(I) The Fermi gas near unitarity

(II) The superfluid-insulator quantum phase transition

(III) Phases of quantum antiferromagnets

(I) The Fermi gas near unitarity

P. Nikolic and S. Sachdev
Physical Review A 75, 033608 (2007)

M. Veillette, A. Lamacraft, E.G. Moon, L. Radzihovsky,
S. Sachdev, and D. Sheehy, to appear....

Outline

1. Spinless fermions with repulsive interactions
2. Bosons with repulsive interactions
3. Spinful fermions with repulsive interactions
4. Spinful fermions with attractive interactions
5. Expansion in $1/N$
6. Imbalanced Fermi gas

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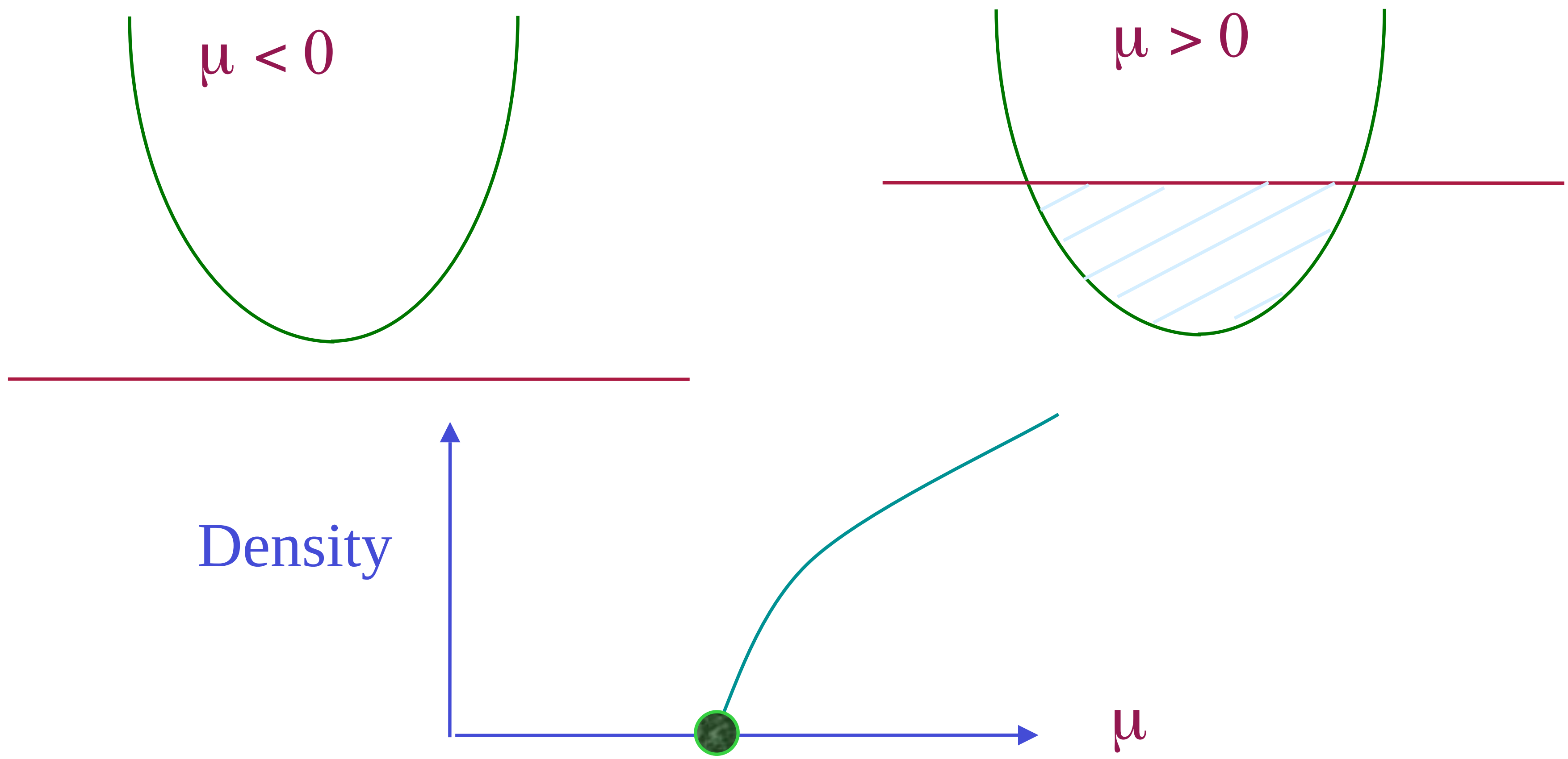
1. Spinless fermions with repulsive interactions

$$\mathcal{S} = \int d\tau d^d x \left[\psi^\dagger \frac{\partial \psi}{\partial \tau} - \frac{\hbar^2}{2m} \psi^\dagger \frac{\partial^2 \psi}{\partial x^2} - \mu \psi^\dagger \psi + \tilde{u} \psi^\dagger \nabla \psi^\dagger \psi \nabla \psi \right]$$



1. Spinless fermions with repulsive interactions

Characteristics of this 'trivial' quantum critical point:

• No "order parameter". "Topological" characterization in the existence of the Fermi surface in one state.

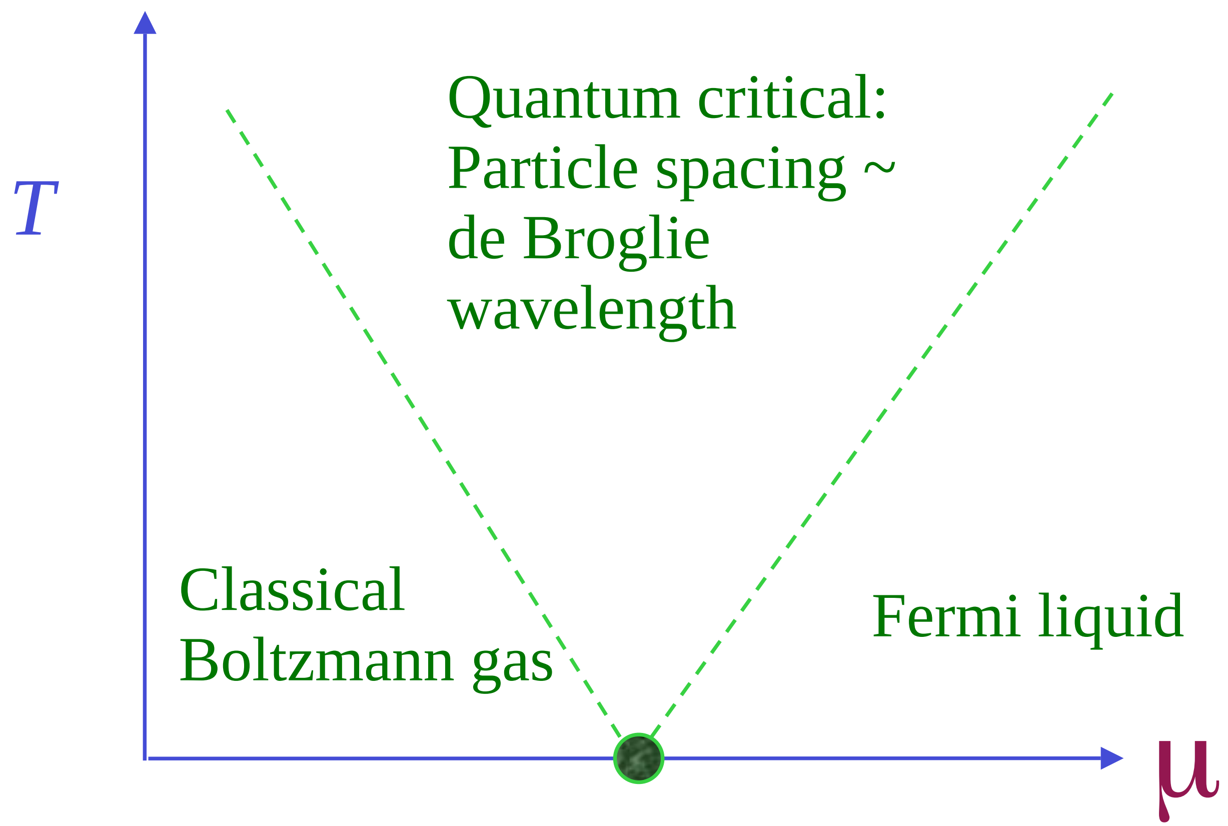
• No transition at $T > 0$.

• Characteristic crossovers at $T > 0$, between quantum criticality, and low T regimes.

• Strong T-dependent scaling in quantum critical regime, with response functions scaling universally as a function of k^z/T and ω/T, where z is the dynamic critical exponent.

1. Spinless fermions with repulsive interactions

Characteristics of this 'trivial' quantum critical point:



1. Spinless fermions with repulsive interactions

RG flow of this fixed point

$$\frac{d\tilde{u}}{d\ell} = -d\tilde{u}$$

Interactions are irrelevant in all d.

Implications: As $\mu \searrow 0$, the free energy density is given by

$$\mathcal{F} = -\mathcal{C}_d \mu \left(\frac{2m\mu}{\hbar^2} \right)^{d/2} \left[1 + \textit{irrelevant corrections which vanish with a positive power of } \mu \right]$$

where $\mathcal{C}_d$ is a universal number. Here, the free Fermi model yields

$$\mathcal{C}_d = - \int_0^1 \frac{d^d p}{(2\pi)^d} (p^2 - 1)$$

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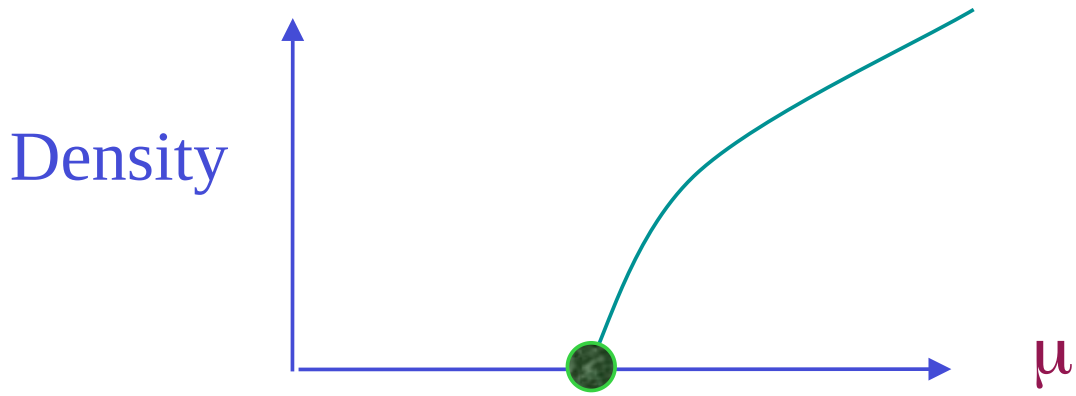
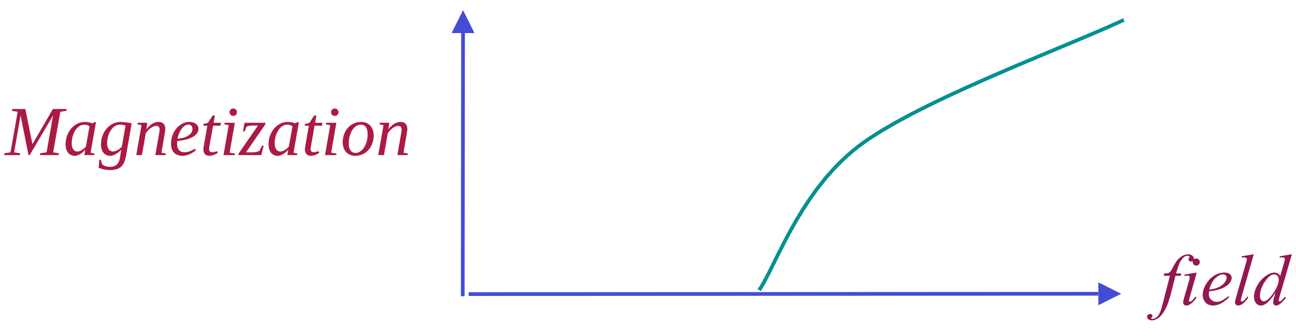
2. Bosons with repulsive interactions

$$\mathcal{S} = \int d\tau d^d x \left[\psi^\dagger \frac{\partial \psi}{\partial \tau} - \frac{\hbar^2}{2m} \psi^\dagger \frac{\partial^2 \psi}{\partial x^2} - \mu \psi^\dagger \psi + \frac{u}{2} (\psi^\dagger \psi)^2 \right]$$

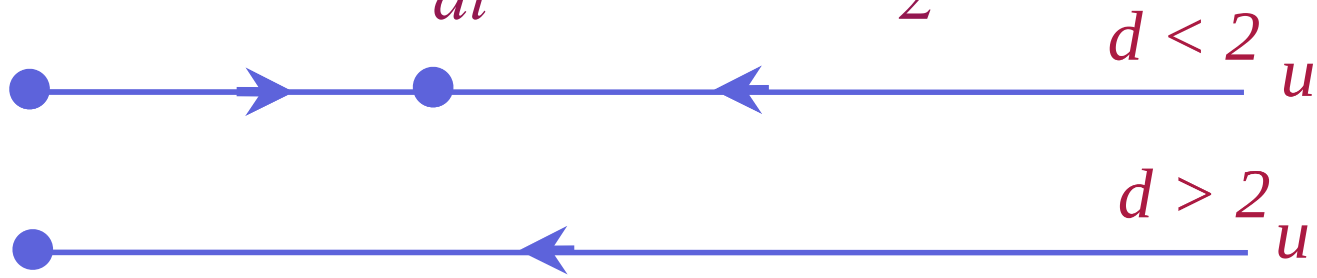


• Describes field-induced magnetization transitions in spin gap compounds



2. Bosons with repulsive interactions

$$\frac{du}{dl} = (2-d)u - \frac{u^2}{2}$$



- Critical theory in $d = 1$ is that of the spinless Fermi gas (Tonks gas), *i.e.* $\mathcal{C}_1$ equals that of the free spinless Fermi gas.

- The interaction u is *dangerously irrelevant* for $d \geq 2$. This leads to violations of universality, and $\mathcal{C}_d$ depends upon microscopic details *e.g.* Lee-Yang theory shows that as $\mu \searrow 0$

$$\mathcal{C}_3 = \sqrt{\frac{\hbar^2}{2m\mu}} \frac{1}{16\pi a}$$

where a is the s-wave scattering length.

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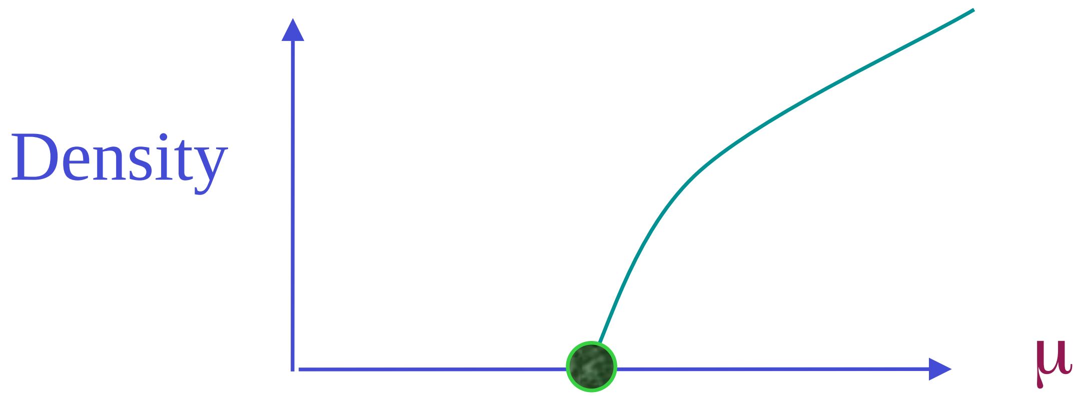

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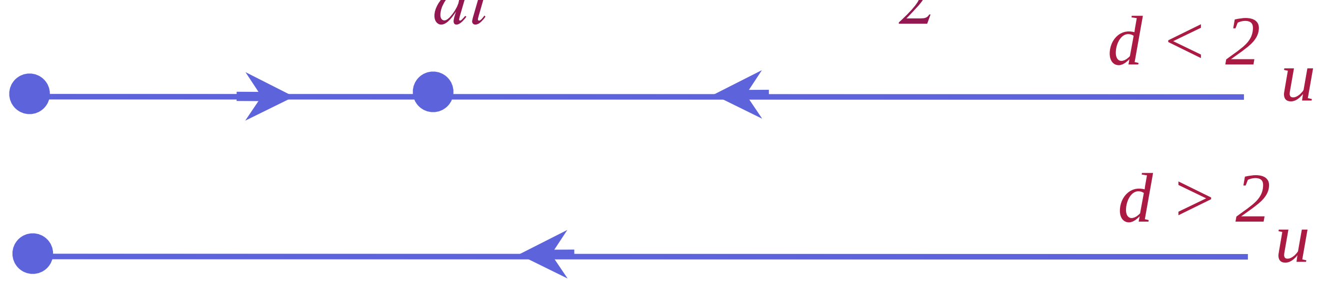
$$\frac{du}{dl} = (2-d)u - \frac{u^2}{2}$$



- Critical theory in $d = 1$ is that of the spinless Fermi gas (Tonks gas), *i.e.* $\mathcal{C}_1$ equals that of the free spinless Fermi gas.

- The interaction u is *irrelevant* for $d \geq 2$. So $\mathcal{C}_d$ is that of the free spinful Fermi gas

$$\mathcal{C}_d = -2 \int_0^1 \frac{d^d p}{(2\pi)^d} (p^2 - 1)$$

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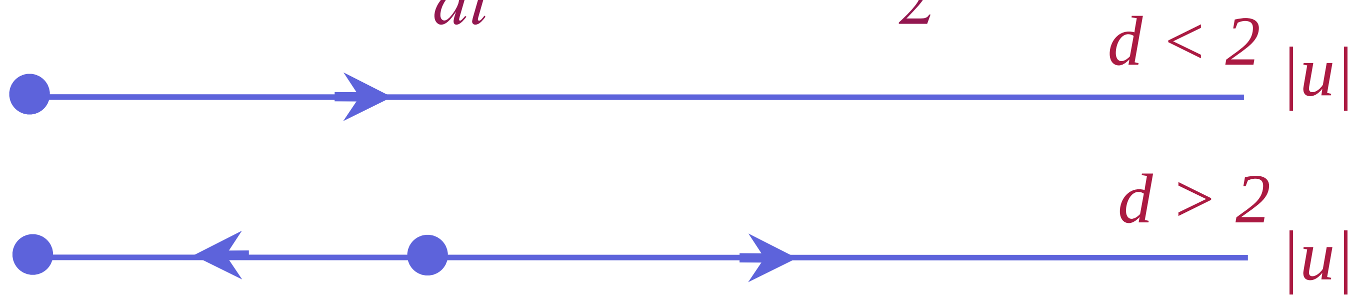
Now, as we will see, the density can be finite even for $\mu < 0$, because energy gained from attraction can overcome the chemical potential

4. Spinful fermions with attractive interactions

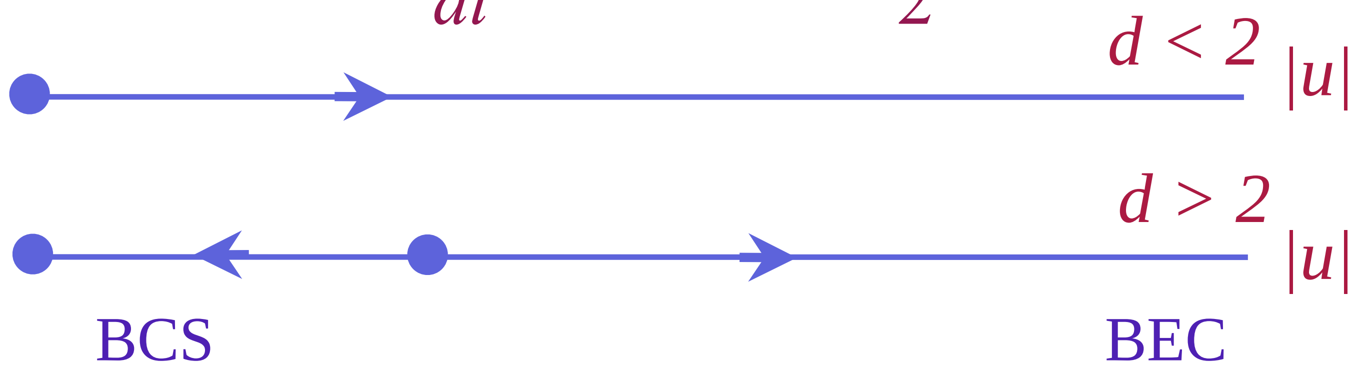
$$\frac{du}{dl} = (2-d)u - \frac{u^2}{2}$$



- For $d < 2$ the fermions form bosonic bound states which repel each other. The resulting 'BEC theory' maps onto the "*bosons with repulsive interactions*" case considered earlier with $\mu \to 2\mu$.

4. Spinful fermions with attractive interactions

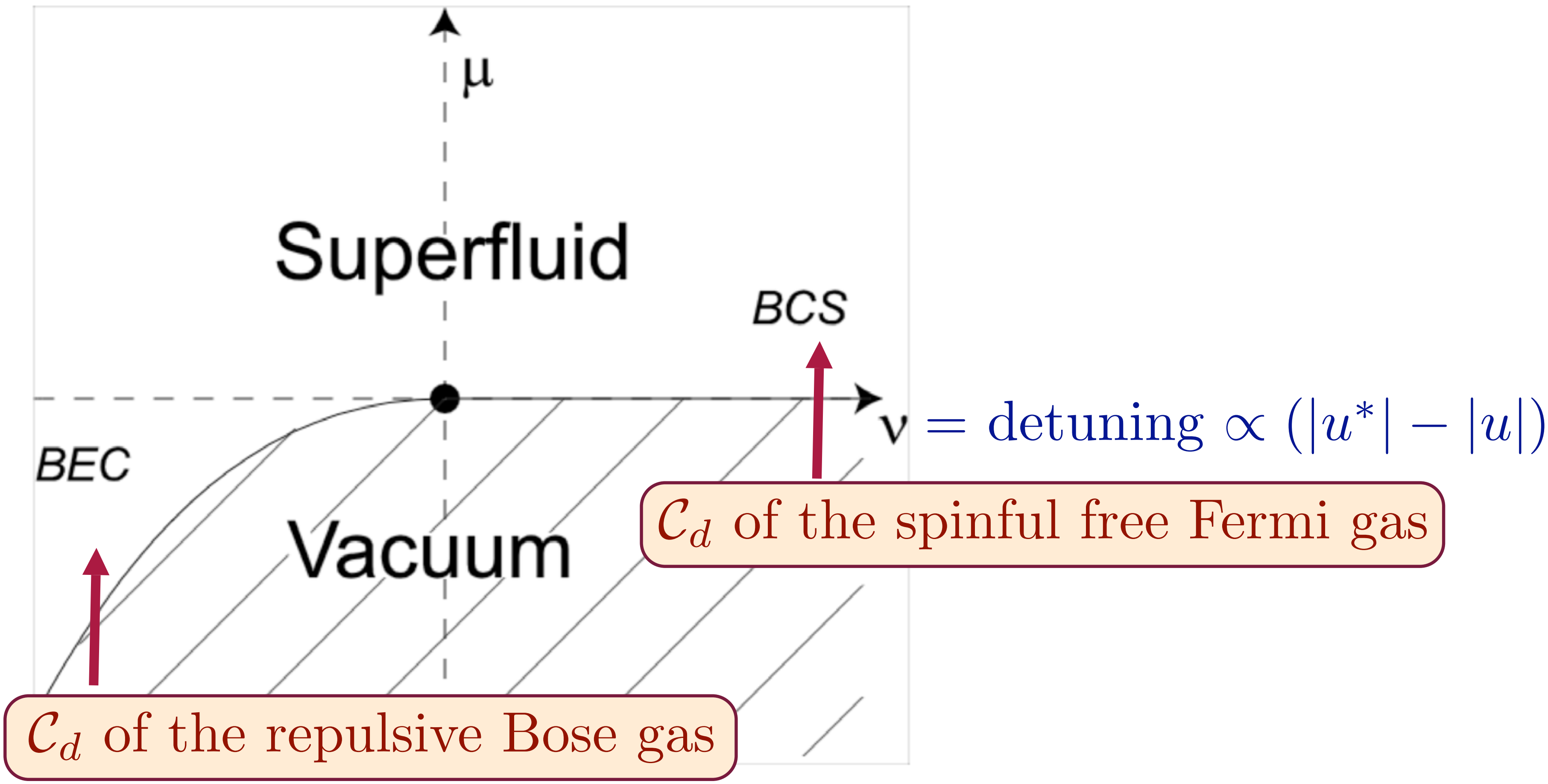
$$\frac{du}{dl} = (2-d)u - \frac{u^2}{2}$$



- For $d > 2$, there is an unstable fixed point which describes **Feshbach resonance**. For attractions larger than the fixed point value, we obtain a 'BEC theory' of molecules, which maps onto the "*fermions with repsulsive interactions*" case considered earlier with $\mu \rightarrow 2\mu$. Weak interactions are formally irrelevant, and we obtain 'BCS theory', in which $\mathcal{C}_d$ is given by the spinful free Fermi gas.

4. Spinful fermions with attractive interactions

$T = 0$ phase diagram for $d > 2$



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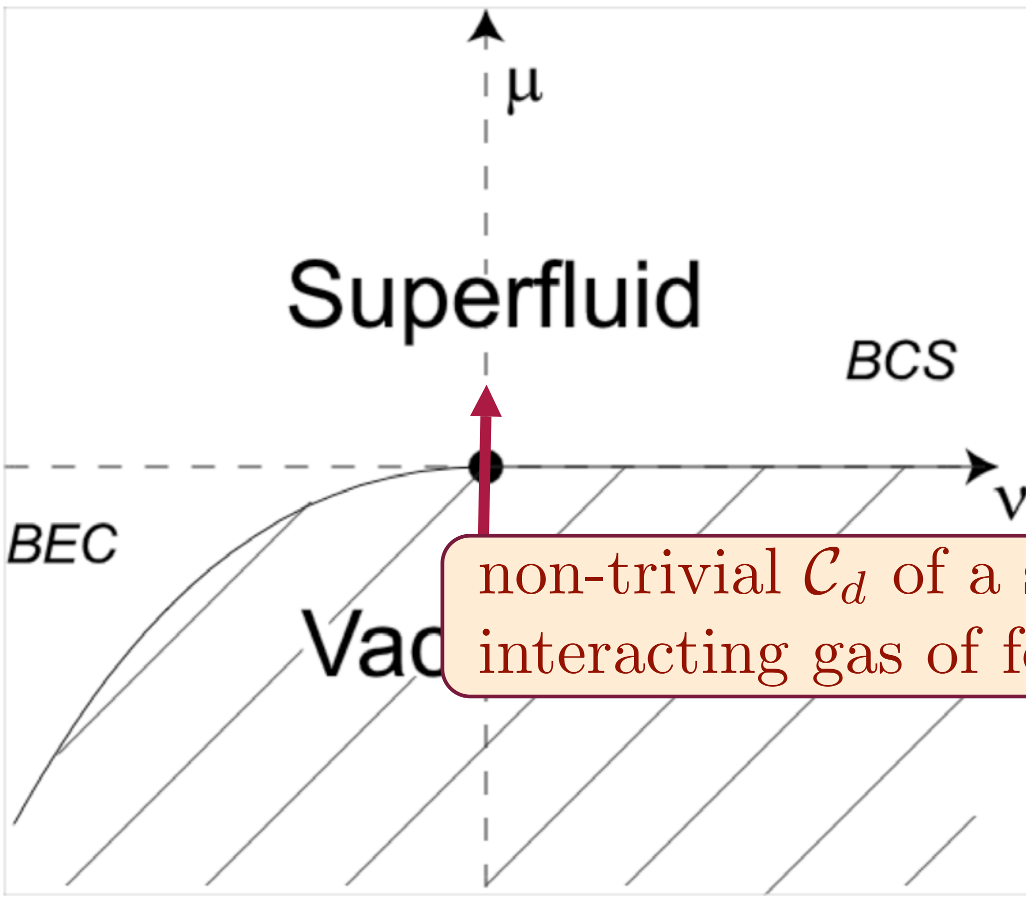
$T = 0$ phase diagram for $d > 2$



ν = detuning $\propto (|u^*| - |u|)$

non-trivial $\mathcal{C}_d$ of a strongly interacting gas of fermions and bosons

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Two channel model

- Critical field theory of interacting atoms and molecules (*s*-wave)

$$\mathcal{S}_c = \int \mathrm{d}\tau \mathrm{d}^d x \Biggl\{ \psi_\sigma^\dagger \left(\frac{\partial}{\partial \tau} - \frac{\nabla^2}{2m} \right) \psi_\sigma + \Phi^\dagger \left(\frac{\partial}{\partial \tau} - \frac{\nabla^2}{4m} + r_{0c} \right) \Phi - g_0 \left(\Phi^\dagger \psi_\uparrow \psi_\downarrow + \Phi \; \psi_\downarrow^\dagger \psi_\uparrow^\dagger \right) \Biggr\}$$

- Relevant perturbations

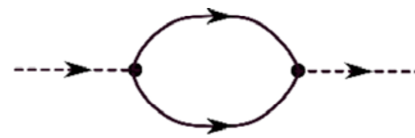
$$\mathcal{S}_p = \int \mathrm{d}\tau \mathrm{d}^d x \Bigl\{ -\mu (\psi_\sigma^\dagger \psi_\sigma + 2\Phi^\dagger \Phi) + \delta \Phi^\dagger \Phi - h \left(\psi_\uparrow^\dagger \psi_\uparrow - \psi_\downarrow^\dagger \psi_\downarrow \right) \Bigr\}$$

$$\delta = \frac{m\nu}{4\pi} g_0^2$$

Feshbach resonance

- Exact renormalization group


= 0


≠ 0

- Renormalization group flow

$$\frac{dg}{d\ell} = \frac{(4-d)}{2}g - \frac{g^3}{2}$$

$$\frac{d\mu}{d\ell} = 2\mu \qquad \frac{dh}{d\ell} = 2h \qquad \frac{d\nu}{d\ell} = (2-g^2)\nu$$

- The critical exponents of the relevant operators are the same as in the one-channel model
- Both models describe the same fixed point => Feshbach resonance

ε-Expansions

- Thermodynamic properties are universal functions at the resonance
  - Expressed as expansions in powers of ε
  - One-channel: $\varepsilon=d-2$; two-channel: $\varepsilon=4-d$

Y. Nishida and D. T. Son, Phys. Rev. Lett. **97**, 050403 (2006)

- Fundamental limitations
  - Perturbation theory in the interaction strength (u_0, g_0)
  - Critical coupling must be small (u_0, $g_0 \sim \varepsilon$)
  - Tractable effects of pairing fluctuations are small

    One-channel => negligible pairing amplitude

    Two-channel => deeply in the superfluid, or 1 Fermi sea

- Large-N expansion allows strong coupling at the resonance...

Sp(2N) generalization

- Sp(2N) generalization of the two-channel model
  - N-pairs of spin-up and spin-down fermions, $\psi_{i\sigma}$ (σ=1..N)
  - Fermions coupled to a single molecule field, Φ

$$\mathcal{S} = \int \mathrm{d}\tau \mathrm{d}^d x \Bigg[\psi_{i\sigma}^{\dagger} \left(\frac{\partial}{\partial \tau} - \frac{\nabla^2}{2m} - \mu + V(\mathbf{x}) \right) \psi_{i\sigma} + h(\psi_{i\uparrow}^{\dagger}\psi_{i\uparrow} - \psi_{i\downarrow}^{\dagger}\psi_{i\downarrow}) + N\frac{m\nu}{4\pi}\Phi^{\dagger}\Phi + \Phi^{\dagger}\psi_{i\downarrow}\psi_{i\uparrow} + \Phi\psi_{i\uparrow}^{\dagger}\psi_{i\downarrow}^{\dagger} \Bigg]$$

- Feshbach resonance at zero density
  - Bare molecule dispersion is irrelevant
  - Relevant perturbations: μ, ν, h, $V(\boldsymbol{x})$...

Large-N expansion

- Renormalization group equations

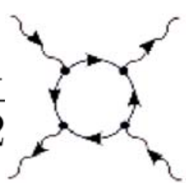
$$\frac{d\mu}{d\ell} = 2\mu \qquad \frac{dg}{d\ell} = \frac{(4-d)}{2}g - Ng^3$$

$$\frac{dh}{d\ell} = 2h \qquad g^* = \text{const.} \times g_0^* \sim \frac{1}{\sqrt{N}}$$

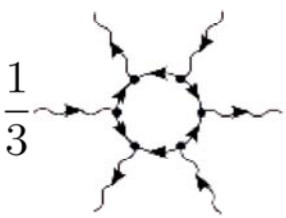
- Effective theory of Cooper pairs/molecules
  - Wiggly lines: Φ and $\Phi^\dagger$
  - New molecule bare propagator: “bubble diagram” ∝ $1/N$
  - New molecule bare vertices: ∝ N
  - Perturbation theory generates a $1/N$ expansion

$$\mathcal{S}_{\text{eff}} = N\frac{m\nu}{4\pi}\int d\tau d^3r \Phi^\dagger \Phi +$$

$$+\frac{1}{2}$$



$$+\frac{1}{3}$$



$$+\cdots$$

Universal mean-field phase diagram

- Free energy of molecules
  - Static uniform superfluid (mean-field Φ=const. at $N=\infty$)
  - Integrate-out fermions (Gaussian)

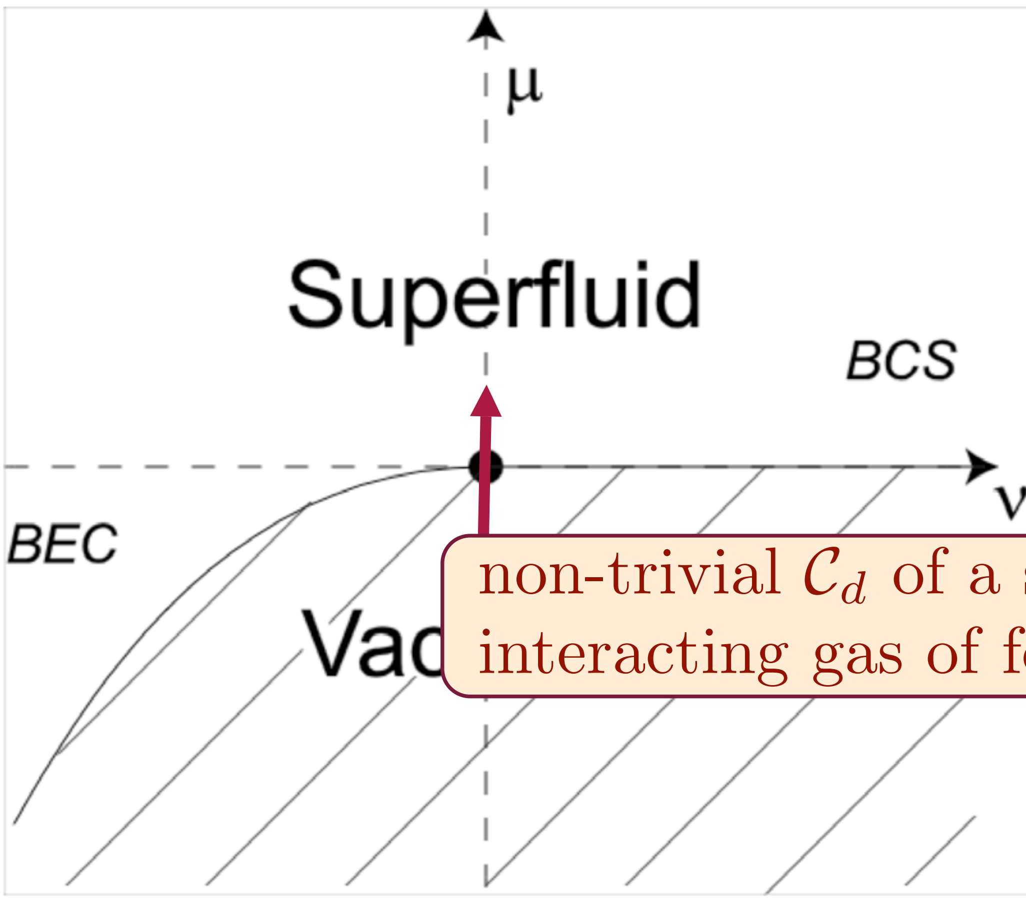
$$\frac{\mathcal{F}}{N} = \frac{m\nu}{4\pi}|\Phi|^2 - \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\left[\sqrt{\left(\frac{p^2}{2m}-\mu\right)^2 + |\Phi|^2} - \left(\frac{p^2}{2m}-\mu\right) - \frac{m}{p^2}|\Phi|^2 \right.$$

$$\left. + T\ln\left(1+e^{-(\sqrt{(p^2/(2m)-\mu)^2+|\Phi|^2}-h)/T}\right) + T\ln\left(1+e^{-(\sqrt{(p^2/(2m)-\mu)^2+|\Phi|^2}+h)/T}\right)\right]$$

  - Obtain |Φ| by minimizing F(|Φ|)
  - Substitute |Φ| in the fermion action to check the Fermi seas

4. Spinful fermions with attractive interactions

$T = 0$ phase diagram for $d > 2$



$\nu = \text{detuning} \propto (|u^*| - |u|)$

non-trivial $\mathcal{C}_d$ of a strongly interacting gas of fermions and bosons

Numerical results from $1/N$ expansion at unitarity

- The universal constant in the $\mathcal{C}_3 = 0.0297$ at $N = \infty$.
- Alternative way to present these results

$$\frac{\mu}{\varepsilon_F} = 0.5906 - \frac{0.312}{N}$$
$$\frac{\Delta}{\varepsilon_F} = 0.6864 - \frac{0.196}{N}$$

where ε_F is the Fermi energy of a free spinful Fermi gas at the same density.

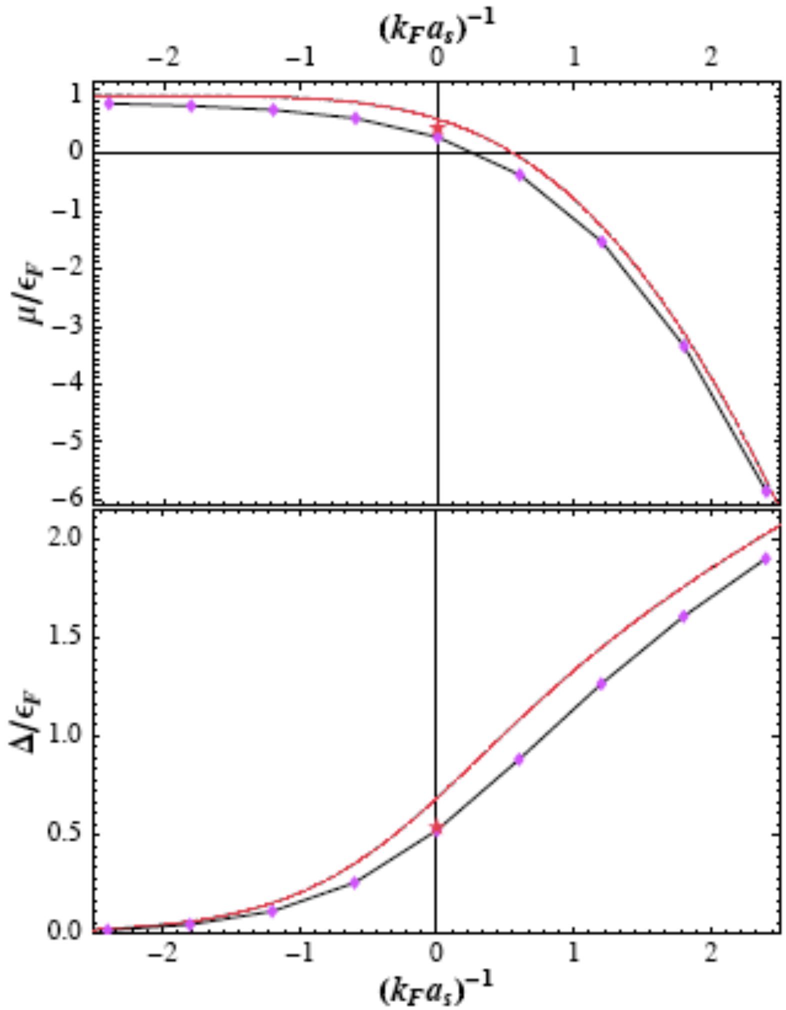


FIG. 1: Chemical potential μ and order parameter Δ as a function of $(k_F a_s)^{-1}$. The dashed lines are the $N \to \infty$ results for μ and Δ. The diamond symbols include the $\mathcal{O}(1/N)$ corrections, evaluated at $N = 1$. The solid lines are a guide to the eye. The star symbols at unitarity are the results of quantum Monte Carlo calculation from Ref. 19.

M. Y. Veillette, D.E. Sheehy, and L. Radzihovsky, Phys. Rev. A **75**, 043614 (2007)

Finite temperatures

- 2[nd] order superluid-normal phase transition at $T=T_c$

	$N=1$	MC
$$\frac{\mu}{T_c} = 1.50448 + \frac{2.785}{N} + \mathcal{O}(1/N^2)$$	4.28948	3.247
$$\frac{\varepsilon_F}{T_c} = 2.01424 + \frac{5.317}{N} + \mathcal{O}(1/N^2)$$	7.33124	6.579
$$\frac{P/N}{(2m)^{3/2}T_c^{5/2}} = 0.13188 + \frac{0.4046}{N} + \mathcal{O}(1/N^2)$$	0.53648	0.776

Monte-Carlo: E.Burovski, N.Prokof'ev, B.Svistunov, M.Troyer *Phys.Rev.Lett.* **96**, 160402 (2006)

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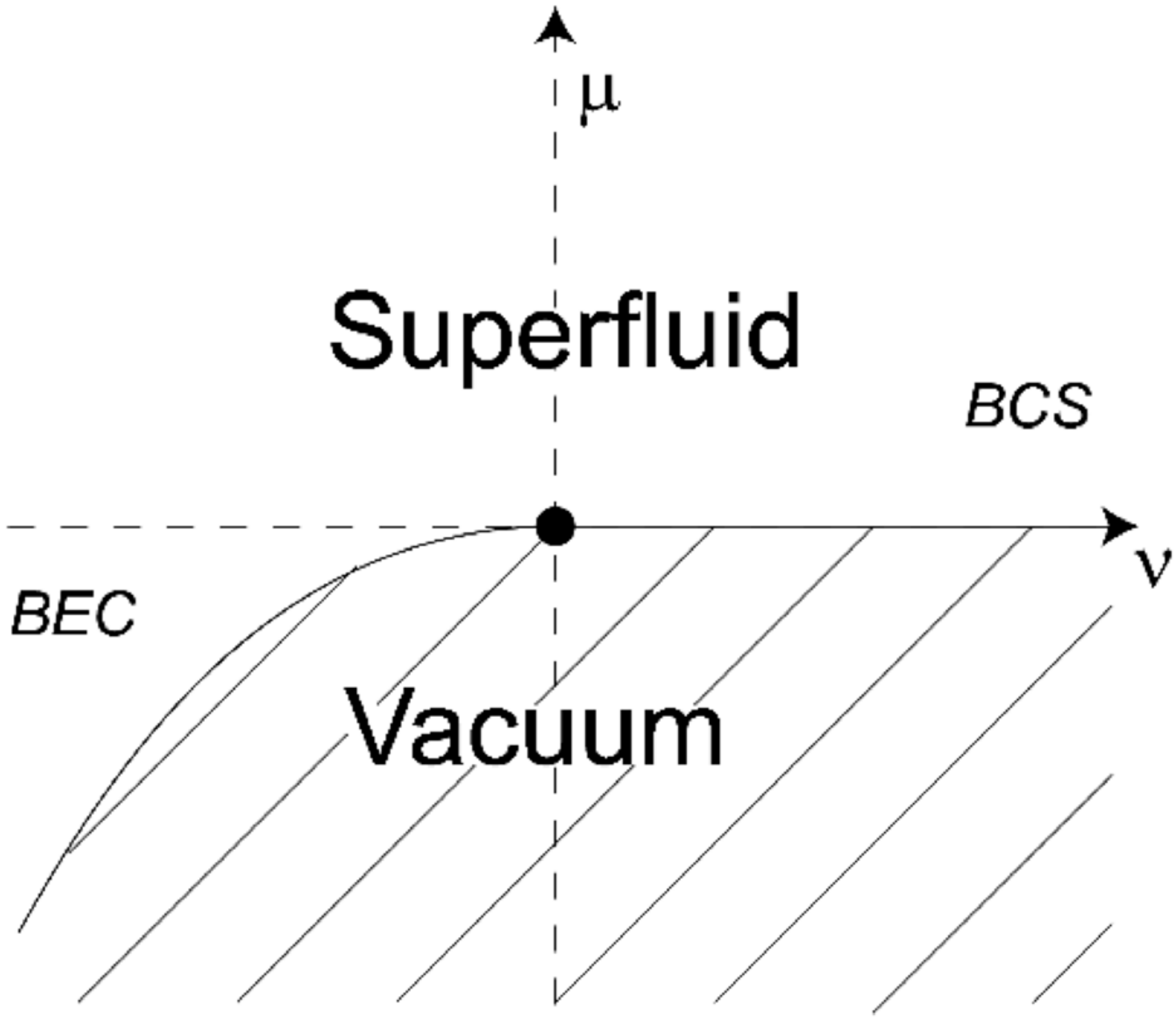
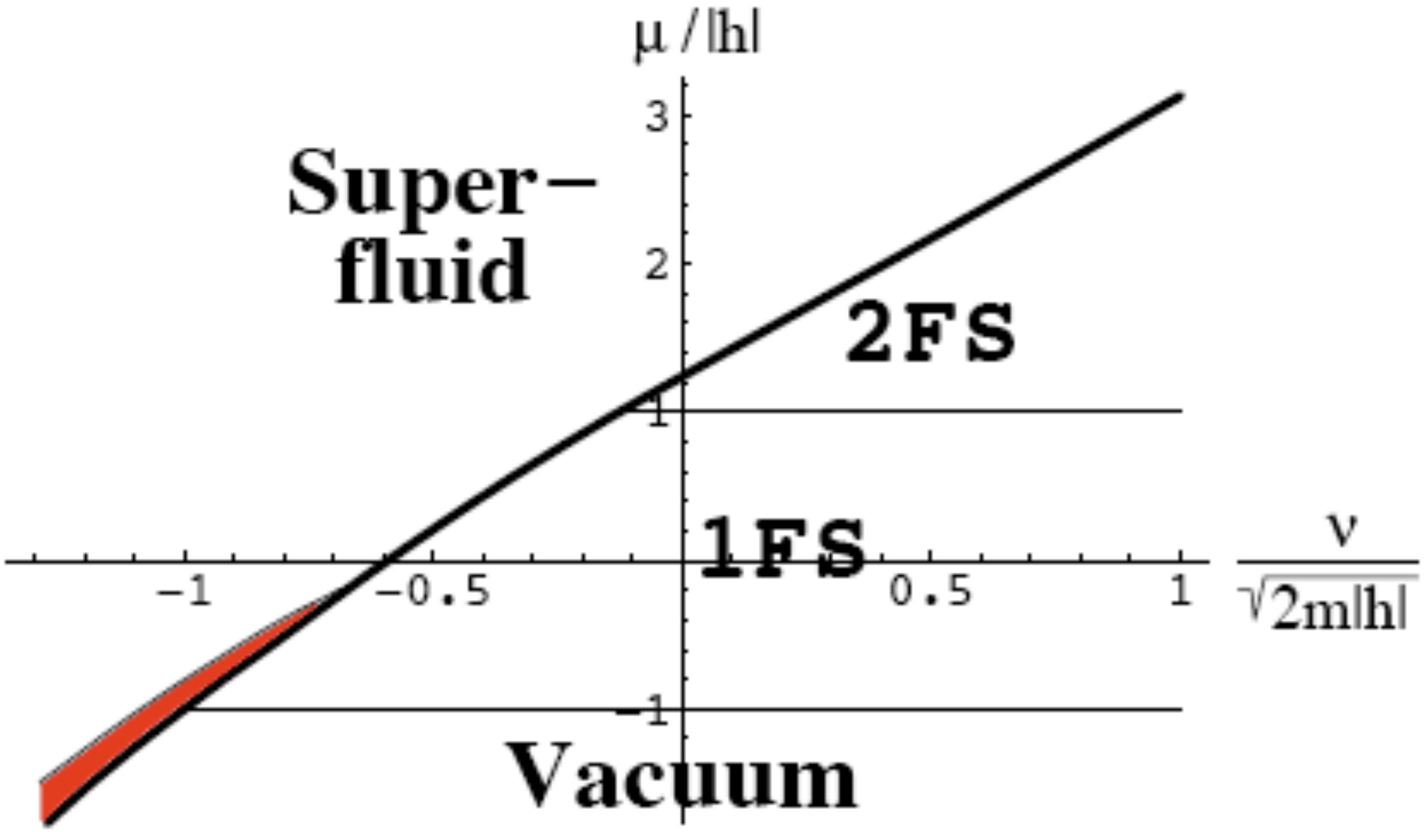
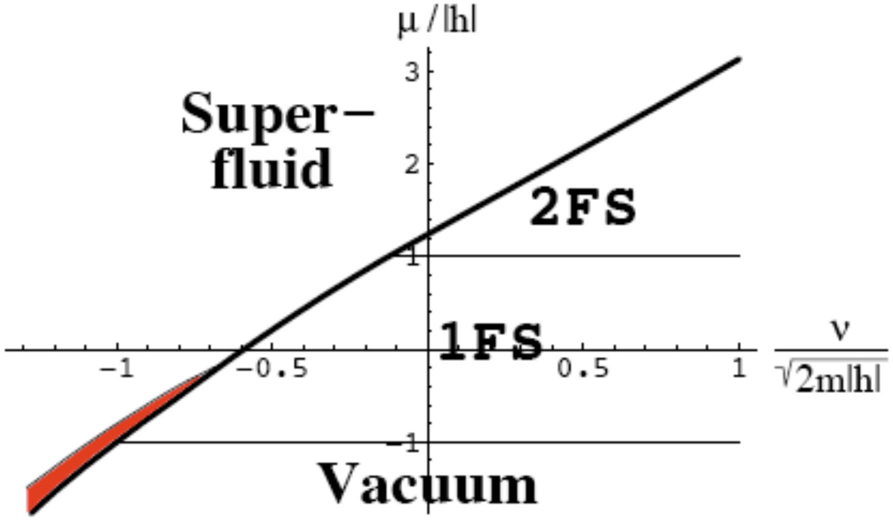
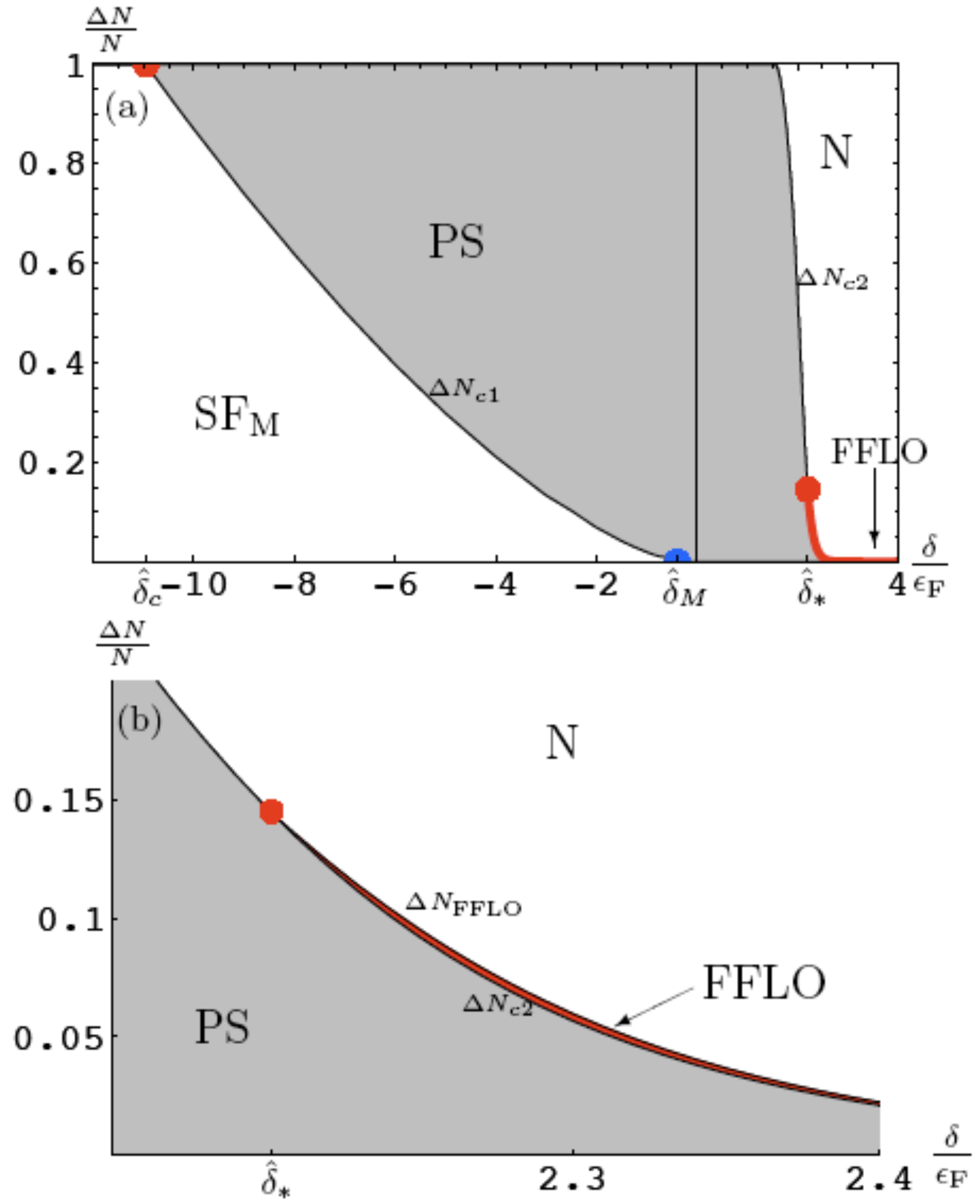
T=0 phase diagram (equal density, h=0)



T=0 phase diagram (unequal density)

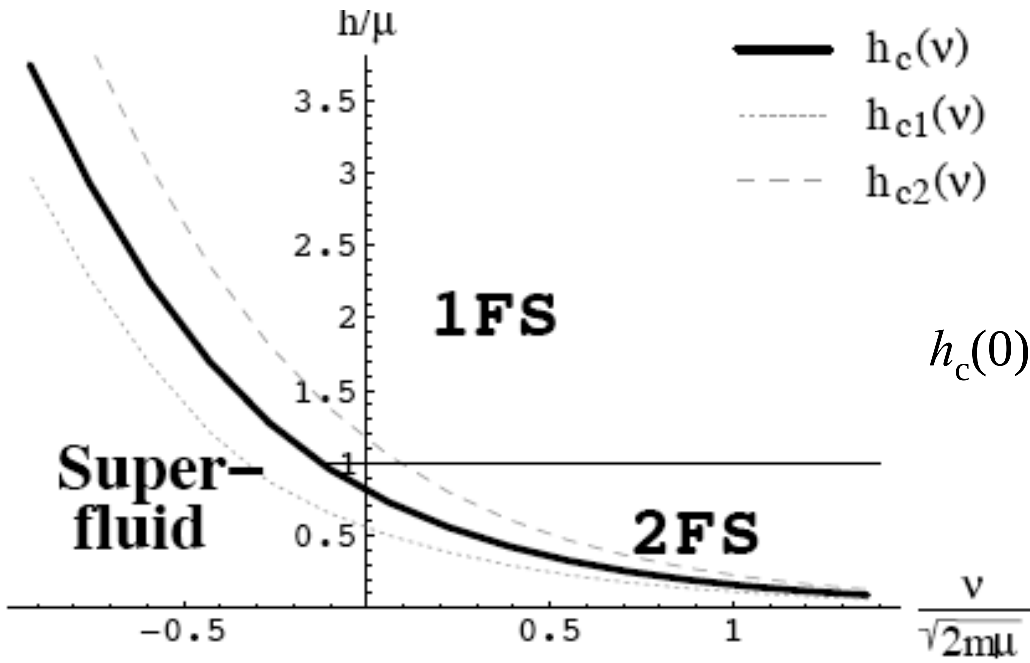






D.E. Sheehy and L. Radzihovsky, Phys. Rev. Lett. **96**, 060401 (2006)

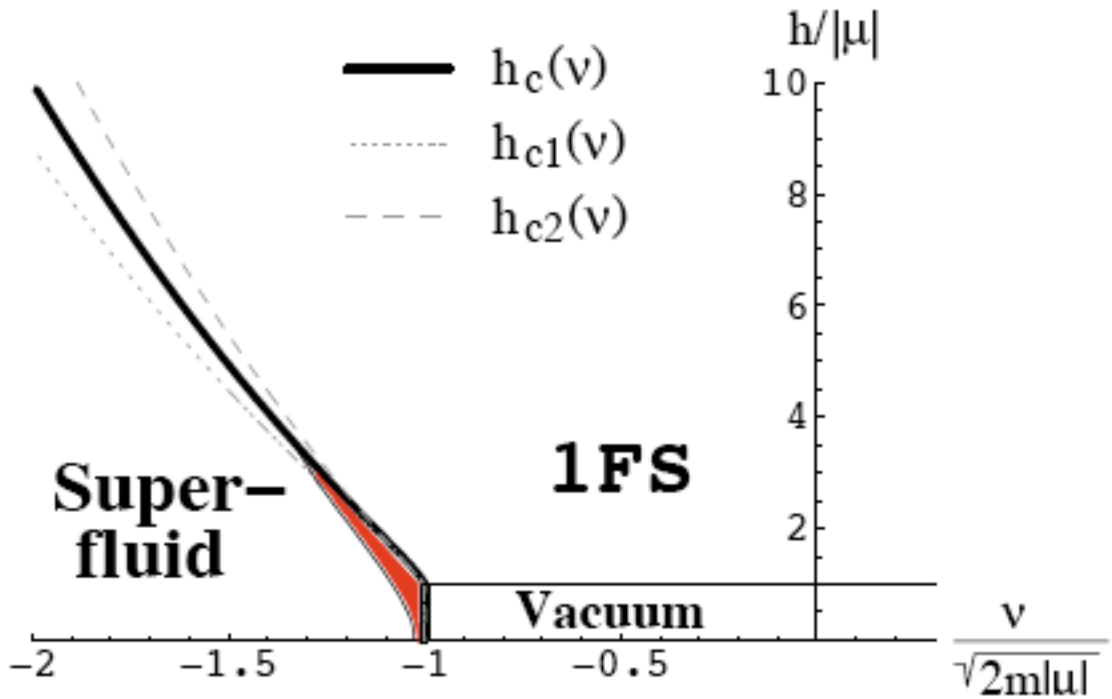
T=0 phases and transitions



$h_c(0)=0.807\mu+O(1/N)$

- Predictions
  - 1st order transition between the superfluid and normal phases
  - Smooth BEC-BCS crossover
  - Uniform magnetized BEC superfluid phase for $\mu<0$
  - Normal phases with one (1N) or two (2N) Fermi seas

T=0 phases and transitions



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Characteristics of the 2FS Normal state

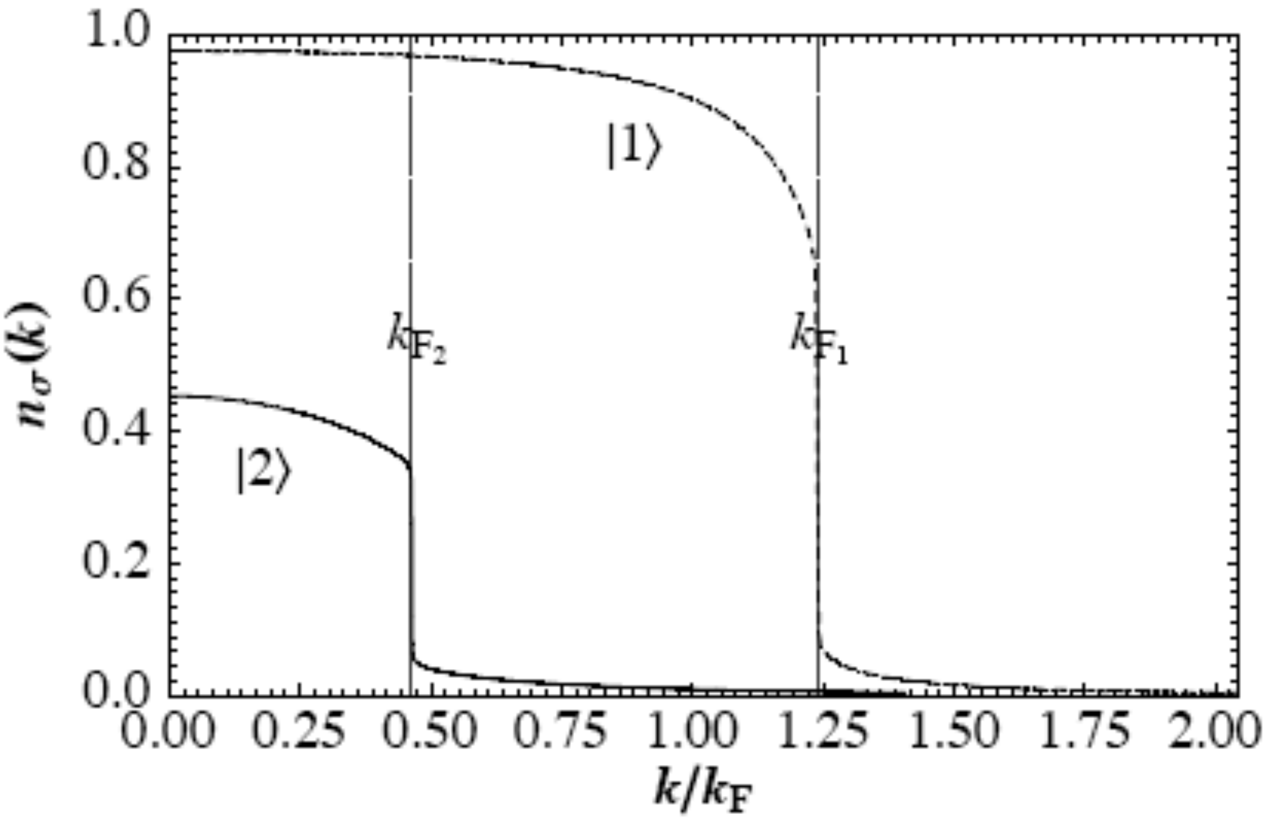

FIG. 1: Momentum distribution, $n_\sigma(k)$ of majority ($|1\rangle$) and minority ($|2\rangle$) atoms for a polarization $P = 0.9$ at zero temperature and at resonance. The residues for the majority and minority atoms are respectively $Z_1 = 0.56$ and $Z_2 = 0.29$.

M. Veillette, A. Lamacraft, E.G. Moon, L. Radzihovsky, S. Sachdev, and D. Sheehy, to appear

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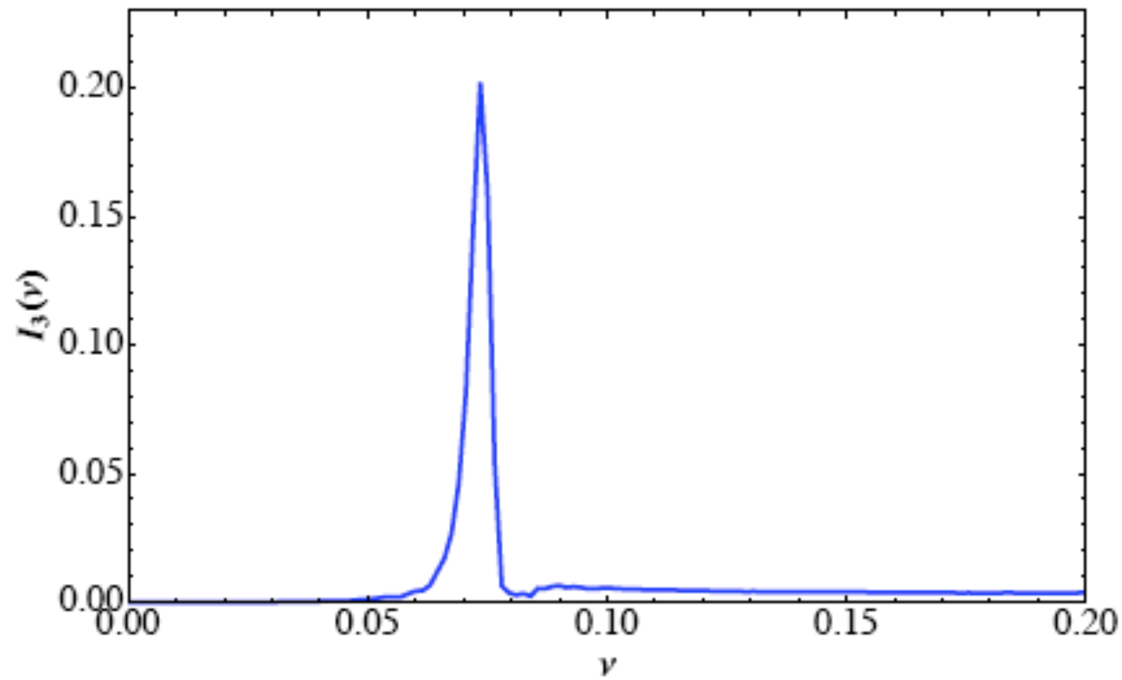

Radio frequency absorption spectrum (preliminary)

FIG. 2: RF spectrum for polarization $P = 0.95$: the intensity $I_3(\nu)$ (arbitrary units) vs. the detuning from the resonance frequency ν measured in units of the Fermi energy ϵ_F. In free space, the resonance would be at $\nu = 0$. The shift in the resonance frequency above is due to strong interactions between fermions in the non-superfluid ground state of a polarized Fermi gas. Our primary claim is that such a shift is present even while the ground state remains a Fermi liquid, with the discontinuities in the momentum distribution function shown in Fig. 1.

M. Veillette, A. Lamacraft, E.G. Moon, L. Radzihovsky, S. Sachdev, and D. Sheehy, to appear

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L . Balents, L. Bartosch, A. Burkoc,
S. Sachdev. and K. Sengupta,
Physical Review B 71, 144508 (2005)

E.G. Moon, P. Nikolic and S. Sachdev
Physical Review Letters 99, 230403 (2007)

Outline

1. Lattice bosons at integer filling

2. Fermions near unitarity at even-integer filling

3. Boson-vortex duality

4. Lattice bosons at fractional filling

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The Superfluid-Insulator transition

Boson Hubbard model

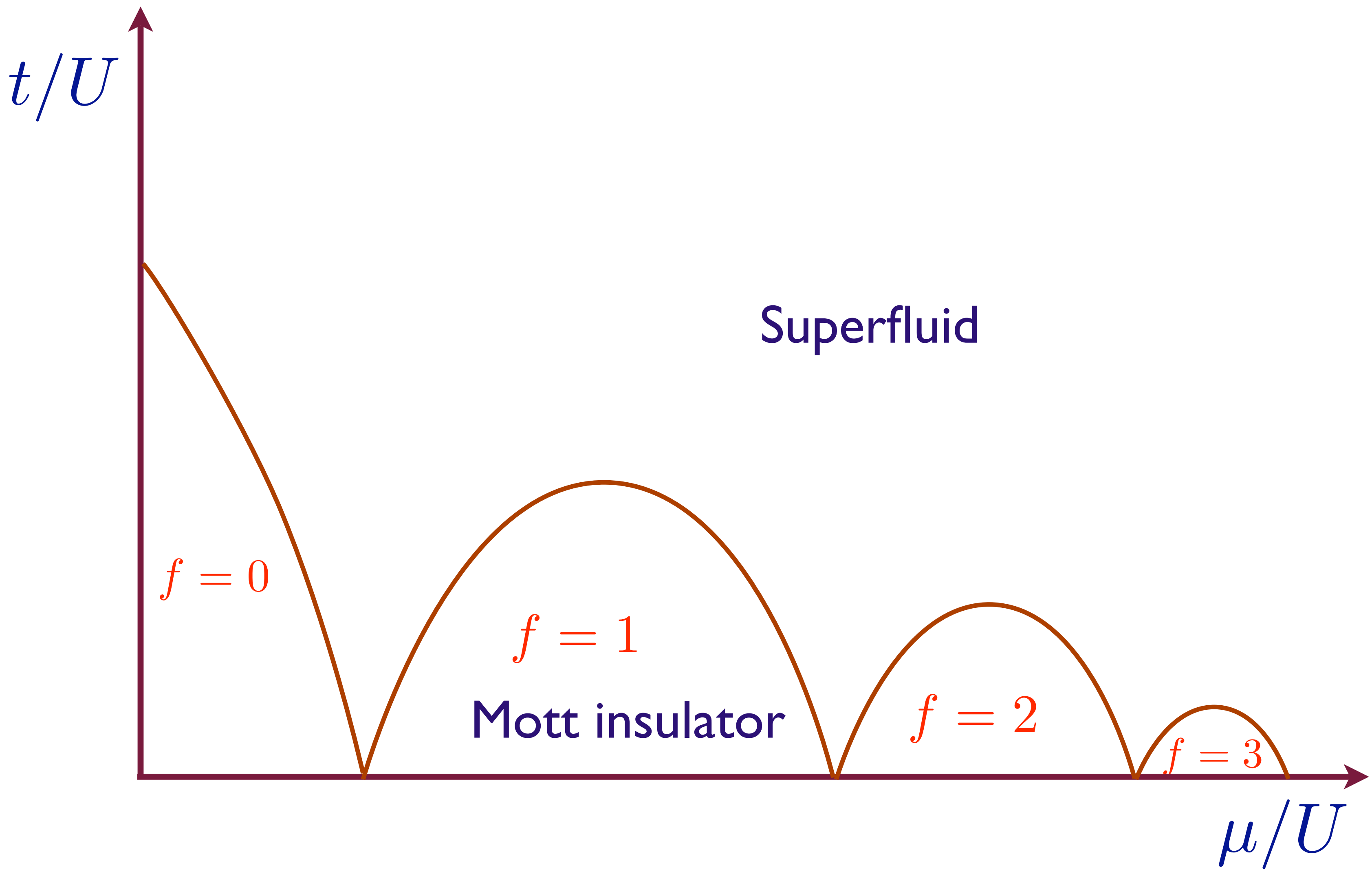
Degrees of freedom: Bosons, $b_j^\dagger$, hopping between the sites, j, of a lattice, with short-range repulsive interactions.

$$H = -t\sum_{\langle ij\rangle} b_i^\dagger b_j - \mu \sum_j n_j + \frac{U}{2}\sum_j n_j(n_j - 1) + \cdots$$

$$n_j \equiv b_j^\dagger b_j$$

M.P. A. Fisher, P.B. Weichmann, G. Grinstein, and D.S. Fisher, *Phys. Rev. B* **40**, 546 (1989).

Mean field theory



LGW theory of the superfluid insulator transition

- Identify order parameter $\Psi(x,\tau) \sim b_j^\dagger$
- Symmetries:

Gauge invariance:	$\Psi \to \Psi e^{i\theta}$
Time reversal	$\tau \to -\tau \quad ; \quad \Psi \to \Psi^*$
Spatial inversion	$x \to -x$

- Write down most general Lagrangian consistent with symmetries

$$\mathcal{Z} = \int \mathcal{D}\Psi(x,\tau) \exp\left(-\int d^d x \int d\tau \mathcal{L}\left[\Psi\right]\right)$$

$$\mathcal{L}\left[\Psi\right] = K\Psi^* \frac{\partial \Psi}{\partial \tau} + |\partial_\tau \Psi|^2 + c^2 \left|\nabla_x \Psi\right|^2 + r|\Psi|^2 + \frac{u}{2}|\Psi|^4 + \ldots$$

- Identify phases at $r \gg 0$ and $r \ll 0$ with the insulator and the superfluid respectively.
- For $K \neq 0$, the particle and hole excitations have different energies.

- Gauge-invariance of the underlying boson Hamiltonian shows that

$$K = -\frac{\partial r}{\partial \mu}$$

- In mean-field theory, the ground state energy, E, across the superfluid-insulator transition has the non-analytic term

$$E = E_0 - \frac{r^2}{2u}\theta(-r)$$

(Beyond mean-field theory, the non-analytic term is $E \sim r^{(d+z)\nu}$).

- Because the density of bosons $= -\partial E/\partial \mu$, this implies a change in the boson density across the transition *unless* $\partial r/\partial \mu = 0$

- A superfluid-insulator transition at fixed boson density must have.

$$K = 0$$

Outline

1. Lattice bosons at integer filling

2. Fermions near unitarity at even-integer filling

3. Boson-vortex duality

4. Lattice bosons at fractional filling

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Add a periodic potential V to the balanced Fermi gas.

$$V(\mathbf{r}) = V\left[3 + \cos\left(\frac{2\pi x}{a_{\rm L}}\right) + \cos\left(\frac{2\pi y}{a_{\rm L}}\right) + \cos\left(\frac{2\pi z}{a_{\rm L}}\right)\right]$$

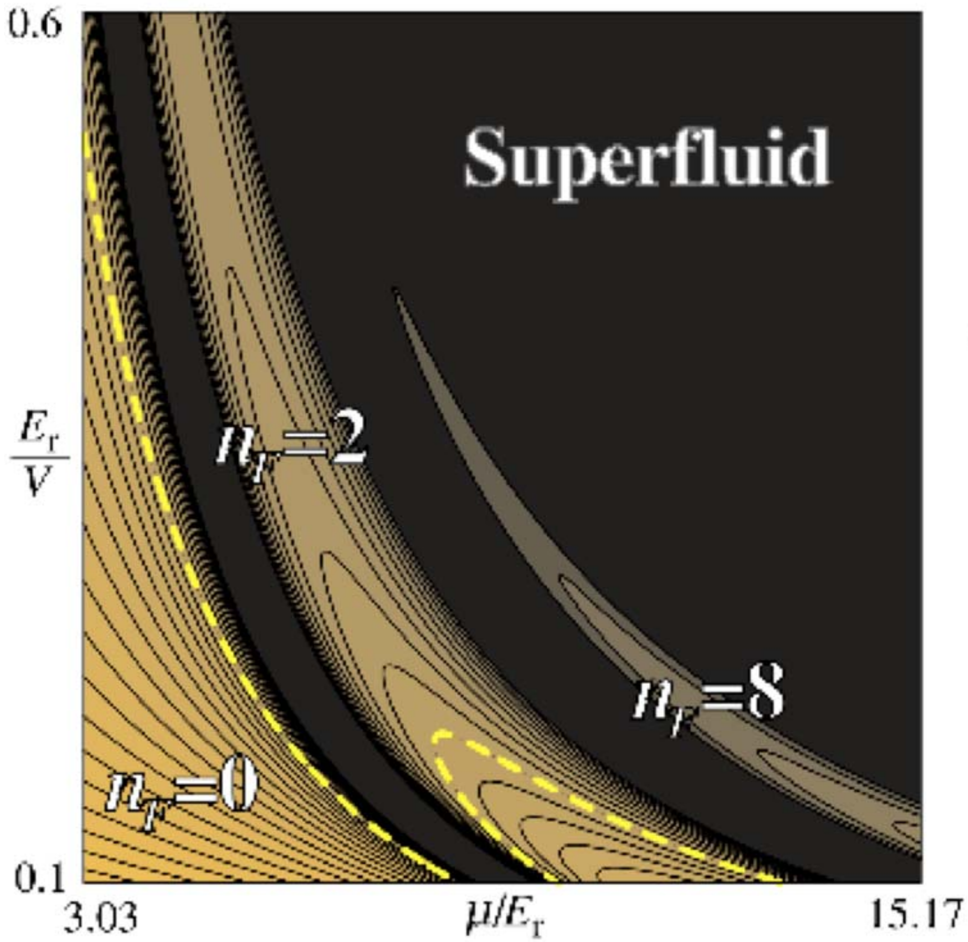
Two new parameters, V and a_L.

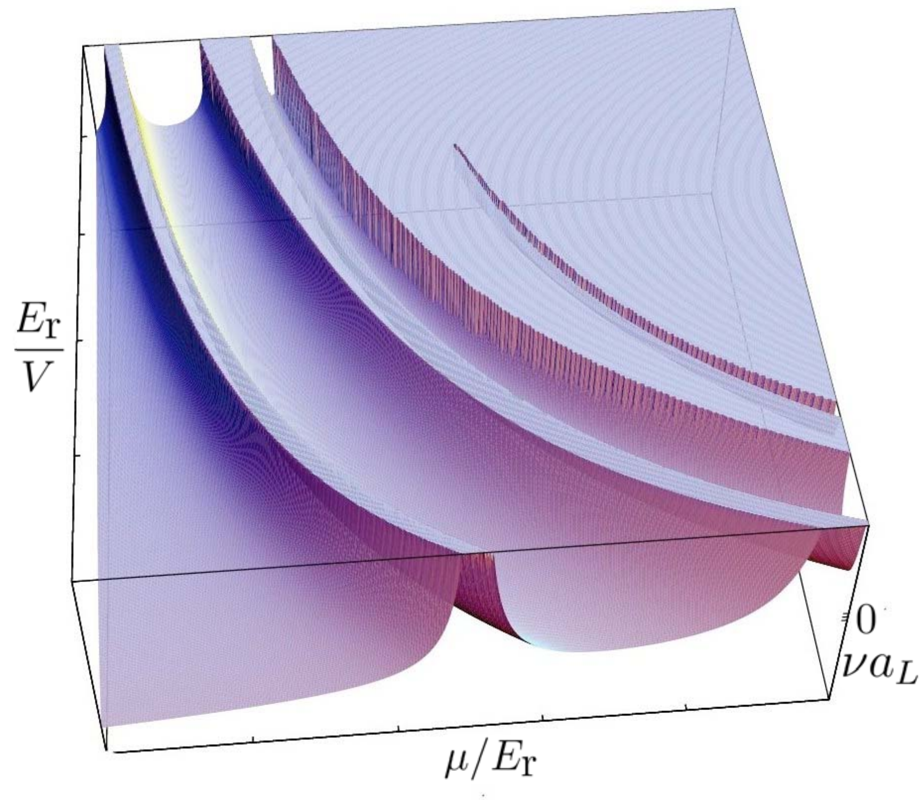
- When the number of fermions per unit cell, n, is an **even integer**, then for sufficiently large V, the ground state can be an insulator. In all other regimes, the ground state is a paired superfluid.
- For large negative detuning (BEC limit), the insulator is a Mott insulator of bosons.
- For large positive detuning (BCS limit), the insulator is a band insulator of fermions.
- Near unitarity, the insulator is neither bosonic nor fermionic. Multiple fermionic Bloch bands are occupied. Bosonic Hubbard model is also not applicable.
- By universality, the critical potential V_c for the superfluid-insulator transition is given by

$$V_c = \frac{\hbar^2}{m a_L^2} F_n(a_L \nu)$$

Add a periodic potential V to the balanced Fermi gas.

$$V(\mathbf{r}) = V\left[3+\cos\left(\frac{2\pi x}{a_{\mathrm{L}}}\right)+\cos\left(\frac{2\pi y}{a_{\mathrm{L}}}\right)+\cos\left(\frac{2\pi z}{a_{\mathrm{L}}}\right)\right]$$



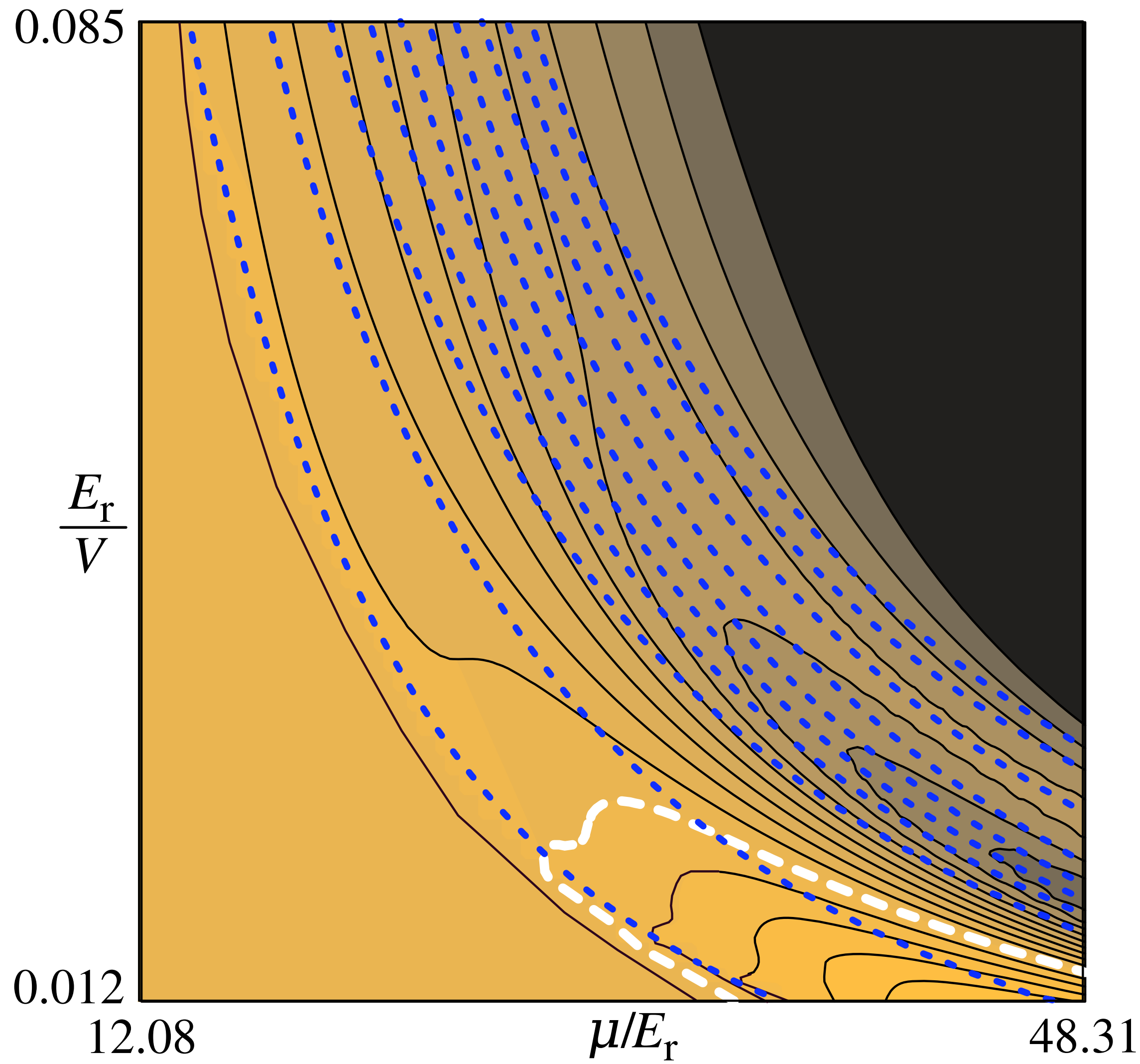


Add a periodic potential V to the balanced Fermi gas.

$$V(\mathbf{r}) = V\left[3 + \cos\left(\frac{2\pi x}{a_L}\right) + \cos\left(\frac{2\pi y}{a_L}\right) + \cos\left(\frac{2\pi z}{a_L}\right)\right]$$

There is a significant discrepancy between our *T=0* theory and measurements of V_c by the MIT group

Possible reason: finite T effects are complex: normal states are possible at all densities



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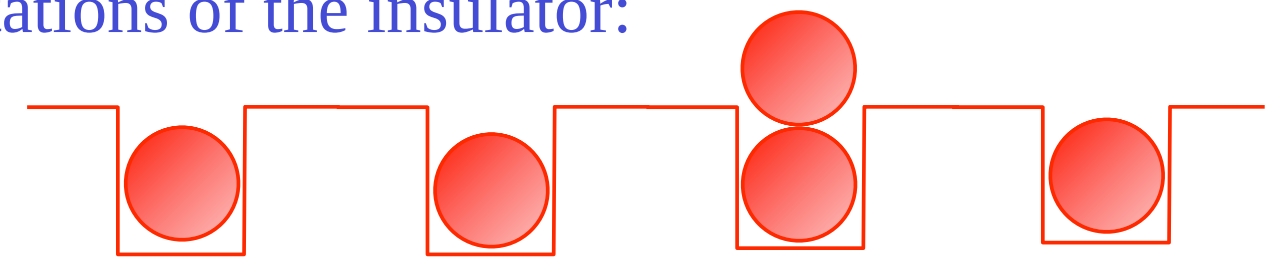
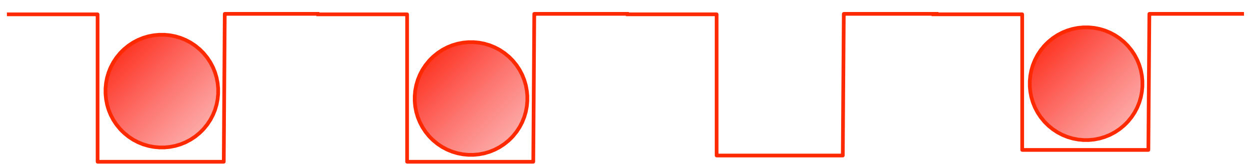
Excitations of the insulator:



Particles ~ $\psi^\dagger$



Holes ~ ψ

Density of particles = density of holes $\Rightarrow$
"relativistic" field theory for ψ:

$$\mathcal{S} = \int d^3x \left[|\partial_\mu \psi|^2 + s|\psi|^2 + \frac{u}{2}|\psi|^4 \right]$$

$$\text{Insulator} \Leftrightarrow \langle \psi \rangle = 0$$

$$\text{Superfluid} \Leftrightarrow \langle \psi \rangle \neq 0$$

Approaching the transition from the superfluid

Excitations of the superfluid: (A) **Spin waves**

With $\psi \sim e^{i\theta}$, action for spin waves is

$$\mathcal{S}_{sw} = \frac{\rho_s}{2} \int d^3x (\partial_\mu \theta)^2$$

Dual form: After a Hubbard-Stratonovich transformation, write

$$\mathcal{S}_{sw} = \int d^3x \left[\frac{1}{2\rho_s} J_\mu^2 + i J_\mu \partial_\mu \theta \right]$$

Integrating over θ yields $\partial_\mu J_\mu = 0$. Solve, by writing

$$J_\mu = \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda$$

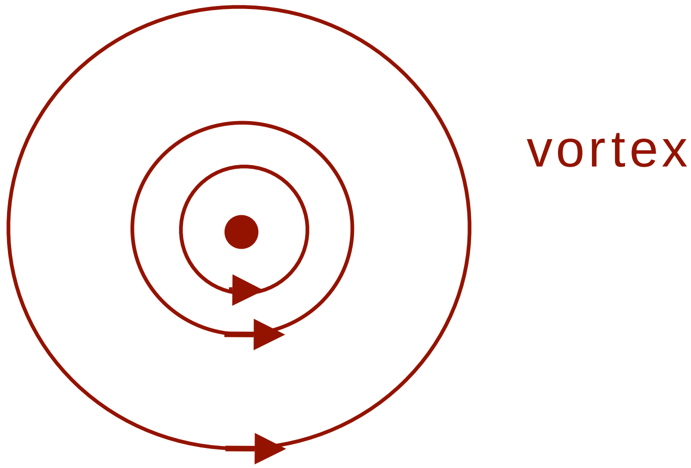
leading to

$$\mathcal{S}_{sw} = \int d^3x \left[\frac{1}{2\rho_s} (\epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda)^2 \right]$$

Spin waves are dual to a U(1) gauge theory in 2+1 dimensions

Approaching the transition from the superfluid

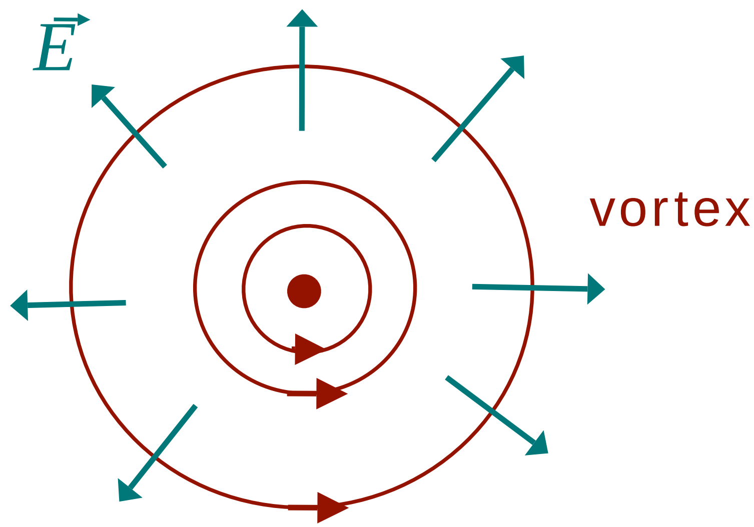
Excitations of the superfluid: (B) **Vortices**



A vortex is a point-like object. We can therefore define a local field operator, φ, which annihilates a vortex.

Approaching the transition from the superfluid

Excitations of the superfluid: (B) **Vortices**



A vortex is a point-like object. We can therefore define a local field operator, φ, which annihilates a vortex.

Each vortex is the source of an 'electric field' $\vec{E}$ associated with the U(1) gauge field A_μ.

Consequently, φ carries $+1$ U(1) gauge charge.

Approaching the transition from the superfluid

Excitations of the superfluid: **Spin wave and vortices**

φ: vortex annihilation operator.

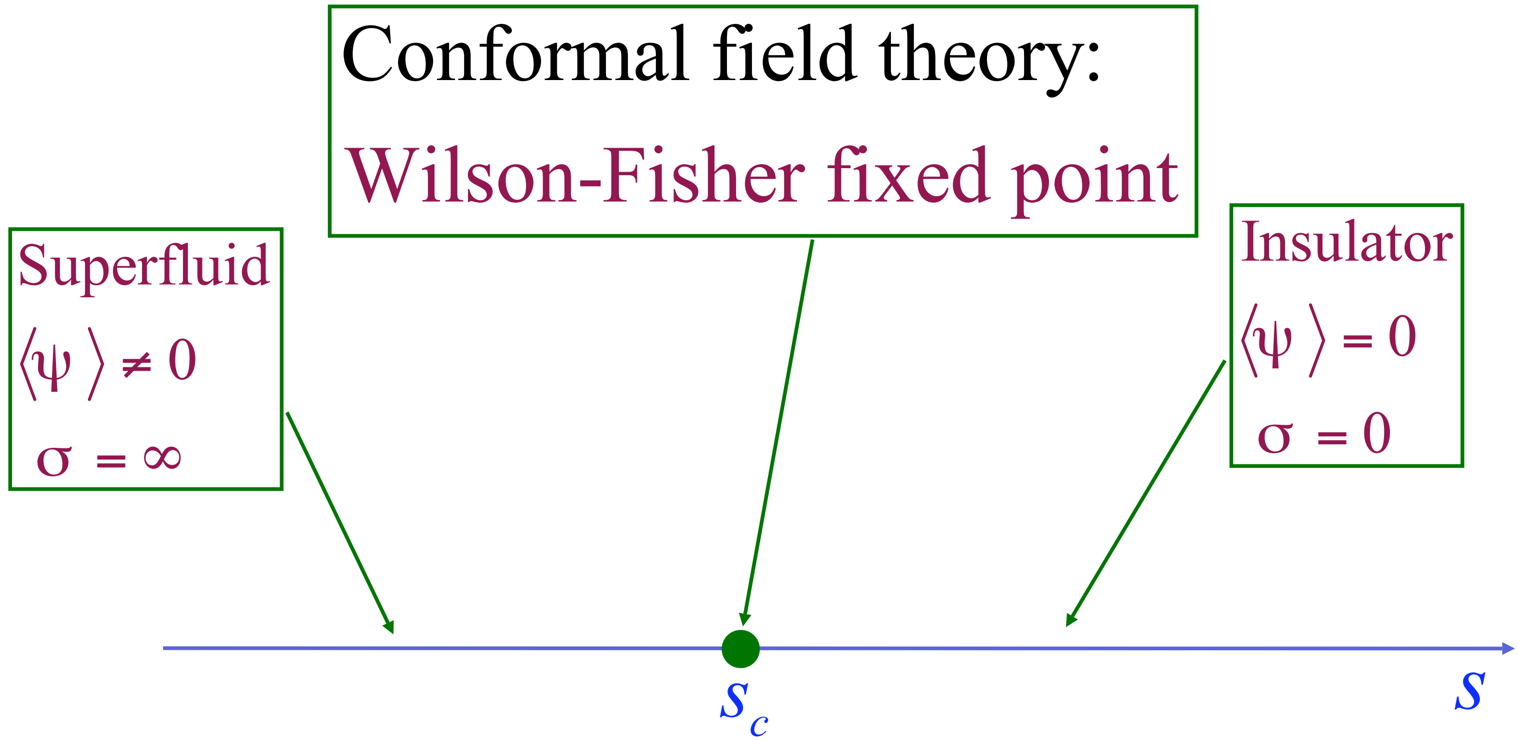
$\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda$: boson current $\sim i\psi^*\partial_\mu\psi - i\partial_\mu\psi^*\psi$.

Density of vortices = density of antivortices $\Rightarrow$
"relativistic" field theory for φ:

$$\mathcal{S}_{\text{dual}} = \int d^3x \Big[|(\partial_\mu - iA_\mu)\varphi|^2 + \tilde{s}|\varphi|^2 + \frac{\tilde{u}}{2}|\varphi|^4 + \frac{1}{2\rho_s}(\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda)^2 \Big]$$

$$\text{Superfluid} \Leftrightarrow \langle\varphi\rangle = 0$$

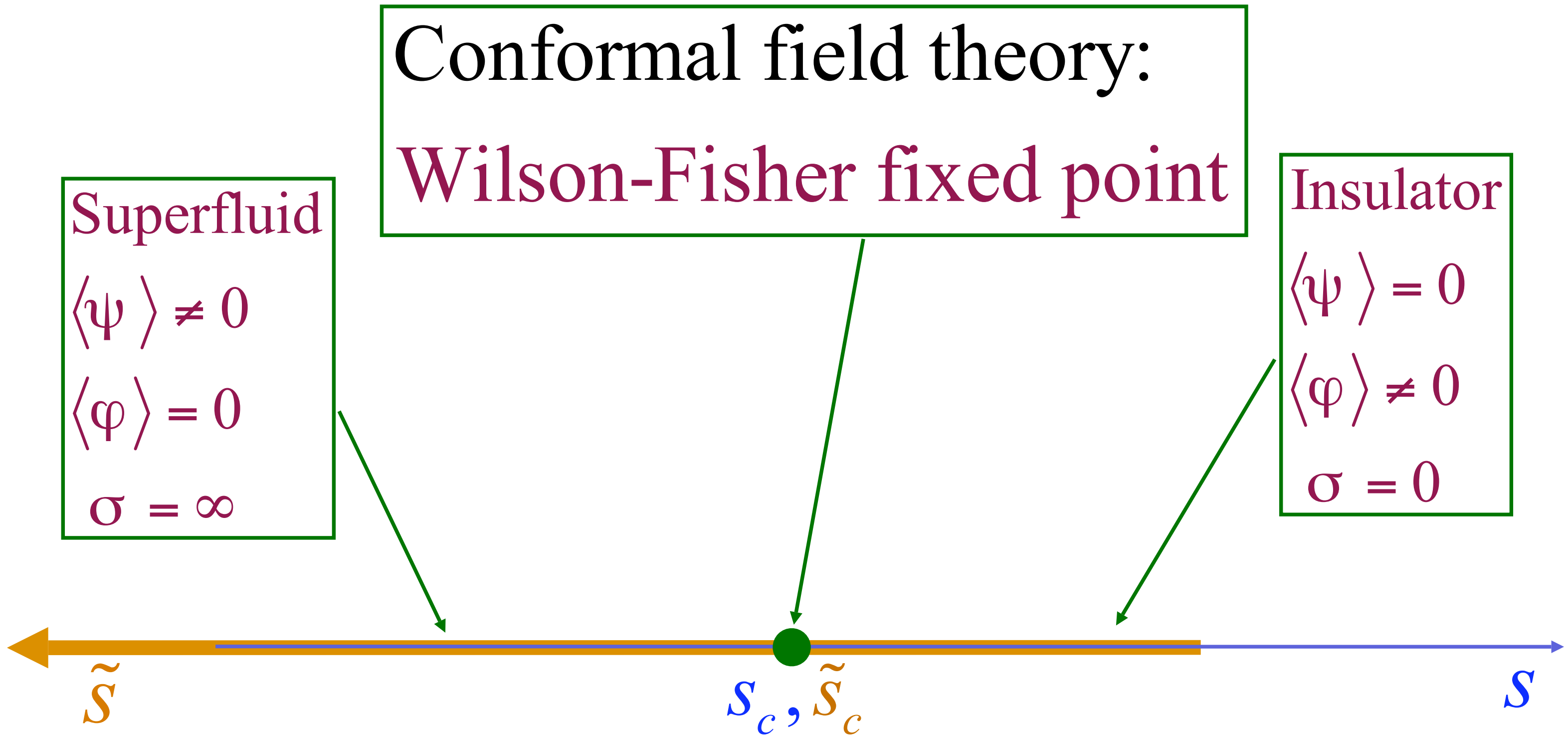
$$\text{Insulator} \Leftrightarrow \langle\varphi\rangle \neq 0$$



Using the boson quasiparticle excitations of the insulator $\sim \psi$

$$\mathcal{S} = \int d^3x \left[|\partial_\mu \psi|^2 + s|\psi|^2 + \frac{u}{2}|\psi|^4 \right]$$

C. Dasgupta and B.I. Halperin, *Phys. Rev. Lett*. **47**, 1556 (1981)



Using the boson quasiparticle excitations of the insulator $\sim \psi$

$$\mathcal{S} = \int d^3x \left[|\partial_\mu \psi|^2 + s|\psi|^2 + \frac{u}{2}|\psi|^4 \right]$$

<u>is dual to</u>

Using the vortex quasiparticle excitations of the superfluid $\sim \varphi$

$$\mathcal{S}_{\rm dual} = \int d^3x \left[|(\partial_\mu - iA_\mu)\varphi|^2 + \tilde{s}|\varphi|^2 + \frac{\tilde{u}}{2}|\varphi|^4 + \frac{1}{2e^2}(\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda)^2 \right]$$

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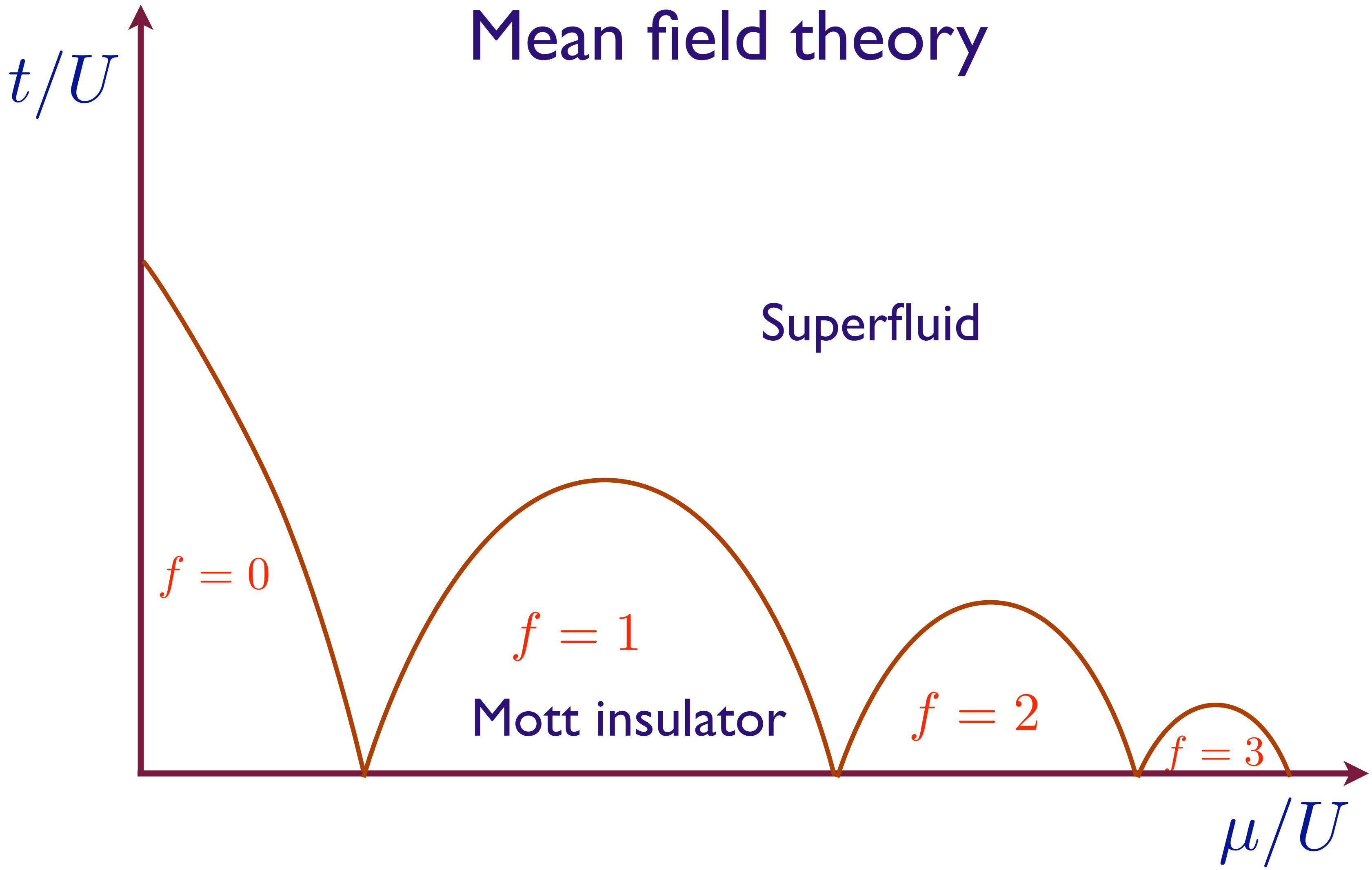
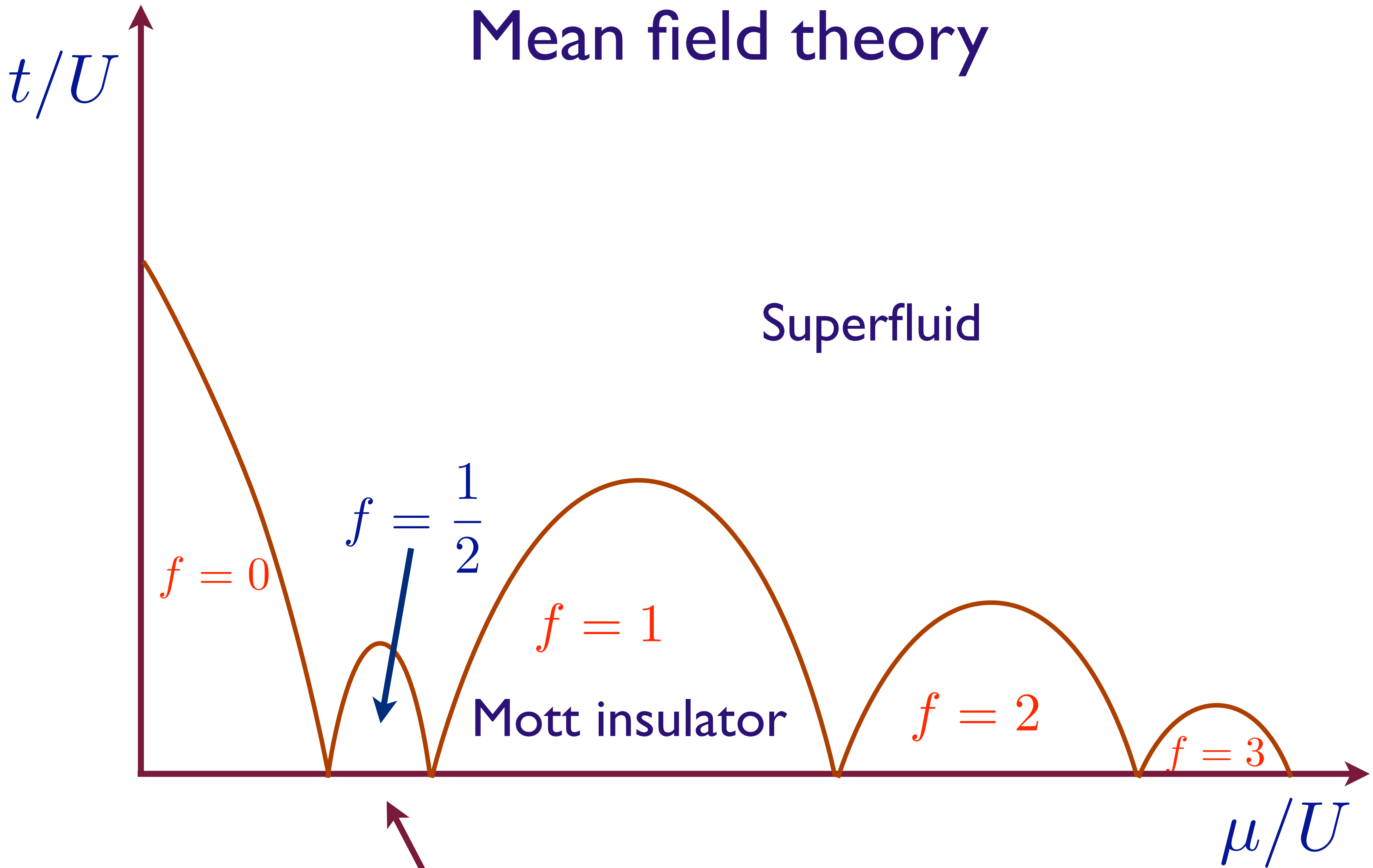
Mean field theory

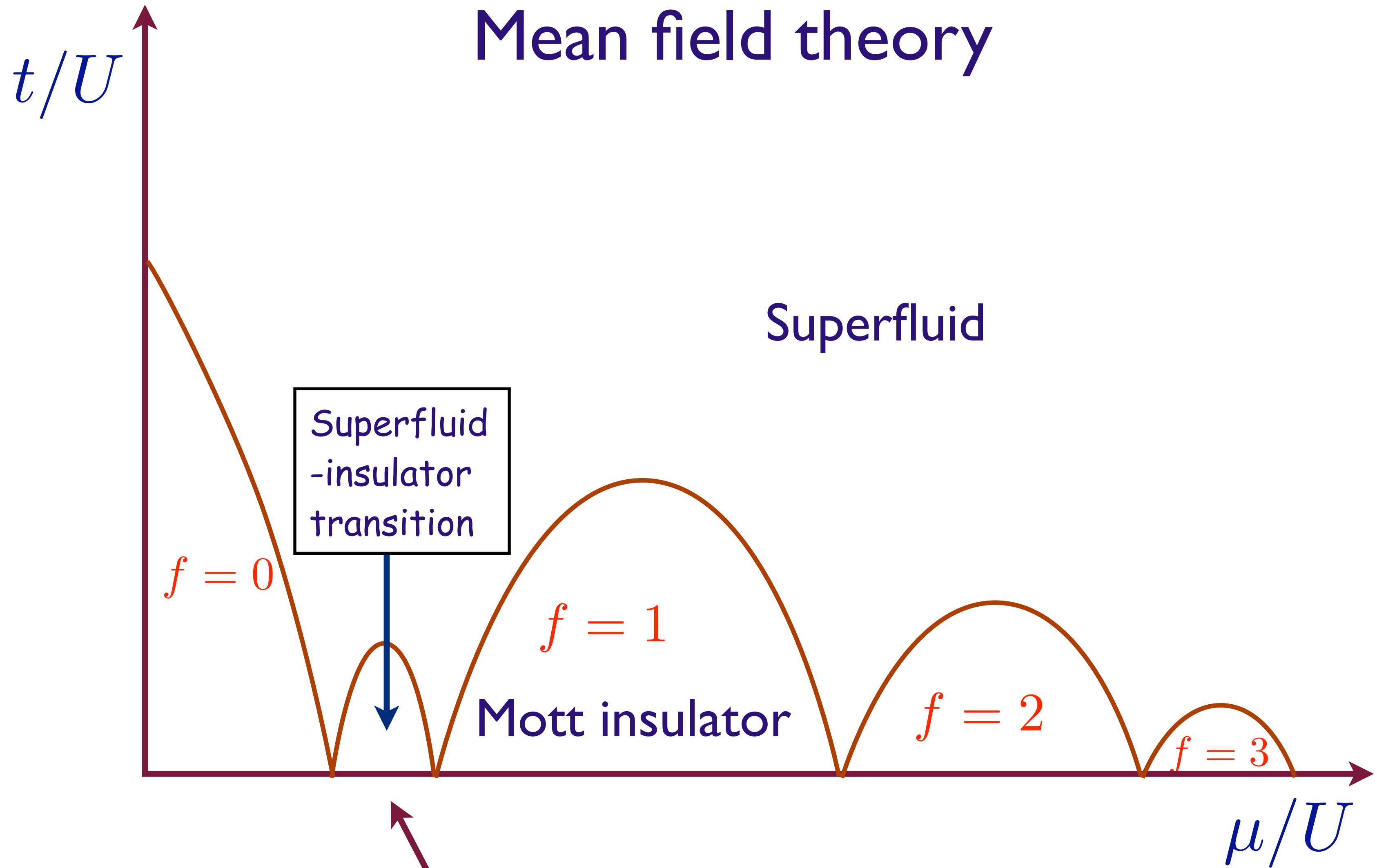


Mean field theory



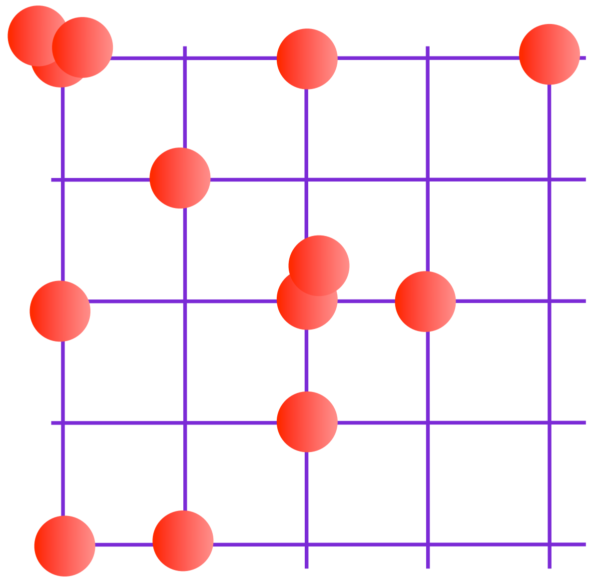
Near f=1/2, and other rational f, off-site interactions can induce additional correlated insulating states

Mean field theory



Near *f*=1/2, and other rational *f*, off-site interactions can induce additional correlated insulating states

Bosons at filling fraction f = 1/2 or *S=1/2 XXZ model*



$$\langle \Psi \rangle \neq 0$$

Weak interactions: superfluidity

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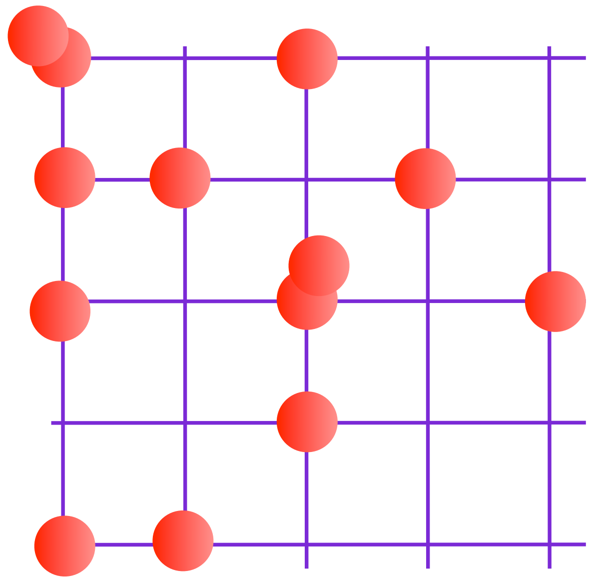

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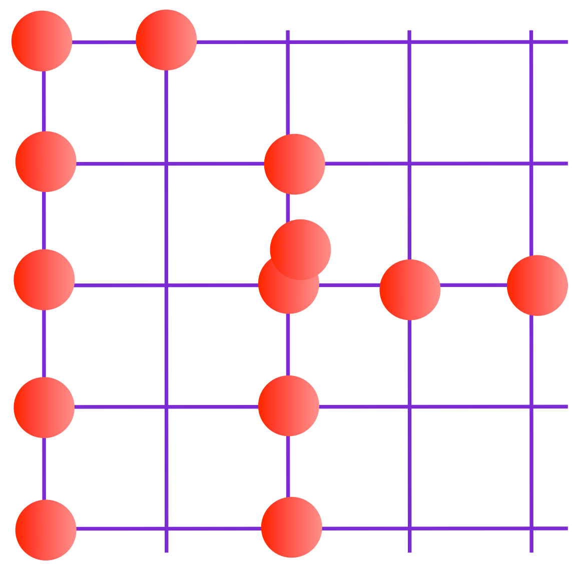

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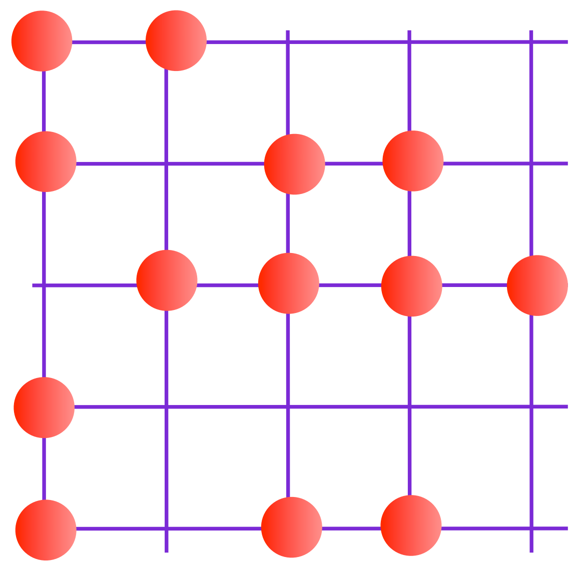

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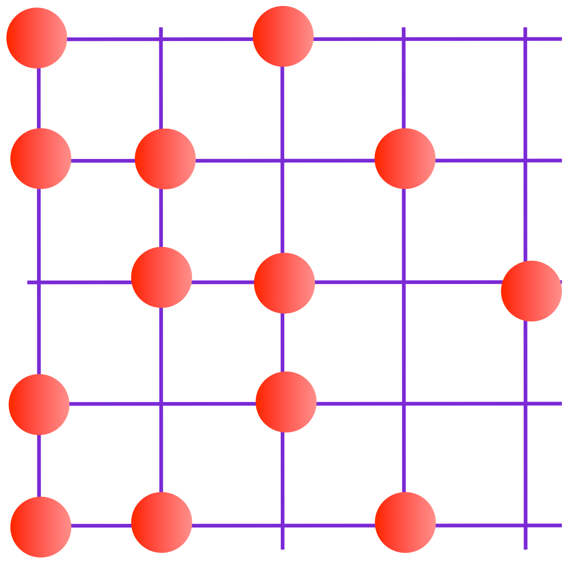

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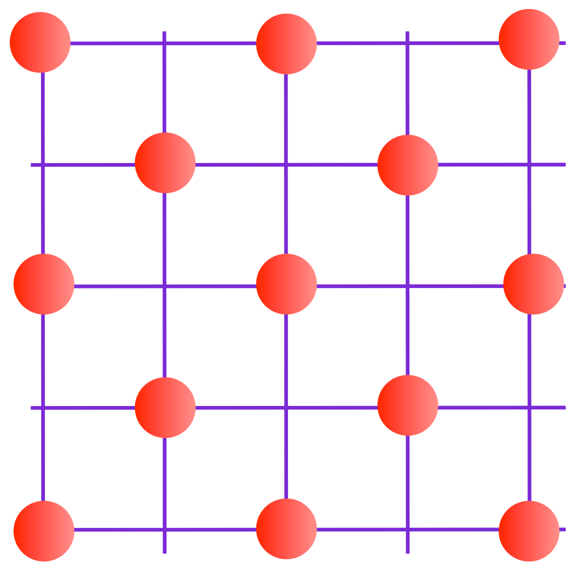

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Strong interactions: insulator

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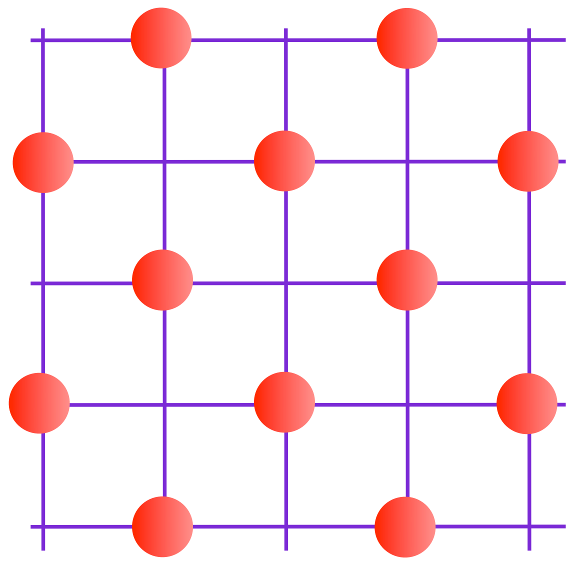

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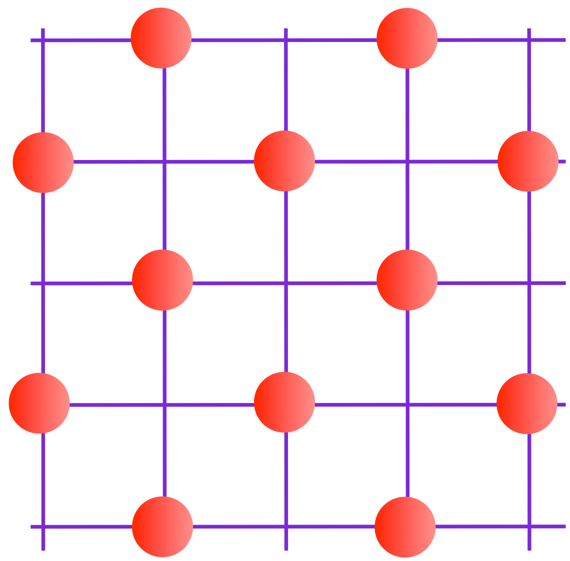

$$\langle \Psi \rangle = 0$$

Strong interactions: insulator

Insulator has "density wave" order

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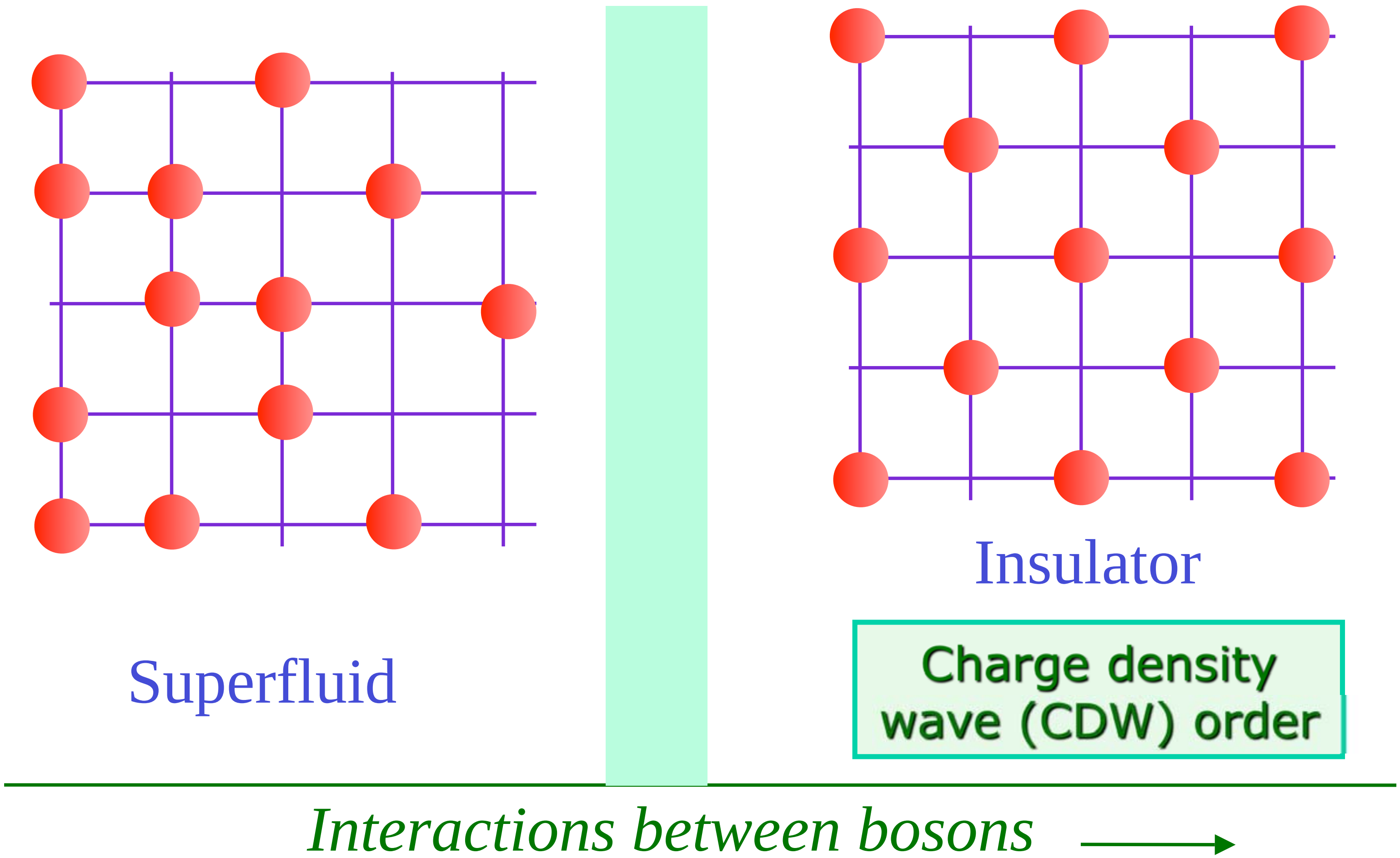

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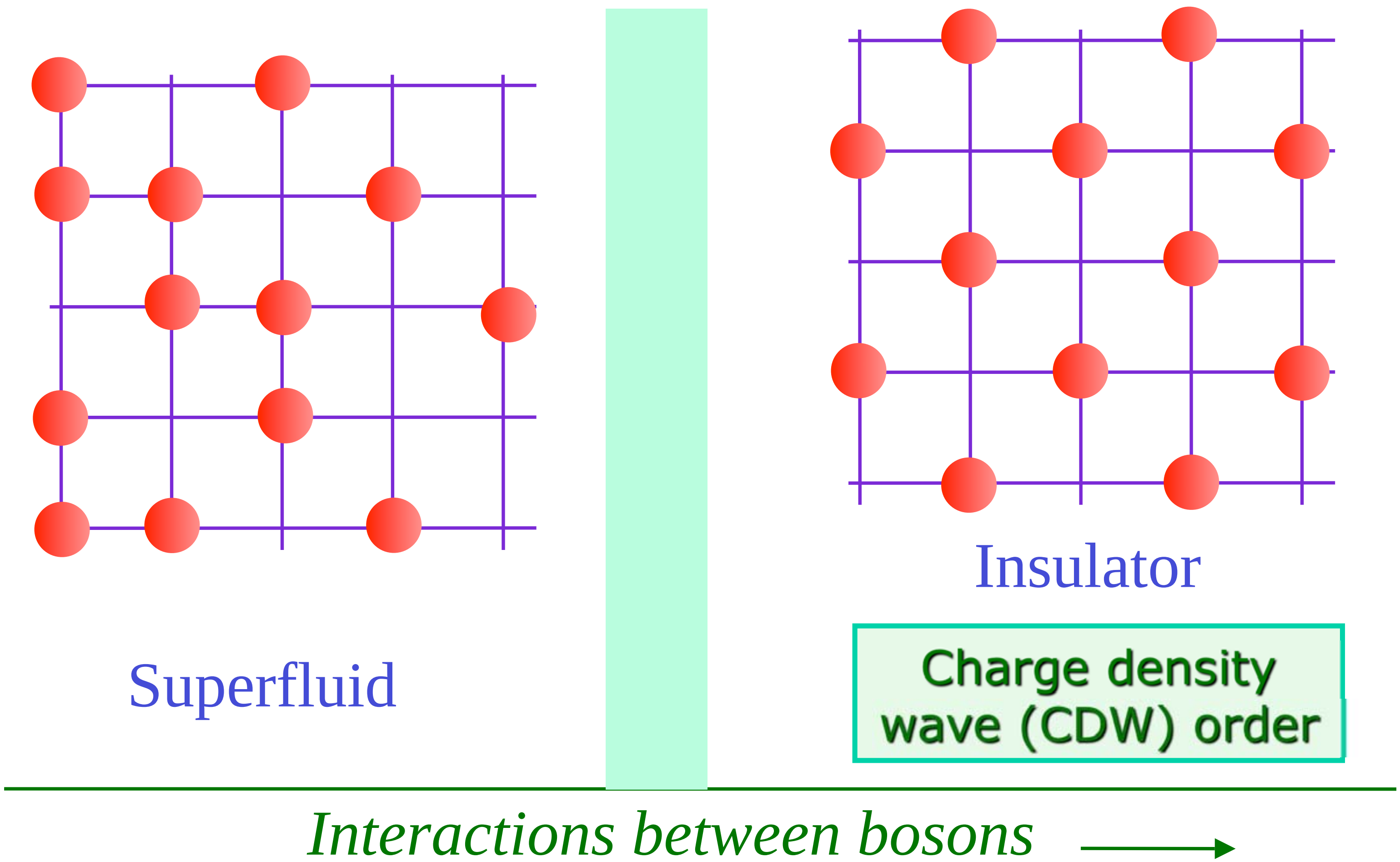

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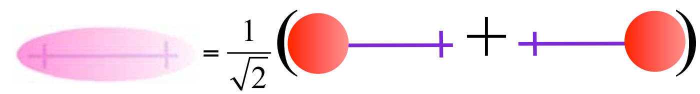
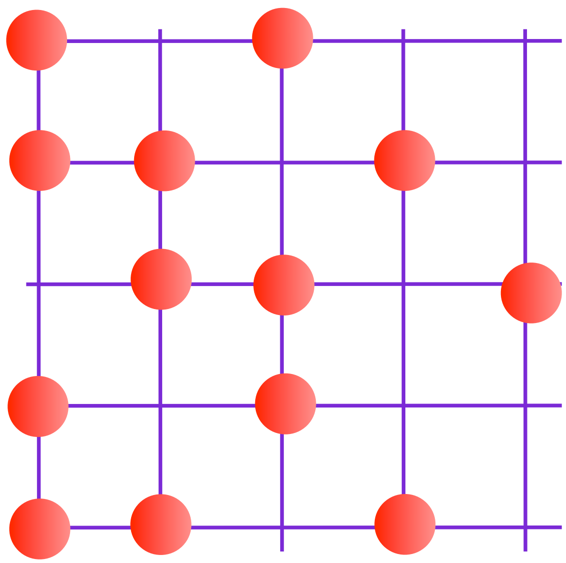
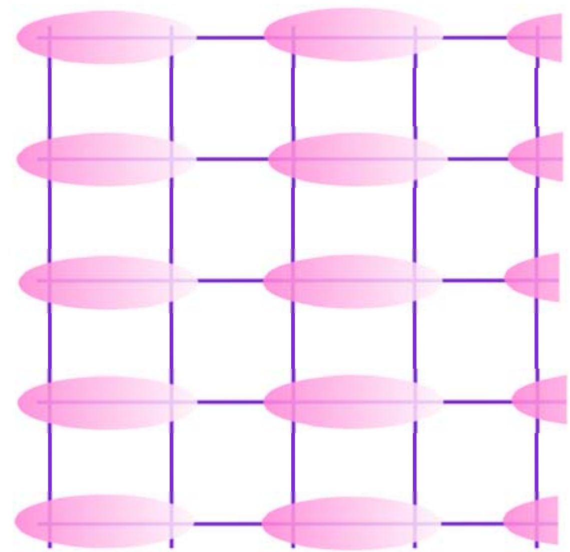




Superfluid



Insulator

Valence bond solid
(VBS) order

Interactions between bosons →

N. Read and S. Sachdev, *Phys. Rev. Lett*. **62**, 1694 (1989).

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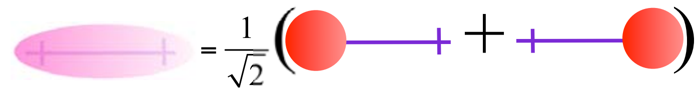

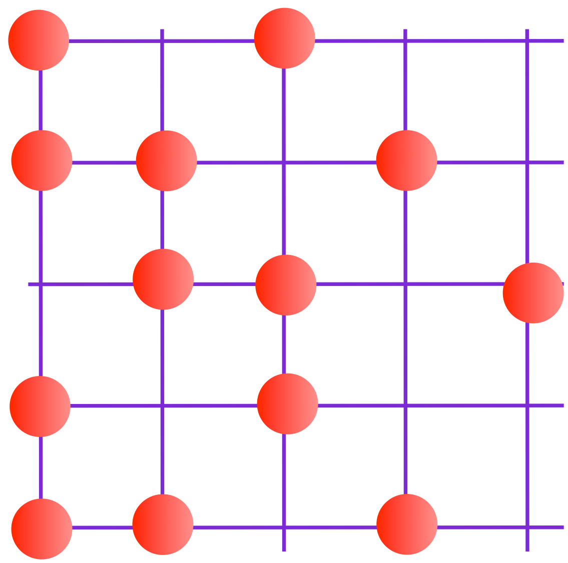

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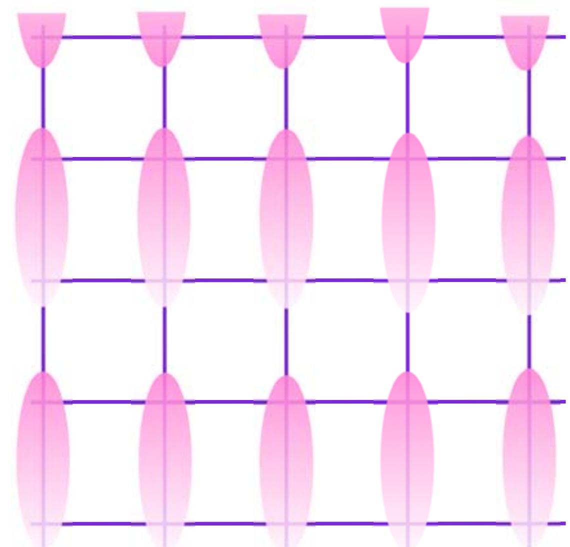

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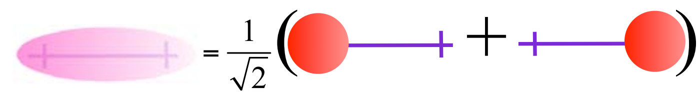

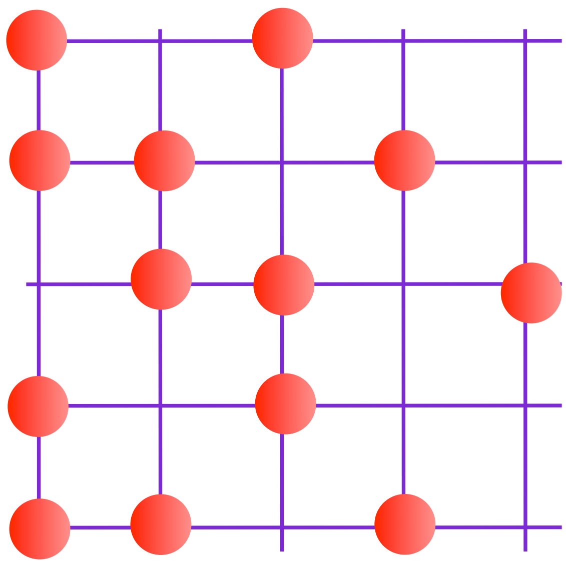

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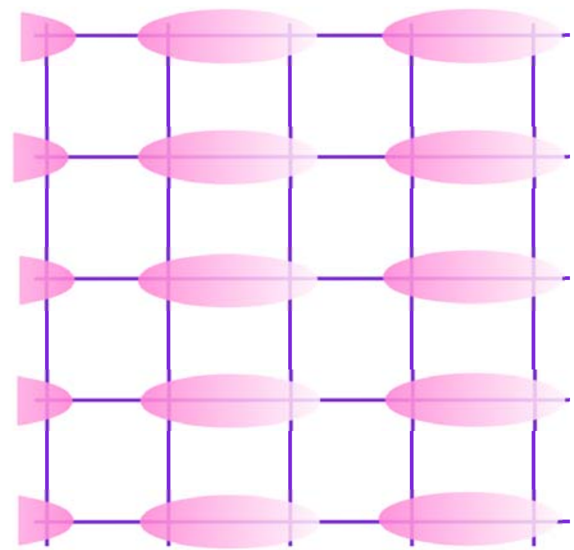

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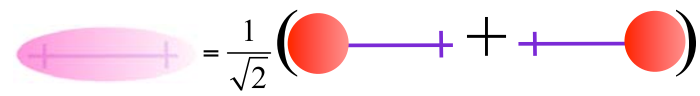

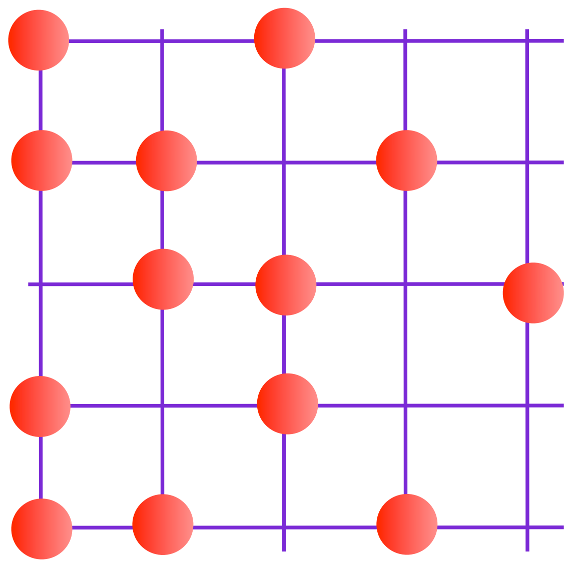

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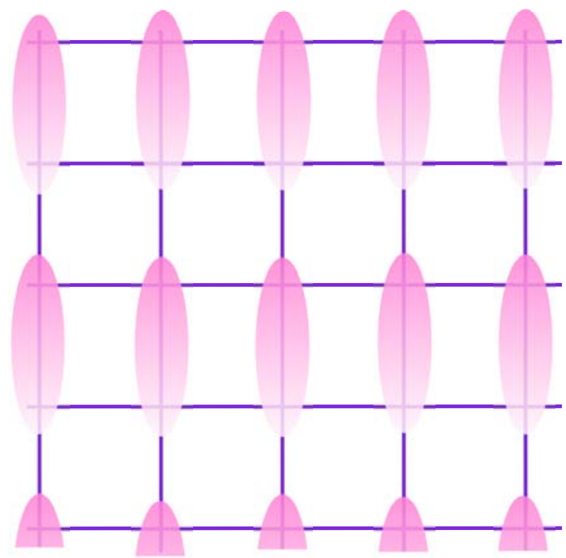

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The superfluid-insulator quantum phase transition

Key difficulty: Multiple order parameters (Bose-Einstein condensate, charge density wave, valence-bond-solid order…) not related by symmetry, but clearly physically connected. Standard methods only predict strong first order transitions (for generic parameters).

The superfluid-insulator quantum phase transition

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Key theoretical tool: *Quantum theory of vortices*

Effective theory of vortices

Recall the relation from the boson-vortex duality transformation:

$$\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda: \text{ boson current} \sim i\psi^*\partial_\mu\psi - i\partial_\mu\psi^*\psi.$$

The boson density acts like a dual "magnetic field" that acts on the vortices.

At filling fraction, *f*, the flux per unit cell of the lattice is *f*.

Influence of the periodic potential on vortex motion

Let the Hamiltonian of a single vortex be $\mathcal{H}_v$.

In general, this is a very complicated object, but we can obtain all needed information by symmetry considerations.

The Hamiltonian $\mathcal{H}_v$ should commute with T_x, the operator which translates the square lattice by one site in the x direction (and similarly for T_y):

$$[T_x, \mathcal{H}_v] = 0$$

$$[T_y, \mathcal{H}_v] = 0$$

However, T_x and T_y do not commute with each other.

Under translation along a distance $\mathbf{s}$, a vortex picks up a a Aharanov-Bohm phase factor $\exp\left(i \int_0^{\mathbf{s}} d\mathbf{r} \cdot \mathbf{A}\right)$.

Consequently

$$\boxed{T_x T_y = \exp\left(i\phi\right) T_y T_x}$$

where ϕ is the dual "flux" through a unit cell, This "flux" has the value

$$\boxed{\phi = 2\pi f}$$

where f is the filling fraction of bosons (Cooper pairs). We will consider the case of rational filling fraction $f = p/q$, where p, q are relatively prime integers.

Bosons on the square lattice at filling fraction f=p/q

$$[T_x, \mathcal{H}_v] = 0$$

$$[T_y, \mathcal{H}_v] = 0$$

$$T_x T_y = \exp\left(2\pi i p/q\right) T_y T_x$$

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$$T_x T_y = \exp\left(2\pi i p/q\right) T_y T_x$$

Theorem:
The ground state of $\mathcal{H}_v$ is at least q-fold degenerate. We can choose a basis, $|m\rangle$ $(m = 0 \ldots (q-1))$, for the ground states such that

$$\begin{aligned} T_x|m\rangle &= |m+1\rangle \\ T_y|m\rangle &= e^{2\pi i m p/q}|m\rangle \end{aligned}$$

Vortices in a superfluid near a Mott insulator at filling $f=p/q$

Simplest representation of magnetic space group by the quantum vortex "particle" with field operator φ

At filling $f=p/q$, there are q species of vortices, φ_ℓ (with $\ell=1\ldots q$), associated with q degenerate minima in the vortex spectrum. These vortices realize the smallest, q-dimensional, representation of the magnetic algebra.

$$T_x : \varphi_\ell \to \varphi_{\ell+1} \quad ; \quad T_y : \varphi_\ell \to e^{2\pi i \ell f} \varphi_\ell$$

$$R : \varphi_\ell \to \frac{1}{\sqrt{q}} \sum_{m=1}^{q} \varphi_m e^{2\pi i \ell m f}$$

Vortices in a superfluid near a Mott insulator at filling $f=p/q$

The wavefunction of the φ_ℓ vortices in flavor space characterizes the density-wave order

Density-wave order:

Status of space group symmetry determined by density operators ρ_Q at wavevectors $\boldsymbol{Q}_{mn} = \frac{2\pi p}{q}(m,n)$

$$\rho_{mn} = e^{i\pi mnf} \sum_{\ell=1}^{q} \varphi_\ell^* \varphi_{\ell+n} e^{2\pi i \ell mf}$$

$$T_x : \rho_Q \to \rho_Q e^{iQ\cdot\hat{x}} \quad ; \qquad T_y : \rho_Q \to \rho_Q e^{iQ\cdot\hat{y}}$$

$$R : \rho(\boldsymbol{Q}) \to \rho(R\boldsymbol{Q})$$

Field theory with projective symmetry

Degrees of freedom:

q complex φ_ℓ vortex fields

1 non-compact U(1) gauge field A_μ

$$\mathcal{S} = \int d^2x d\tau \left[\sum_\ell \left\{ |(\partial_\mu - iA_\mu)\varphi_\ell|^2 + s|\varphi_\ell|^2 \right\} + \frac{1}{2e^2}(\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda)^2 + \sum_{\ell mn} \gamma_{mn}\varphi^*_\ell \varphi^*_{\ell+m}\varphi_{\ell+n}\varphi_{\ell+m-n} \right]$$

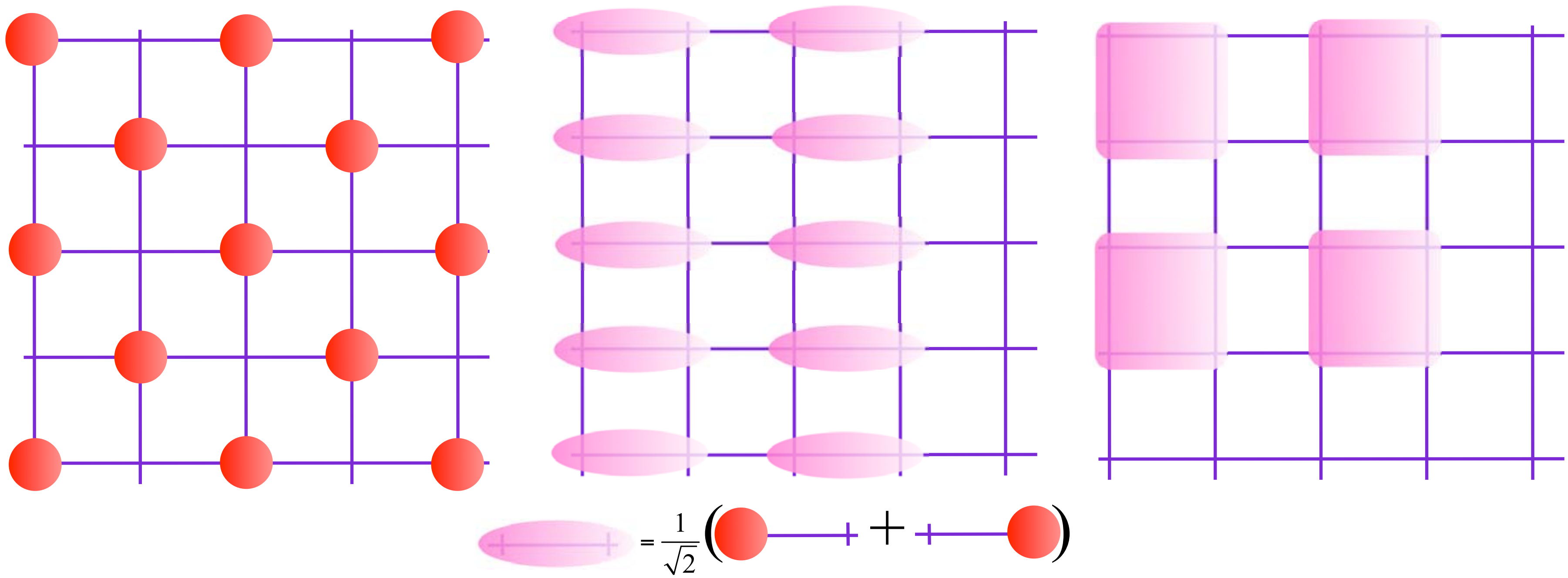
The projective symmetries constrain the couplings γ_{mn} to obey

$$\gamma_{mn} = \gamma_{-m,-n} \;\; ; \;\; \gamma_{mn} = \gamma_{m,m-n} \;\; ; \;\; \gamma_{mn} = \gamma_{m-2n,-n}$$

$$\gamma_{\bar{m}\bar{n}} = \frac{1}{q}\sum_{mn} \gamma_{mn} e^{-2\pi i f[n(\bar{m}-\bar{n})+\bar{n}(m-n)]}$$

Field theory with projective symmetry

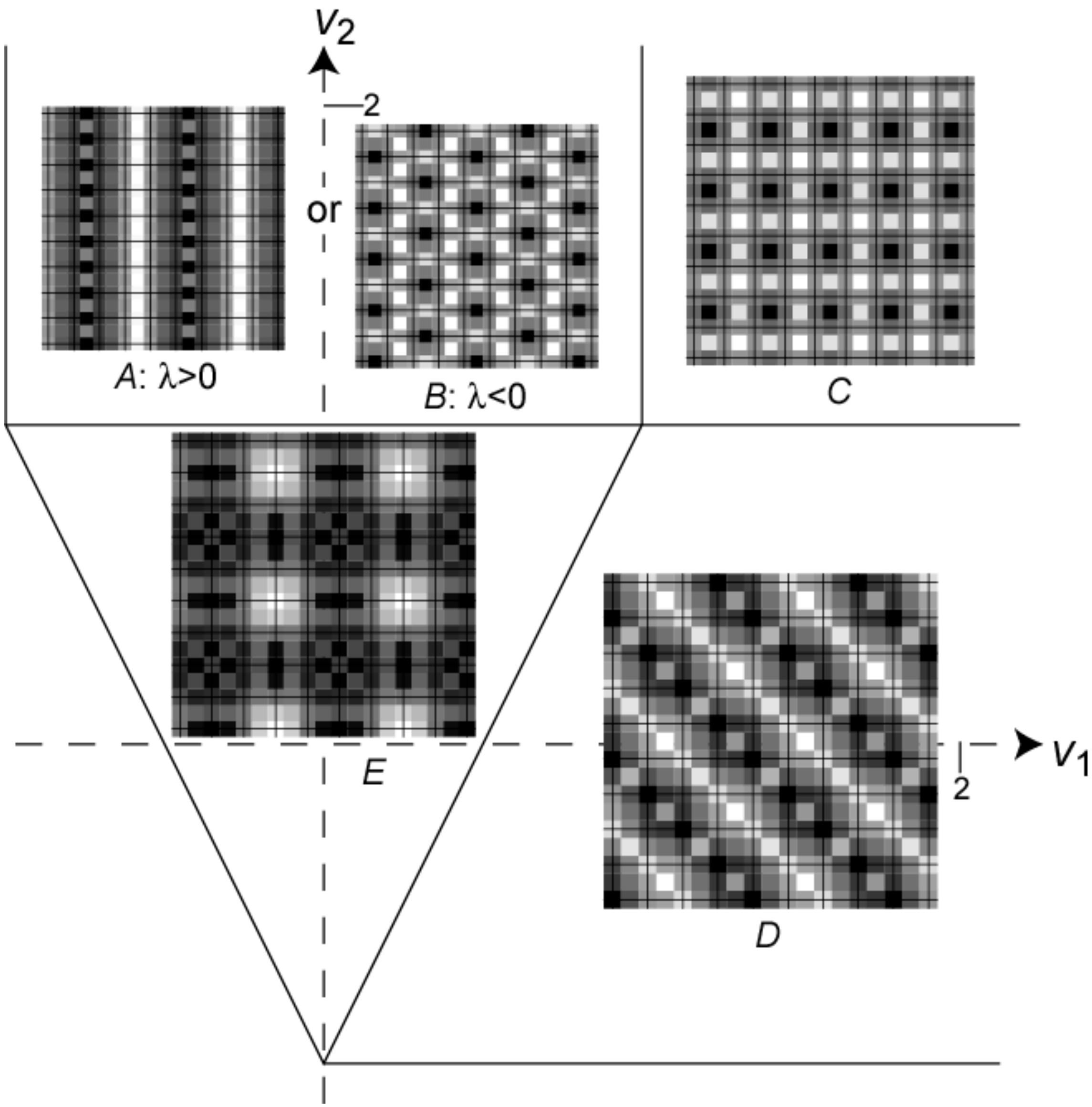
Spatial structure of insulators for q=2 (f=1/2)



All insulating phases have density-wave order $\rho(\boldsymbol{r}) = \sum_{\boldsymbol{Q}} \rho_{\boldsymbol{Q}} e^{i\boldsymbol{Q}.\boldsymbol{r}}$ with $\langle \rho_{\boldsymbol{Q}} \rangle \neq 0$

Field theory with projective symmetry

Spatial structure of insulators for q=4 (f=1/4 or 3/4)



$a \times b$ unit cells;

q/a, q/b, ab/q,

all integers

Vortices in a superfluid near a Mott insulator at filling $f=p/q$

- The excitations of the superfluid are described by the quantum mechanics of q flavors of low energy vortices moving in zero dual "magnetic" field.

- The orientation of the vortex in flavor space implies a particular configuration of VBS order in its vicinity.

- Any pinned vortex must pick an orientation in flavor space: this induces a halo of VBS order in its vicinity

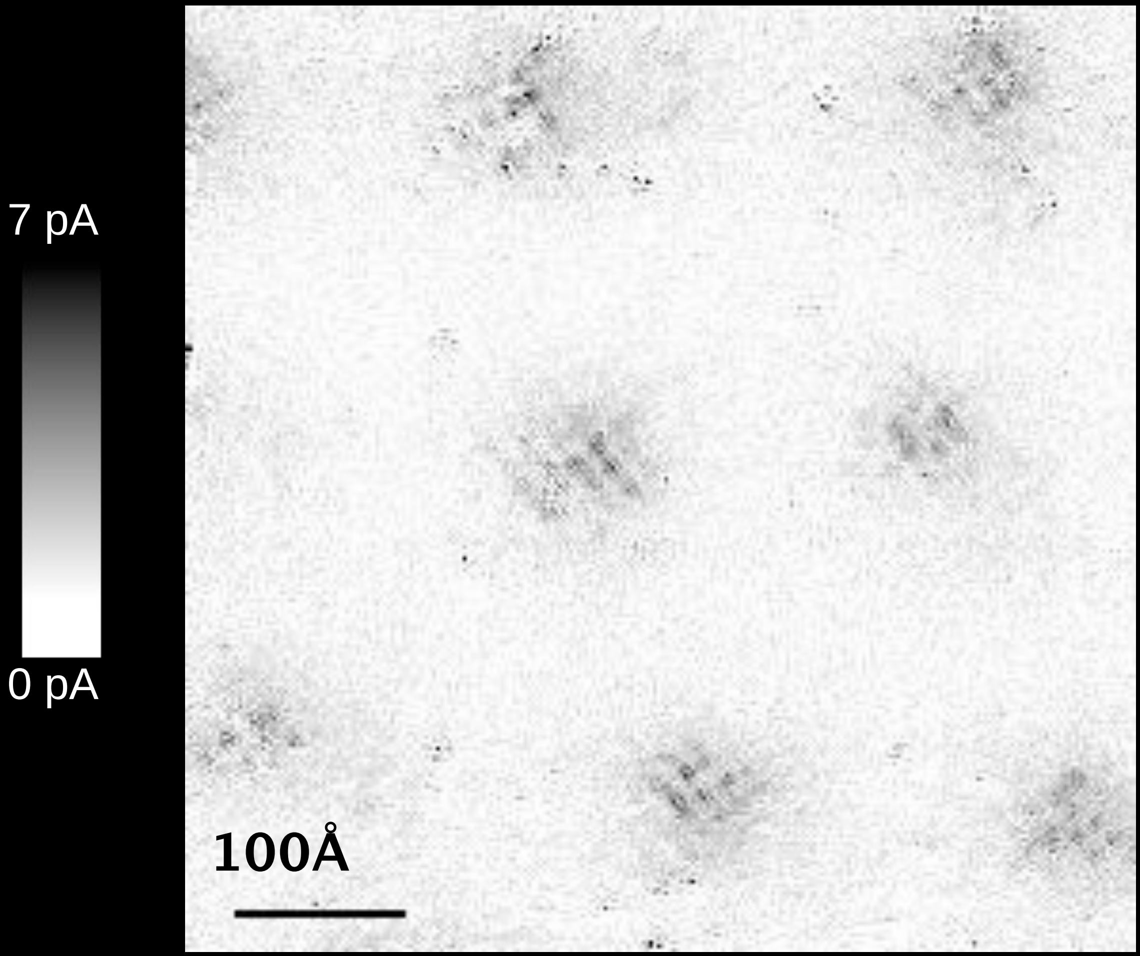
Field theory with projective symmetry

Density operators $\rho_{\boldsymbol{Q}}$ at wavevectors $\boldsymbol{Q}_{mn} = \frac{2\pi p}{q}(m,n)$

$$\rho_{mn} = e^{i\pi mnf} \sum_{\ell=1}^{q} \varphi_{\ell}^{*} \varphi_{\ell+n} e^{2\pi i \ell mf}$$

Each pinned vortex in the superfluid has a halo of density wave order over a length scale ≈ the zero-point quantum motion of the vortex. This scale diverges upon approaching the insulator

Vortex-induced LDOS of $Bi_2Sr_2CaCu_2O_{8+\delta}$ integrated from 1meV to 12meV at 4K



Vortices have halos with LDOS modulations at a period ≈ 4 lattice spacings

J. Hoffman, E. W. Hudson, K. M. Lang, V. Madhavan, S. H. Pan, H. Eisaki, S. Uchida, and J. C. Davis, *Science* 295, 466 (2002).

Prediction of VBS order near vortices: K. Park and S. Sachdev, Phys. Rev. B **64**, 184510 (2001).

(I) The Fermi gas near unitarity

(II) The superfluid-insulator quantum phase transition

(III) Phases of quantum antiferromagnets

(III) Phases of Quantum antiferromagnets

Review articles:

arXiv:0711.3015, Nature Physics to appear

cond-mat/0401041

Outline

1. Quantum "disordering" magnetic order

   Collinear order and confinement

2. Z_2 spin liquids

   Noncollinear order and fractionalization

3. U(1) spin liquids

   Valence bond solid (VBS) order

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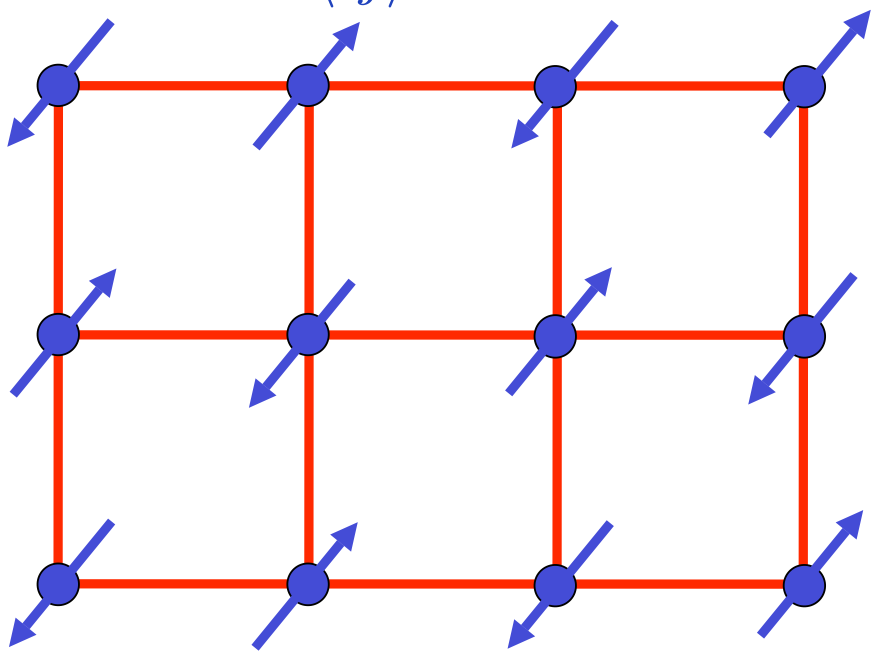
Square lattice antiferromagnet

$$H = \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j$$



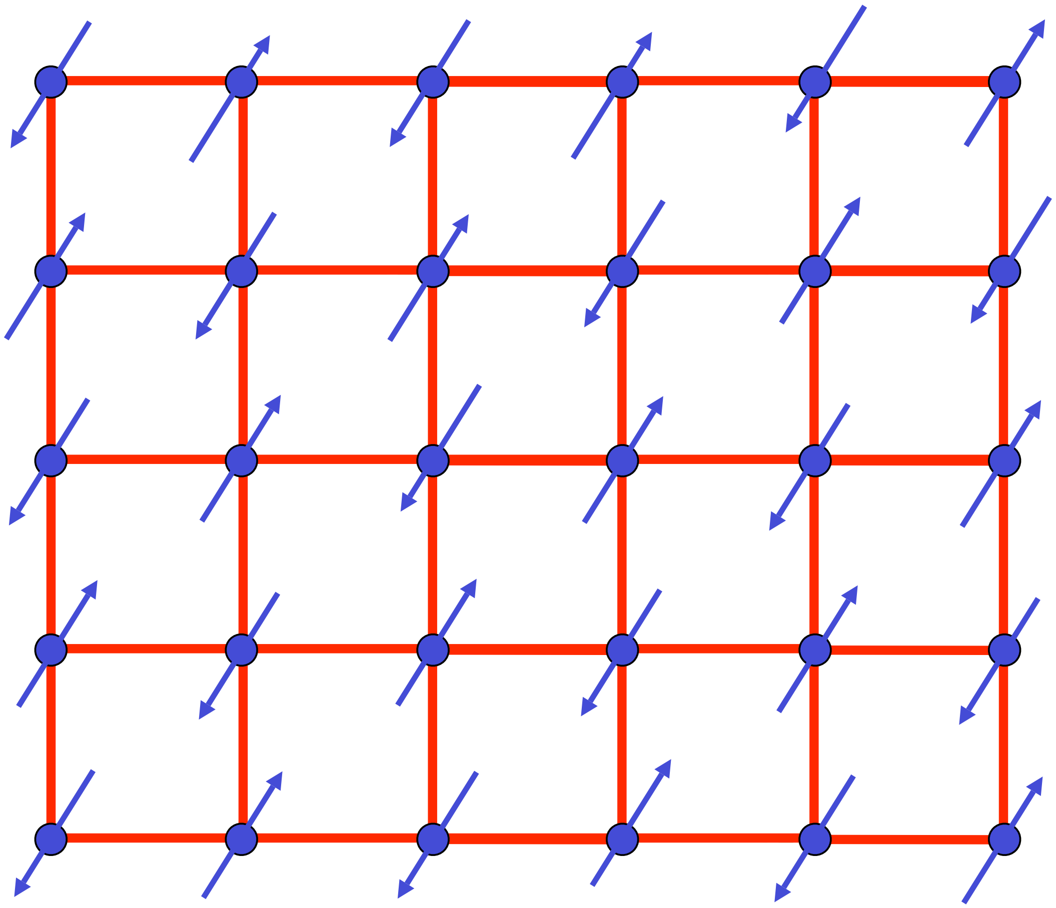
Ground state has long-range Néel order

Order parameter is a single vector field $\vec{\varphi} = \eta_i \vec{S}_i$

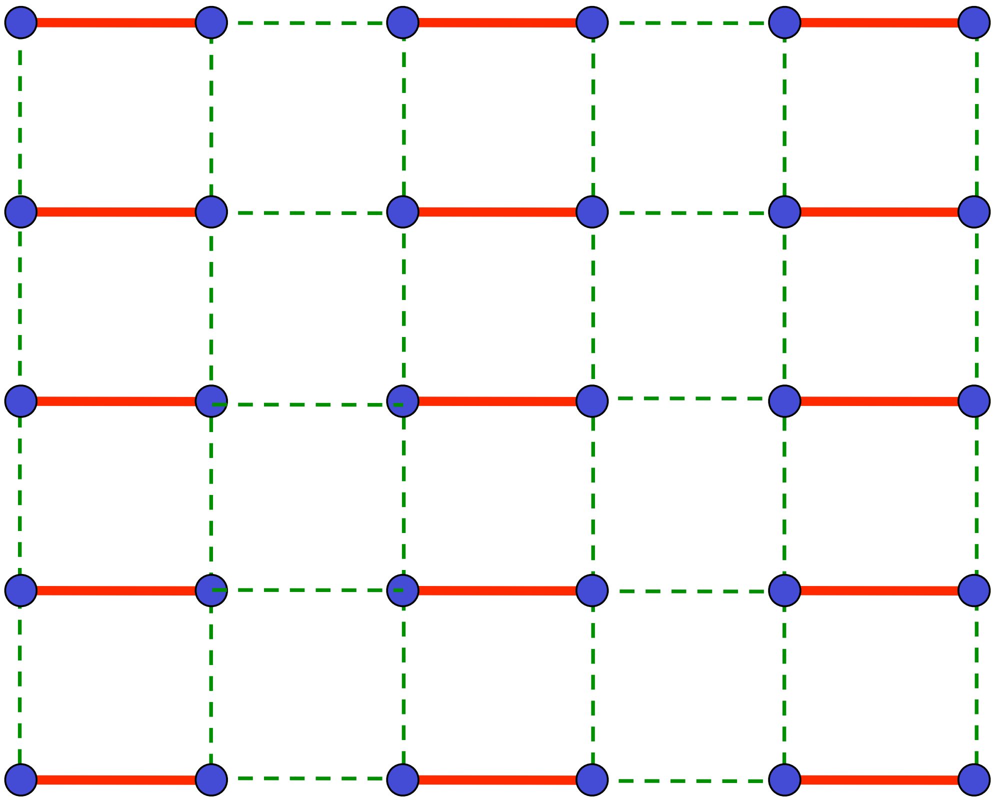
$\eta_i = \pm 1$ on two sublattices

$\langle \vec{\varphi} \rangle \neq 0$ in Néel state.

Antiferromagnetic (Neel) order in the insulator

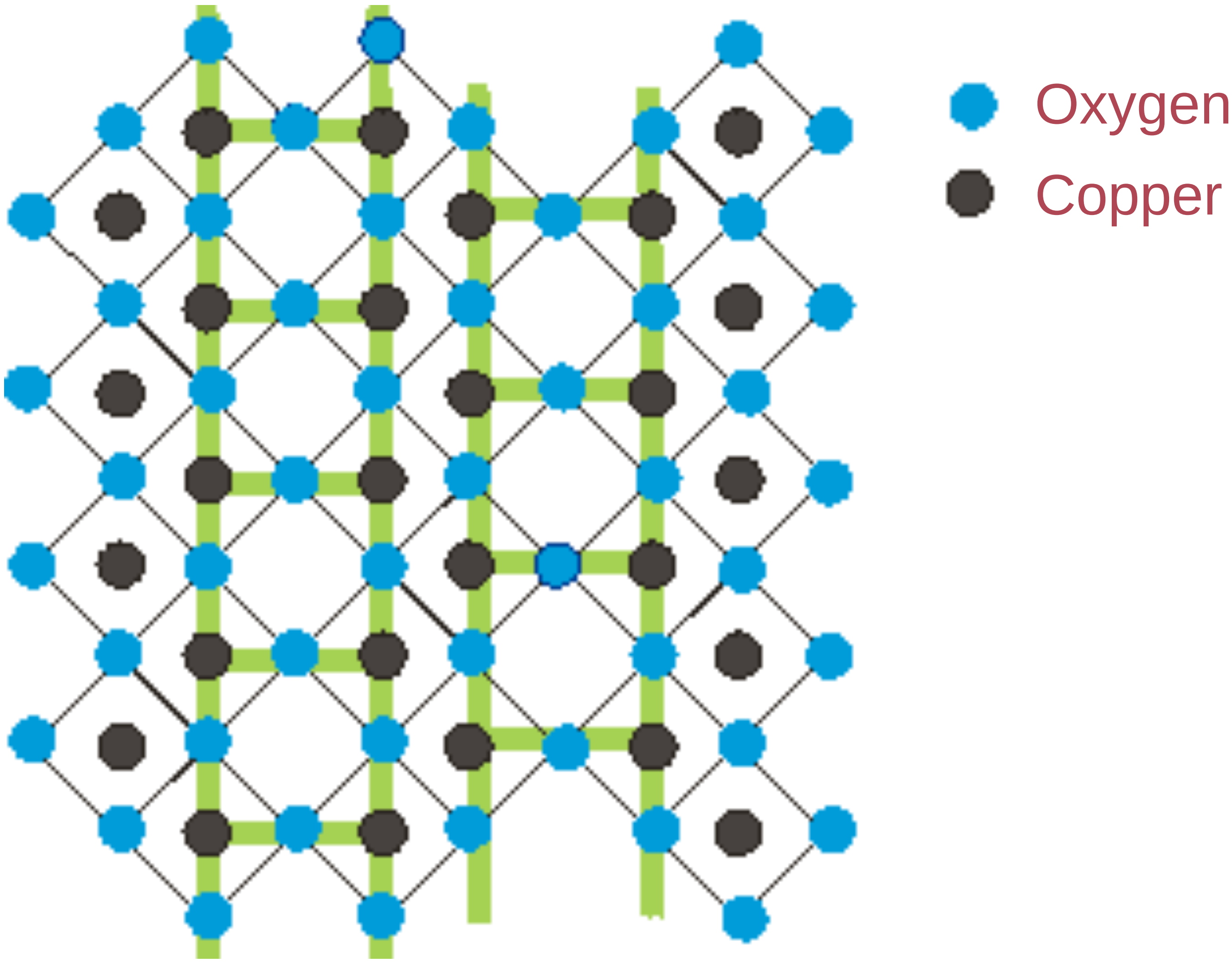


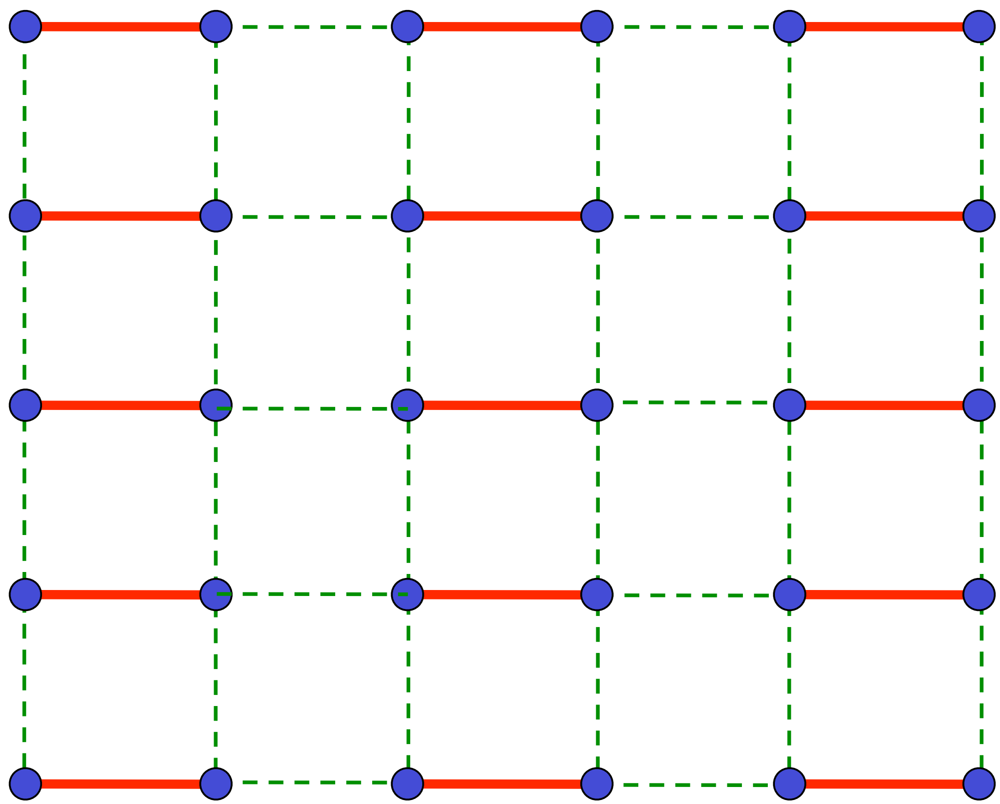
No entanglement of spins

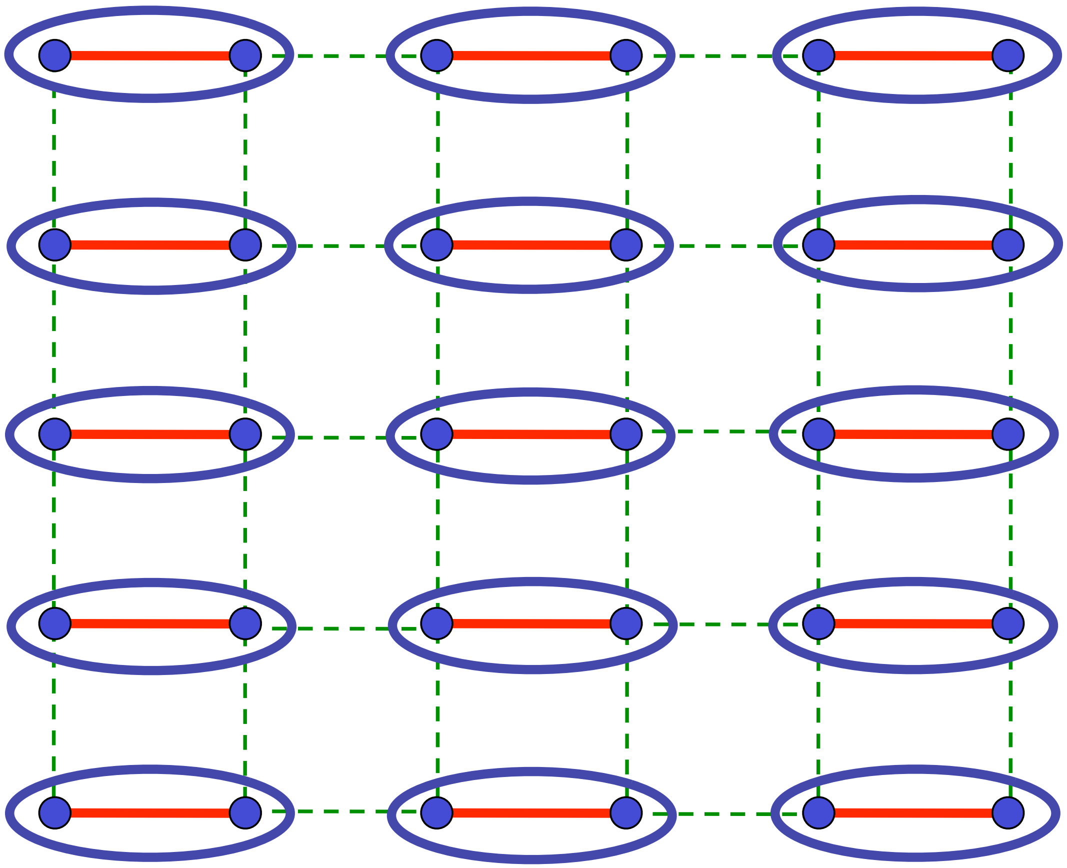


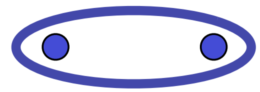
Weaken some bonds to induce spin entanglement in a new quantum phase

$SrCu_2O_3$









$$= \frac{1}{\sqrt{2}}\left(\left|\uparrow\downarrow\right\rangle - \left|\downarrow\uparrow\right\rangle \right)$$

Ground state is a product of pairs
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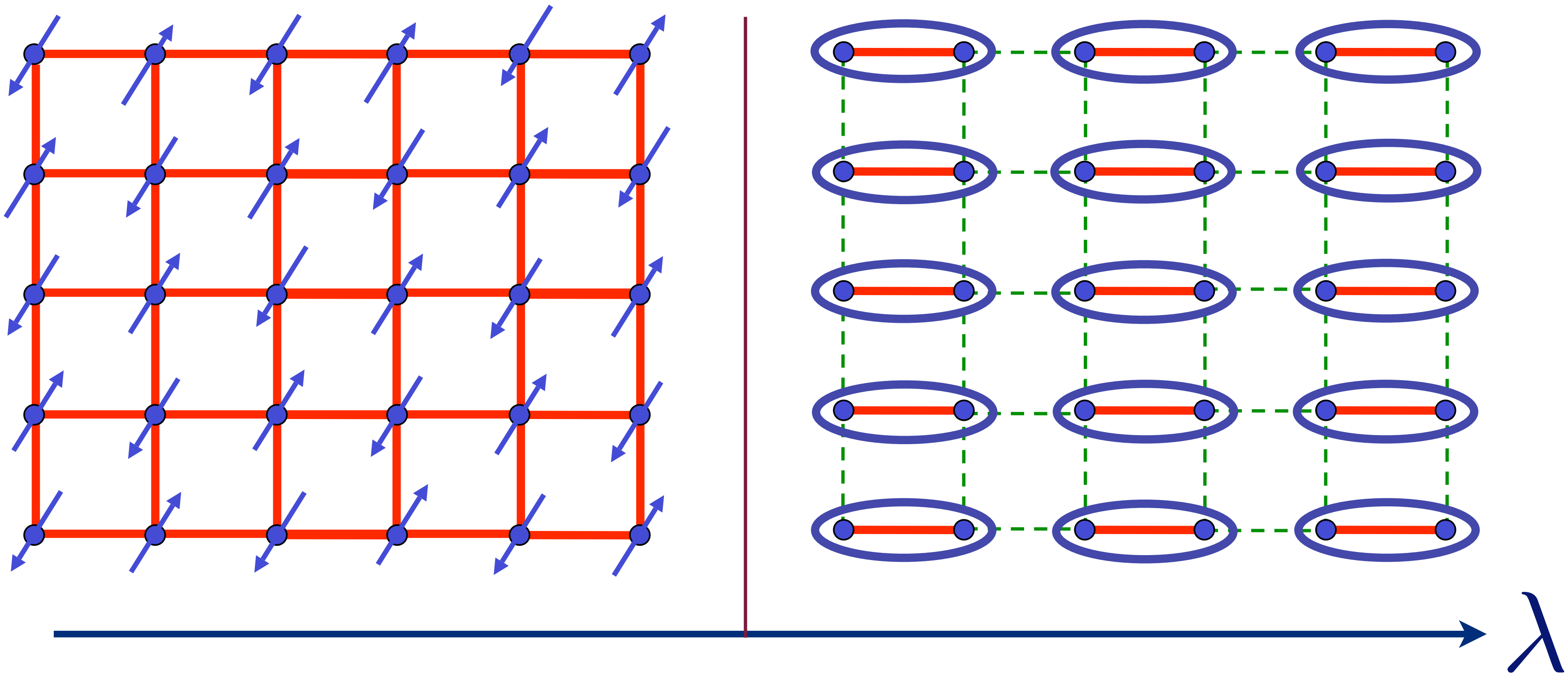
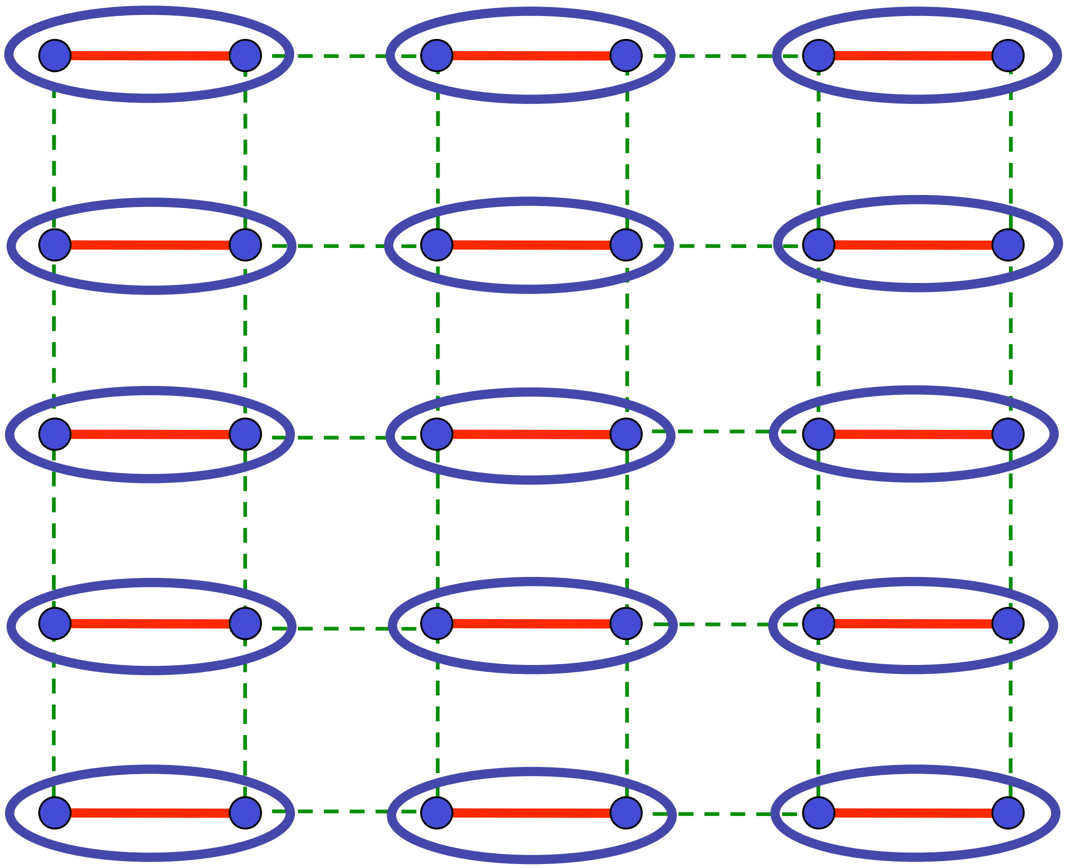
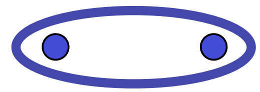
Phase diagram as a function of the ratio of exchange interactions, λ







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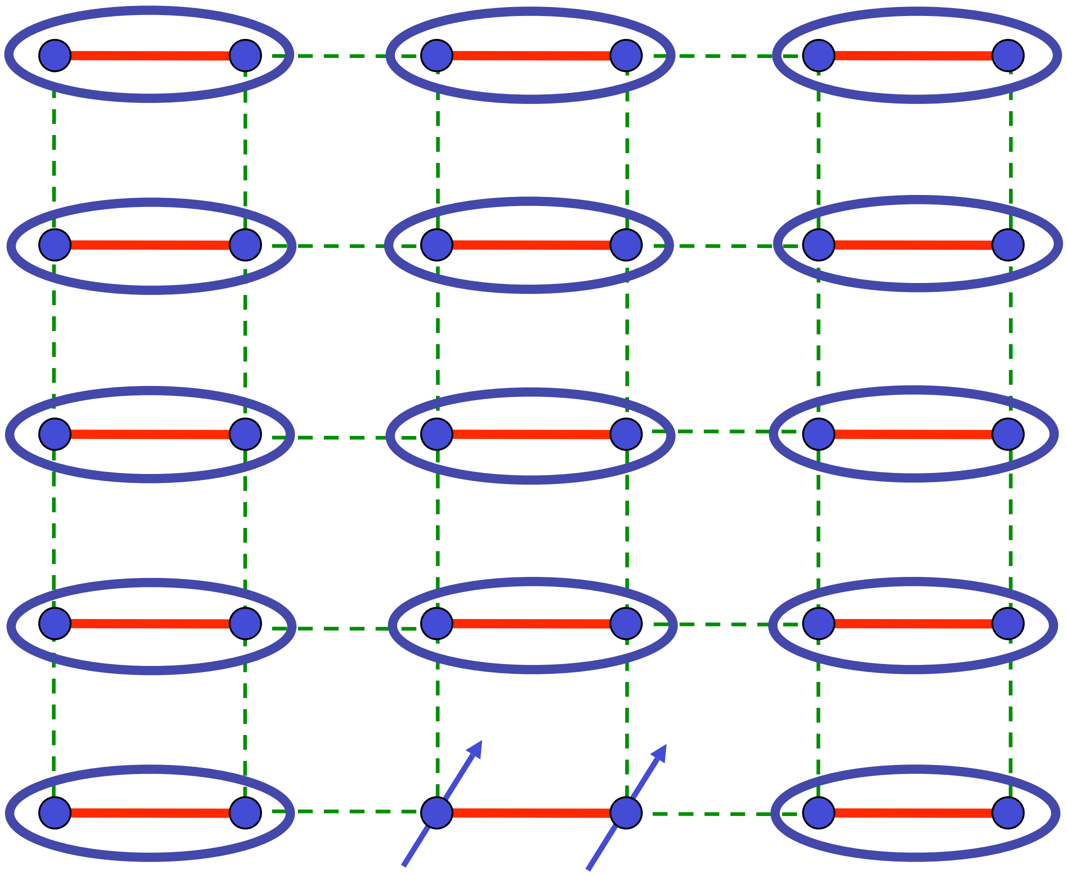

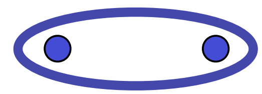

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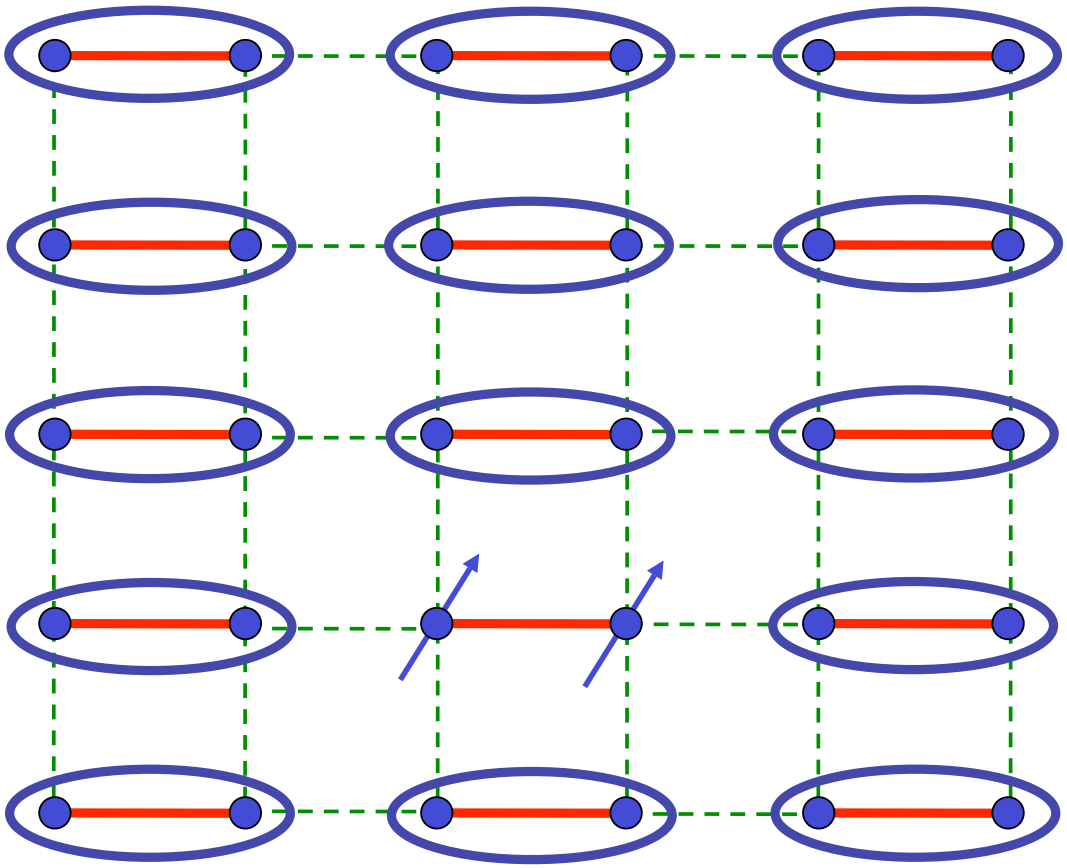
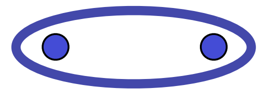
Excitation: S=1 ***triplon***





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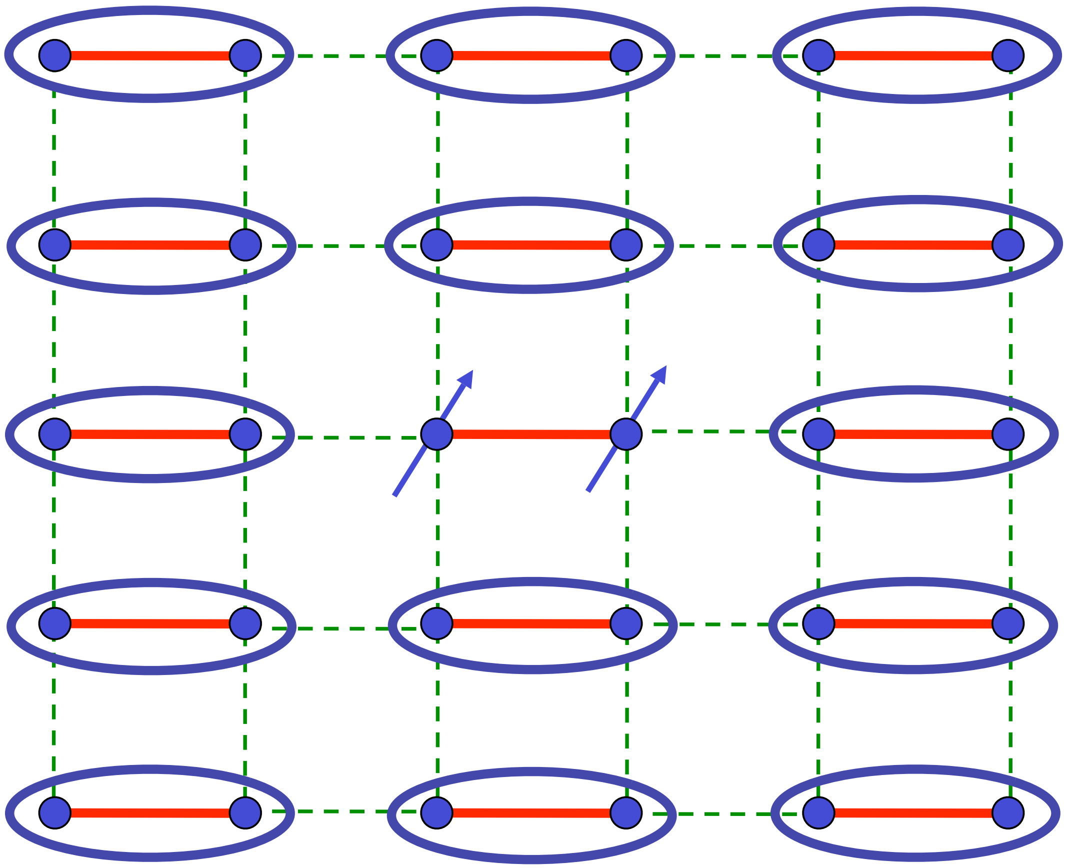

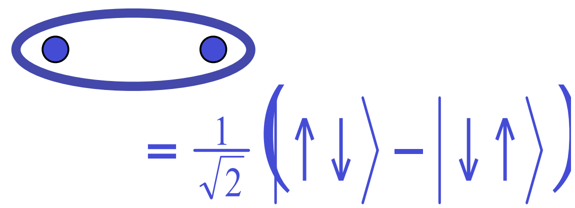

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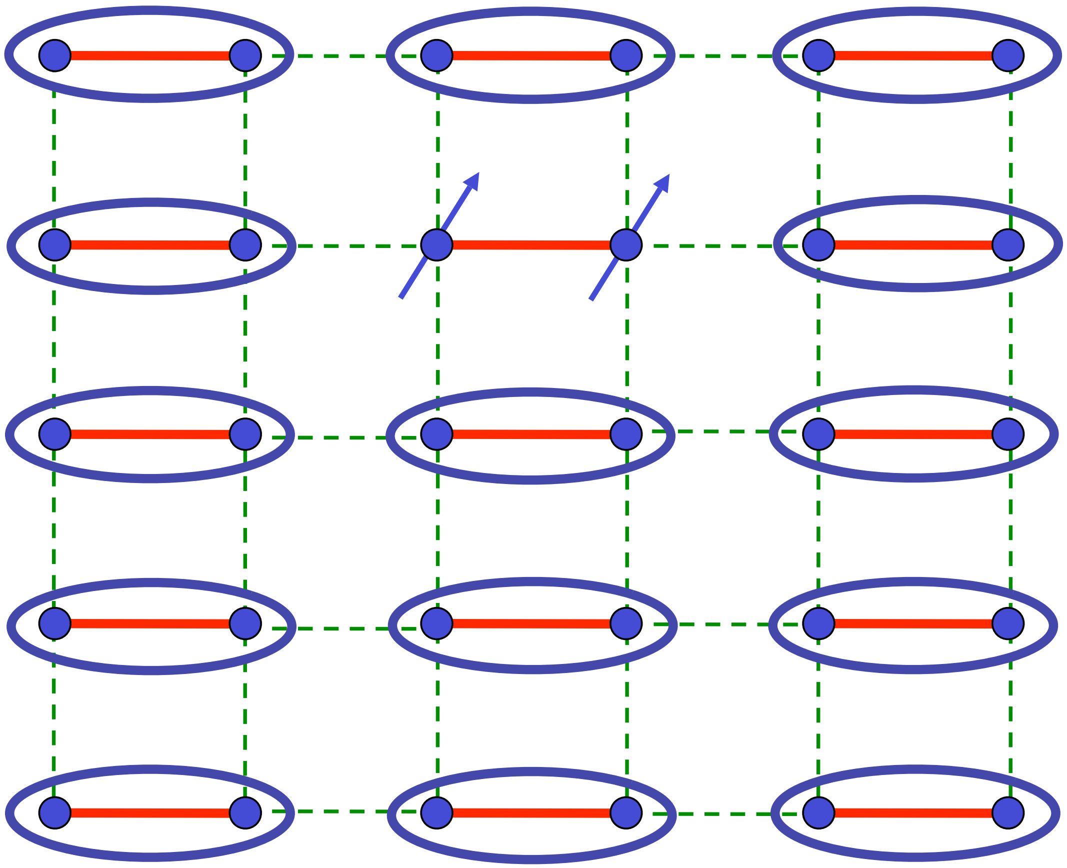

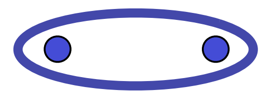

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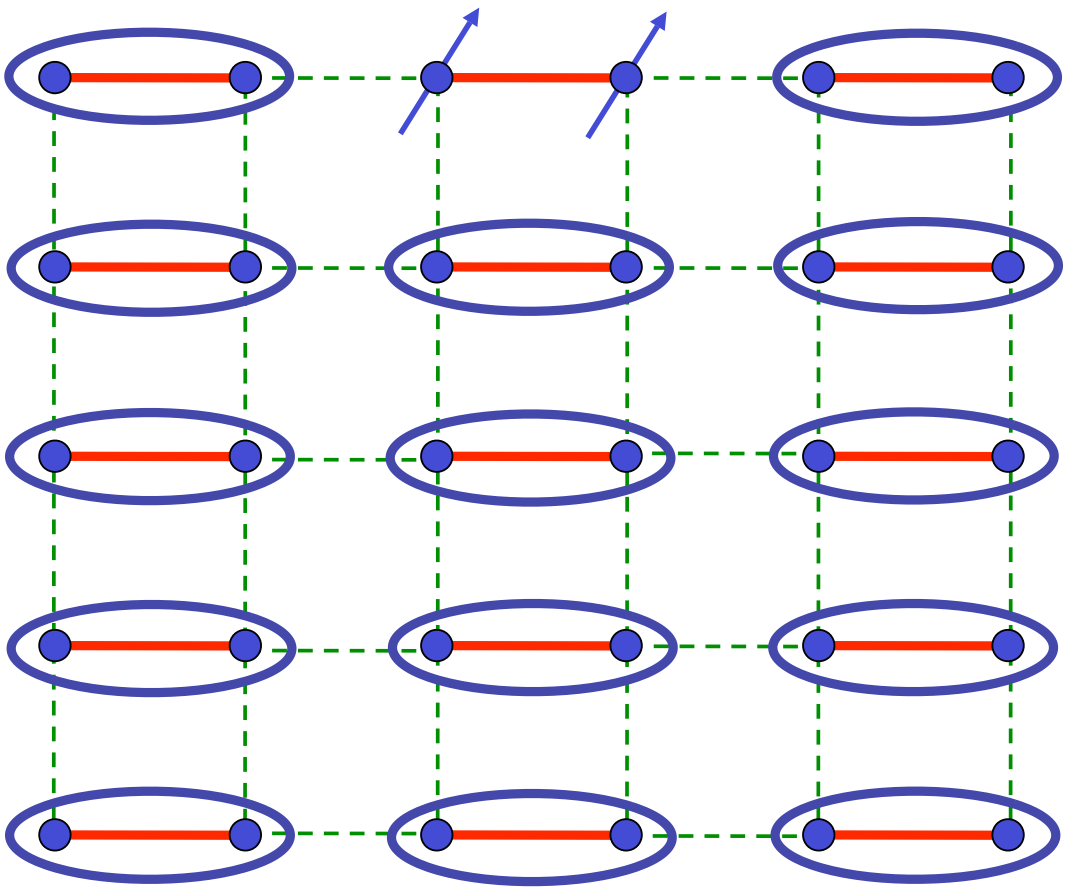

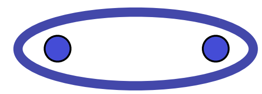

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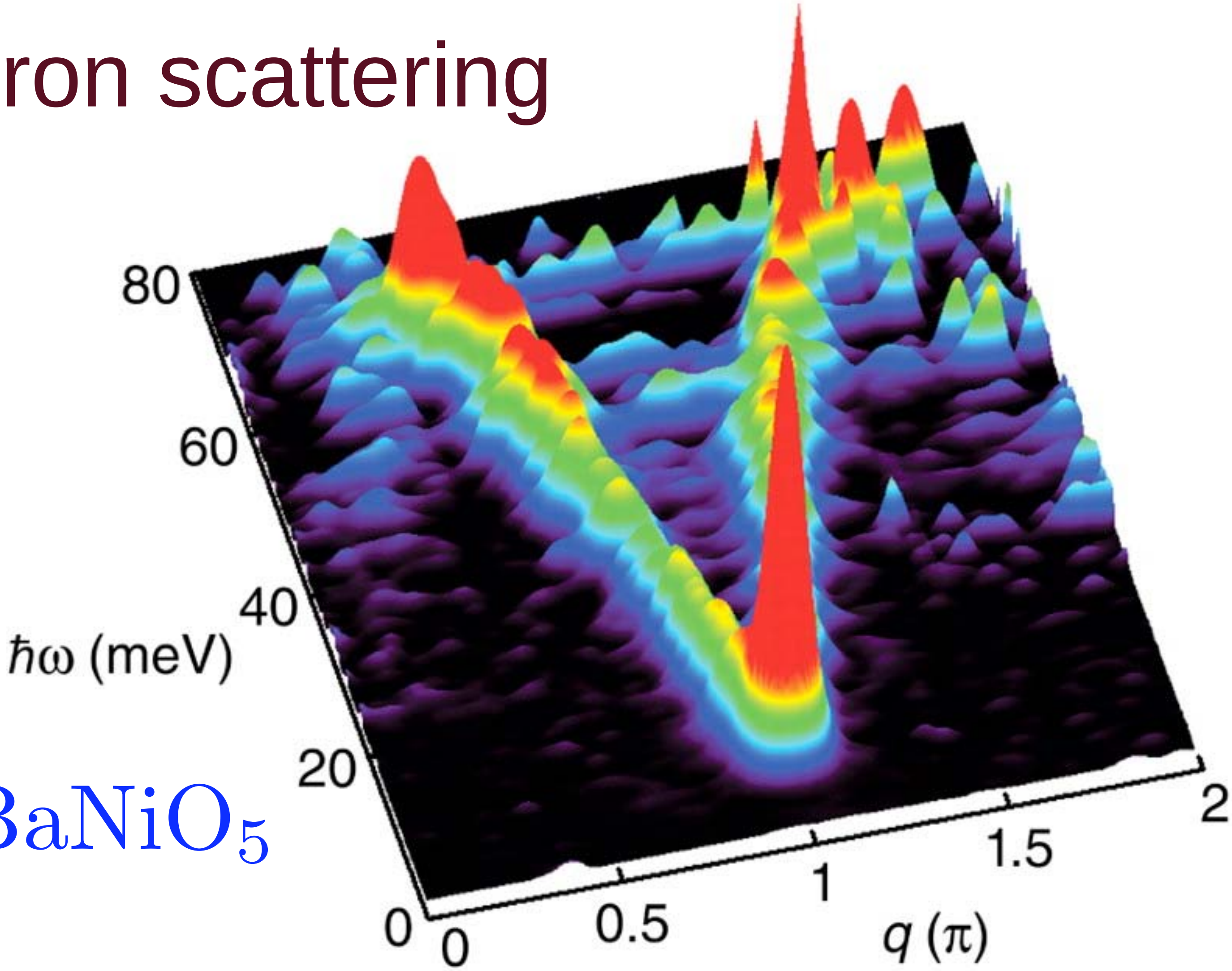
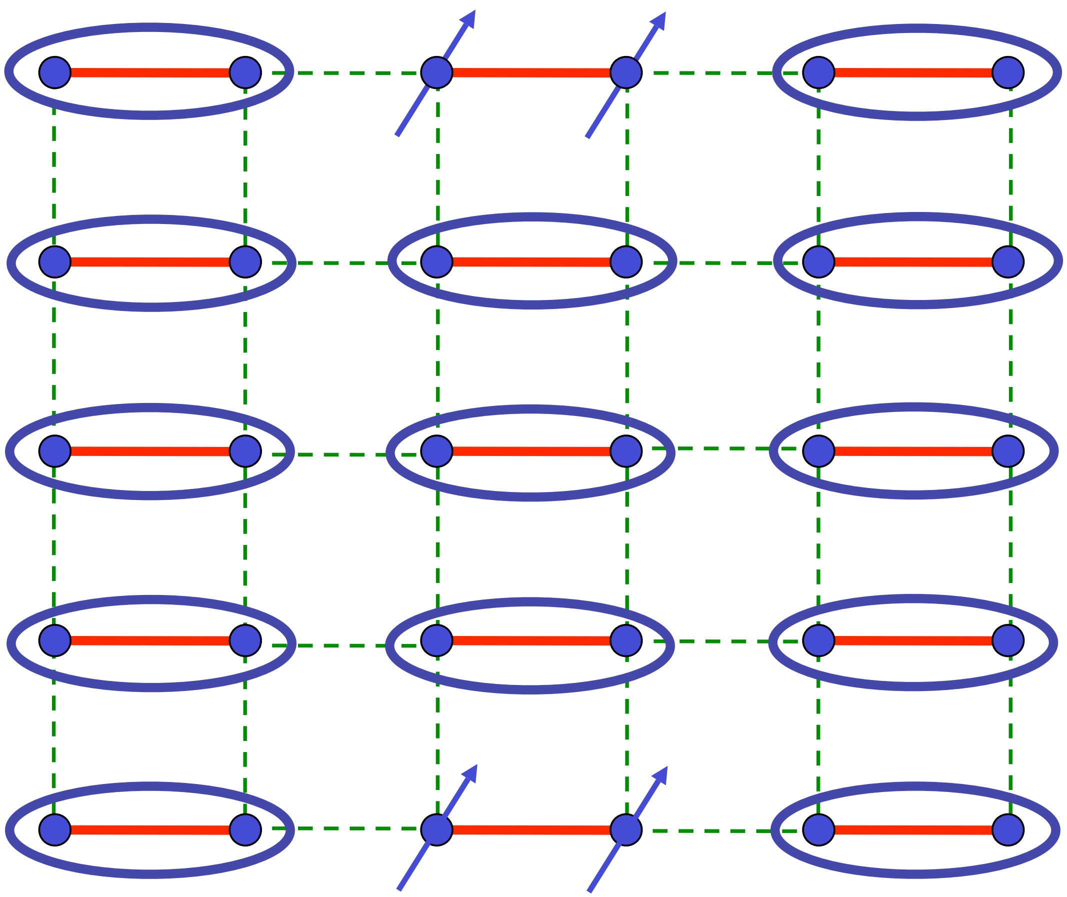
Neutron scattering

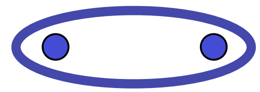


Y_2BaNiO_5

G. Xu, C. Broholm, Yeong-Ah Soh, G. Aeppli, J. F. DiTusa, Y. Chen, M. Kenzelmann, C. D. Frost, T. Ito, K. Oka, and H. Takagi, Science **317**, 1049 (2007).

Collision of triplons





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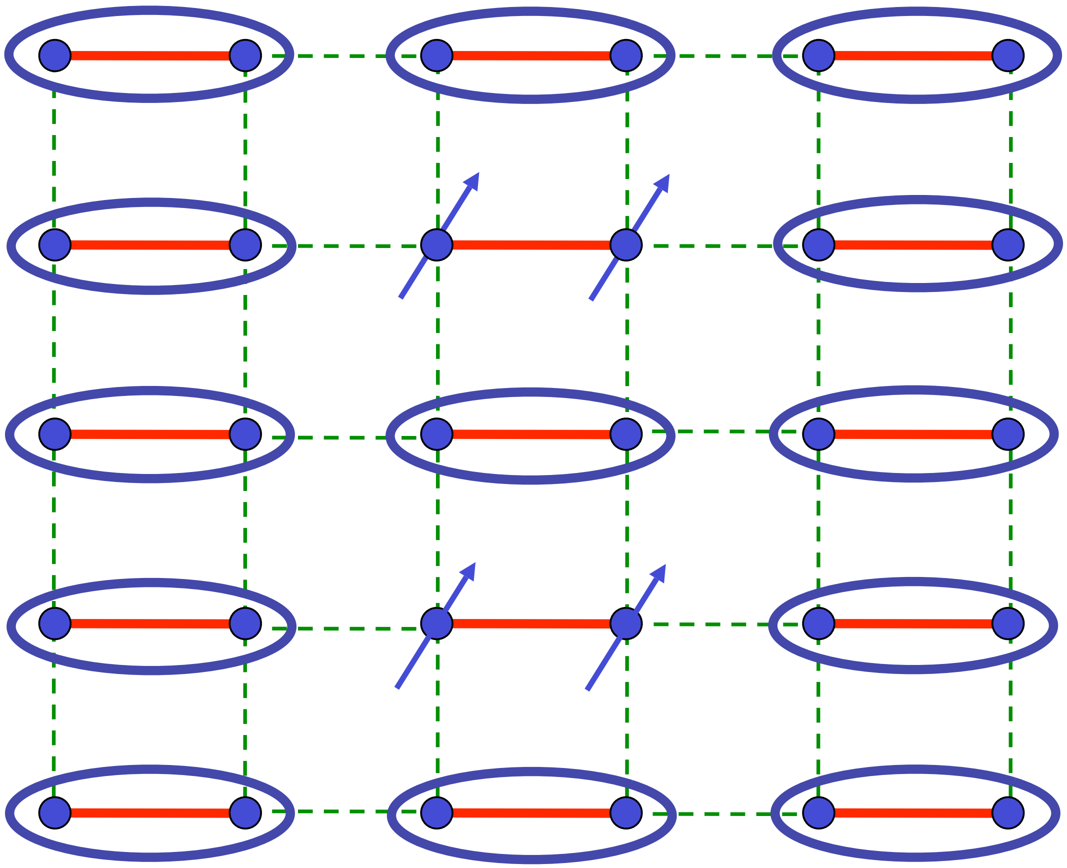

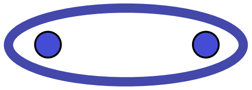

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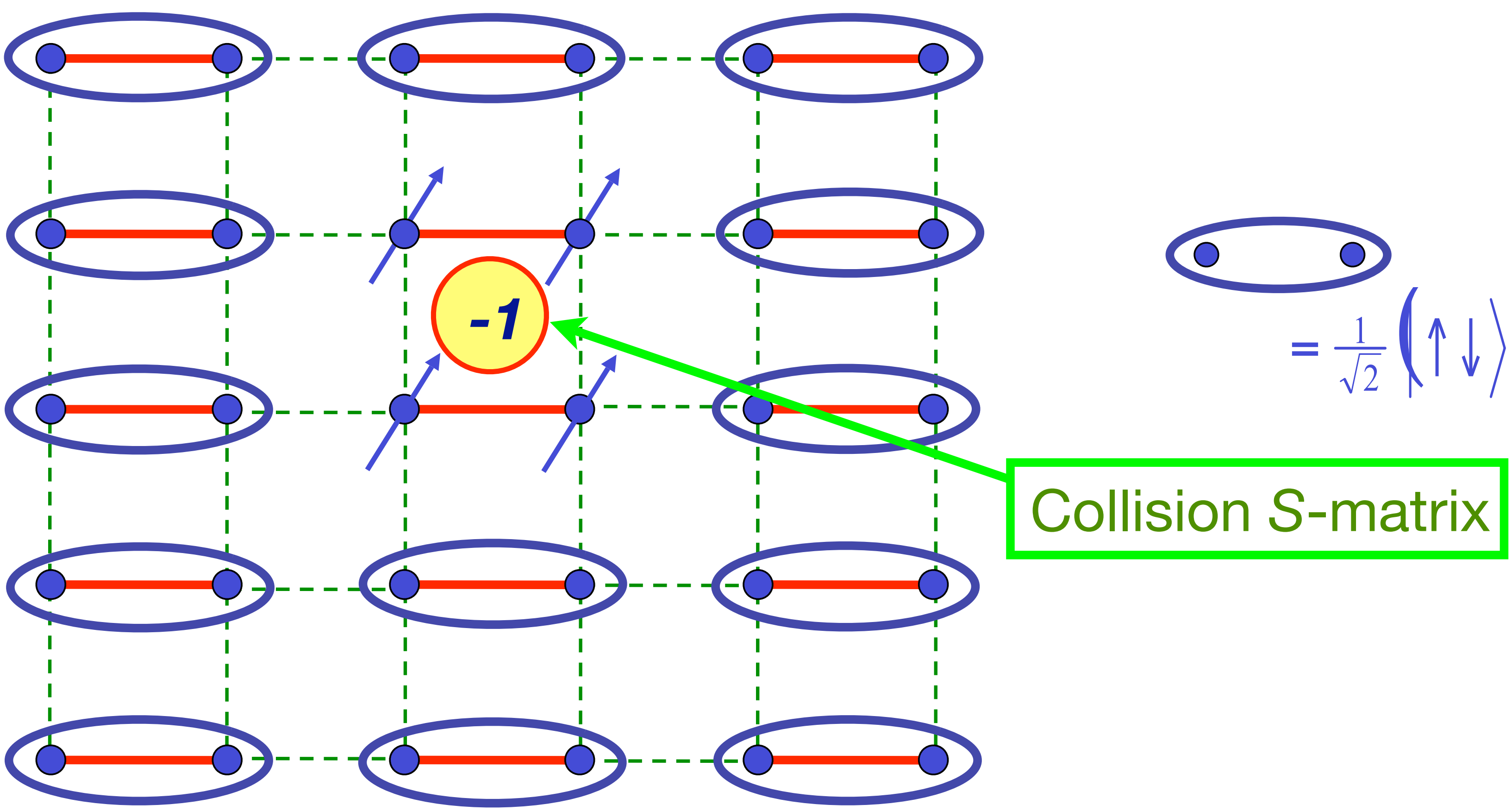

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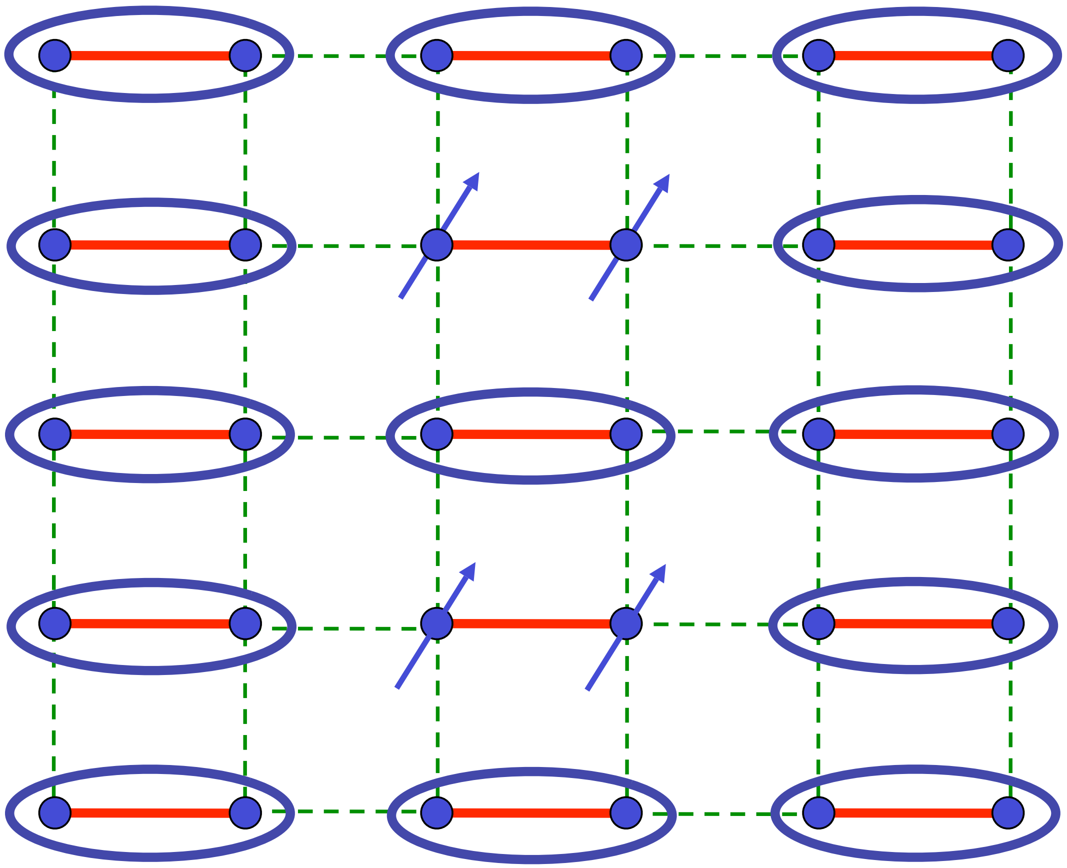

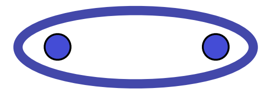

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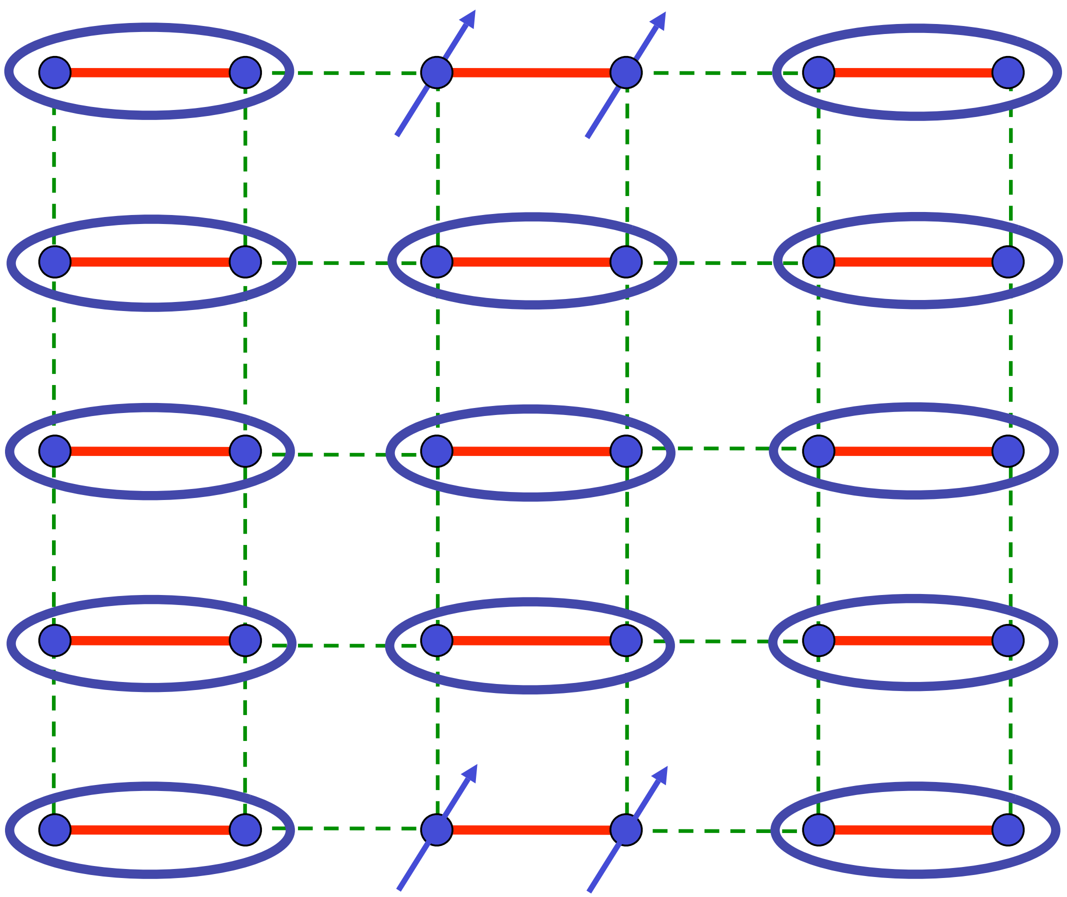

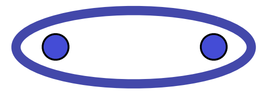

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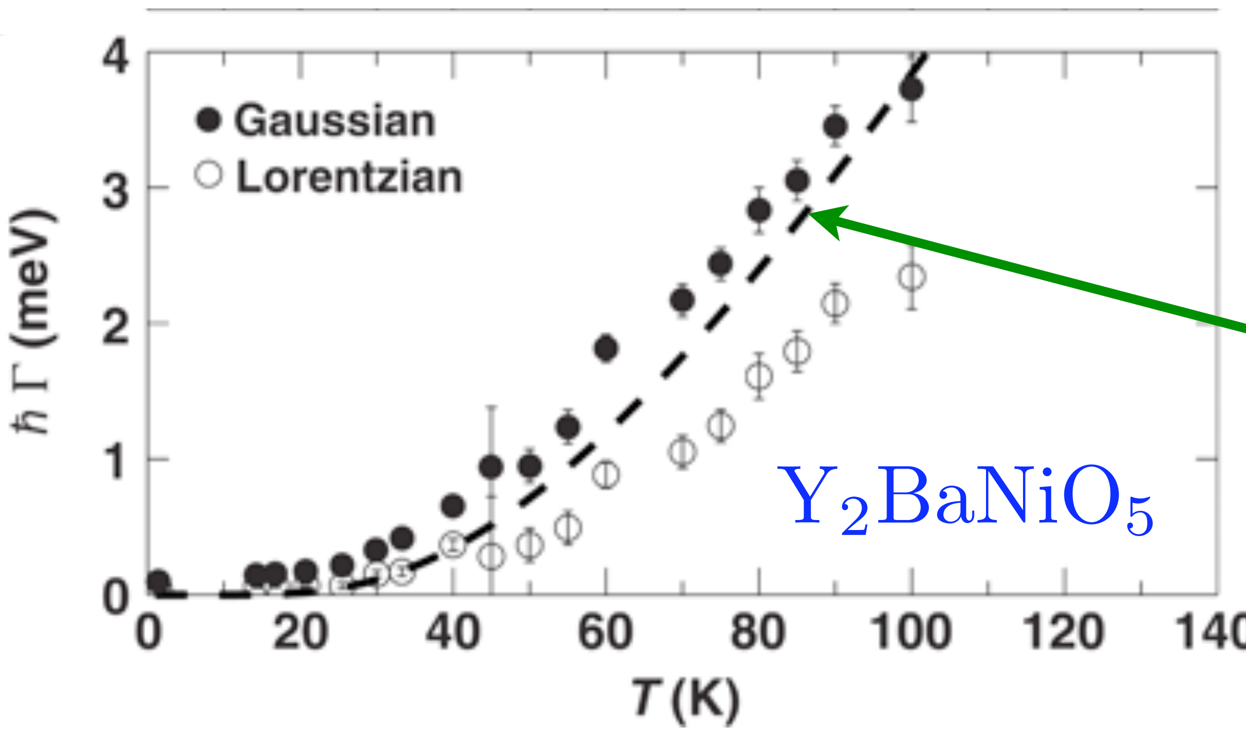
Neutron scattering linewidth



Parameter free prediction by K. Damle and S. Sachdev, Phys. Rev. B **57**, 8307 (1998) from multiple collisions with universal, quantum *S*-matrices

G. Xu, C. Broholm, Yeong-Ah Soh, G. Aeppli, J. F. DiTusa, Y. Chen, M. Kenzelmann, C. D. Frost, T. Ito, K. Oka, and H. Takagi, Science **317**, 1049 (2007).

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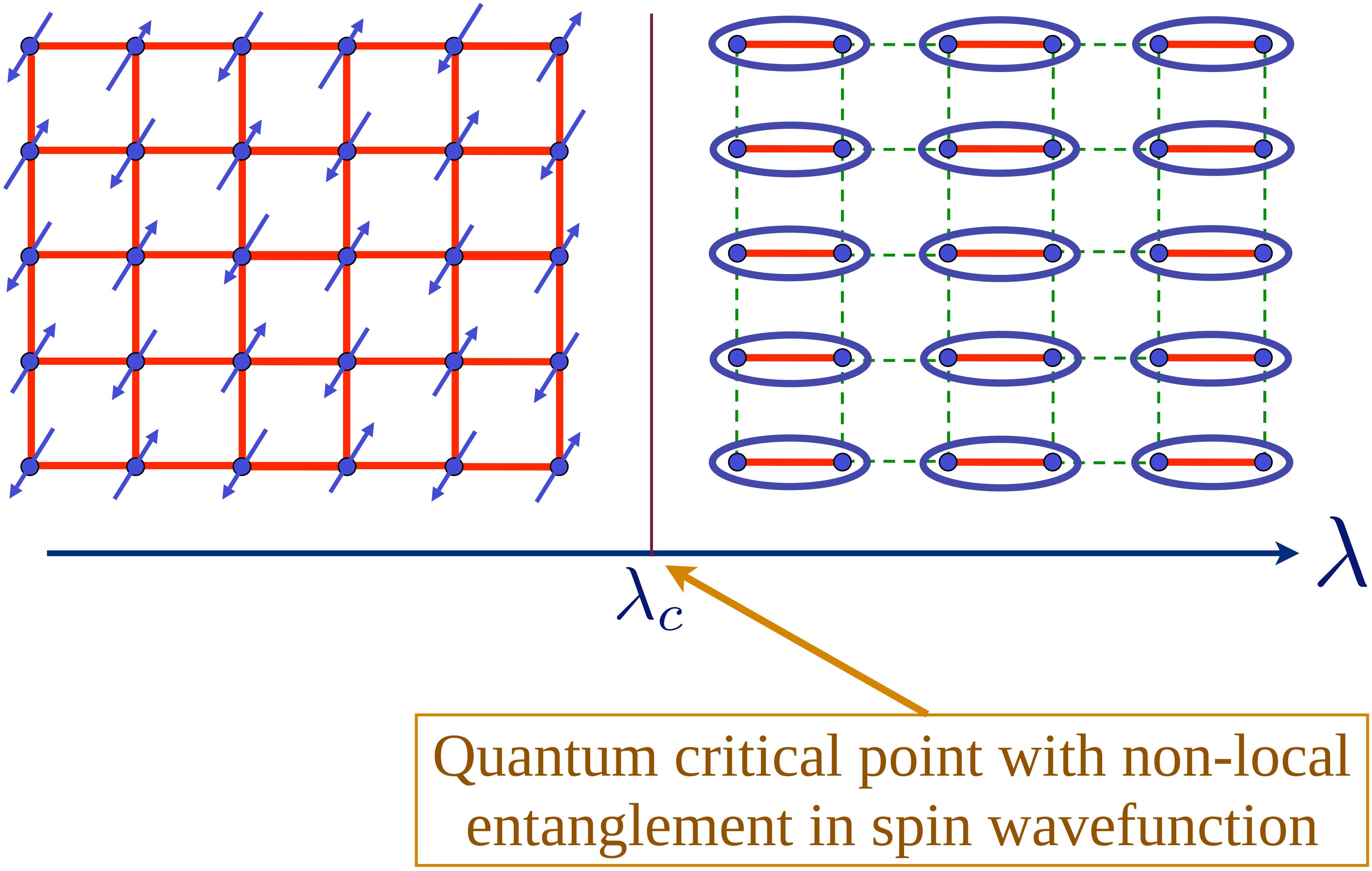

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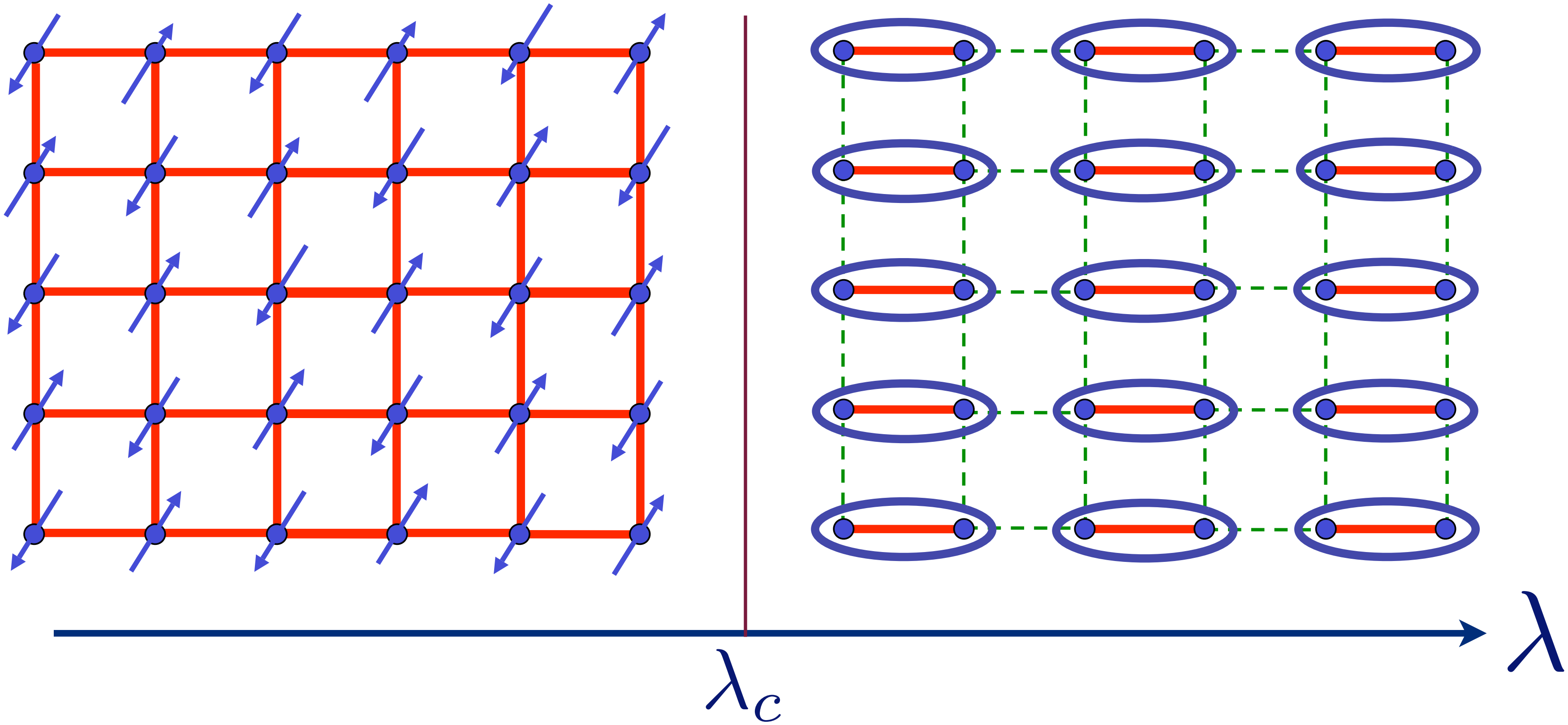

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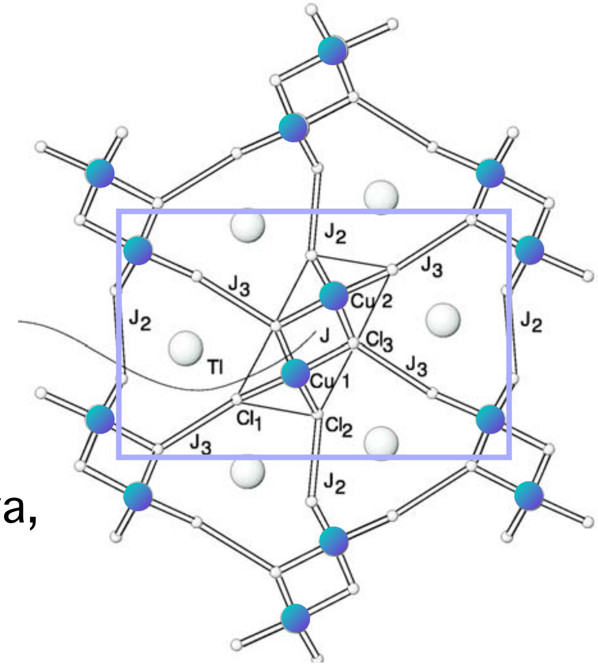
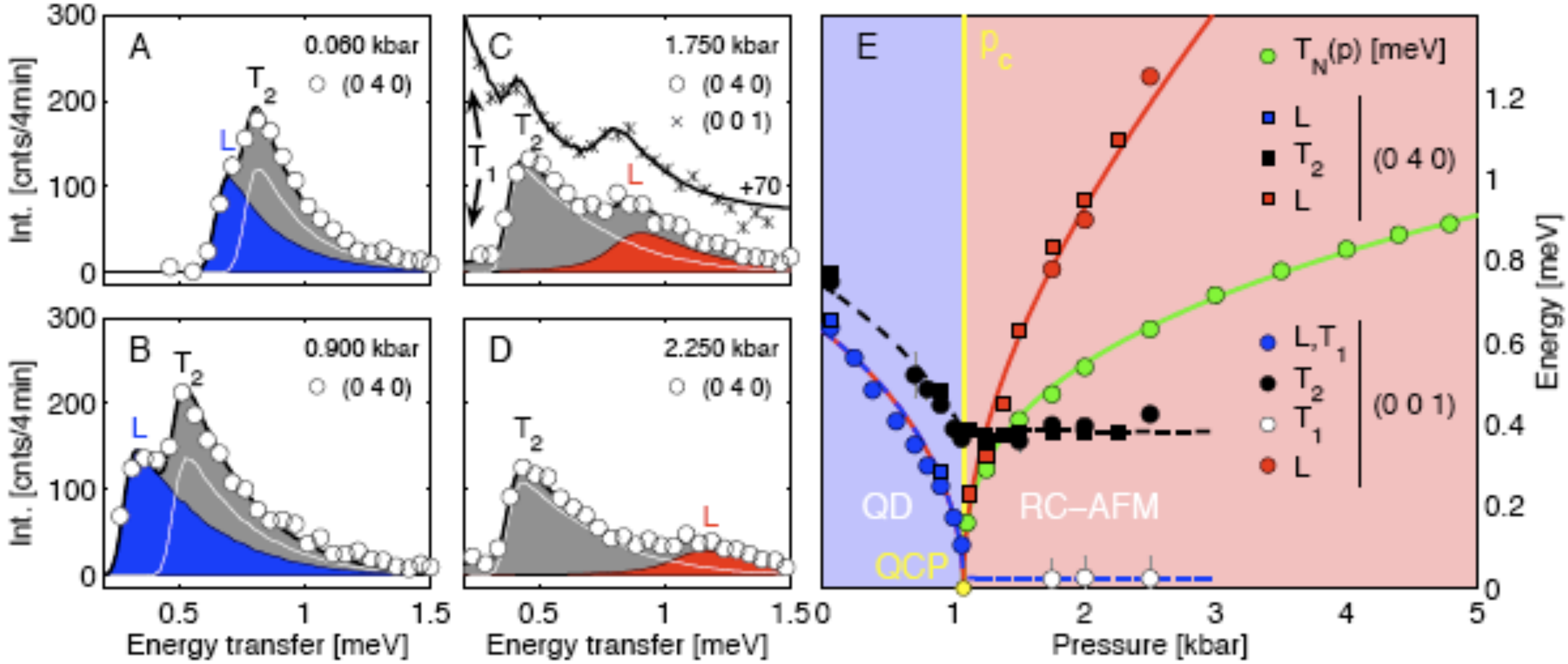
Landau-Ginzburg-Wilson Theory

Observation of longitudinal mode in $TlCuCl_3$

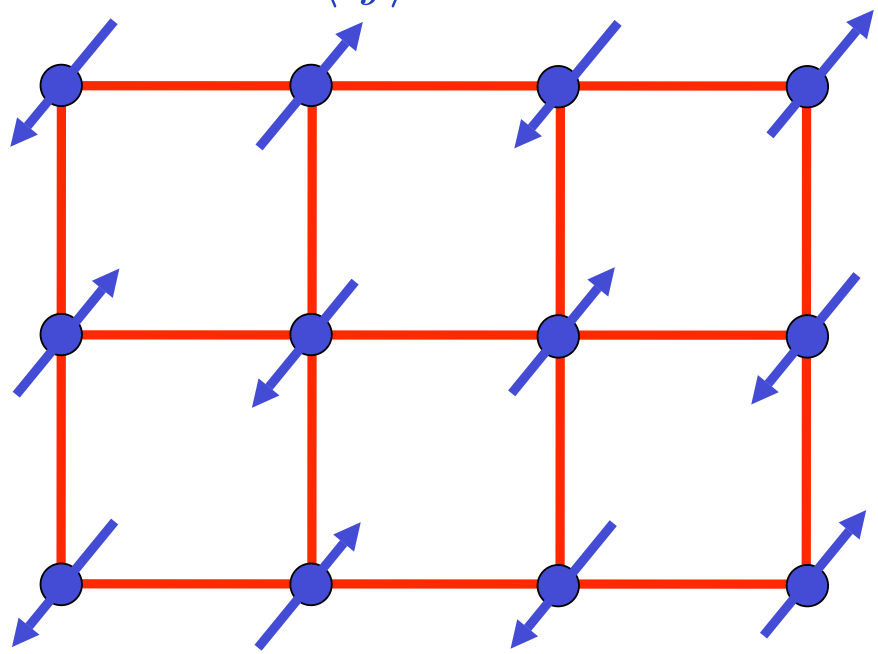
Christian Ruegg, Bruce Normand, Masashige Matsumoto, Albert Furrer, Desmond McMorrow, Karl Kramer, Hans-Ulrich Gudel, Severian Gvasaliya, Hannu Mutka, and Martin Boehm





Square lattice antiferromagnet

$$H = \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j$$



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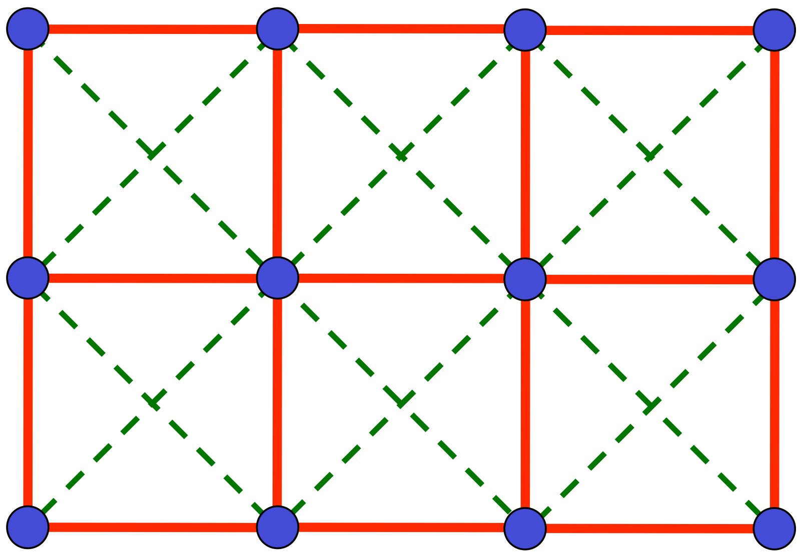

Destroy Neel order by perturbations which preserve full square lattice symmetry *e.g.* second-neighbor or ring exchange.

What is the state with $\langle \vec{\varphi} \rangle = 0$?

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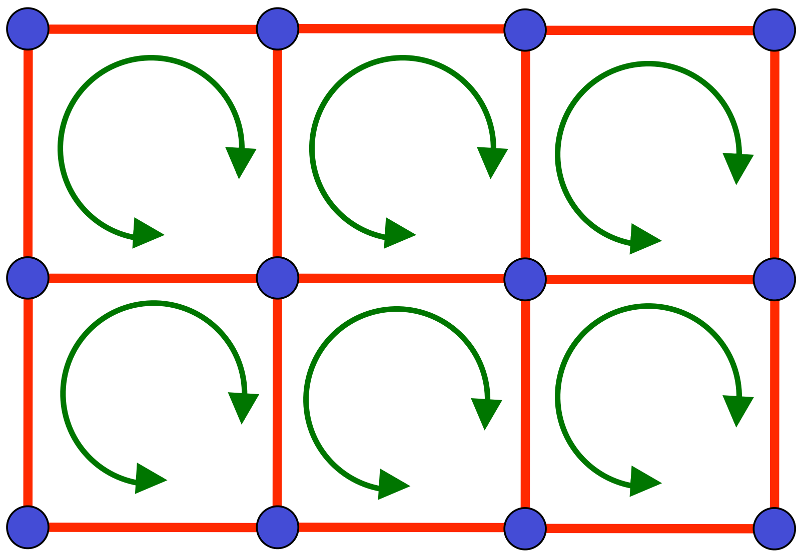

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LGW theory for quantum criticality

Landau-Ginzburg-Wilson theory: write down an effective action for the antiferromagnetic order parameter $\vec{\varphi}$ by expanding in powers of $\vec{\varphi}$ and its spatial and temporal derivatives, while preserving all symmetries of the microscopic Hamiltonian

$$\mathcal{S}_\varphi = \int d^2x d\tau \left[\frac{1}{2} \left(c^2 \left(\nabla_x \vec{\varphi} \right)^2 + \left(\partial_\tau \vec{\varphi} \right)^2 + s \vec{\varphi}^2 \right) + u \left(\vec{\varphi}^2 \right)^2 \right]$$

S. Chakravarty, B.I. Halperin, and D.R. Nelson, *Phys. Rev.* B **39**, 2344 (1989)

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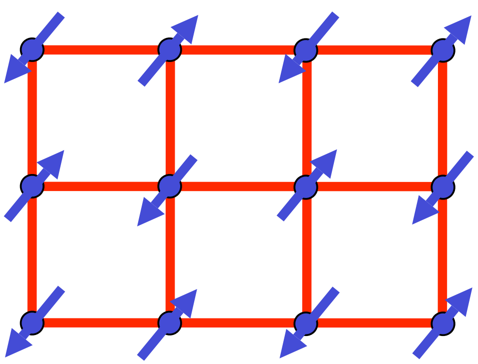
A.V. Chubukov, S. Sachdev, and J.Ye, *Phys. Rev.* B **49**, 11919 (1994)



$\langle \vec{\varphi} \rangle \neq 0$

Néel state

State with no broken symmetries. Fluctuations of $\vec{\varphi}$ about $\vec{\varphi} = 0$ realize a *stable* $S = 1$ quasiparticle with energy $\varepsilon_k = \sqrt{s + c^2 k^2}$

$\langle \vec{\varphi} \rangle = 0$

s_c $\qquad$ s

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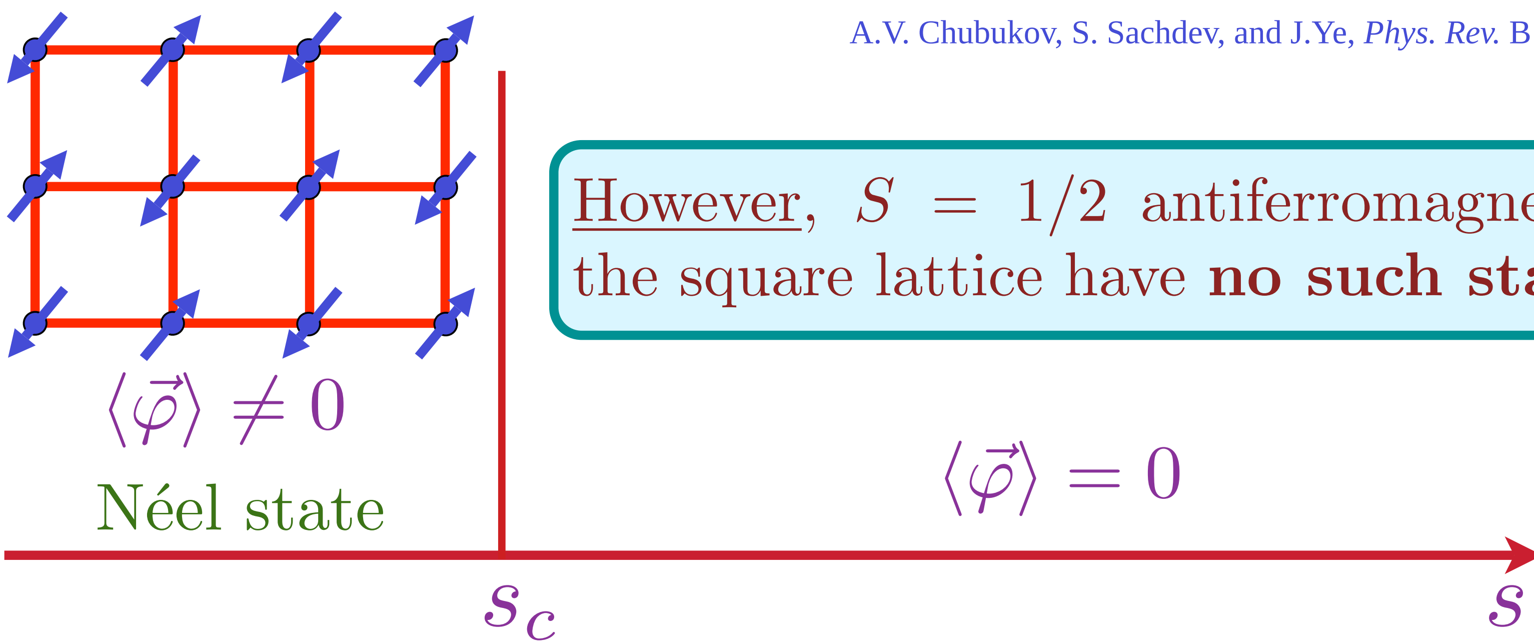

However, $S = 1/2$ antiferromagnets on the square lattice have **no such state**.

There is no state with a gapped, stable S=1 quasiparticle and no broken symmetries

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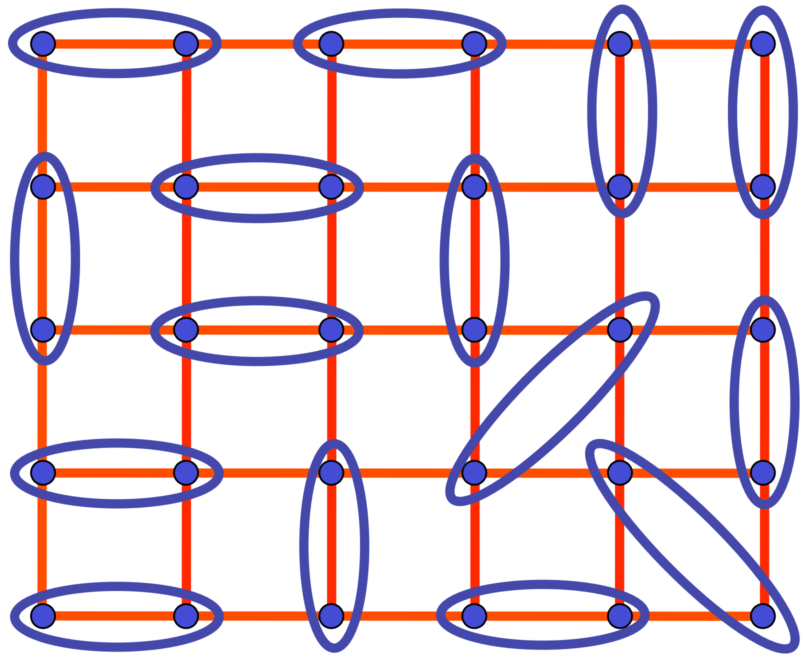

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"Liquid" of valence bonds has fractionalized *S=1/2* excitations

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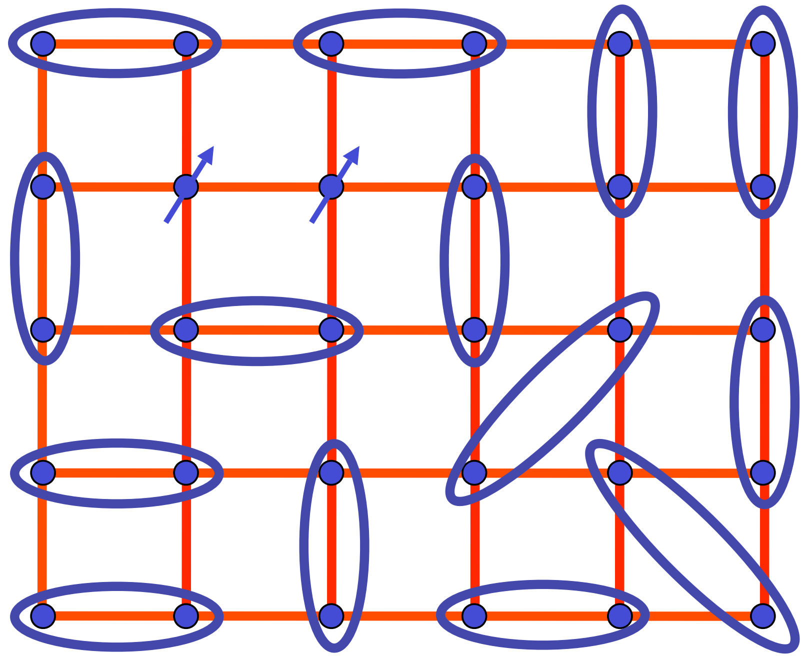


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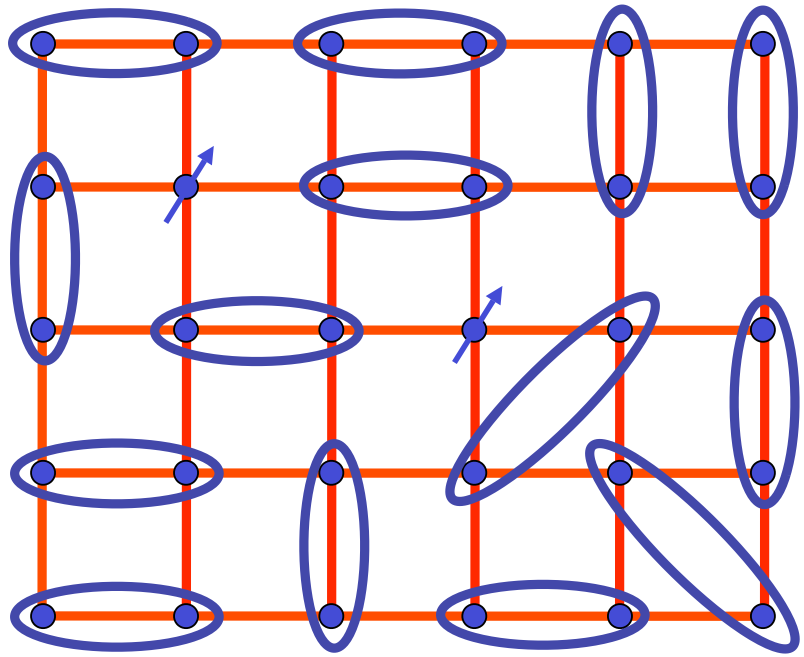

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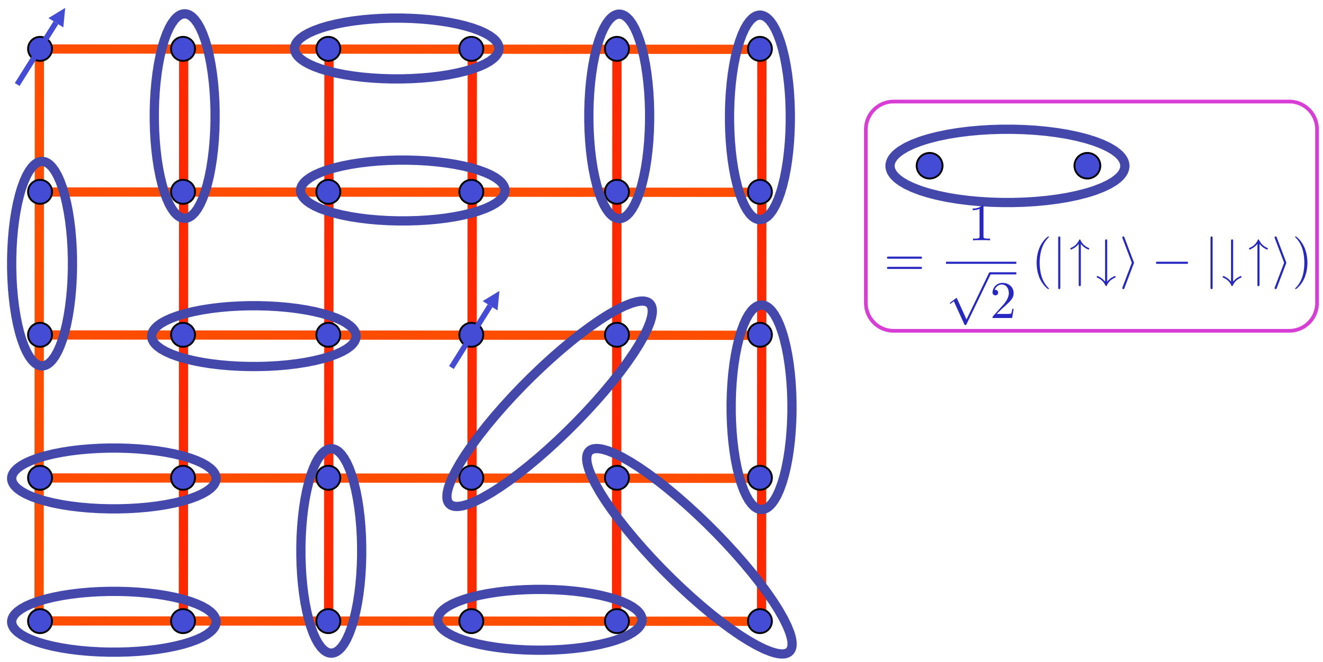

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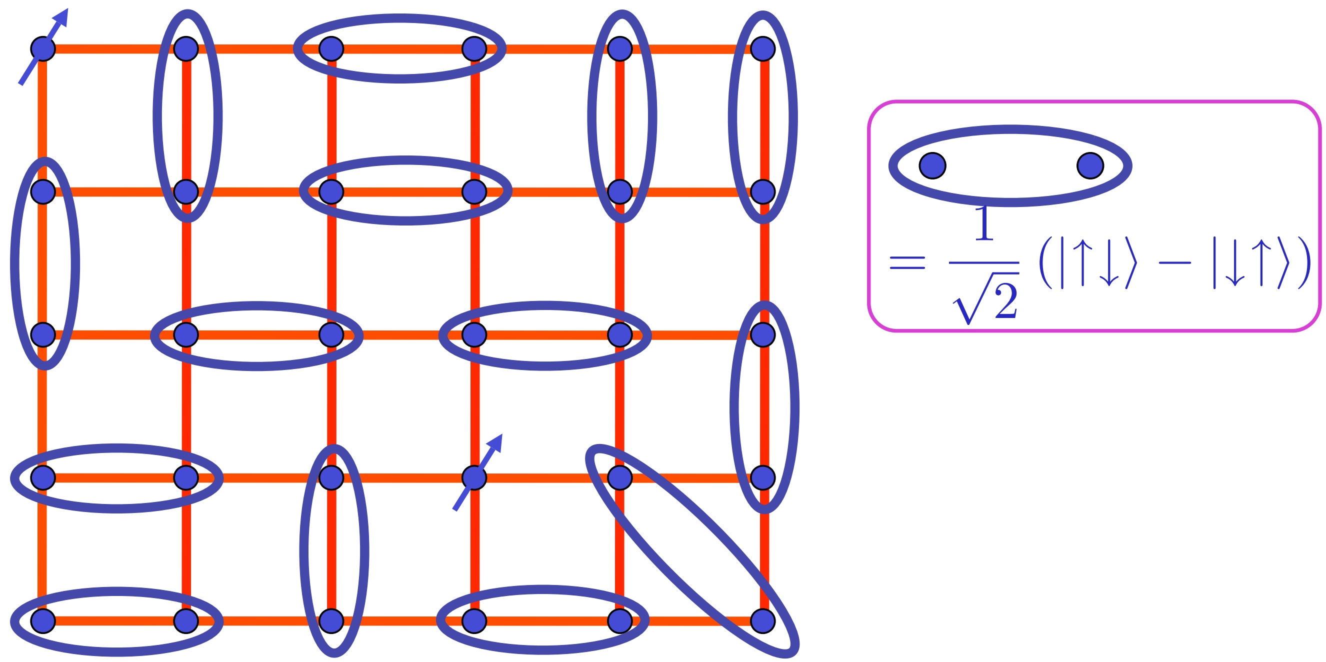

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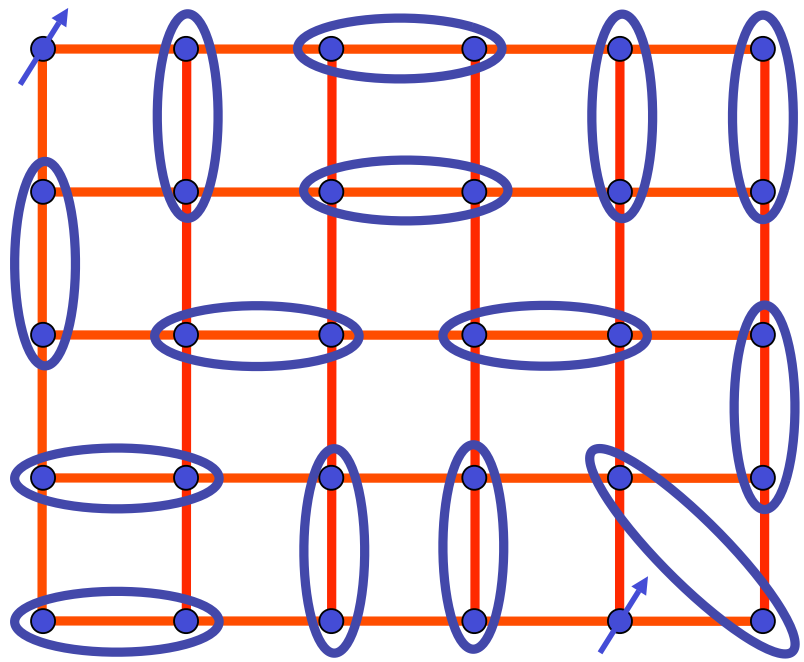


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Possible theory for fractionalization and topological order

Decompose the Néel order parameter into *spinors*

$$\vec{\varphi} = z_\alpha^* \vec{\sigma}_{\alpha\beta} z_\beta$$

where $\vec{\sigma}$ are Pauli matrices, and z_α are complex spinors which carry spin $S = 1/2$.

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Effective theory for spinons must be invariant under the U(1) gauge transformation

$$z_\alpha \rightarrow e^{i\theta} z_\alpha$$

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Naive expectation: Low energy spinon theory for "quantum disordering" a Néel state is

$$\mathcal{S}_z = \int d^2x d\tau \Big[c^2 \left|(\nabla_x - iA_x)z_\alpha\right|^2 + \left|(\partial_\tau - iA_\tau)z_\alpha\right|^2 + s\left|z_\alpha\right|^2 + u\left(|z_\alpha|^2\right)^2 + \frac{1}{4e^2}(\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda)^2 \Big]$$

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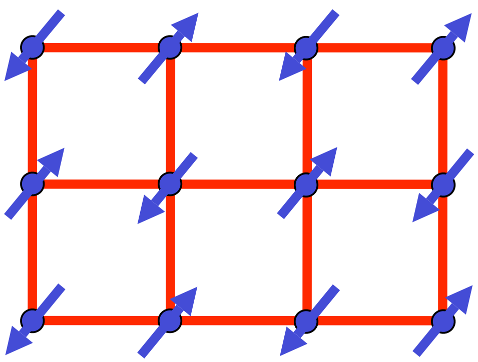

$\langle z_\alpha \rangle \neq 0$

Néel state

Spin liquid state with stable $S = 1/2$ z_α spinons, and a gapless U(1) photon A_μ representing the topological order.

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s_c s

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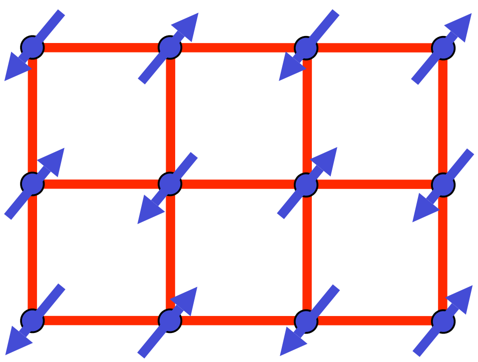

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Valence bond solid order order and spinon confinement eventually appear at the longest scales

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s

Outline

1. Quantum "disordering" magnetic order

   Collinear order and confinement

2. Z_2 spin liquids

   Noncollinear order and fractionalization

3. U(1) spin liquids

   Valence bond solid (VBS) order

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- Find a collective excitation Φ with the gauge transformation

$$\Phi \to e^{2i\theta}\Phi$$

- Higgs state with $\langle\Phi\rangle \neq 0$ is described by the fractionalized phase of a Z_2 gauge theory in the which the spinons z_α carry Z_2 gauge charges (E. Fradkin and S. Shenker, Phys. Rev. D **19**, 3682 (1979)).

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- *What is Φ in the antiferromagnet* ? Its physical interpretation becomes clear from its allowed coupling to the spinons:

$$\mathcal{S}_{z,\Phi} = \int d^2 r d\tau \left[\lambda \Phi^* \epsilon_{\alpha\beta} z_\alpha \partial_x z_\beta + \text{c.c.}\right]$$

From this coupling it follows that the states with $\langle\Phi\rangle \neq 0$ have **coplanar spin correlations**.

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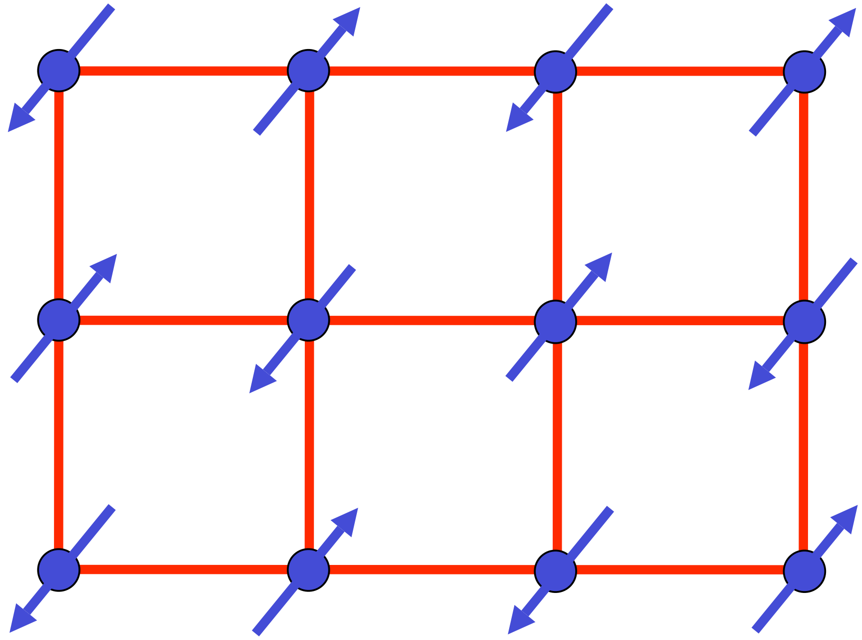
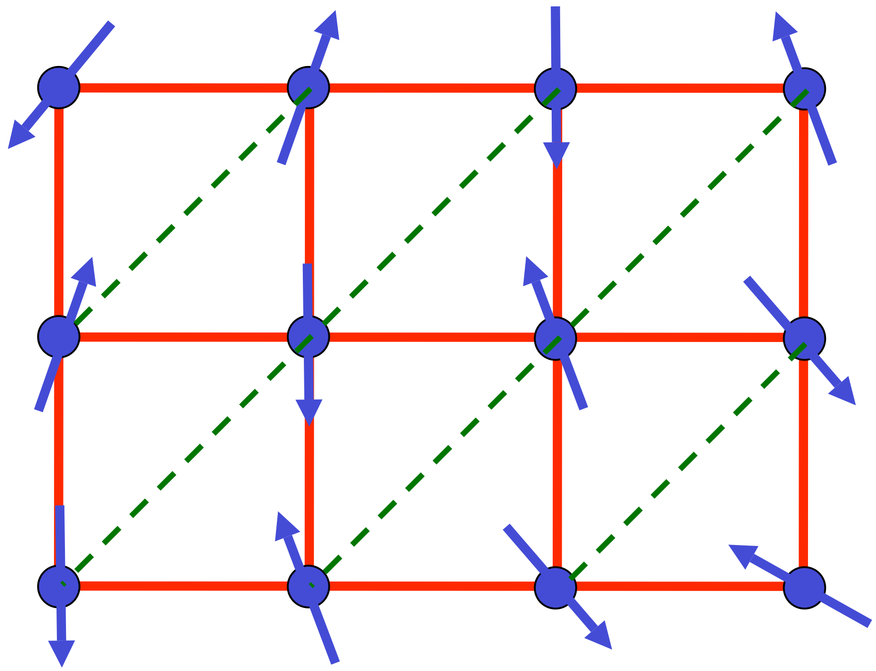


Collinear magnetic order with $\langle \Phi \rangle = 0$.

A spin density wave:

$$\langle \vec{S}_i \rangle \propto (\cos(\mathbf{K} \cdot \mathbf{r}_i), \sin(\mathbf{K} \cdot \mathbf{r}_i), 0)$$

$$\mathbf{K} = (\pi, \pi).$$



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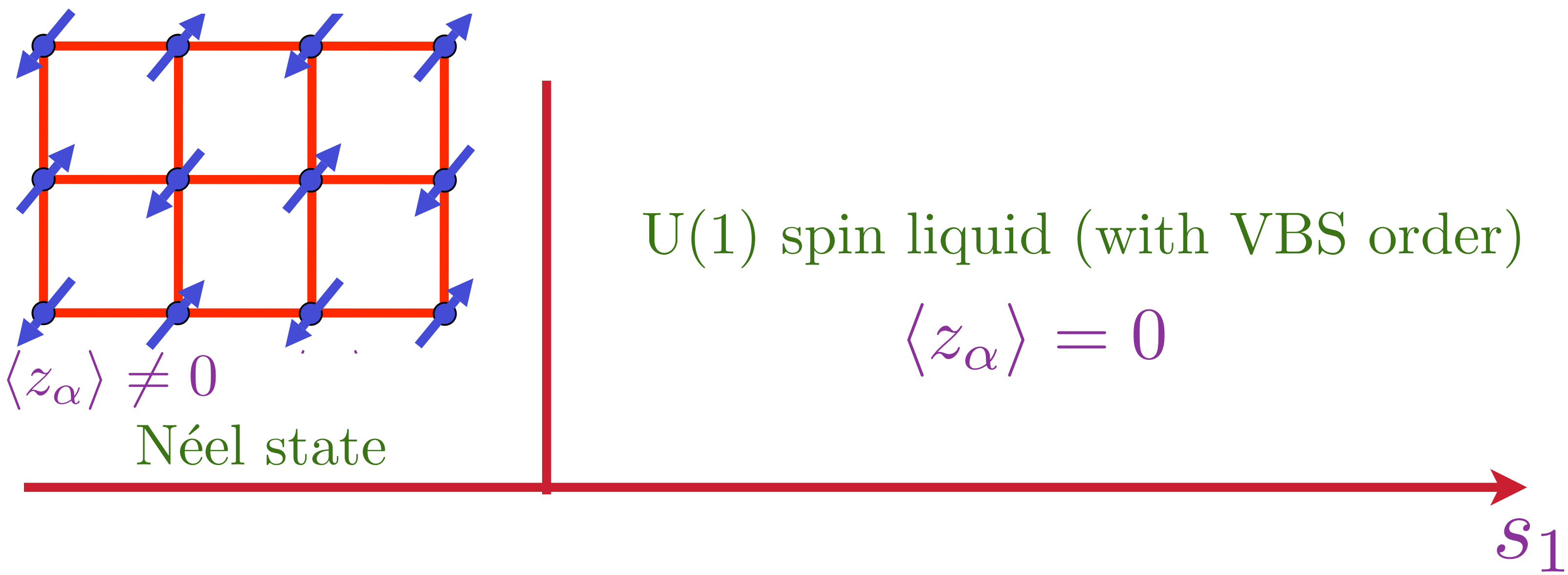
Experimental realization: $CsCuCl_3$

Phase diagram of gauge theory of spinons

$$\mathcal{S}_z = \int d^2x d\tau \left[\left|(\partial_\mu - iA_\mu) z_\alpha\right|^2 + s_1 \left|z_\alpha\right|^2 + u \left(\left|z_\alpha\right|^2\right)^2 + \frac{1}{4e^2}(\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda)^2 \right.$$



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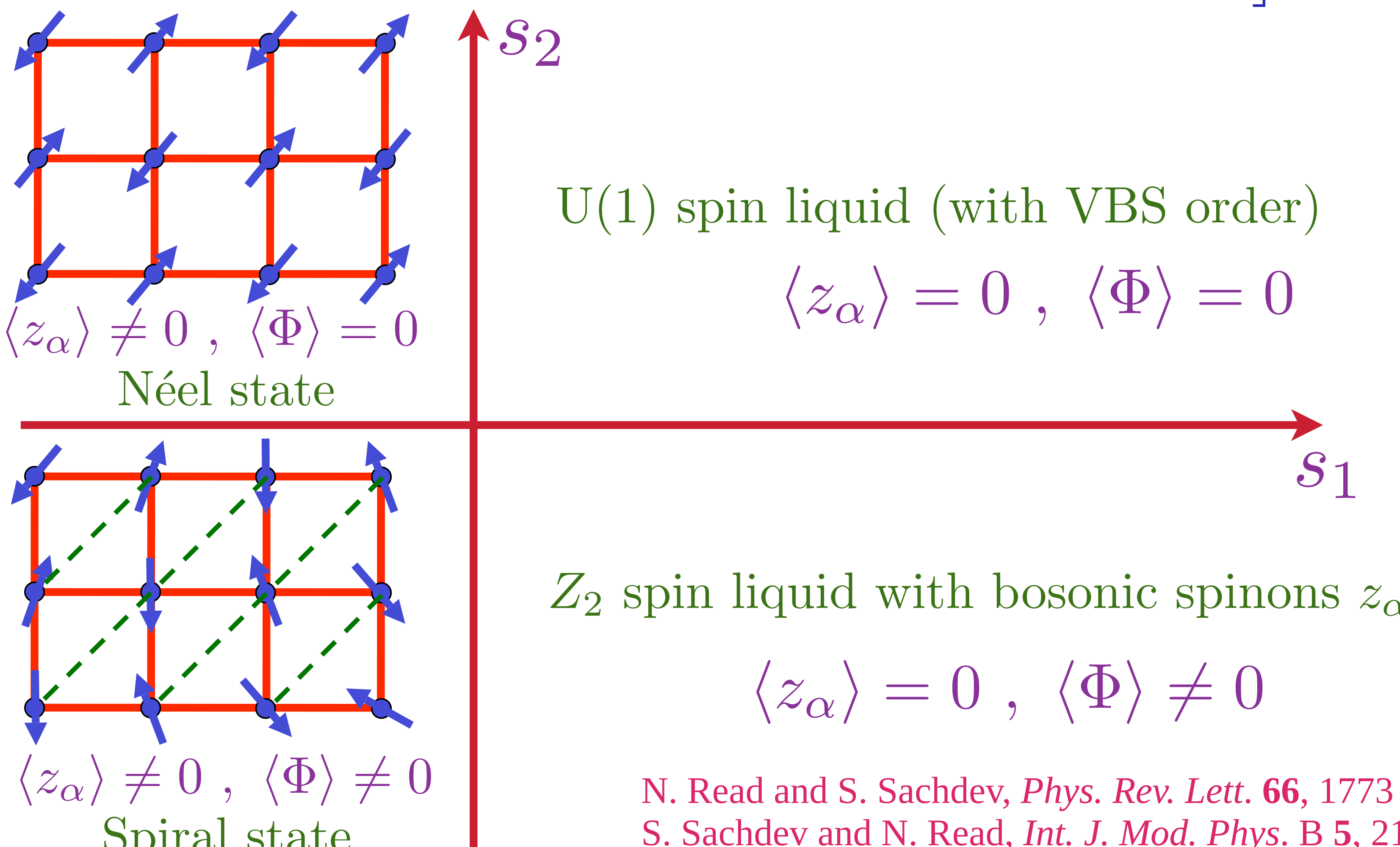

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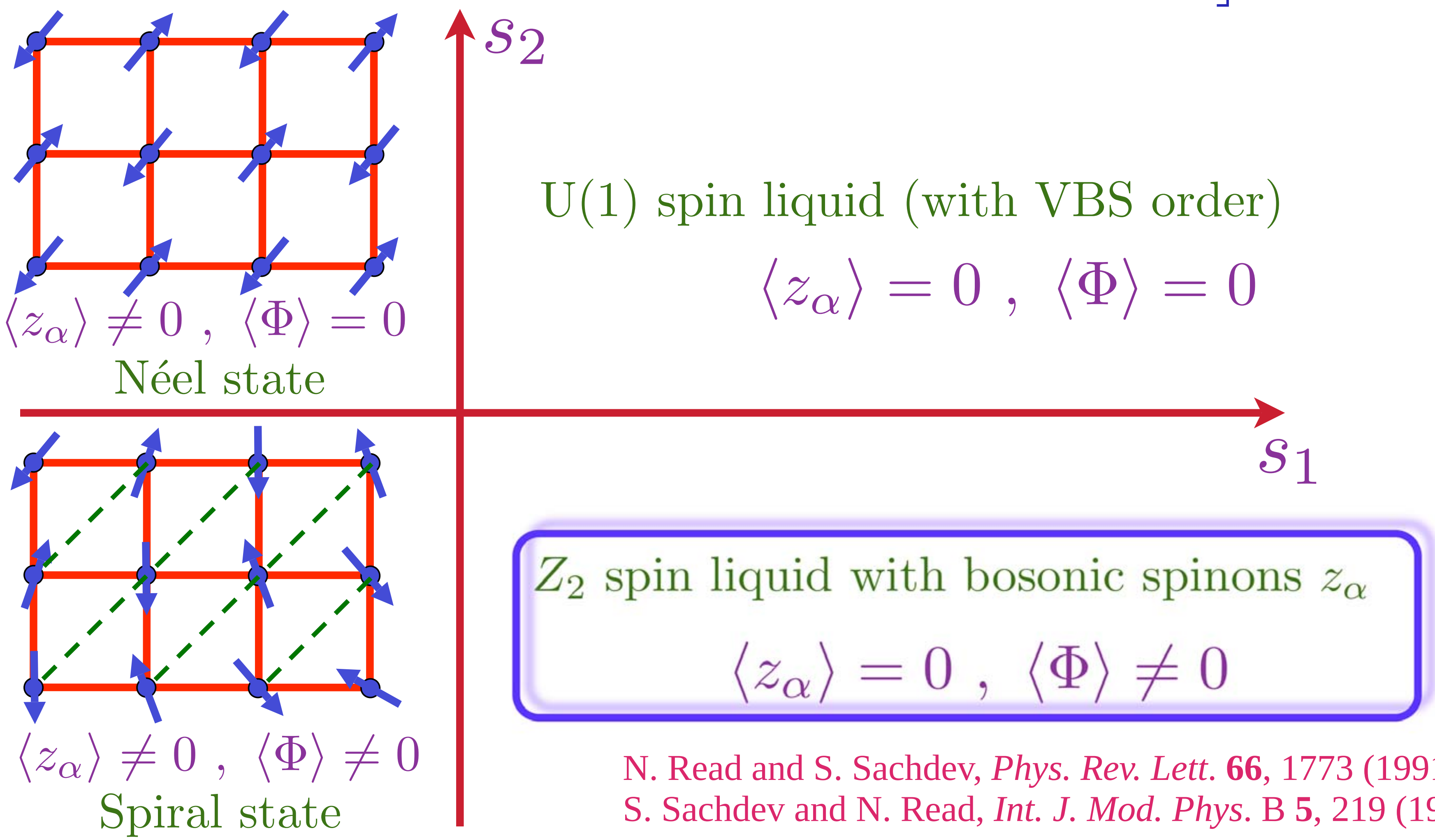

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Characteristics of Z_2 spin liquid

- Two classes of gapped excitations:
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  - Z_2 vortex associated with $2\pi n$ winding in phase of Φ. This vortex appears as a π flux-tubes to spinons

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- Same states (without spinons) and Z_2 gauge theories found to describe liquid phases of quantum dimer models (R. Moessner and S. L. Sondhi, Phys. Rev. Lett. **86**, 1881 (2001).

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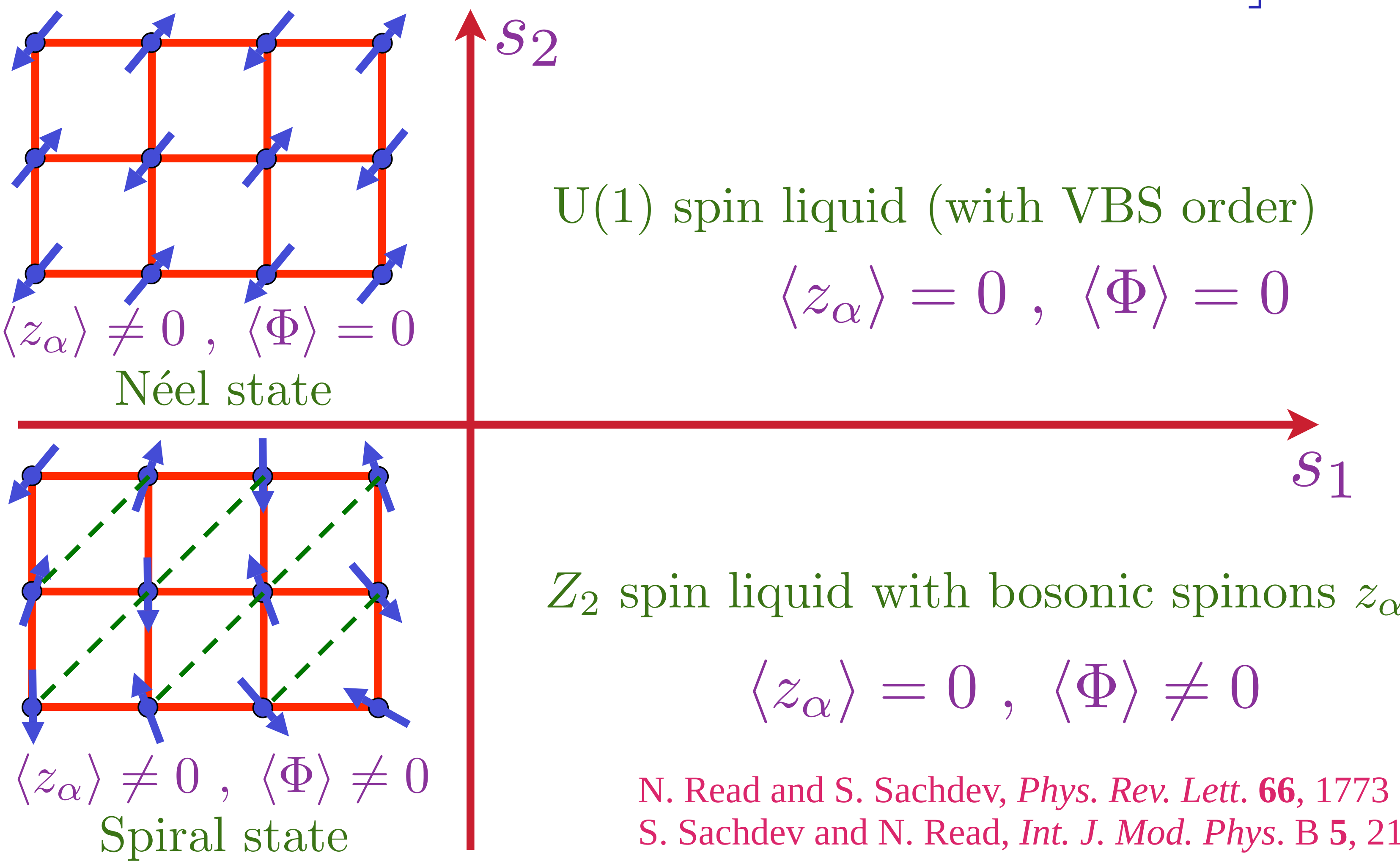

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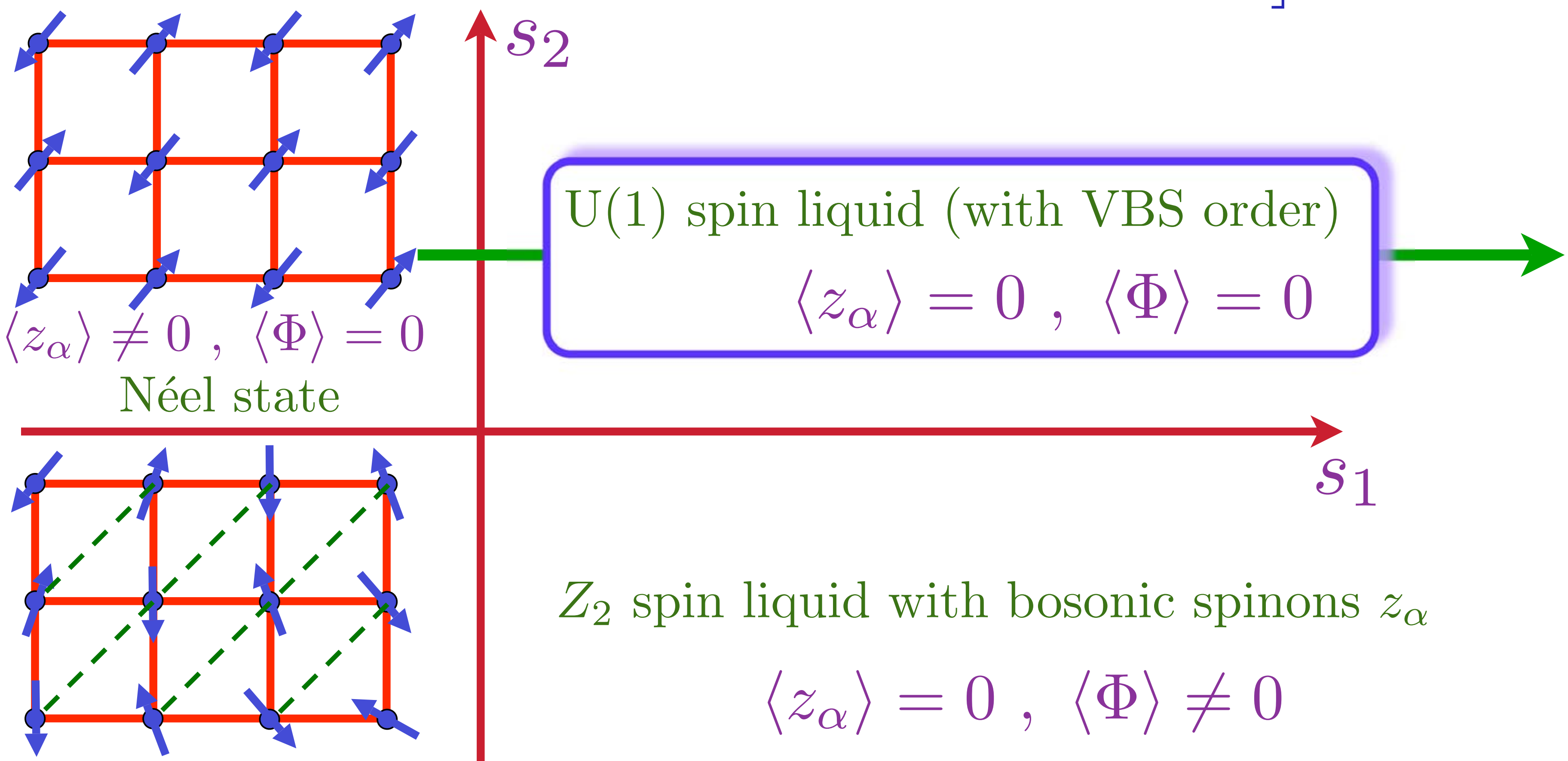

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Duality mapping:
The low energy theory for the U(1) spin liquid state is obtained by neglecting the gapped spinons. So the low energy theory is contains just a free "photon":

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We decouple this to

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Integrate over A_μ to obtain constraint $\epsilon_{\mu\nu\lambda} \partial_\nu J_\lambda = 0$. Solve this constraint by $J_\mu = \partial_\mu \chi$ to obtain the **dual theory**

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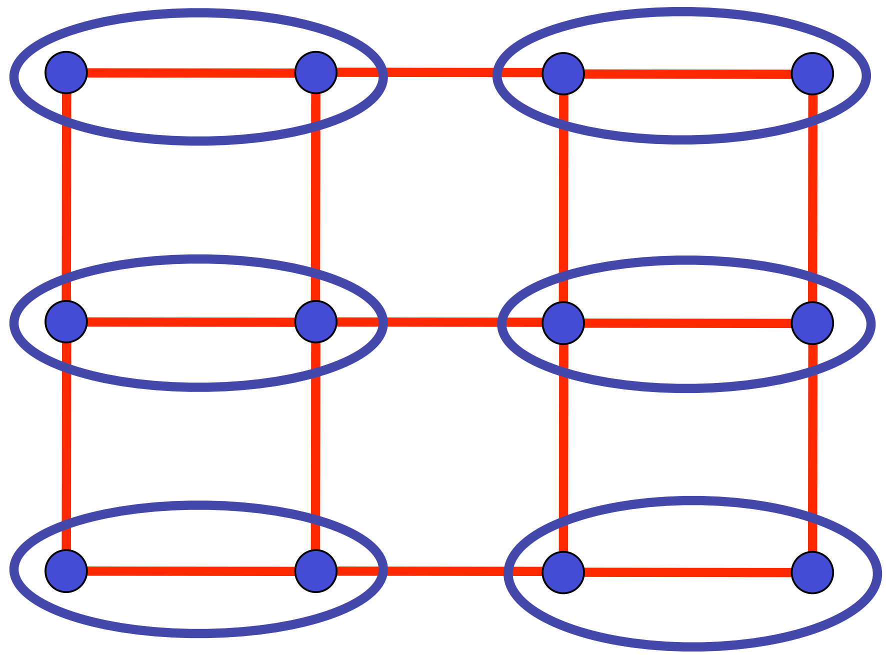
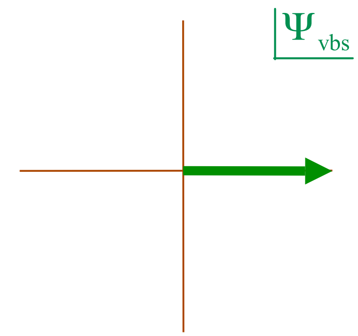
This theory has a global shift symmetry $\chi \to \chi + \text{constant}$.

What is the physical interpretation of this shift symmetry ?

Characterization of VBS state with $\langle \vec{\varphi} \rangle = 0$





Such a state breaks the symmetry of rotations by $n\pi/2$ about lattice sites, and has $\langle \Psi_{\text{vbs}} \rangle \neq 0$, where Ψ_{vbs} is the ***VBS order parameter***

$$\Psi_{\text{vbs}}(i) = \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j e^{i \arctan(\mathbf{r}_j - \mathbf{r}_i)}$$

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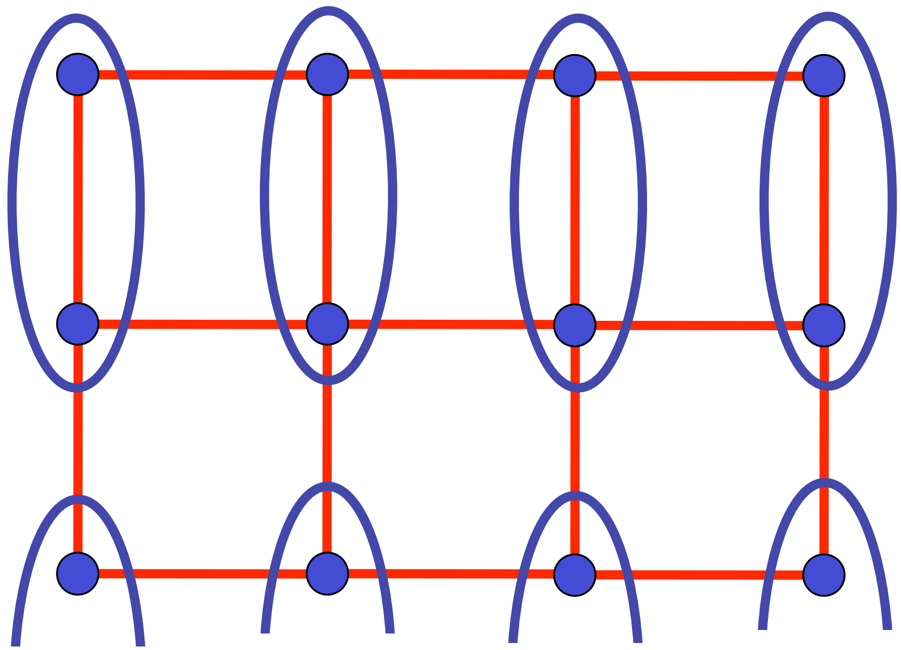

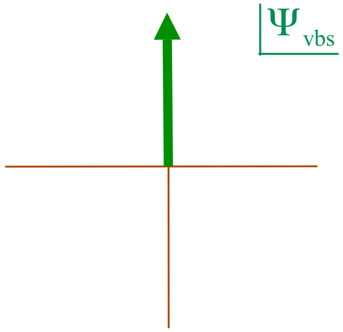

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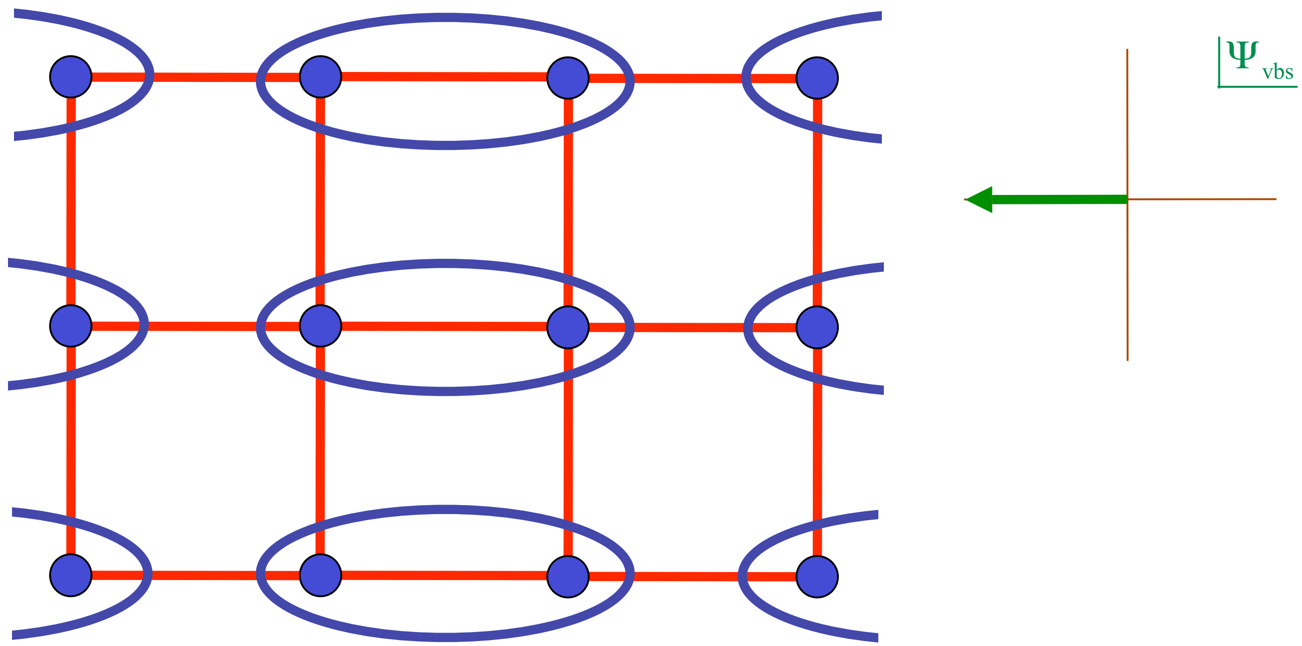

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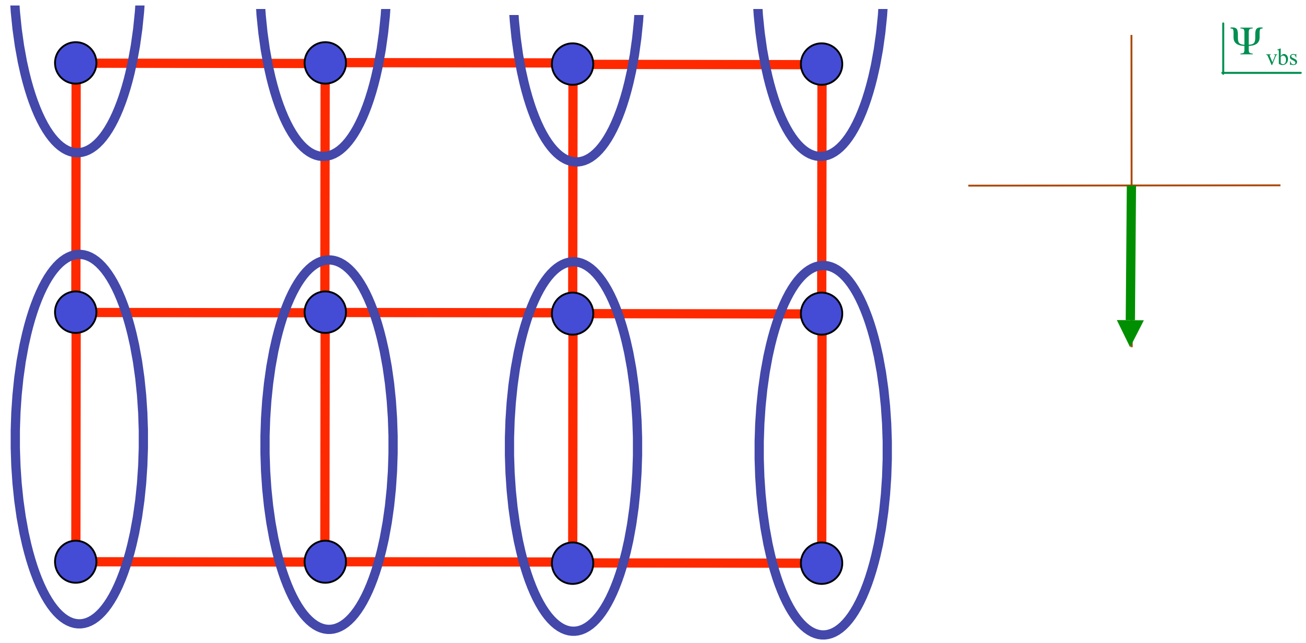

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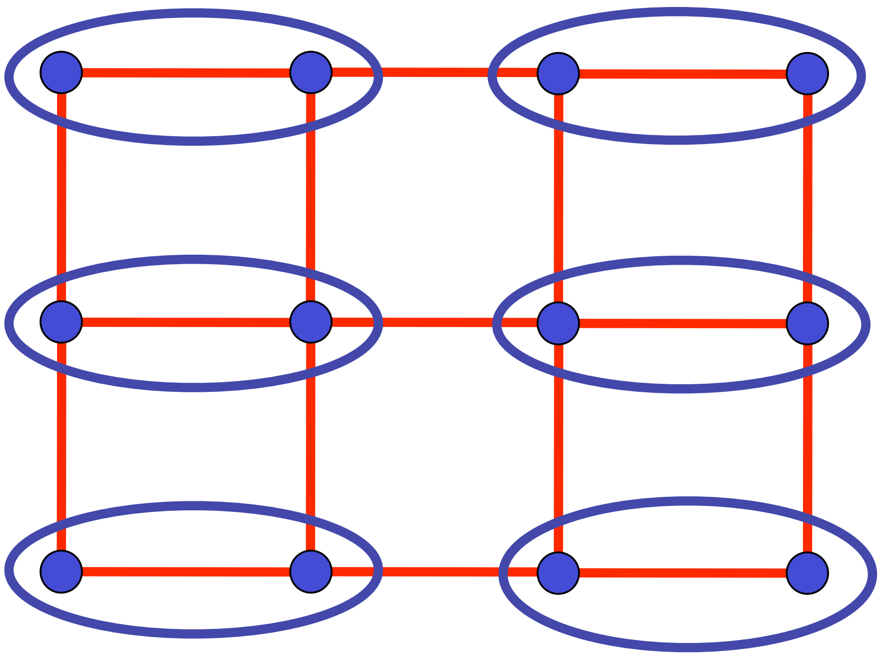

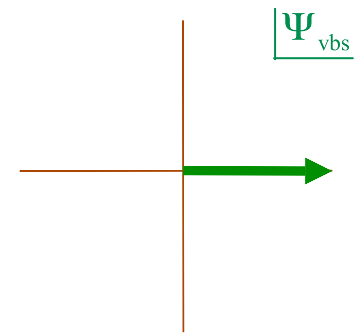

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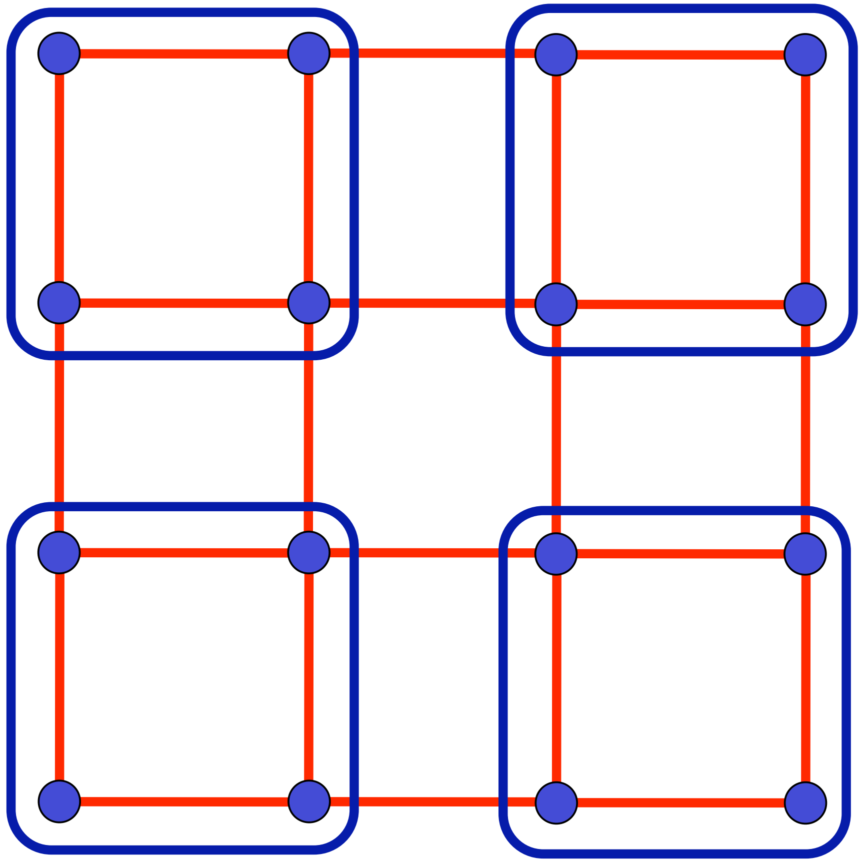

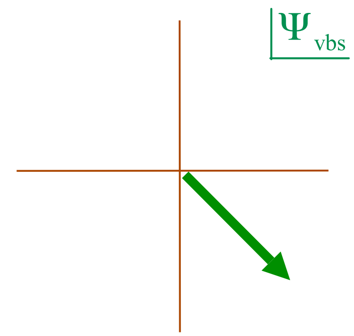

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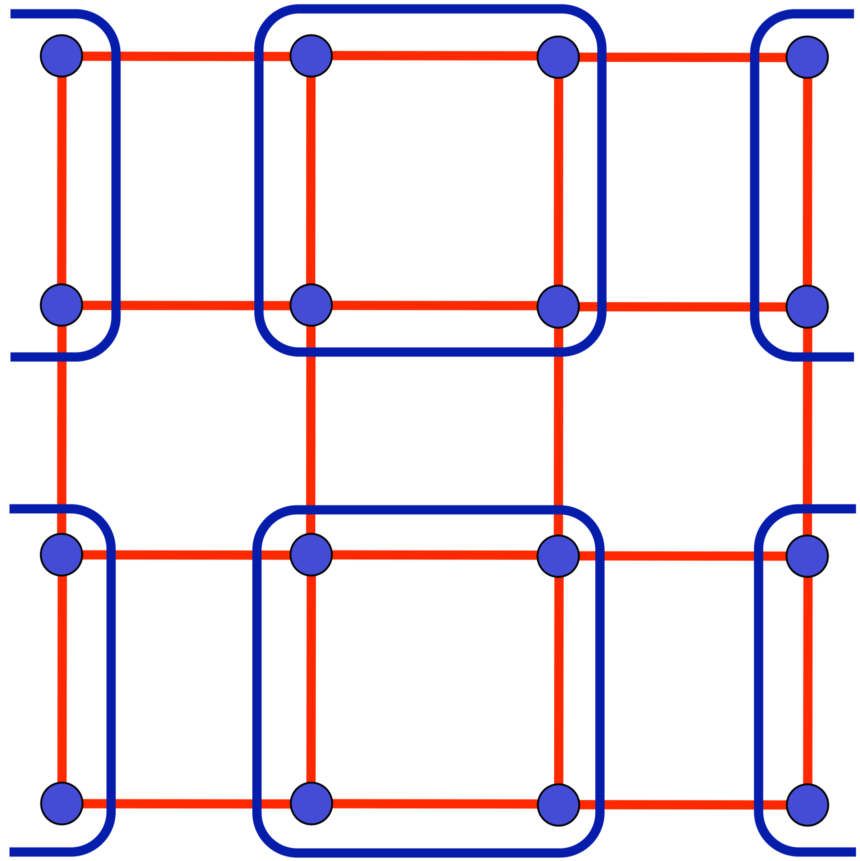

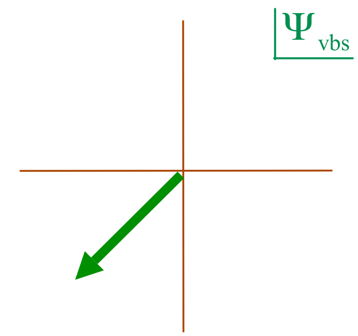

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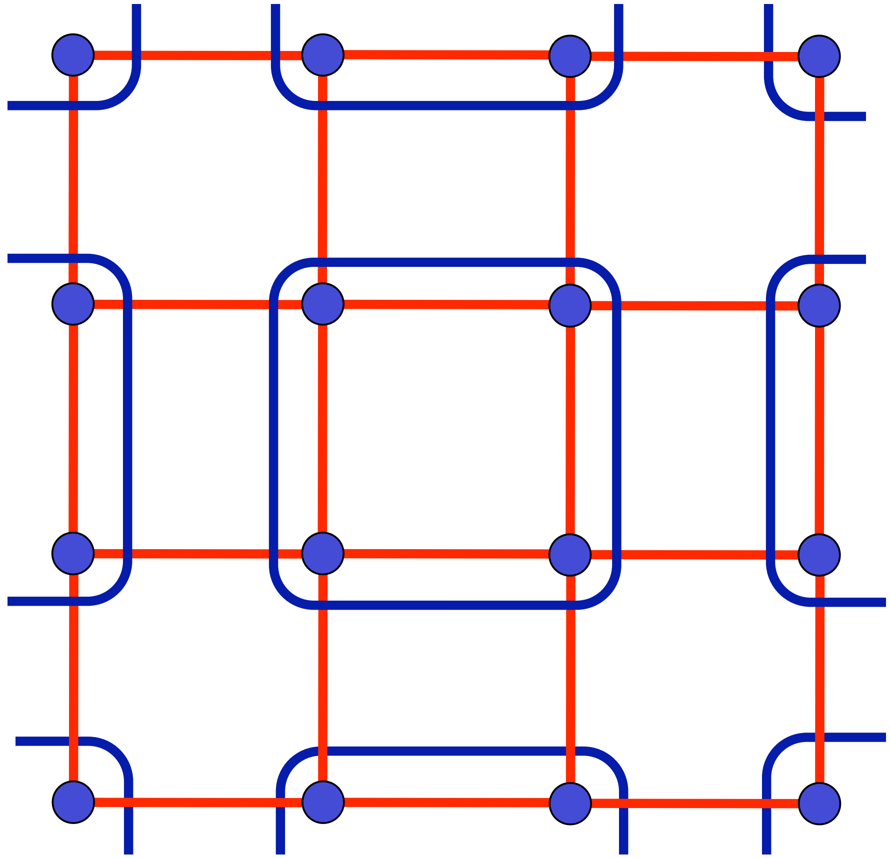
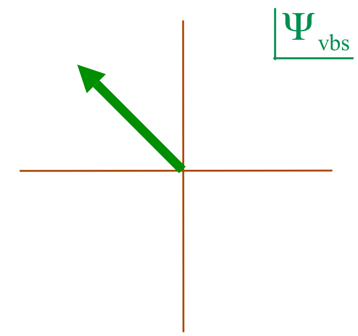
Characterization of VBS state with $\langle \vec{\varphi} \rangle = 0$





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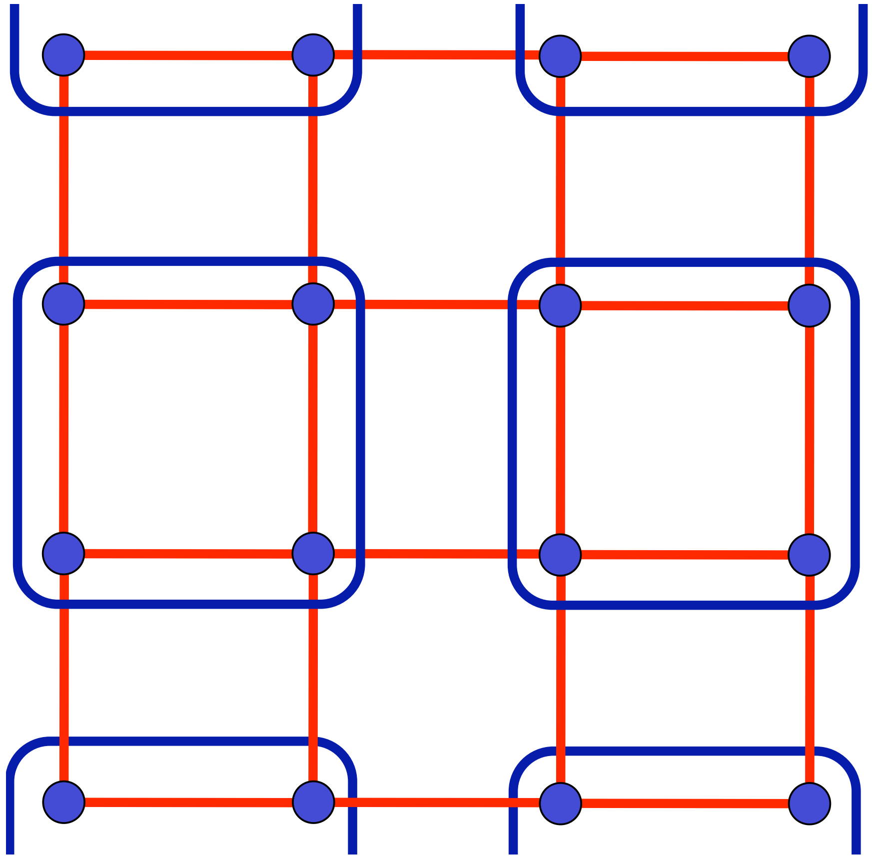

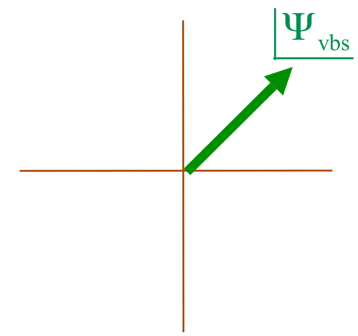

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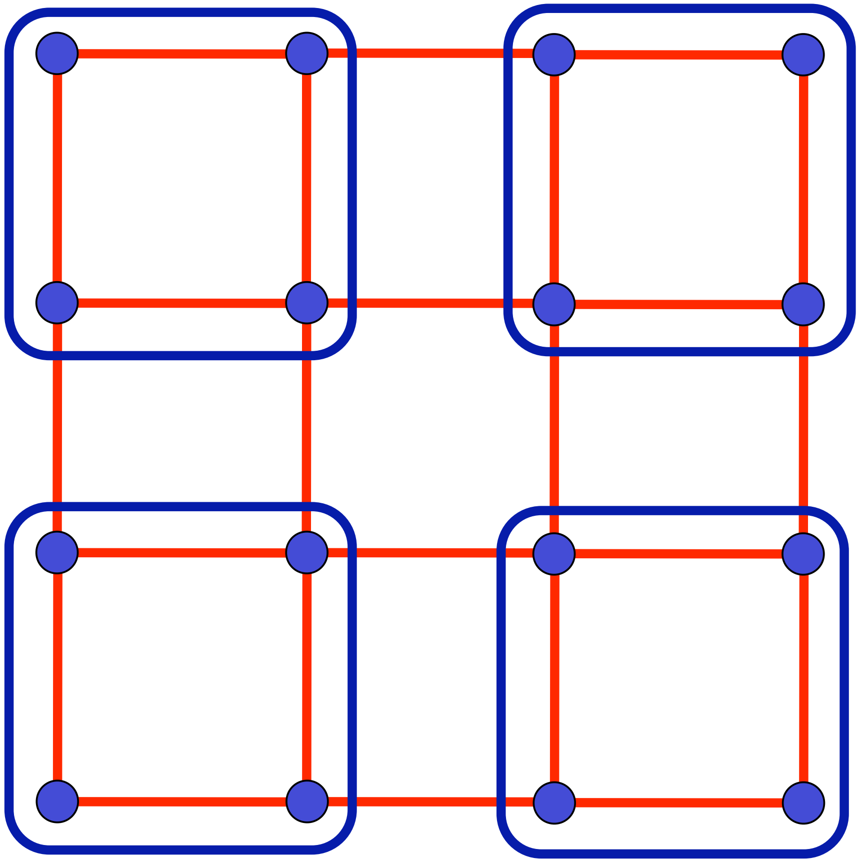

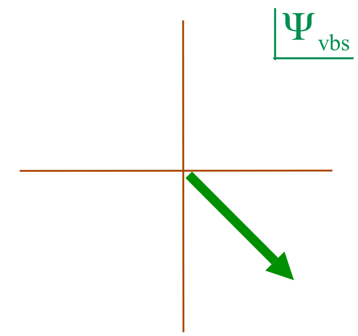

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Consequences of Berry phases:

- We obtain the fundamental equivalence

$$\boxed{\exp\left(i\frac{2\pi\chi}{e_0^2}\right) \sim \Psi_{\text{vbs}}.}$$

- The continuous shift symmetry $\chi \to \chi + c$ is an enlargement of the Z_4 lattice rotation symmetry of the square lattice.

N. Read and S. Sachdev, *Phys. Rev. Lett*. **62**, 1694 (1990)
T. Senthil, A. Vishwanath, L. Balents, S. Sachdev and M.P.A. Fisher, *Science* **303**, 1490 (2004).

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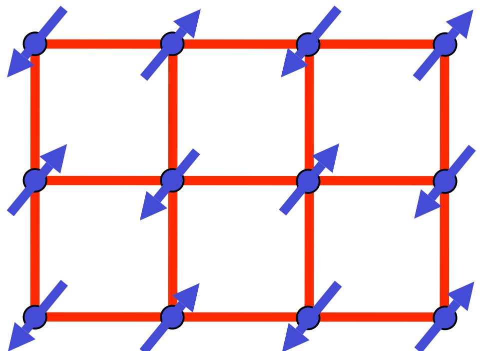
Phase diagram of gauge theory of spinons

$$\mathcal{S}_{z,\Phi} = \int d^2x d\tau \Big[\left|(\partial_\mu - iA_\mu)z_\alpha\right|^2 + s_1 \left|z_\alpha\right|^2 + u \left(|z_\alpha|^2\right)^2 + \frac{1}{4e^2}(\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda)^2$$

$$+ |(\partial_\mu - 2iA_\mu)\Phi|^2 + s_2|\Phi|^2 + \tilde{u}|\Phi|^4 + \lambda\Phi^*\epsilon_{\alpha\beta}z_\alpha\partial_x z_\beta + \text{c.c.} \Big]$$

s_2



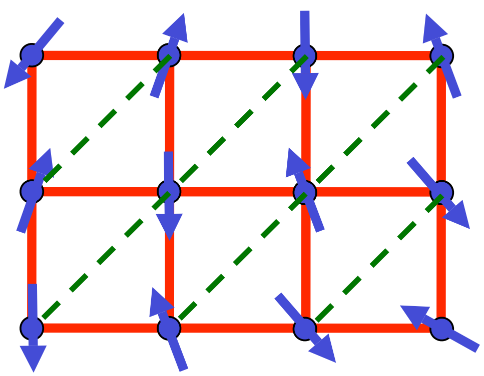
$\langle z_\alpha \rangle \neq 0 \; , \; \langle \Phi \rangle = 0$

Néel state

U(1) spin liquid (with VBS order)

$\langle z_\alpha \rangle = 0 \; , \; \langle \Phi \rangle = 0$

s_1



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Spiral state

Z_2 spin liquid with bosonic spinons z_α

$\langle z_\alpha \rangle = 0 \; , \; \langle \Phi \rangle \neq 0$

N. Read and S. Sachdev, *Phys. Rev. Lett.* **66**, 1773 (1991)

S. Sachdev and N. Read, *Int. J. Mod. Phys.* B **5**, 219 (1991)

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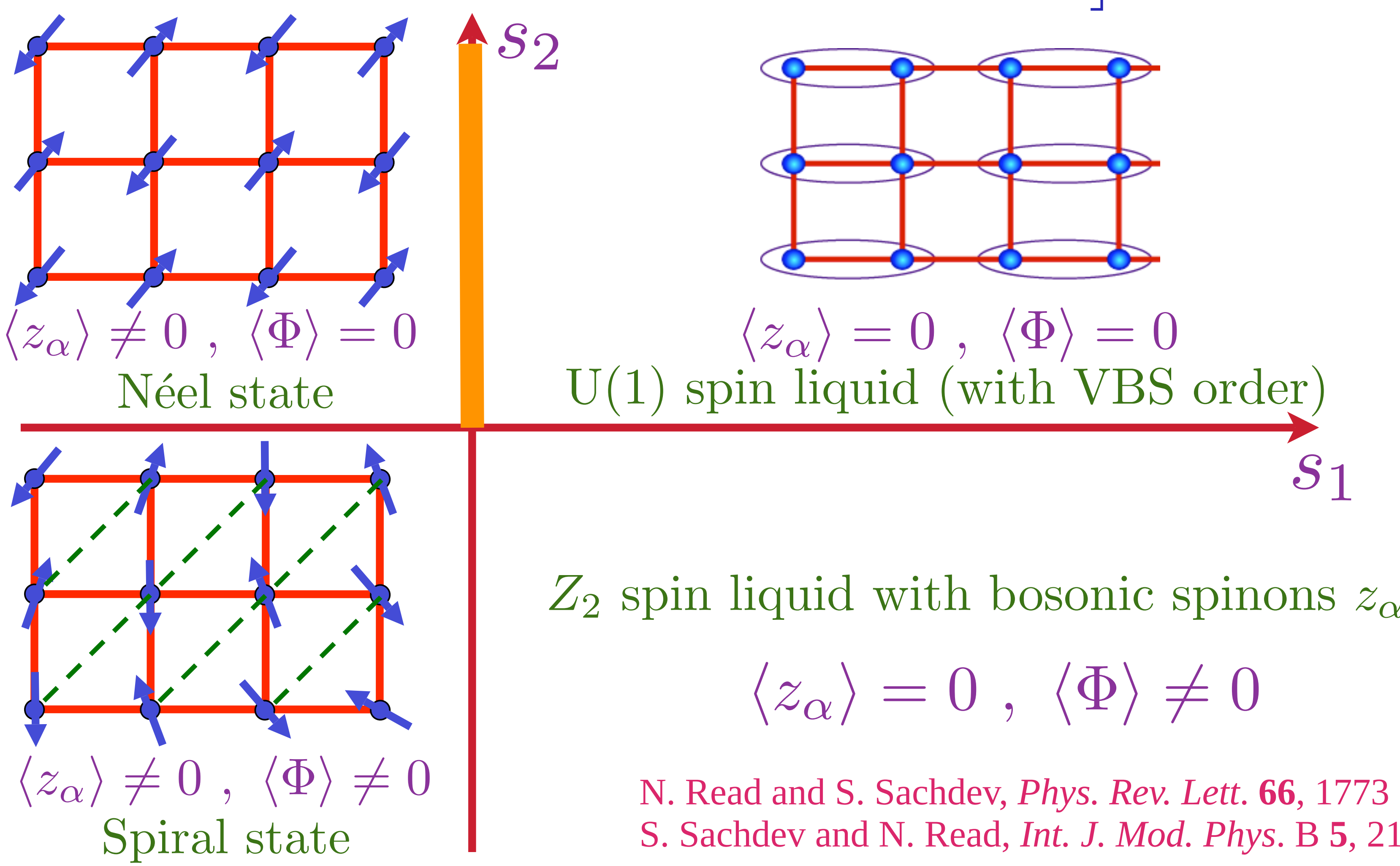

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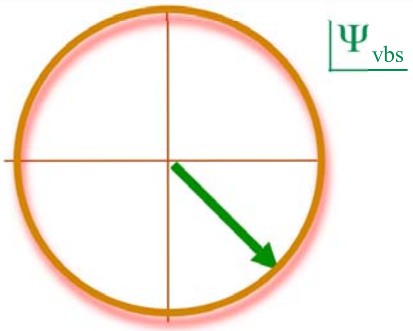
- **Key consequences**

  - The Z_4 lattice rotation symmetry is spontaneously broken, leading to long-range VBS order
  - Near the critical point, where the photon is (nearly) gapless, the fluctuations of $\Psi_{\rm vbs}$ have a *circular symmetry.*



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$$\mathcal{H}_{\mathrm{SU}(2)} = J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j - Q \sum_{\langle ijkl \rangle} \left(\mathbf{S}_i \cdot \mathbf{S}_j - \tfrac{1}{4}\right) \left(\mathbf{S}_k \cdot \mathbf{S}_l - \tfrac{1}{4}\right)$$

Quantum Monte Carlo simulations display convincing evidence for a transition from a

Neel state at small Q

to a

VBS state at large Q

A.W. Sandvik, *Phys. Rev. Lett.* **98**, 2272020 (2007).
R.G. Melko and R.K. Kaul, arXiv:0707.2961;
F.-J. Jiang, M. Nyfeler, S. Chandrasekharan, and U.-J. Wiese, arXiv:0710.3926

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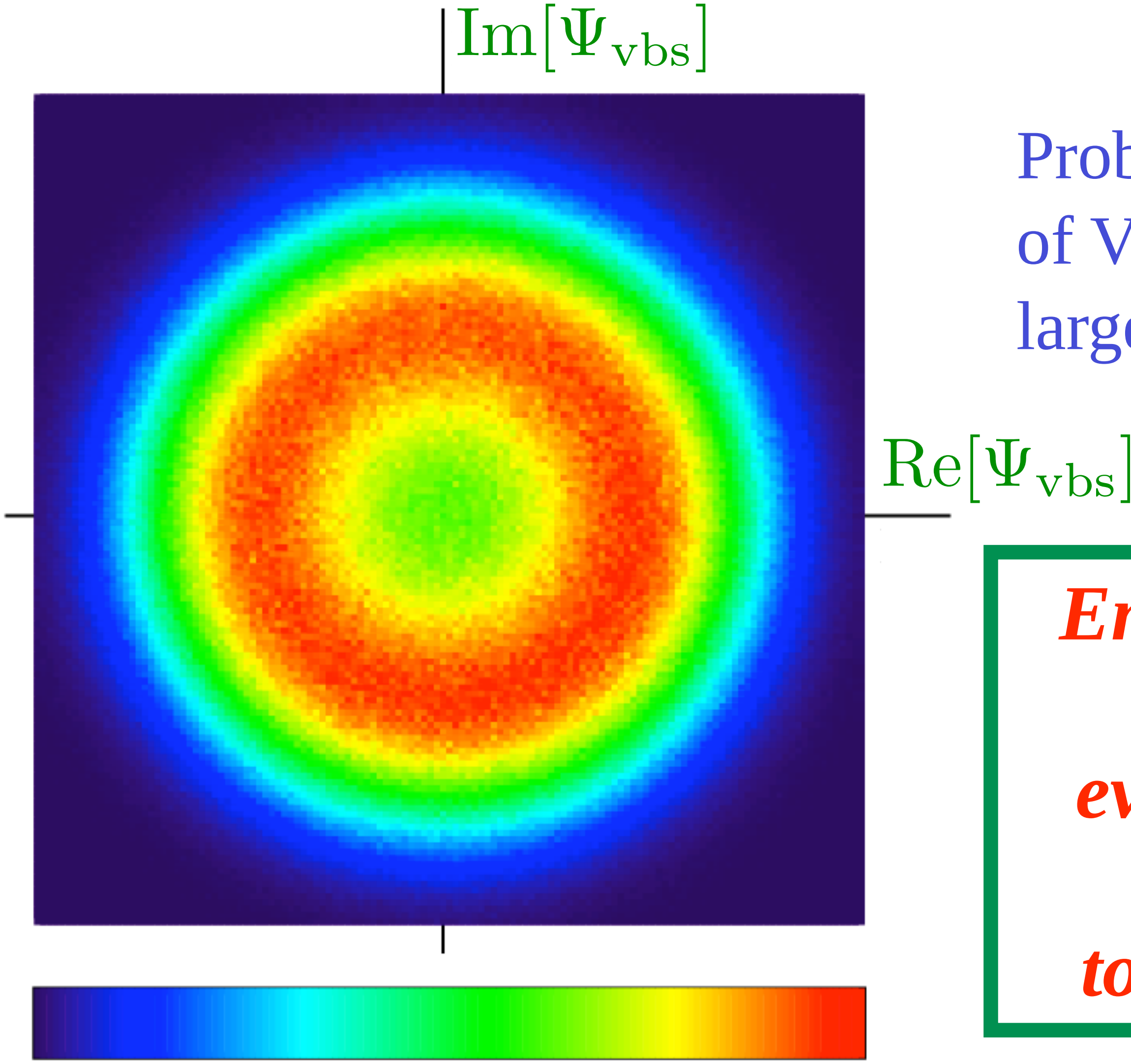

Probability distribution of VBS order Ψ_{vbs} at large Q

Emergent circular symmetry is evidence for U(1) photon and topological order

A.W. Sandvik, *Phys. Rev. Lett.* **98**, 2272020 (2007).

Conclusions