Quantum critical transport and AdS/CFT

Lars Fritz, Harvard
Sean Hartnoll, Harvard
Christopher Herzog, Princeton
Pavel Kovtun, Victoria
Markus Mueller, Trieste
Joerg Schmalian, Iowa
Dam Son, Washington

PHYSICS
VE RI TAS
HARVARD

1. Quantum-critical transport

   Collisionless-to-hydrodynamic crossover of CFT3s

2. Exact solution from AdS/CFT

   Constraints from duality relations

3. Generalized magnetohydrodynamics

   Quantum criticality and dyonic black holes

4. Experiments

   Graphene and the cuprate superconductors

1. Quantum-critical transport

Collisionless-to-hydrodynamic crossover of CFT3s

2. Exact solution from AdS/CFT

Constraints from duality relations

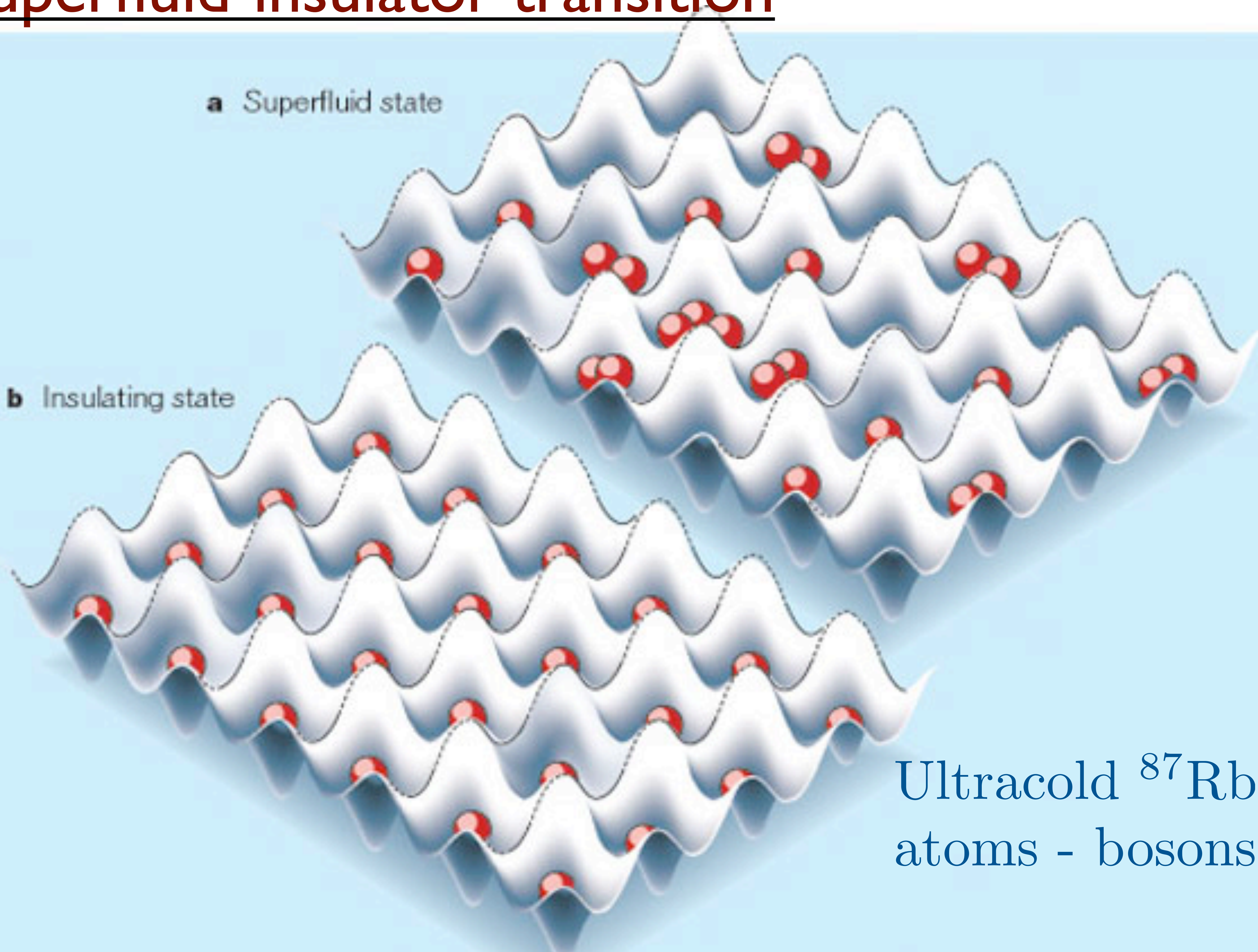
3. Generalized magnetohydrodynamics

Quantum criticality and dyonic black holes

4. Experiments

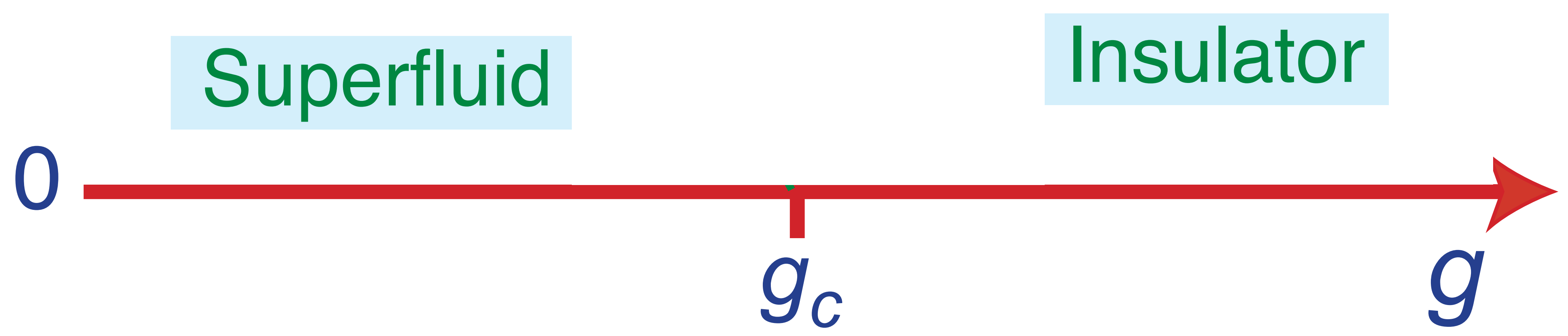
Graphene and the cuprate superconductors

Superfluid-insulator transition



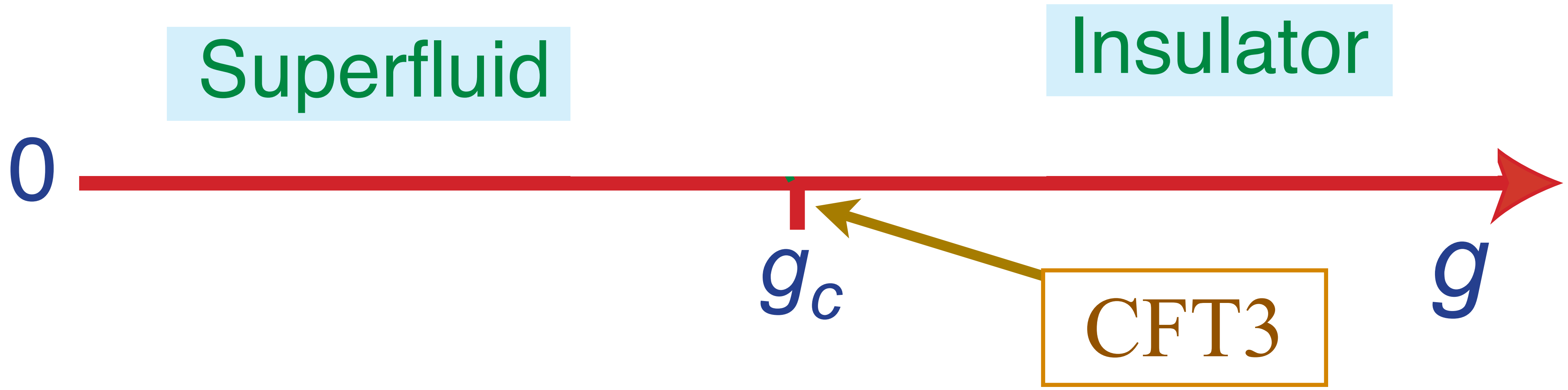
Ultracold ^{87}Rb atoms - bosons

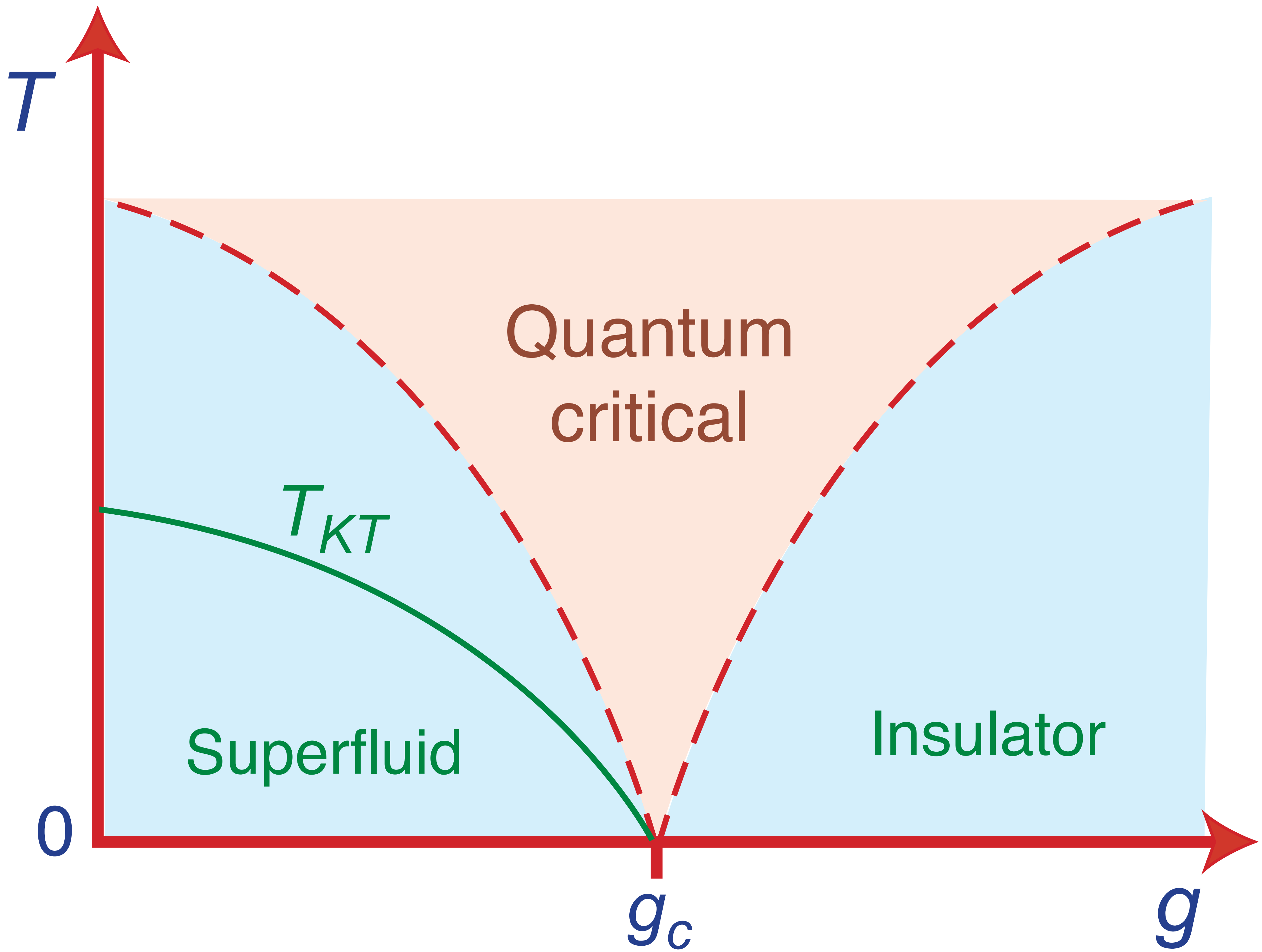
M. Greiner, O. Mandel, T. Esslinger, T. W. Hänsch, and I. Bloch, *Nature* **415**, 39 (2002).

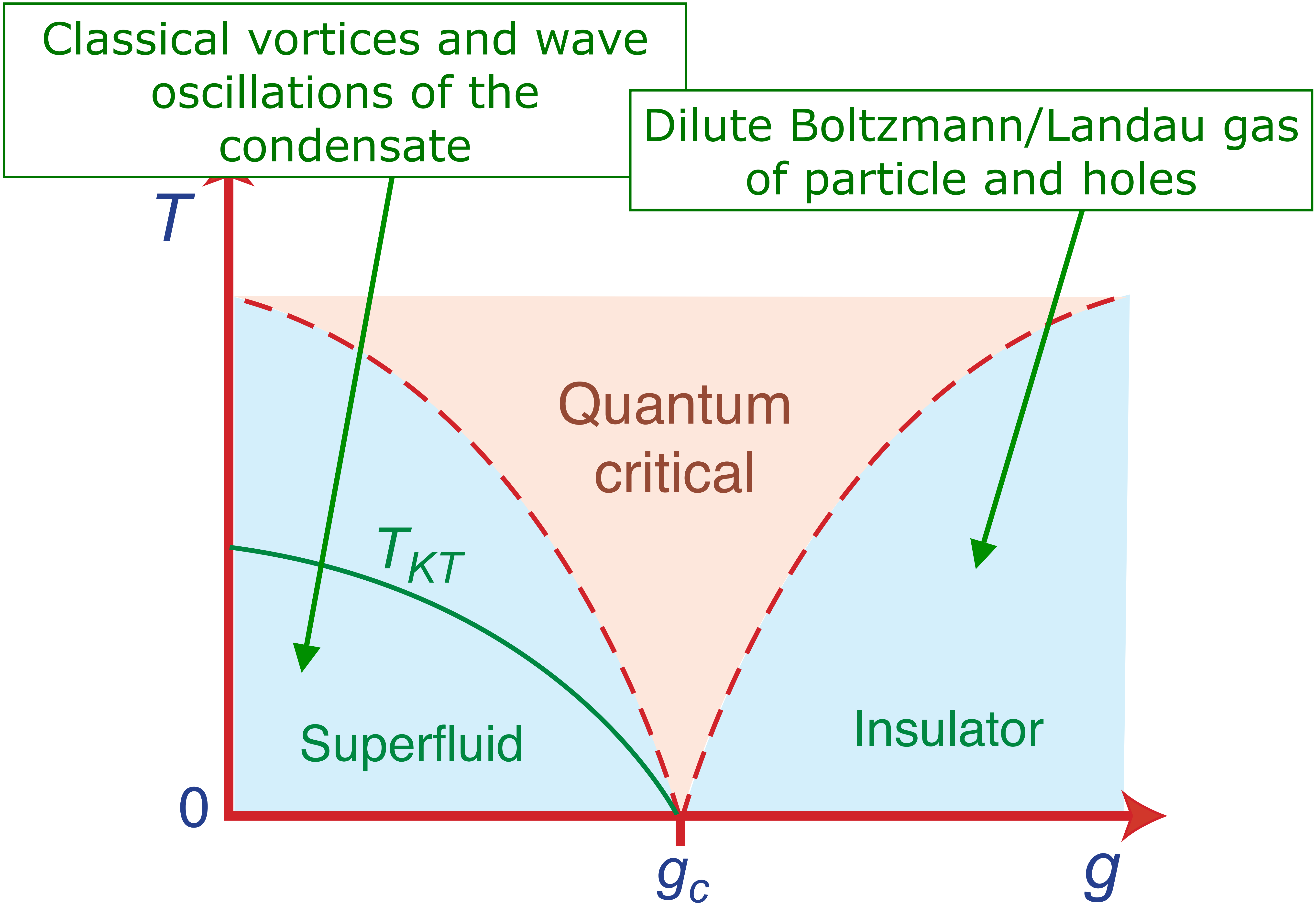

Superfluid
Insulator
0
g_c
g

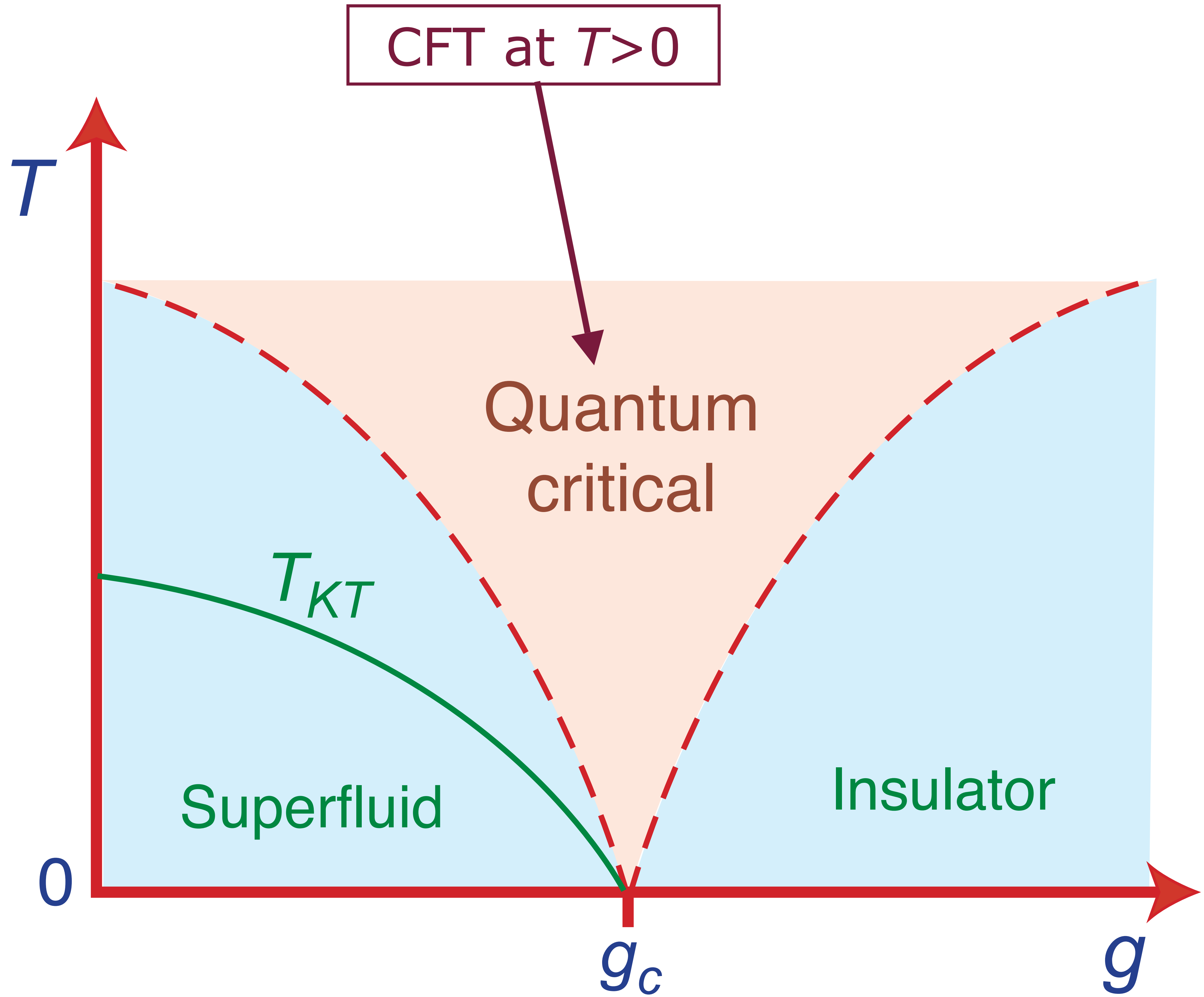
$$\mathcal{S} = \int d^2r d\tau \left[|\partial_\tau \psi|^2 + v^2 |\vec{\nabla} \psi|^2 + (g - g_c)|\psi|^2 + \frac{u}{2}|\psi|^4 \right]$$

$$\langle \psi \rangle \neq 0 \qquad\qquad \langle \psi \rangle = 0$$




T
Quantum
critical
T_KT
Superfluid
Insulator
0
g_c
g


Classical vortices and wave oscillations of the condensate
Dilute Boltzmann/Landau gas of particle and holes
T
Quantum critical
T_{KT}
Superfluid
Insulator
0
g_c
g


CFT at T>0
T
Quantum
critical
T_KT
Superfluid
Insulator
0
g_c
g

Resistivity of Bi films

Conductivity σ

$$\sigma_{\text{Superconductor}}(T \to 0) = \infty$$

$$\sigma_{\text{Insulator}}(T \to 0) = 0$$

$$\sigma_{\text{Quantum critical point}}(T \to 0) \approx \frac{4e^2}{h}$$

D. B. Haviland, Y. Liu, and A. M. Goldman, *Phys. Rev. Lett.* **62**, 2180 (1989)

M. P. A. Fisher, *Phys. Rev. Lett.* **65**, 923 (1990)

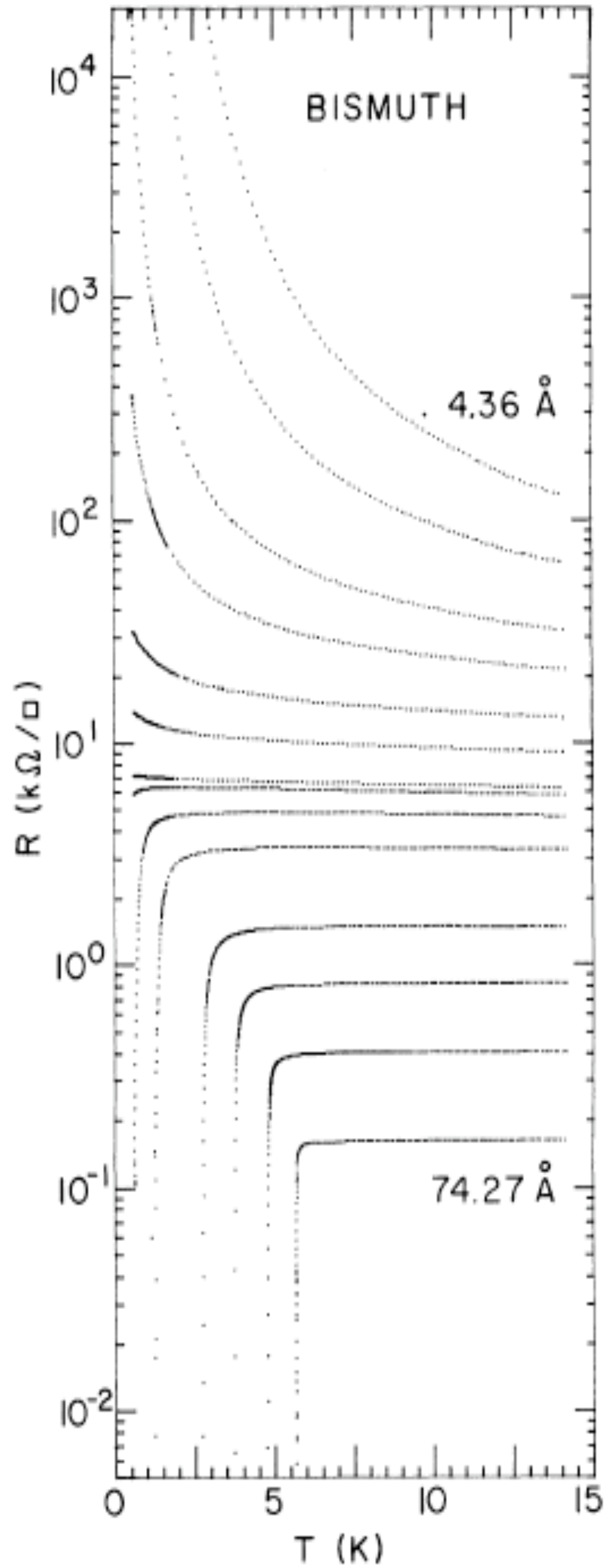


FIG. 1. Evolution of the temperature dependence of the sheet resistance $R(T)$ with thickness for a Bi film deposited onto Ge. Fewer than half of the traces actually acquired are shown. Film thicknesses shown range from 4.36 to 74.27 Å.

Quantum critical transport

Quantum *"perfect fluid"*
with shortest possible
relaxation time, τ_R

$$\tau_R \gtrsim \frac{\hbar}{k_B T}$$

S. Sachdev, *Quantum Phase Transitions,* Cambridge (1999).

Quantum critical transport

Transport co-oefficients not determined by collision rate, but by universal constants of nature

Electrical conductivity

$$\sigma = \frac{e^2}{h} \times [\text{Universal constant } \mathcal{O}(1)\ ]$$

K. Damle and S. Sachdev, *Phys. Rev. B* **56**, 8714 (1997).

Quantum critical transport

Transport co-oefficients not determined by collision rate, but by universal constants of nature

Momentum transport

$$\frac{\eta}{s} \equiv \frac{\text{viscosity}}{\text{entropy density}}$$

$$= \frac{\hbar}{k_B} \times [\text{Universal constant } \mathcal{O}(1)\ ]$$

P. Kovtun, D. T. Son, and A. Starinets, *Phys. Rev. Lett.* **94,** 11601 (2005)

Density correlations in CFTs at $T > 0$

Two-point density correlator, $\chi(k, \omega)$

Kubo formula for conductivity $\sigma(\omega) = \lim_{k \to 0} \frac{-i\omega}{k^2} \chi(k, \omega)$

For *all* CFT2s, at all $\hbar\omega / k_B T$

$$\chi(k, \omega) = \frac{4e^2}{h} K \frac{vk^2}{v^2 k^2 - \omega^2} \quad ; \quad \sigma(\omega) = \frac{4e^2}{h} \frac{Kv}{-i\omega}$$

where K is a universal number characterizing the CFT2 (the level number), and v is the velocity of "light".

This follows from the conformal mapping of the plane to the cylinder, which relates correlators at $T = 0$ to those at $T > 0$.

CFT correlator of $\mathrm{U}(1)$ current J_μ in 1+1 dimensions

Charge density correlation at $T = 0$:

$$\left\langle J_R(x,\tau) J_R(0) \right\rangle \sim \frac{1}{(\tau + ix)^2}$$

$$\left\langle J_t(k,\omega) J_t(-k,-\omega) \right\rangle \sim \frac{k^2}{k^2 - \omega^2}$$

CFT correlator of $\mathrm{U}(1)$ current J_μ in 1+1 dimensions

Charge density correlation at $T \geq 0$:

$$\left\langle J_R(x,\tau) J_R(0) \right\rangle \sim \frac{\pi^2 T^2}{\sin^2\left(\pi T(\tau + ix)\right)}$$

$$\left\langle J_t(k, i\omega_n) J_t(-k, -i\omega_n) \right\rangle \sim \frac{k^2}{k^2 + \omega_n^2}$$

Conformal mapping of plane to cylinder with circumference *1/T*

CFT correlator of $U(1)$ current J_μ in 1+1 dimensions

Charge density correlation at $T \geq 0$:

$$\left\langle J_R(x,\tau) J_R(0) \right\rangle \sim \frac{\pi^2 T^2}{\sin^2\left(\pi T(\tau + ix)\right)}$$

$$\left\langle J_t(k, i\omega_n) J_t(-k, -i\omega_n) \right\rangle \sim \frac{k^2}{k^2 + \omega_n^2}$$

$$\left\langle J_t(k,\omega) J_t(-k,-\omega) \right\rangle \sim \frac{k^2}{k^2 - \omega^2}$$

Conformal mapping of plane to cylinder with circumference *1/T*

Density correlations in CFTs at $T > 0$

Two-point density correlator, $\chi(k,\omega)$

Kubo formula for conductivity $\sigma(\omega) = \lim_{k\to 0} \frac{-i\omega}{k^2}\chi(k,\omega)$

For *all* CFT2s, at all $\hbar\omega/k_B T$

$$\chi(k,\omega) = \frac{4e^2}{h} K \frac{vk^2}{v^2k^2 - \omega^2} \quad ; \quad \sigma(\omega) = \frac{4e^2}{h}\frac{Kv}{-i\omega}$$

where K is a universal number characterizing the CFT2 (the level number), and v is the velocity of "light".
This follows from the conformal mapping of the plane to the cylinder, which relates correlators at $T = 0$ to those at $T > 0$.

No hydrodynamics in CFT2s.

Density correlations in CFTs at $T > 0$

Two-point density correlator, $\chi(k, \omega)$

Kubo formula for conductivity $\sigma(\omega) = \lim_{k \to 0} \frac{-i\omega}{k^2} \chi(k, \omega)$

For *all* CFT3s, at $\hbar\omega \gg k_B T$

$$\chi(k, \omega) = \frac{4e^2}{h} K \frac{k^2}{\sqrt{v^2 k^2 - \omega^2}} \quad ; \quad \sigma(\omega) = \frac{4e^2}{h} K$$

where K is a universal number characterizing the CFT3, and v is the velocity of "light".

Density correlations in CFTs at $T > 0$

Two-point density correlator, $\chi(k,\omega)$

Kubo formula for conductivity $\sigma(\omega) = \lim_{k\to 0} \frac{-i\omega}{k^2}\chi(k,\omega)$

However, for *all* CFT3s, at $\hbar\omega \ll k_B T$, we have the Einstein relation

$$\chi(k,\omega) = 4e^2\chi_c \frac{Dk^2}{Dk^2 - i\omega} \quad ; \quad \sigma(\omega) = 4e^2 D\chi_c = \frac{4e^2}{h}\Theta_1\Theta_2$$

where the **compressibility**, χ_c, and the **diffusion constant** D obey

$$\chi = \frac{k_B T}{(hv)^2}\Theta_1 \quad ; \quad D = \frac{hv^2}{k_B T}\Theta_2$$

with Θ_1 and Θ_2 universal numbers characteristic of the CFT3

K. Damle and S. Sachdev, *Phys. Rev. B* **56**, 8714 (1997).

Density correlations in CFTs at $T > 0$

In CFT3s collisions are "phase" randomizing, and lead to relaxation to local thermodynamic equilibrium. So there is a crossover from collisionless behavior for $\hbar\omega \gg k_B T$, to hydrodynamic behavior for $\hbar\omega \ll k_B T$.

$$\sigma(\omega) = \begin{cases} \dfrac{4e^2}{h} K & , \quad \hbar\omega \gg k_B T \\ \dfrac{4e^2}{h} \Theta_1 \Theta_2 \equiv \sigma_Q & , \quad \hbar\omega \ll k_B T \end{cases}$$

and in general we expect $K \neq \Theta_1 \Theta_2$ (verified for Wilson-Fisher fixed point).

K. Damle and S. Sachdev, *Phys. Rev. B* **56**, 8714 (1997).

1. Quantum-critical transport

Collisionless-to-hydrodynamic crossover of CFT3s

2. Exact solution from AdS/CFT

Constraints from duality relations

3. Generalized magnetohydrodynamics

Quantum criticality and dyonic black holes

4. Experiments

Graphene and the cuprate superconductors

1. Quantum-critical transport

Collisionless-to-hydrodynamic crossover of CFT3s

2. Exact solution from AdS/CFT

Constraints from duality relations

3. Generalized magnetohydrodynamics

Quantum criticality and dyonic black holes

4. Experiments

Graphene and the cuprate superconductors

SU(N) SYM3 with $\mathcal{N} = 8$ supersymmetry

- Has a single dimensionful coupling constant, e_0, which flows to a strong-coupling fixed point $e_0 = e_0^*$ in the infrared.
- The CFT3 describing this fixed point resembles "critical spin liquid" theories.
- This CFT3 is the low energy limit of string theory on an M2 brane. The AdS/CFT correspondence provides a dual description using 11-dimensional supergravity on $\text{AdS}_4 \times S_7$.
- The CFT3 has a global SO(8) R symmetry, and correlators of the SO(8) charge density can be computed exactly in the large N limit, even at $T > 0$.

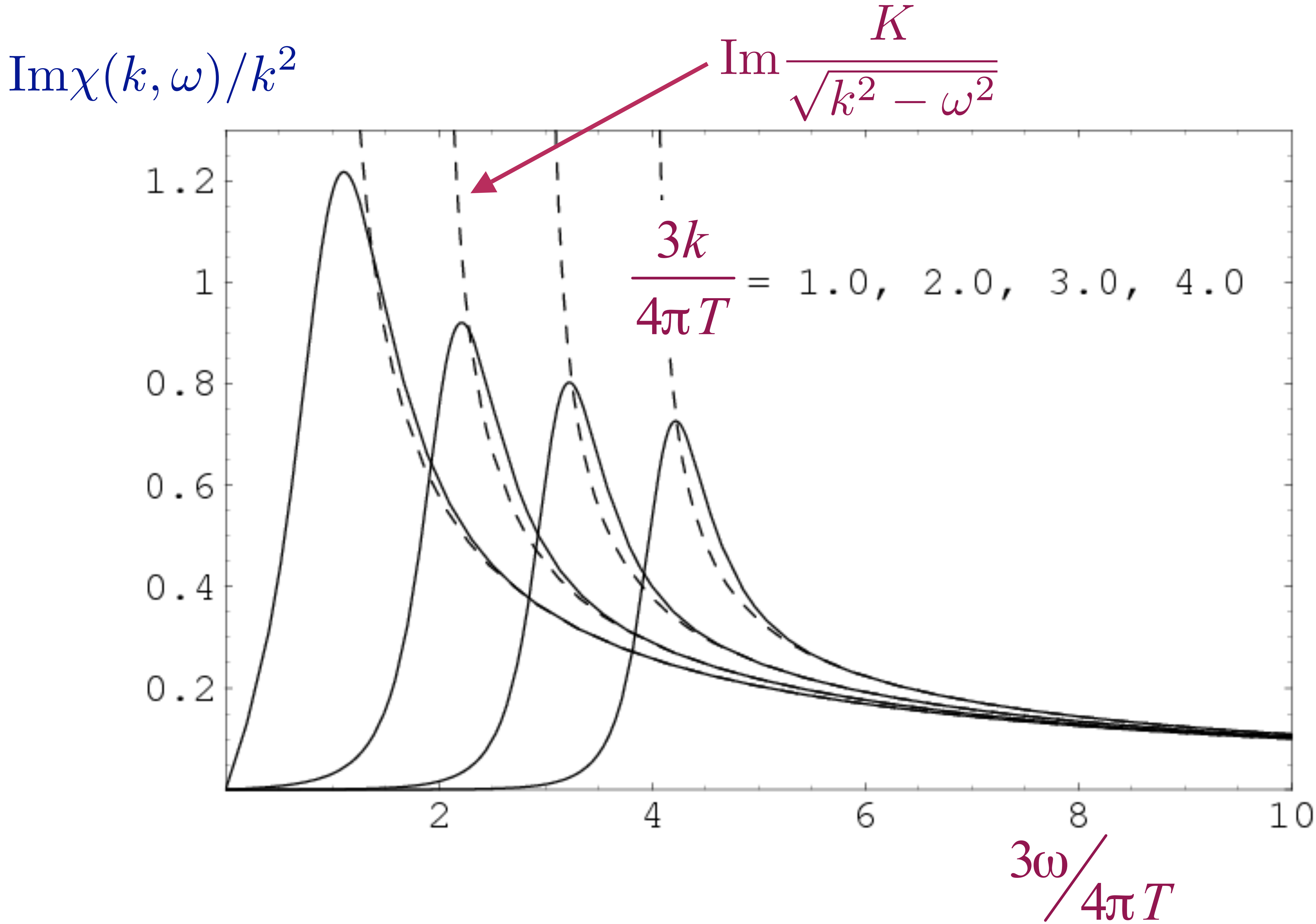
SU(N) SYM3 with $\mathcal{N} = 8$ supersymmetry

- The SO(8) charge correlators of the CFT3 are given by the usual AdS/CFT prescription applied to the following gauge theory on AdS4:

$$\mathcal{S} = -\frac{1}{4g_{4D}^2}\int d^4x\sqrt{-g}g^{MA}g^{NB}F^a_{MN}F^a_{AB}$$

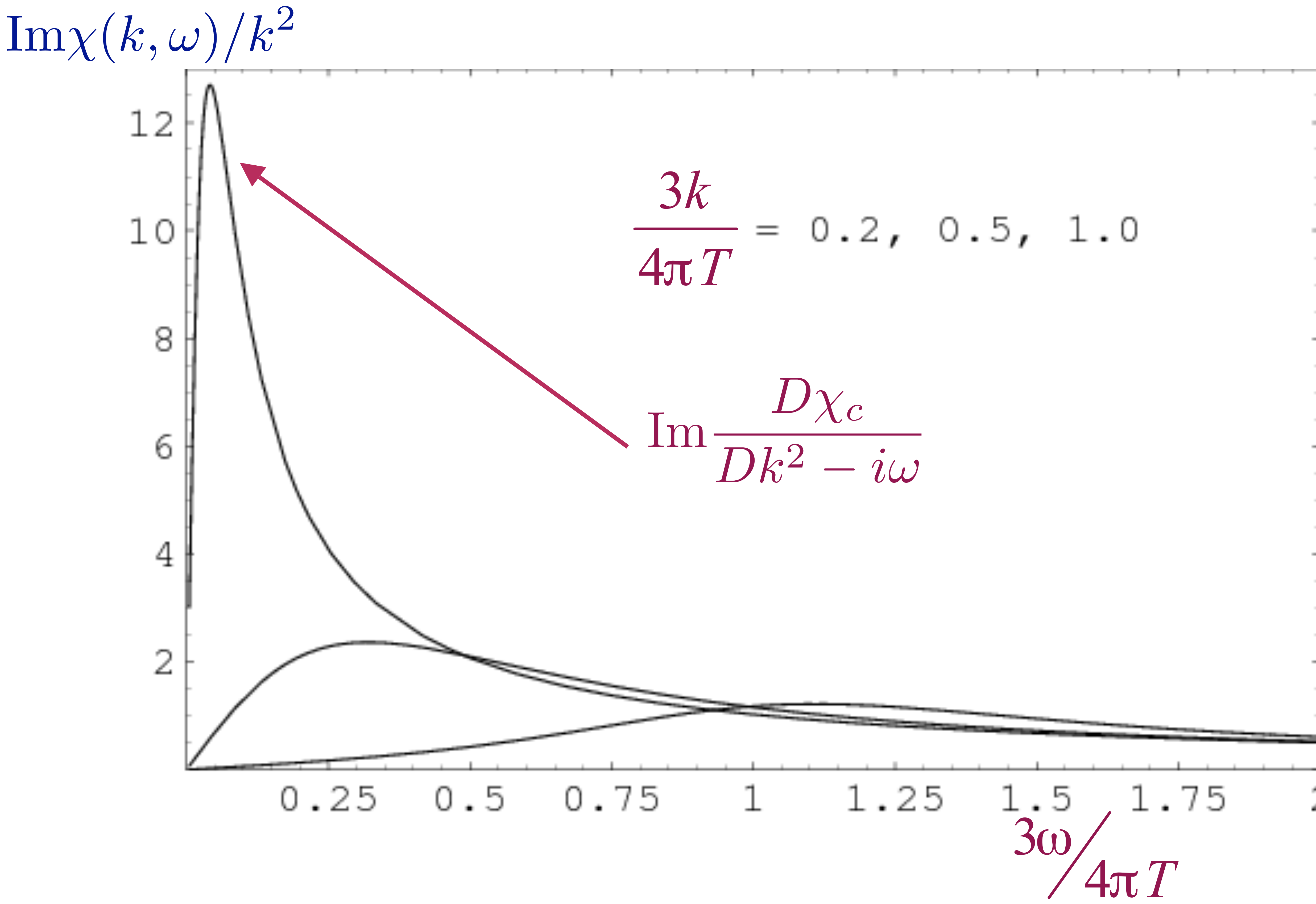
  where $a = 1 \ldots 28$ labels the generators of SO(8). Note that in large N theory, this looks like 28 copies of an Abelian gauge theory.

Collisionless to hydrodynamic crossover of SYM3



P. Kovtun, C. Herzog, S. Sachdev, and D.T. Son, Phys. Rev. D **75**, 085020 (2007)

Collisionless to hydrodynamic crossover of SYM3



P. Kovtun, C. Herzog, S. Sachdev, and D.T. Son, Phys. Rev. D **75**, 085020 (2007)

Universal constants of SYM3

$$\chi_c = \frac{k_B T}{(hv)^2}\Theta_1$$

$$D = \frac{hv^2}{k_B T}\Theta_2$$

$$\sigma(\omega) = \begin{cases} \frac{4e^2}{h}K & , \quad \hbar\omega \gg k_B T \\ \frac{4e^2}{h}\Theta_1\Theta_2 & , \quad \hbar\omega \ll k_B T \end{cases}$$

$$K = \frac{\sqrt{2}N^{3/2}}{3}$$

$$\Theta_1 = \frac{8\pi^2\sqrt{2}N^{3/2}}{9}$$

$$\Theta_2 = \frac{3}{8\pi^2}$$

C. Herzog, JHEP **0212**, 026 (2002)

P. Kovtun, C. Herzog, S. Sachdev, and D.T. Son, Phys. Rev. D **75**, 085020 (2007)

Electromagnetic self-duality

- Unexpected result, $K = \Theta_1 \Theta_2$.
- This is traced to a *four*-dimensional electromagnetic self-duality of the theory on AdS_4. In the large N limit, the SO(8) currents decouple into 28 U(1) currents with a Maxwell action for the U(1) gauge fields on AdS_4.
- This special property is not expected for generic CFT3s.

CFT correlator of $\mathrm{U}(1)$ current J_μ at $T = 0$

$$\left\langle J_\mu(p) J_\nu(-p) \right\rangle = K\sqrt{p^2}\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)$$

K: a universal number analogous to the level number of the Kac-Moody algebra in 1+1 dimensions

Application of Kubo formula shows that

$$\sigma\left(\omega/T = \infty\right) = \frac{4e^2}{h} 2\pi K$$

CFT correlator of $\mathrm{U}(1)$ current J_μ at $T > 0$

$$\left\langle J_\mu(k,\omega) J_\nu(-k,-\omega)\right\rangle = \sqrt{k^2-\omega^2}\left(P^T_{\mu\nu} K^T(k,\omega) + P^L_{\mu\nu} K^L(k,\omega)\right)$$

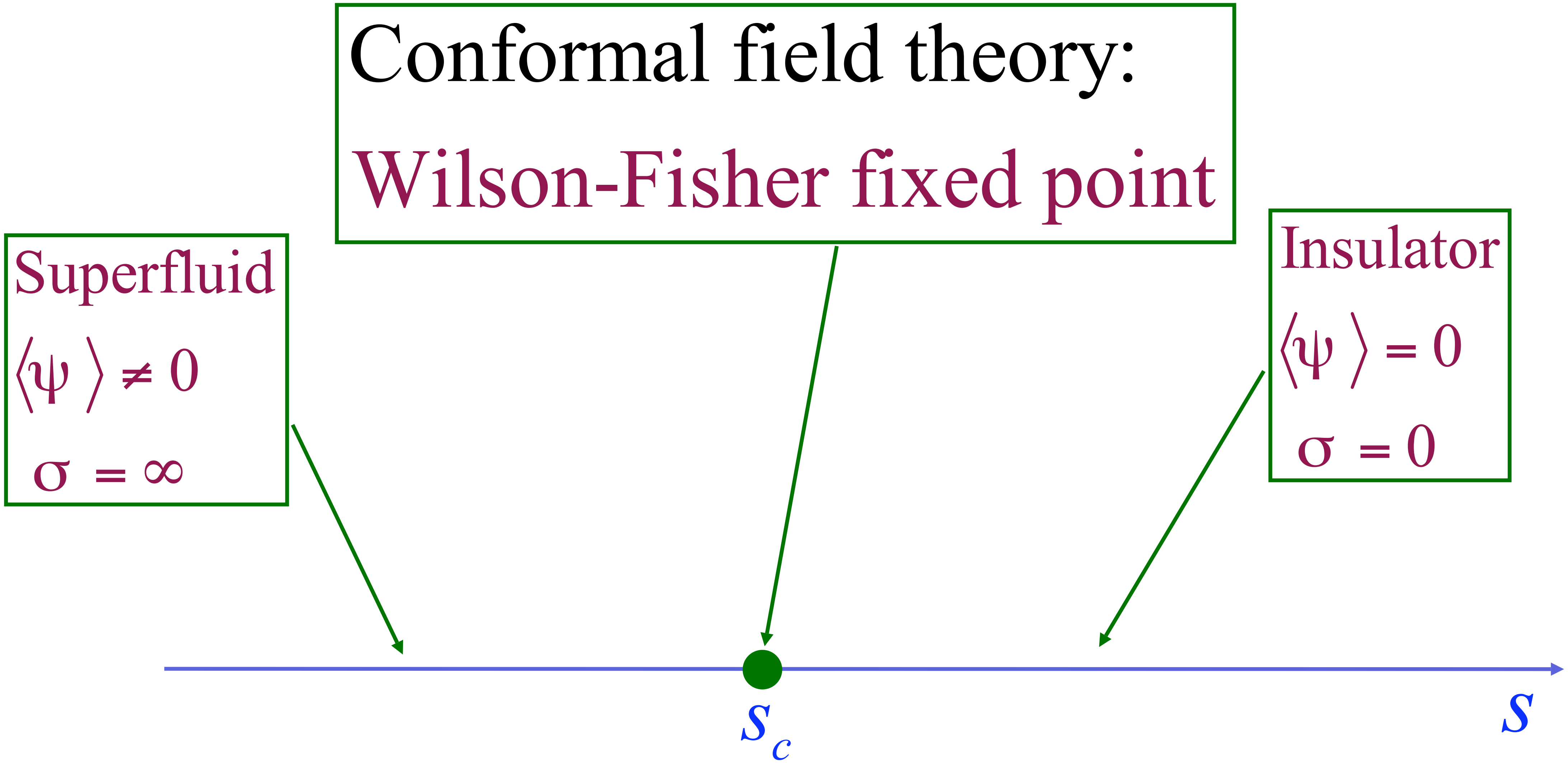
The projectors are defined by

$$P^T_{ij} = \delta_{ij} - \frac{k_i k_j}{k^2} \text{ and } P^L_{\mu\nu} = \eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} - P^T_{\mu\nu} \quad ; \quad p = (k,\omega)$$

while $K^{L,T}(k,\omega)$ are universal functions of ω/T and k/T

Application of Kubo formula shows that

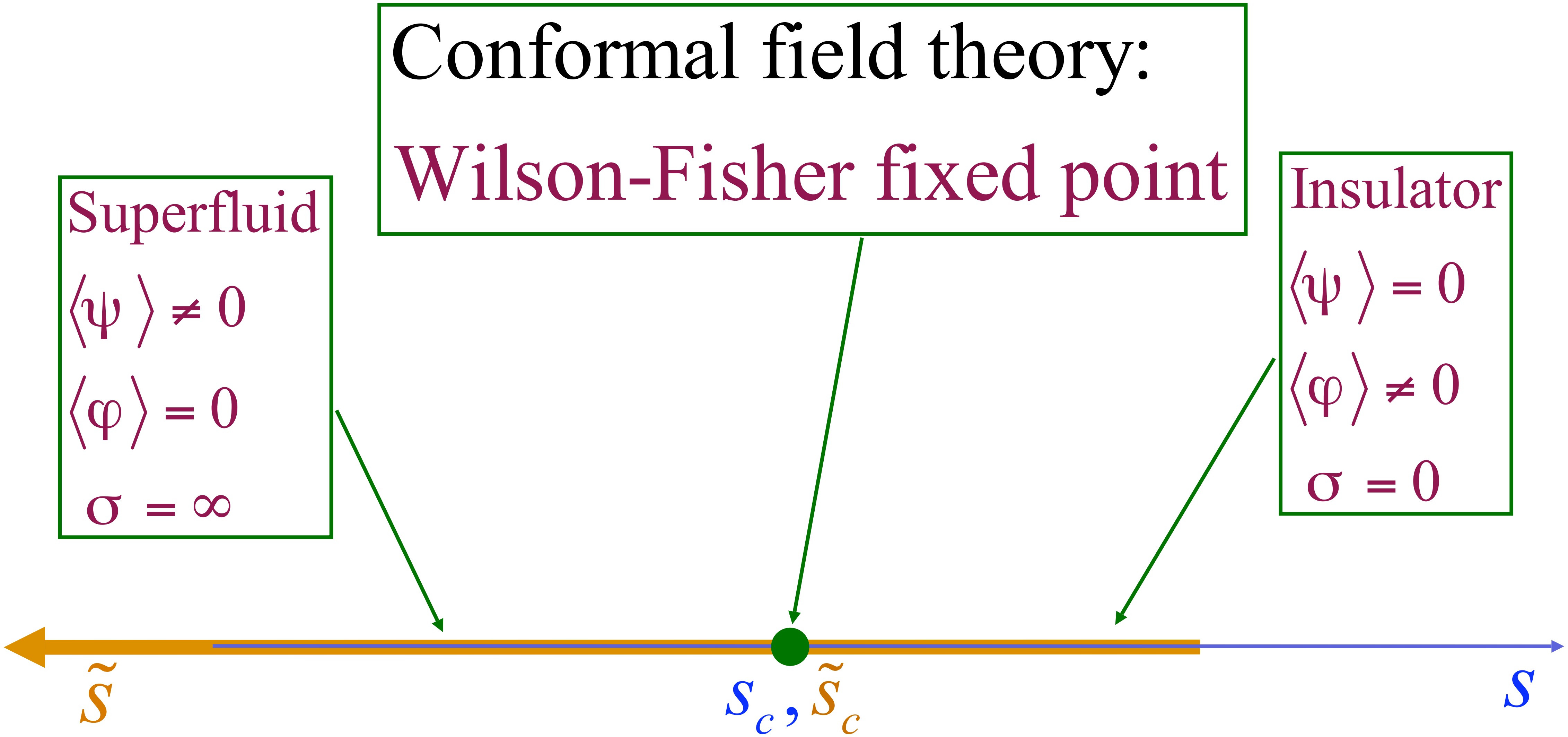
$$\sigma\left(\omega/T\right) = \frac{4e^2}{h} 2\pi K^T(0,\omega) = \frac{4e^2}{h} 2\pi K^L(0,\omega)$$



Using the boson quasiparticle excitations of the insulator $\sim \psi$

$$\mathcal{S} = \int d^3x \left[|\partial_\mu \psi|^2 + s|\psi|^2 + \frac{u}{2}|\psi|^4 \right]$$

C. Dasgupta and B.I. Halperin, *Phys. Rev. Lett.* **47**, 1556 (1981)



Using the boson quasiparticle excitations of the insulator $\sim \psi$

$$\mathcal{S} = \int d^3x \left[|\partial_\mu \psi|^2 + s|\psi|^2 + \frac{u}{2}|\psi|^4 \right]$$

<u>is dual to</u>

Using the vortex quasiparticle excitations of the superfluid $\sim \varphi$

$$\mathcal{S}_{\rm dual} = \int d^3x \left[|(\partial_\mu - iA_\mu)\varphi|^2 + \tilde{s}|\varphi|^2 + \frac{\tilde{u}}{2}|\varphi|^4 + \frac{1}{2e^2}(\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda)^2 \right]$$

C. Dasgupta and B.I. Halperin, *Phys. Rev. Lett.* **47**, 1556 (1981)

Consequences of duality on CFT correlators of $U(1)$ currents

$$\langle J_\mu(k,\omega) J_\nu(k,\omega)\rangle_{\mathcal{S}} = \sqrt{k^2-\omega^2}\left(P^T_{\mu\nu}K^T(k,\omega) + P^L_{\mu\nu}K^L(k,\omega)\right)$$

$$\left\langle \widetilde{J}_\mu(k,\omega) \widetilde{J}_\nu(k,\omega)\right\rangle_{\mathcal{S}_{\rm dual}} = \sqrt{k^2-\omega^2}\left(P^T_{\mu\nu}\widetilde{K}^T(k,\omega) + P^L_{\mu\nu}\widetilde{K}^L(k,\omega)\right)$$

$$K^L(k,\omega)\widetilde{K}^T(k,\omega) = \frac{1}{4\pi^2}$$
$$K^T(k,\omega)\widetilde{K}^L(k,\omega) = \frac{1}{4\pi^2}$$

Application of Kubo formula shows that

$$\sigma\left(\omega/T\right) = \frac{4e^2}{h}2\pi K^T(0,\omega) = \frac{4e^2}{h}2\pi K^L(0,\omega)$$

C. Herzog, P. Kovtun, S. Sachdev, and D.T. Son, hep-th/0701036

Correlations of SO(8) currents of the SYM_3 SCFT at $T > 0$

$$\left\langle J_\mu^a(k,\omega) J_\nu^b(-k,-\omega) \right\rangle = \delta^{ab}\sqrt{k^2-\omega^2}\left(P_{\mu\nu}^T K^T(k,\omega) + P_{\mu\nu}^L K^L(k,\omega)\right)$$

The self-duality of the 4D abelian gauge fields leads to

$$K^L(k,\omega) K^T(k,\omega) = \frac{N^3}{18\pi^2}$$

C. Herzog, P. Kovtun, S. Sachdev, and D.T. Son, hep-th/0701036

Correlations of SO(8) currents of the SYM_3 SCFT at $T > 0$

$$\left\langle J_\mu^a(k,\omega) J_\nu^b(-k,-\omega) \right\rangle = \delta^{ab} \sqrt{k^2 - \omega^2} \left(P_{\mu\nu}^T K^T(k,\omega) + P_{\mu\nu}^L K^L(k,\omega) \right)$$

The self-duality of the 4D abelian gauge fields leads to

$$\boxed{K^L(k,\omega) K^T(k,\omega) = \frac{N^3}{18\pi^2}}$$

Analyticity of correlations at $T > 0$ implies

$$K^T(0,\omega) = K^L(0,\omega),$$

and so the conductivity

$$\sigma(\omega/T) = K^T(0,\omega) = K^L(0,\omega) = \sqrt{\frac{N^3}{72\pi^2}}$$

is frequency independent.

C. Herzog, P. Kovtun, S. Sachdev, and D.T. Son, hep-th/0701036

Electromagnetic self-duality

- Unexpected result, $K = \Theta_1 \Theta_2$.

- This is traced to a *four*-dimensional electromagnetic self-duality of the theory on AdS_4. In the large N limit, the SO(8) currents decouple into 28 U(1) currents with a Maxwell action for the U(1) gauge fields on AdS_4.

- This special property is not expected for generic CFT3s.

1. Quantum-critical transport

Collisionless-to-hydrodynamic crossover of CFT3s

2. Exact solution from AdS/CFT

Constraints from duality relations

3. Generalized magnetohydrodynamics

Quantum criticality and dyonic black holes

4. Experiments

Graphene and the cuprate superconductors

1. Quantum-critical transport

Collisionless-to-hydrodynamic crossover of CFT3s

2. Exact solution from AdS/CFT

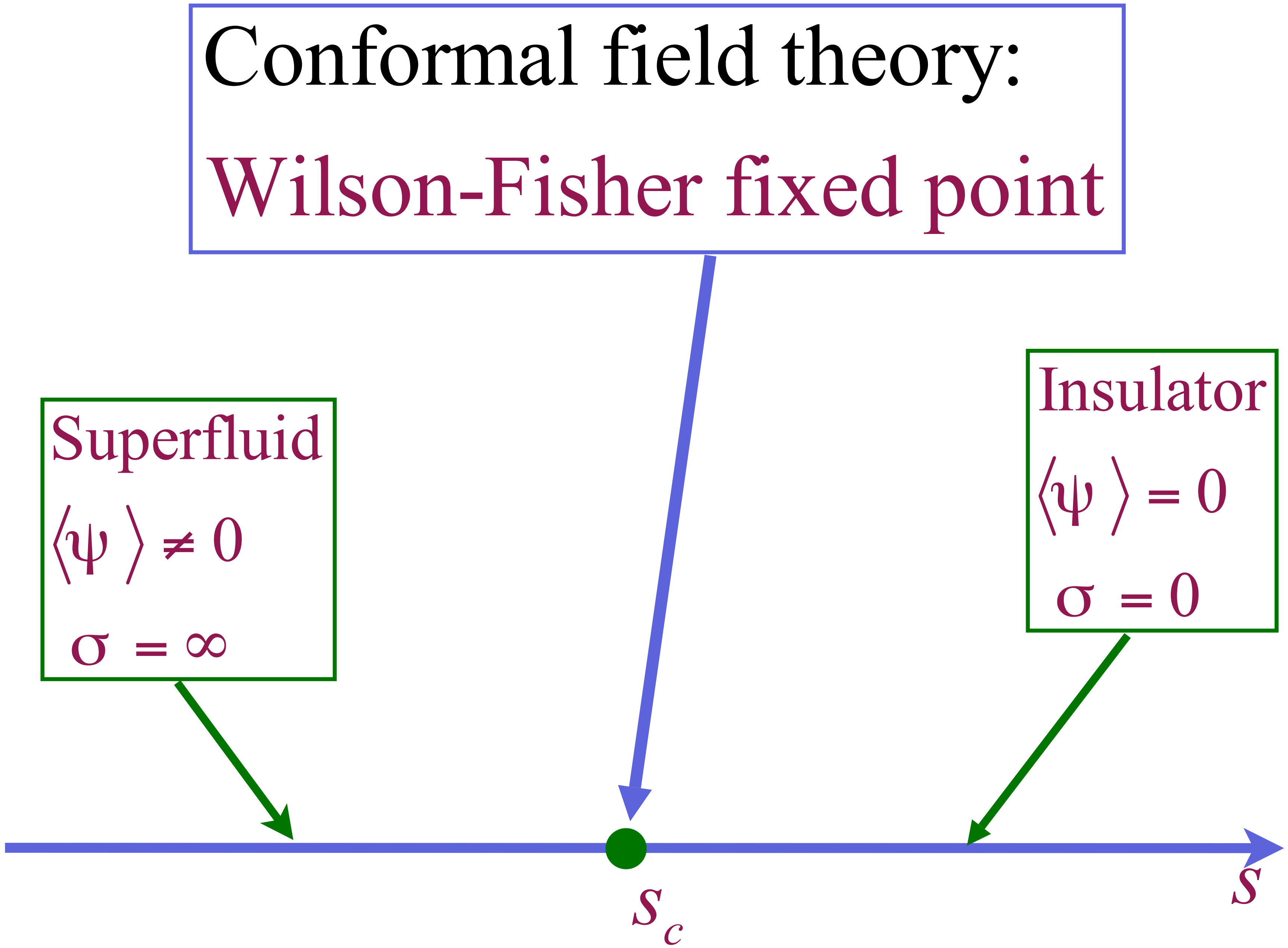
Constraints from duality relations

3. Generalized magnetohydrodynamics

Quantum criticality and dyonic black holes

4. Experiments

Graphene and the cuprate superconductors

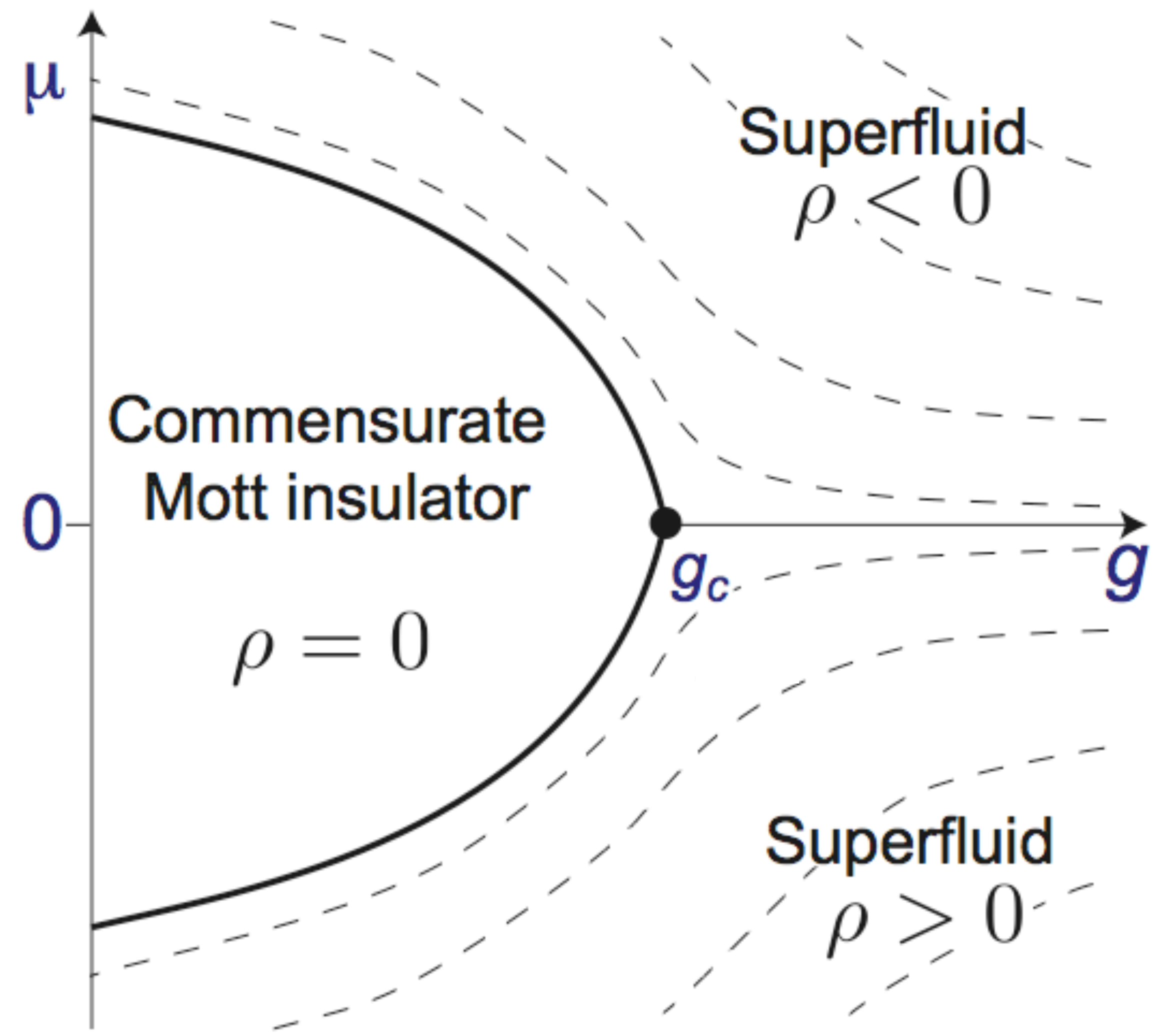


$$\mathcal{S} = \int d^2 r d\tau \left[|\partial_\tau \psi|^2 + c^2 |\vec{\nabla}\psi|^2 + s|\psi|^2 + \frac{u}{2}|\psi|^4 \right]$$

For experimental applications, we must move away from the ideal CFT

- A chemical potential μ

- A magnetic field B



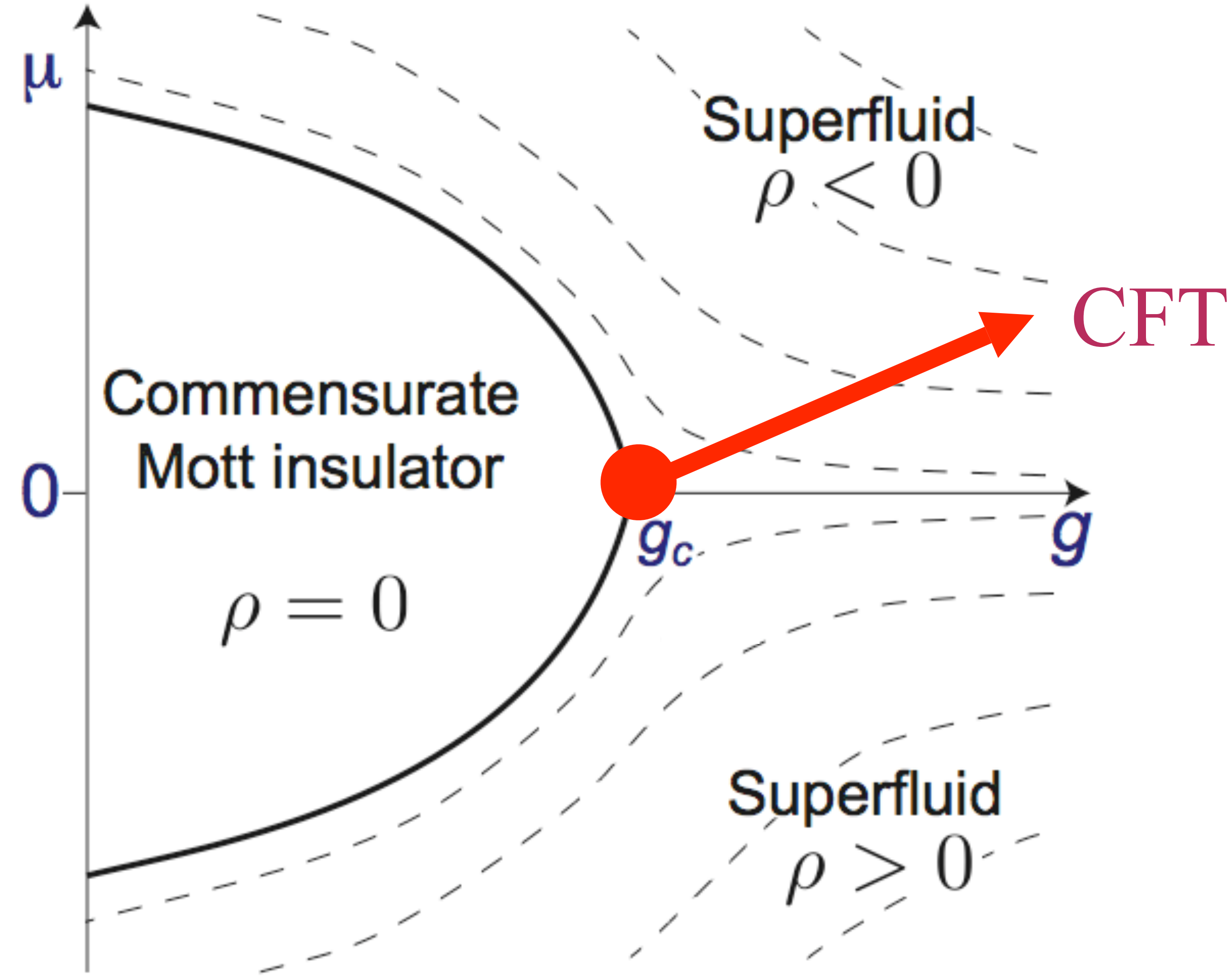
e.g.

$$\mathcal{S} = \int d^2r d\tau \left[|(\partial_\tau - \mu)\psi|^2 + v^2 \left| (\vec{\nabla} - i\vec{A})\psi \right|^2 - g|\psi|^2 + \frac{u}{2}|\psi|^4 \right]$$

$$\nabla \times \vec{A} = B$$

For experimental applications, we must move away from the ideal CFT

- A chemical potential μ
- A magnetic field B

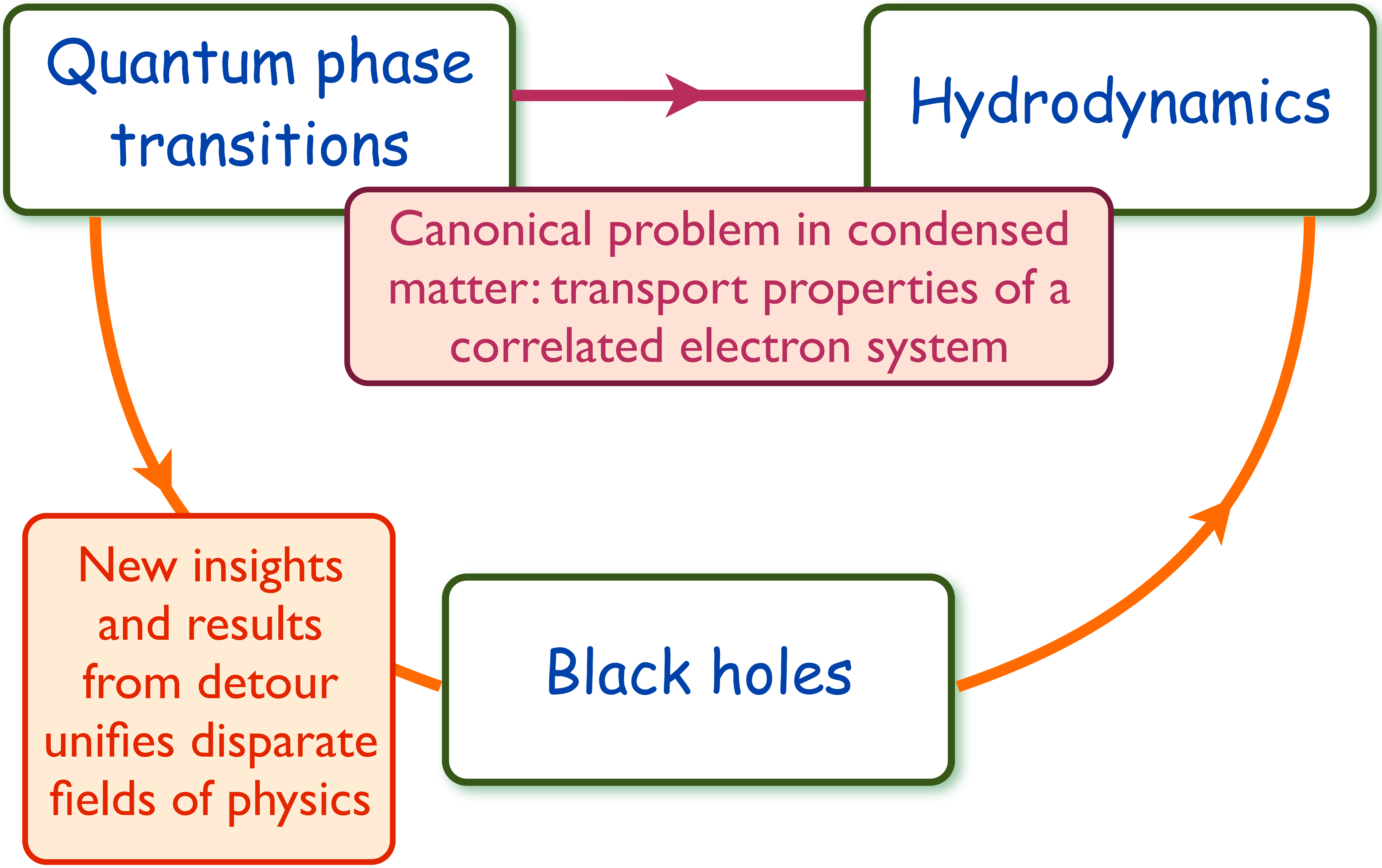


e.g.

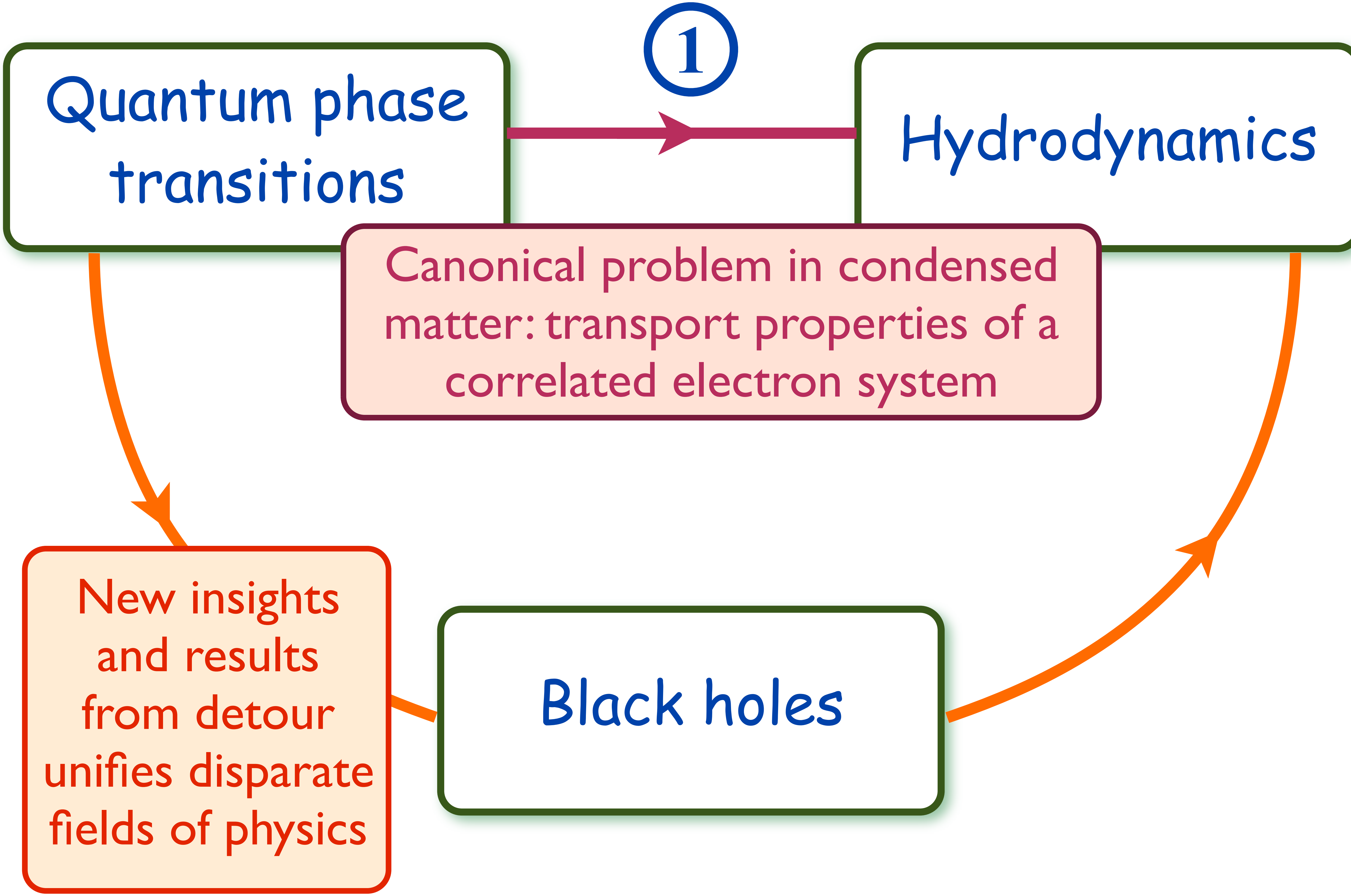
$$\mathcal{S} = \int d^2r d\tau \left[|(\partial_\tau - \mu)\psi|^2 + v^2 \left|(\vec{\nabla} - i\vec{A})\psi\right|^2 - g|\psi|^2 + \frac{u}{2}|\psi|^4 \right]$$

$$\nabla \times \vec{A} = B$$

Three foci of modern physics



Three foci of modern physics



Hydrodynamics of quantum critical systems

1. Use quantum field theory + quantum transport equations + classical hydrodynamics

   Uses physical model but strong-coupling makes explicit solution difficult

In the regime $\hbar\omega \ll k_B T$, we can use the principles of hydrodynamics:

- Describe system in terms of local state variables which obey the equation of state
- Express conserved currents in terms of gradients of state variables using transport co-efficients. These are restricted by demanding that the system relaxes to *local equilibrium i.e.* entropy production is positive.
- The conservation laws are the equations of motion.

The variables entering the hydrodynamic theory are

- the external magnetic field $F^{\mu\nu}$,

$$F^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & B \\ 0 & -B & 0 \end{pmatrix},$$

- $T^{\mu\nu}$, the stress energy tensor,
- J^{μ}, the current,
- ρ, the **difference** in density from the Mott insulator.
- ε, the local energy
- P, the local pressure,
- u^{μ}, the local velocity, and
- σ_Q, a universal conductivity, which is the **single transport co-efficient**.

The dependence of ε, P, σ_Q on T and v follows from simple scaling arguments

Lorentz invariance and positivity of entropy production lead to the hydrodynamic equations of motion and constitutive relations:

S.A. Hartnoll, P.K. Kovtun, M. Müller, and S. Sachdev, *Phys. Rev.* B **76** 144502 (2007)

Lorentz invariance and positivity of entropy production lead to the hydrodynamic equations of motion and constitutive relations:

$$\begin{aligned} \partial_\mu J^\mu &= 0 \\ \partial_\mu T^{\mu\nu} &= F^{\mu\nu} J_\nu \end{aligned}$$

Conservation laws/equations of motion

S.A. Hartnoll, P.K. Kovtun, M. Müller, and S. Sachdev, *Phys. Rev.* B **76** 144502 (2007)

Lorentz invariance and positivity of entropy production lead to the hydrodynamic equations of motion and constitutive relations:

$$
\begin{aligned}
\partial_\mu J^\mu &= 0 \\
\partial_\mu T^{\mu\nu} &= F^{\mu\nu} J_\nu \\
T^{\mu\nu} &= (\varepsilon + P) u^\mu u^\nu + P g^{\mu\nu} \\
J^\mu &= \rho u^\mu
\end{aligned}
$$

Constitutive relations which follow from Lorentz transformation to moving frame

S.A. Hartnoll, P.K. Kovtun, M. Müller, and S. Sachdev, *Phys. Rev.* B **76** 144502 (2007)

Lorentz invariance and positivity of entropy production lead to the hydrodynamic equations of motion and constitutive relations:

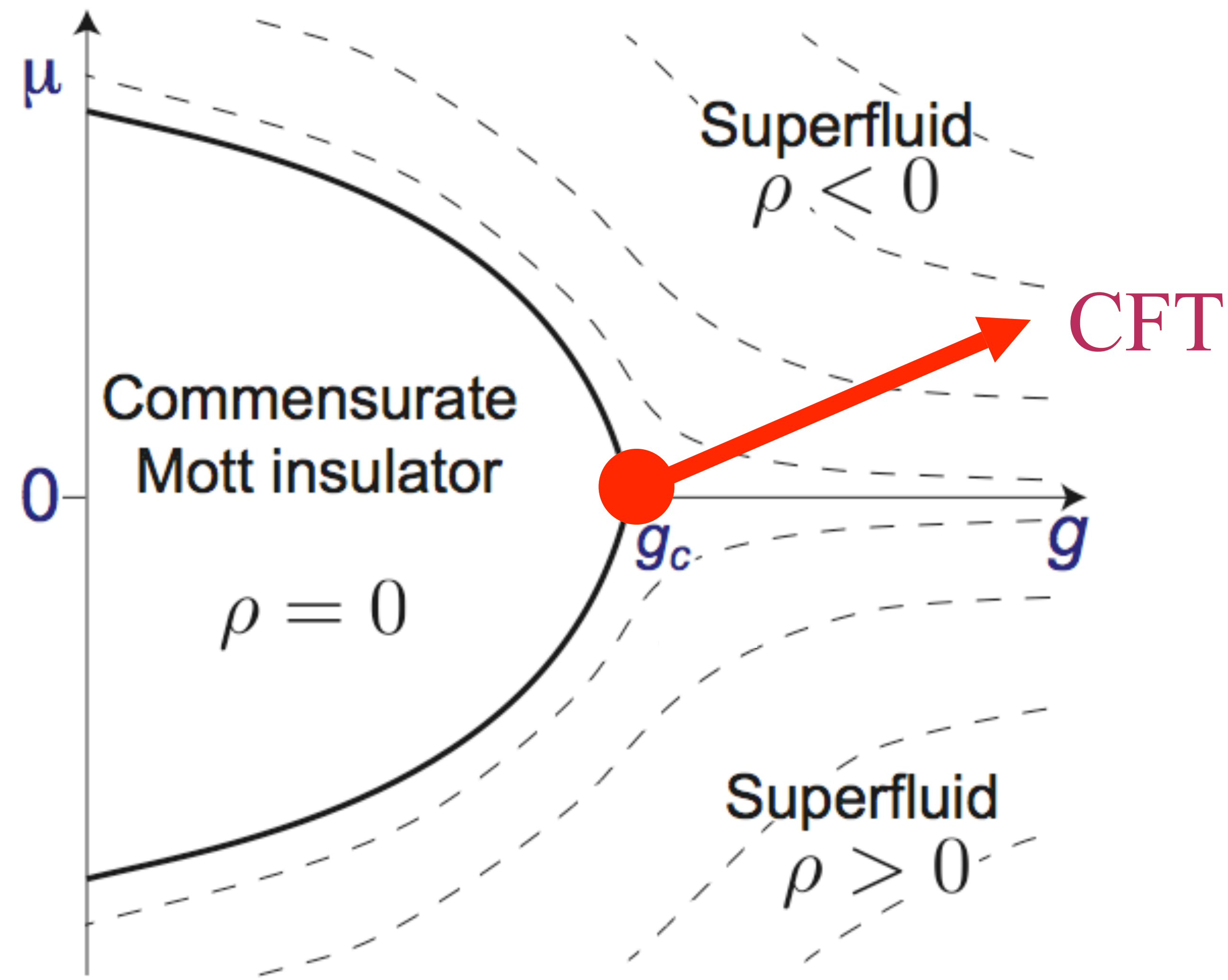
$$
\begin{aligned}
\partial_\mu J^\mu &= 0 \\
\partial_\mu T^{\mu\nu} &= F^{\mu\nu} J_\nu \\
T^{\mu\nu} &= (\varepsilon + P) u^\mu u^\nu + P g^{\mu\nu} \\
J^\mu &= \rho u^\mu + \sigma_Q (g^{\mu\nu} + u^\mu u^\nu) \left[(-\partial_\nu \mu + F_{\nu\lambda} u^\lambda) + \mu \frac{\partial_\mu T}{T} \right]
\end{aligned}
$$

Single dissipative term allowed by requirement of positive entropy production. There is only one independent transport co-efficient

S.A. Hartnoll, P.K. Kovtun, M. Müller, and S. Sachdev, *Phys. Rev.* B **76** 144502 (2007)

For experimental applications, we must move away from the ideal CFT

- A chemical potential μ
- A magnetic field B
- An impurity scattering rate $1/\tau_{imp}$ (its T dependence follows from scaling arguments)



e.g.

$$\mathcal{S} = \int d^2 r d\tau \left[|(\partial_\tau - \mu)\psi|^2 + v^2 \left| (\vec{\nabla} - i\vec{A})\psi \right|^2 - g|\psi|^2 + V(r)|\psi|^2 + \frac{u}{2}|\psi|^4 \right]$$

$$\nabla \times \vec{A} = B \quad , \quad \overline{V(r)} = 0 \quad , \quad \overline{V(r)V(r')} = V_{\text{imp}}^2 \delta^2(r - r')$$

Lorentz invariance and positivity of entropy production lead to the hydrodynamic equations of motion and constitutive relations:

$$
\begin{aligned}
\partial_\mu J^\mu &= 0 \\
\partial_\mu T^{\mu\nu} &= F^{\mu\nu} J_\nu + \frac{1}{\tau_{\rm imp}} (\delta^\mu_\nu + u^\mu u_\nu) T^{\nu\gamma} u_\gamma \\
T^{\mu\nu} &= (\varepsilon + P) u^\mu u^\nu + P g^{\mu\nu} \\
J^\mu &= \rho u^\mu + \sigma_Q (g^{\mu\nu} + u^\mu u^\nu) \left[(-\partial_\nu \mu + F_{\nu\lambda} u^\lambda) + \mu \frac{\partial_\mu T}{T} \right]
\end{aligned}
$$

S.A. Hartnoll, P.K. Kovtun, M. Müller, and S. Sachdev, *Phys. Rev.* B **76** 144502 (2007)

Lorentz invariance and positivity of entropy production lead to the hydrodynamic equations of motion and constitutive relations:

$$
\begin{aligned}
\partial_\mu J^\mu &= 0 \\
\partial_\mu T^{\mu\nu} &= F^{\mu\nu} J_\nu + \frac{1}{\tau_{\text{imp}}} \left(\delta^\mu_\nu + u^\mu u_\nu\right) T^{\nu\gamma} u_\gamma \\
T^{\mu\nu} &= (\varepsilon + P) u^\mu u^\nu + P g^{\mu\nu} \\
J^\mu &= \rho u^\mu + \sigma_Q \left(g^{\mu\nu} + u^\mu u^\nu\right) \left[\left(-\partial_\nu \mu + F_{\nu\lambda} u^\lambda\right) + \mu \frac{\partial_\mu T}{T} \right]
\end{aligned}
$$

Solve initial value problem and relate results to response functions (Kadanoff+Martin)

S.A. Hartnoll, P.K. Kovtun, M. Müller, and S. Sachdev, *Phys. Rev.* B **76** 144502 (2007)

Hydrodynamic cyclotron resonance at a frequency

$$\omega_c = \frac{e^* B \rho v^2}{c(\varepsilon + P)}$$

and with width

$$\gamma = \sigma_Q \frac{B^2 v^2}{c^2(\varepsilon + P)}$$

where B = magnetic field, ρ = charge density away from density of CFT, ε = energy density, P = pressure, v = velocity of "light" in CFT, and $\sigma_Q e^2/h$ is the universal conductivity of the CFT.

S.A. Hartnoll, P.K. Kovtun, M. Müller, and S. Sachdev, *Phys. Rev.* B **76** 144502 (2007)

From these relations, we obtained results for the transport co-efficients, expressed in terms of a "cyclotron" frequency and damping:

$$\omega_c = \frac{2eB\rho v^2}{c(\varepsilon + P)} \quad , \quad \gamma = \sigma_Q \frac{B^2 v^2}{c^2(\varepsilon + P)}$$

Longitudinal conductivity

$$\sigma_{xx} = \sigma_Q \left[\frac{(\omega + i/\tau_{\rm imp})(\omega + i\gamma + i\omega_c^2/\gamma + i/\tau_{\rm imp})}{(\omega + i\gamma + i/\tau_{\rm imp})^2 - \omega_c^2} \right] .$$

S.A. Hartnoll, P.K. Kovtun, M. Müller, and S. Sachdev, *Phys. Rev.* B **76** 144502 (2007)

From these relations, we obtained results for the transport co-efficients, expressed in terms of a "cyclotron" frequency and damping:

$$\omega_c = \frac{2eB\rho v^2}{c(\varepsilon + P)} \quad , \quad \gamma = \sigma_Q \frac{B^2 v^2}{c^2(\varepsilon + P)}$$

Longitudinal conductivity

$$\sigma_{xx} = \sigma_Q \left[\frac{(\omega + i/\tau_{\text{imp}})(\omega + i\gamma + i\omega_c^2/\gamma + i/\tau_{\text{imp}})}{(\omega + i\gamma + i/\tau_{\text{imp}})^2 - \omega_c^2} \right].$$

$$= \sigma_Q + \frac{4e^2\rho^2 v^2}{(\varepsilon + P)} \frac{1}{(-i\omega + 1/\tau_{\text{imp}})} \quad \text{as } B \to 0$$

S.A. Hartnoll, P.K. Kovtun, M. Müller, and S. Sachdev, *Phys. Rev.* B **76** 144502 (2007)

From these relations, we obtained results for the transport co-efficients, expressed in terms of a "cyclotron" frequency and damping:

$$\omega_c = \frac{2eB\rho v^2}{c(\varepsilon + P)} \quad , \quad \gamma = \sigma_Q \frac{B^2 v^2}{c^2(\varepsilon + P)}$$

Thermal conductivity

$$\begin{aligned}\kappa_{xx} &= \sigma_Q \left(\frac{k_B^2 T}{4e^2}\right) \left(\frac{\varepsilon + P}{k_B T \rho}\right)^2 \left[\frac{(\omega_c^2/\gamma)(\omega_c^2/\gamma + 1/\tau_{\text{imp}})}{(\omega_c^2/\gamma + 1/\tau_{\text{imp}})^2 + \omega_c^2}\right] \\ &= \frac{1}{\sigma_Q} k_B^2 T \left(\frac{c(\varepsilon + P)}{k_B T B}\right)^2 \left[\frac{\gamma(\omega_c^2/\gamma + 1/\tau_{\text{imp}})}{(\omega_c^2/\gamma + 1/\tau_{\text{imp}})^2 + \omega_c^2}\right]\end{aligned}$$

S.A. Hartnoll, P.K. Kovtun, M. Müller, and S. Sachdev, *Phys. Rev.* B **76** 144502 (2007)

From these relations, we obtained results for the transport co-efficients, expressed in terms of a "cyclotron" frequency and damping:

$$\omega_c = \frac{2eB\rho v^2}{c(\varepsilon + P)} \quad , \quad \gamma = \sigma_Q \frac{B^2 v^2}{c^2(\varepsilon + P)}$$

Thermal conductivity

$$\kappa_{xx} = \sigma_Q \left(\frac{k_B^2 T}{4e^2}\right) \left(\frac{\varepsilon + P}{k_B T \rho}\right)^2 \quad \boxed{\rightarrow 1 \text{ as } B \rightarrow 0}$$

$$= \frac{1}{\sigma_Q} k_B^2 T \left(\frac{c(\varepsilon + P)}{k_B T B}\right)^2 \left[\frac{\gamma(\omega_c^2/\gamma + 1/\tau_{\text{imp}})}{(\omega_c^2/\gamma + 1/\tau_{\text{imp}})^2 + \omega_c^2}\right]$$

S.A. Hartnoll, P.K. Kovtun, M. Müller, and S. Sachdev, *Phys. Rev.* B **76** 144502 (2007)

From these relations, we obtained results for the transport co-efficients, expressed in terms of a "cyclotron" frequency and damping:

$$\omega_c = \frac{2eB\rho v^2}{c(\varepsilon + P)} \quad , \quad \gamma = \sigma_Q \frac{B^2 v^2}{c^2(\varepsilon + P)}$$

Thermal conductivity

$$\kappa_{xx} = \sigma_Q \left(\frac{k_B^2 T}{4e^2}\right) \left(\frac{\varepsilon + P}{k_B T \rho}\right)^2 \left[\frac{(\omega_c^2/\gamma)(\omega_c^2/\gamma + 1/\tau_{\mathrm{imp}})}{(\omega_c^2/\gamma + 1/\tau_{\mathrm{imp}})^2 + \omega_c^2}\right]$$

$$= \frac{1}{\sigma_Q} k_B^2 T \left(\frac{c(\varepsilon + P)}{k_B T B}\right)^2 \quad \boxed{\rightarrow 1 \text{ as } \rho \rightarrow 0}$$

S.A. Hartnoll, P.K. Kovtun, M. Müller, and S. Sachdev, *Phys. Rev.* B **76** 144502 (2007)

From these relations, we obtained results for the transport co-efficients, expressed in terms of a "cyclotron" frequency and damping:

$$\omega_c = \frac{2eB\rho v^2}{c(\varepsilon + P)} \quad , \quad \gamma = \frac{B^2 v^2}{c^2(\varepsilon + P)}$$

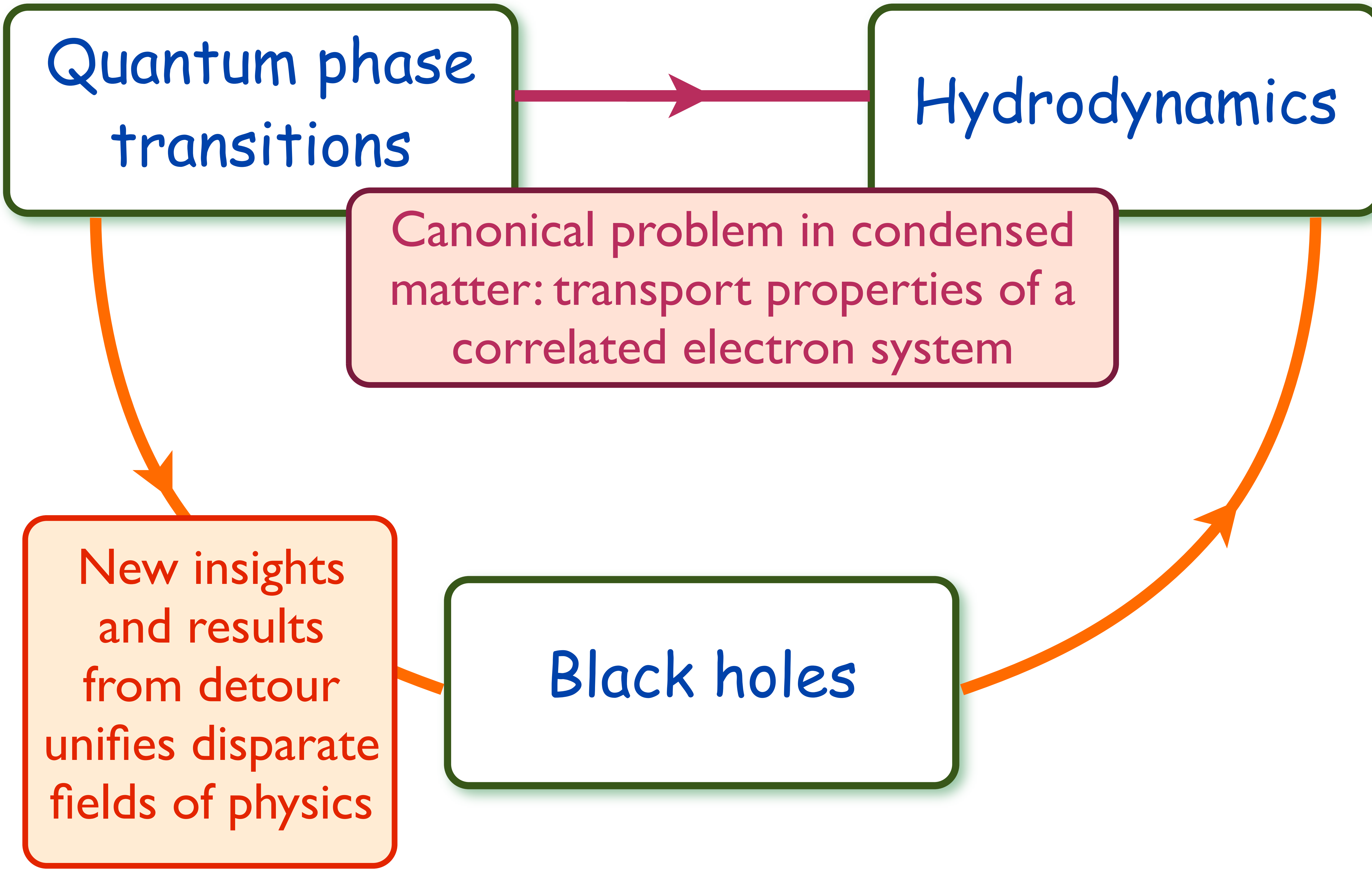
Nernst signal

$$e_N = \left(\frac{k_B}{2e}\right)\left(\frac{\varepsilon + P}{k_B T \rho}\right)\left[\frac{\omega_c/\tau_{\text{imp}}}{(\omega_c^2/\gamma + 1/\tau_{\text{imp}})^2 + \omega_c^2}\right]$$

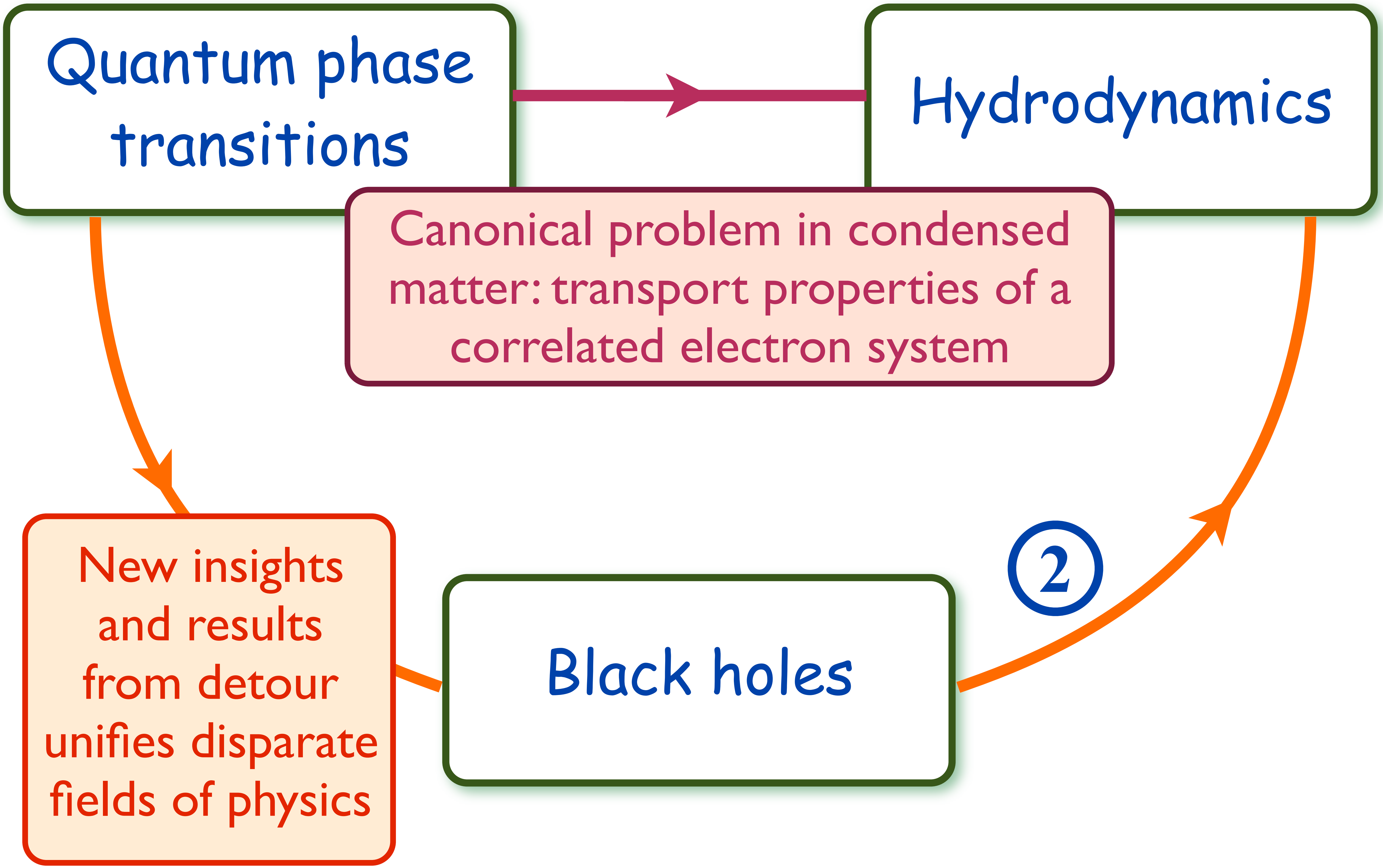
$$\frac{k_B}{2e} = 43.086 \mu\text{V/K}$$

S.A. Hartnoll, P.K. Kovtun, M. Müller, and S. Sachdev, *Phys. Rev.* B **76** 144502 (2007)

Three foci of modern physics



Three foci of modern physics



Hydrodynamics of quantum critical systems

1. Use quantum field theory + quantum transport equations + classical hydrodynamics

   Uses physical model but strong-coupling makes explicit solution difficult

2. Solve Einstein-Maxwell equations in the background of a black hole in AdS space

   Yields hydrodynamic relations which apply to general classes of quantum critical systems. First exact numerical results for transport co-efficients (for supersymmetric systems).

Exact Results

To the solvable supersymmetric, Yang-Mills theory CFT, we add

- A chemical potential μ
- A magnetic field B

After the AdS/CFT mapping, we obtain the Einstein-Maxwell theory of a black hole with

- An electric charge
- A magnetic charge

S.A. Hartnoll, P.K. Kovtun, M. Müller, and S. Sachdev, *Phys. Rev.* B **76** 144502 (2007)

Exact Results

To the solvable supersymmetric, Yang-Mills theory CFT, we add

- A chemical potential μ
- A magnetic field B

After the AdS/CFT mapping, we obtain the Einstein-Maxwell theory of a black hole with

- An electric charge
- A magnetic charge

The exact results are found to be in *precise* accord with *all* hydrodynamic results presented earlier

S.A. Hartnoll, P.K. Kovtun, M. Müller, and S. Sachdev, *Phys. Rev.* B **76** 144502 (2007)

1. Quantum-critical transport

   Collisionless-to-hydrodynamic crossover of CFT3s

2. Exact solution from AdS/CFT

   Constraints from duality relations

3. Generalized magnetohydrodynamics

   Quantum criticality and dyonic black holes

4. Experiments

   Graphene and the cuprate superconductors

1. Quantum-critical transport

Collisionless-to-hydrodynamic crossover of CFT3s

2. Exact solution from AdS/CFT

Constraints from duality relations

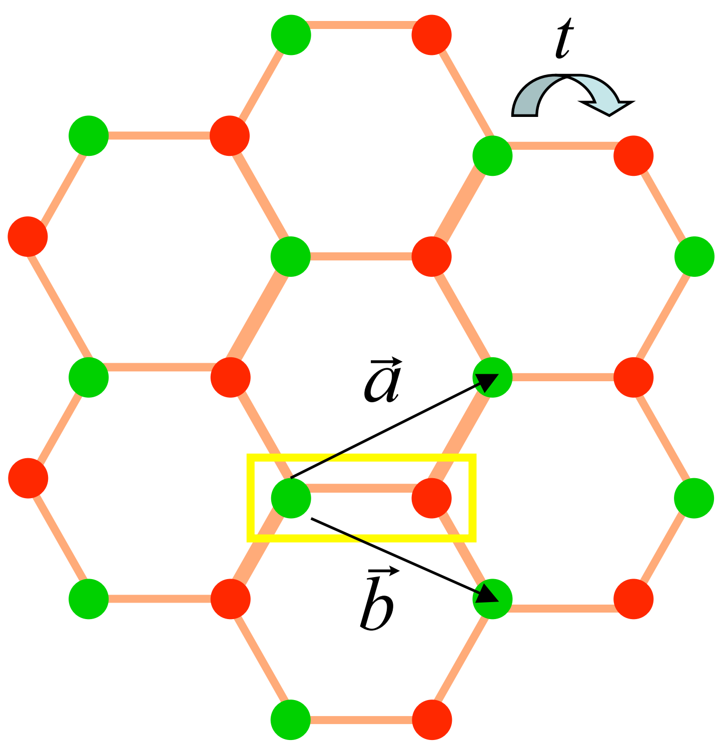
3. Generalized magnetohydrodynamics

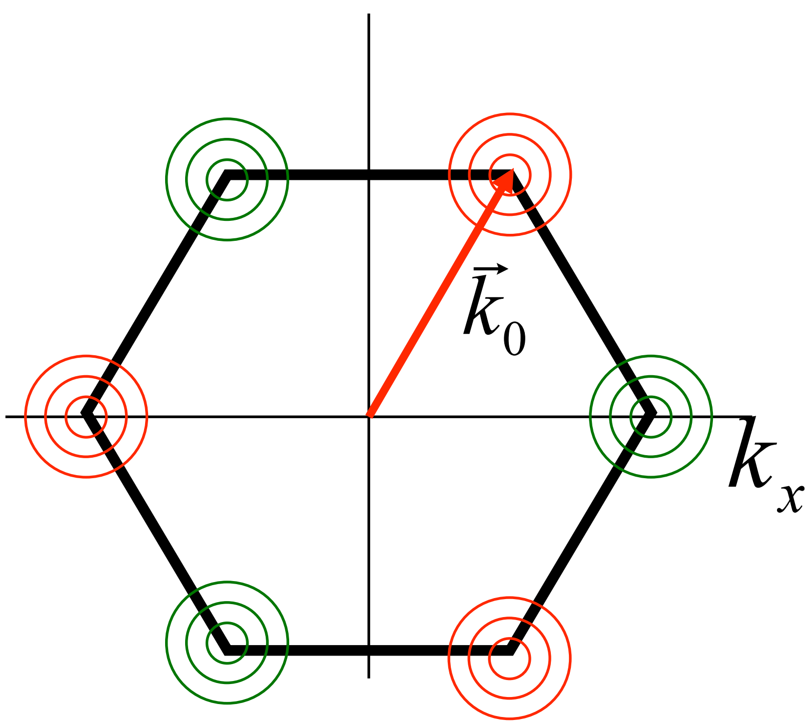
Quantum criticality and dyonic black holes

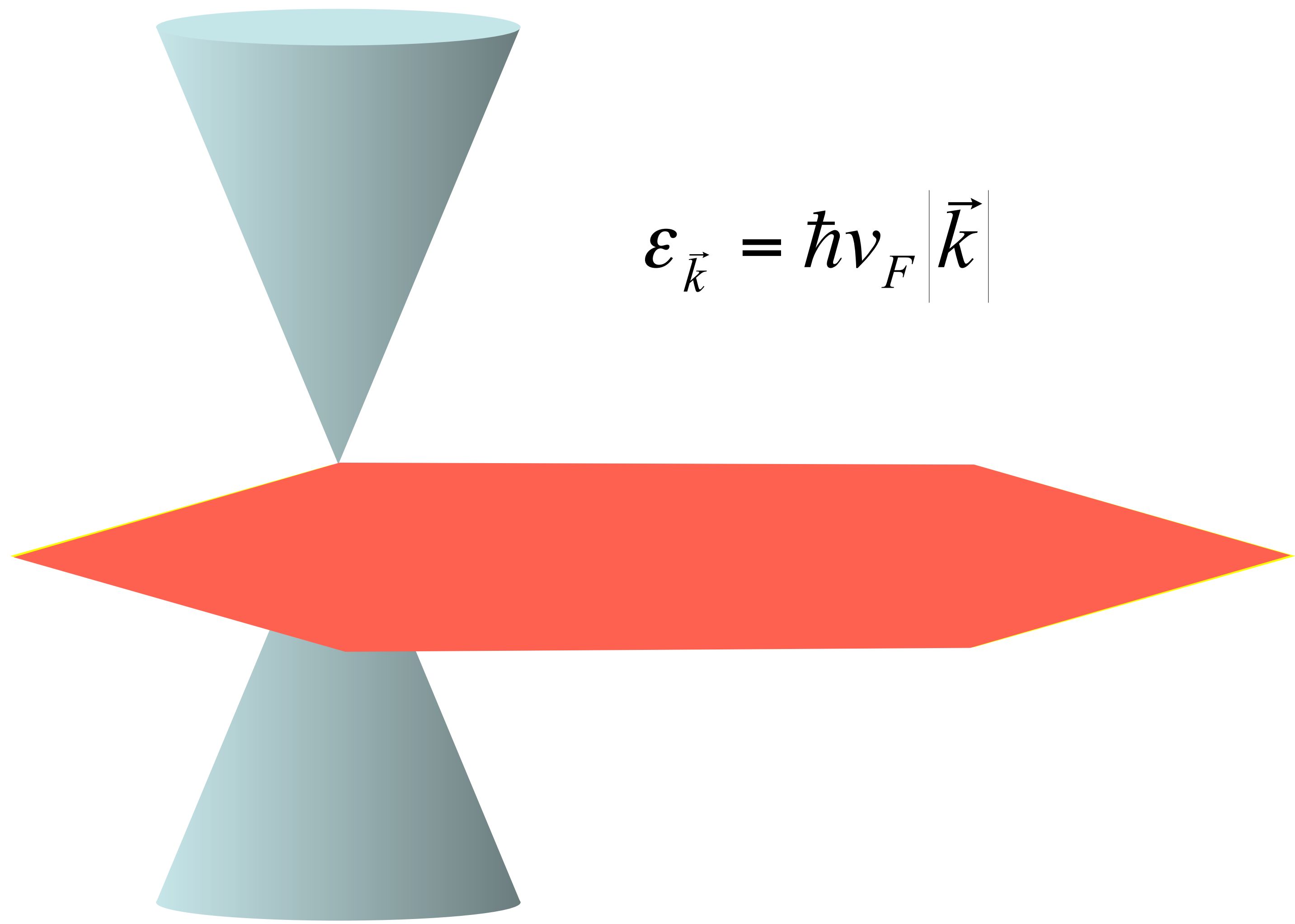
4. Experiments

Graphene and the cuprate superconductors

Graphene







$$\varepsilon_{\vec{k}} = \hbar v_F \left|\vec{k}\right|$$

Graphene

Low energy theory has 4 two-component Dirac fermions, ψ_σ, $\sigma = 1 \ldots 4$, interacting with a $1/r$ Coulomb interaction

$$\begin{aligned} \mathcal{S} &= \int d^2 r d\tau \psi_\sigma^\dagger \left(\partial_\tau - i v_F \vec{\sigma} \cdot \vec{\nabla} \right) \psi_\sigma \\ &\quad + \frac{e^2}{2} \int d^2 r d^2 r' d\tau \psi_\sigma^\dagger \psi_\sigma(r) \frac{1}{|r - r'|} \psi_{\sigma'}^\dagger \psi_{\sigma'}(r') \end{aligned}$$

Graphene

Low energy theory has 4 two-component Dirac fermions, ψ_σ, $\sigma = 1 \ldots 4$, interacting with a $1/r$ Coulomb interaction

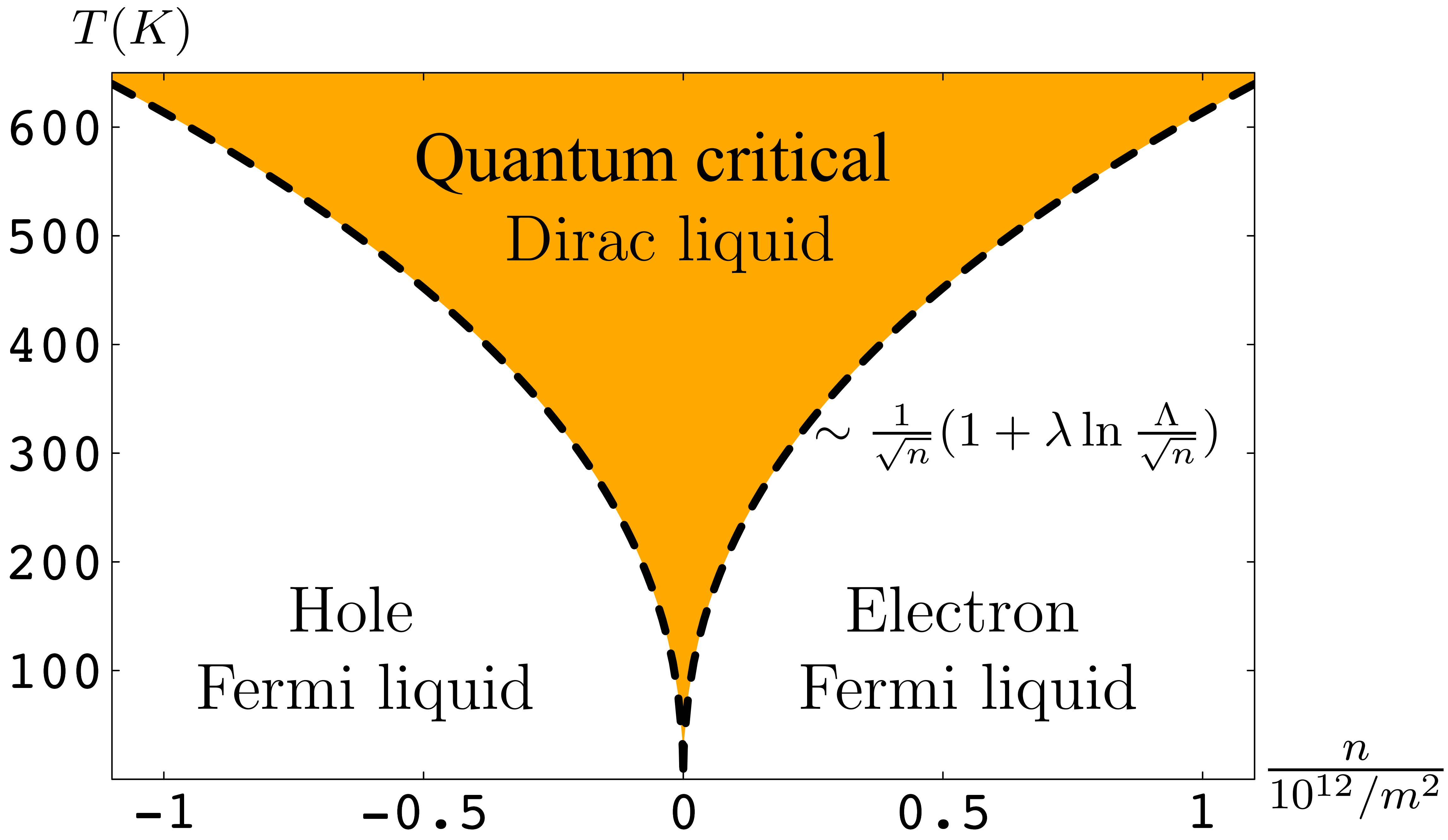
$$\begin{aligned} \mathcal{S} &= \int d^2 r d\tau \psi_\sigma^\dagger \left(\partial_\tau - i v_F \vec{\sigma} \cdot \vec{\nabla} \right) \psi_\sigma \\ &\quad + \frac{e^2}{2} \int d^2 r d^2 r' d\tau \psi_\sigma^\dagger \psi_\sigma(r) \frac{1}{|r - r'|} \psi_{\sigma'}^\dagger \psi_{\sigma'}(r') \end{aligned}$$

Dimensionless "fine-structure" constant $\alpha = e^2/(\hbar v_F)$.
RG flow of α:

$$\frac{d\alpha}{d\ell} = -\alpha^2 + \ldots$$

Behavior is similar to a conformal field theory (CFT) in 2+1 dimensions with $\alpha \sim 1/\ln(\text{scale})$

Graphene



$T(K)$
600
500
400
300
200
100
Quantum critical
Dirac liquid
$\sim \frac{1}{\sqrt{n}}(1+\lambda \ln \frac{\Lambda}{\sqrt{n}})$
Hole
Fermi liquid
Electron
Fermi liquid
-1
-0.5
0
0.5
1
$\frac{n}{10^{12}/m^2}$

Solve quantum Boltzmann equation for graphene

M. Müller, L. Fritz, and S. Sachdev, *Physical Review B* **78**, 115406 (2008)

Solve quantum Boltzmann equation for graphene

The results are found to be in *precise* accord with *all* hydrodynamic results presented earlier, and many results are extended beyond hydrodynamic regime.

M. Müller, L. Fritz, and S. Sachdev, *Physical Review B* **78**, 115406 (2008)

Collisionless-hydrodynamic crossover in pure, undoped, graphene

$$\sigma_Q(\omega) = \begin{cases} \dfrac{e^2}{h}\left[\dfrac{\pi}{2} + \mathcal{O}\left(\dfrac{1}{\ln(\Lambda/\omega)}\right)\right] & , \quad \hbar\omega \gg k_B T \\ \dfrac{e^2}{h\alpha^2(T)}\left[0.760 + \mathcal{O}\left(\dfrac{1}{|\ln(\alpha(T))|}\right)\right] & , \quad \hbar\omega \ll k_B T \alpha^2(T) \end{cases}$$

I. Herbut, V. Juricic, and O. Vafek, *Phys. Rev. Lett.* **100**, 046403 (2008).

where $\alpha(T)$ is the T-dependent fine structure constant which obeys

$$\alpha(T) = \frac{\alpha}{1 + (\alpha/4)\ln(\Lambda/T)} \stackrel{T\to 0}{\sim} \frac{4}{\ln(\Lambda/T)}$$

L. Fritz, M. Mueller, J. Schmalian and S. Sachdev, *Physical Review B* **78**, 085416 (2008)

See also A. Kashuba, arXiv:0802.2216

Universal conductivity σ_Q : graphene

L. Fritz, J. Schmalian, M. Mueller, and S. Sachdev, arXiv:0802.4289

General doping:

Lightly disordered system:

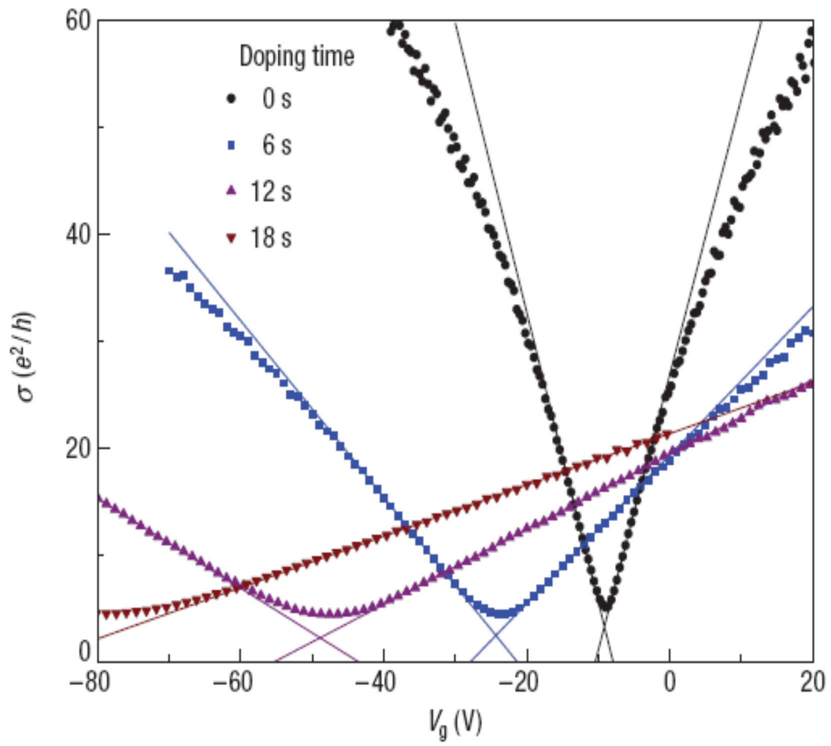
$$\sigma_{xx}(\omega;\mu,\Delta) = \frac{e^2}{\tau_{\rm imp}^{-1} - i\omega}\frac{\rho^2 v_F^2}{\varepsilon + P} + \sigma_Q + \delta\sigma(\Delta,\omega,\mu)$$

$$\delta\sigma(\Delta,\omega,\mu) = \mathcal{O}(\Delta/\alpha^2)$$

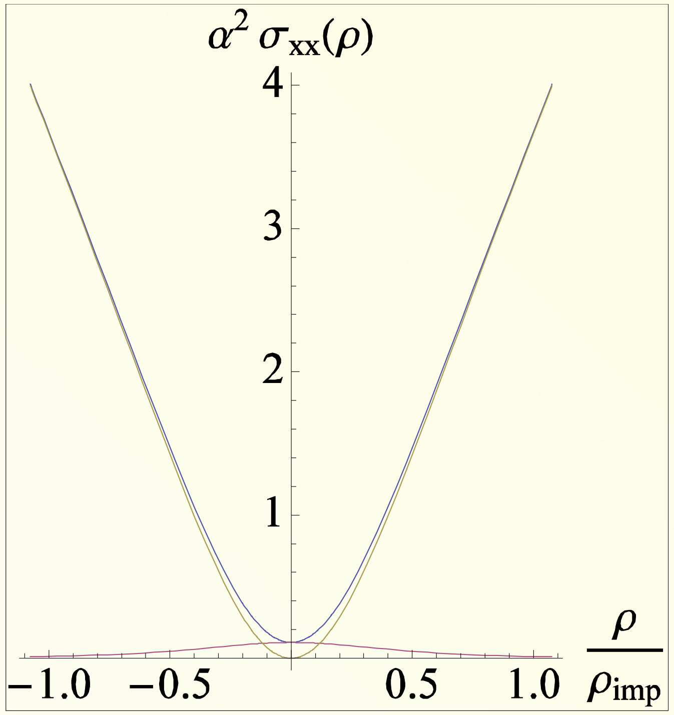
Fermi liquid regime:

$$\sigma_{xx}(\omega = 0;\mu \gg T) \approx \frac{e^2\rho^2 v_F^2 \tau_{\rm imp}}{\varepsilon + P} = \frac{2}{\pi}\frac{1}{(Z\alpha)^2}\frac{e^2}{h}\frac{\rho}{\rho_{\rm imp}}$$

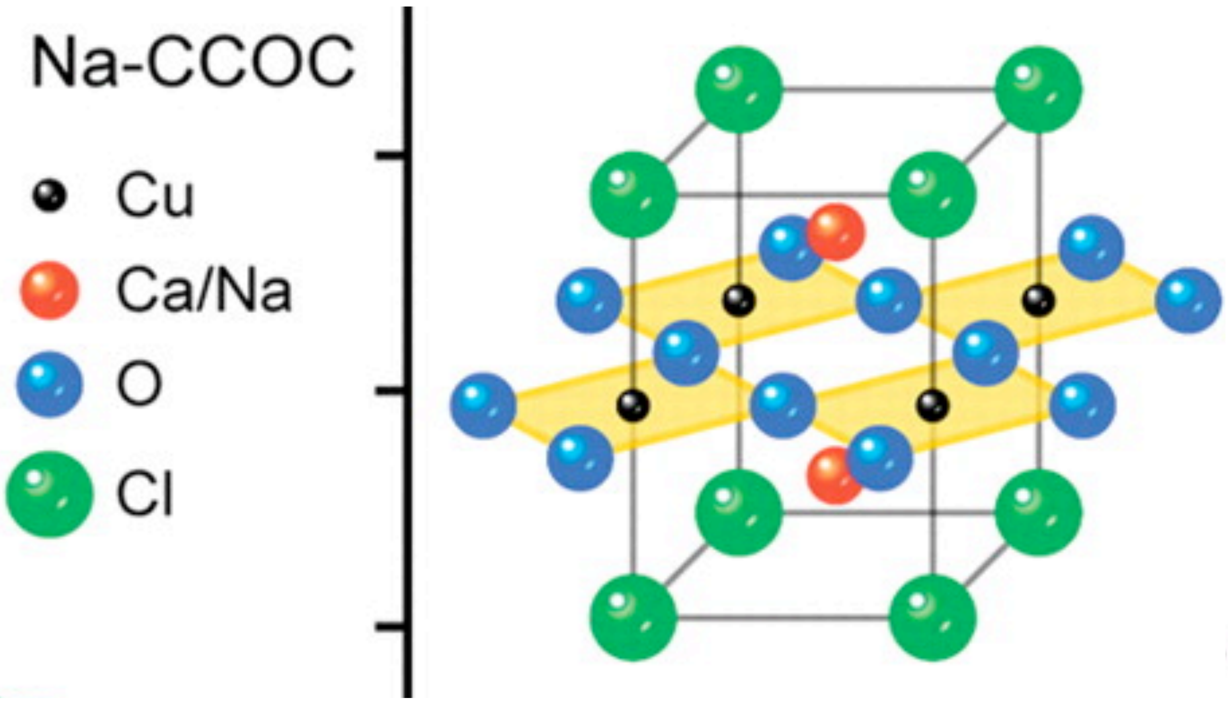
J.-H. Chen et al. Nat. Phys. ***4****, 377 (2008).*

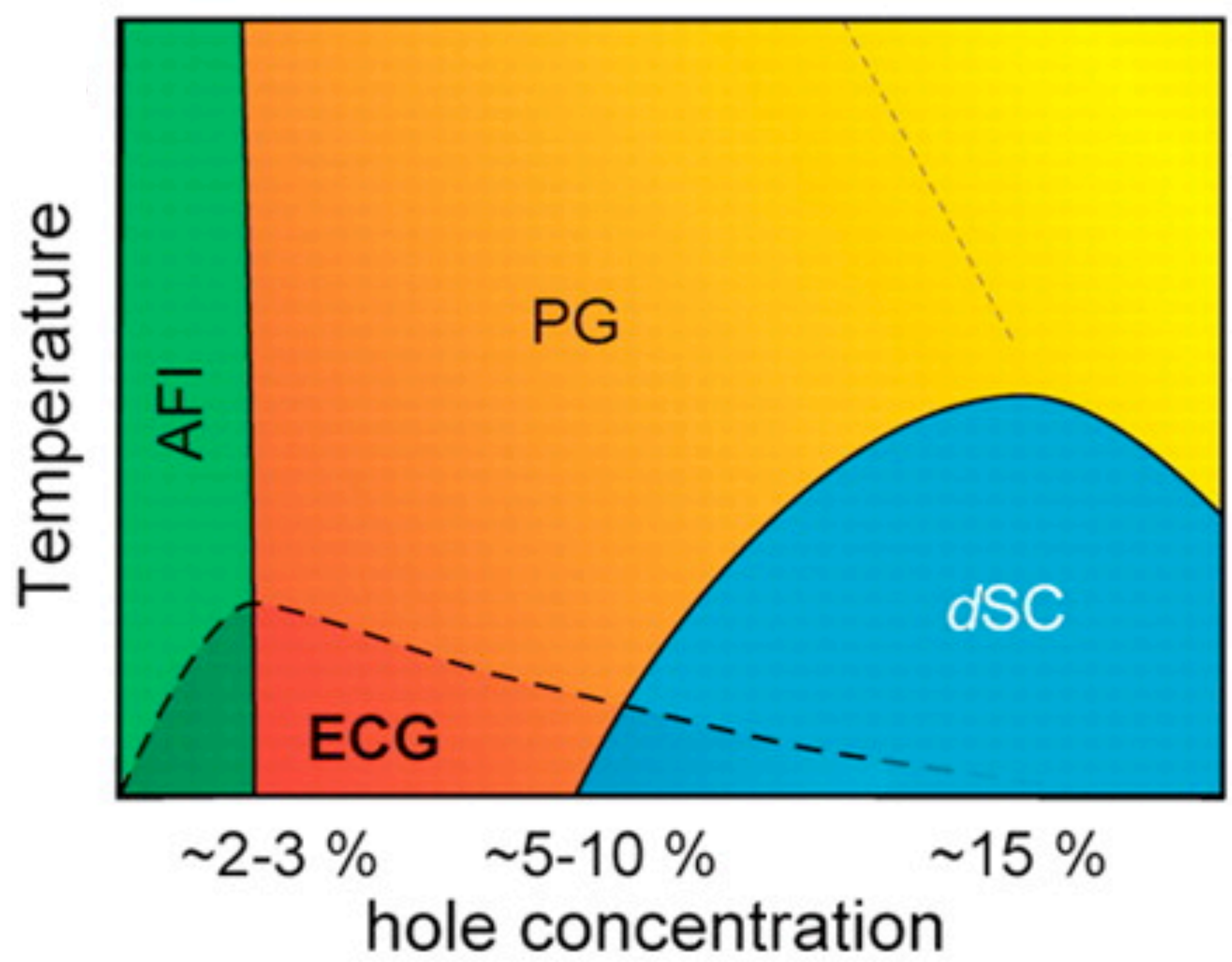


↔

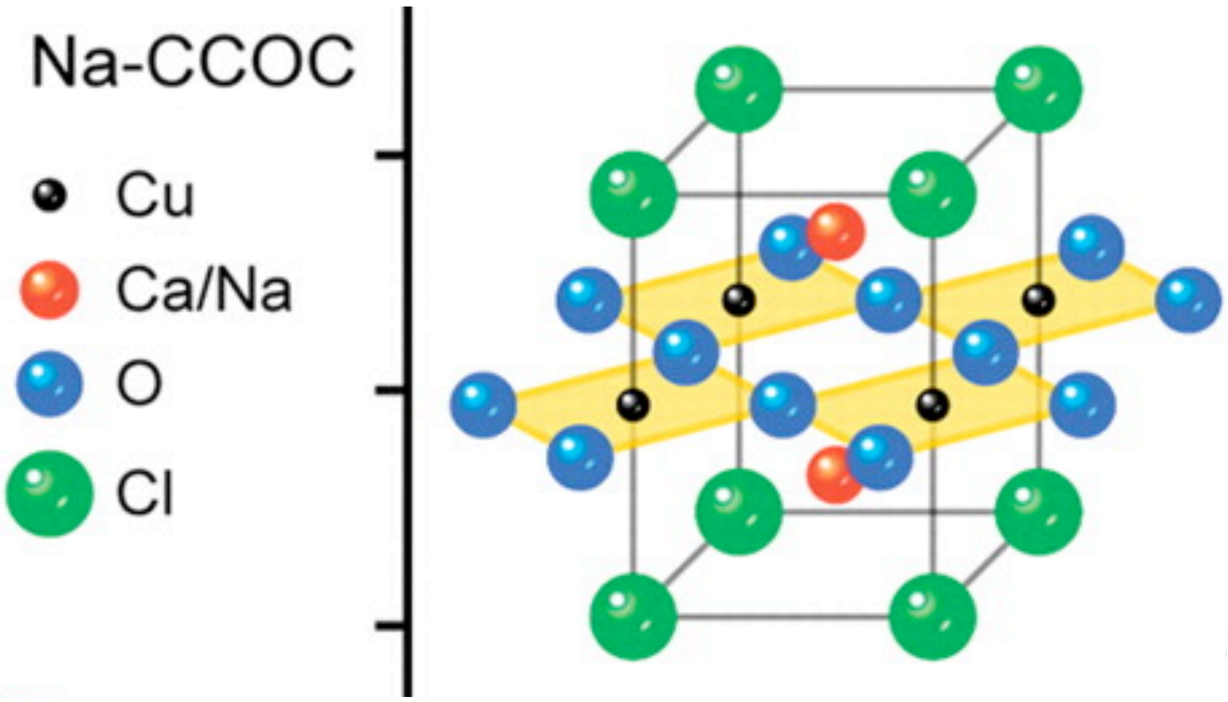


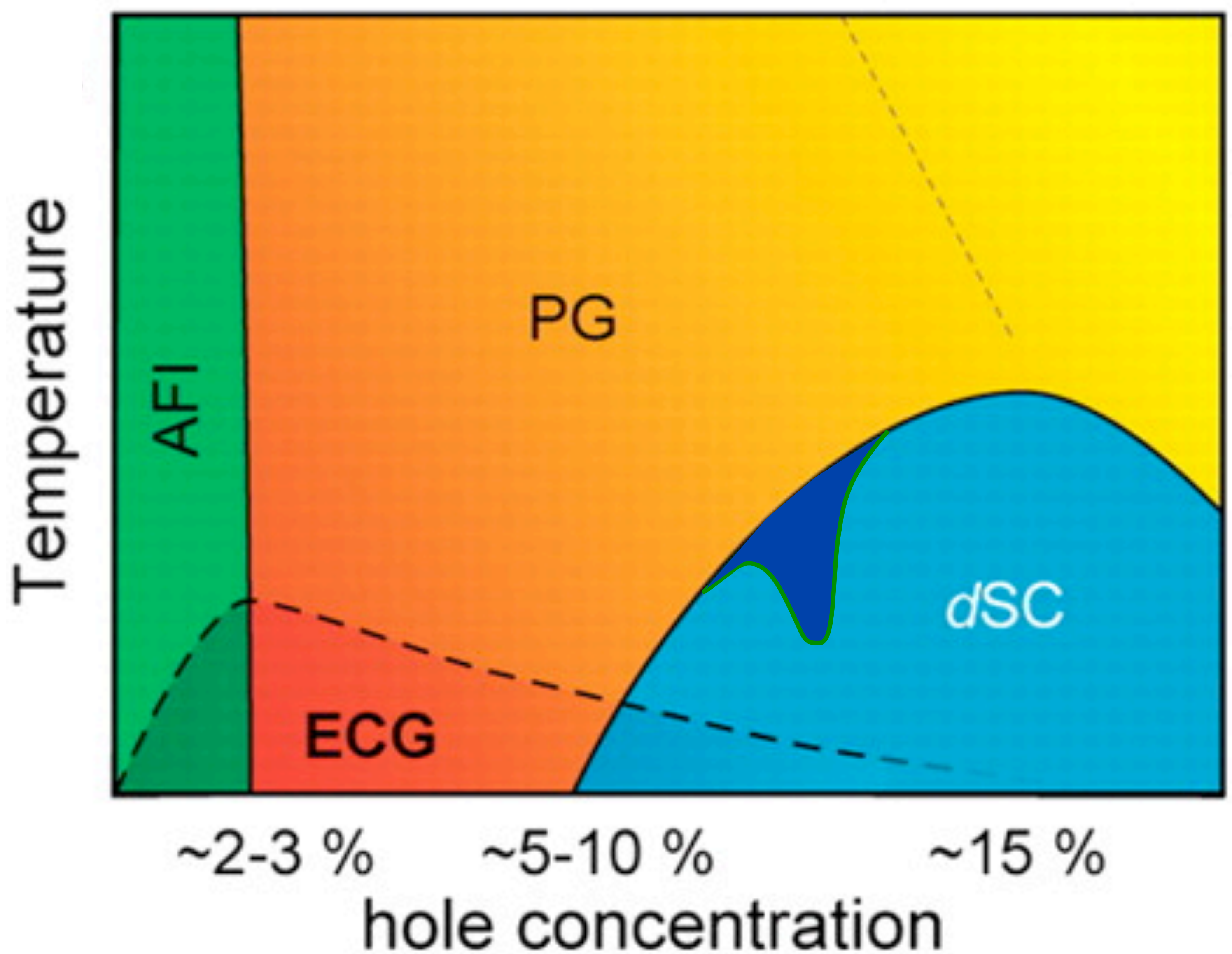
The cuprate superconductors





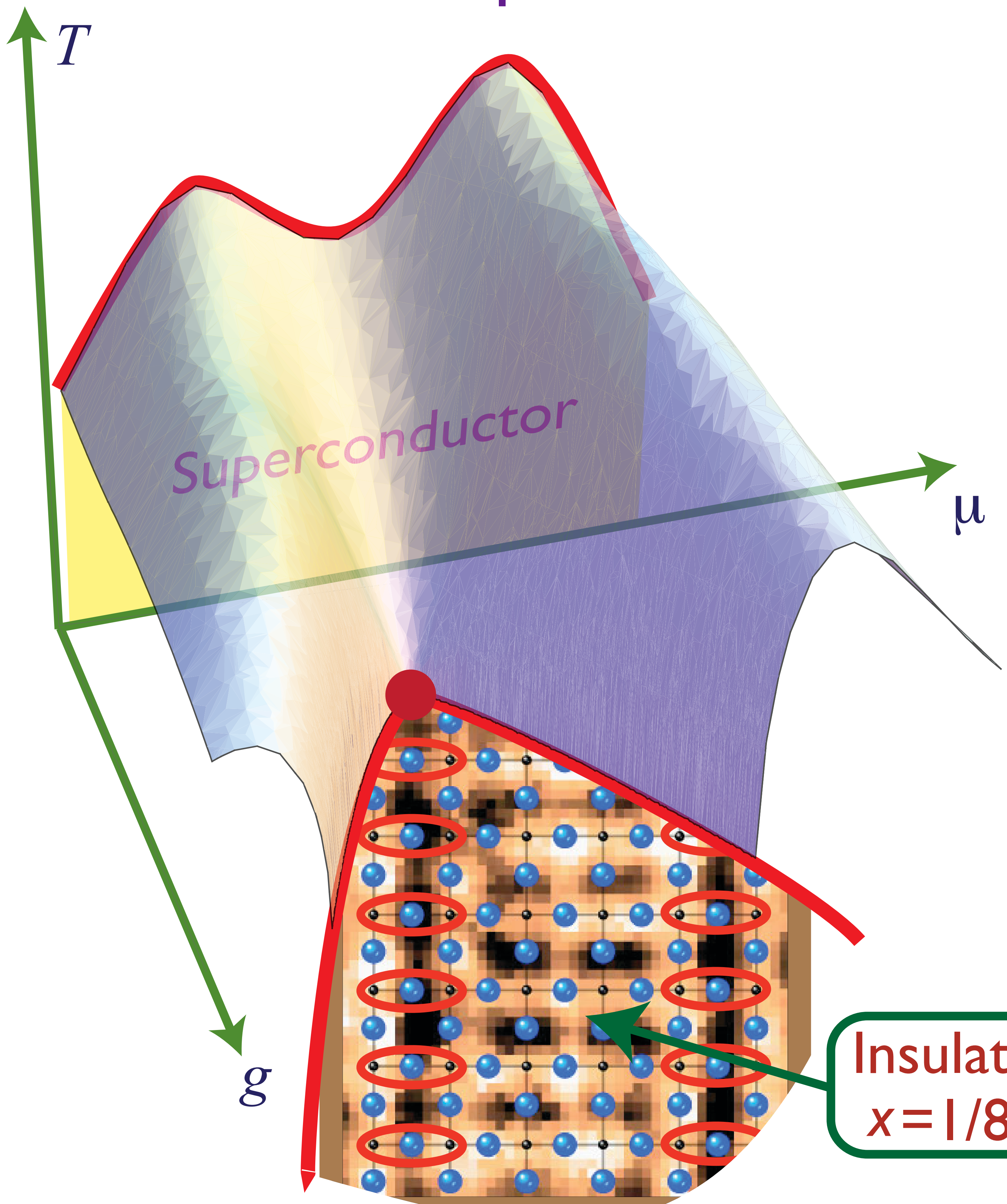
The cuprate superconductors



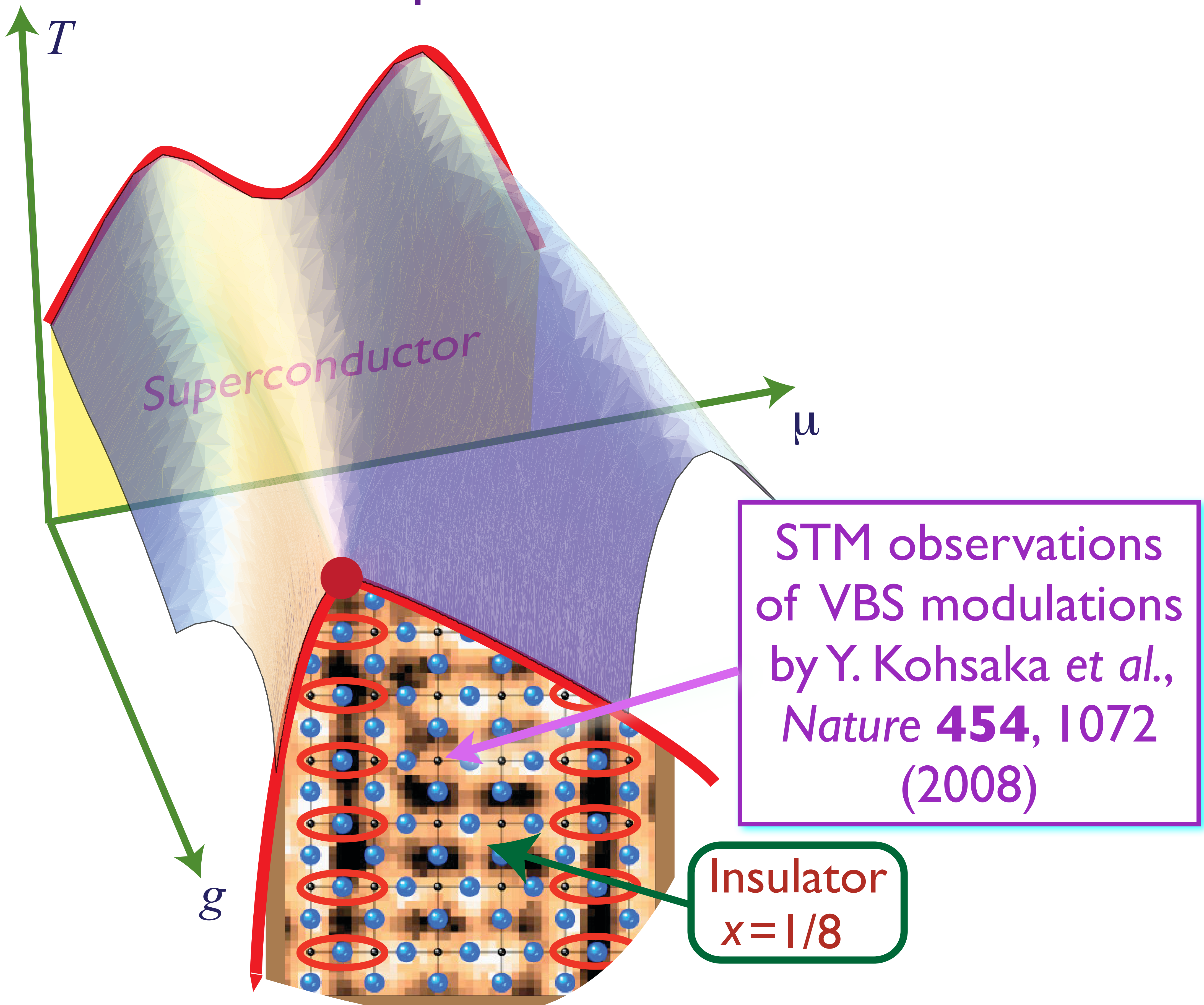


Proximity to an insulator at 12.5% hole concentration

Cuprates

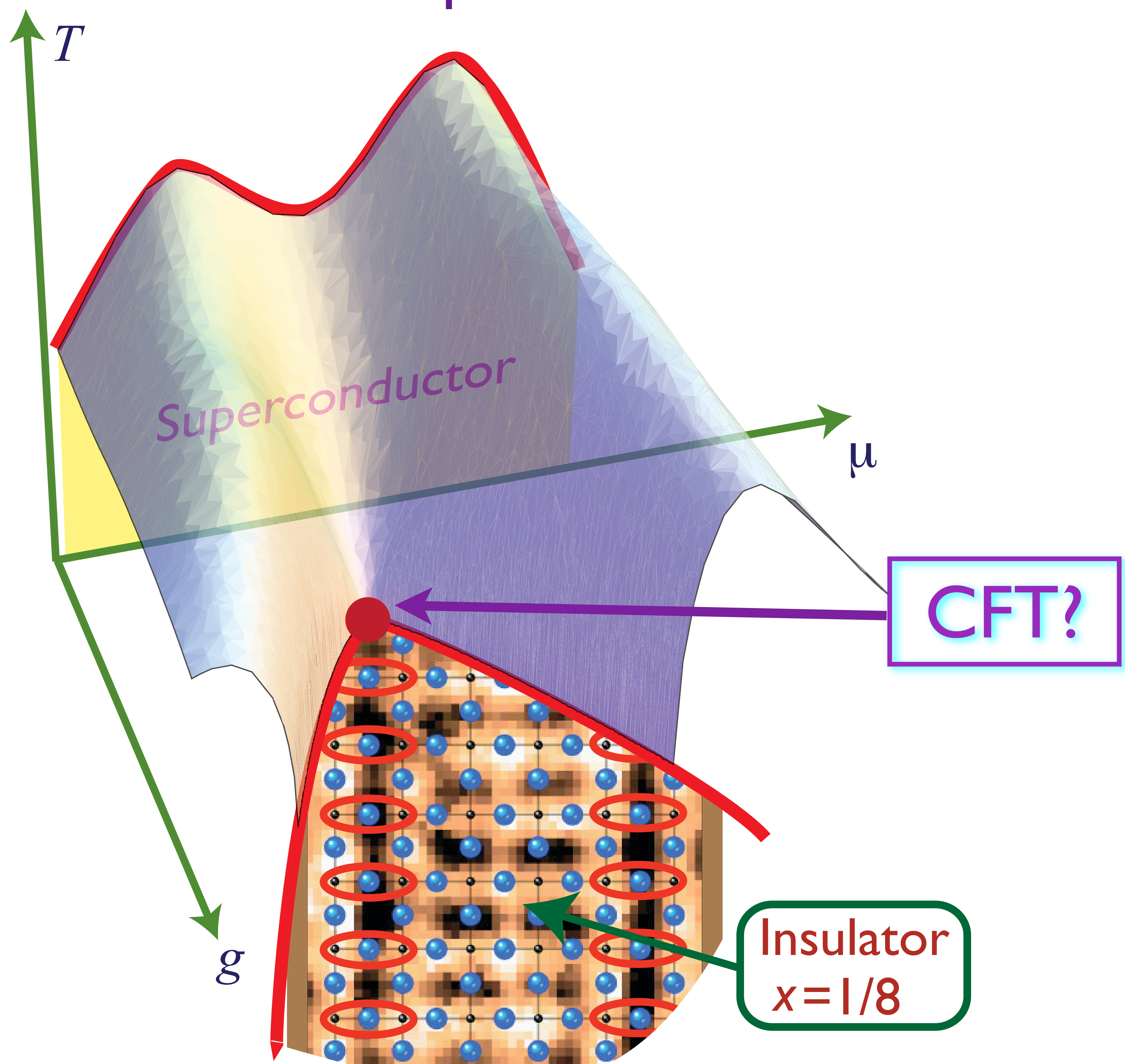


Cuprates

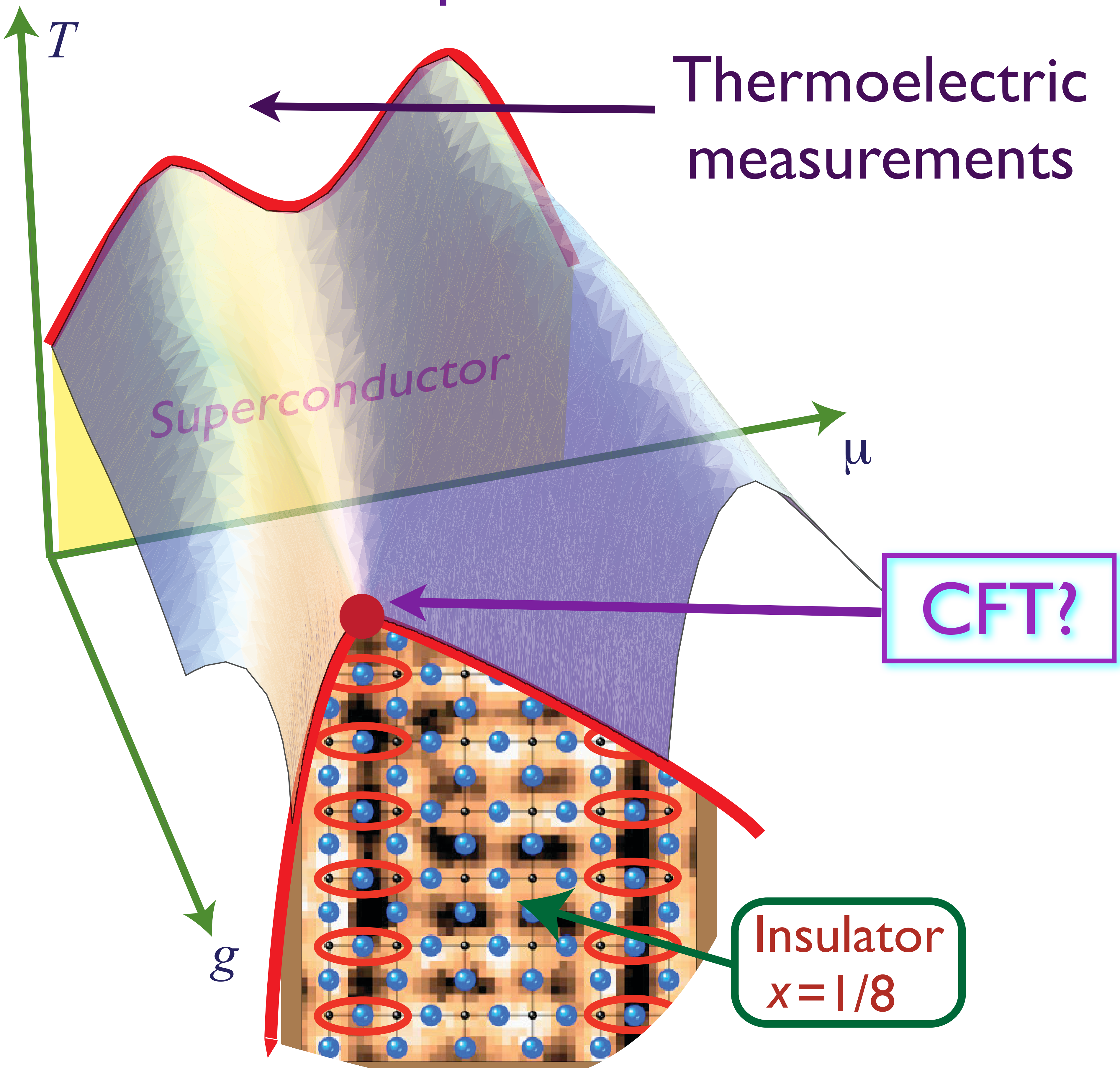


T
μ
g
Superconductor
STM observations of VBS modulations by Y. Kohsaka *et al.*, *Nature* **454**, 1072 (2008)
Insulator
x=1/8

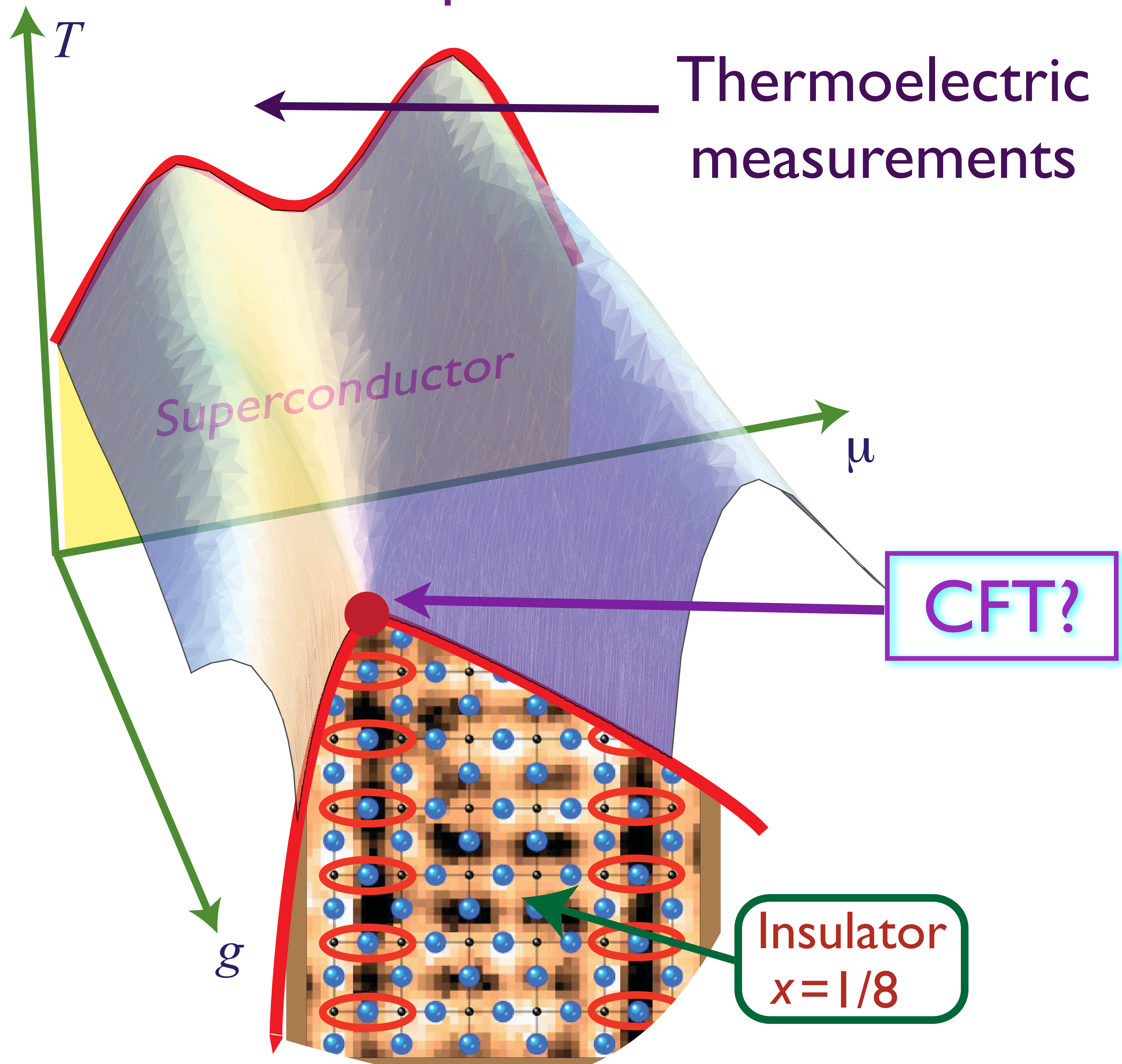
Cuprates



Cuprates

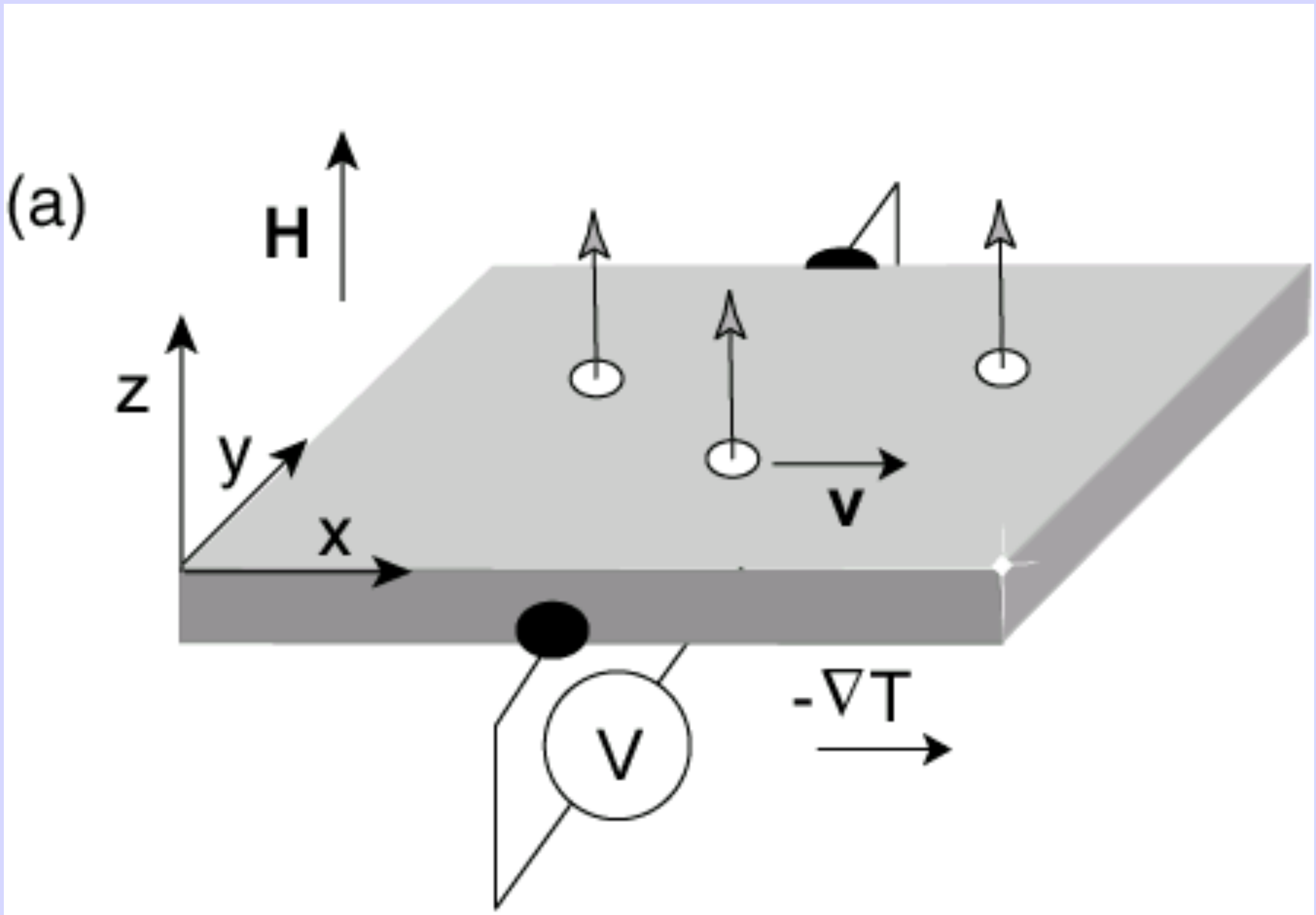

T
Thermoelectric measurements
Superconductor
μ
CFT?
g
Insulator
x=1/8

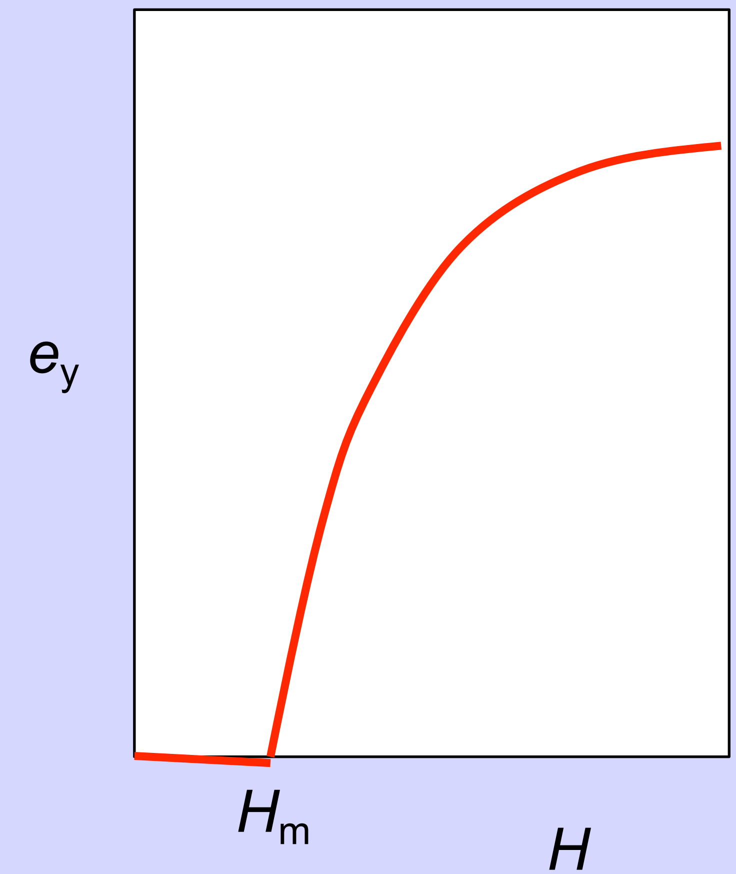
Cuprates



T
Thermoelectric
measurements
Superconductor
μ
CFT?
g
Insulator
x=1/8

Nernst experiment





Nernst signal (transverse thermoelectric response)

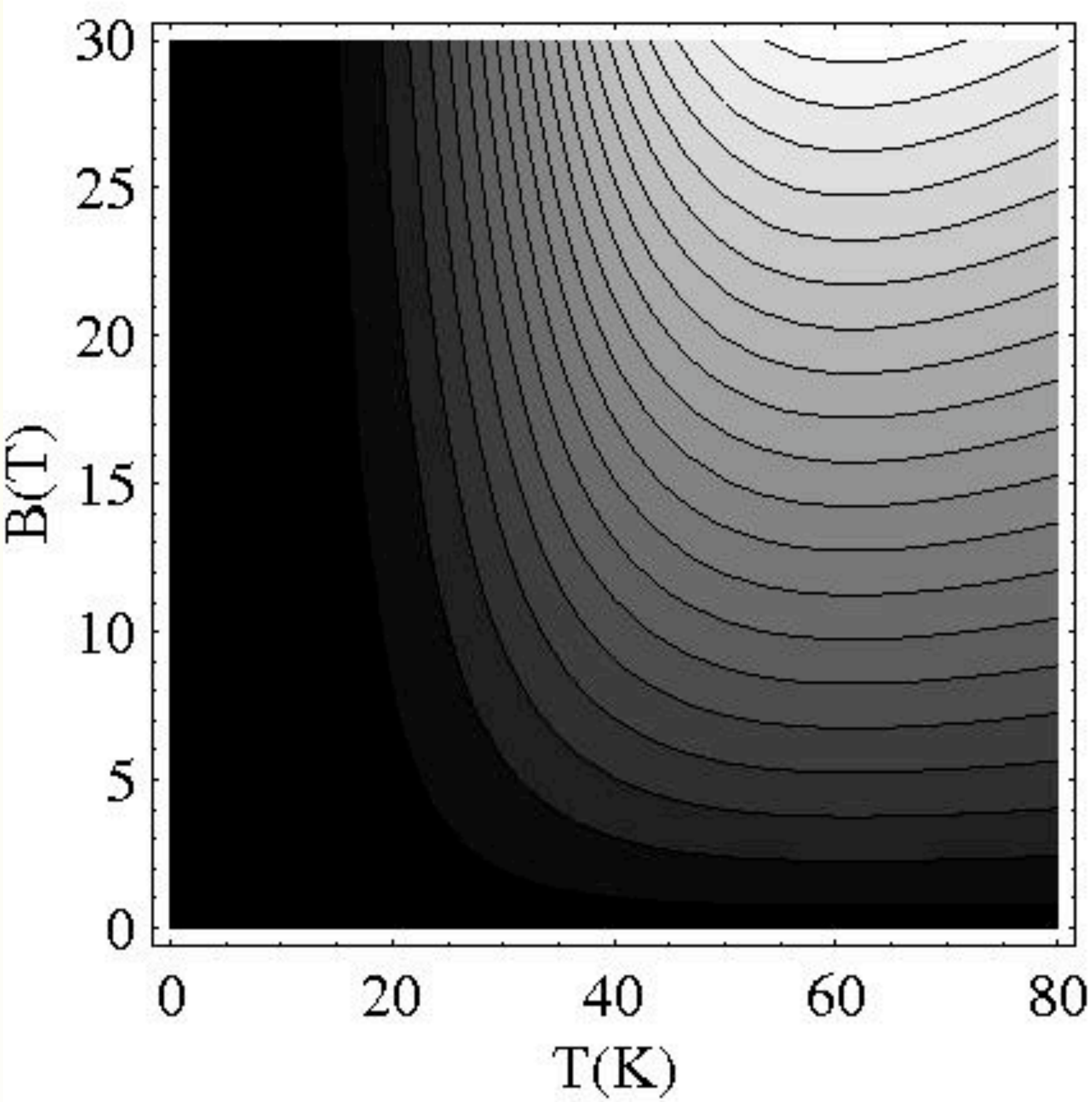
$$e_N = \left(\frac{k_B}{e^*}\right)\left(\frac{\varepsilon + P}{k_B T \rho}\right)\left[\frac{\omega_c/\tau_{\rm imp}}{(\omega_c^2/\gamma + 1/\tau_{\rm imp})^2 + \omega_c^2}\right]$$

where $\tau_{\rm imp}$ is the momentum relaxation time due to impurities or umklapp scattering.

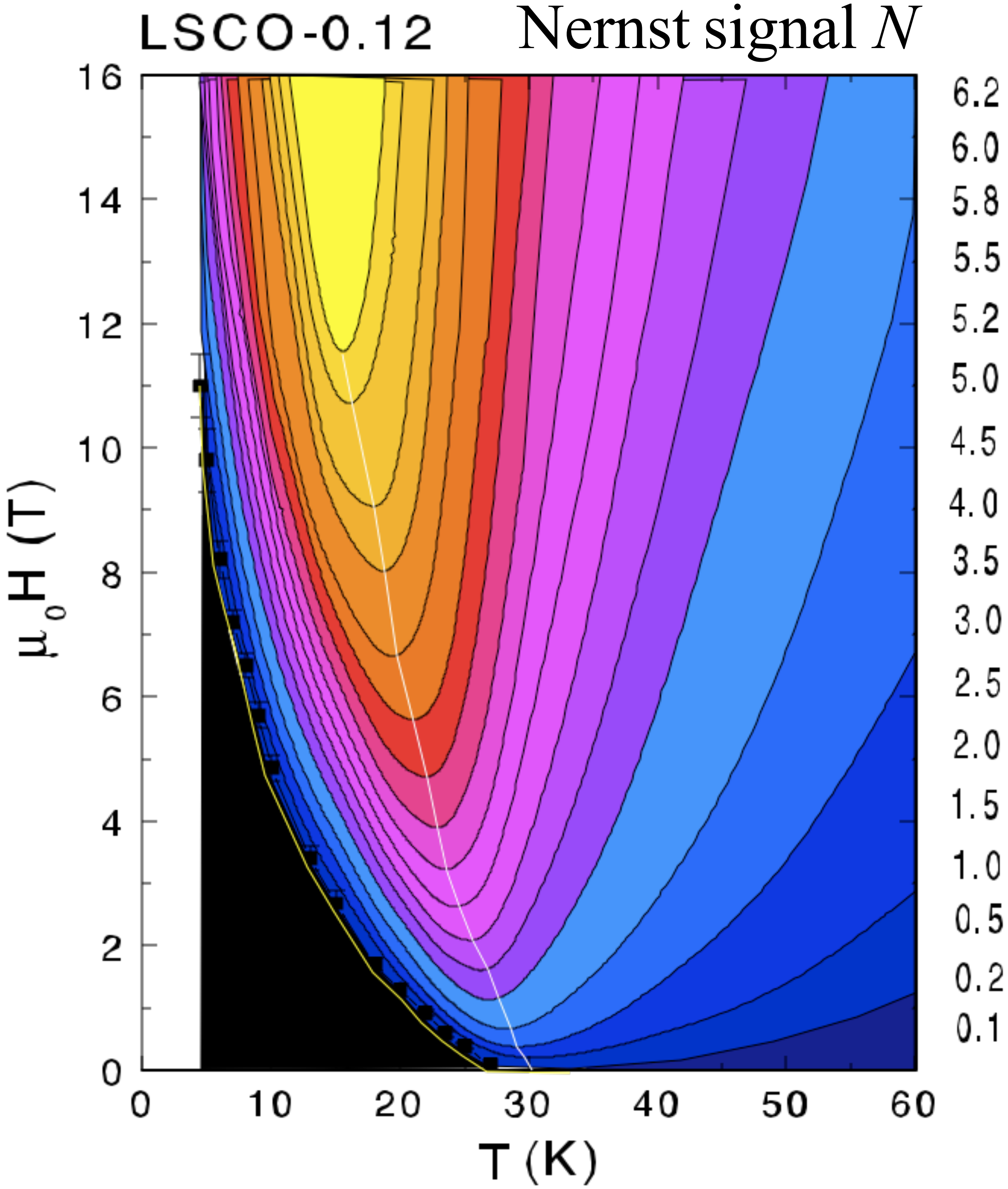
S.A. Hartnoll, P.K. Kovtun, M. Müller, and S. Sachdev, *Phys. Rev.* B **76** 144502 (2007)

LSCO Experiments

Theory for N



Nernst signal N



Y. Wang, L. Li, and N. P. Ong, *Phys. Rev.* B **73**, 024510 (2006).
S.A. Hartnoll, P.K. Kovtun, M. Müller, and S. Sachdev, *Phys. Rev.* B **76** 144502 (2007)

B and T dependencies are in semi-quantitative agreement with observations on cuprates, with reasonable values for only 2 adjustable parameters, $\tau_{\rm imp}$ and v.

B and T dependencies are in semi-quantitative agreement with observations on cuprates, with reasonable values for only 2 adjustable parameters, τ_{imp} and v.

Similar results apply to electronic transport in graphene, where the the relativistic Dirac spectrum of the electrons leads to analogies with the hydrodynamics of CFTs. We have made specific quantitative predictions for THz experiments on graphene at room temperature in the presence of a modest applied magnetic field.

Conclusions

- Theory for transport near quantum phase transitions in superfluids and antiferromagnets
- Exact solutions via black hole mapping have yielded first exact results for transport co-efficients in interacting many-body systems, and were valuable in determining general structure of hydrodynamics.
- Theory of Nernst effect near the superfluid-insulator transition, and connection to cuprates.
- Quantum-critical magnetotransport in graphene.