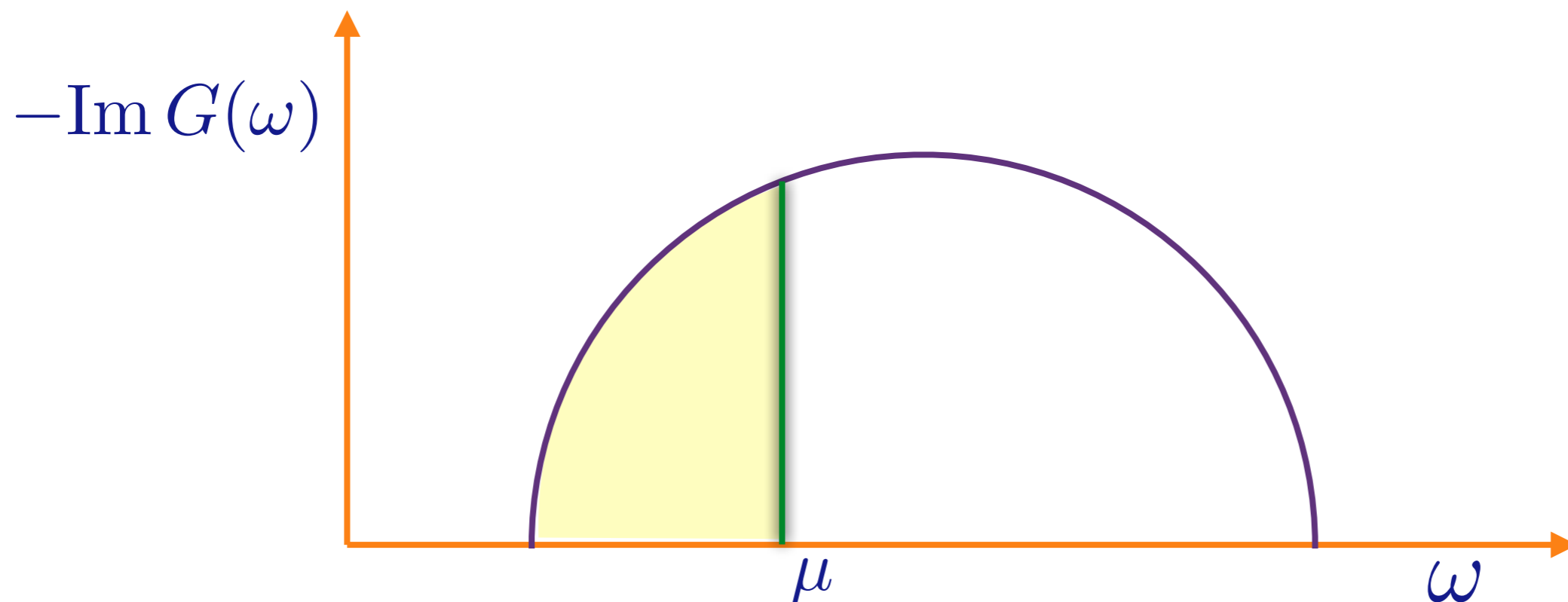
Fermions occupying the eigenstates of a  
 $N \times N$  random matrix

# Infinite-range model with quasiparticles

Feynman graph expansion in  $t_{ij..}$ , and graph-by-graph average, yields exact equations in the large  $N$  limit:

$$G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = t^2 G(\tau)$$
$$G(\tau = 0^-) = Q.$$

$G(\omega)$  can be determined by solving a quadratic equation.



# Infinite-range model with quasiparticles

Now add weak interactions

$$H = \frac{1}{(N)^{1/2}} \sum_{i,j=1}^N t_{ij} c_i^\dagger c_j + \frac{1}{(2N)^{3/2}} \sum_{i,j,k,\ell=1}^N J_{ij;kl} c_i^\dagger c_j^\dagger c_k c_\ell$$

$J_{ij;kl}$  are independent random variables with  $\overline{J_{ij;kl}} = 0$  and  $|\overline{J_{ij;kl}}|^2 = J^2$ . We compute the lifetime of a quasiparticle,  $\tau_\alpha$ , in an exact eigenstate  $\psi_\alpha(i)$  of the free particle Hamiltonian with energy  $E_\alpha$ . By Fermi's Golden rule, for  $E_\alpha$  at the Fermi energy

$$\begin{aligned} \frac{1}{\tau_\alpha} &= \pi J^2 \rho_0^3 \int dE_\beta dE_\gamma dE_\delta f(E_\beta)(1 - f(E_\gamma))(1 - f(E_\delta)) \delta(E_\alpha + E_\beta - E_\gamma - E_\delta) \\ &= \frac{\pi^3 J^2 \rho_0^3}{4} T^2 \end{aligned}$$

where  $\rho_0$  is the density of states at the Fermi energy.

Fermi liquid state: Two-body interactions lead to a scattering time of quasiparticle excitations from in (random) single-particle eigenstates which diverges as  $\sim T^{-2}$  at the Fermi level.

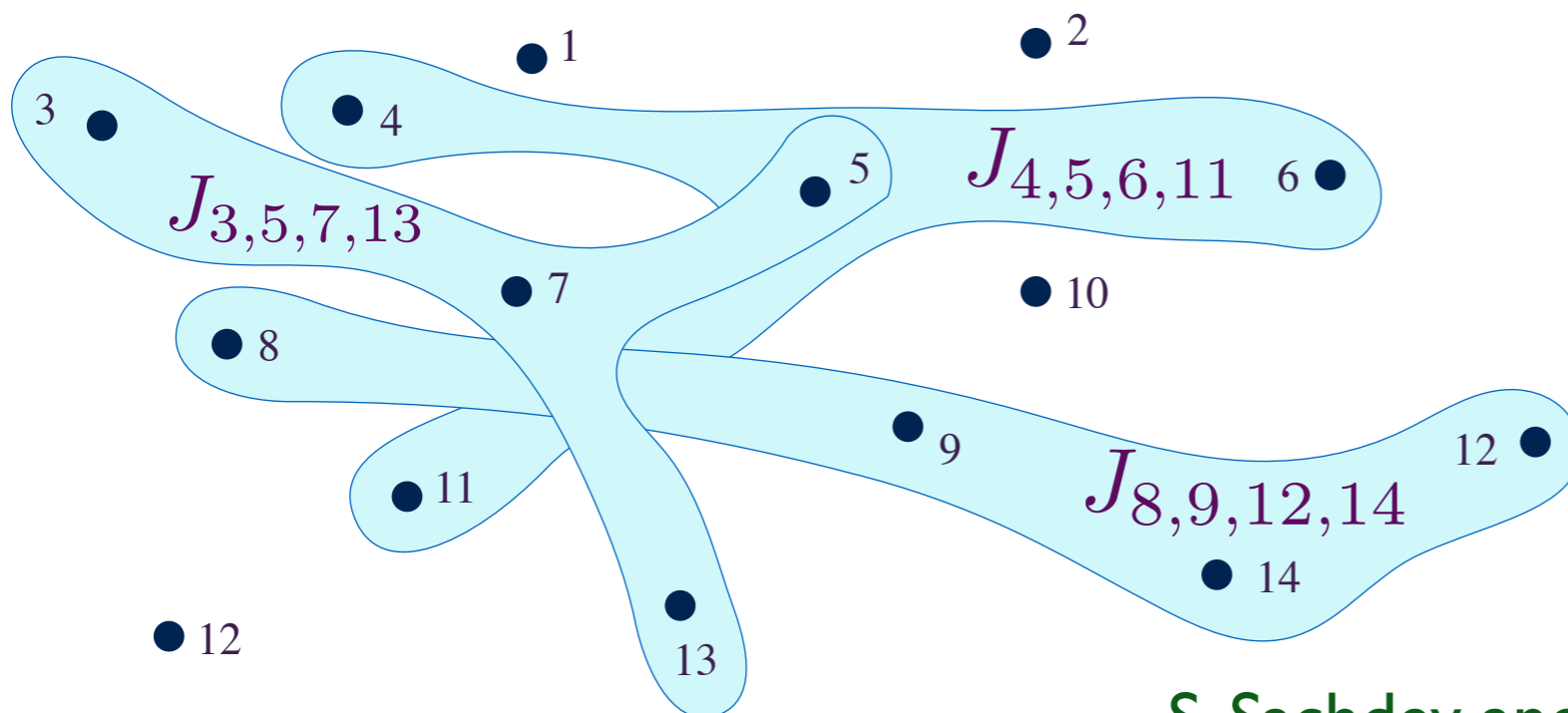
# SYK model

To obtain a non-Fermi liquid, we set  $t_{ij} = 0$ :

$$H_{\text{SYK}} = \frac{1}{(2N)^{3/2}} \sum_{i,j,k,\ell=1}^N J_{ij;kl} c_i^\dagger c_j^\dagger c_k c_\ell - \mu \sum_i c_i^\dagger c_i$$

$$Q = \frac{1}{N} \sum_i c_i^\dagger c_i$$

$H_{\text{SYK}}$  is similar, and has identical properties, to the SY model.



A fermion can move only by entangling with another fermion: the Hamiltonian has “nothing but entanglement”.

S. Sachdev and J. Ye, Phys. Rev. Lett. **70**, 3339 (1993)

A. Kitaev, unpublished; S. Sachdev, PRX **5**, 041025 (2015)

# SYK model

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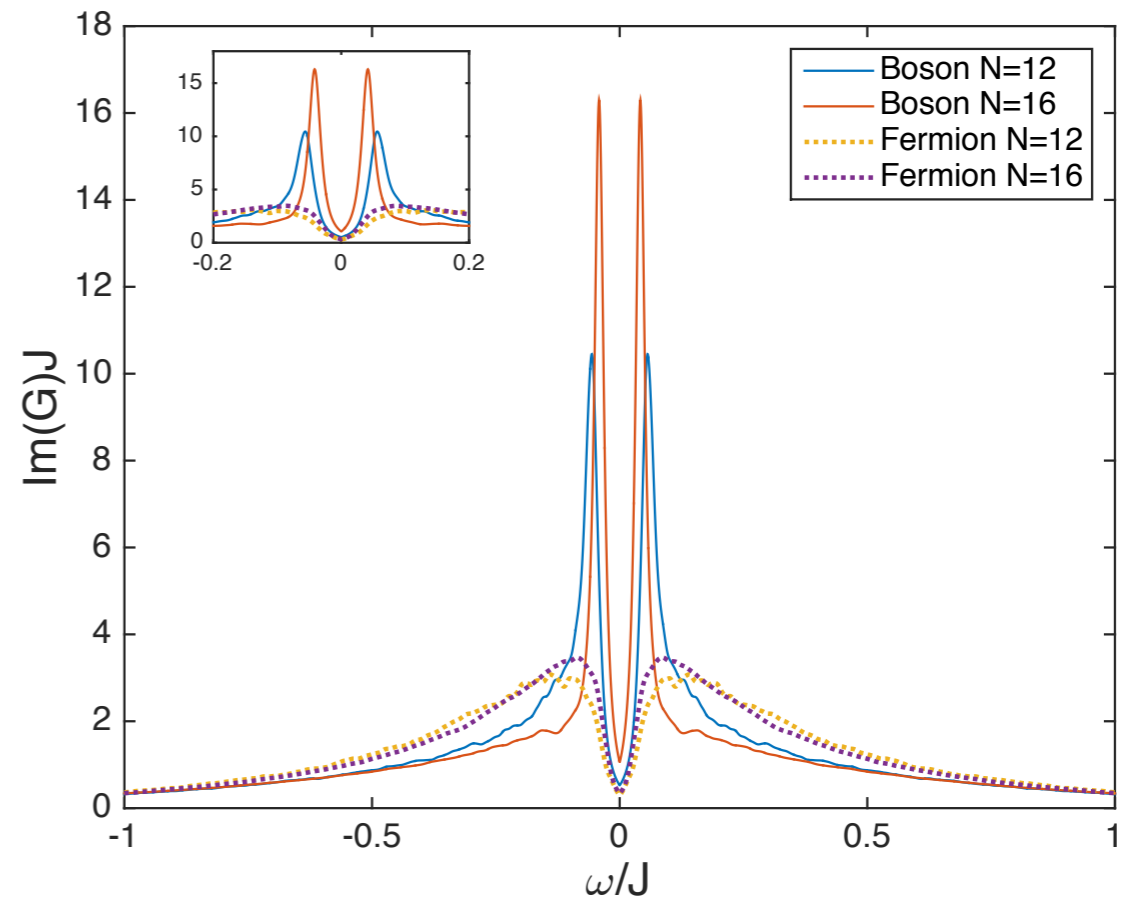
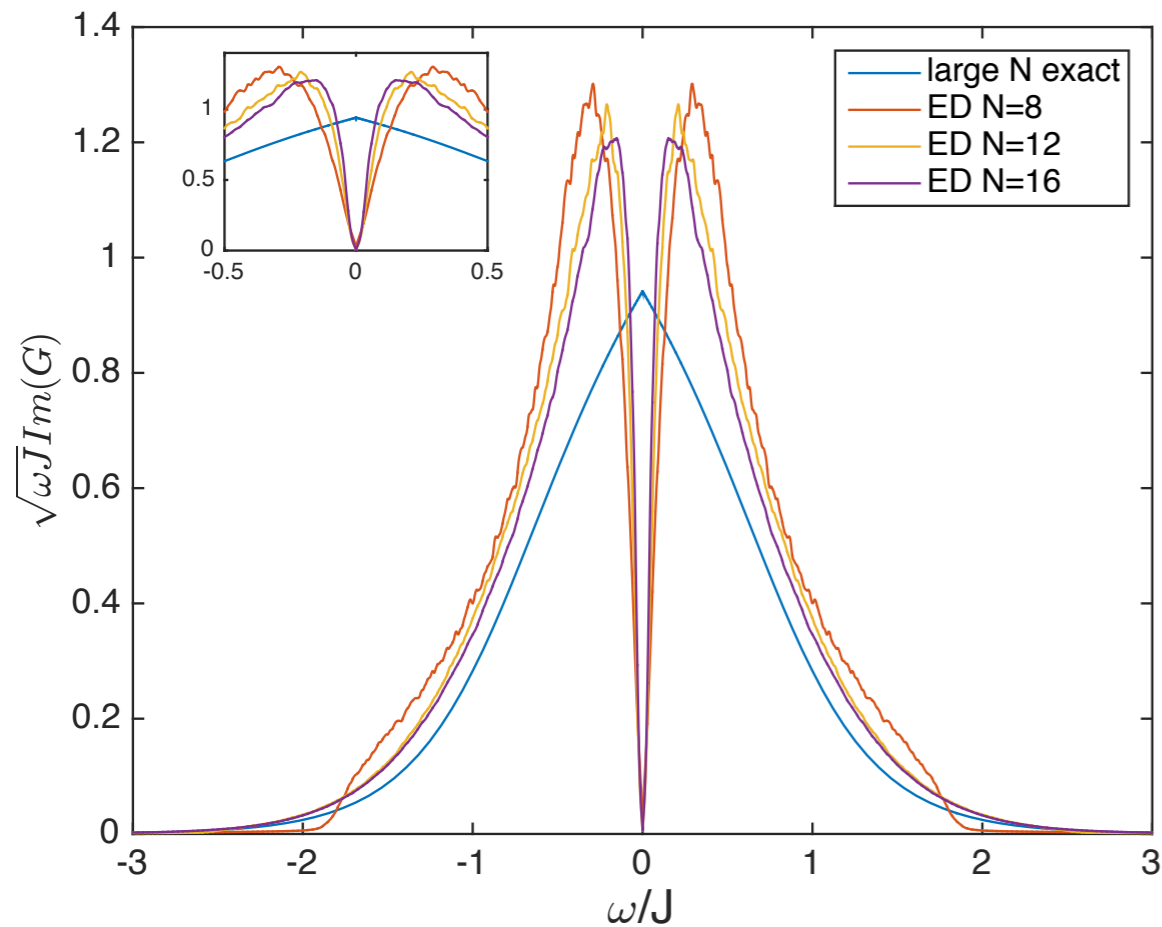
$$G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = -J^2 G^2(\tau) G(-\tau)$$
$$G(\tau = 0^-) = Q.$$

Low frequency analysis shows that the solutions must be gapless and obey

$$\Sigma(z) = \mu - \frac{1}{A} \sqrt{z} + \dots \quad , \quad G(z) = \frac{A}{\sqrt{z}}$$

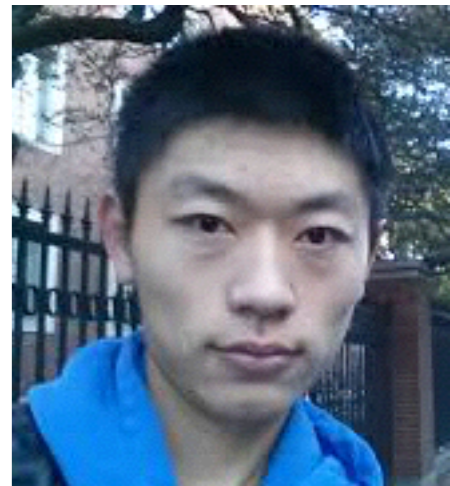
for some complex  $A$ . The ground state is a non-Fermi liquid, with a continuously variable density  $Q$ .

# SYK model



Large  $N$  solution of equations for  $G$  and  $\Sigma$  agree well with exact diagonalization of the finite  $N$  Hamiltonian  $\Rightarrow$  no spin-glass order

However, exact diagonalization of the same model with hard-core bosons indicates the presence of spin-glass order in the ground state.



# SYK model

- $T = 0$  Green's function  $G \sim 1/\sqrt{\tau}$

S. Sachdev and J. Ye, Phys. Rev. Lett. **70**, 3339 (1993)

# SYK model

- $T = 0$  Green's function  $G \sim 1/\sqrt{\tau}$
- $T > 0$  Green's function implies conformal invariance  
 $G \sim 1/(\sin(\pi T \tau))^{1/2}$

A. Georges and O. Parcollet PRB **59**, 5341 (1999)

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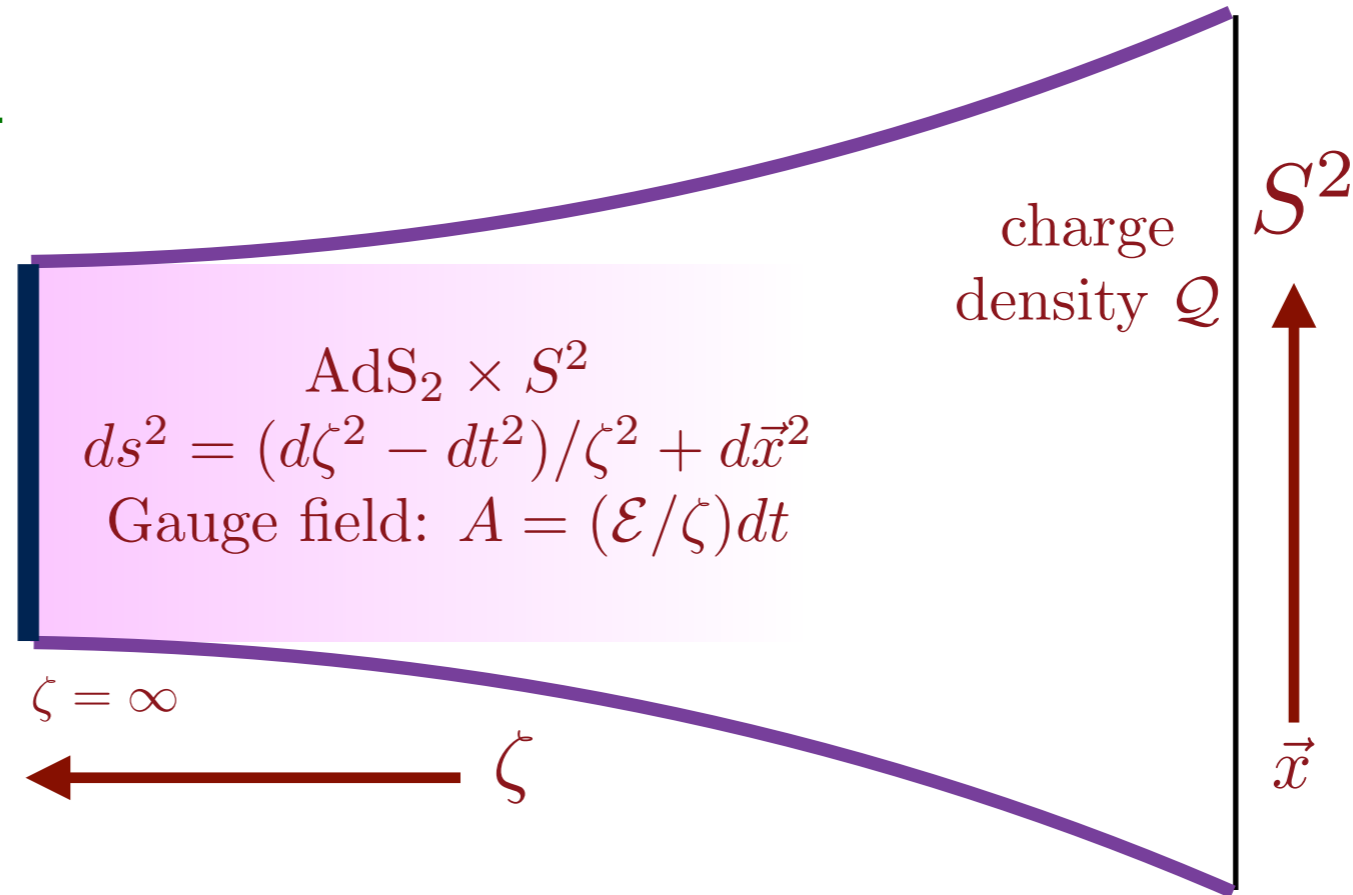
A. Georges, O. Parcollet, and S. Sachdev, Phys. Rev. B **63**, 134406 (2001)

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- Non-zero entropy as  $T \rightarrow 0$ ,  $S(T \rightarrow 0) = NS_0 + \dots$
- These features indicate that the SYK model is dual to the low energy limit of a quantum gravity theory of black holes with  $\text{AdS}_2$  near-horizon geometry. The Bekenstein-Hawking entropy is  $NS_0$ .

S. Sachdev, PRL **105**, 151602 (2010)

# SYK and AdS<sub>2</sub>



PHYSICAL REVIEW LETTERS **105, 151602 (2010)**



## Holographic Metals and the Fractionalized Fermi Liquid

Subir Sachdev

*Department of Physics, Harvard University, Cambridge, Massachusetts 02138, USA*

(Received 23 June 2010; published 4 October 2010)

We show that there is a close correspondence between the physical properties of holographic metals near charged black holes in anti-de Sitter (AdS) space, and the fractionalized Fermi liquid phase of the lattice Anderson model. The latter phase has a “small” Fermi surface of conduction electrons, along with a spin liquid of local moments. This correspondence implies that certain mean-field gapless spin liquids are states of matter at nonzero density realizing the near-horizon,  $\text{AdS}_2 \times \mathbb{R}^2$  physics of Reissner-Nordström black holes.

# SYK model

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- $T > 0$  Green's function implies conformal invariance  
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- These features indicate that the SYK model is dual to the low energy limit of a quantum gravity theory of black holes with  $\text{AdS}_2$  near-horizon geometry. The Bekenstein-Hawking entropy is  $NS_0$ .
- There is a scalar zero mode associated with the breaking of reparameterization invariance down to  $\text{SL}(2, \mathbb{R})$ . The same pattern of symmetries is present in gravity theories on  $\text{AdS}_2$ .

## SYK model

- The dependence of  $S_0$  on the density  $Q$  matches the behavior of the Wald-Bekenstein-Hawking entropy of  $\text{AdS}_2$  horizons in a large class of gravity theories.

S. Sachdev PRX 5, 041025 (2015)

## SYK model

- The dependence of  $S_0$  on the density  $\mathcal{Q}$  matches the behavior of the Wald-Bekenstein-Hawking entropy of  $\text{AdS}_2$  horizons in a large class of gravity theories.
- The scalar zero mode leads to a linear-in- $T$  specific heat

$$S(T \rightarrow 0) = NS_0 + N\gamma T + \dots$$

An identical scalar zero mode is also present in the low energy limit of theories of quantum gravity on  $\text{AdS}_2$ .

J. Maldacena and D. Stanford, arXiv:1604.07818

# SYK model

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- The Lyapunov time to quantum chaos saturates the lower bound both in the SYK model and in quantum gravity.

$$\tau_L = \frac{1}{2\pi} \frac{\hbar}{k_B T}$$

A. Kitaev, KITP talk, 2015

J. Maldacena and D. Stanford, arXiv:1604.07818

# SYK model

After integrating the fermions, the partition function can be written as a path integral with an action  $S$  analogous to a Luttinger-Ward functional

$$Z = \int \mathcal{D}G(\tau_1, \tau_2) \mathcal{D}\Sigma(\tau_1, \tau_2) \exp(-NS)$$

A. Georges, O. Parcollet, and S. Sachdev,  
Phys. Rev. B **63**, 134406 (2001)

$$S = \ln \det [\delta(\tau_1 - \tau_2)(\partial_{\tau_1} + \mu) - \Sigma(\tau_1, \tau_2)] \\ + \int d\tau_1 d\tau_2 \Sigma(\tau_1, \tau_2) [G(\tau_2, \tau_1) + (J^2/2)G^2(\tau_2, \tau_1)G^2(\tau_1, \tau_2)]$$

At frequencies  $\ll J$ , the time derivative in the determinant is less important, and without it the path integral is invariant under the reparametrization and gauge transformations

A. Georges and O. Parcollet  
PRB **59**, 5341 (1999)

A. Kitaev, unpublished  
S. Sachdev, PRX **5**, 041025 (2015)

$$\tau = f(\sigma)$$

$$G(\tau_1, \tau_2) = [f'(\sigma_1)f'(\sigma_2)]^{-1/4} \frac{g(\sigma_1)}{g(\sigma_2)} G(\sigma_1, \sigma_2)$$

$$\Sigma(\tau_1, \tau_2) = [f'(\sigma_1)f'(\sigma_2)]^{-3/4} \frac{g(\sigma_1)}{g(\sigma_2)} \Sigma(\sigma_1, \sigma_2)$$

where  $f(\sigma)$  and  $g(\sigma)$  are arbitrary functions.

# SYK model

Let us write the large  $N$  saddle point solutions of  $S$  as

$$G_s(\tau_1 - \tau_2) \sim (\tau_1 - \tau_2)^{-1/2} \quad , \quad \Sigma_s(\tau_1 - \tau_2) \sim (\tau_1 - \tau_2)^{-3/2}.$$

These are not invariant under the reparametrization symmetry but are invariant only under a  $SL(2, \mathbb{R})$  subgroup under which

$$f(\tau) = \frac{a\tau + b}{c\tau + d} \quad , \quad ad - bc = 1.$$

So the (approximate) reparametrization symmetry is spontaneously broken.

## Reparametrization zero mode

Expand about the saddle point by writing

$$G(\tau_1, \tau_2) = [f'(\tau_1)f'(\tau_2)]^{1/4} G_s(f(\tau_1) - f(\tau_2))$$

(and similarly for  $\Sigma$ ) and obtain an effective action for  $f(\tau)$ . This action does not vanish because of the time derivative in the determinant which is not reparameterization invariant.

J. Maldacena and D. Stanford, arXiv:1604.07818

See also A. Kitaev, unpublished, and J. Polchinski and V. Rosenhaus, arXiv:1601.06768

## SYK model

However the effective action must vanish for  $SL(2,R)$  transformations because  $G_s, \Sigma_s$  are invariant under it. In this manner we obtain the effective action as a Schwarzian

$$NS_{\text{eff}} = -\frac{N\gamma}{4\pi^2} \int d\tau \{f, \tau\} \quad , \quad \{f, \tau\} \equiv \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2 ,$$

where the specific heat,  $\mathcal{C} = \gamma T$ .

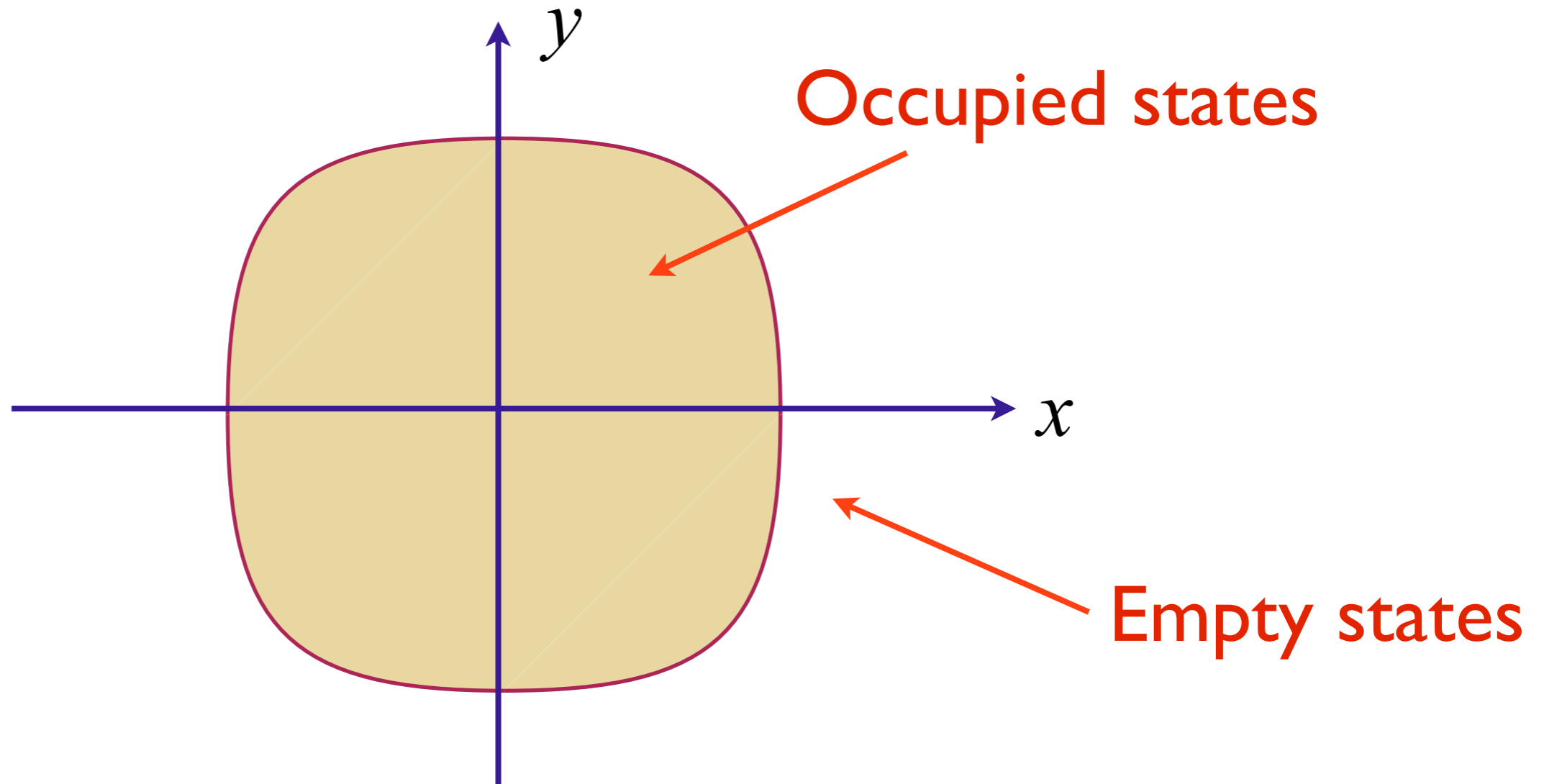
The Schwarzian effective action implies that the SYK model *saturates* the lower bound on the Lyapunov time

$$\tau_L = \frac{1}{2\pi} \frac{\hbar}{k_B T}$$

# Theories of non-Fermi liquids

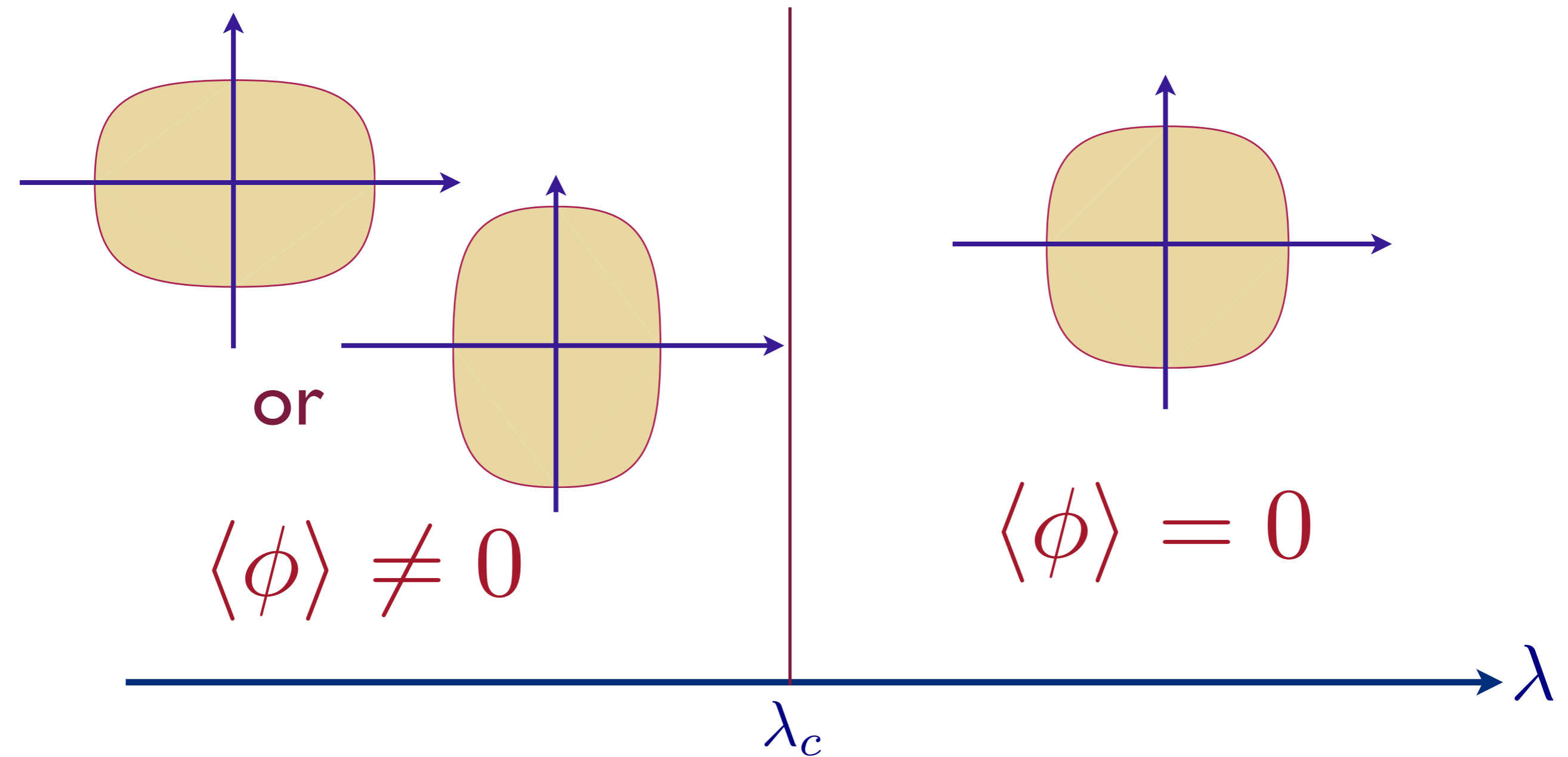
- Sachdev-Ye-Kitaev (SYK) model
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# Quantum criticality of Ising-nematic ordering in a metal



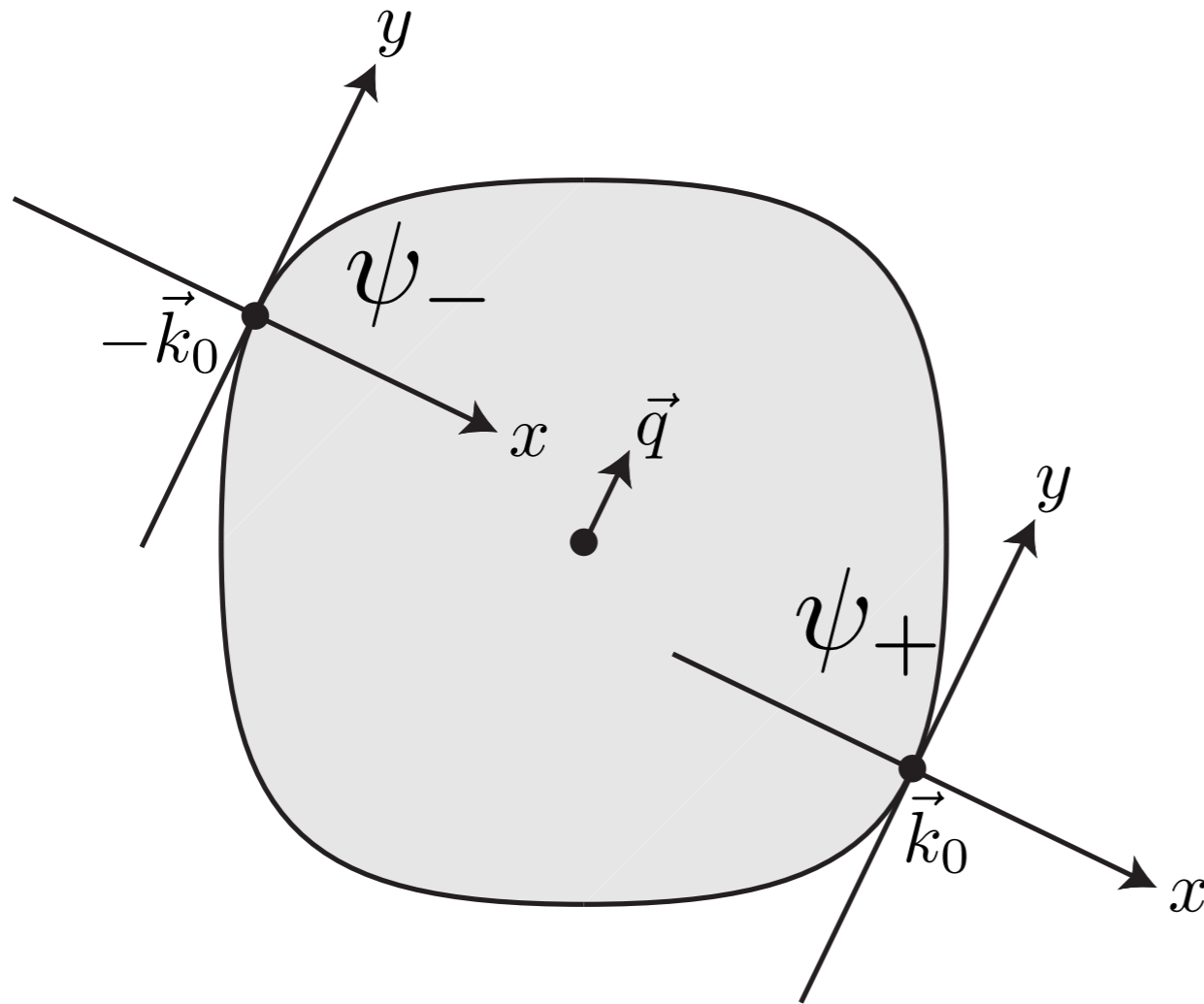
A metal with a Fermi surface  
with full square lattice symmetry

# Quantum criticality of Ising-nematic ordering in a metal



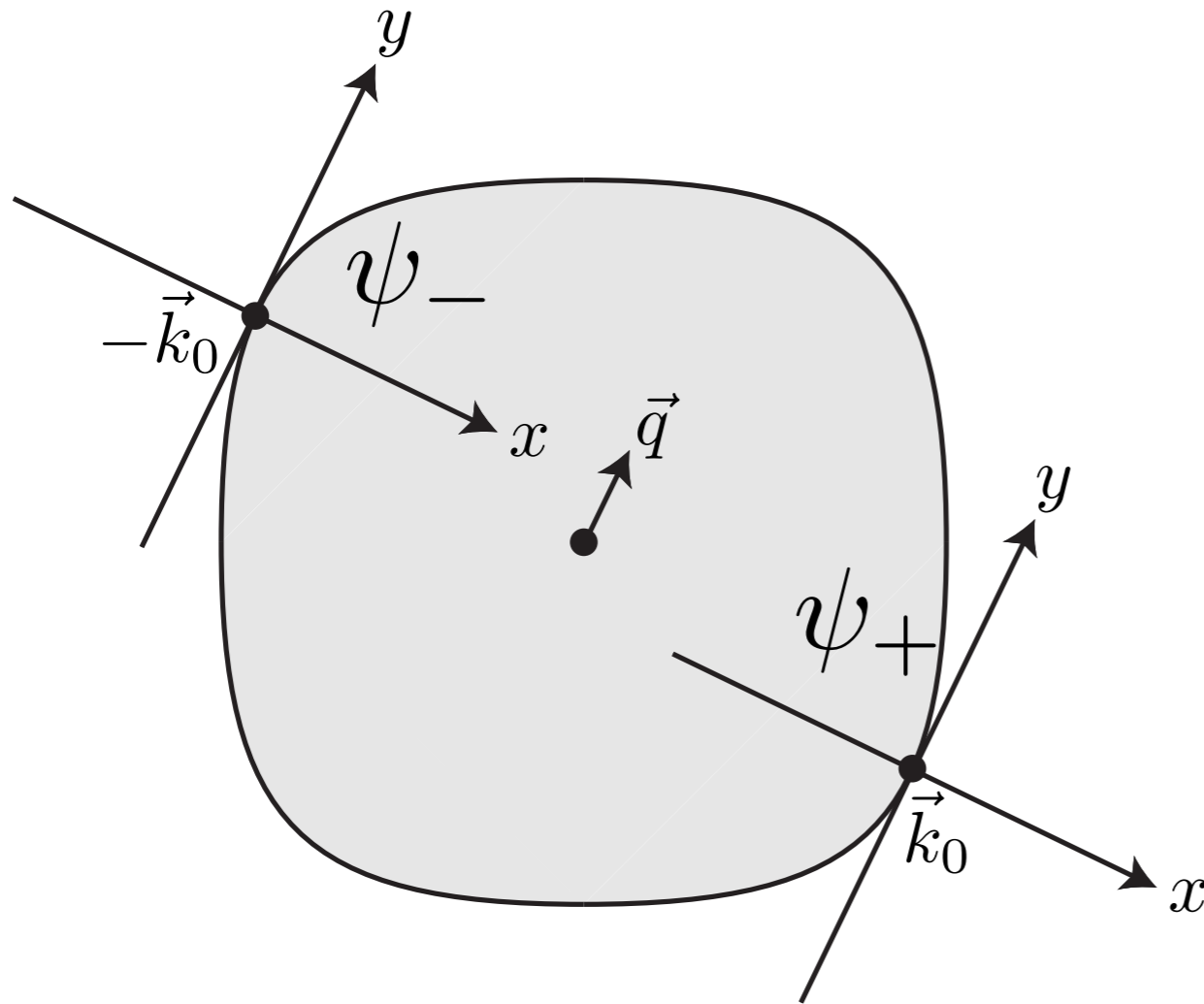
Pomeranchuk instability as a function of coupling  $\lambda$

# Quantum criticality of Ising-nematic ordering in a metal



- $\phi$  fluctuation at wavevector  $\vec{q}$  couples most efficiently to fermions near  $\pm\vec{k}_0$ .
- Expand fermion kinetic energy at wavevectors about  $\pm\vec{k}_0$  and boson ( $\phi$ ) kinetic energy about  $\vec{q} = 0$ .

# Quantum criticality of Ising-nematic ordering in a metal



- Exact solution for some exponents for the ‘one-sided’ chiral model (Shouvik Sur and Sung-Sik Lee, PRB **90**, 045121 (2014)).
- Expansion in  $\epsilon = 5/2 - d$  (D. Dalidovich and Sung-Sik Lee, PRB **88**, 245106 (2013)).

$$\mathcal{L}[\psi_{\pm}, \phi] =$$

$$\psi_+^\dagger (\partial_\tau - i\partial_x - \partial_y^2) \psi_+ + \psi_-^\dagger (\partial_\tau + i\partial_x - \partial_y^2) \psi_-$$

$$- \phi \left( \psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- \right) + \frac{1}{2g^2} (\partial_y \phi)^2$$

# Quantum criticality of Ising-nematic ordering in a metal

The entropy density,  $s$ , obeys

$$s \sim T^{(d-\theta)/z} \sim T^{2/3}$$

Y. B. Kim, A. Furusaki, X.-G. Wen, and P.A. Lee, PRB 50, 17917 (1994)

where  $z = 3/2$  is the dynamic critical exponent for fermionic excitations dispersing normal to the Fermi surface, and  $d - \theta = 1$  is the number of dimensions normal to the Fermi surface.

A RG analysis using a dimensionality expansion below  $d = 5/2$  shows that the optical conductivity obeys

$$\sigma \sim \omega^{(d-\theta-2)/z} \sim \omega^{-2/3}$$

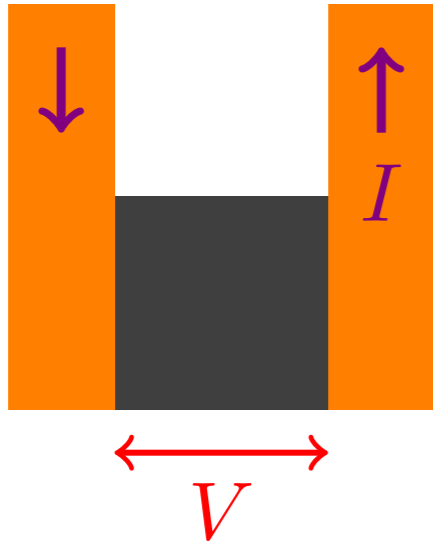
A. Eberlein, I. Mandal, and S. Sachdev, PRB 94, 045133 (2016)

We also computed the shear viscosity and found

$$\eta \sim T^{(d-\theta-2)/z} \sim T^{-2/3}$$

A.A. Patel, A. Eberlein, and S. Sachdev, arXiv:1607.03894

Note that  $\eta/s \sim T^{-2/z}$  does not scale to a constant: so for the viscosity we do not have a system of reduced dimensionality, a consequence of the anisotropic scaling between the directions normal and parallel to the Fermi surface.



$$V = IR \quad R \sim \frac{1}{\sigma}$$

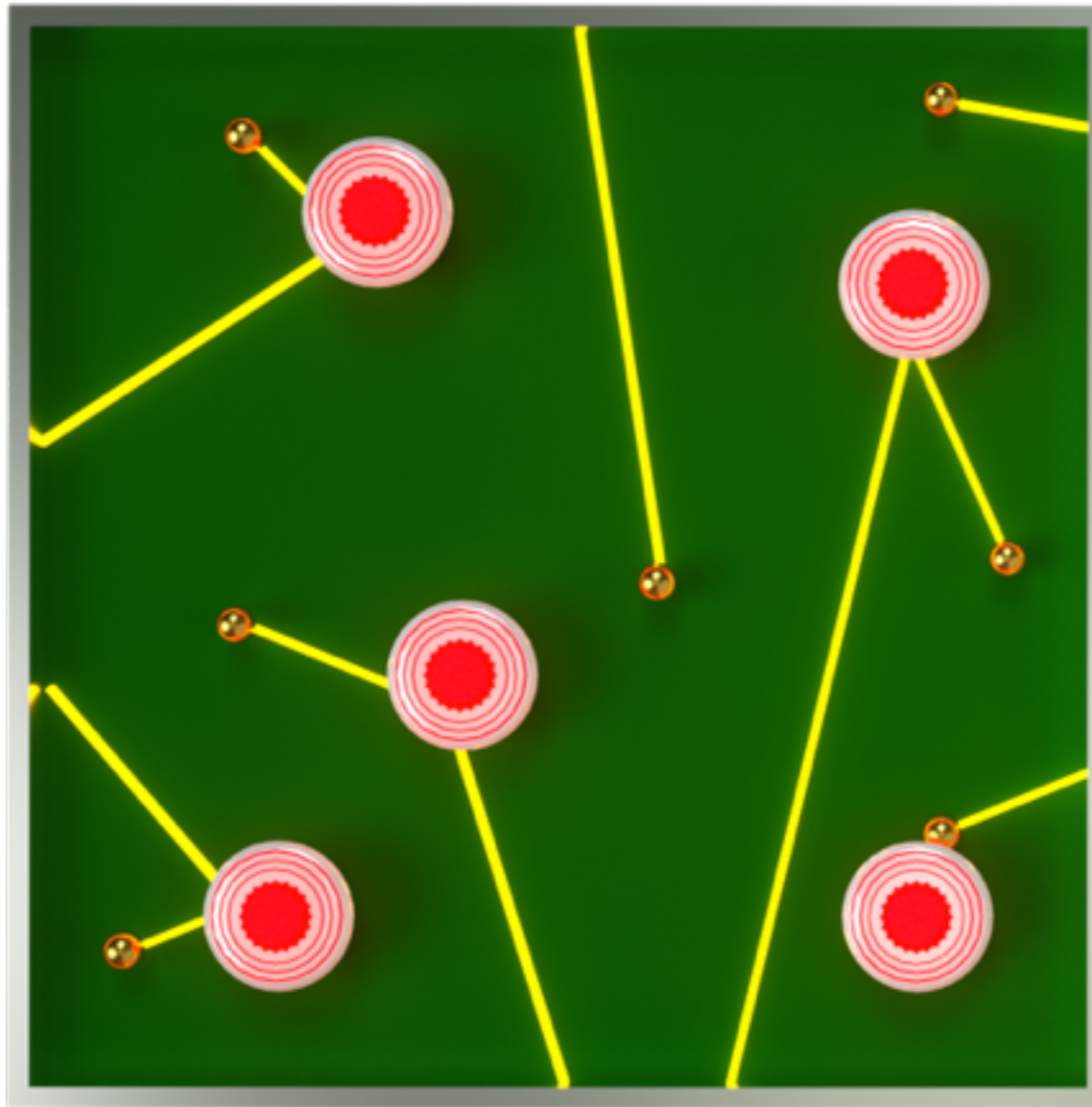
- more generally, measure thermoelectric transport:

$$\begin{pmatrix} \delta J_i \\ \delta Q_i \end{pmatrix} = \begin{pmatrix} \sigma_{ij} & \alpha_{ij} \\ T\bar{\alpha}_{ij} & \bar{\kappa}_{ij} \end{pmatrix} \begin{pmatrix} \delta E_j \\ -\partial_j \delta T \equiv T\delta\zeta_j \end{pmatrix}.$$

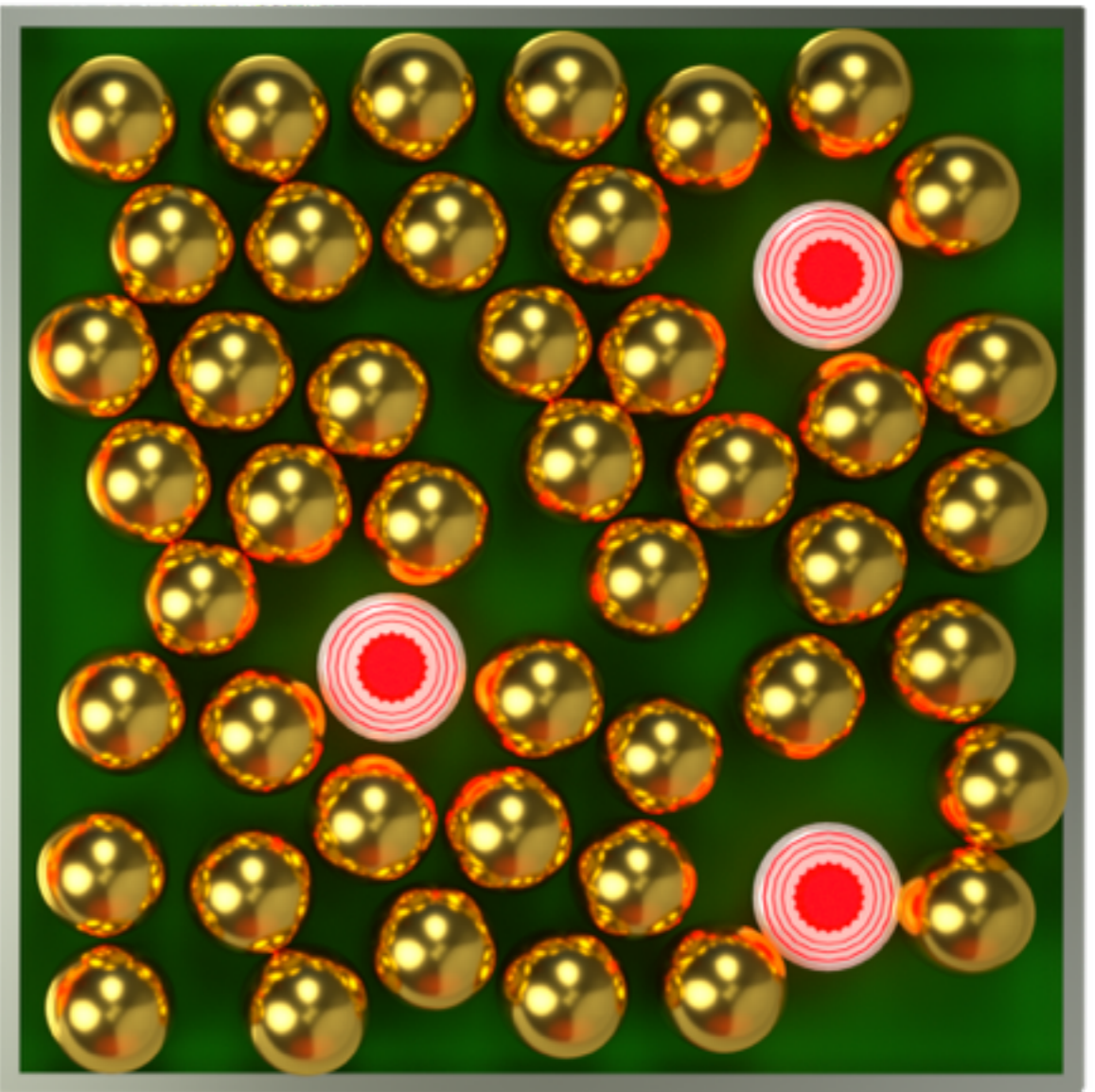
# Quantum criticality of Ising-nematic ordering in a metal

- Early computations of fermions scattering off  $\phi$  fluctuations obtained  $\sigma \sim T^{-4/3}$ .
- However, this ignores “momentum drag”: the  $\phi$  momentum is ultimately given back to the fermions, and this happens rapidly because the fermion- $\phi$  coupling is (infinitely) strong.  

D. L. Maslov, V.I. Yudson, A.V. Chubukov,  
PRL **106**, 106403 (2011)
- The critical theories of the clean non-Fermi liquid can always be defined in the continuum in a manner so that there is a conserved “momentum” operator  $\vec{P}$ , so that  $[\vec{P}, \mathcal{O}(x)] = i\nabla_x \mathcal{O}(x)$ .



Fermi liquids: quasiparticles moving ballistically between impurity (red circles) scattering events



Strange metals: electrons scatter frequently off each other, so there is no regime of ballistic quasiparticle motion. The electron "liquid" then "flows" around impurities

# Thermoelectric transport coefficients

Transport has two components: a “momentum drag” term, and a “quantum critical” term.

$$\begin{aligned}\sigma &= \frac{Q^2}{\mathcal{M}} \pi \delta(\omega) + \sigma_Q(\omega) \\ \alpha &= \frac{SQ}{\mathcal{M}} \pi \delta(\omega) + \alpha_Q(\omega) \\ \bar{\kappa} &= \frac{T\mathcal{S}^2}{\mathcal{M}} \pi \delta(\omega) + \bar{\kappa}_Q(\omega)\end{aligned}$$

with entropy density  $\mathcal{S}$ ,  $Q \equiv \chi_{J_x, P_x}$ , and  $\mathcal{M} \equiv \chi_{P_x, P_x}$ .

In theories which are relativistic at high energies (including graphene),  $T\alpha_Q(\omega) = -\mu\sigma_Q(\omega)$ ,  $T\bar{\kappa}_Q(\omega) = \mu^2\sigma_Q(\omega)$ ,  $\mathcal{M} = T\mathcal{S} + \mu Q = \mathcal{H}$  the enthalpy density, and  $Q = n$  the electron density

S.A. Hartnoll, P. K. Kovtun, M. Müller, and S. Sachdev, PRB **76**, 144502 (2007)

A. Lucas and S. Sachdev, PRB **91**, 195122 (2015)

# Translational symmetry breaking

Momentum relaxation by an external source  $h$  coupling to the operator  $\mathcal{O}$

$$H = H_0 - \int d^d x h(x) \mathcal{O}(x).$$

Leads to an additional term in equations of motion:

$$\partial_\mu T^{\mu i} = \dots - \frac{T^{it}}{\tau_{\text{imp}}} + \dots$$

“Memory function” methods yield an explicit expression for  $\tau_{\text{imp}}$ :

$$\frac{\mathcal{M}}{\tau_{\text{imp}}} = \lim_{\omega \rightarrow 0} \int d^d q |h(q)|^2 q_x^2 \frac{\text{Im} (G_{\mathcal{O}\mathcal{O}}^{\text{R}}(q, \omega))_{H_0}}{\omega} + \dots$$

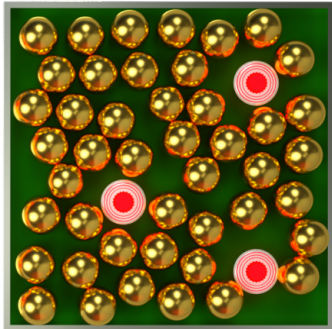
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$$\alpha = \frac{SQ}{\mathcal{M}} \frac{1}{(-i\omega + 1/\tau_{\text{imp}})} + \alpha_Q(\omega)$$
$$\bar{\kappa} = \frac{TS^2}{\mathcal{M}} \frac{1}{(-i\omega + 1/\tau_{\text{imp}})} + \bar{\kappa}_Q(\omega)$$



with entropy density  $\mathcal{S}$ ,  $Q \equiv \chi_{J_x, P_x}$ , and  $\mathcal{M} \equiv \chi_{P_x, P_x}$ .

Obtained in hydrodynamics, and by memory functions

S.A. Hartnoll, P. K. Kovtun, M. Müller, and S. Sachdev, PRB **76**, 144502 (2007)

A. Lucas and S. Sachdev, PRB **91**, 195122 (2015)

# Quantum criticality of Ising-nematic ordering in a metal

- If we choose  $\mathcal{O}$  = number or energy density, then the results are similar to obtained from solving hydrodynamic equations in the presence of impurities.

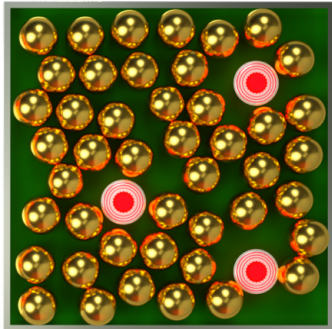
R.A. Davison, K. Schalm, and Jan Zaanen, PRB 89, 245116 (2014)

- For the Ising-nematic critical point, the strongest scattering arises from the choice  $\mathcal{O} = \phi$ , with the disorder coupling to the local orientation of the nematic ordering. This leads to

$$\text{Resistivity } \rho \sim [T \ln(1/T)]^{-1/2}$$

- If Landau damping of the  $\phi$  fluctuations can be neglected, then

$$\text{Resistivity } \rho \sim T$$

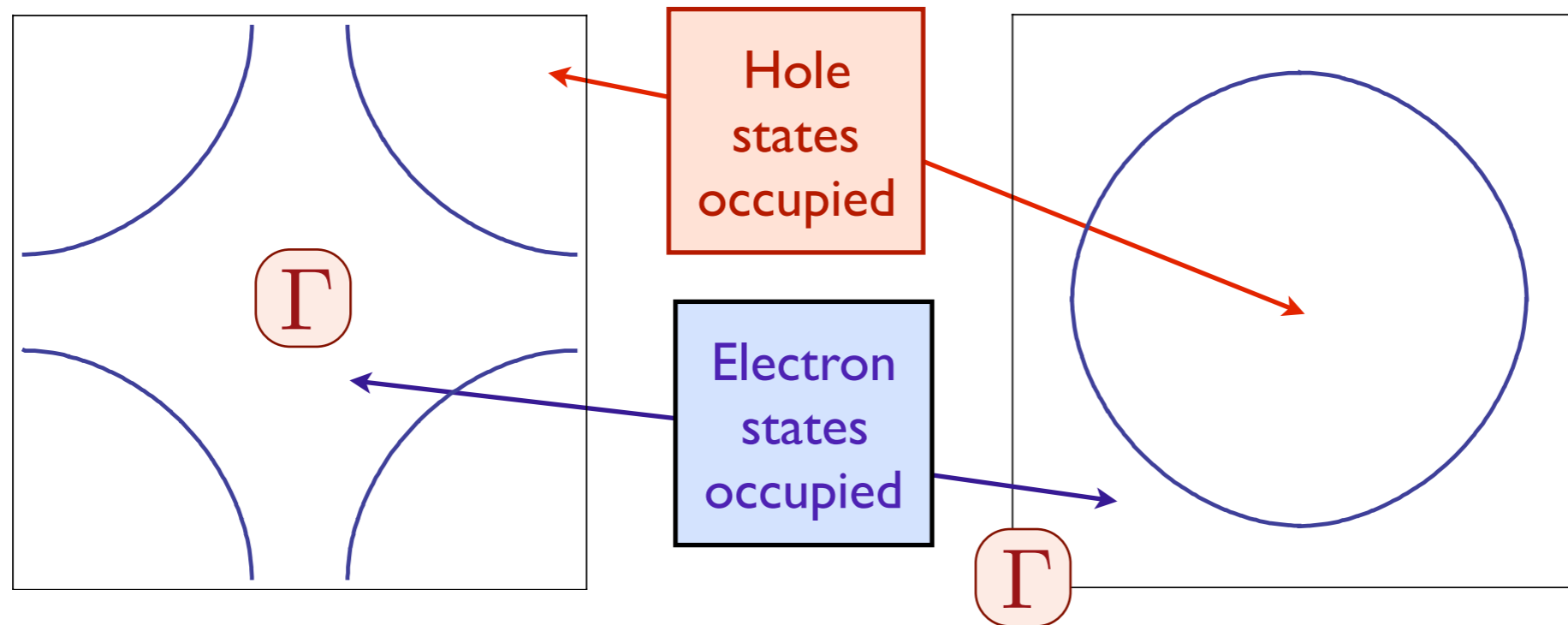


S.A. Hartnoll, R. Mahajan, M. Punk, and S. Sachdev, PRB 89, 155130 (2014)

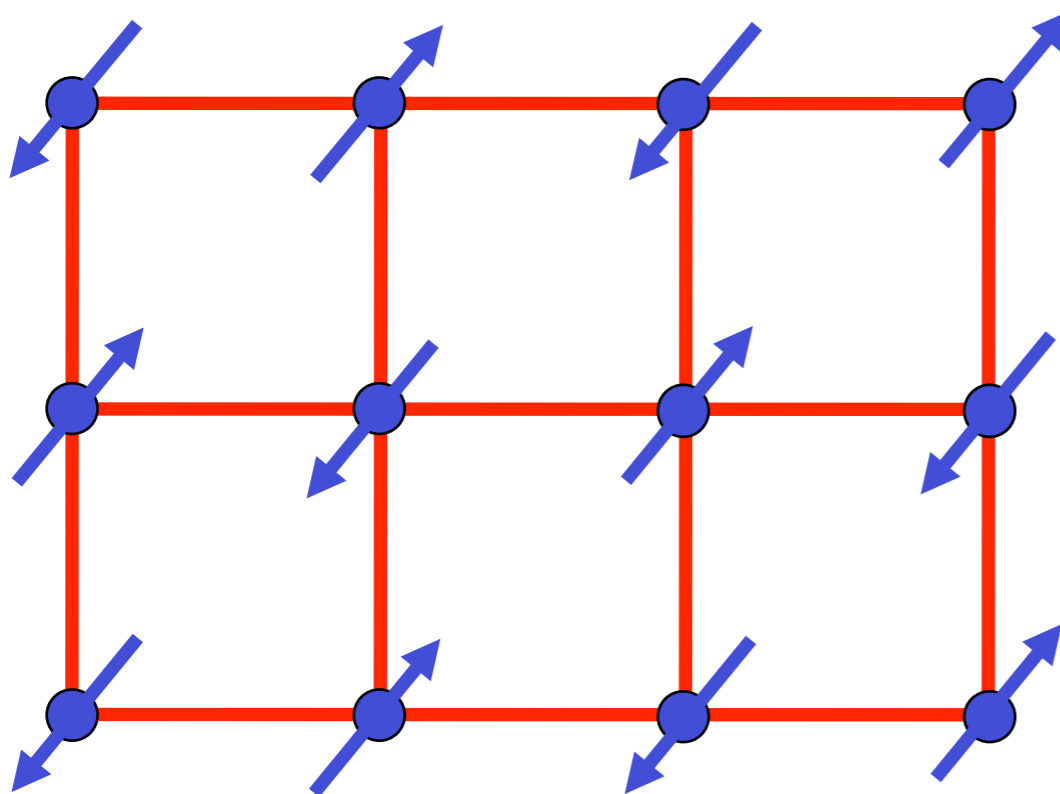
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# Fermi surface+antiferromagnetism



+



The electron spin polarization obeys

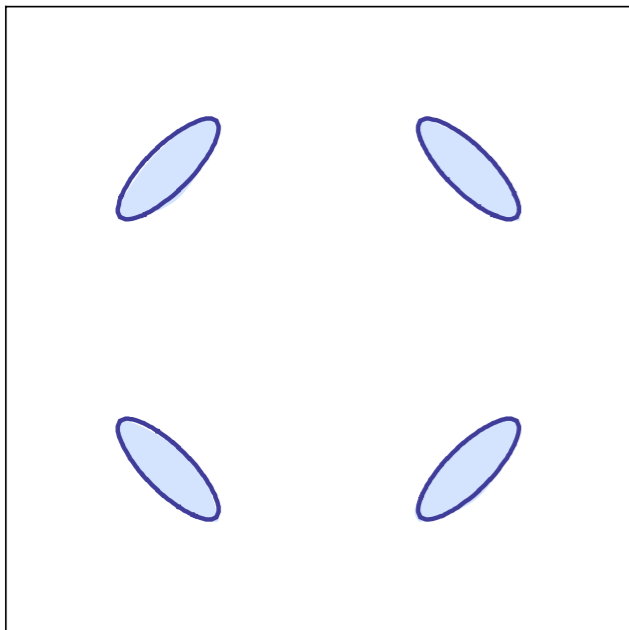
$$\langle \vec{S}(\mathbf{r}, \tau) \rangle = \vec{\varphi}(\mathbf{r}, \tau) e^{i\mathbf{K} \cdot \mathbf{r}}$$

where  $\mathbf{K}$  is the ordering wavevector.

# Square lattice Hubbard model with hole doping

$$\langle \vec{\varphi} \rangle \neq 0$$

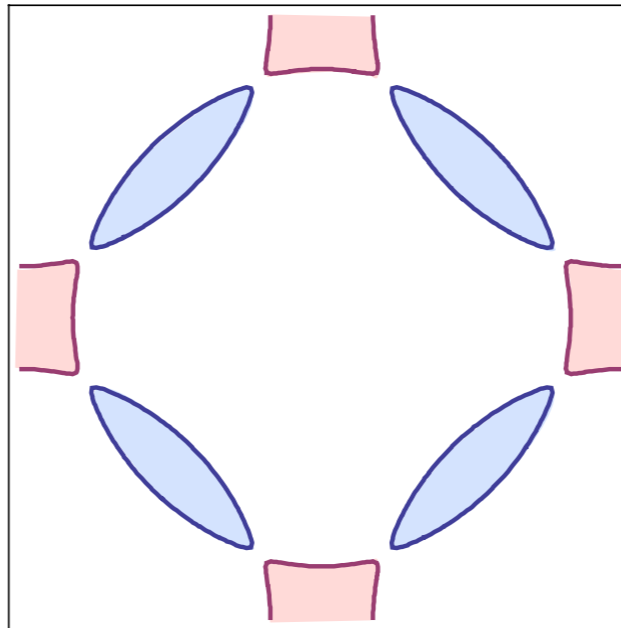
and large



Metal with  
hole pockets

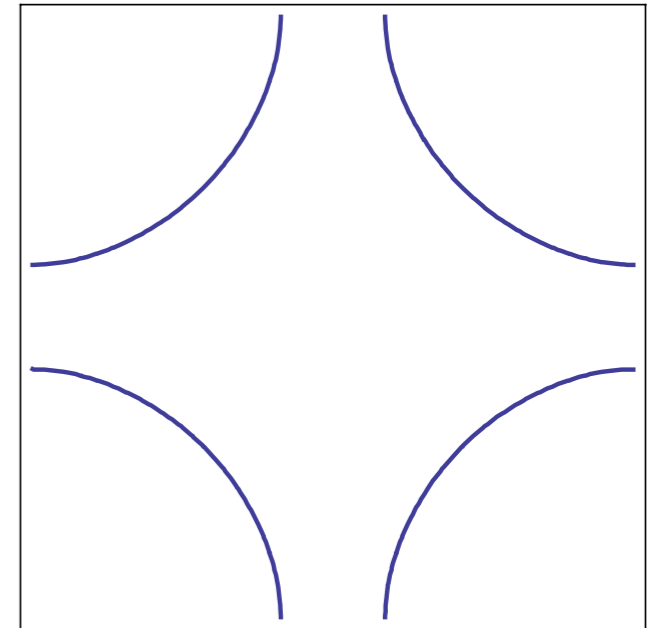
$$\langle \vec{\varphi} \rangle \neq 0$$

and small



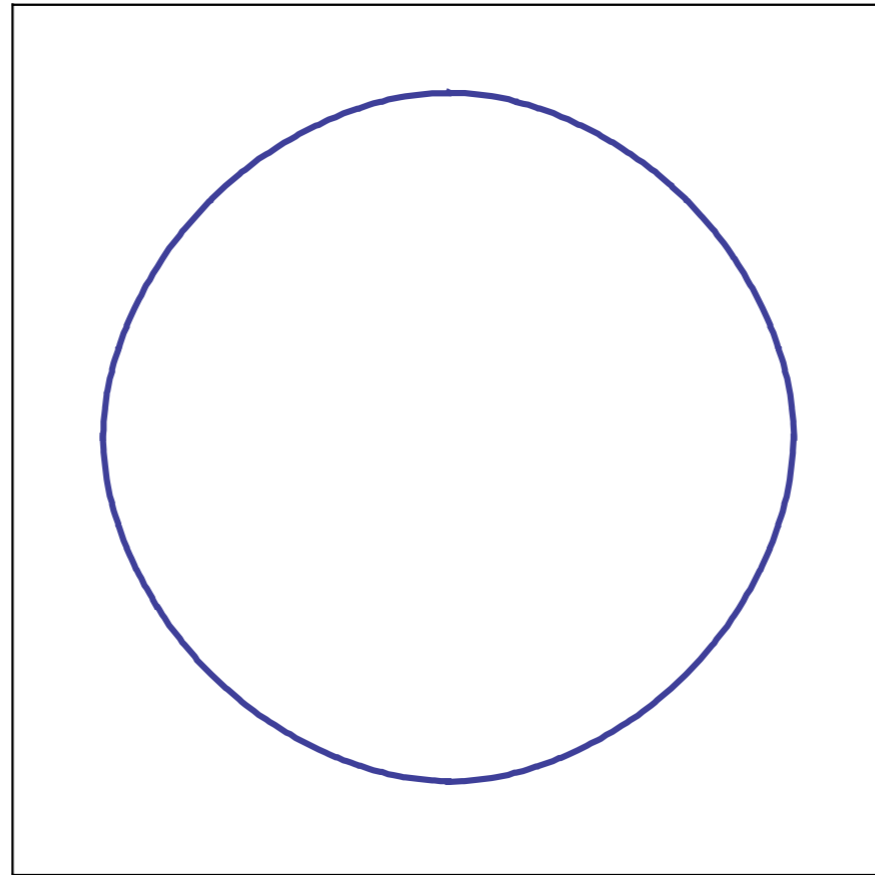
Metal with  
electron and  
hole pockets

$$\langle \vec{\varphi} \rangle = 0$$



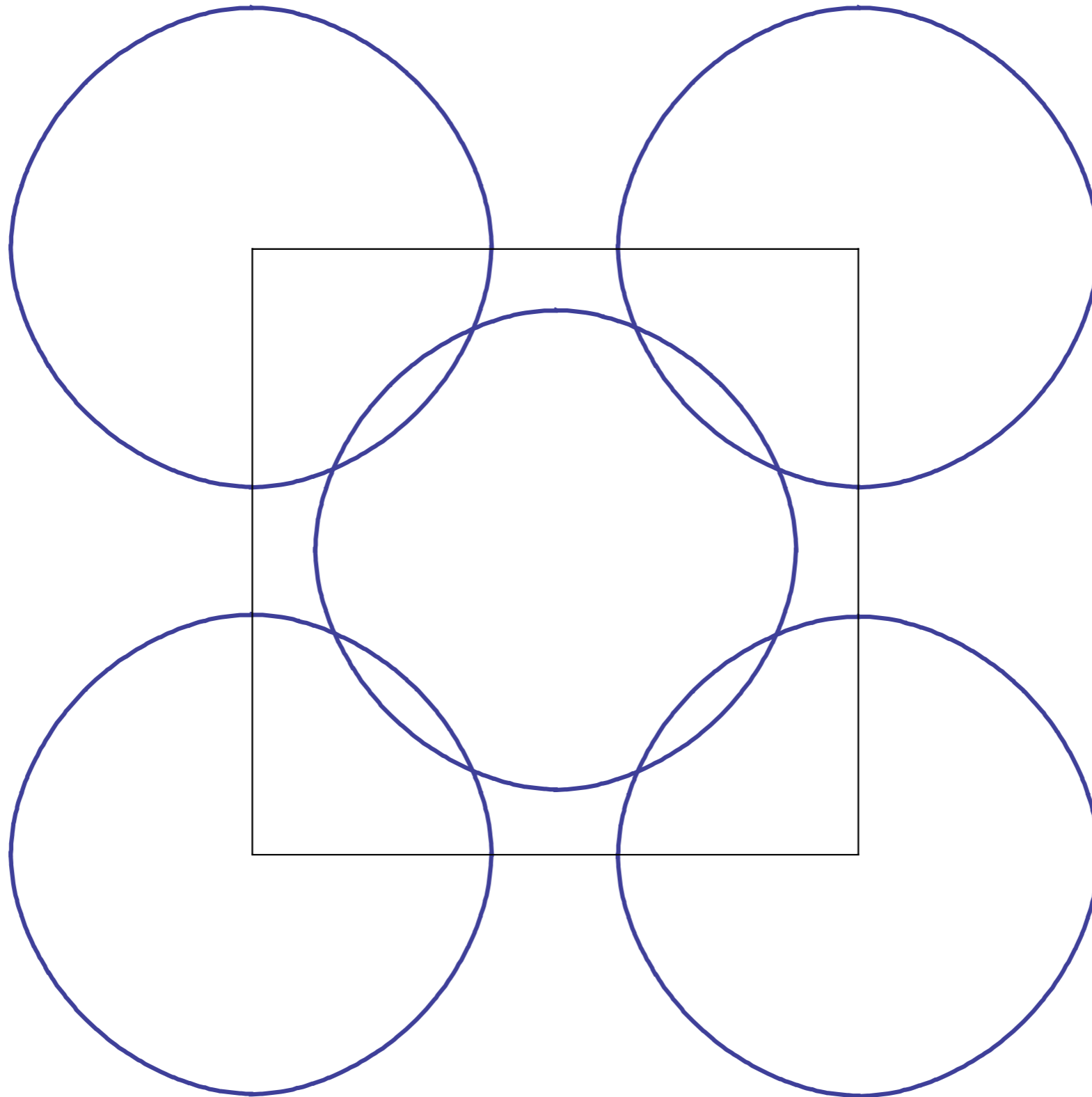
Metal with  
“large” Fermi  
surface

# Fermi surface+antiferromagnetism



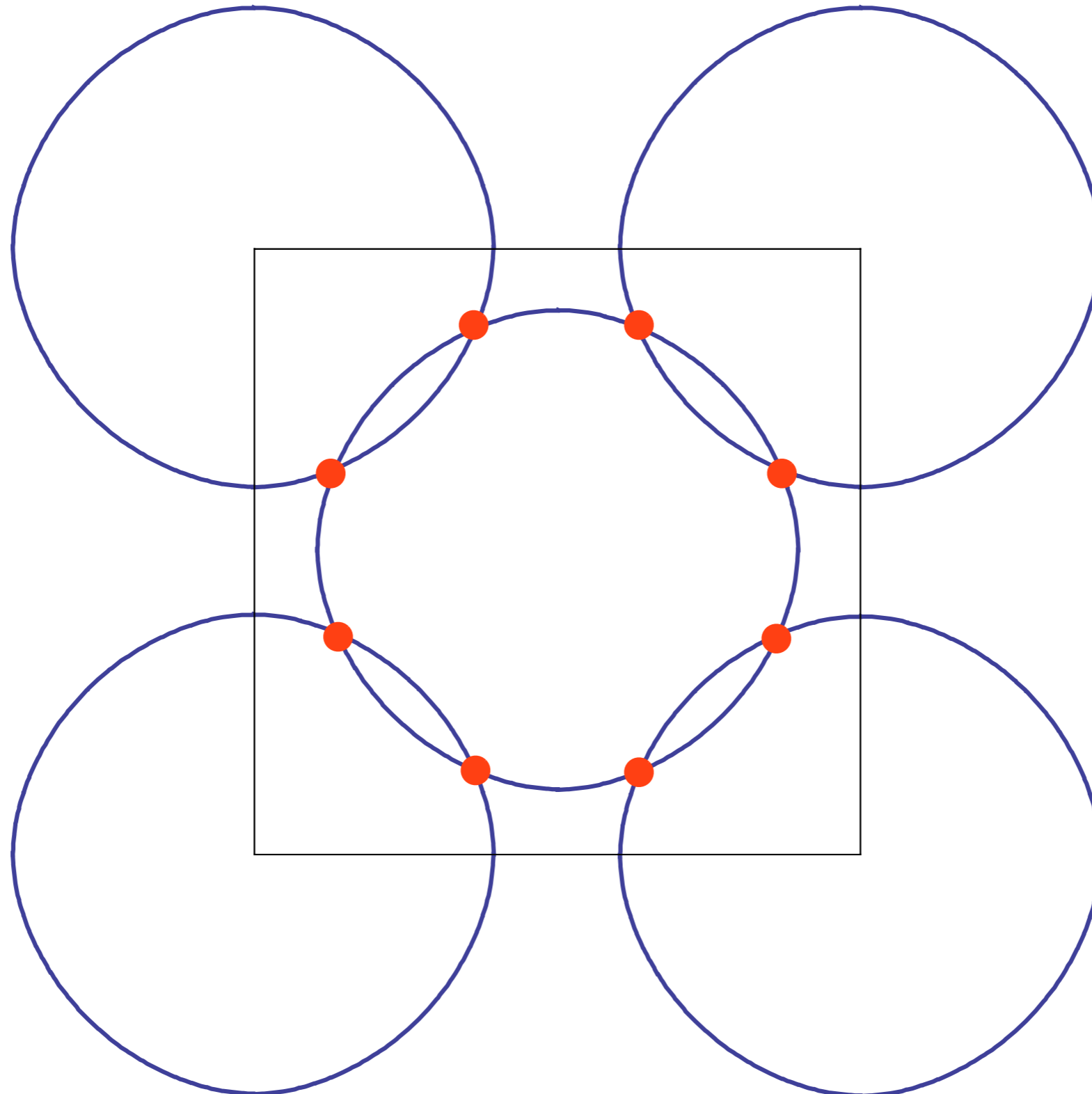
**Metal with “large” Fermi surface**

# Fermi surface+antiferromagnetism

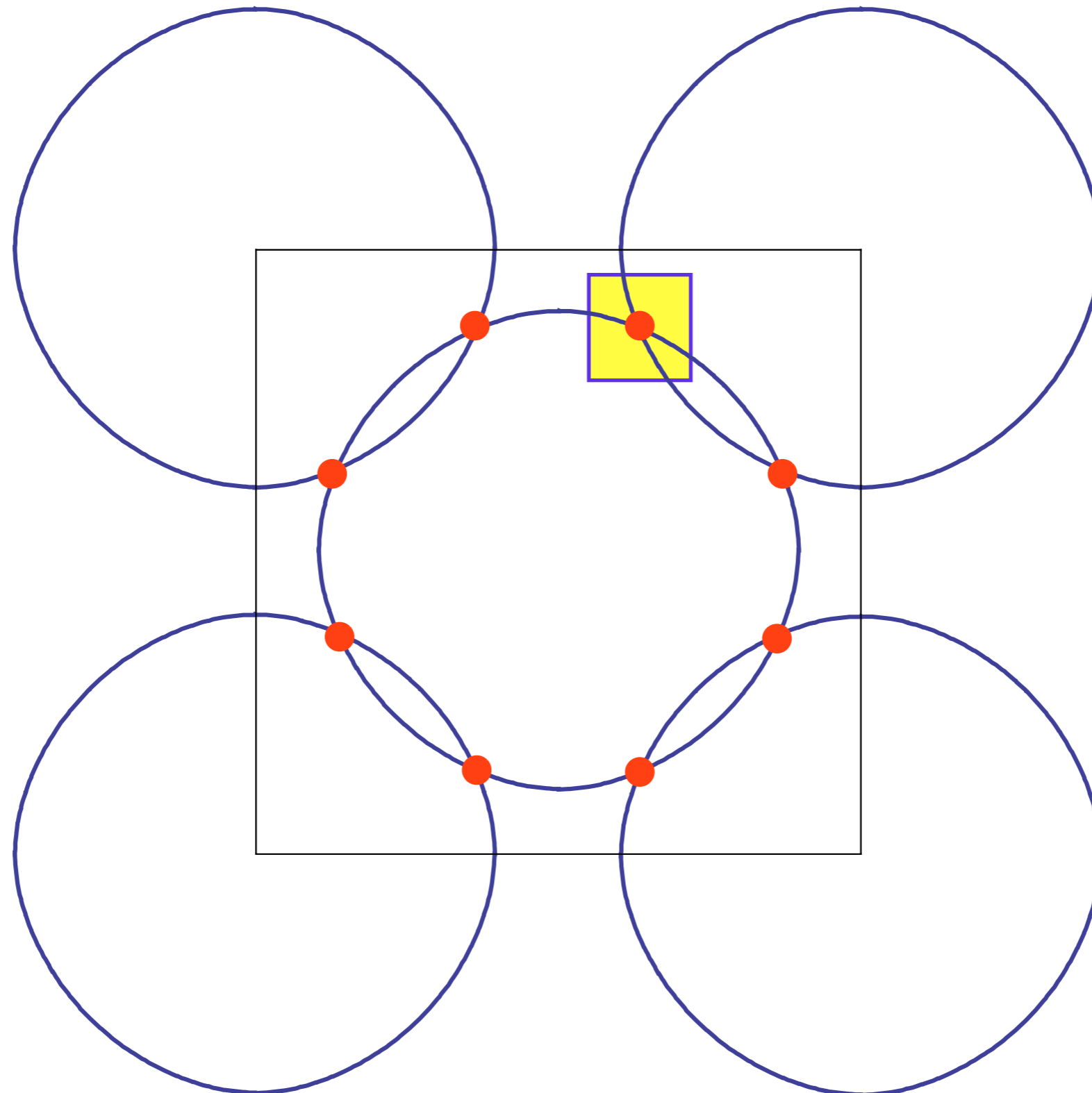


Fermi surfaces translated by  $\mathbf{K} = (\pi, \pi)$ .

# Fermi surface+antiferromagnetism

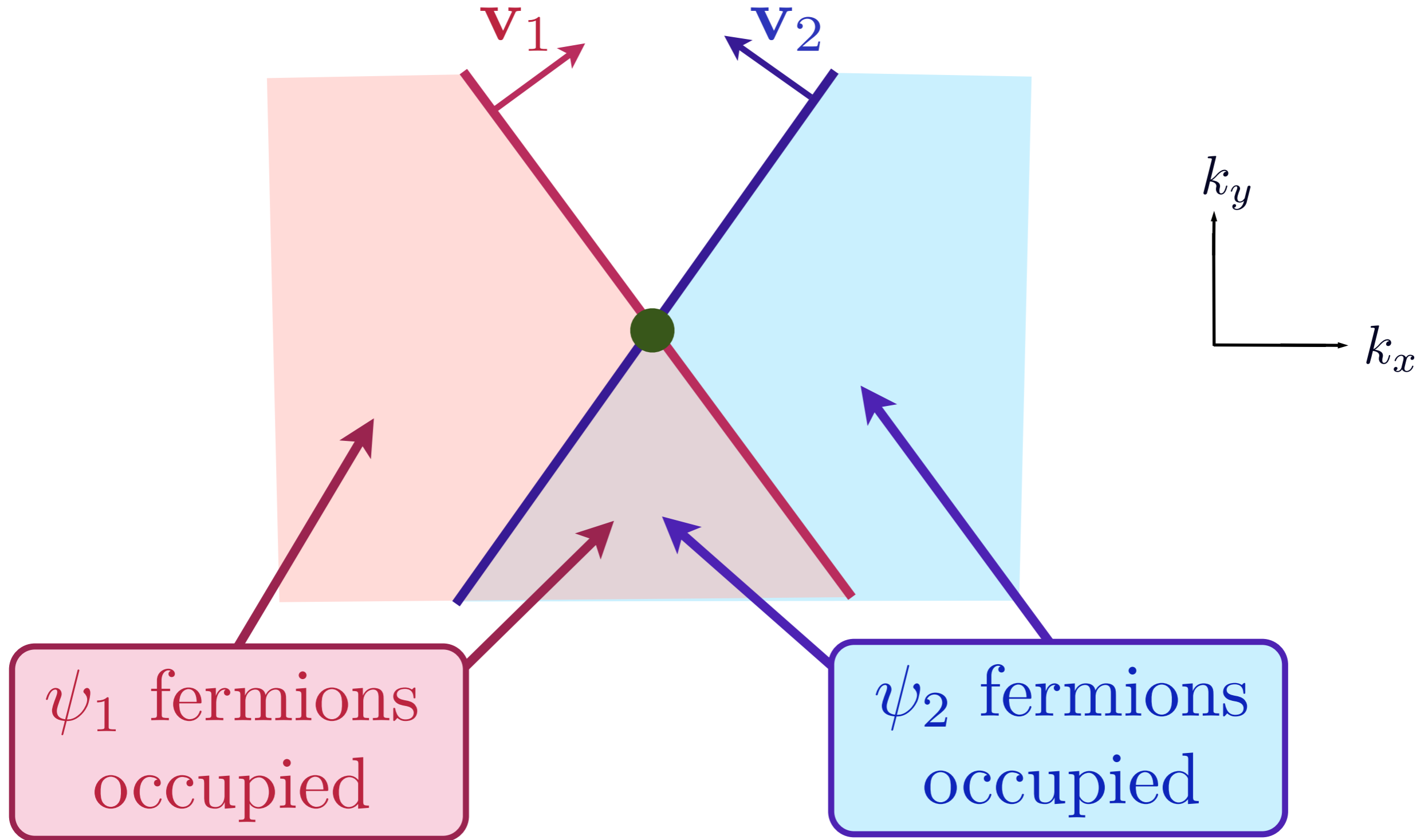


“Hot” spots



**Low energy theory for critical point near hot spots**

Theory has fermions  $\psi_{1,2}$  (with Fermi velocities  $\mathbf{v}_{1,2}$ )  
and boson order parameter  $\vec{\varphi}$ ,  
interacting with coupling  $\lambda$



$$\mathcal{L}_f = \psi_{1\alpha}^\dagger (\partial_\tau - i\mathbf{v}_1 \cdot \nabla_r) \psi_{1\alpha} + \psi_{2\alpha}^\dagger (\partial_\tau - i\mathbf{v}_2 \cdot \nabla_r) \psi_{2\alpha}$$

Order parameter:  $\mathcal{L}_\varphi = \frac{1}{2} (\nabla_r \vec{\varphi})^2 + \frac{1}{2} (\partial_\tau \vec{\varphi})^2 + \frac{s}{2} \vec{\varphi}^2 + \frac{u}{4} \vec{\varphi}^4$

“Yukawa” coupling:  $\mathcal{L}_c = -\lambda \vec{\varphi} \cdot \left( \psi_{1\alpha}^\dagger \vec{\sigma}_{\alpha\beta} \psi_{2\beta} + \psi_{2\alpha}^\dagger \vec{\sigma}_{\alpha\beta} \psi_{1\beta} \right)$

# Quantum criticality of spin density wave ordering in a metal

Because the hotspots are localized at points in momentum space, the fermion excitations are effectively two-dimensional, the “violation of hyperscaling” exponent  $\theta = 0$ . So both the entropy density and the optical conductivity of the hot-spot critical theory obey the conventional scaling

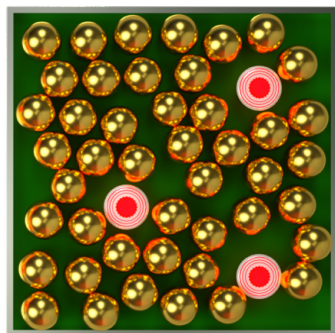
A.A. Patel, P. Strack, and S. Sachdev, PRB **92**, 165105 (2015)

$$s \sim T^{d/z} \sim T^{2/z} \text{ and } \sigma \sim \omega^{(d-2)/z} \sim \text{constant.}$$

For the d.c. transport, a novel hydrodynamic regime is obtained if we assume the hot spots and the cold regions *both* equilibrate with each other on a time scale shorter than the impurity scattering time. Then we find, via memory functions, that the d.c. resistivity is dominated by momentum relaxation near the hot spots. Disorder coupling to  $\vec{\varphi}^2$  (*i.e.* random variations in the local value of the critical coupling) leads to a resistivity

$$\rho \sim T \text{ up to logarithms}$$

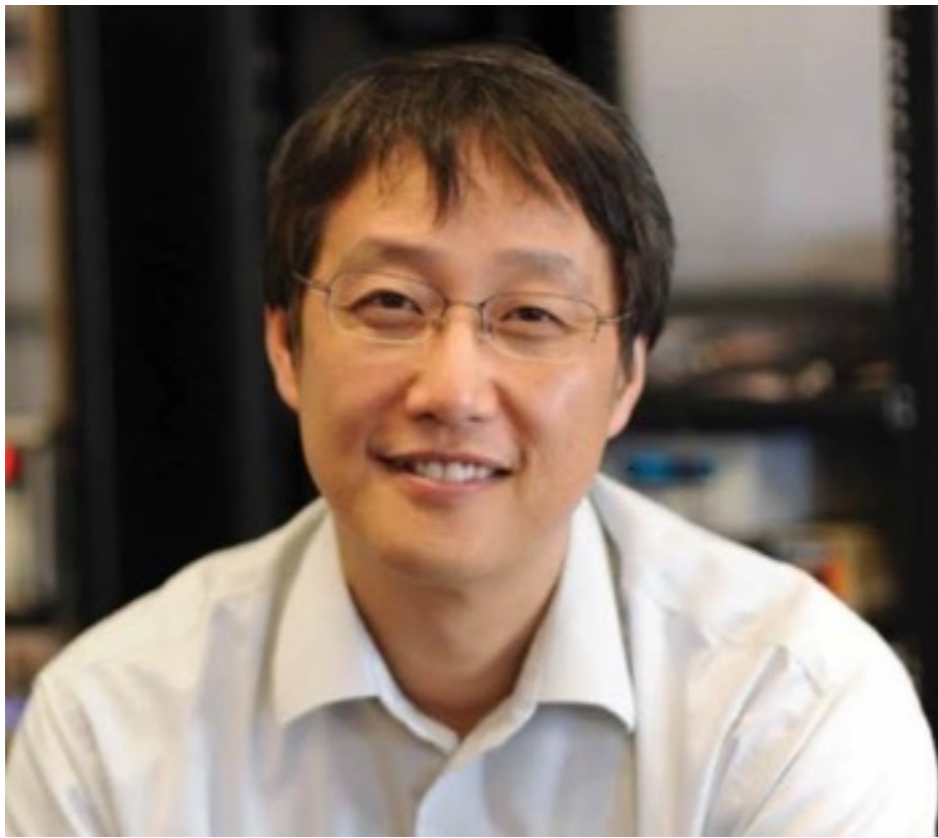
with  $z = 2$ .



A.A. Patel and S. Sachdev, PRB **90**, 165146 (2014)

# Theories of non-Fermi liquids

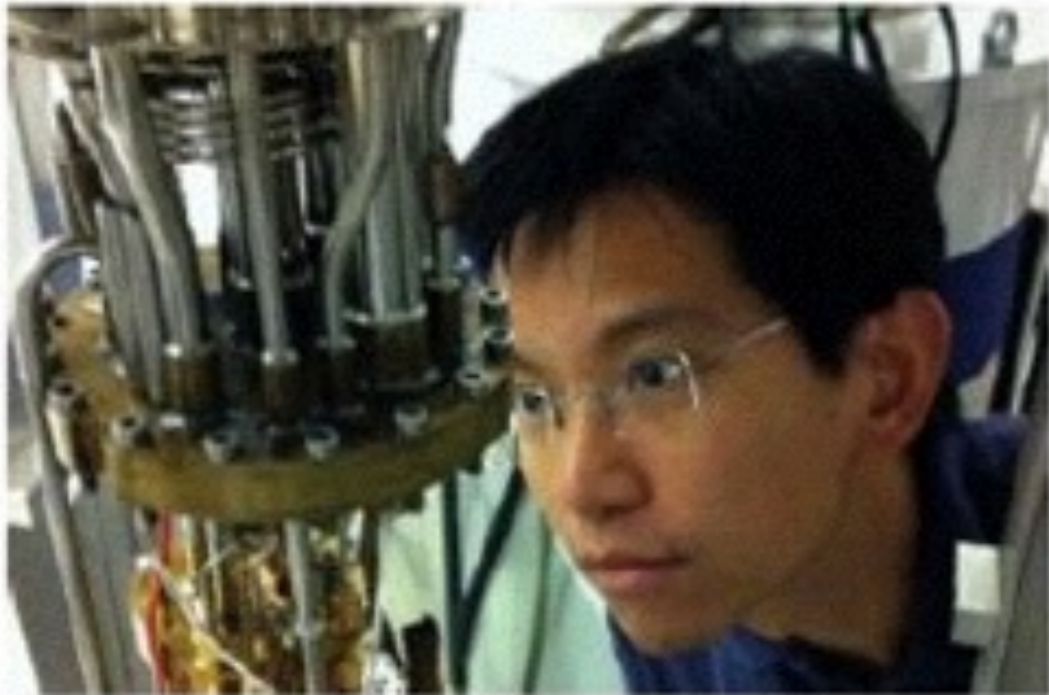
- Sachdev-Ye-Kitaev (SYK) model
- Ising-nematic criticality in  $d=2$
- Spin density wave criticality in  $d=2$
- Graphene



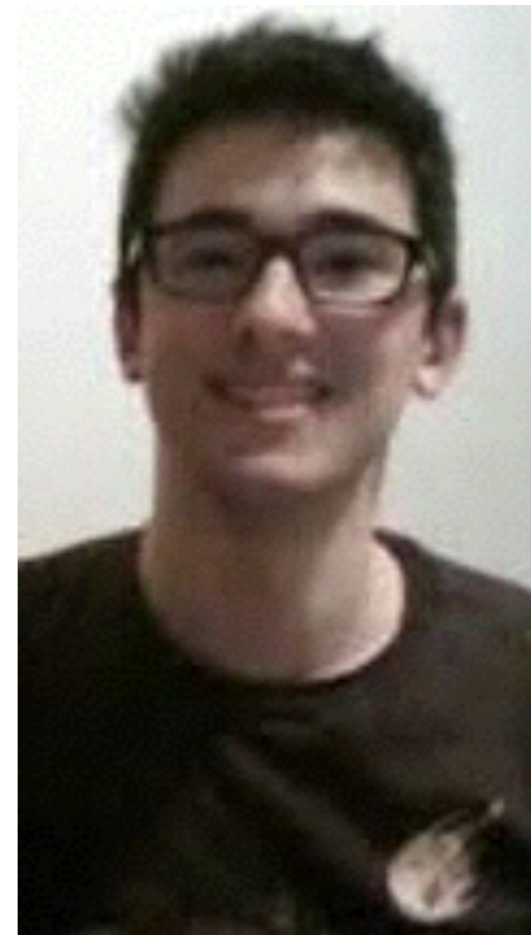
Philip Kim



Jesse Crossno

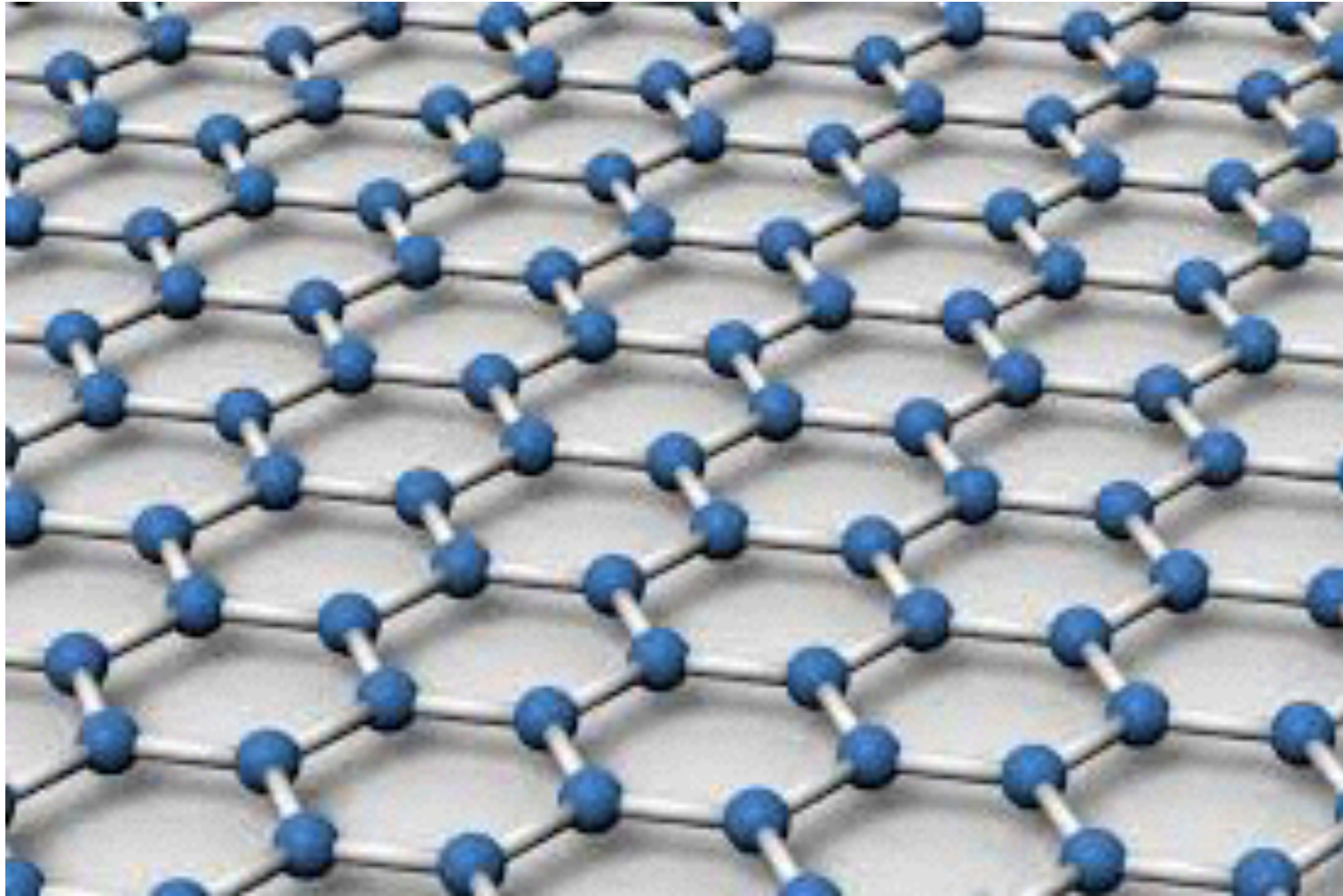


Kin Chung Fong

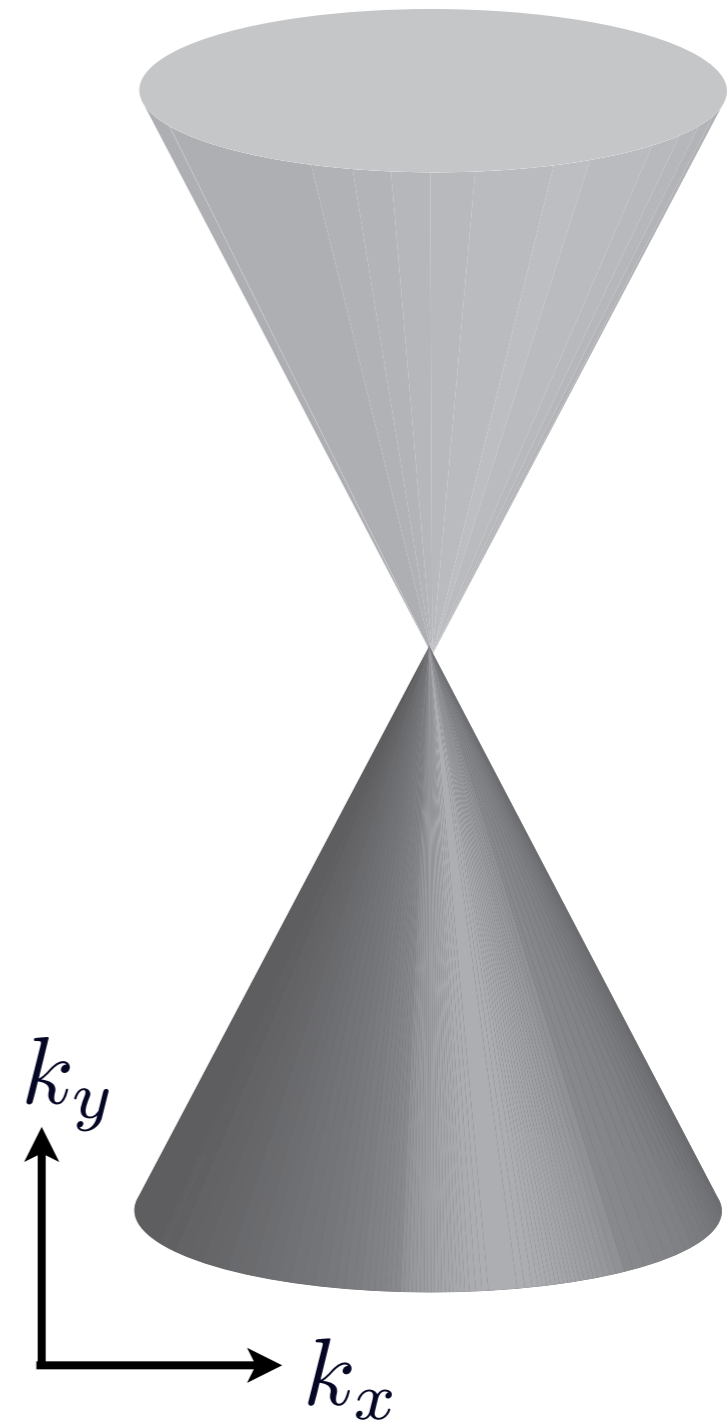


Andrew Lucas

# Graphene

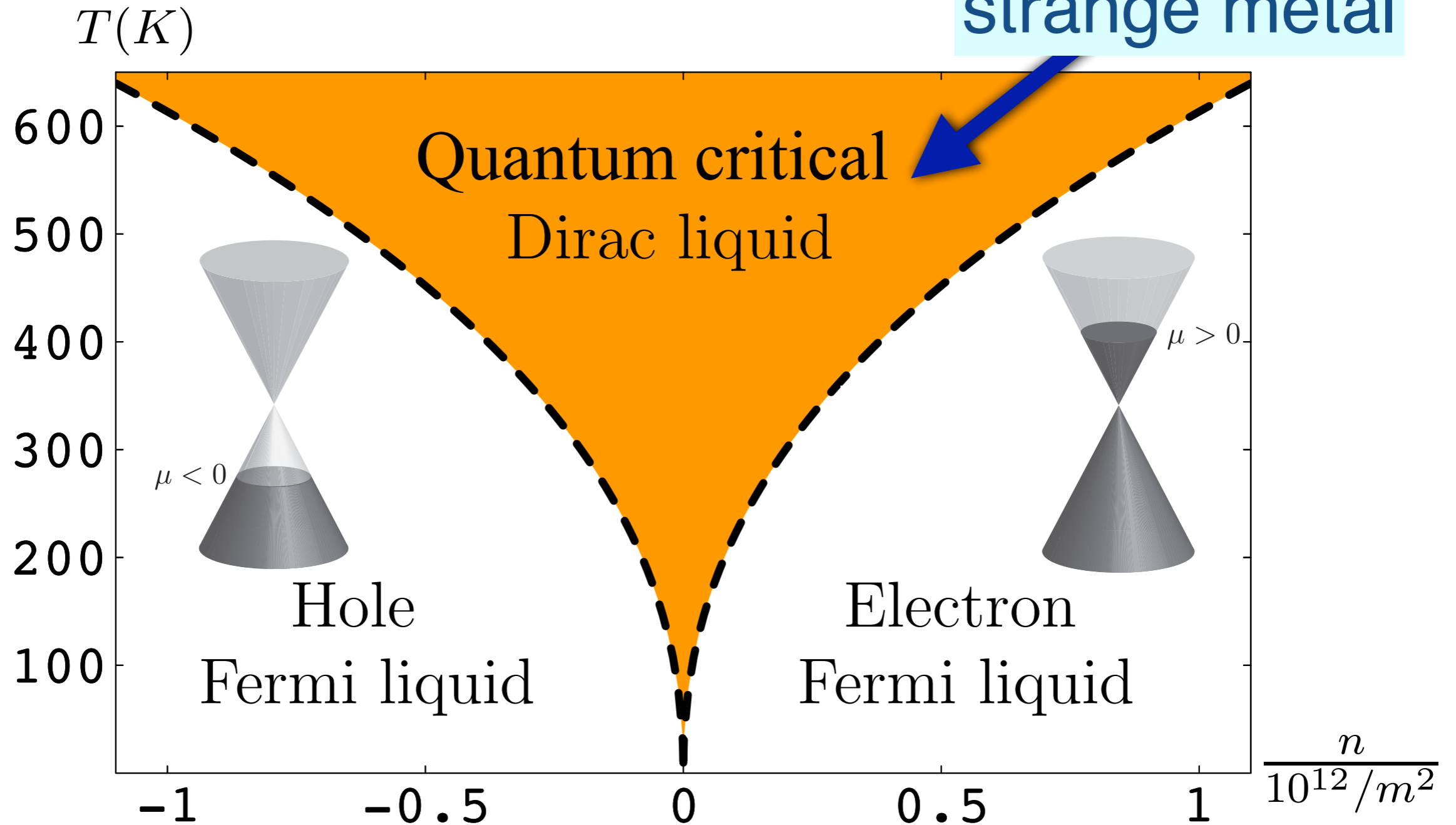


Same “Hubbard” model as for ultracold atoms, but for electrons on the honeycomb lattice



# Graphene

Predicted  
strange metal

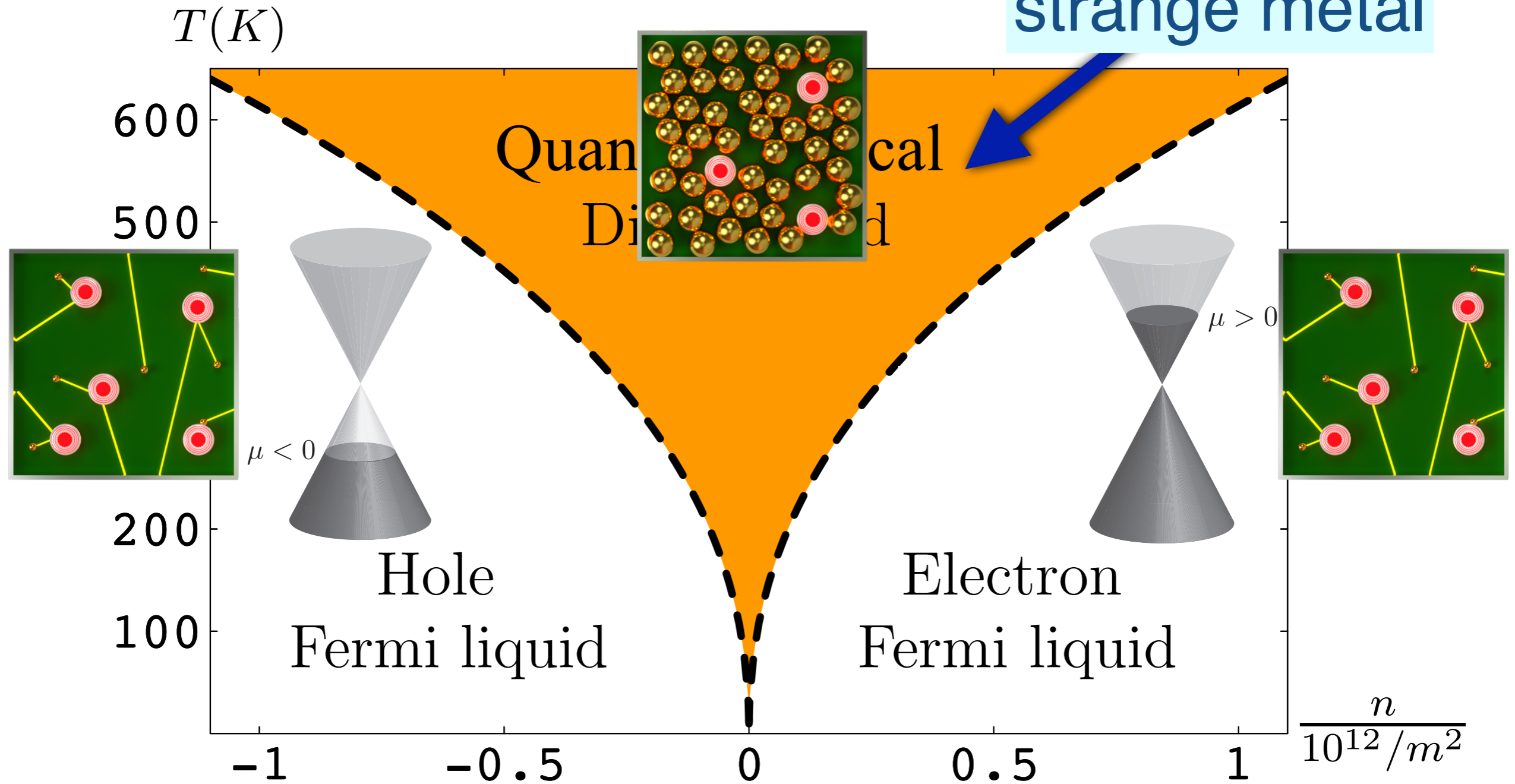


M. Müller, L. Fritz, and S. Sachdev, PRB **78**, 115406 (2008)

M. Müller and S. Sachdev, PRB **78**, 115419 (2008)

# Graphene

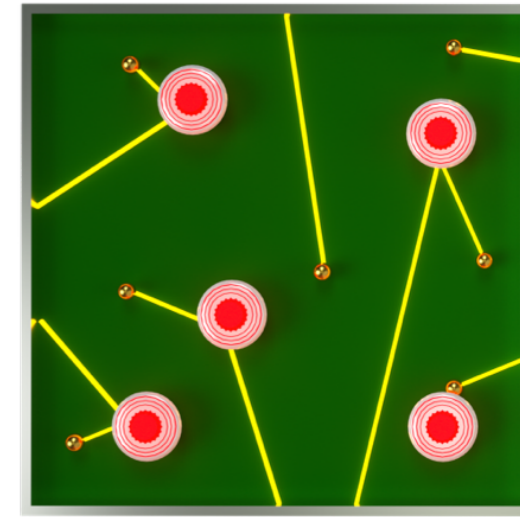
Predicted  
strange metal



M. Müller, L. Fritz, and S. Sachdev, PRB **78**, 115406 (2008)

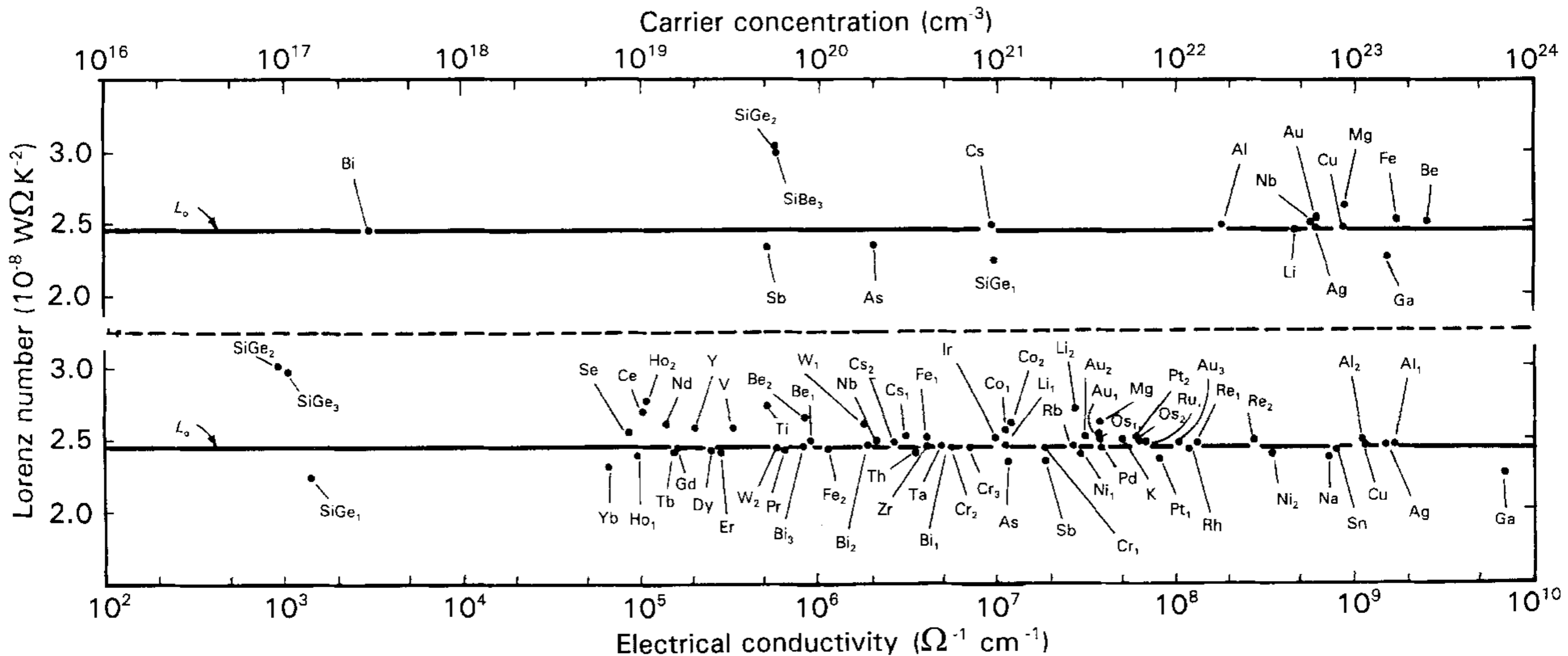
M. Müller and S. Sachdev, PRB **78**, 115419 (2008)

# Thermal and electrical conductivity with quasiparticles



- Wiedemann-Franz law in a Fermi liquid:

$$L_0 = \frac{\kappa}{\sigma T} \approx \frac{\pi^2 k_B^2}{3e^2} \approx 2.45 \times 10^{-8} \frac{\text{W} \cdot \Omega}{\text{K}^2}.$$



# Prediction for transport in the graphene strange metal

For a strange metal with a “relativistic” Hamiltonian, hydrodynamic, holographic, and memory function methods yield for the Lorentz ratio  $L = \kappa/(T\sigma)$

$$\sigma = \sigma_Q \left( 1 + \frac{e^2 v_F^2 Q^2 \tau_{\text{imp}}}{\mathcal{H} \sigma_Q} \right), \quad \kappa = \frac{v_F^2 \mathcal{H} \tau_{\text{imp}}}{T} \left( 1 + \frac{e^2 v_F^2 Q^2 \tau_{\text{imp}}}{\mathcal{H} \sigma_Q} \right)^{-1}$$

$$L = \frac{v_F^2 \mathcal{H} \tau_{\text{imp}}}{T^2 \sigma_Q} \left( 1 + \frac{e^2 v_F^2 Q^2 \tau_{\text{imp}}}{\mathcal{H} \sigma_Q} \right)^{-2},$$

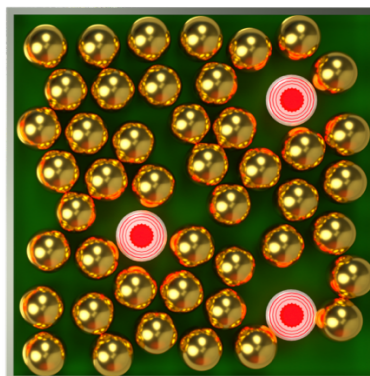
where  $\mathcal{H}$  is the enthalpy density,  $\tau_{\text{imp}}$  is the momentum relaxation time (from impurities), while  $\sigma = \sigma_Q$ , an intrinsic, finite, “quantum critical” conductivity.

- For  $Q = 0$ , as  $\tau_{\text{imp}} \rightarrow \infty$ ,  $\sigma$  remains finite, while  $\kappa \rightarrow \infty$ , and so  $L \rightarrow \infty$ .
- For  $Q \neq 0$ , as  $\tau_{\text{imp}} \rightarrow \infty$ ,  $\sigma \rightarrow \infty$ , while  $\kappa$  remains finite, and so  $L \rightarrow 0$ .

Prediction:  $L$  diverges as  $1/Q^4$  near  $Q = 0$  in clean graphene.

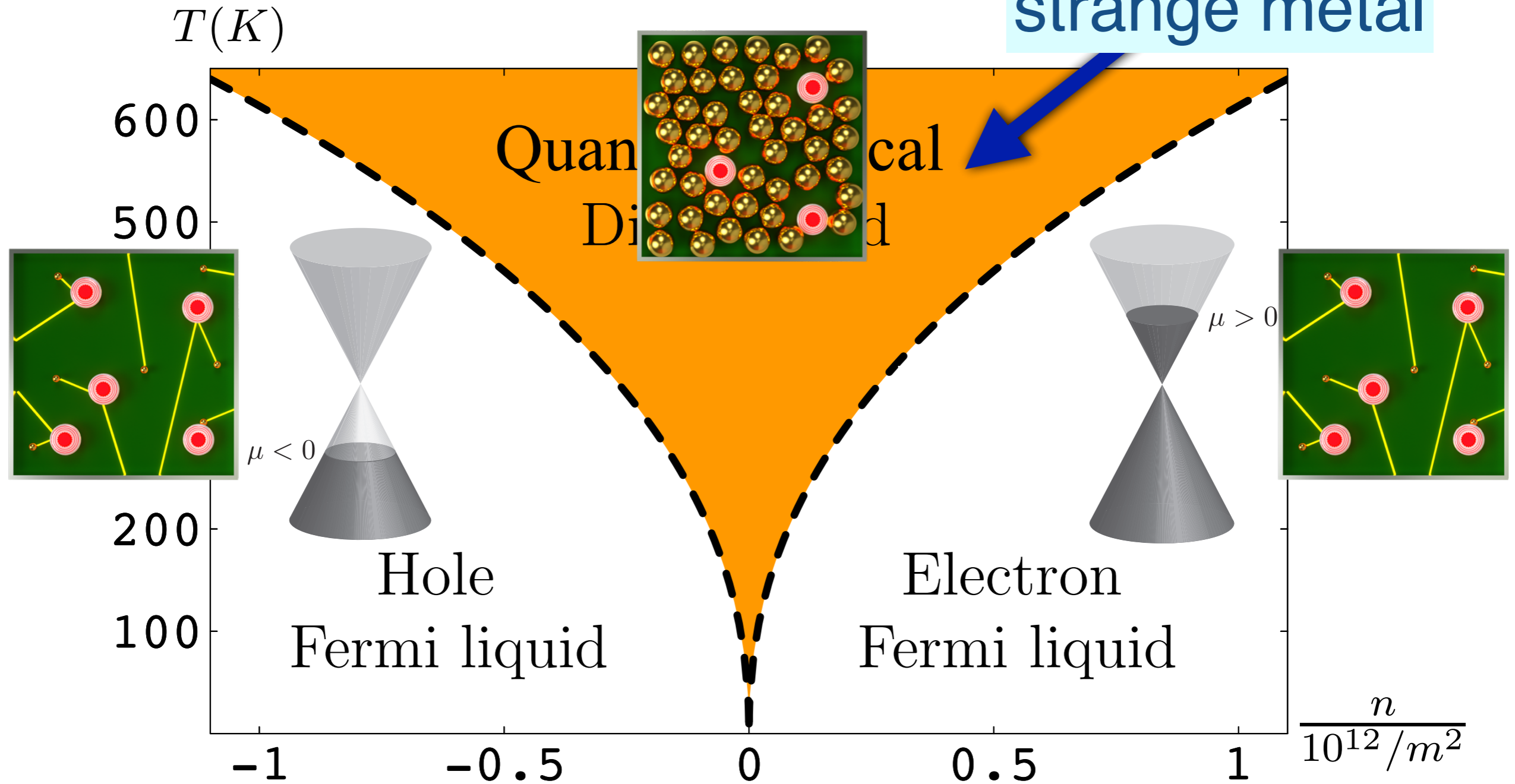
S. A. Hartnoll, P. K. Kovtun, M. Müller, and S. Sachdev, PRB **76**, 144502 (2007)

M. Müller and S. Sachdev, PRB **78**, 115419 (2008)



# Graphene

Predicted  
strange metal

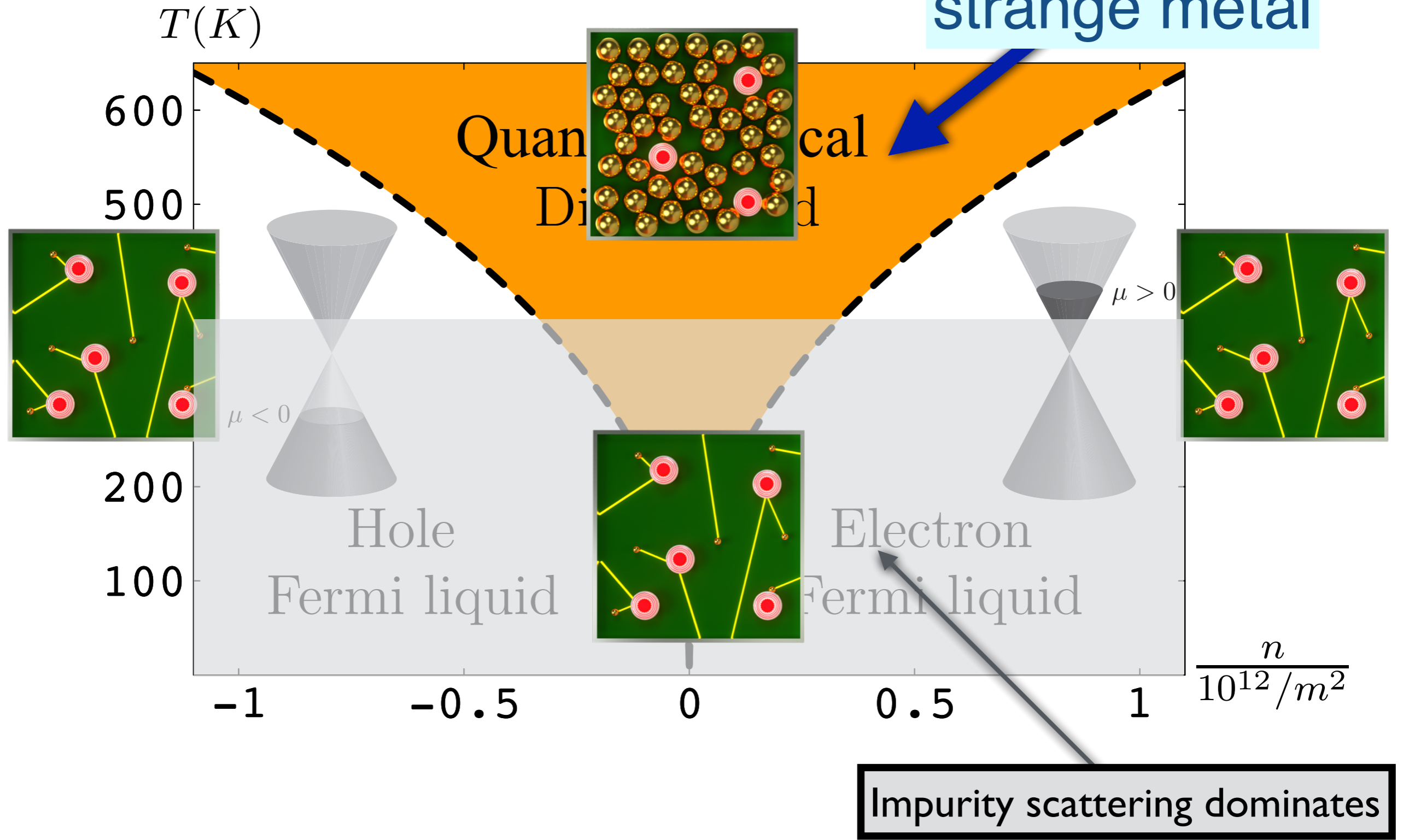


M. Müller, L. Fritz, and S. Sachdev, PRB **78**, 115406 (2008)

M. Müller and S. Sachdev, PRB **78**, 115419 (2008)

# Graphene

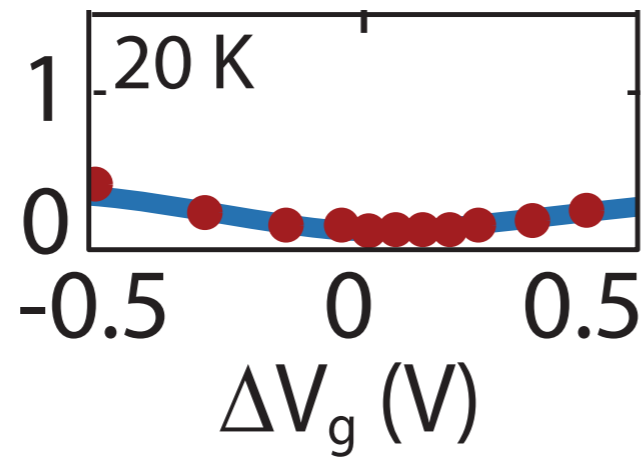
Predicted  
strange metal



M. Müller, L. Fritz, and S. Sachdev, PRB **78**, 115406 (2008)

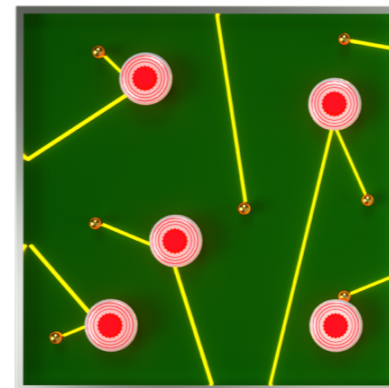
M. Müller and S. Sachdev, PRB **78**, 115419 (2008)

Thermal Conductivity (nW/K)

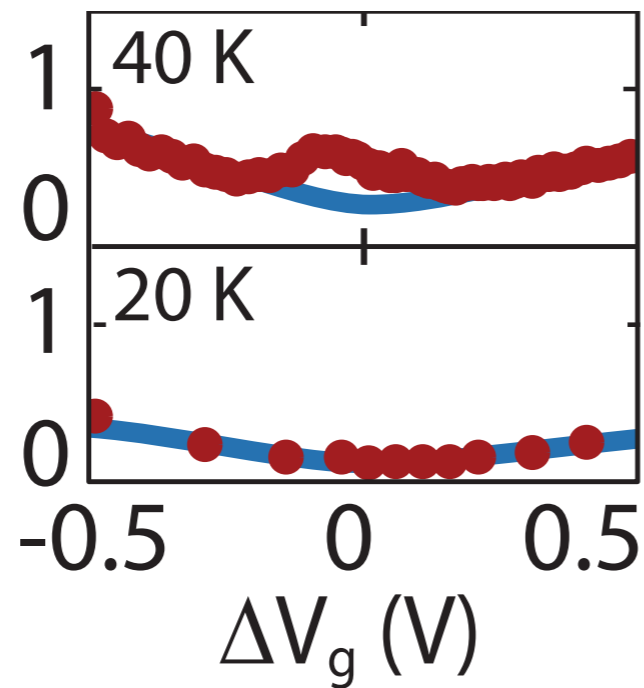


Red dots: data

Blue line: value for  $L = L_0$

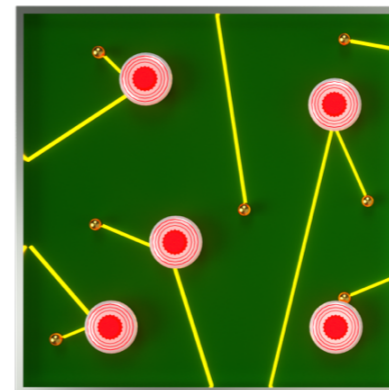


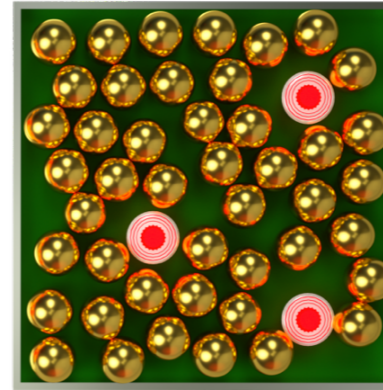
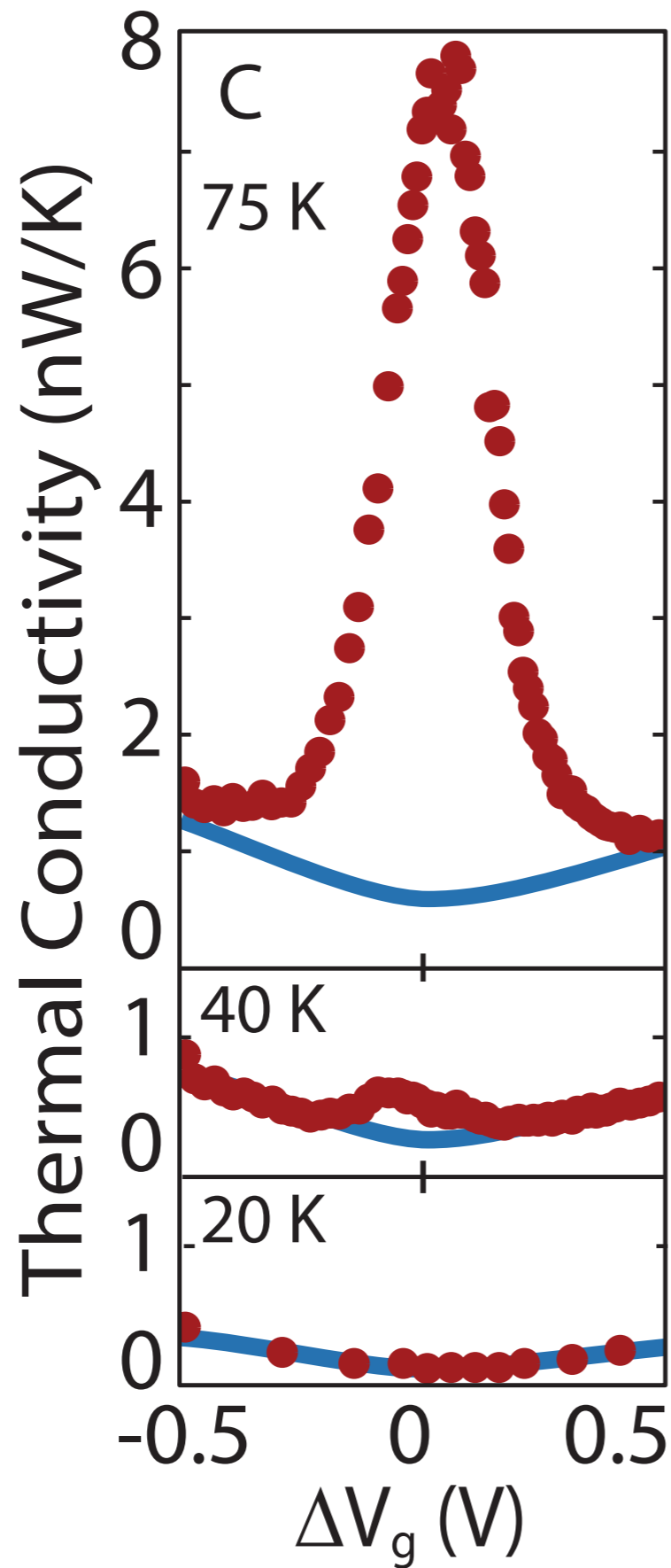
Thermal Conductivity (nW/K)



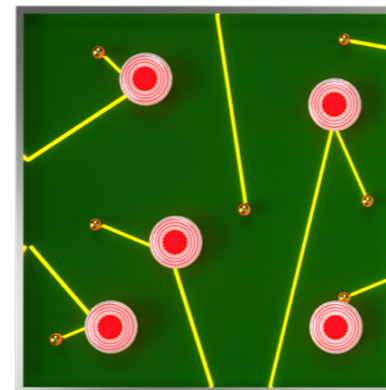
Red dots: data

Blue line: value for  $L = L_0$



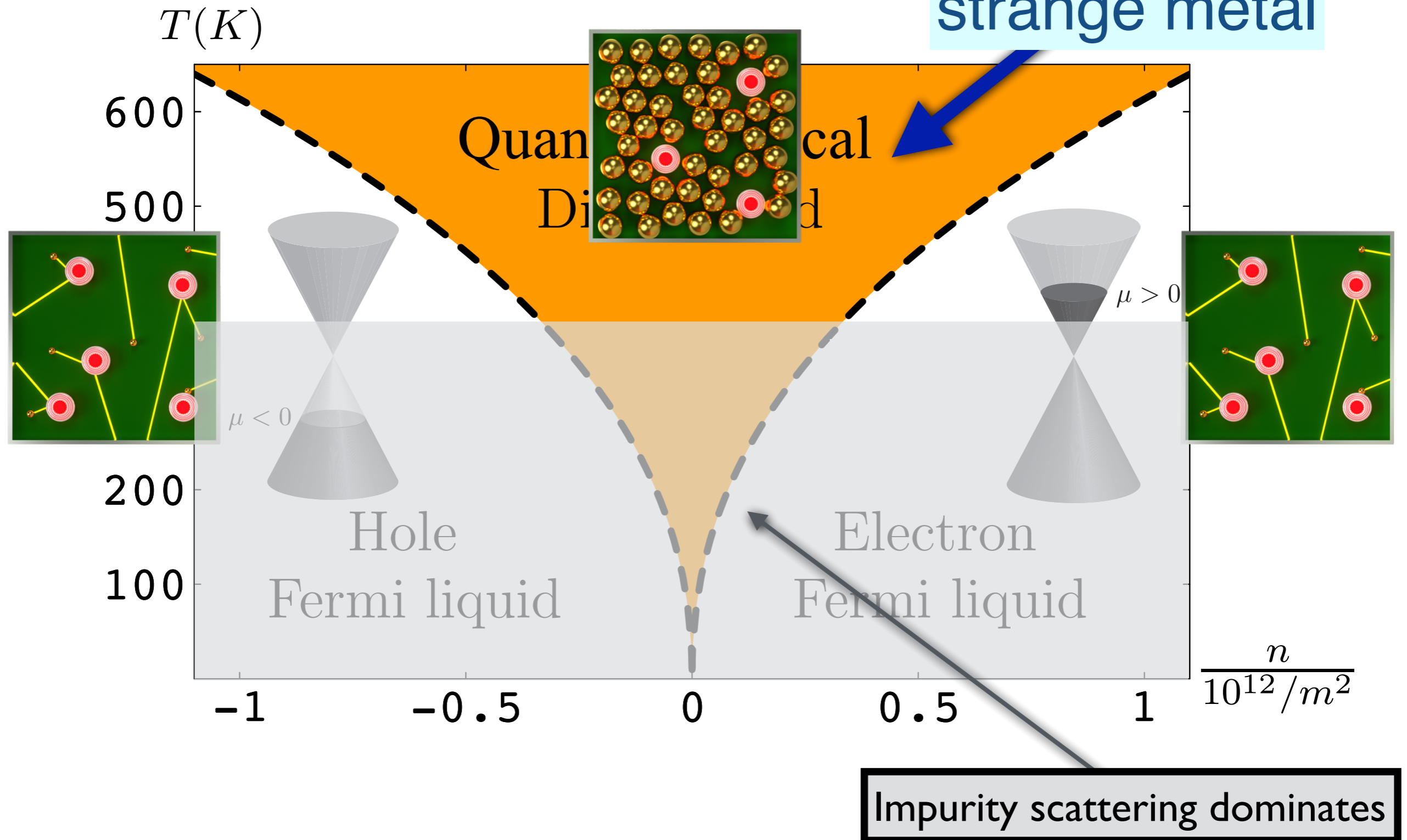


Red dots: data  
Blue line: value for  $L = L_0$



# Graphene

Predicted  
strange metal

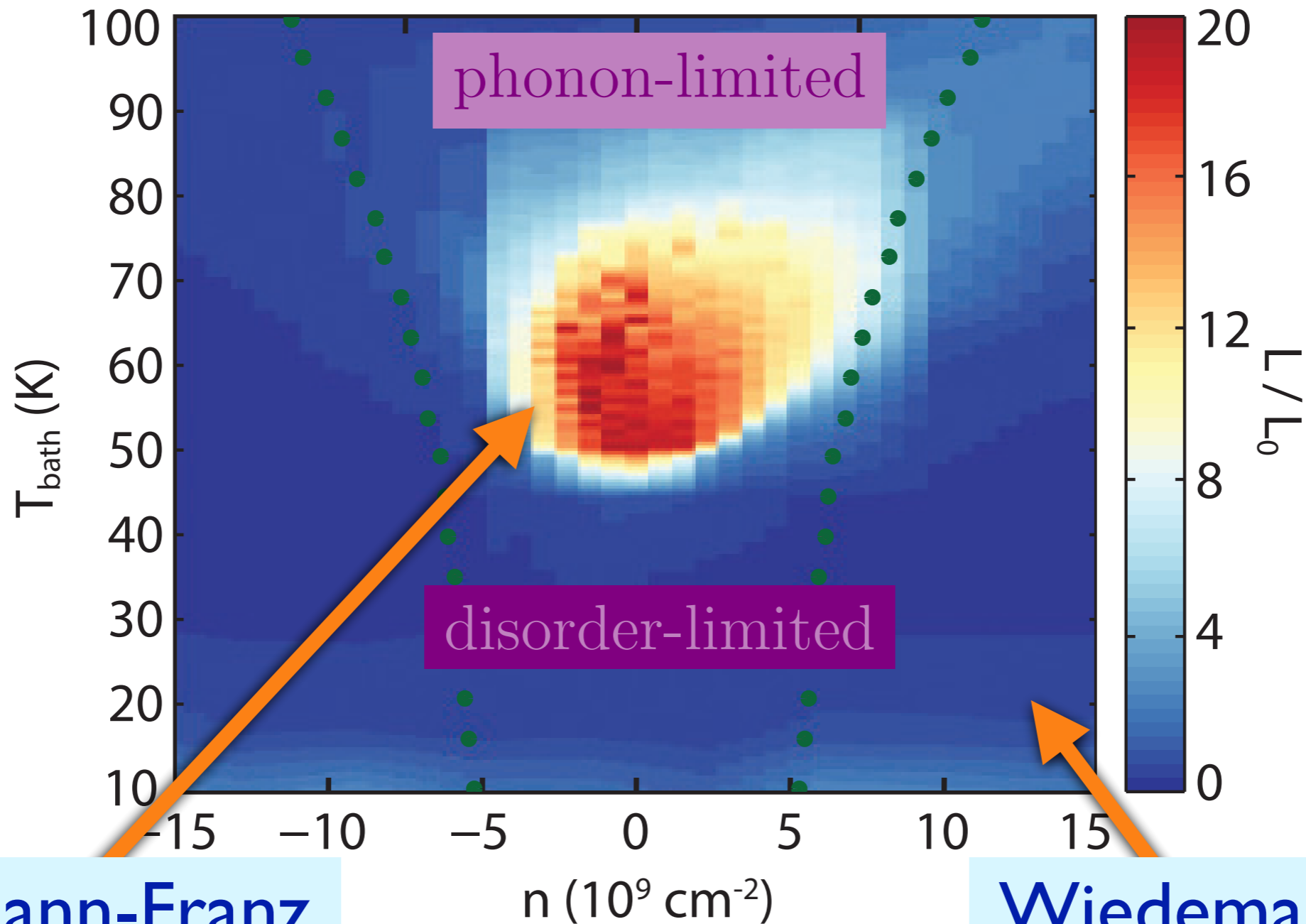
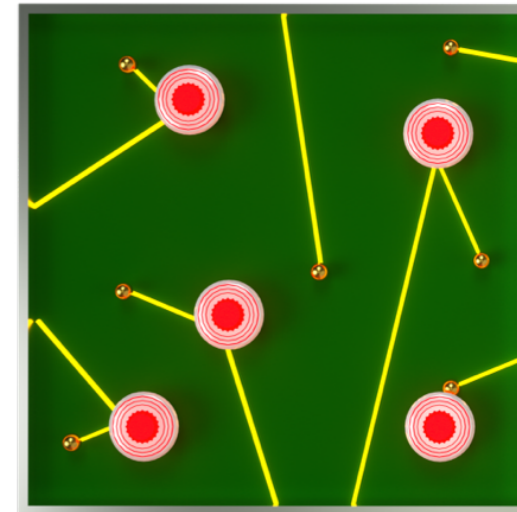


M. Müller, L. Fritz, and S. Sachdev, PRB **78**, 115406 (2008)

M. Müller and S. Sachdev, PRB **78**, 115419 (2008)

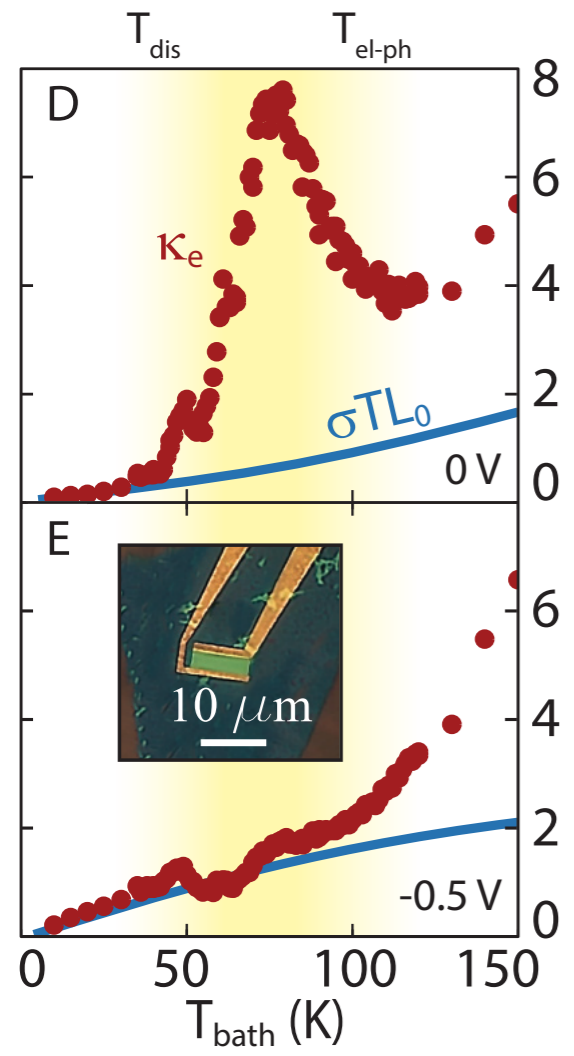
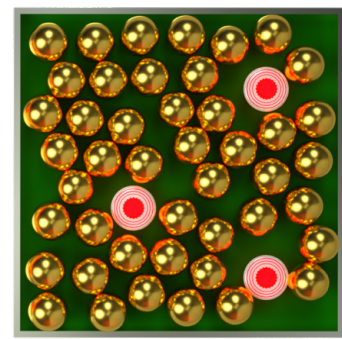
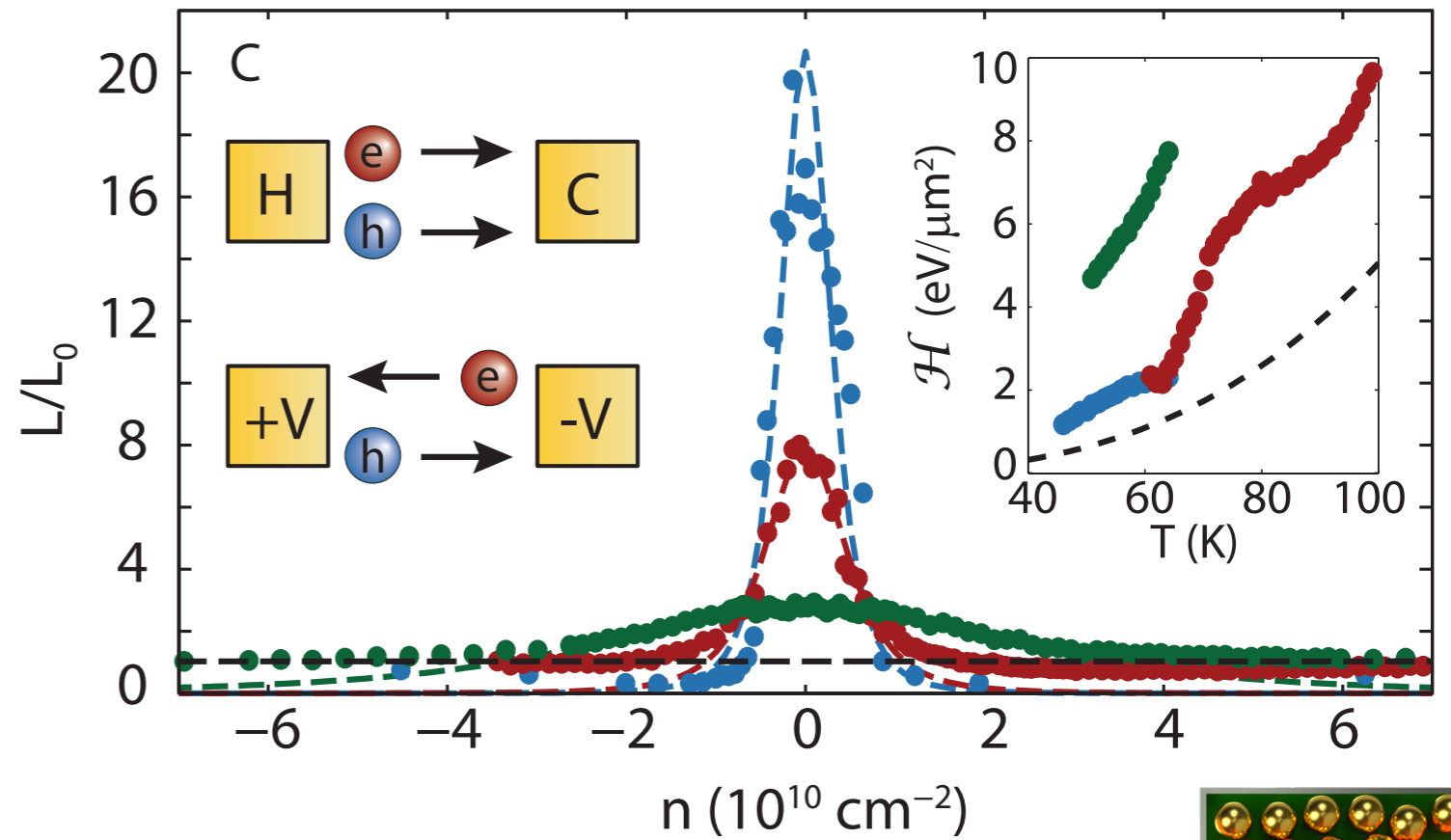
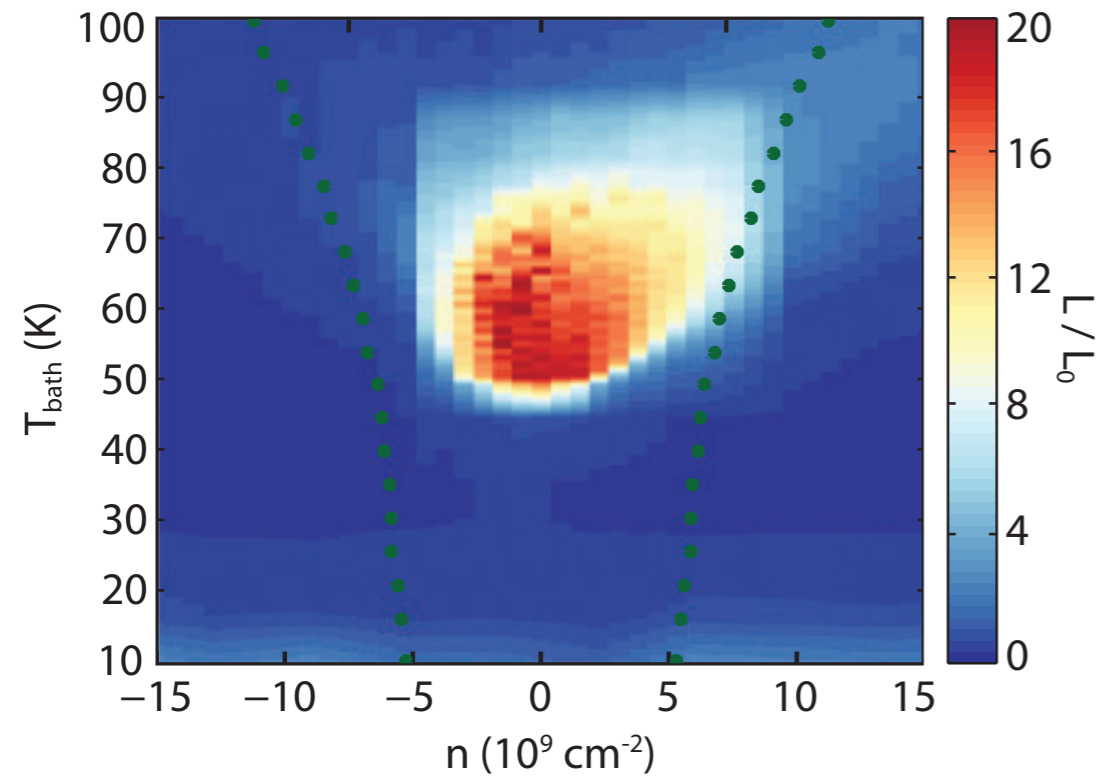
J. Crossno et al., Science **351**, 1058 (2016)

# Strange metal in graphene



Wiedemann-Franz  
violated !

Wiedemann-Franz  
obeyed



Lorentz ratio  $L = \kappa / (T\sigma)$

$$= \frac{v_F^2 \mathcal{H} \tau_{\text{imp}}}{T^2 \sigma_Q} \frac{1}{(1 + e^2 v_F^2 Q^2 \tau_{\text{imp}} / (\mathcal{H} \sigma_Q))^2}$$

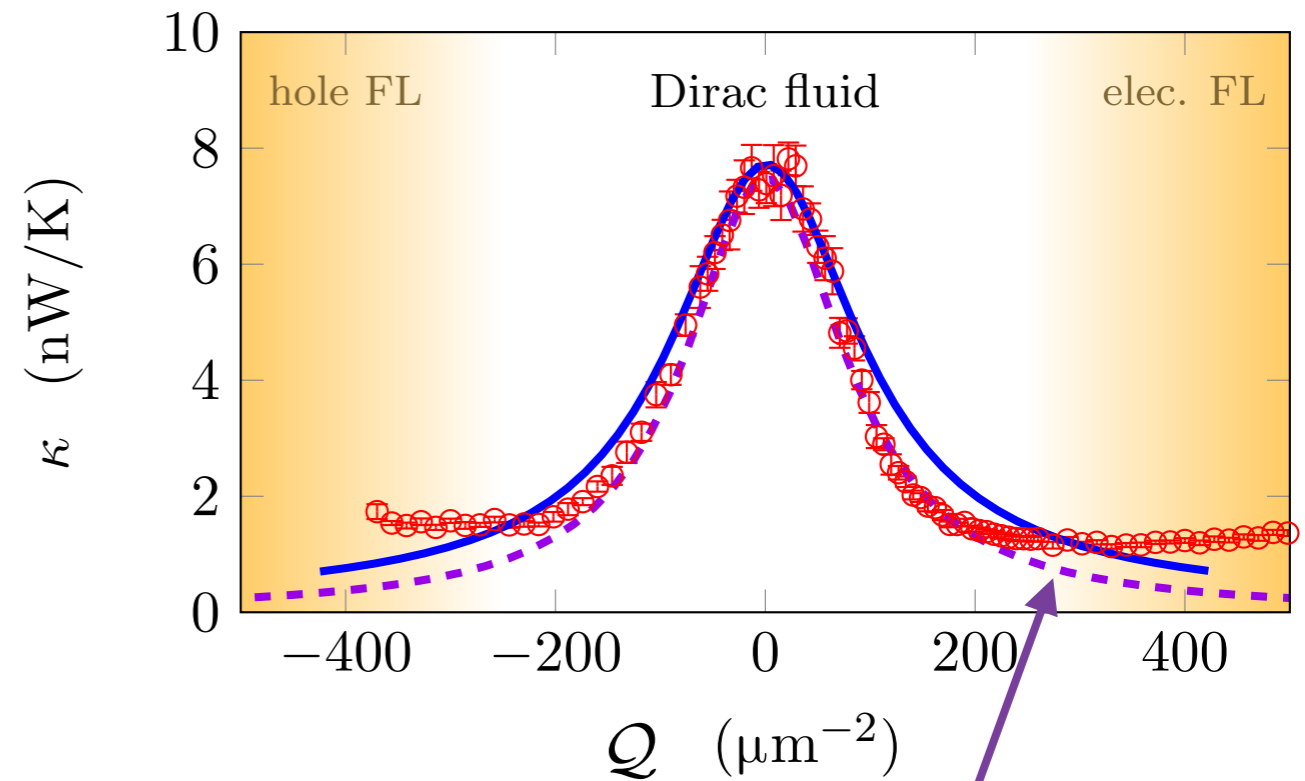
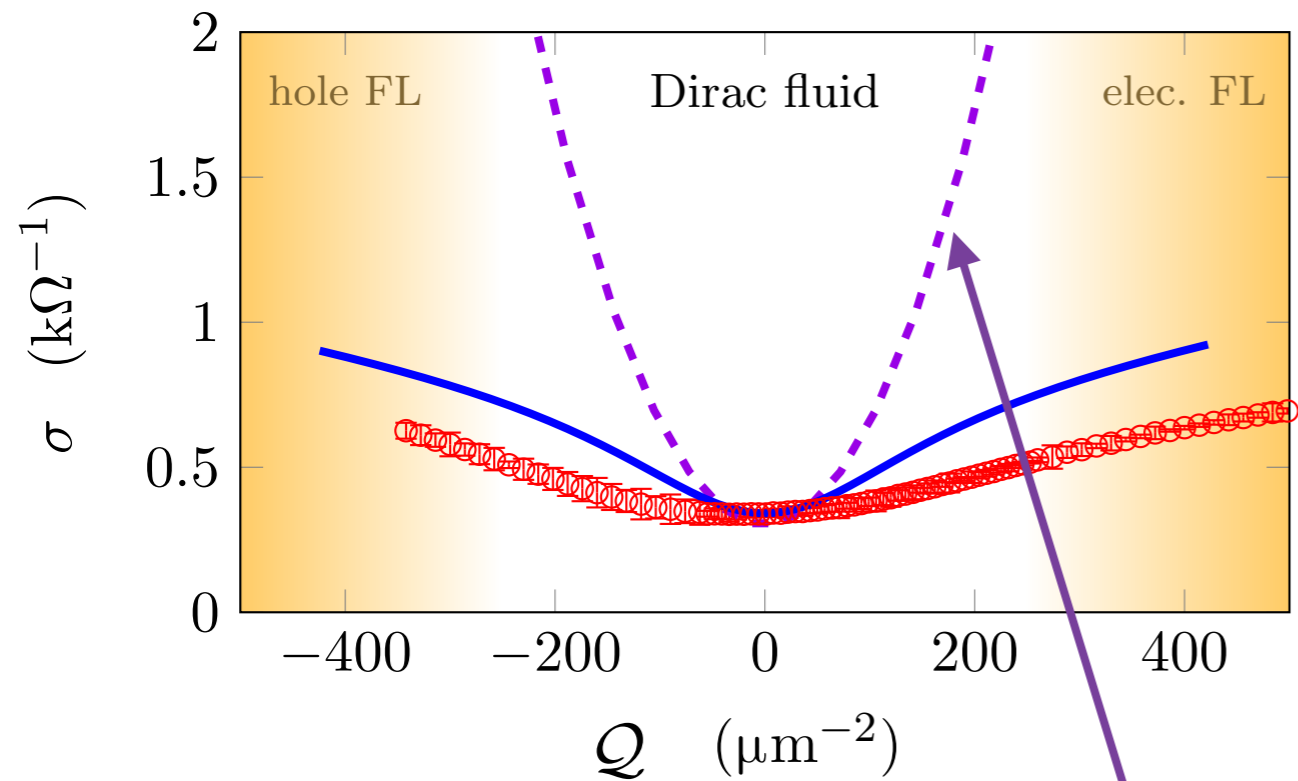
$Q \rightarrow$  electron density;  $\mathcal{H} \rightarrow$  enthalpy density

$\sigma_Q \rightarrow$  quantum critical conductivity

$\tau_{\text{imp}} \rightarrow$  momentum relaxation time from impurities

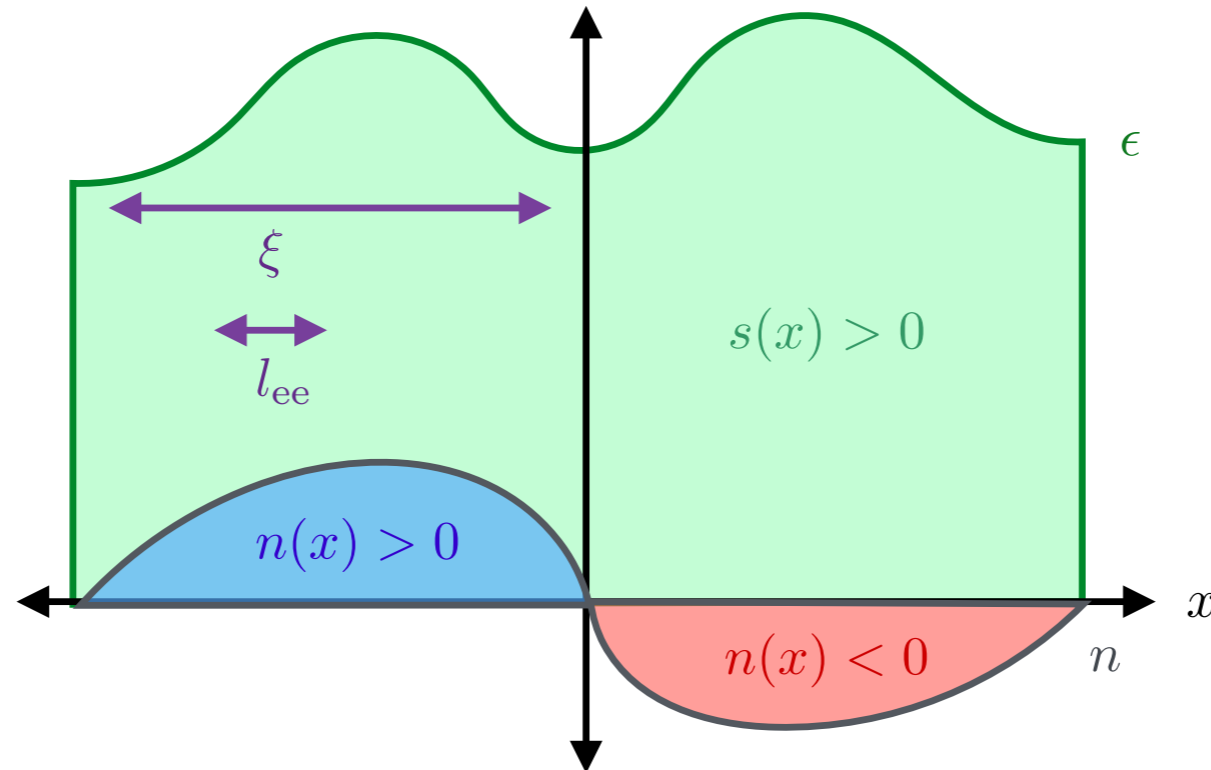
S. A. Hartnoll, P. K. Kovtun, M. Müller, and S. Sachdev, PRB **76**, 144502 (2007)

J. Crossno et al., Science **351**, 1058 (2016)



Comparison to theory with a single momentum relaxation time  $\tau_{\text{imp}}$ . Best fit of density dependence to thermal conductivity does not capture the density dependence of electrical conductivity

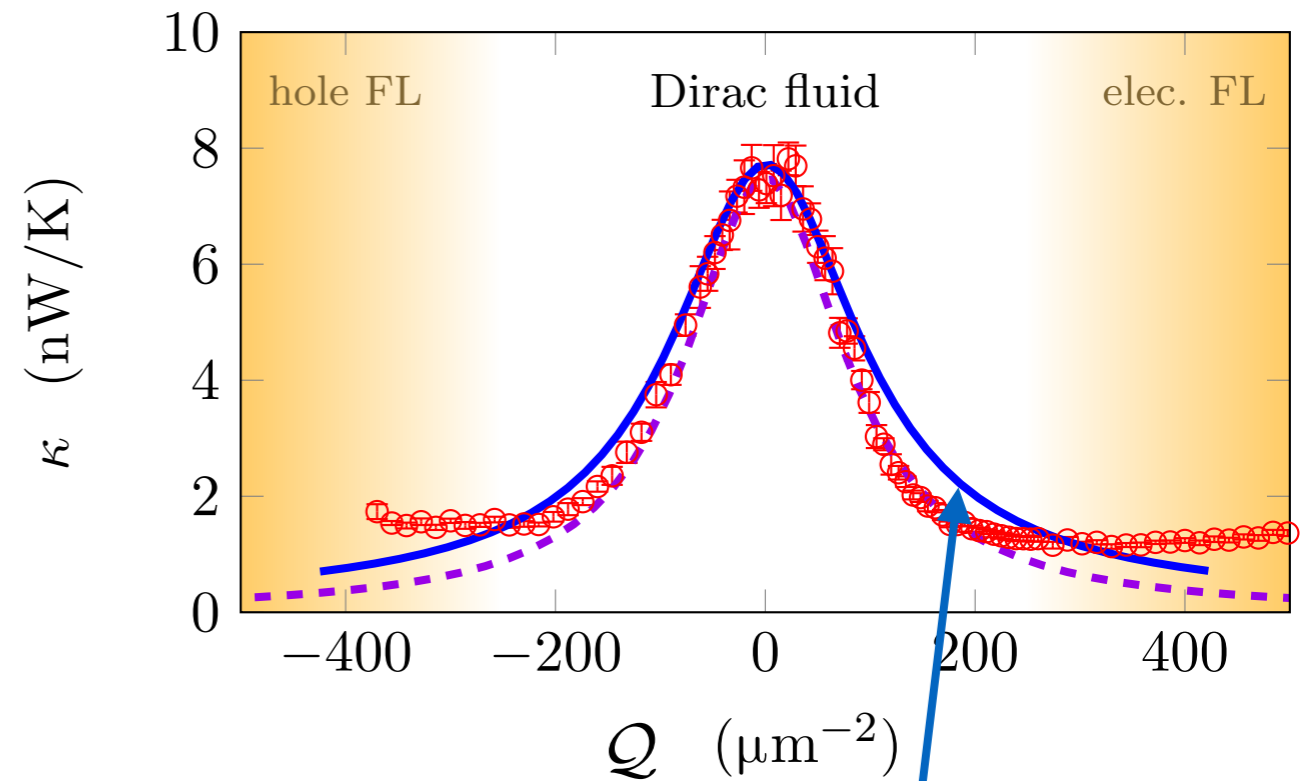
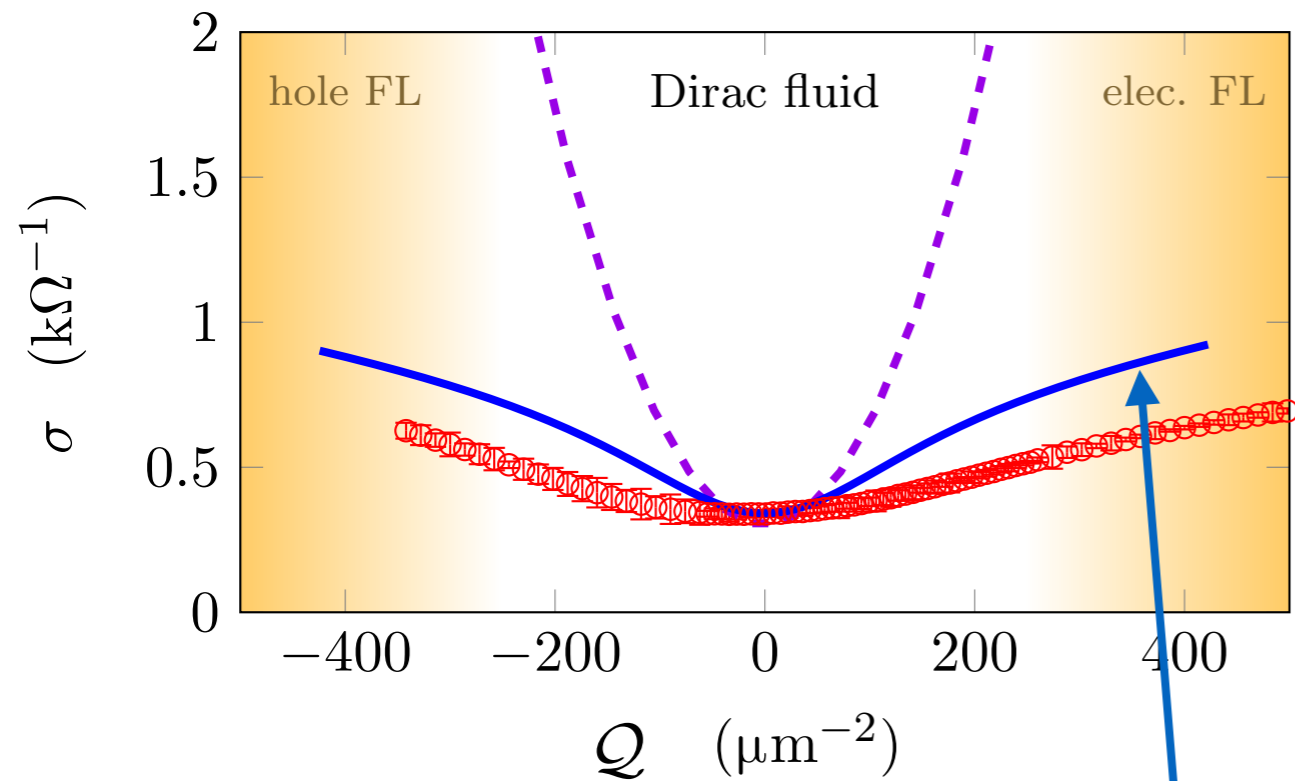
# Non-perturbative treatment of disorder



Note  
 $n \equiv Q$

**Figure 3:** A cartoon of a nearly quantum critical fluid where our hydrodynamic description of transport is sensible. The local chemical potential  $\mu(\mathbf{x})$  always obeys  $|\mu| \ll k_B T$ , and so the entropy density  $s/k_B$  is much larger than the charge density  $|n|$ ; both electrons and holes are everywhere excited, and the energy density  $\epsilon$  does not fluctuate as much relative to the mean. Near charge neutrality the local charge density flips sign repeatedly. The correlation length of disorder  $\xi$  is much larger than  $l_{ee}$ , the electron-electron interaction length.

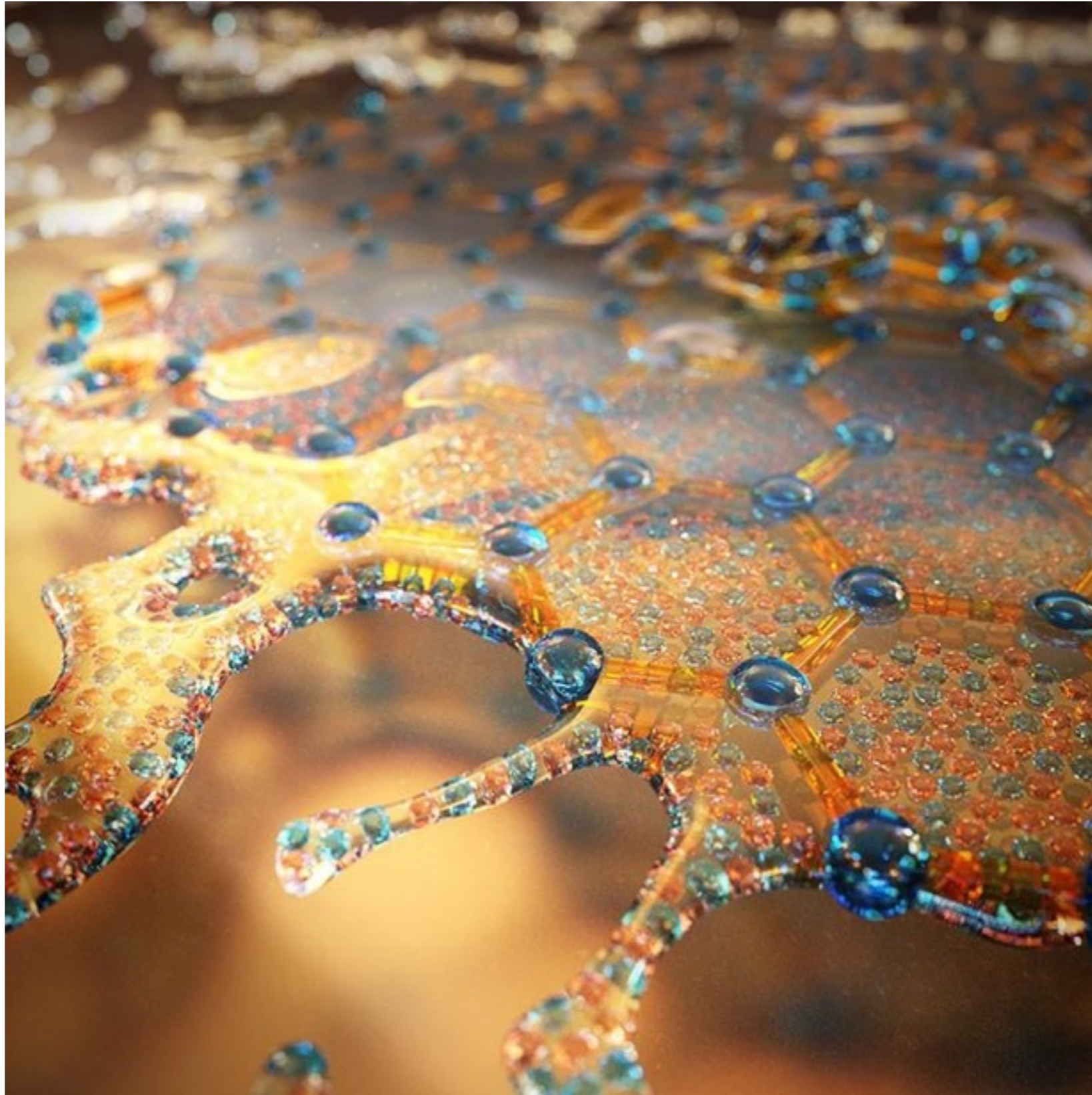
Numerically solve the hydrodynamic equations in the presence of a  $x$ -dependent chemical potential. The thermoelectric transport properties will then depend upon the value of the shear viscosity,  $\eta$ .



Solution of the hydrodynamic equations in the presence of a space-dependent chemical potential.

Best fit of density dependence to thermal conductivity now gives a better fit to the density dependence of the electrical conductivity (for  $\eta/s \approx 10$ ). The  $T$  dependencies of other parameters also agree well with expectation.

# Graphene: “a metal that behaves like water”



# Non-Fermi liquids

- Shortest possible “phase coherence” time, fastest possible local equilibration time, or fastest possible Lyapunov time towards quantum chaos, all of order  $\frac{\hbar}{k_B T}$
- Realization in solvable SYK model, which saturates the lower bound on the Lyapunov time. Its properties have some similarities to non-rational, large central charge CFT2s.
- Remarkable match between SYK and quantum gravity of black holes with AdS<sub>2</sub> horizons, including a SL(2,R)-invariant Schwarzian effective action for thermal energy fluctuations.
- Transport of non-Fermi liquids described by collective flow of fluid around impurities. Non-Fermi liquids with a critical Fermi surface have a divergent  $\eta/s$  as  $T \rightarrow 0$ . This is a consequence of the spatial anisotropy in the vicinity of a Fermi surface point, and unlike existing holographic models.
- Experiments on graphene agree well with predictions of a theory of a nearly relativistic quantum liquid without quasiparticles.