Topological order in insulators and metals

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New Horizons in Quantum Matter
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Subir Sachdev

Talk online: sachdev.physics.harvard.edu

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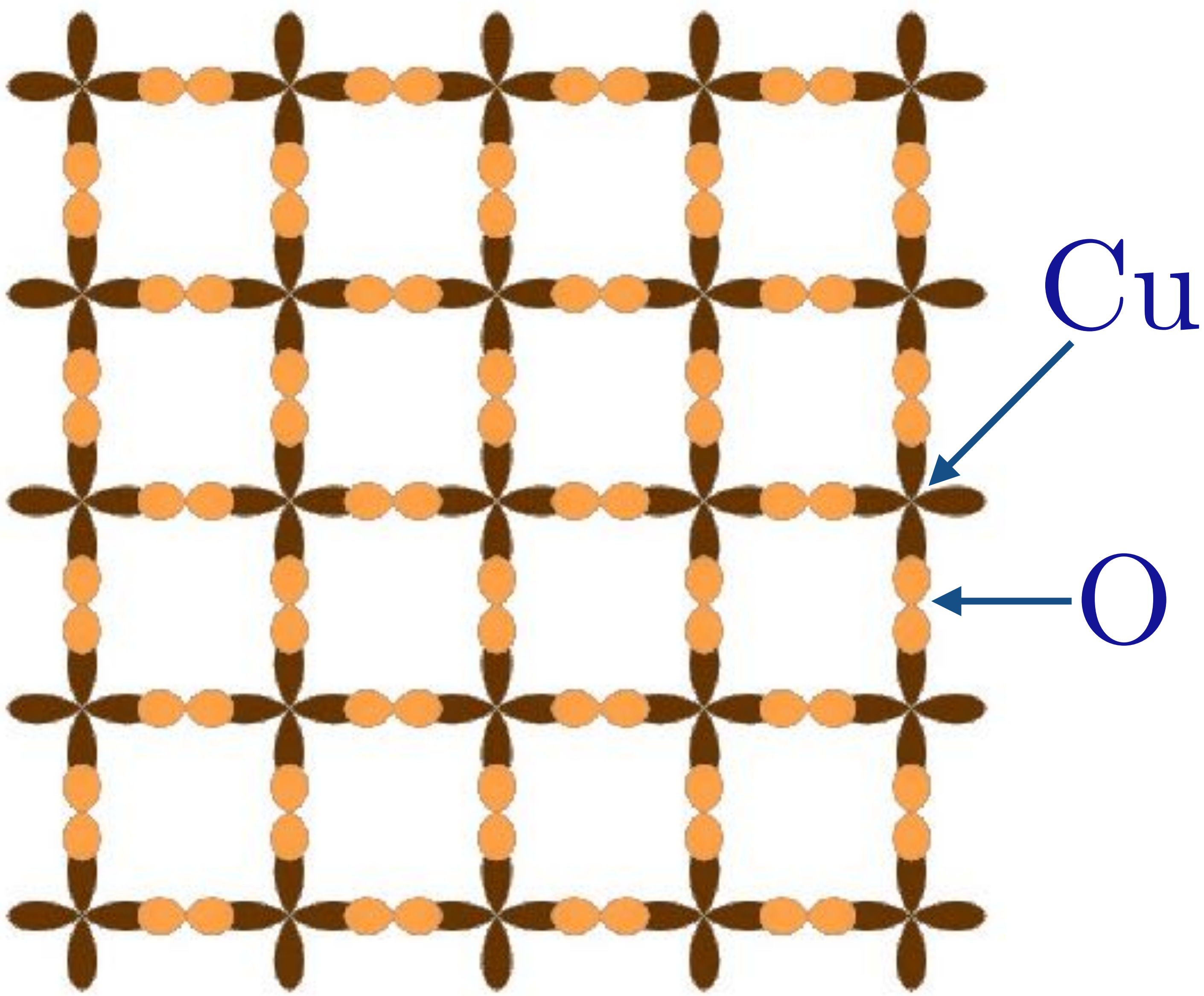
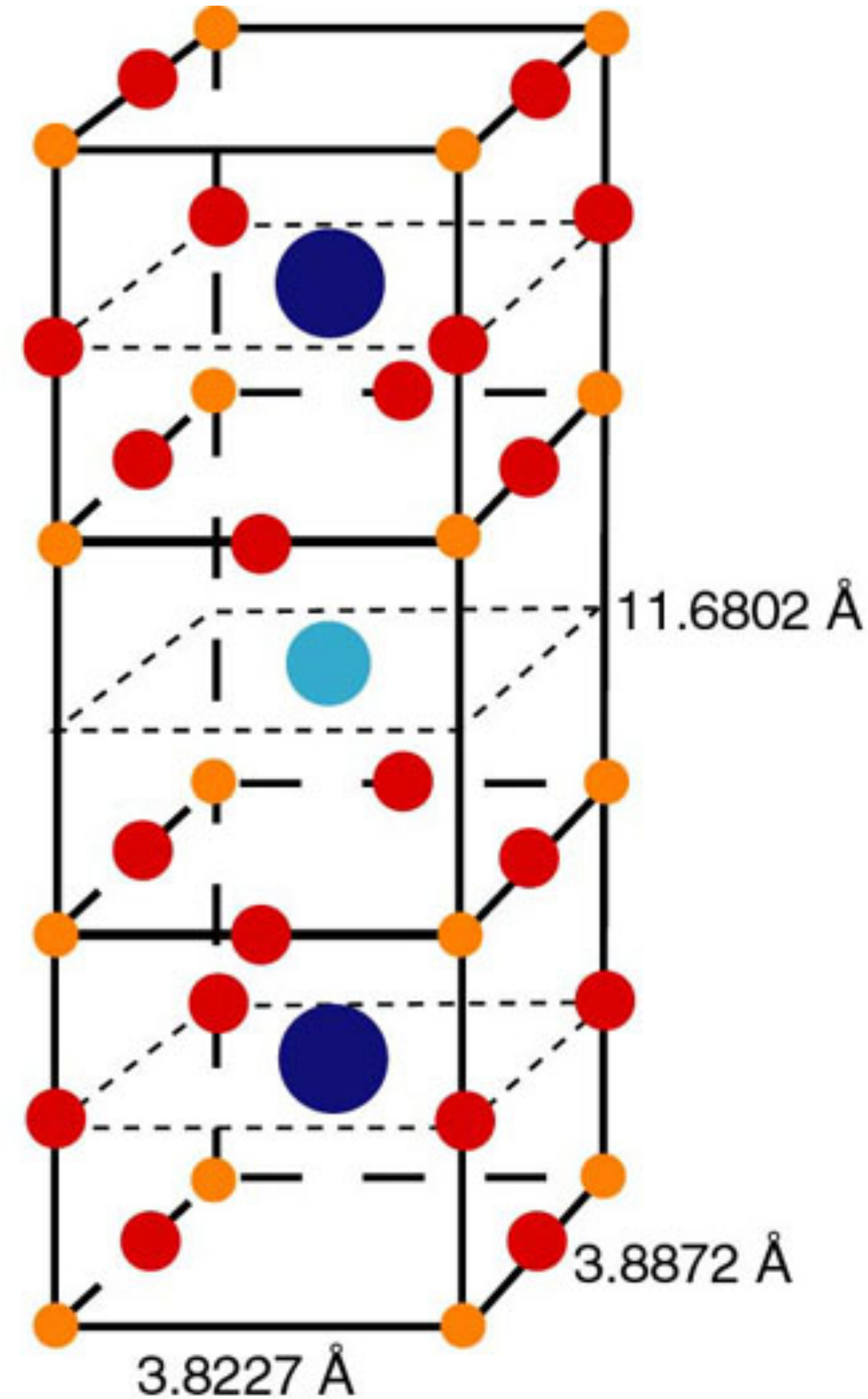
High temperature superconductors

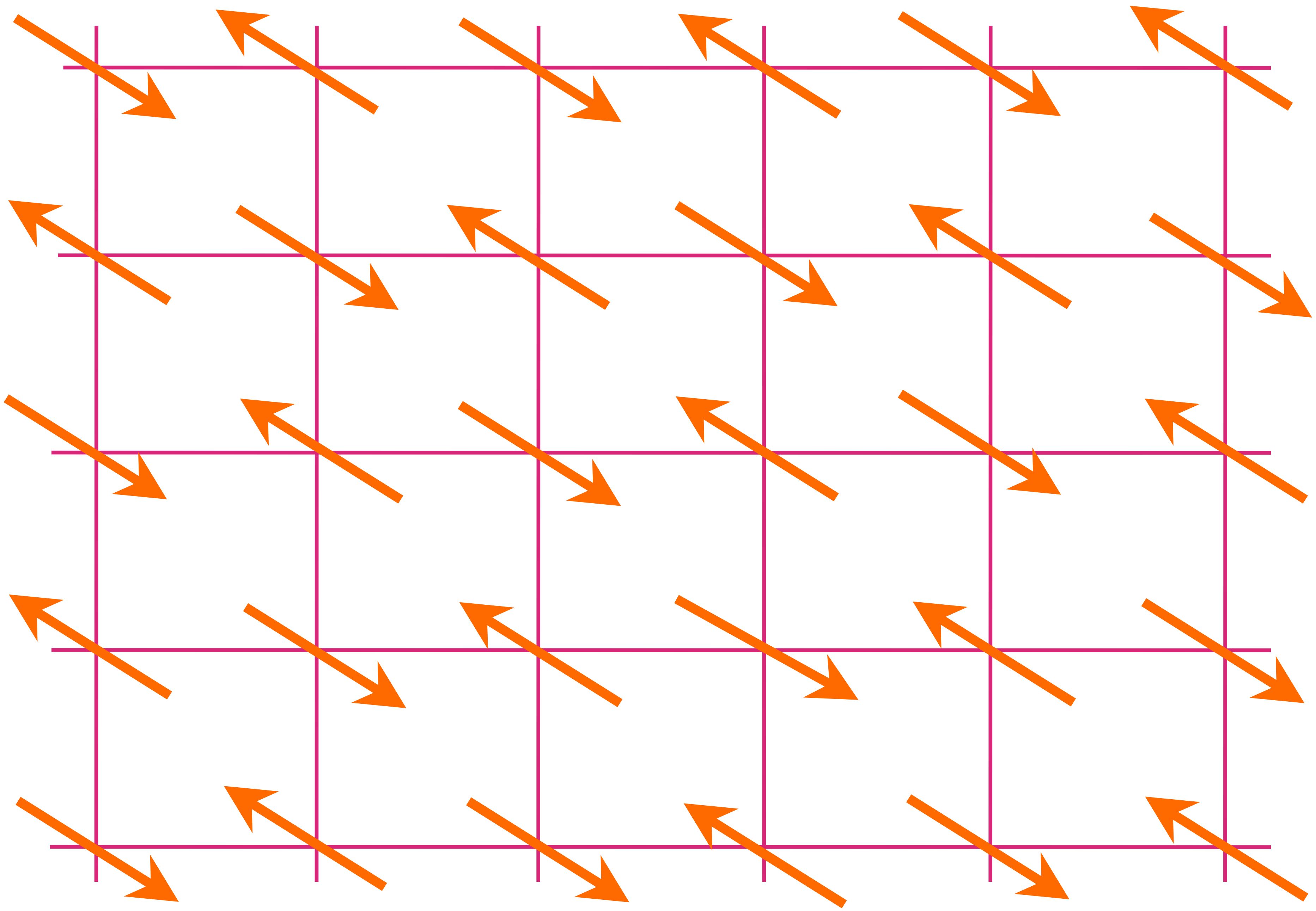


CuO_2 plane



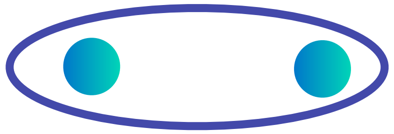
$YBa_2Cu_3O_{6+x}$

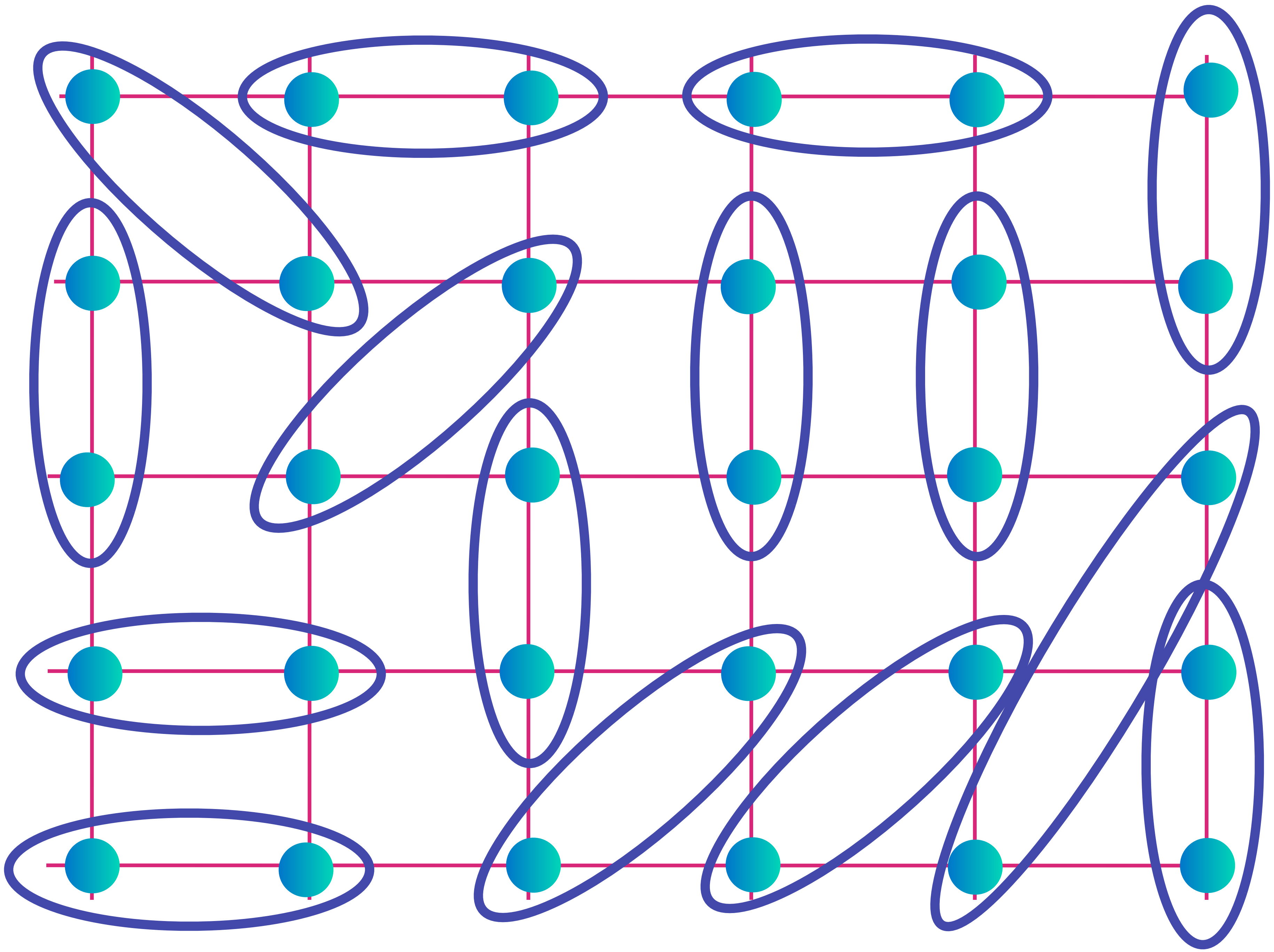
$$H = J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j$$



"Undoped" insulating anti-ferromagnet

Insulating spin liquid

 $= \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) / \sqrt{2}$



Lattice realization of a topological quantum field theory (TQFT)

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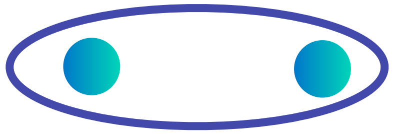
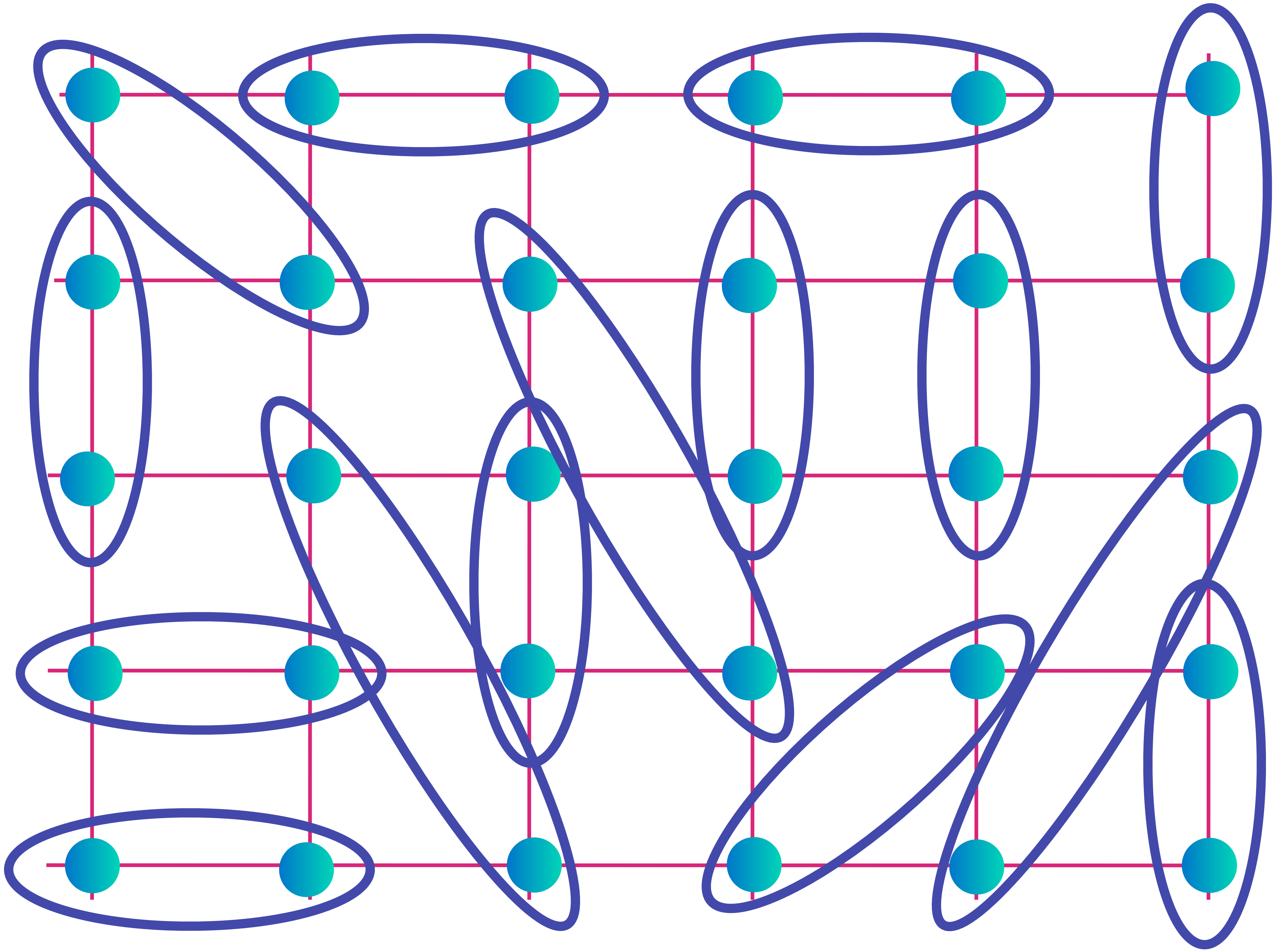
 $= \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)/\sqrt{2}$



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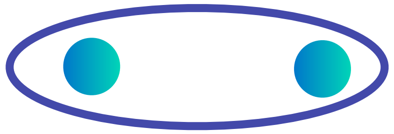
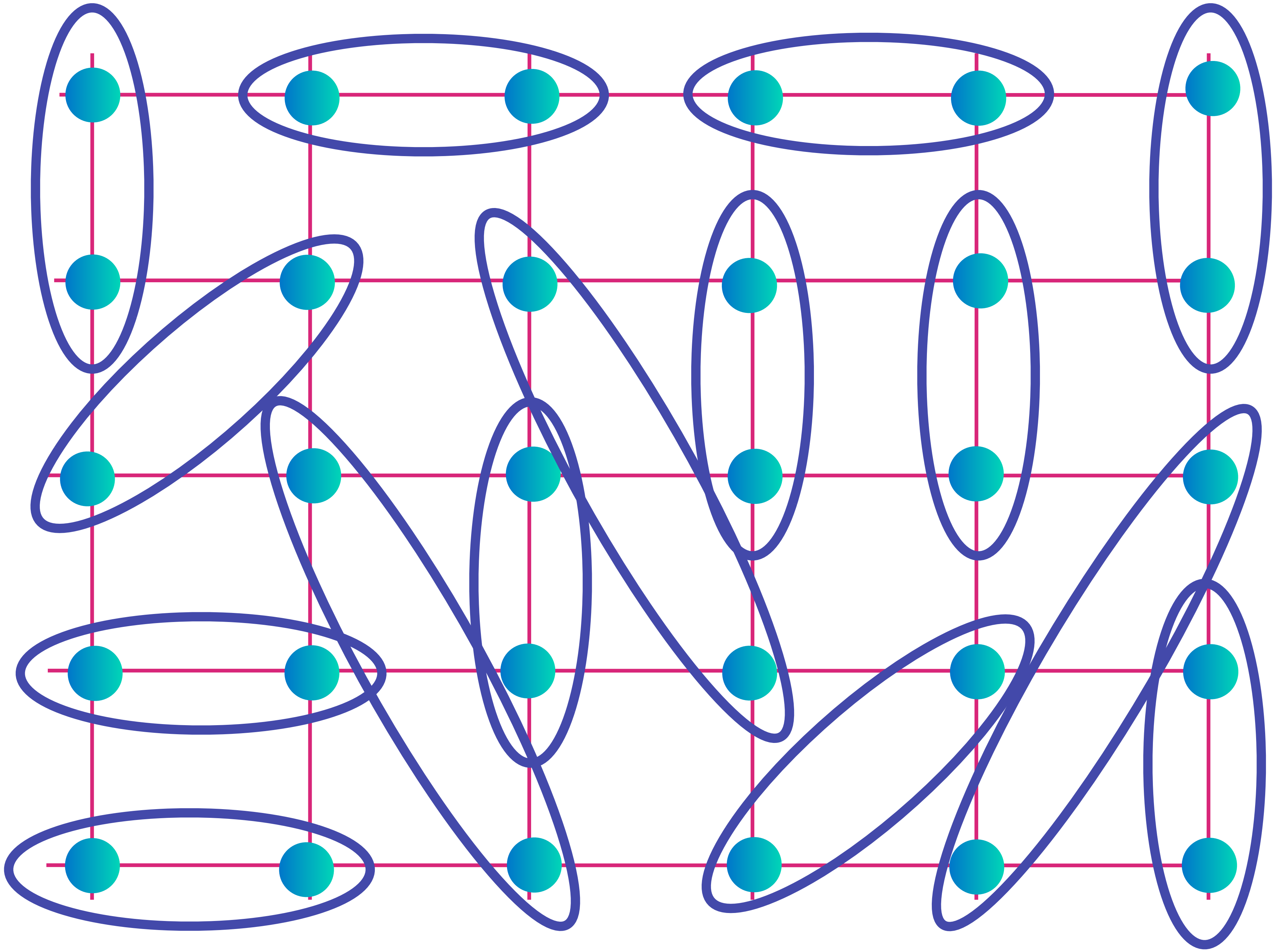
 $= \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right) / \sqrt{2}$



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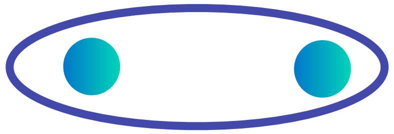

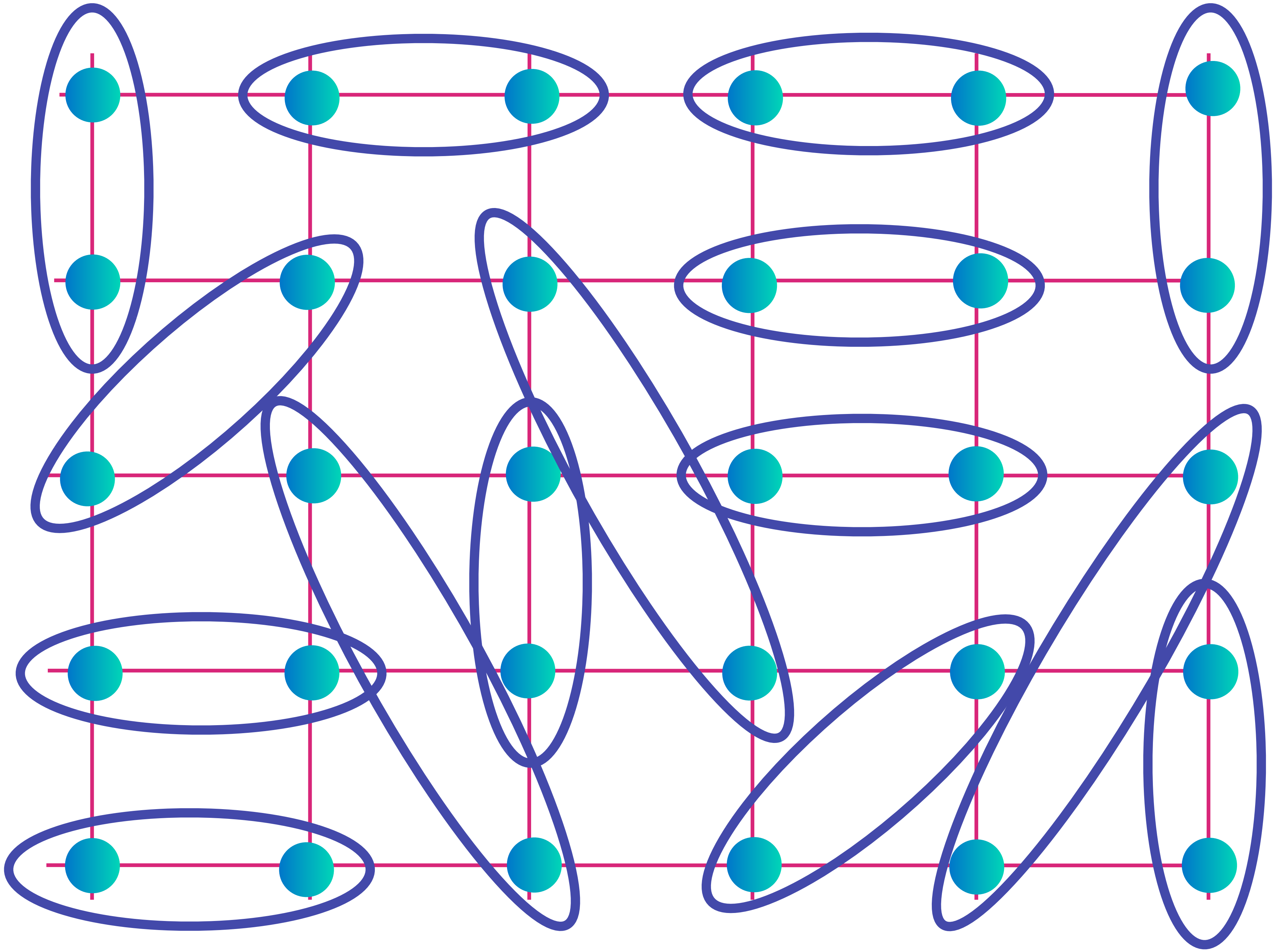

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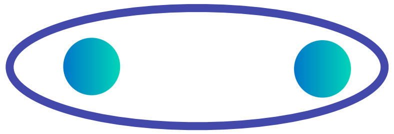

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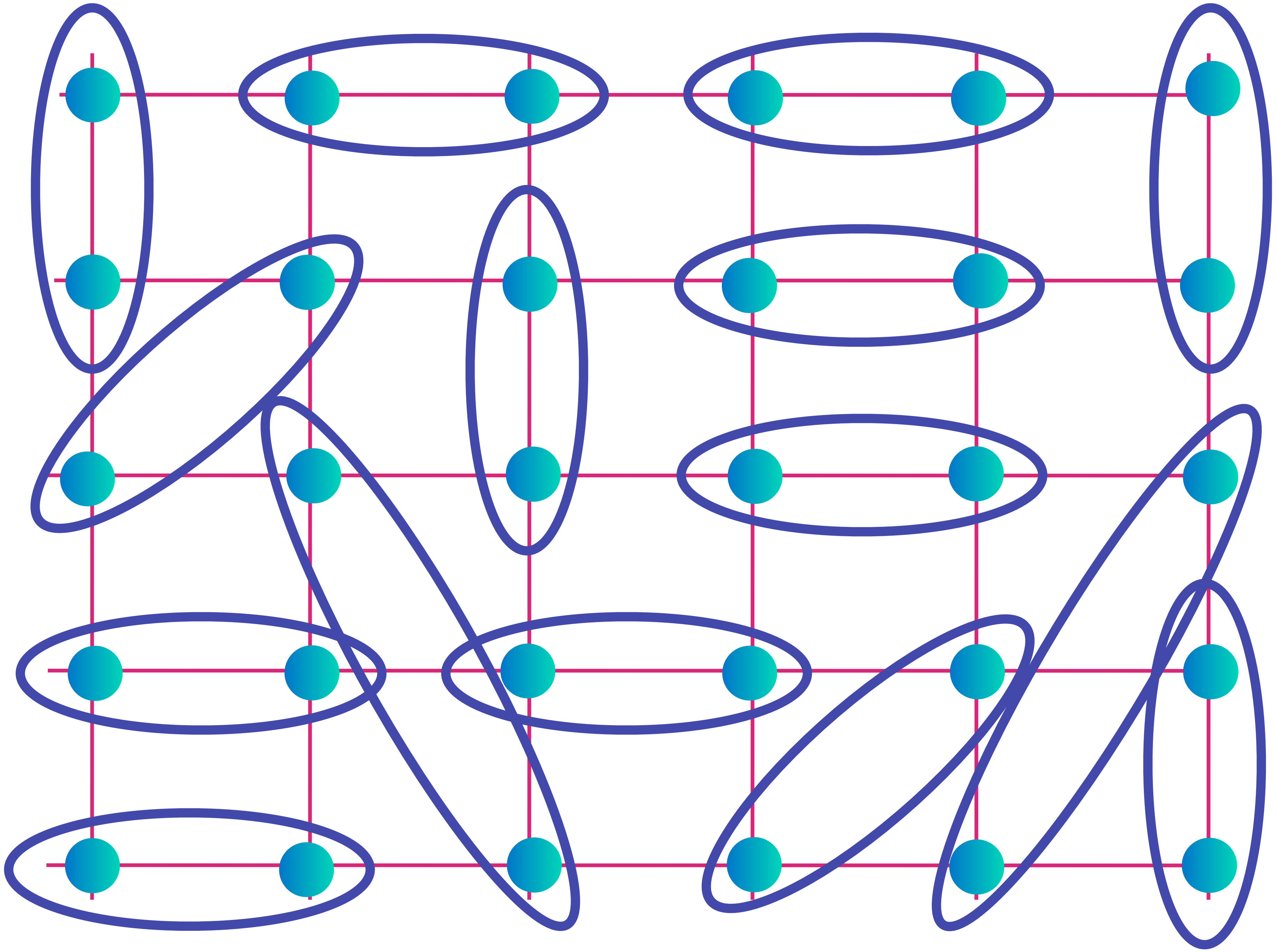

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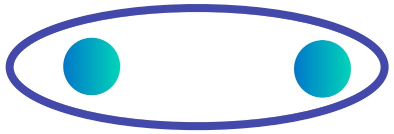
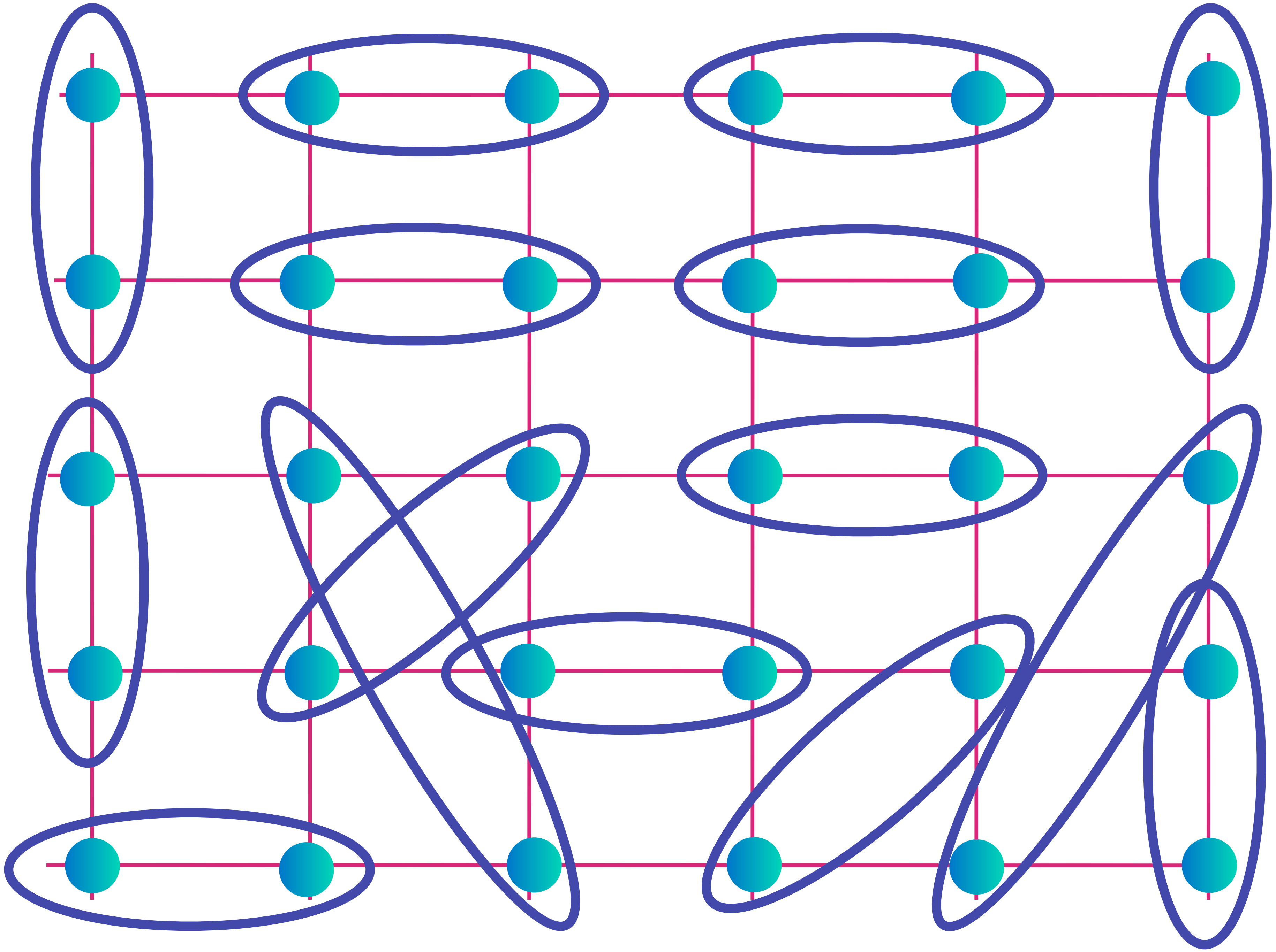
 $= \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)/\sqrt{2}$



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Why is this a TQFT ?

Place insulator on a torus:

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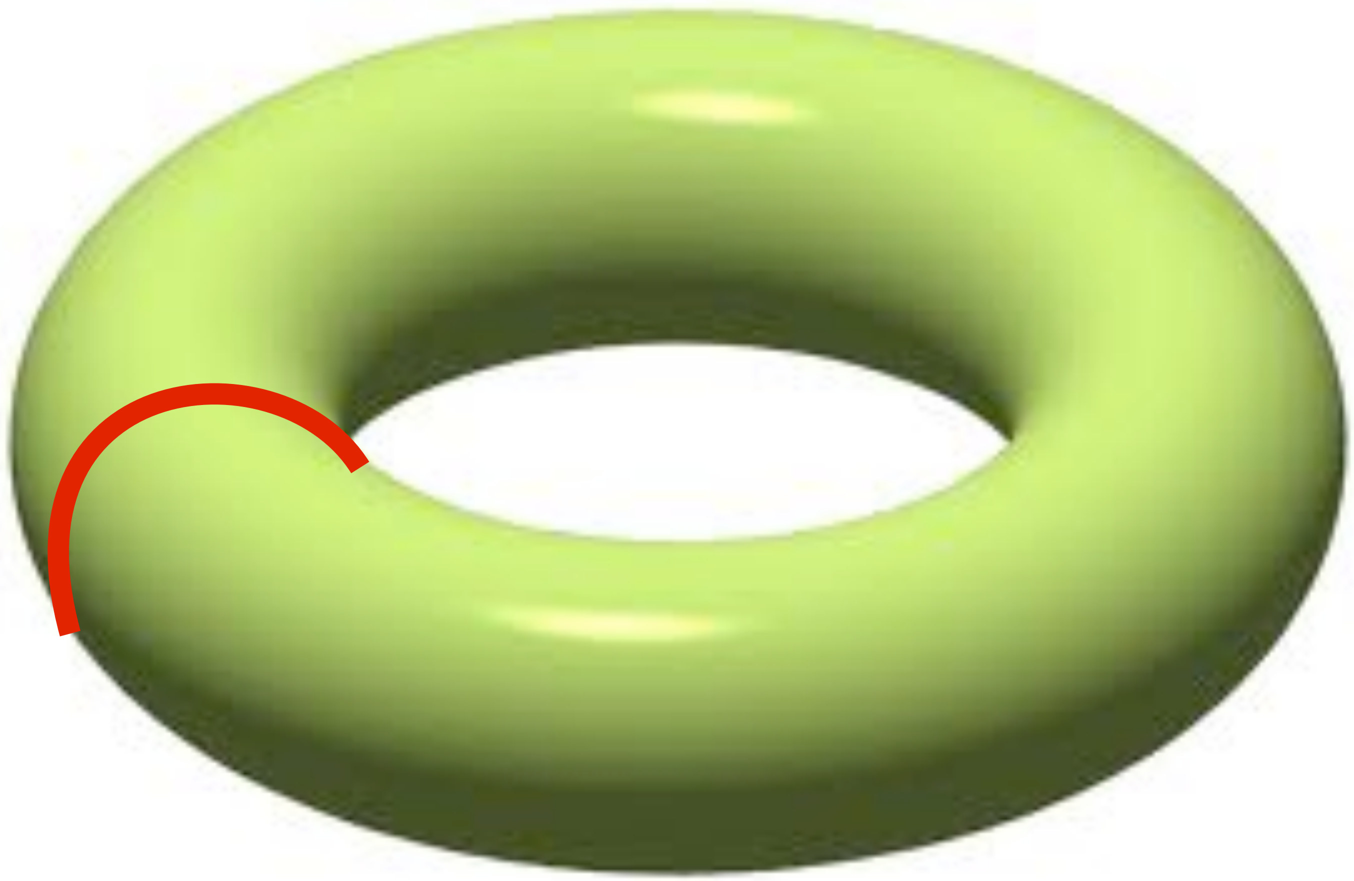

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Number of dimers crossing "branch-cut" is conserved modulo 2

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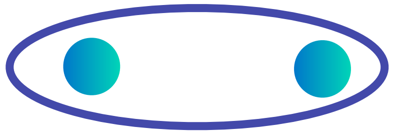
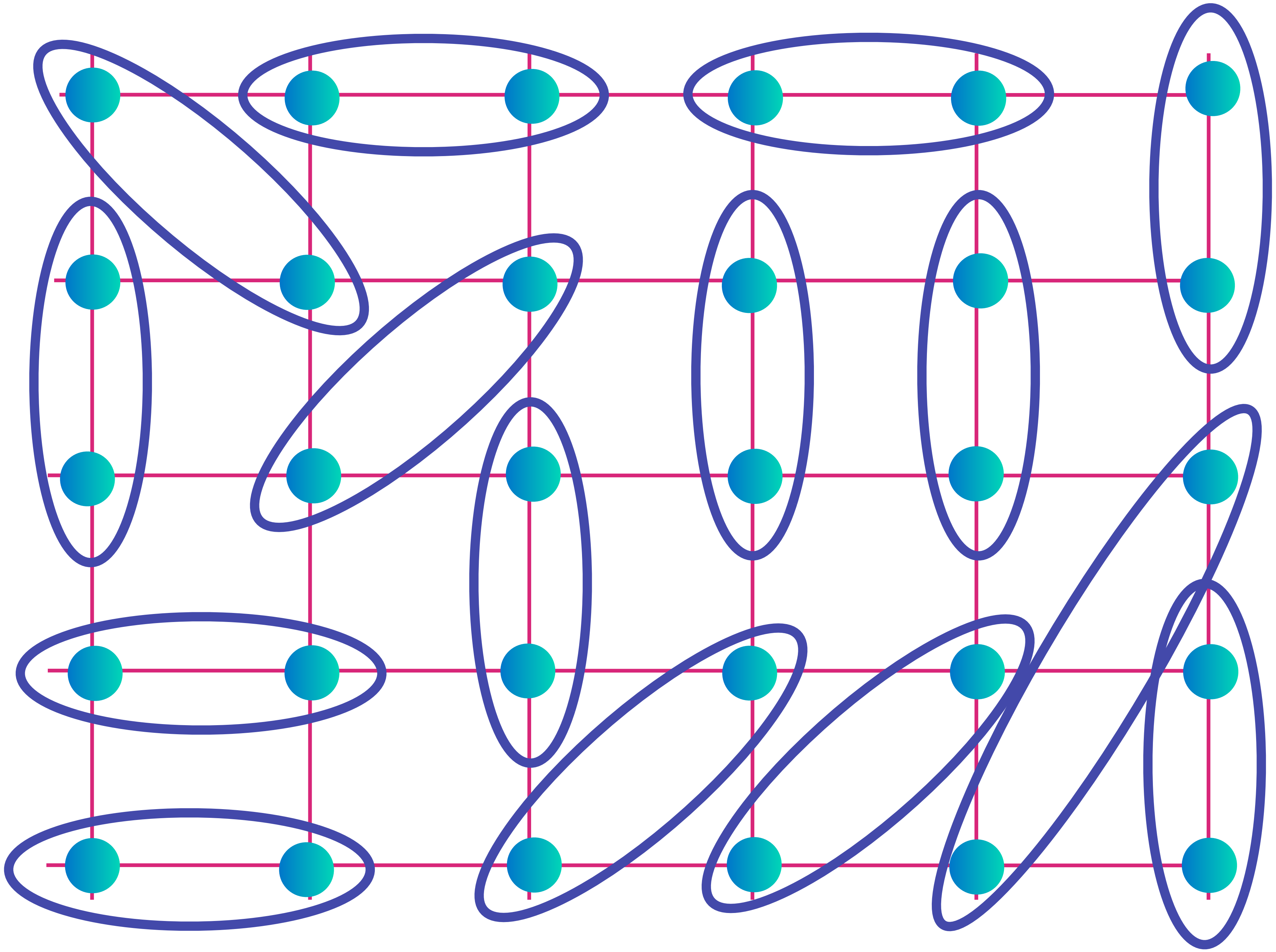
 $= \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)/\sqrt{2}$



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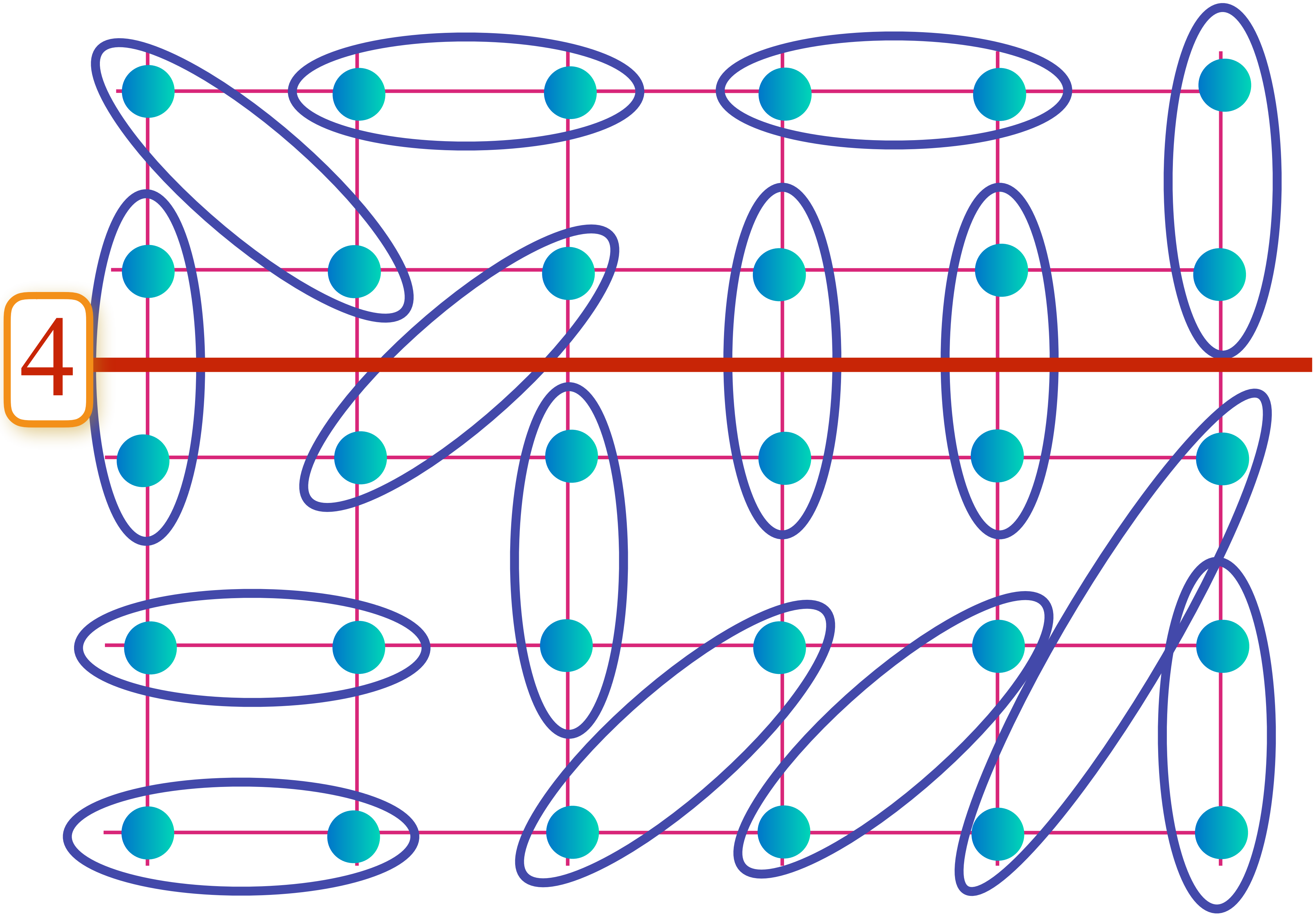

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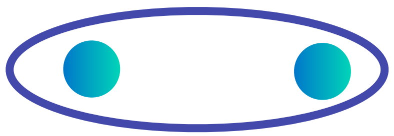
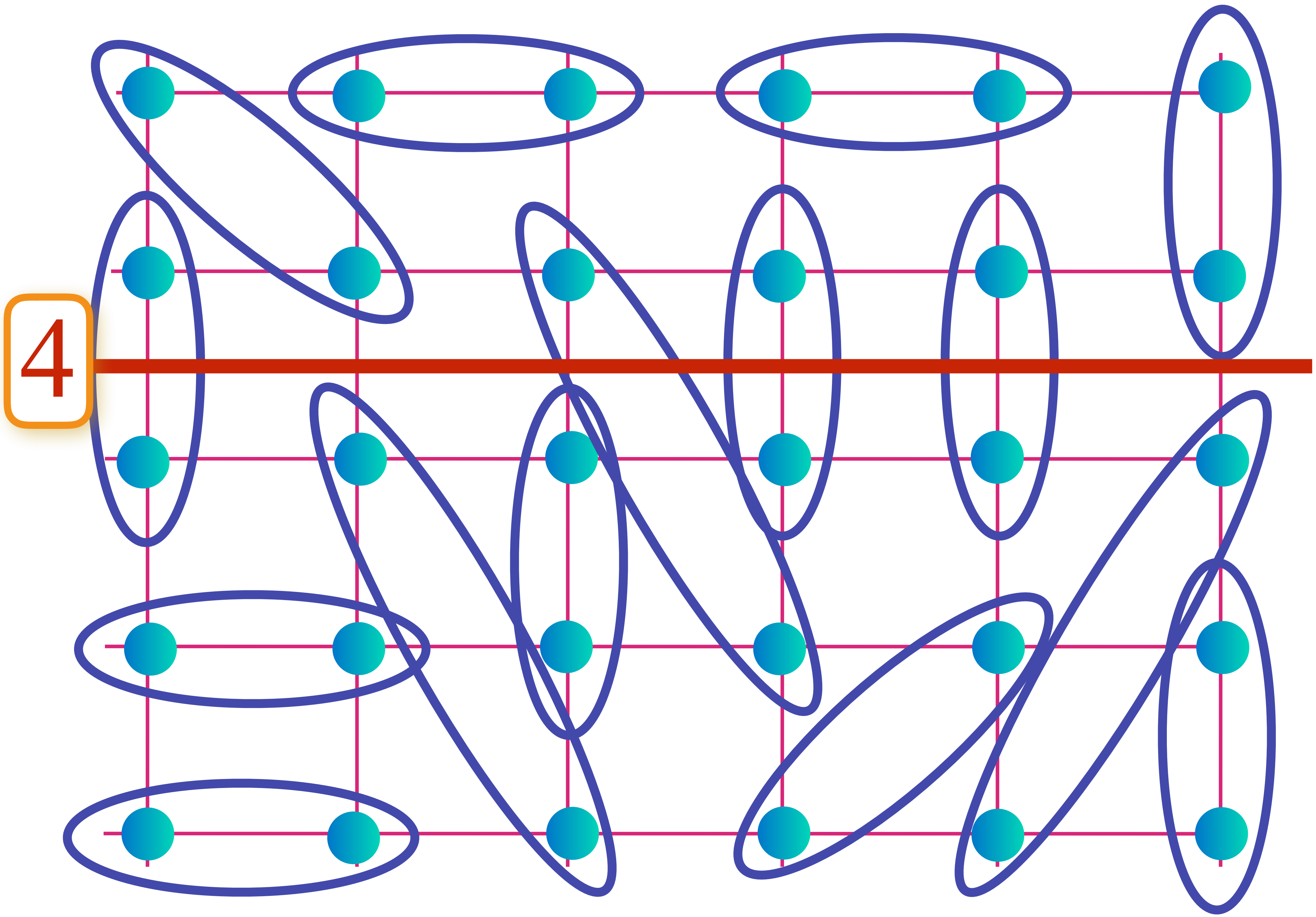
 $= \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)/\sqrt{2}$



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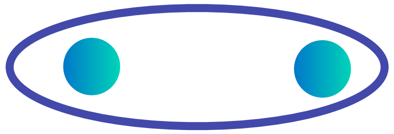

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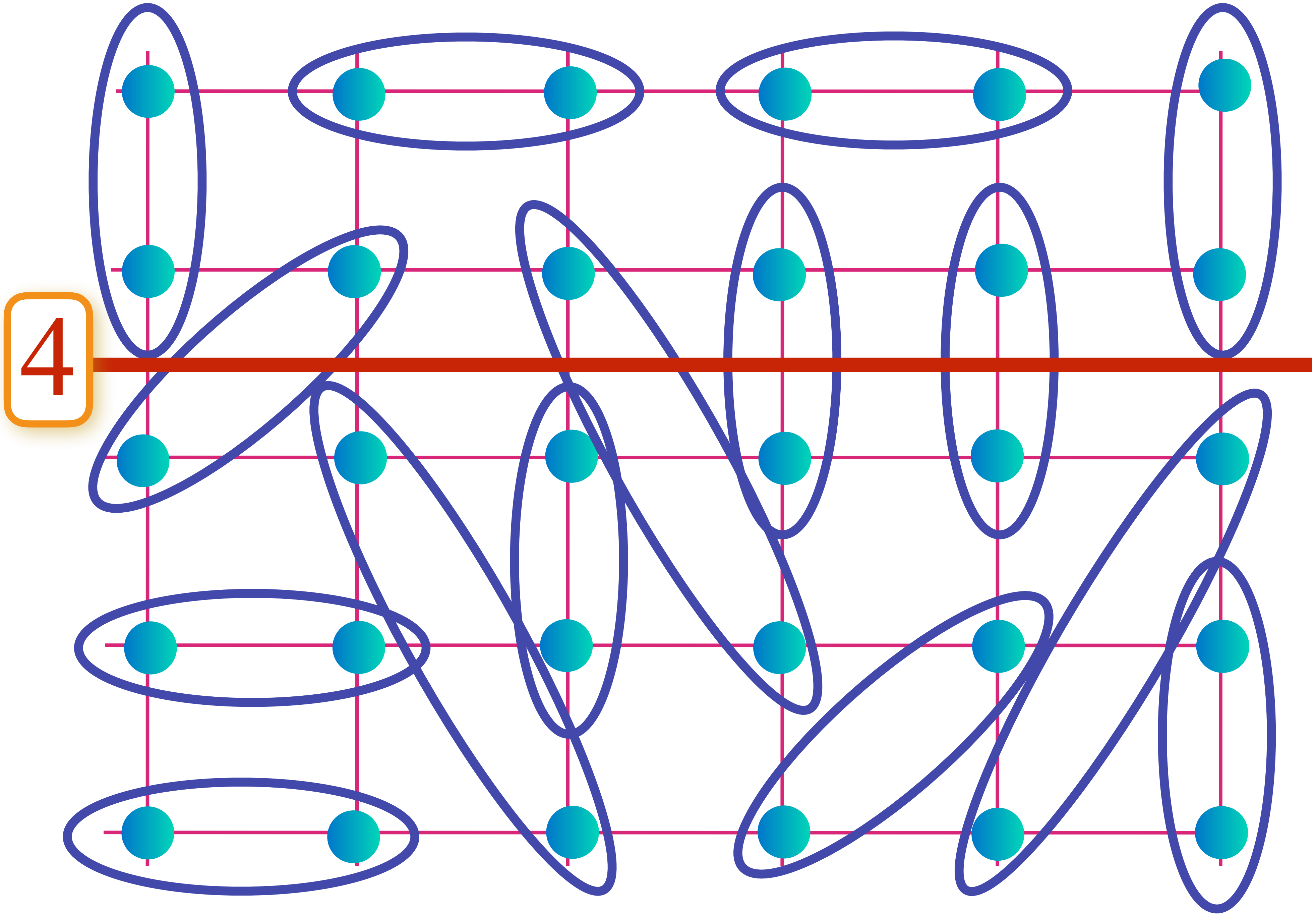

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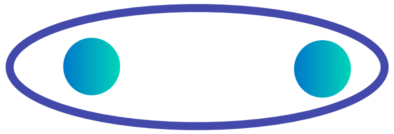
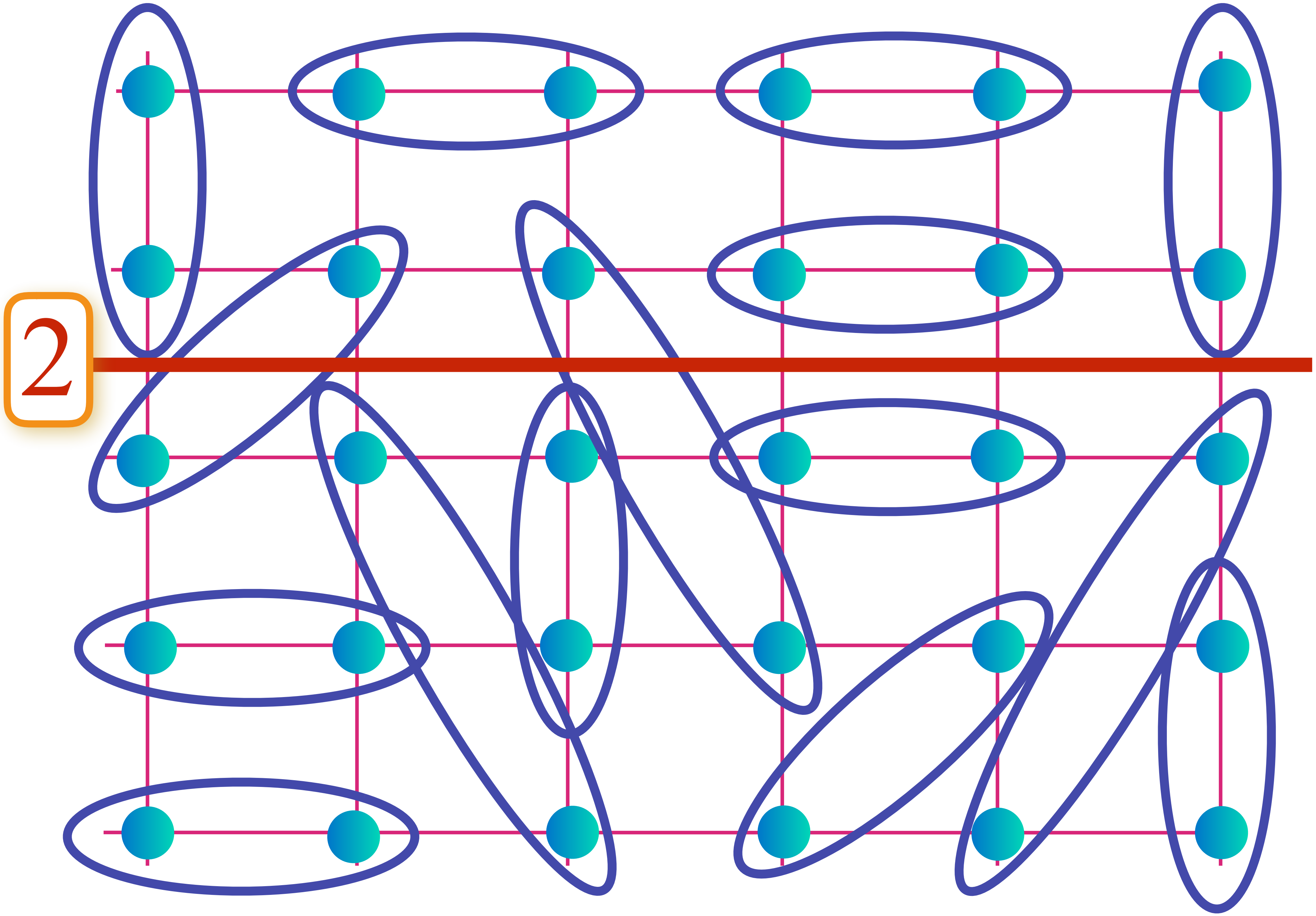
 $= \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)/\sqrt{2}$



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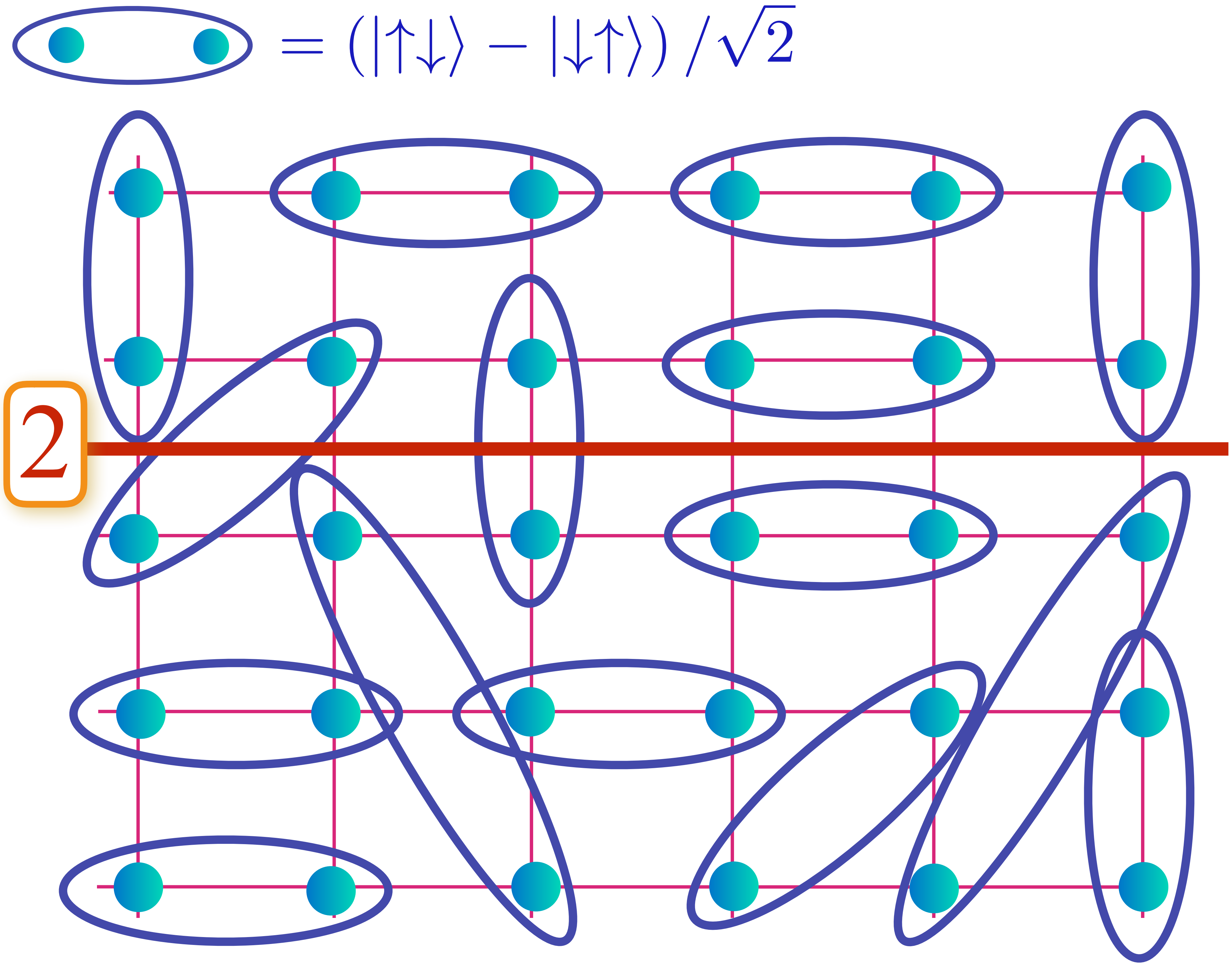

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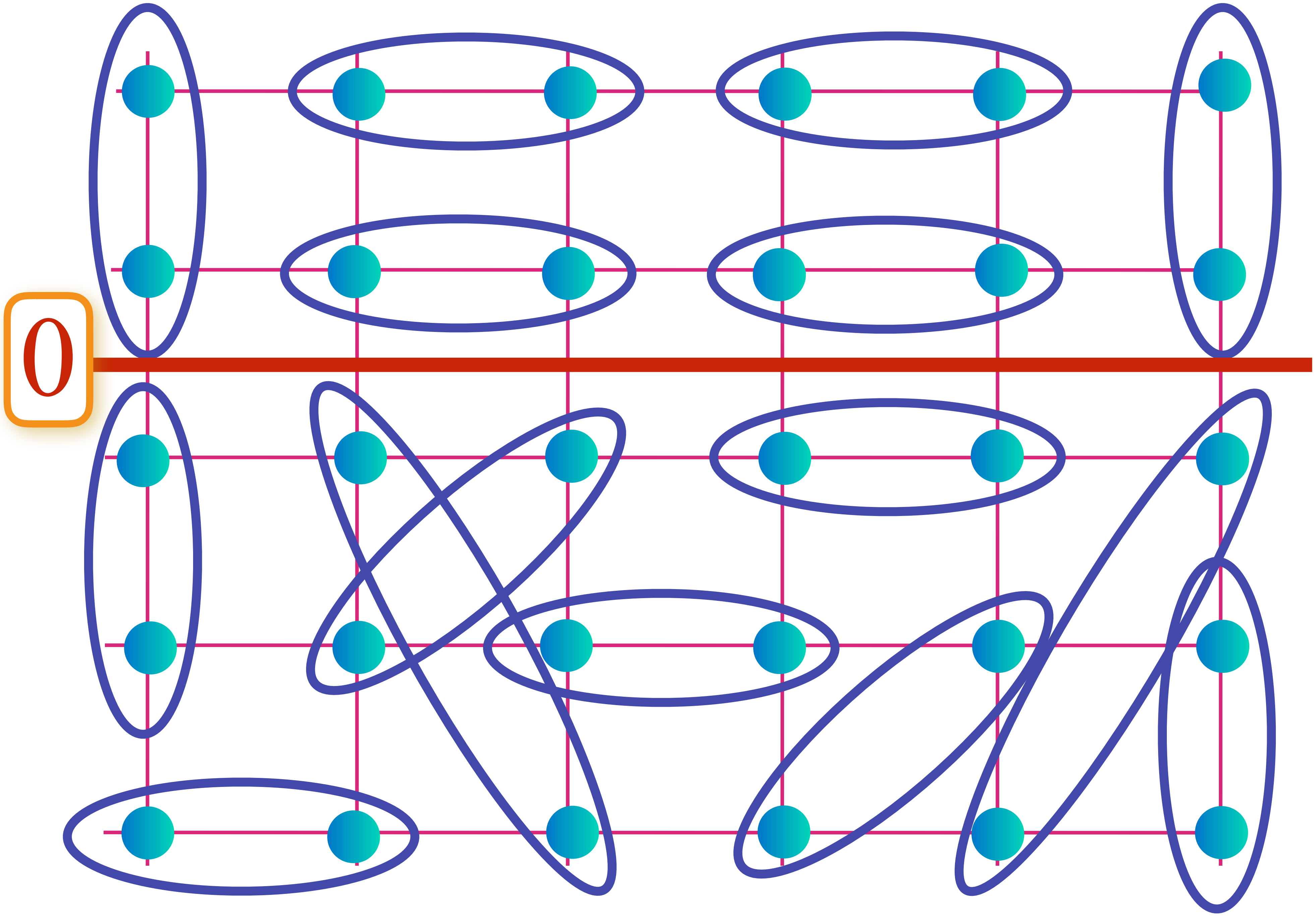

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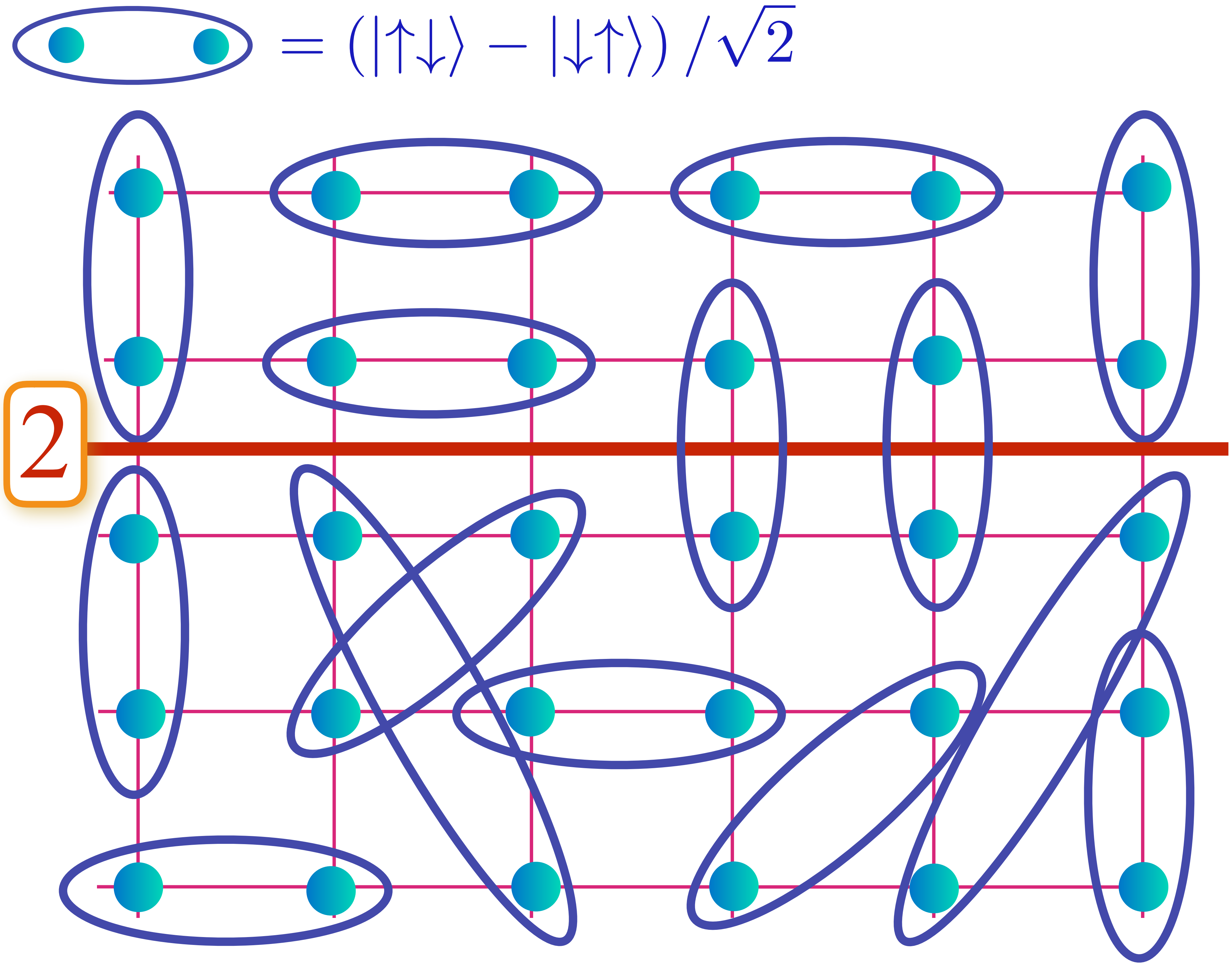

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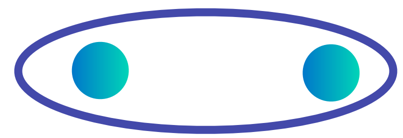
Introduce a pseudo spin, τ_ℓ^z on every link ℓ.



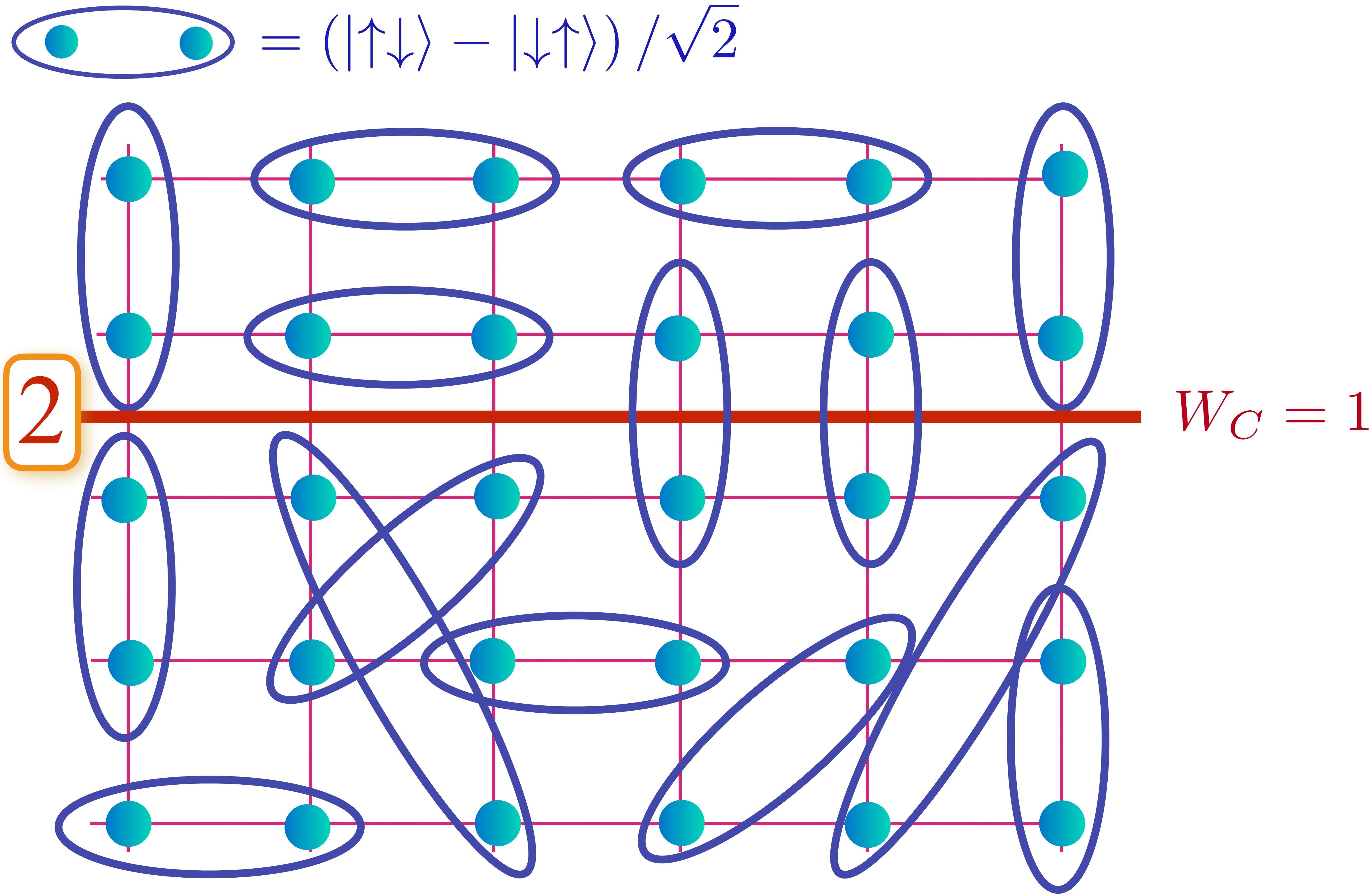
$= |-1\rangle$

$= |+1\rangle$

'Odd' dimer constraint: $$\prod_{\ell \text{ on site } i} \tau_\ell^z = -1.$$

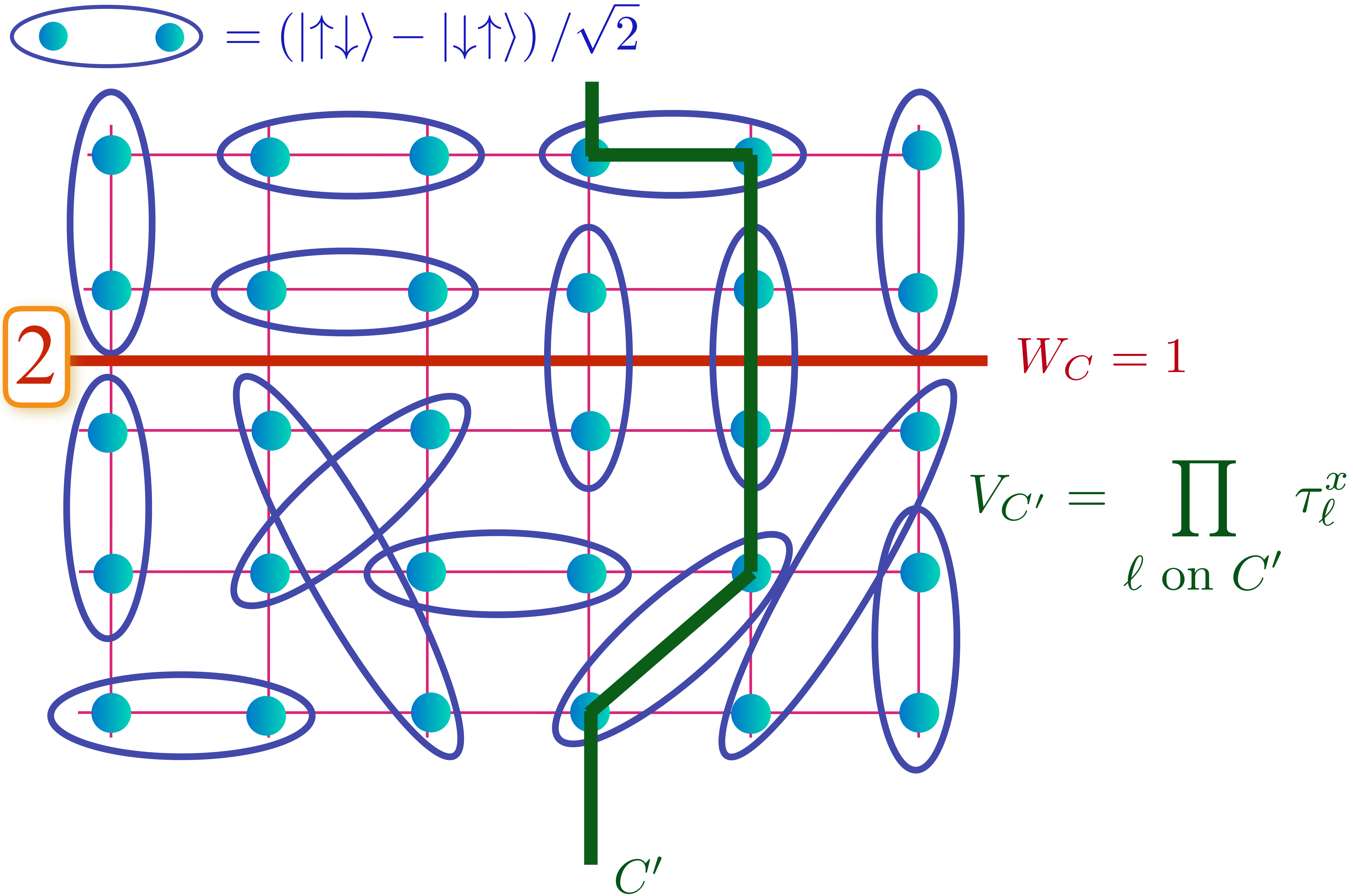
Topologically conserved charge:

$$W_C = \prod_{\ell \text{ cuts contour } C} \tau_\ell^z = \pm 1.$$



$= (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$
2
$W_C = 1$

Operator to change W_C



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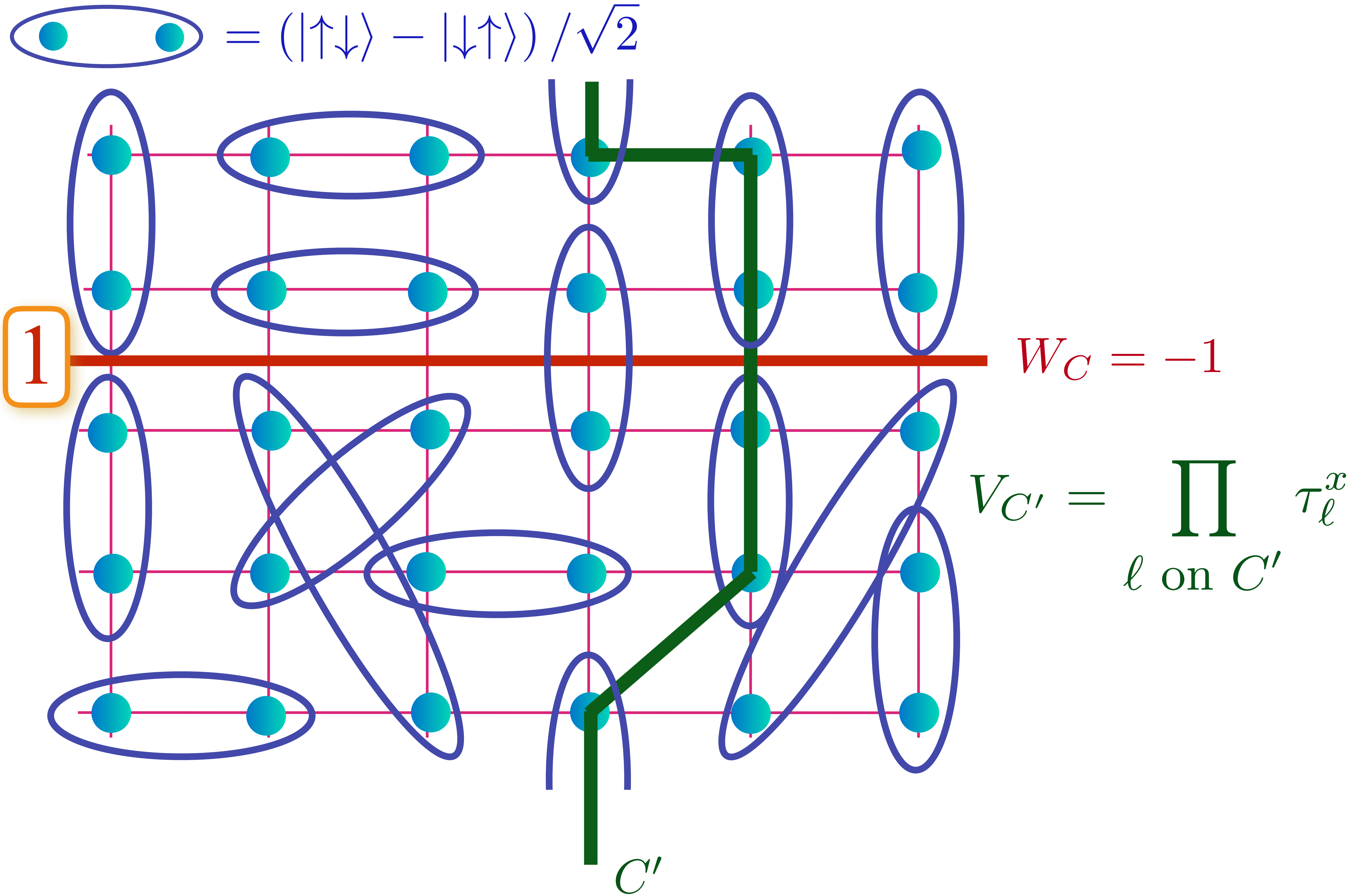

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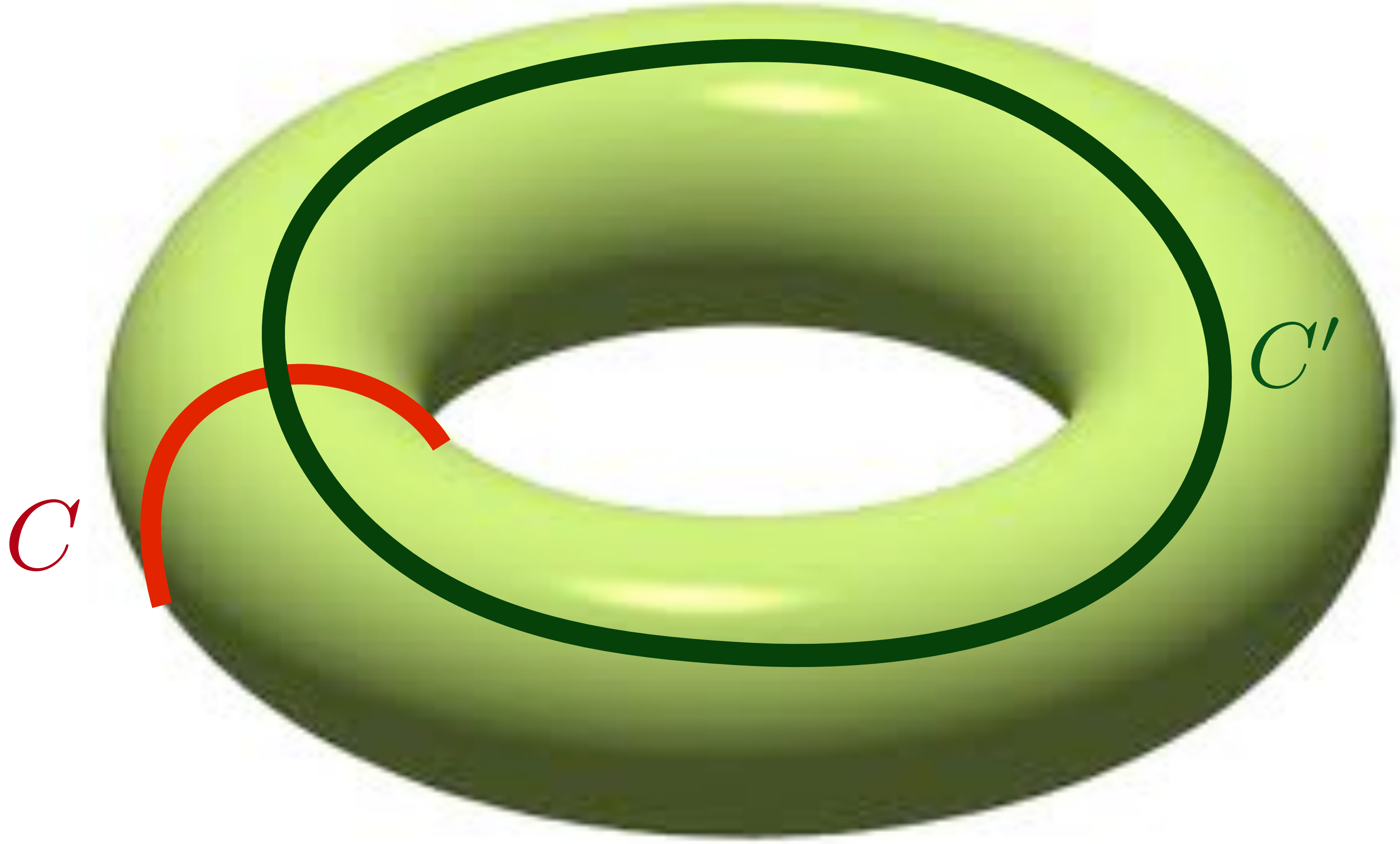


$$
\begin{aligned}
W_C V_{C'} &= -V_{C'} W_C \\
V_C W_{C'} &= -W_{C'} V_C
\end{aligned}
$$

The TQFT

The $\mathbb{Z}_2$ spin liquid: Described by the simplest, non-trivial, topological field theory with time-reversal symmetry:

$$\mathcal{L} = \frac{1}{4\pi} K_{IJ} \int d^3x \, a^I \wedge da^J$$

where a^I, $I = 1, 2$ are U(1) gauge connections, and the K matrix is

$$K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

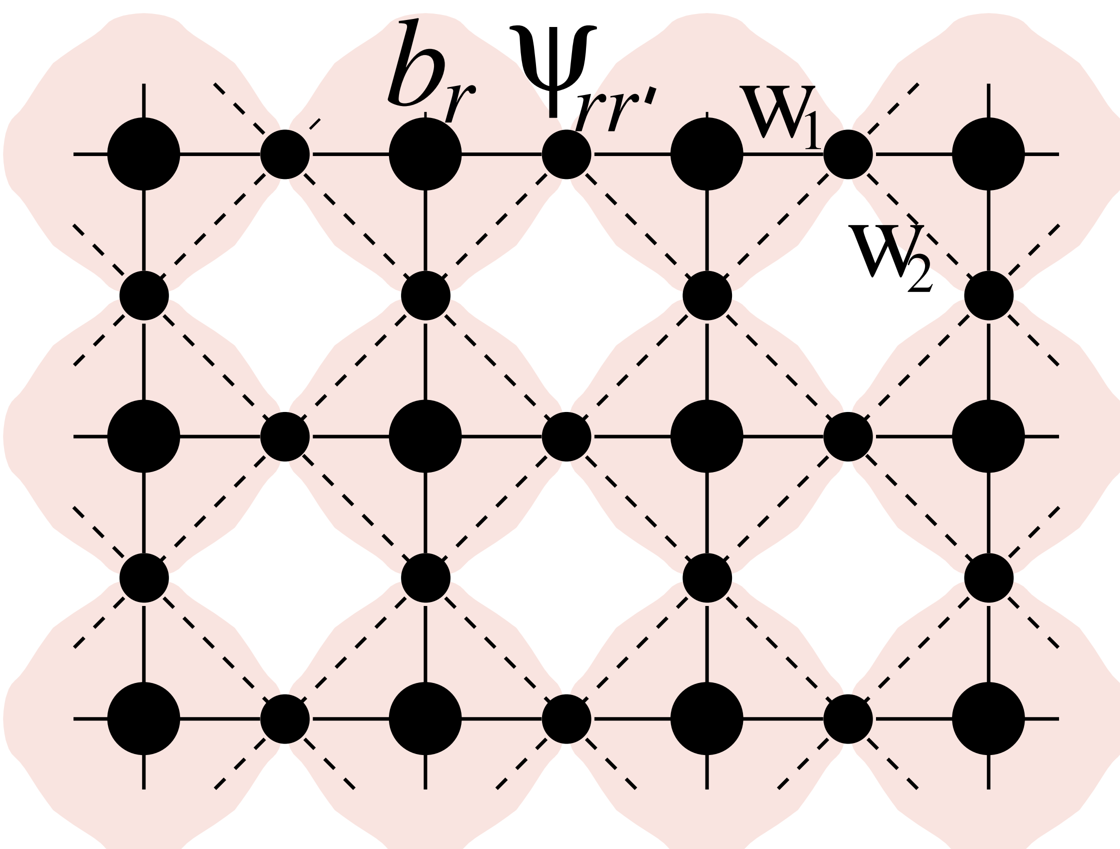
The Wilson loops

$$V_C = \exp\left(i \int_C dx \cdot a^1 \right) \quad , \quad W_C = \exp\left(i \int_C dx \cdot a^2 \right)$$

obey $W_C V_{C'} = -V_{C'} W_C$ when C and C' wrap separate cycles of the torus.

N. Read and S. Sachdev, Phys. Rev. Lett. 66, 1773 (1991); X.-G. Wen, Phys. Rev. B 44, 2664 (1991);
S. Sachdev and M. Vojta, Journal of the Physical Society of Japan **69,** Suppl. B, 1 (2000);
M. Freedman, C. Nayak, K. Shtengel, K. Walker, and Z. Wang, Annals of Physics **310**, 428 (2004)
T. H. Hansson, Vadim Oganesyan, S. L. Sondhi, Annals of Physics **313**, 497 (2004)

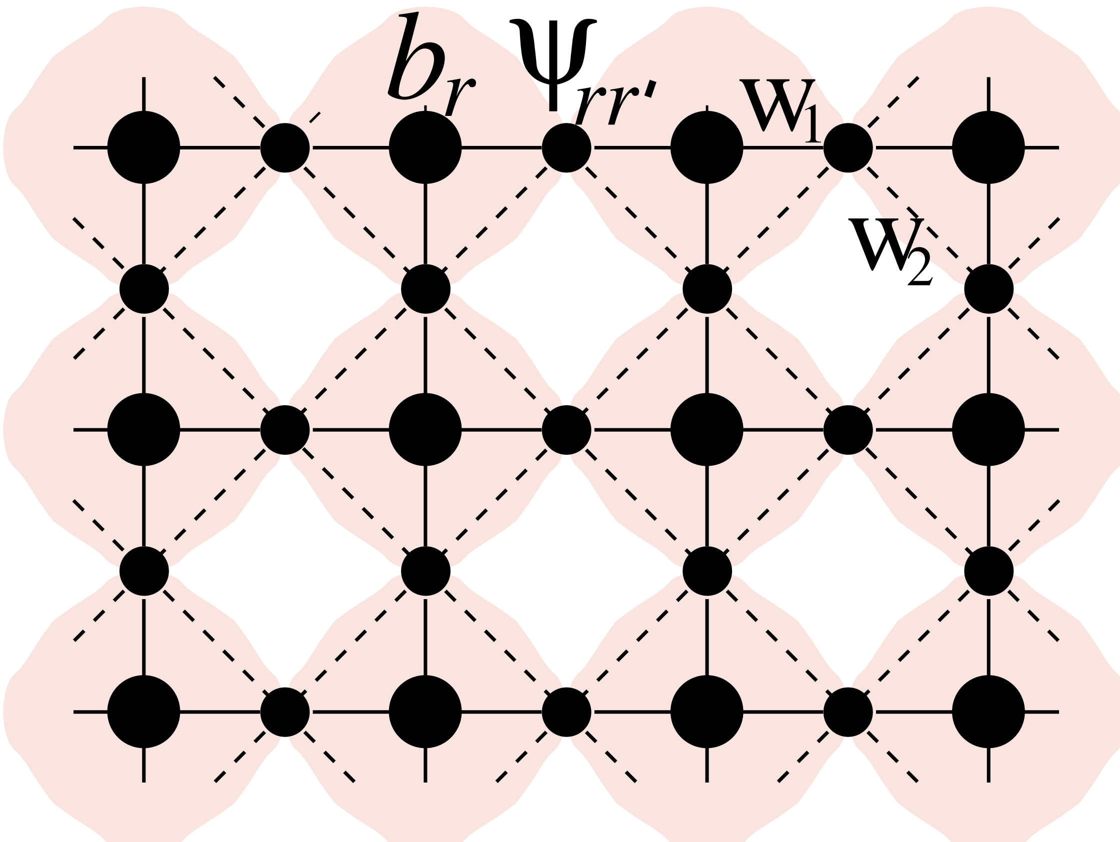
$$
\begin{aligned}
H = & -w_1 \sum_{r, r' \in r} (b_r^\dagger \psi_{rr'} + \text{H.c.}) \\
& - w_2 \sum_{[rr'r'']} (\psi_{rr'}^\dagger \psi_{r'r''} + \text{H.c.}) + u_b \sum_r (n_r^b)^2 \\
& + u_\psi \sum_{\langle rr' \rangle} (n_{rr'}^\psi)^2 + U \sum_r N_r^2 .
\end{aligned}
$$

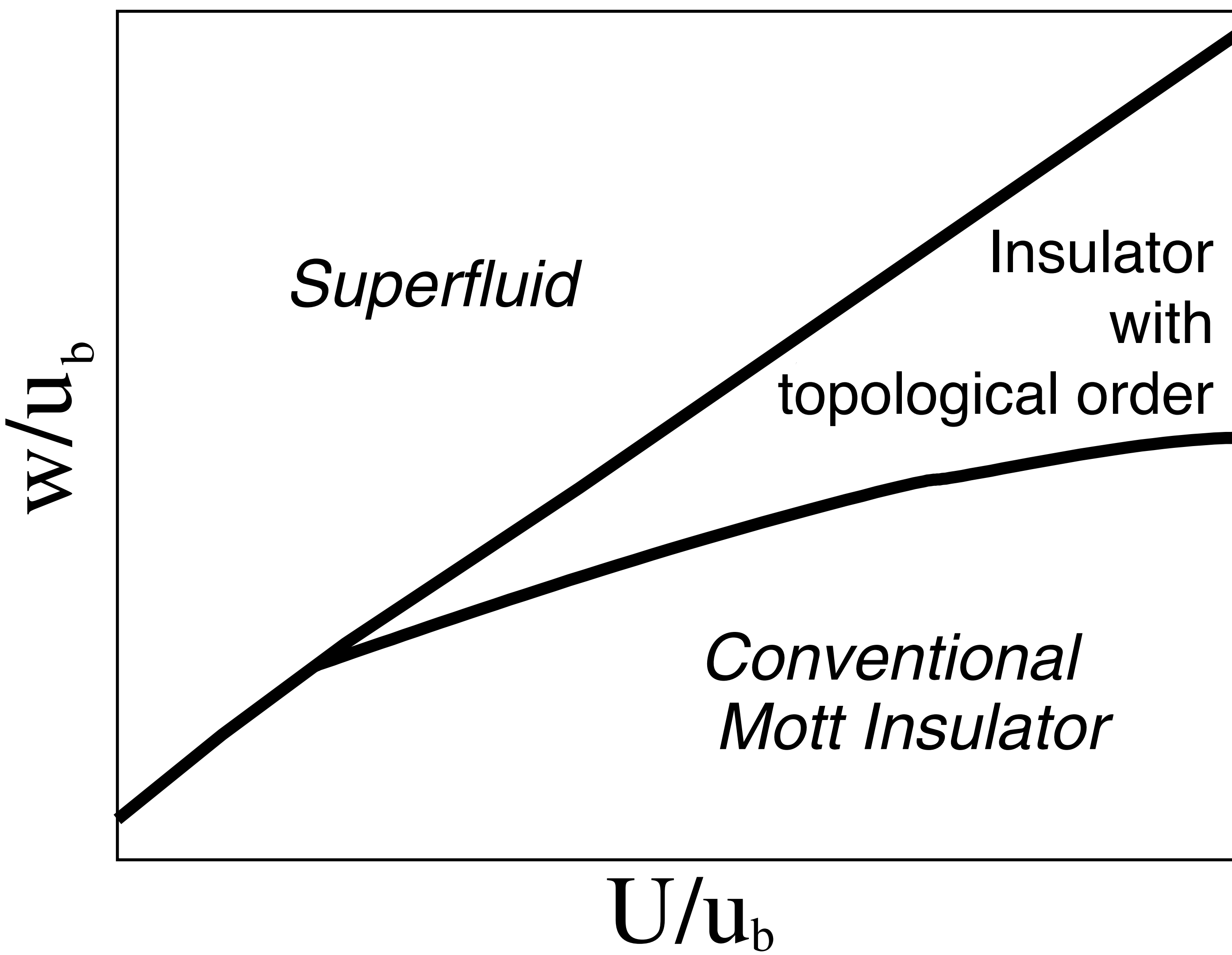


$$
2 N_r = 2 n_r^b + \sum_{r' \in r} n_{rr'}^\psi .
$$

Average of one boson per site: $\langle N_r \rangle = 1$

R. Jalabert and S. Sachdev Phys. Rev. B 44, 686 (1991); S. Sachdev and M. Vojta, Journal of the Physical Society of Japan **69,** Suppl. B, 1 (2000); O.I. Motrunich and T. Senthil, PRL **89**, 277004 (2002).





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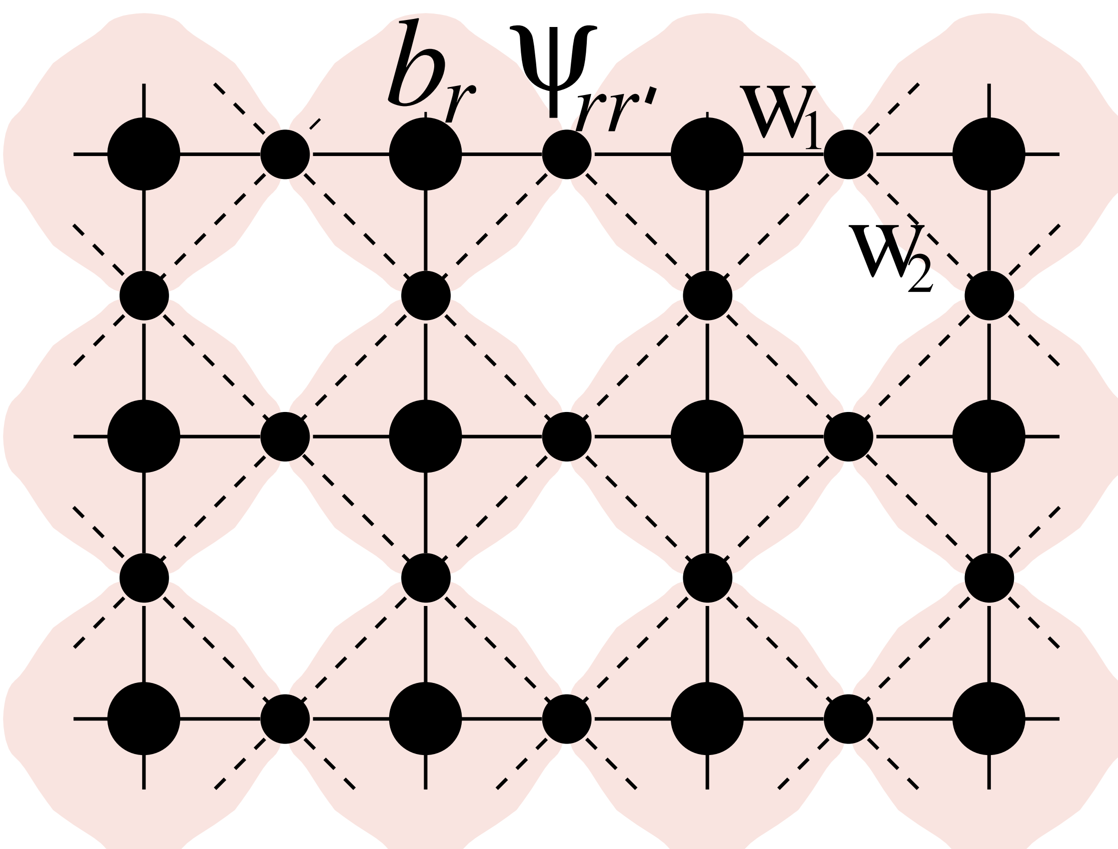

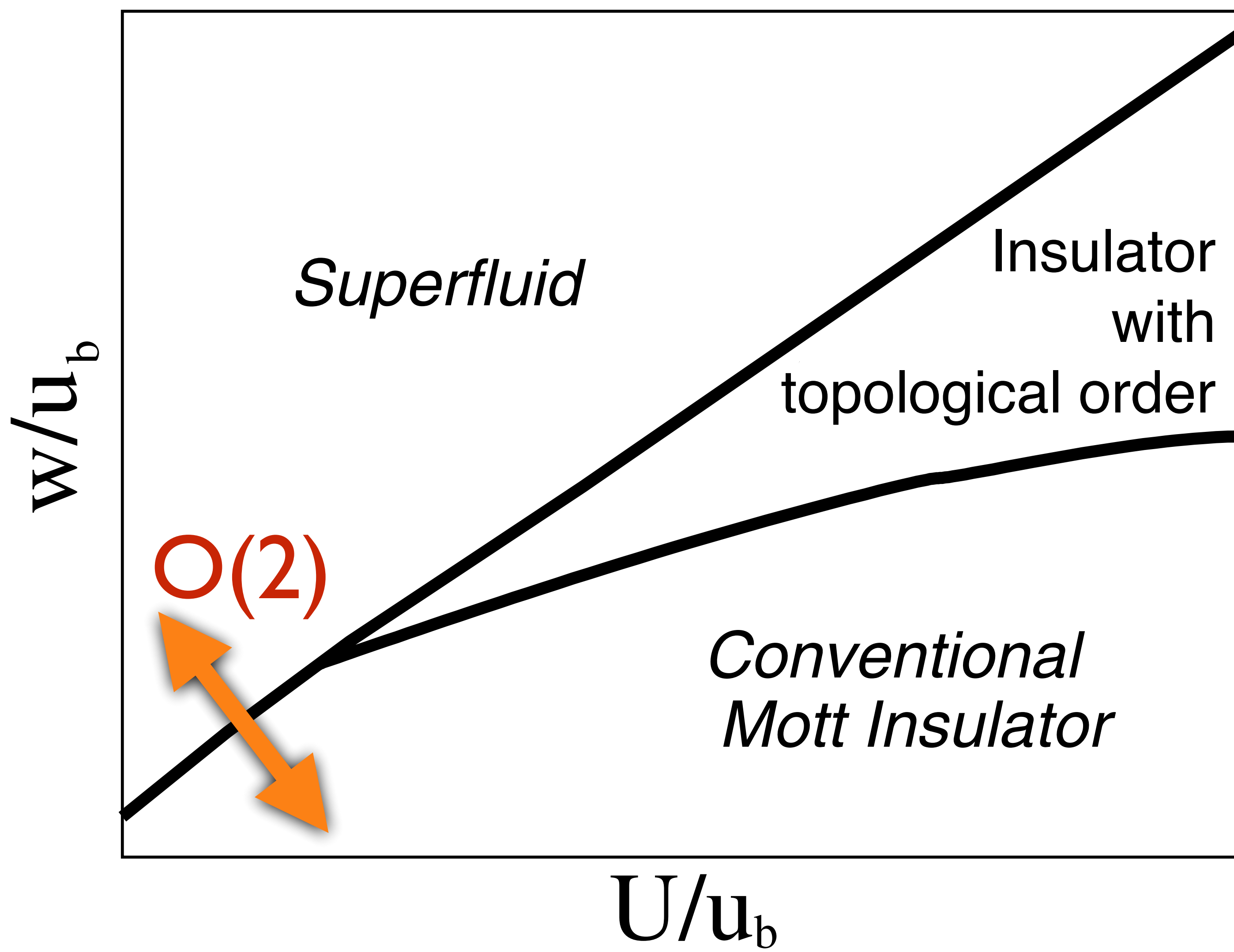

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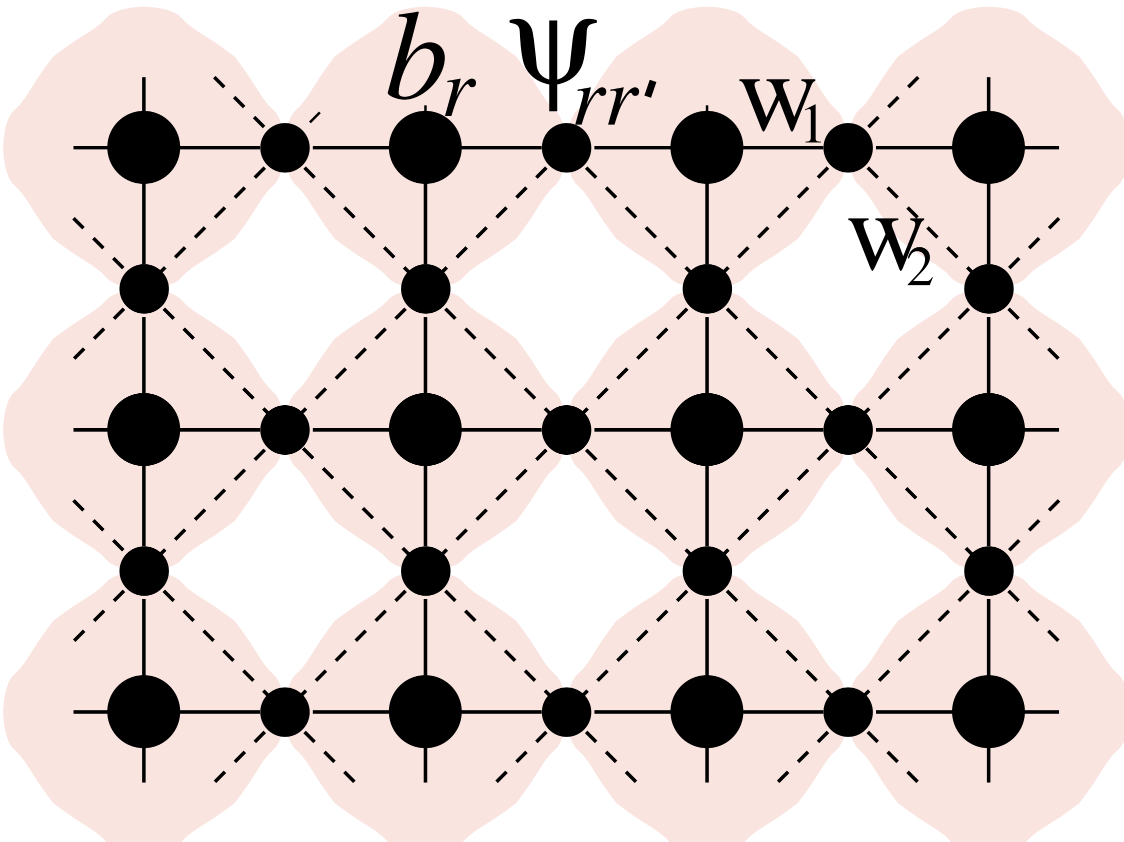
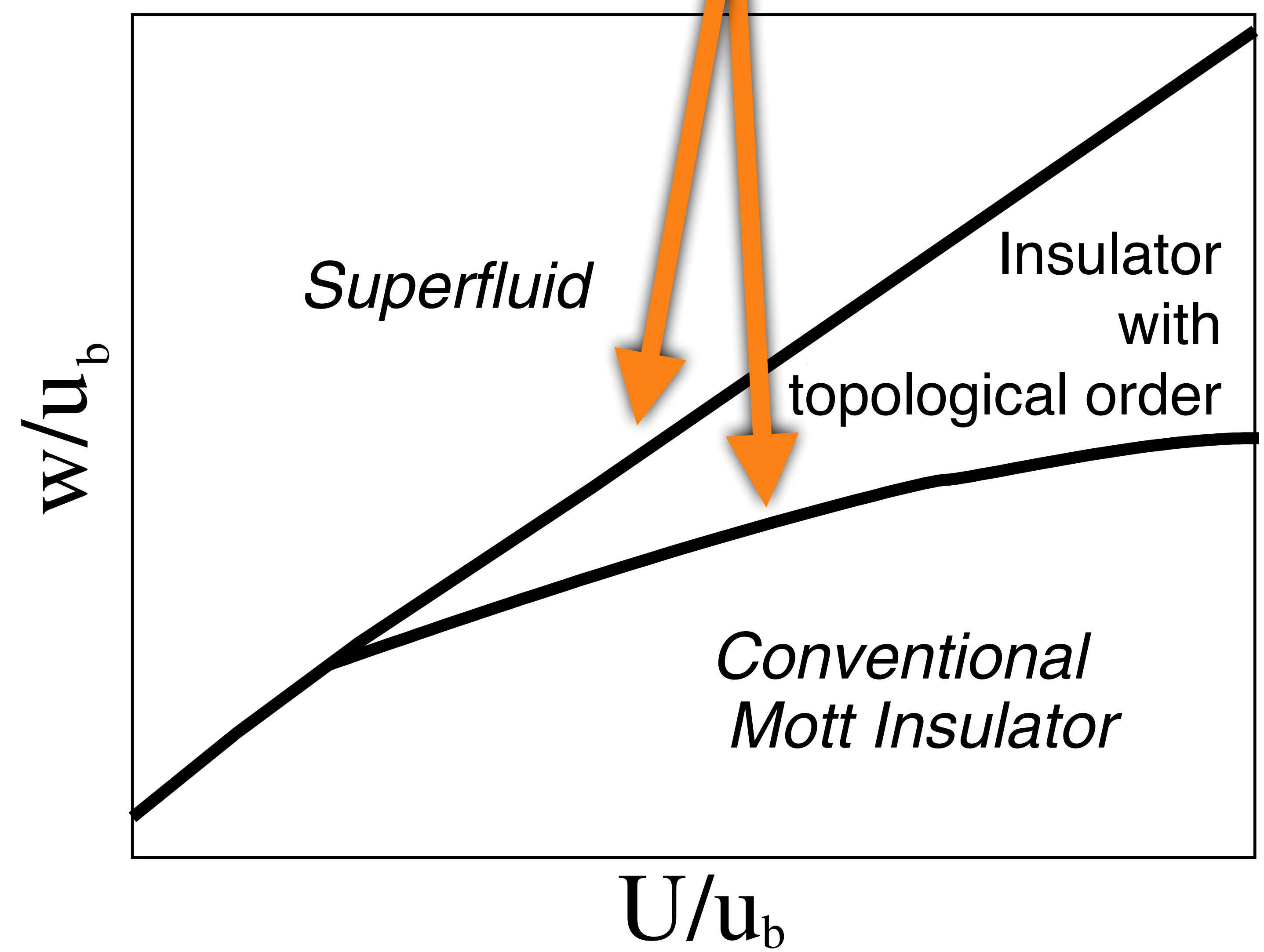
Topological phase transitions





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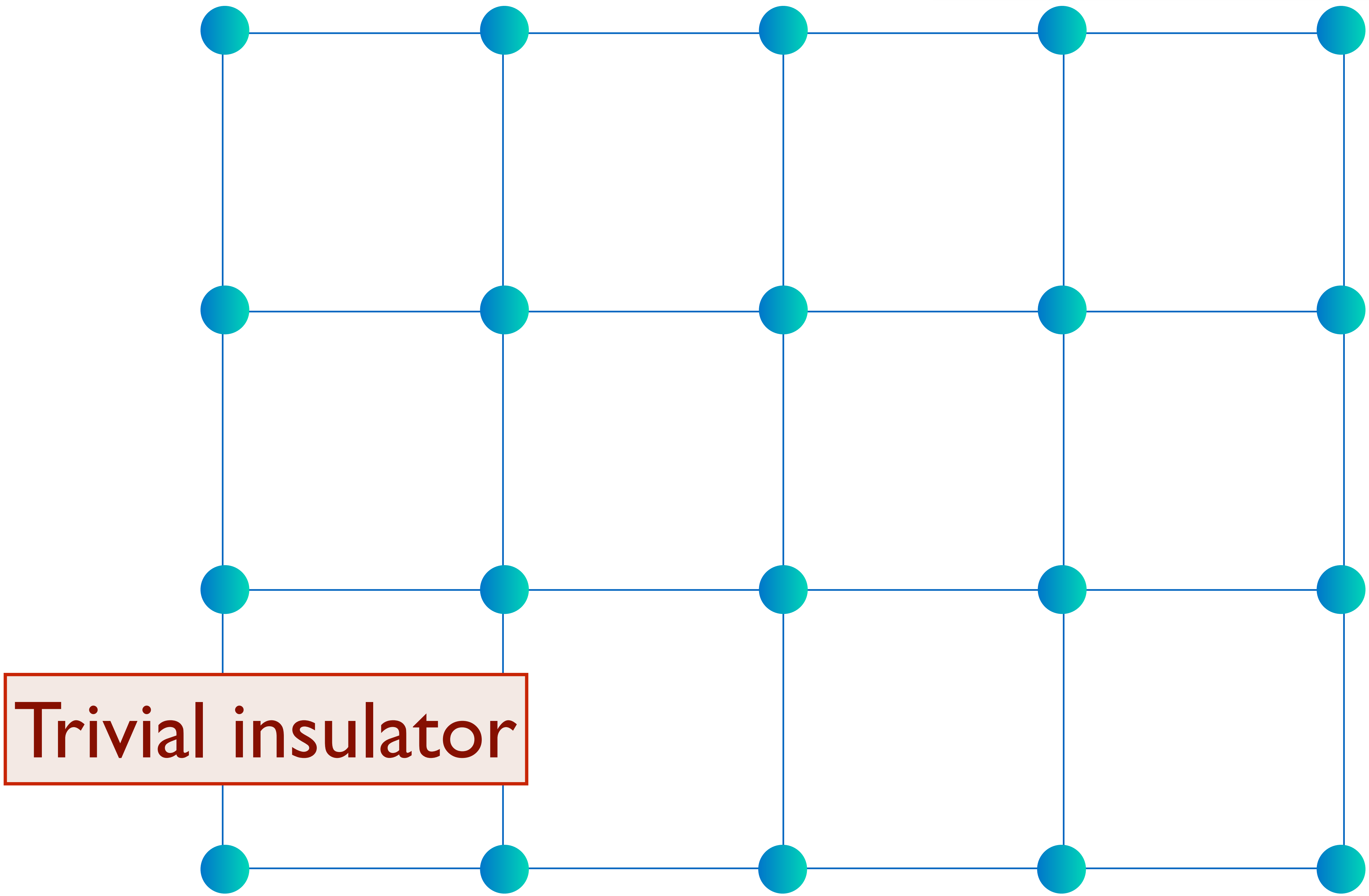
$= b^\dagger$;

$N_r = 1$ for all r



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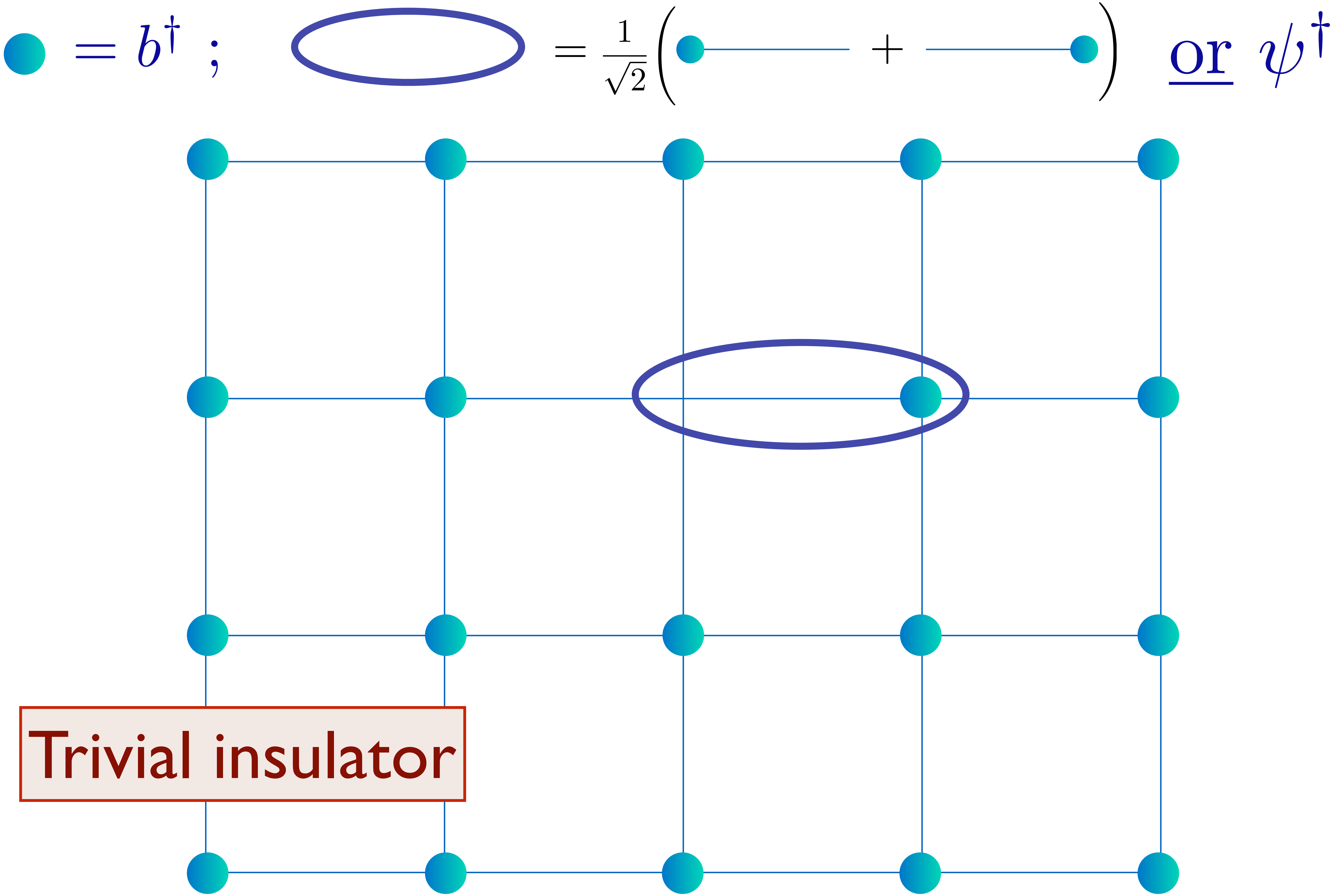

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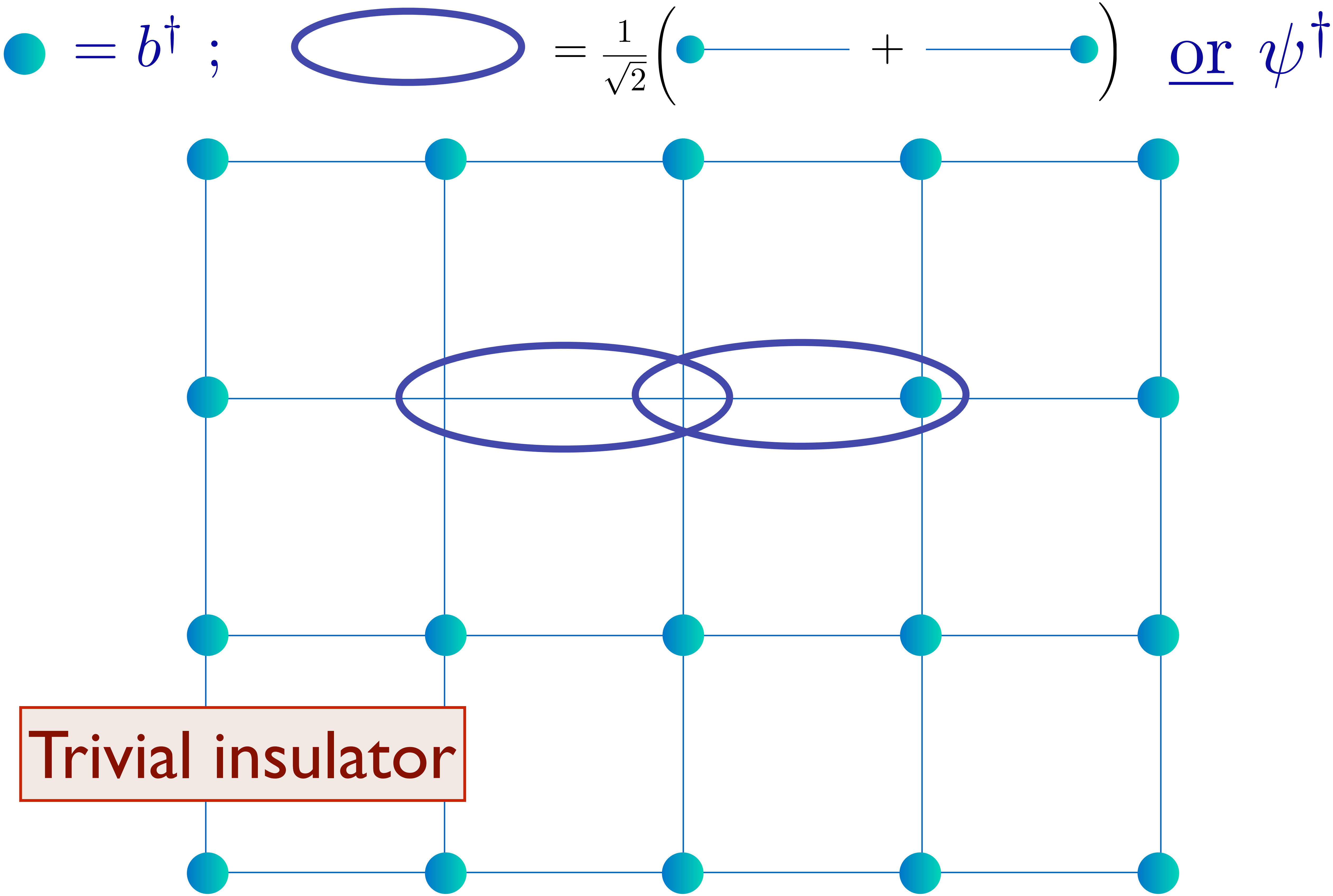

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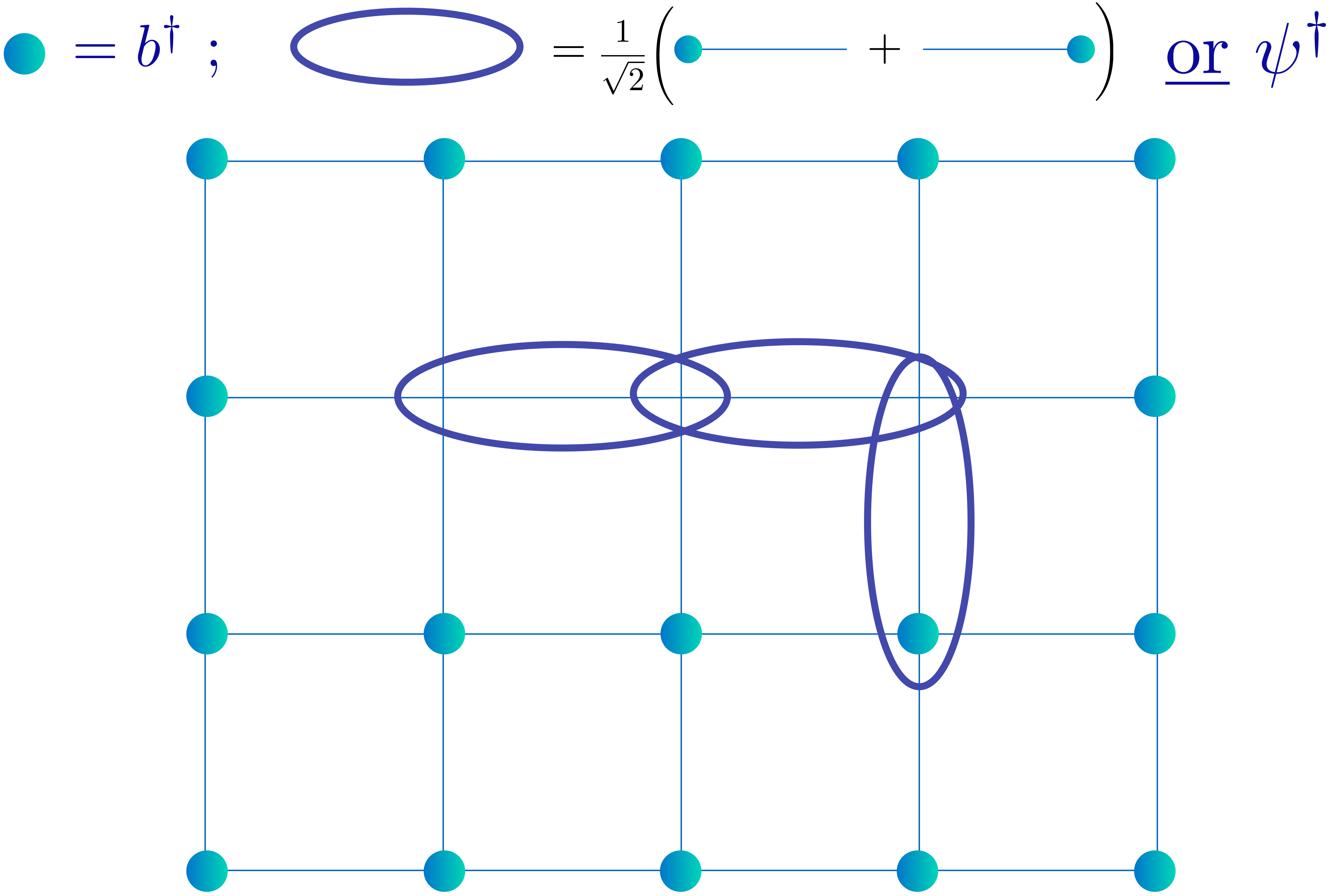
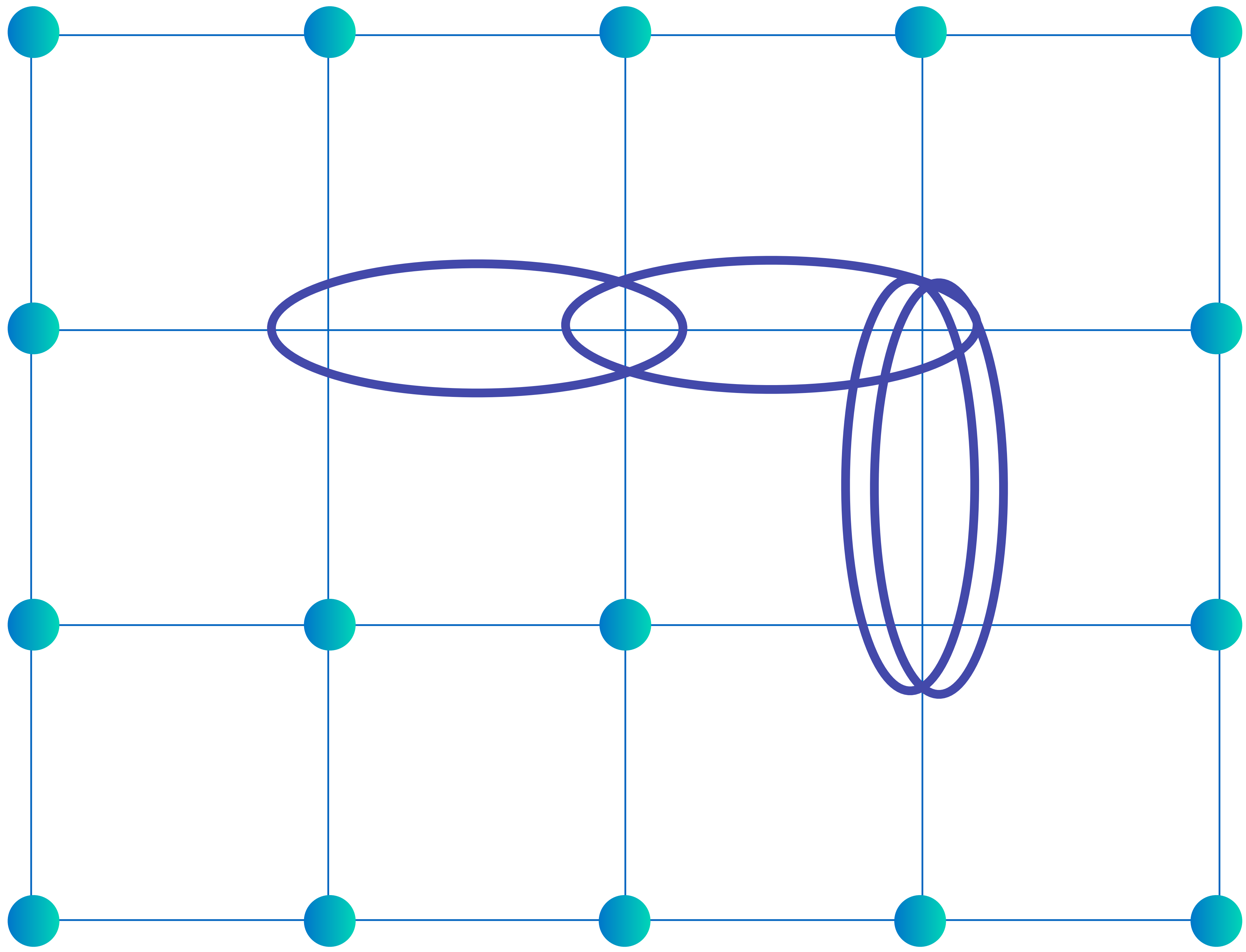


Average of one boson per site: $\langle N_r \rangle = 1$

$\bullet = b^\dagger$; $\quad \bigcirc = \frac{1}{\sqrt{2}}\left(\bullet\!\!-\!\!-\!\!-\; + \;-\!\!-\!\!-\!\!\bullet\right)$ or $\psi^\dagger$



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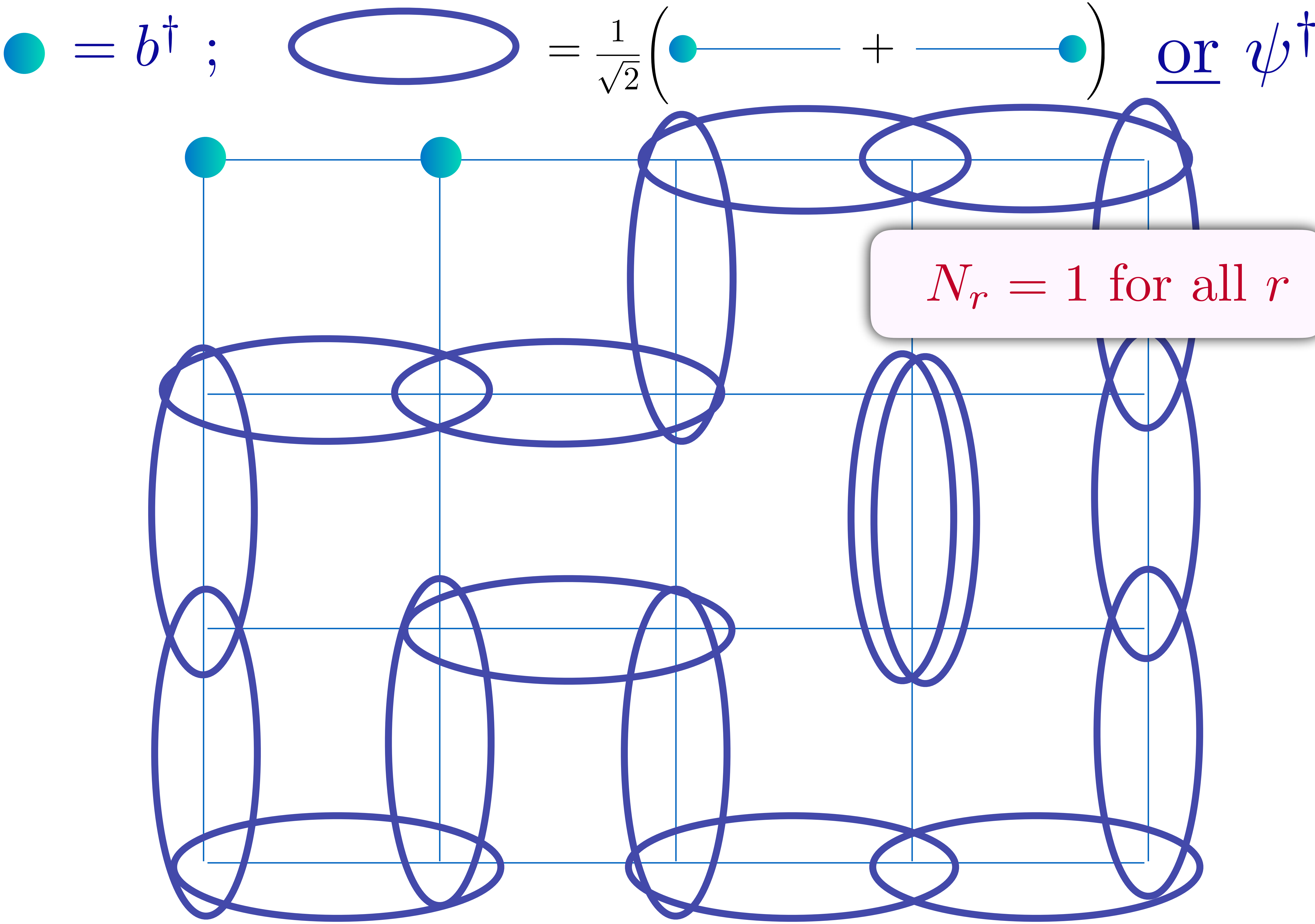
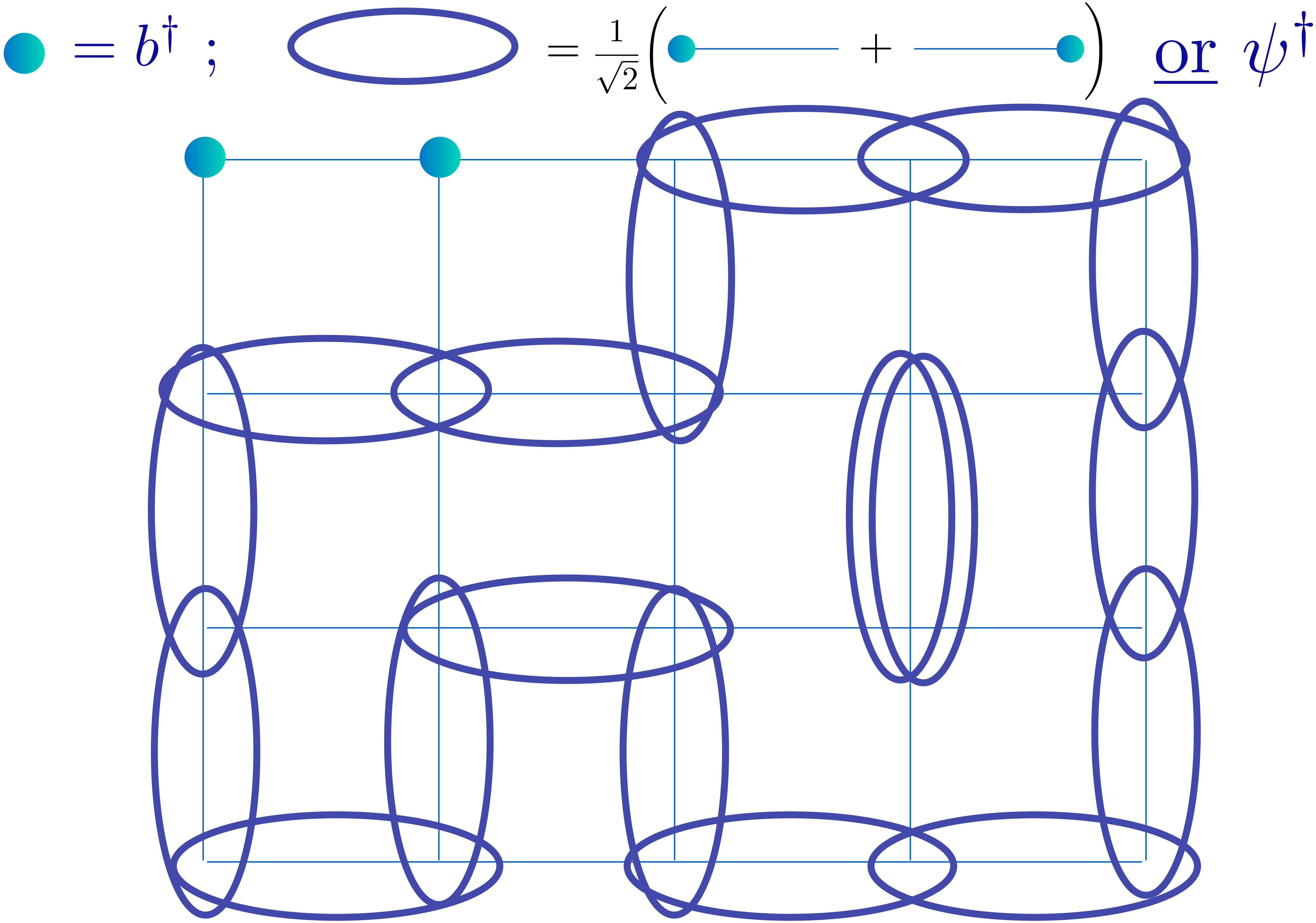


Average of one boson per site: $\langle N_r \rangle = 1$

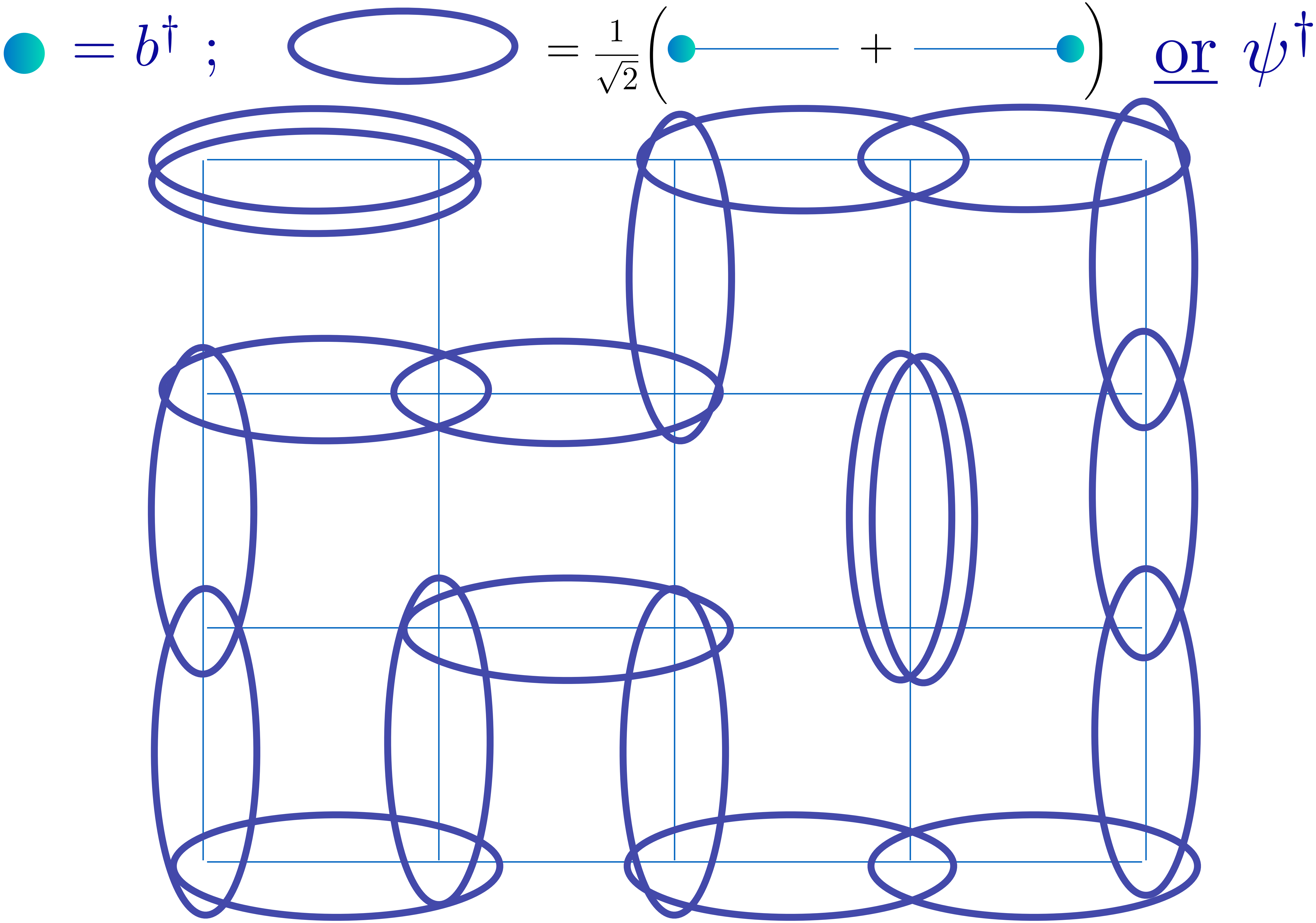
In the limit of large U, we expect to prefer states in which $N_r = 1$ at all r. By the usual procedure of perturbation theory in $1/U$, we obtain an effective Hamiltonian within the subspace of states with $N_r = 1$. To order $1/U$, this effective Hamiltonian is

$$
\begin{aligned}
H_{\text{eff}}^{(0)} = \quad & - J_{\text{bond}} \sum_{\langle rr' \rangle} [(\psi_{rr'}^{\dagger})^2 b_r b_{r'} + \text{H.c.}] \\
& - K_{\text{ring}} \sum_{\square} (\psi_{12}^{\dagger} \psi_{23} \psi_{34}^{\dagger} \psi_{41} + \text{H.c.}), \\
& + u_{\psi} \sum_{\langle rr' \rangle} (n_{rr'}^{\psi})^2 \quad + u_b \sum_r (n_r^b)^2
\end{aligned}
$$

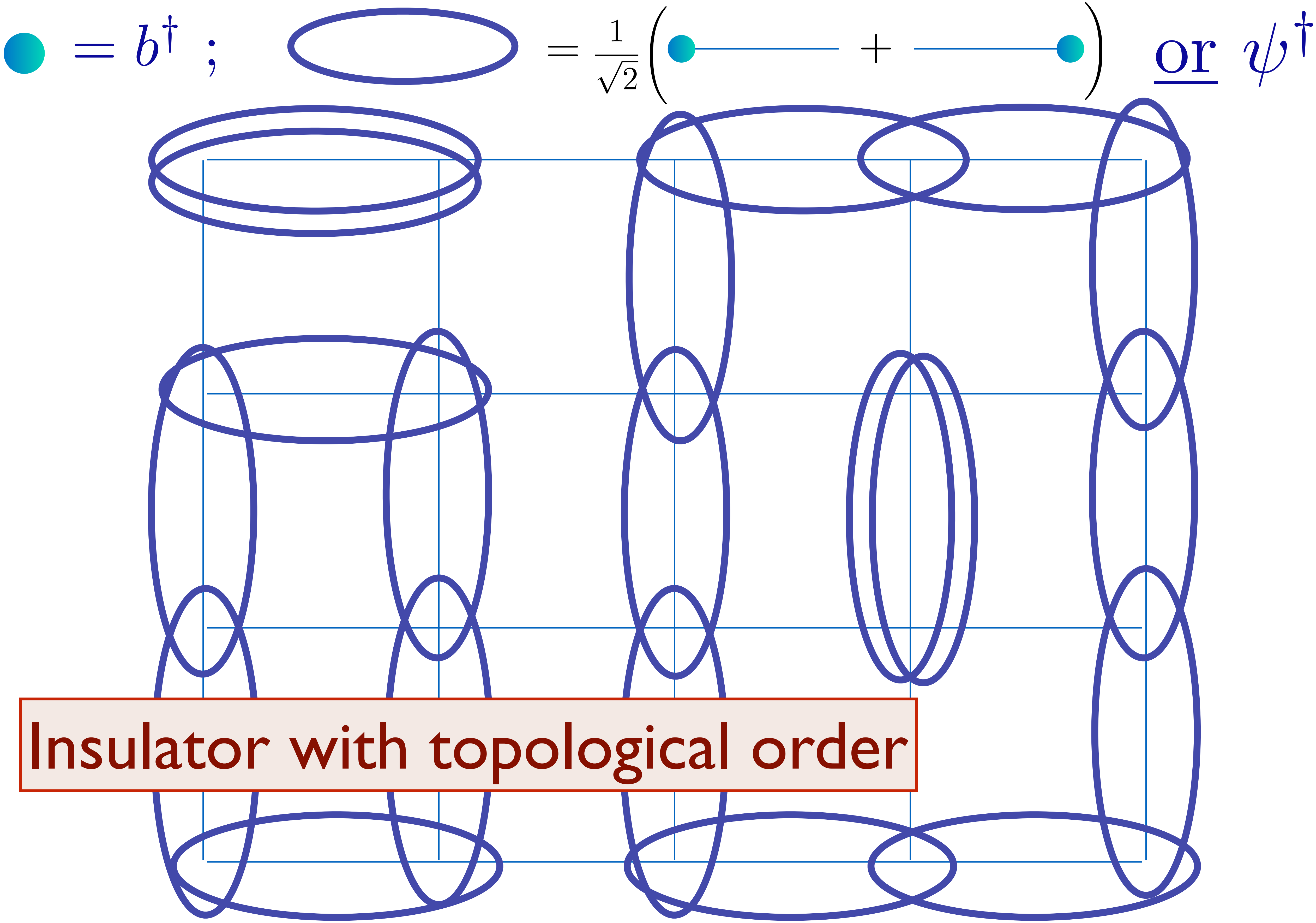
Bosons at unit density on the square lattice; $N_r = 1$ for all r



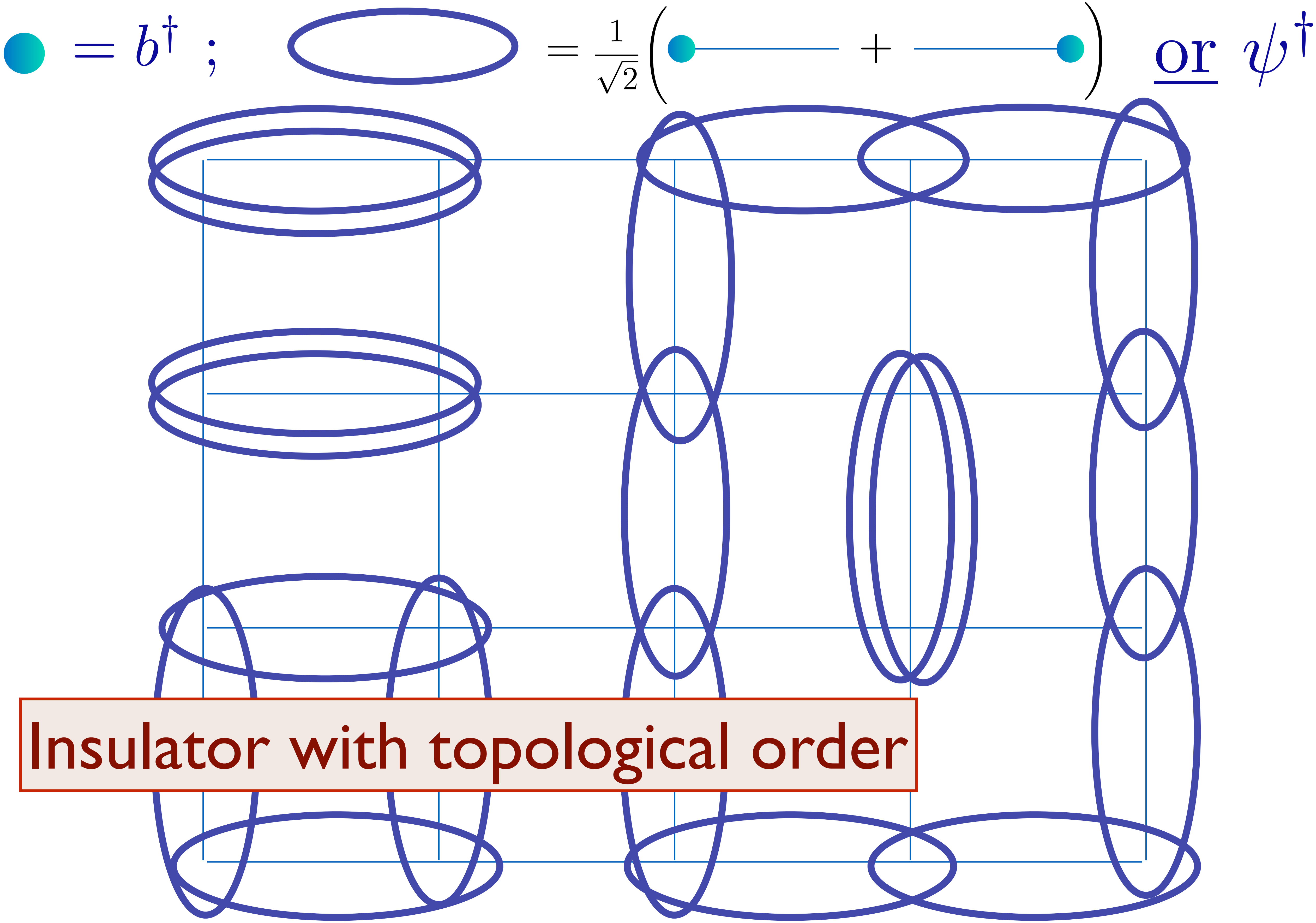
Bosons at unit density on the square lattice; $N_r = 1$ for all r



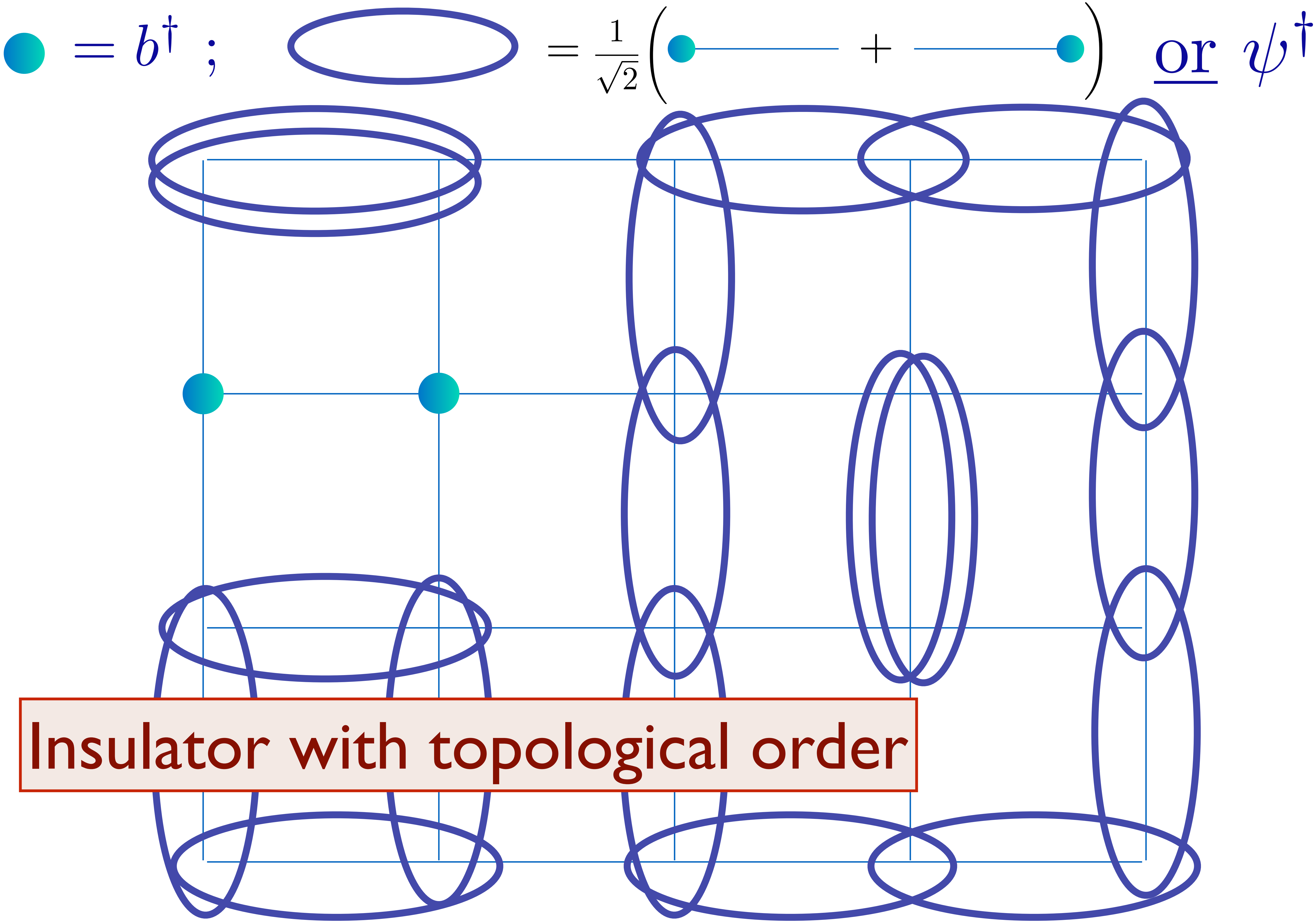
Bosons at unit density on the square lattice; $N_r = 1$ for all r



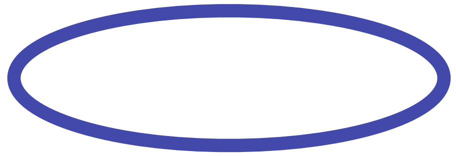
Bosons at unit density on the square lattice; $N_r = 1$ for all r

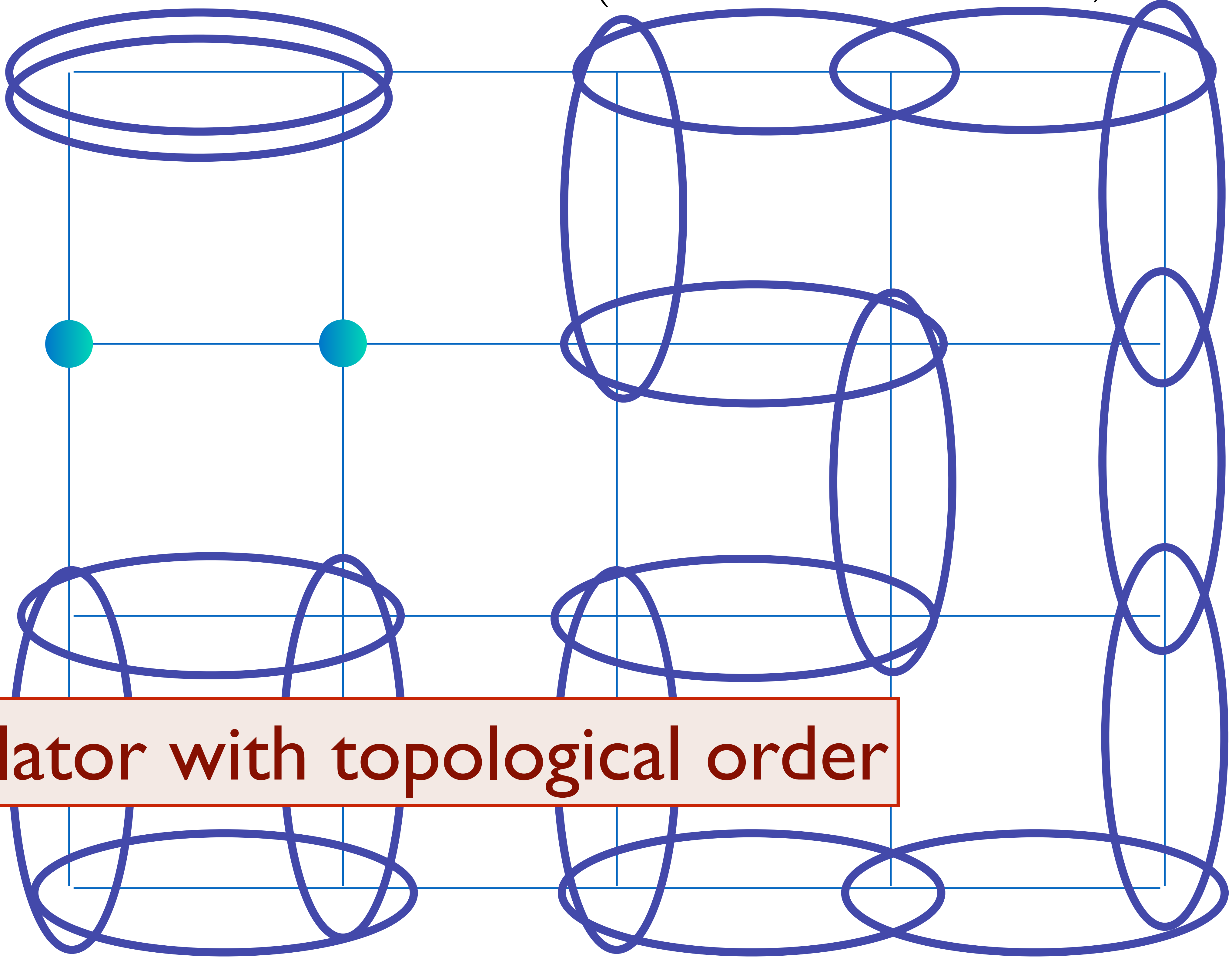


Bosons at unit density on the square lattice; $N_r = 1$ for all r



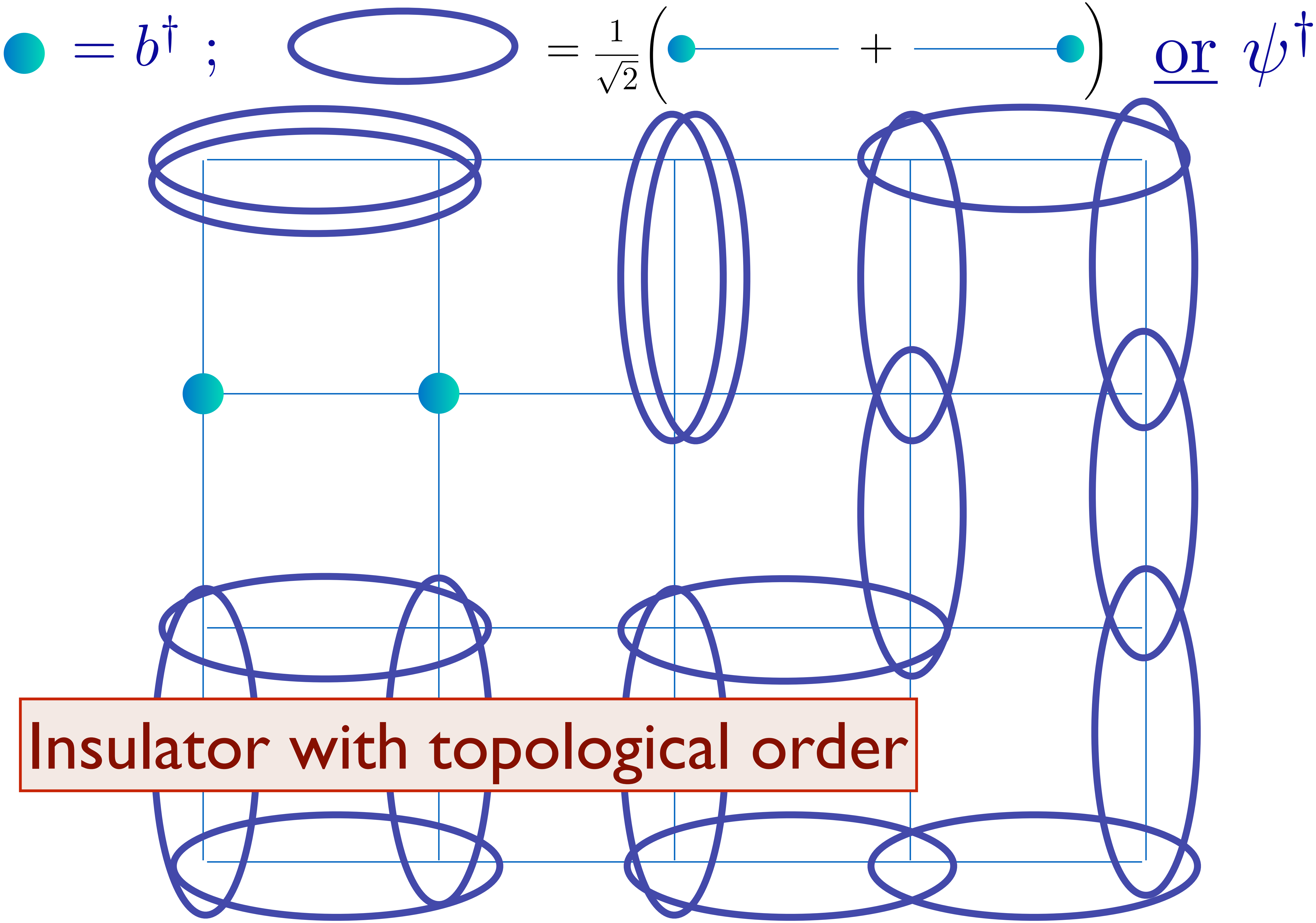
Bosons at unit density on the square lattice; $N_r = 1$ for all r

$\bullet = b^\dagger$;  $= \frac{1}{\sqrt{2}}\left(\bullet\text{———} + \text{———}\bullet \right)$ or $\psi^\dagger$



Insulator with topological order

Bosons at unit density on the square lattice; $N_r = 1$ for all r



Bosons at unit density on the square lattice; $N_r = 1$ for all r

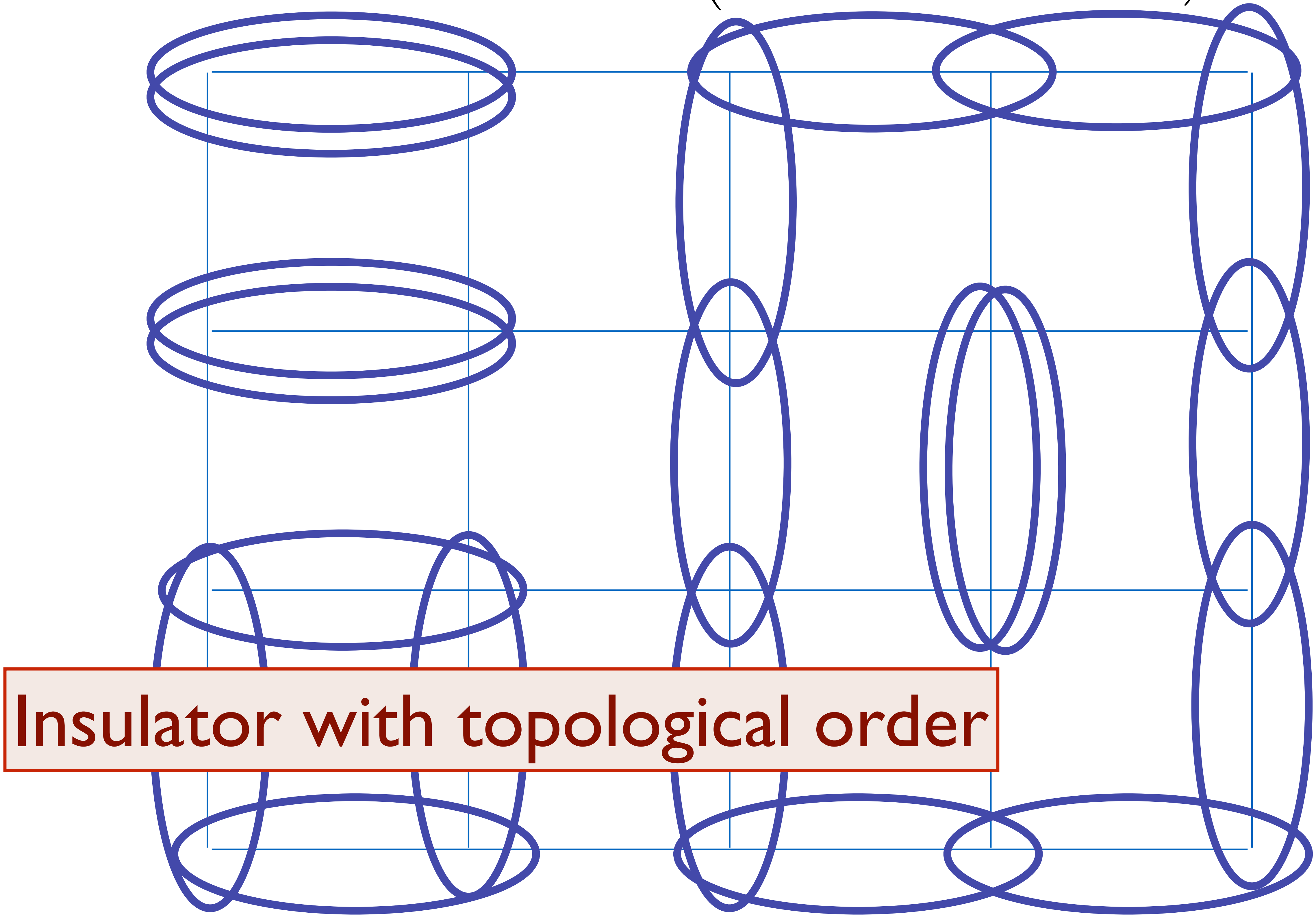
$$
\begin{aligned}
H_{\text{eff}}^{(0)} = \quad & - J_{\text{bond}} \sum_{\langle rr' \rangle} [(\psi_{rr'}^{\dagger})^2 b_r b_{r'} + \text{H.c.}] \\
& - K_{\text{ring}} \sum_{\square} (\psi_{12}^{\dagger} \psi_{23} \psi_{34}^{\dagger} \psi_{41} + \text{H.c.}), \\
& + u_{\psi} \sum_{\langle rr' \rangle} (n_{rr'}^{\psi})^2 \quad + u_b \sum_{r} (n_r^b)^2
\end{aligned}
$$

This effective Hamiltonian has exactly the form of a U(1) lattice gauge theory. We define $b_r \sim e^{i\varepsilon_r \phi_r}$ where $\varepsilon_r = \pm 1$ on the two sublattices, and $\psi_{rr'} \sim e^{i\varepsilon_r a_{r\alpha}}$, where $r' = r + \hat{e}_\alpha$, $\alpha = x, y$. Then the above theory can be written on cubic spacetime lattice in a "relativistic" from with action

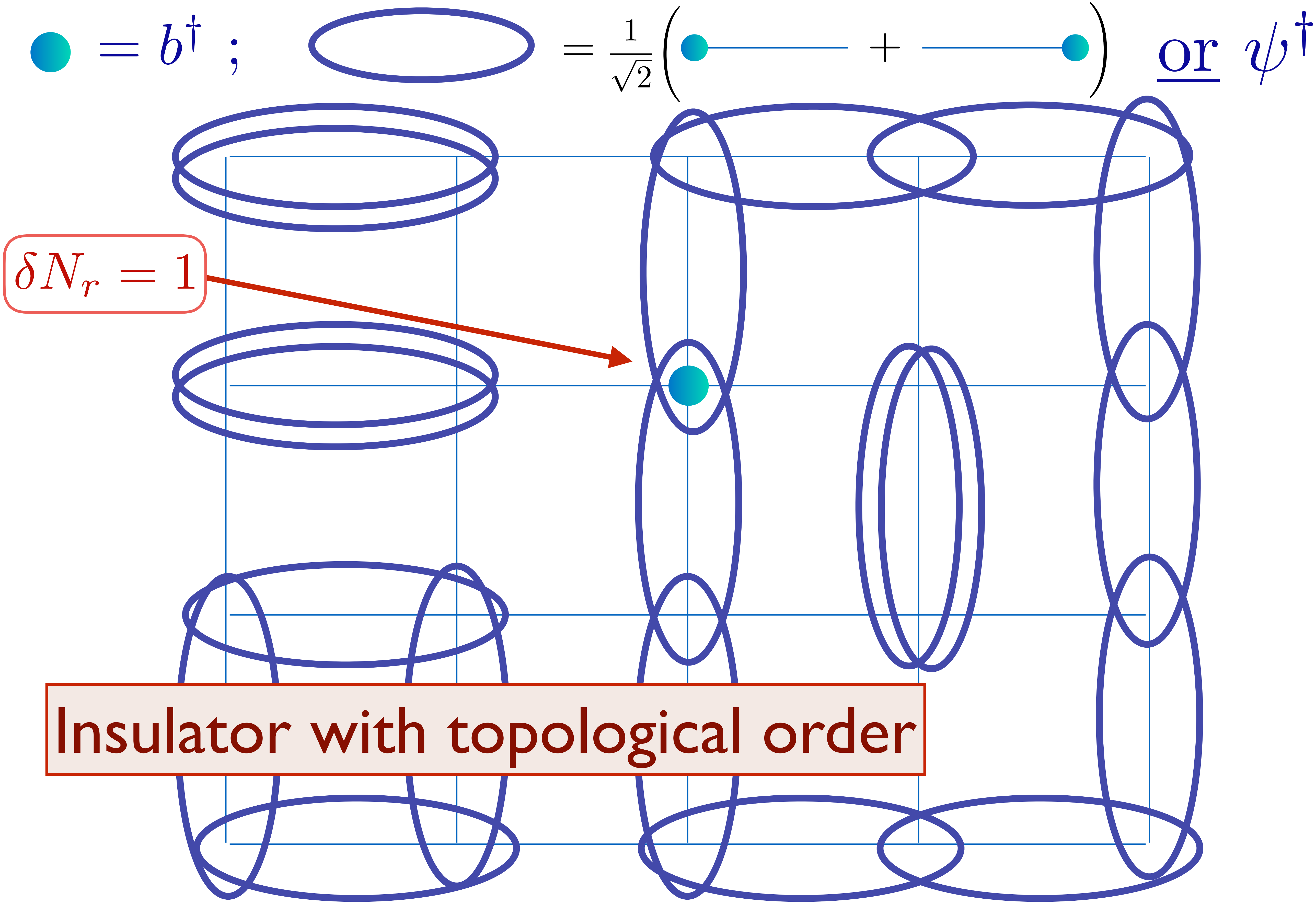
$$
\begin{aligned}
S \quad = \quad & -J \sum_{r} \cos(\Delta_\mu \phi_r - 2a_{r\mu}) \\
& - K \sum_{\square} \cos(\epsilon_{\mu\nu\lambda} \Delta_\nu a_\lambda)
\end{aligned}
$$

The boson $e^{i\phi_r}$ has U(1) gauge charge 2. The Gauss law for this lattice gauge theory is equivalent to the constraint $N_r = 1$ at all r.

$\bullet = b^\dagger$; $\bigcirc = \frac{1}{\sqrt{2}}\left(\bullet\!\!-\!\!-\!\!- \; + \; -\!\!-\!\!-\!\!\bullet \right)$ or $\psi^\dagger$

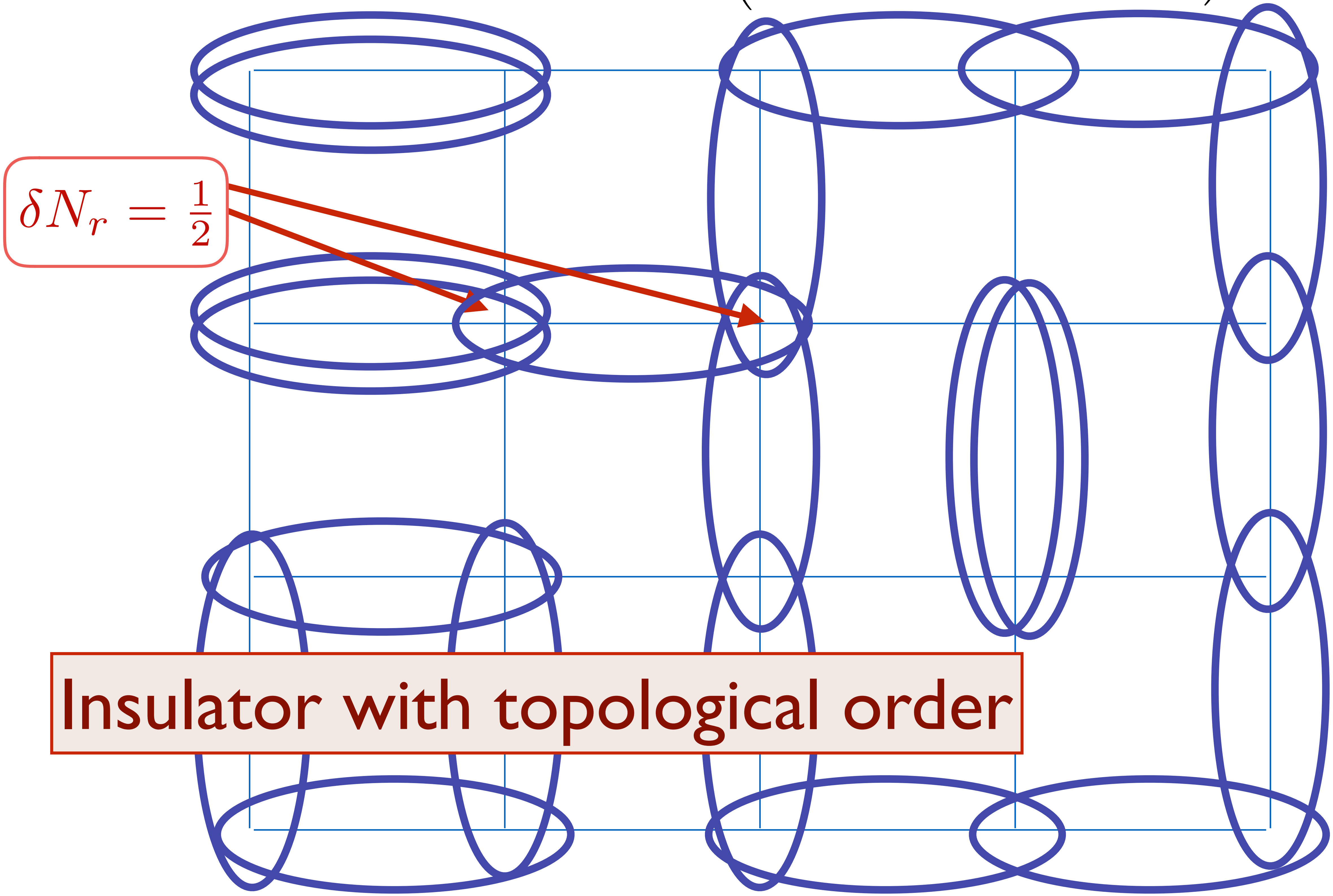


Bosons at unit density on the square lattice; $N_r = 1$ for all r

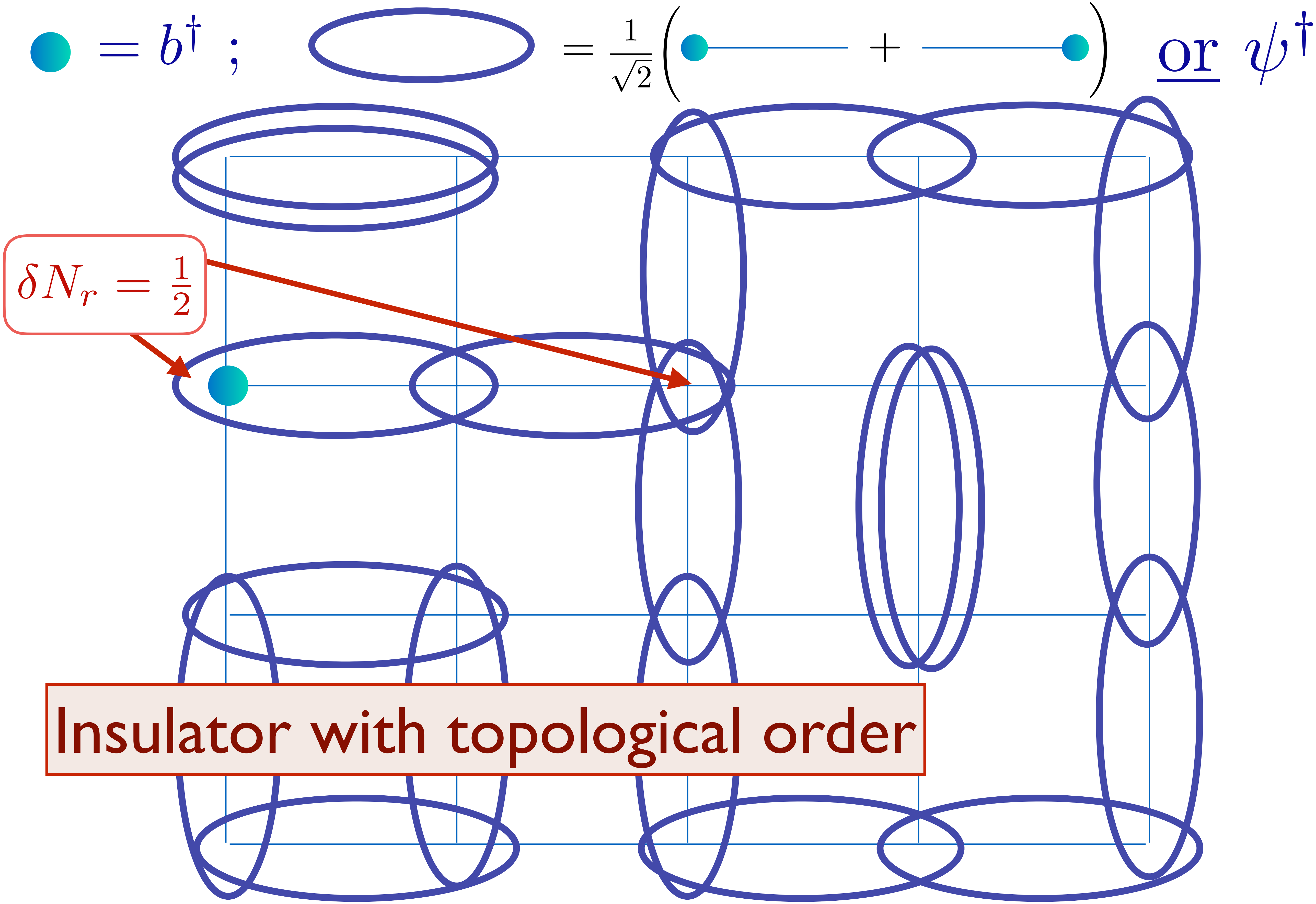


Add a boson on site.

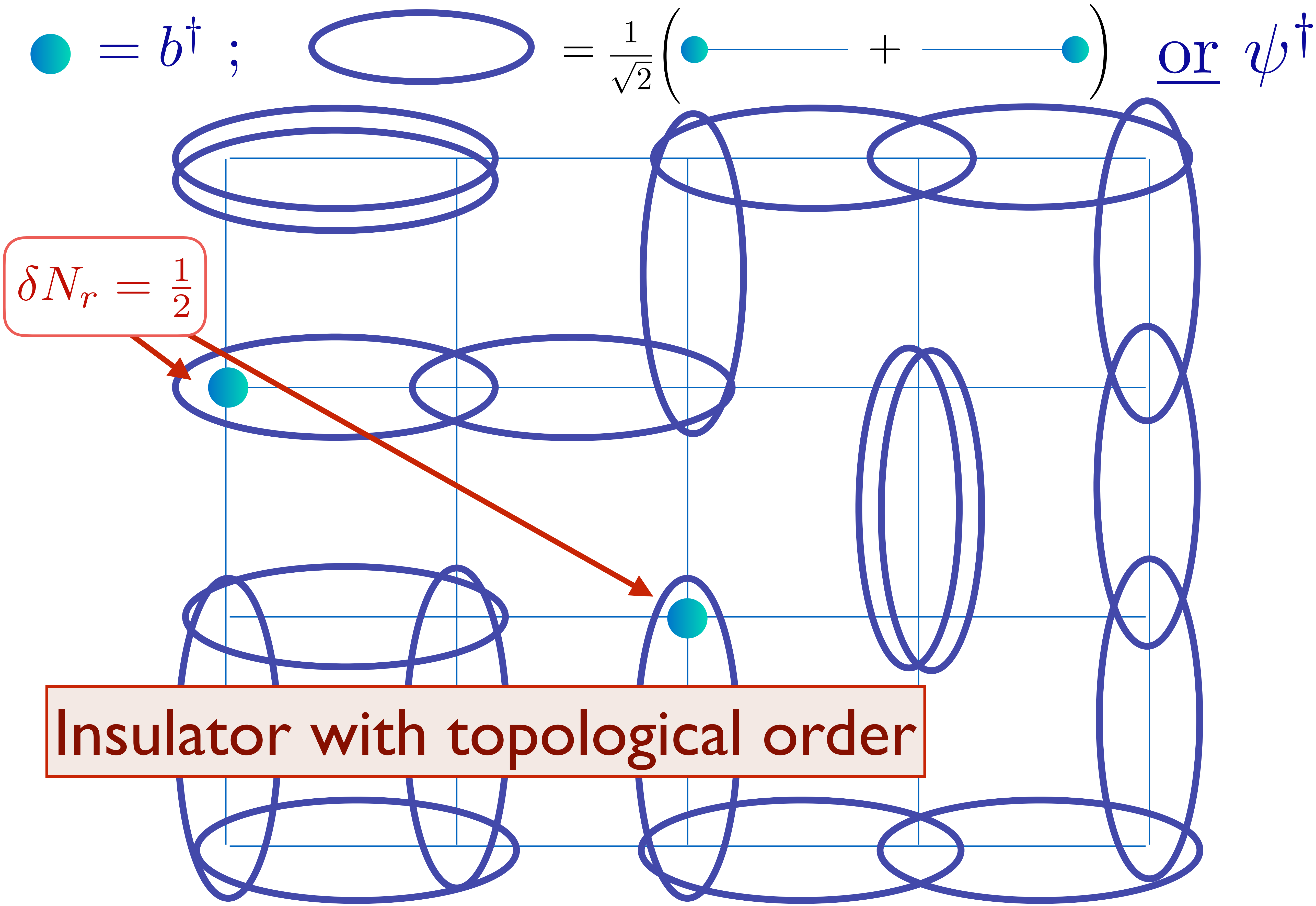
$\bullet = b^\dagger$; (ellipse) $= \frac{1}{\sqrt{2}}\left(\bullet\!\!-\!\!-\!\!- \;+\; -\!\!-\!\!-\!\!\bullet \right)$ or $\psi^\dagger$



At large U, energy is lowered when the boson to splits into 2 half bosons.



The half charge bosons can then move freely through the lattice



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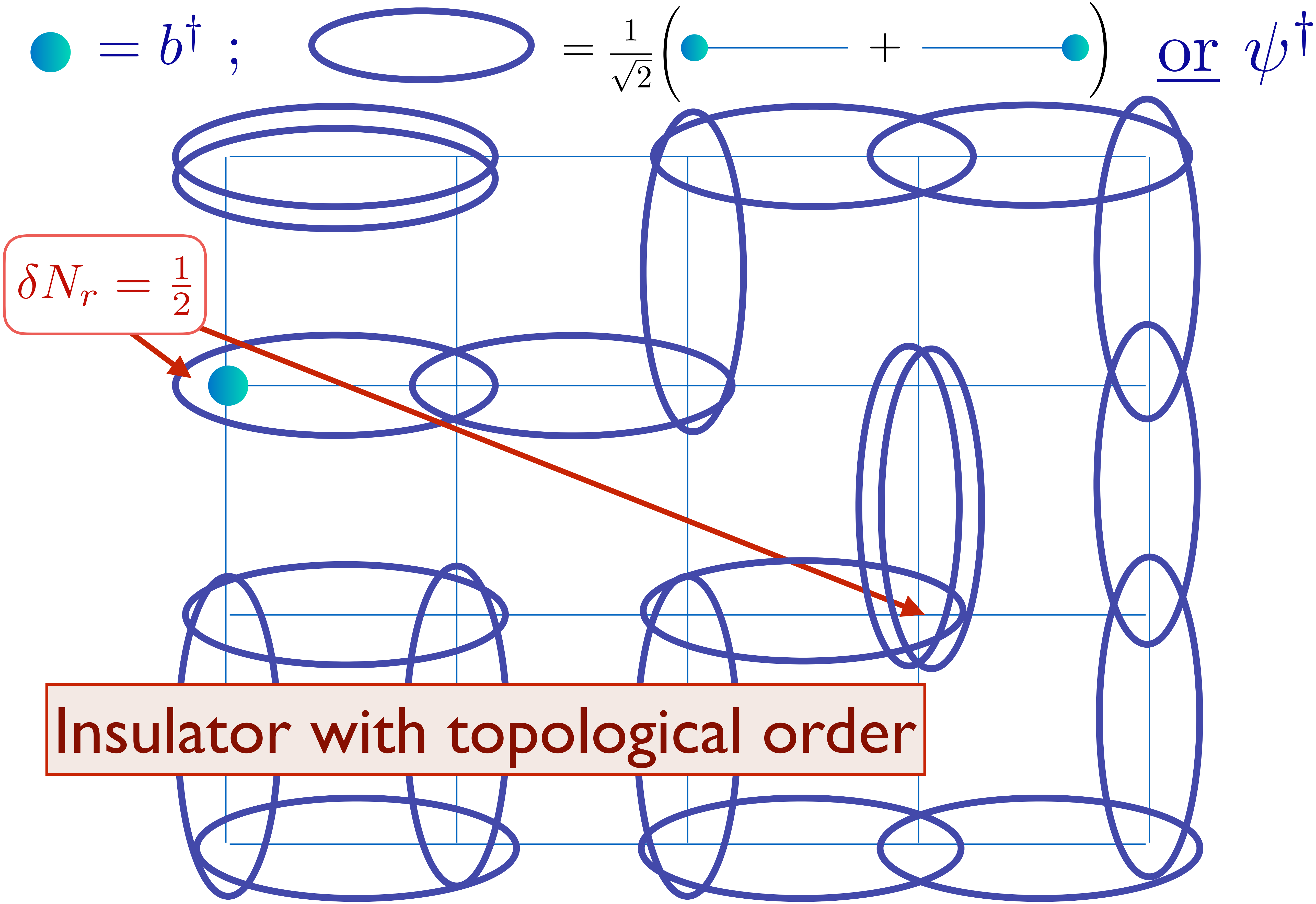

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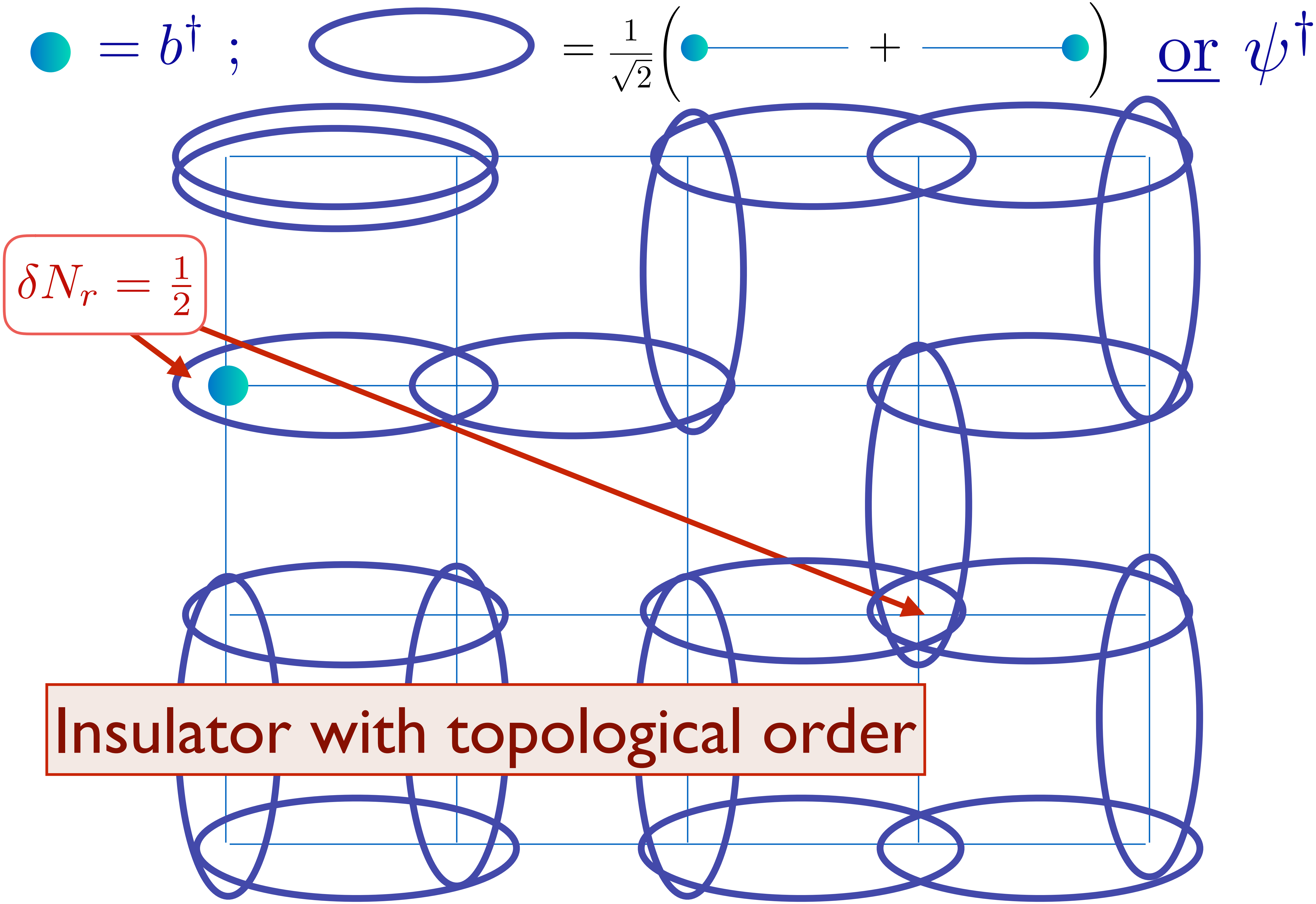
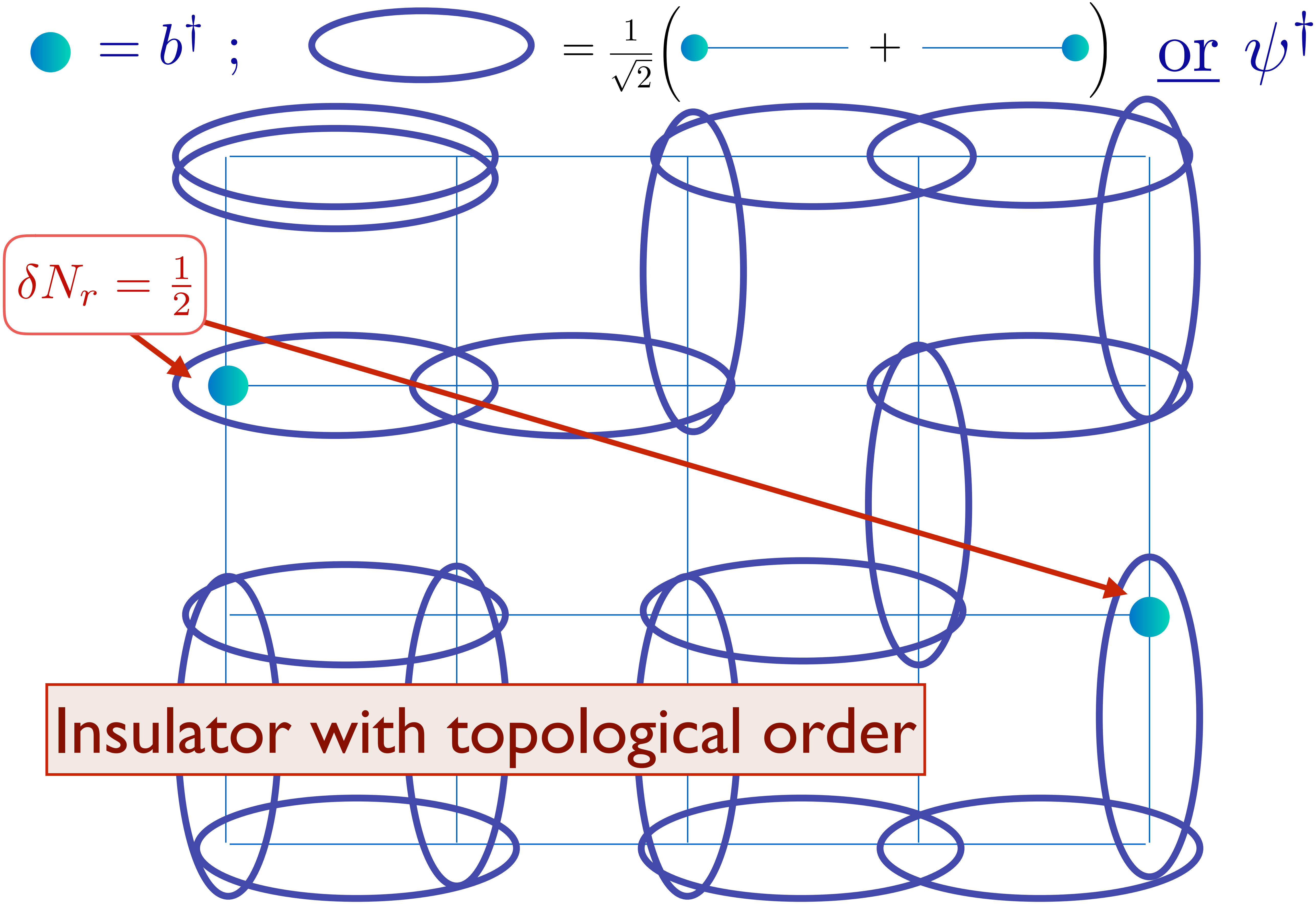


The half charge bosons can then move freely through the lattice



The half charge bosons can then move freely through the lattice

We can move beyond the $N_r = 1$ subspace, and account for these half-charged states, by introducing an operator $e^{i\bar{\theta}_r}$ which has U(1) gauge charge 1, and global boson number charge $\varepsilon_r/2$. Then gauge-invariant boson operators for the site and bond bosons now become

$$b_r = e^{i\varepsilon_r(\phi_r - 2\bar{\theta}_r)} \quad , \quad \psi_{r\alpha} = e^{i\varepsilon_r(a_{r\alpha} - \Delta_\alpha \bar{\theta}_r)}$$

It is easy to verify that these new representations leave the J and K terms in the previous spacetime lattice action for $N_r = 1$ independent of $\bar{\theta}_r$. We can also write the half-charge hopping terms illustrated in the previous slides in this formulation. The final form of the action so obtained is simplest in terms of a new field θ_r in the mapping

$$\bar{\theta}_r = \theta_r \text{ when } \varepsilon_r = 1 \quad , \quad \bar{\theta}_r = -\theta_r + \phi_r \text{ when } \varepsilon_r = -1$$

Note that $e^{i\theta_r}$ which has U(1) gauge charge 1, and global boson number charge 1/2

Bosons at unit density on the square lattice

Collecting these transformations, we obtain the complete action for the full phase diagram is

$$\boxed{\begin{aligned} S \;=\; & -t\sum_r \cos(\Delta_\mu\theta_r - a_{r\mu} - A_{r\mu}/2) \\ & - J\sum_r \cos(\Delta_\mu\phi_r - 2a_{r\mu}) \\ & - K\sum_\square \cos(\epsilon_{\mu\nu\lambda}\Delta_\nu a_\lambda) \end{aligned}}$$

In this form, $e^{i\theta_r}$ has U(1) gauge charge 1, and boson number charge 1/2; $e^{i\phi_r}$ has U(1) gauge charge 2, and boson number charge 0. We have also included an external (fixed) gauge field A_μ, which couples to the boson number. The gauge-invariant boson operator is

$$\boxed{b_r \sim e^{-i2\theta_r + i\phi_r}}$$

and this only has A_μ charge 1/2.

Bosons at unit density on the square lattice

What we have achieved so far:

We started with a model of bosons, b_i on the square lattice, at a density of one boson per site, with short-range interactions. This model has a global U(1) symmetry under which $b \to b\, e^{i\alpha}$.

We showed that this model can be rewritten (under certain conditions) in terms of half-charged boson $h \sim e^{i\theta}$ and a Higgs field $\Phi \sim e^{i\phi}$, so that

$$b \sim \Phi (h^*)^2$$

This theory has the additional *gauge invariance* under which

$$\Phi \to \Phi\, e^{2i\vartheta(x)} \quad , \quad h \to h\, e^{i\vartheta(x)}$$

Then the phases of the theory can be described by the Lagrangian

$$\begin{aligned} \mathcal{L} &= |(\partial_\mu - i a_\mu - i A_\mu/2) h|^2 + m_h^2 |h|^2 + u_h |h|^4 \\ &\quad + |(\partial_\mu - 2i a_\mu)\Phi|^2 + m_\Phi^2 |\Phi|^2 + u_\Phi |\Phi|^4 - K \cos(\epsilon_{\mu\nu\lambda} \Delta_\nu a_\lambda) \ldots \end{aligned}$$

Bosons at unit density on the square lattice

The phases are:

(1) **Superfluid**: $\langle \Phi \rangle \neq 0$, $\langle h \rangle \neq 0$. The global U(1) symmetry is broken, and so there is Goldstone mode. The gauge fluctuations are completely Higgsed.

(2) **Trivial insulator**: $\langle \Phi \rangle = 0$, $\langle h \rangle = 0$ <u>and</u> $\langle \Phi \rangle = 0$, $\langle h \rangle \neq 0$: Strong gauge fluctuations confine Φ and h into the composite $b \sim \Phi^* h^2$. The b quanta are gapped excitations. Such a description is obtained in the confining phase of the compact U(1) gauge theory where there is no Higgs condensate. However, it is *also* obtained in the Higgs phase where there is a h condensate, and such a Higgs phase is smoothly connected to the confining phase.

Bosons at unit density on the square lattice

(3) **Insulator with topological order**: $\langle \Phi \rangle \neq 0$, $\langle h \rangle = 0$. Now the Φ field is condensed and this gaps the U(1) gauge excitations, and h quanta are deconfined gapped excitations. However, there is another gapped excitation: the analog of the Abrikosov vortex in Φ, which we denote by v. We perform a particle-vortex duality transform to obtain an effective action for the gapped field v. Then we obtain the field theory with the continuum Lagrangian (we are being a little sloppy about the role of monopoles here)

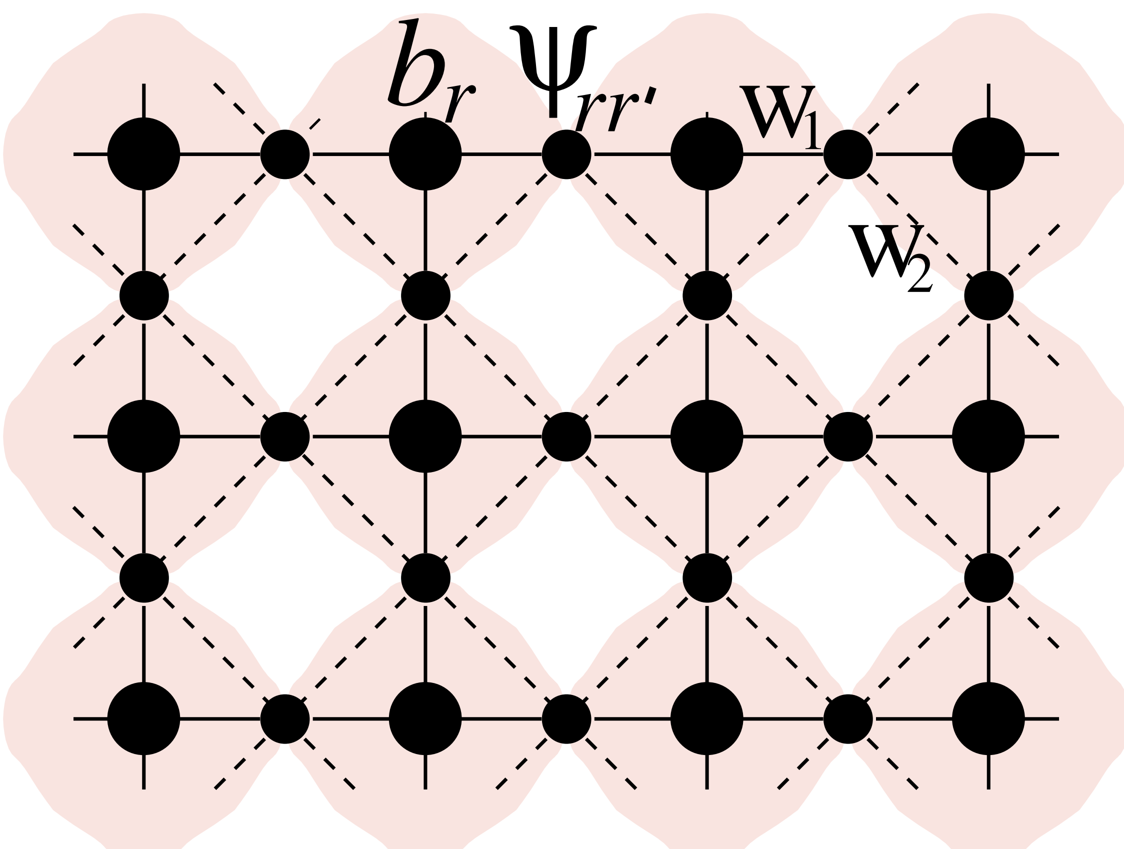
$$
\begin{aligned}
\mathcal{L} \;\; = \;\; & |(\partial_\mu - i a_\mu) h|^2 + m_h^2 |h|^2 + u_h^2 |h|^4 \\
& + |(\partial_\mu - i b_\mu) v|^2 + m_v^2 |v|^2 + u_v^2 |v|^4 \\
& + \frac{i}{\pi} \epsilon_{\mu\nu\lambda} a_\mu \partial_\nu b_\lambda + \frac{i}{2\pi} \epsilon_{\mu\nu\lambda} A_\mu \partial_\nu b_\lambda
\end{aligned}
$$

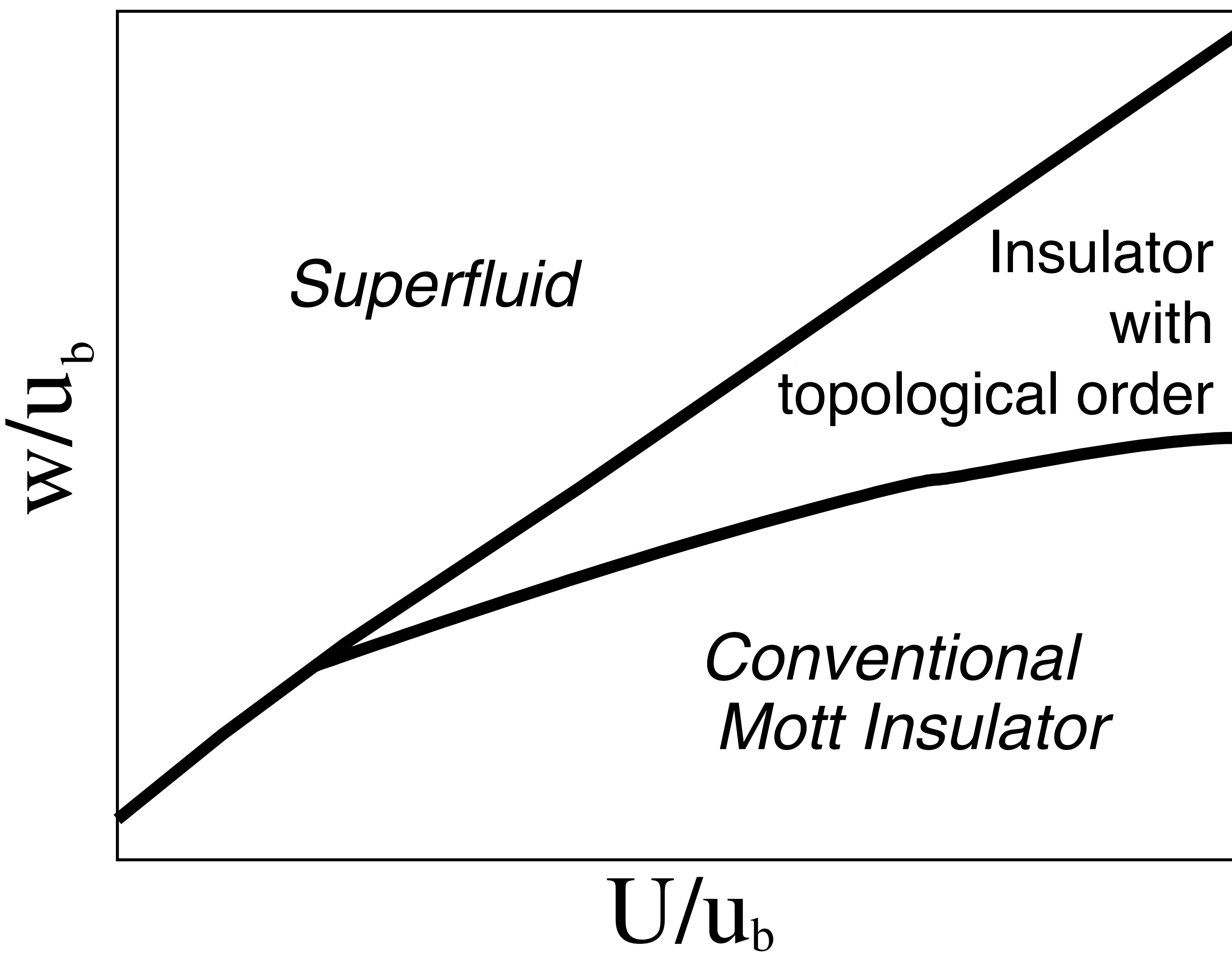
Bosons at unit density on the square lattice

(3) **Insulator with topological order**: $\langle \Phi \rangle \neq 0$, $\langle h \rangle = 0$. Now the Φ field is condensed and this gaps the U(1) gauge excitations, and h quanta are deconfined gapped excitations. However, there is another gapped excitation: the analog of the Abrikosov vortex in Φ, which we denote by v. We perform a particle-vortex duality transform to obtain an effective action for the gapped field v. Then we obtain the field theory with the continuum Lagrangian (we are being a little sloppy about the role of monopoles here)

$$\begin{aligned} \mathcal{L} \;=\; & |(\partial_\mu - i a_\mu) h|^2 + m_h^2 |h|^2 + u_h^2 |h|^4 \\ & + |(\partial_\mu - i b_\mu) v|^2 + m_v^2 |v|^2 + u_v^2 |v|^4 \\ & \boxed{+ \frac{i}{\pi} \epsilon_{\mu\nu\lambda} a_\mu \partial_\nu b_\lambda + \frac{i}{2\pi} \epsilon_{\mu\nu\lambda} A_\mu \partial_\nu b_\lambda} \quad \text{TQFT} \end{aligned}$$

Bosons at unit density on the square lattice

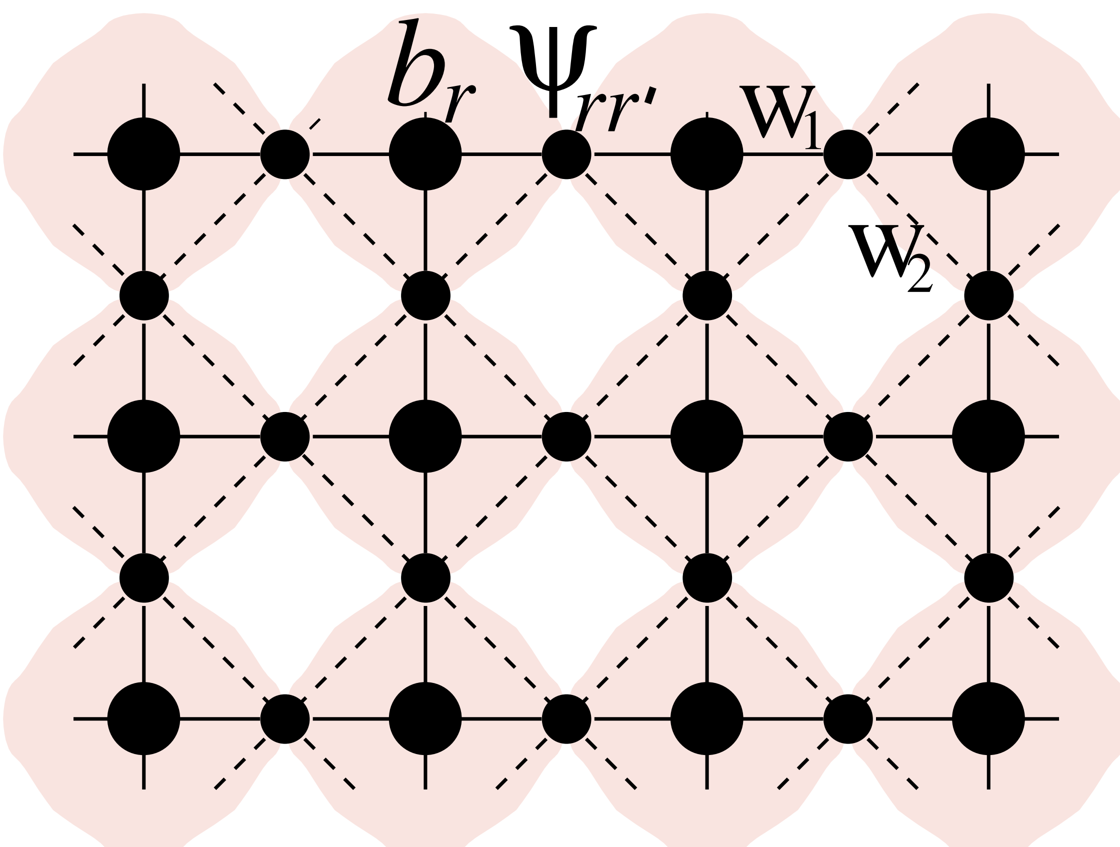


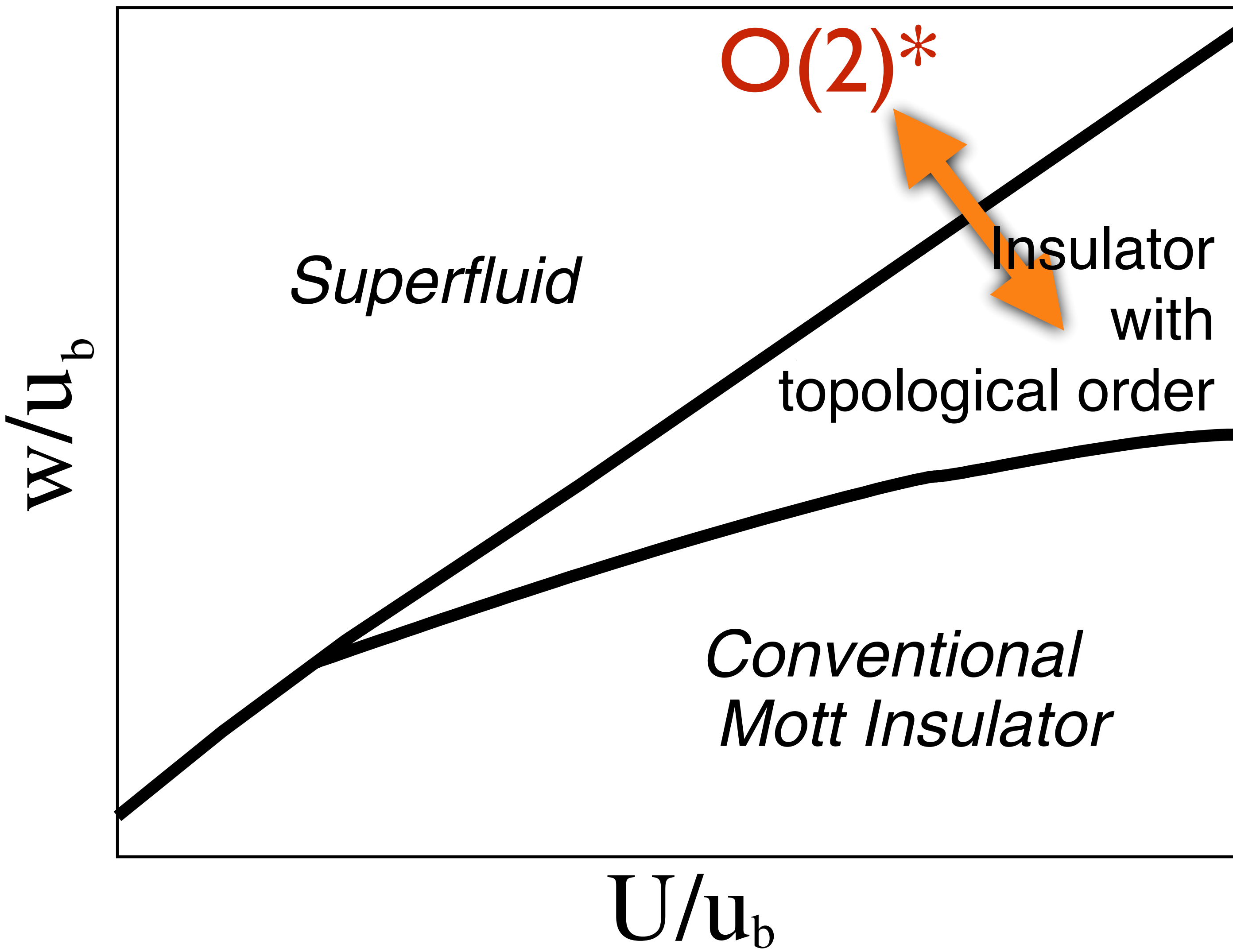


Average of one boson per site: $\langle N_r \rangle = 1$

R. Jalabert and S. Sachdev Phys. Rev. B 44, 686 (1991); S. Sachdev and M. Vojta, Journal of the Physical Society of Japan **69,** Suppl. B, 1 (2000); O.I. Motrunich and T. Senthil, PRL **89**, 277004 (2002).

A.V. Chubukov, T. Senthil and S. Sachdev, PRL **72**, 2089 (1994)





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Transition from superfluid to insulator with $\mathbb{Z}_2$ topological order

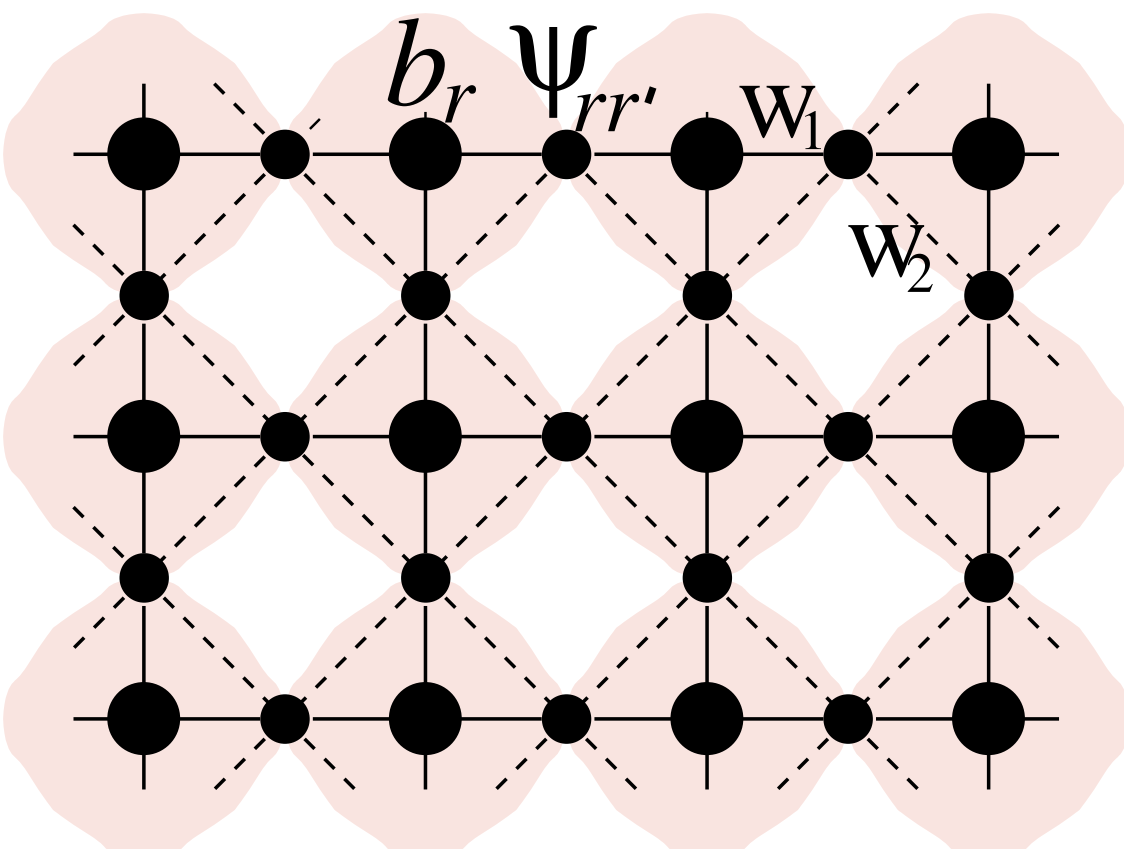
The boson field $\Phi \sim e^{i\phi}$ is condensed in both phases. This gaps out the a_μ field. So the quantum criticality is just the critical theory of the half-charged boson $h \sim e^{i\theta}$

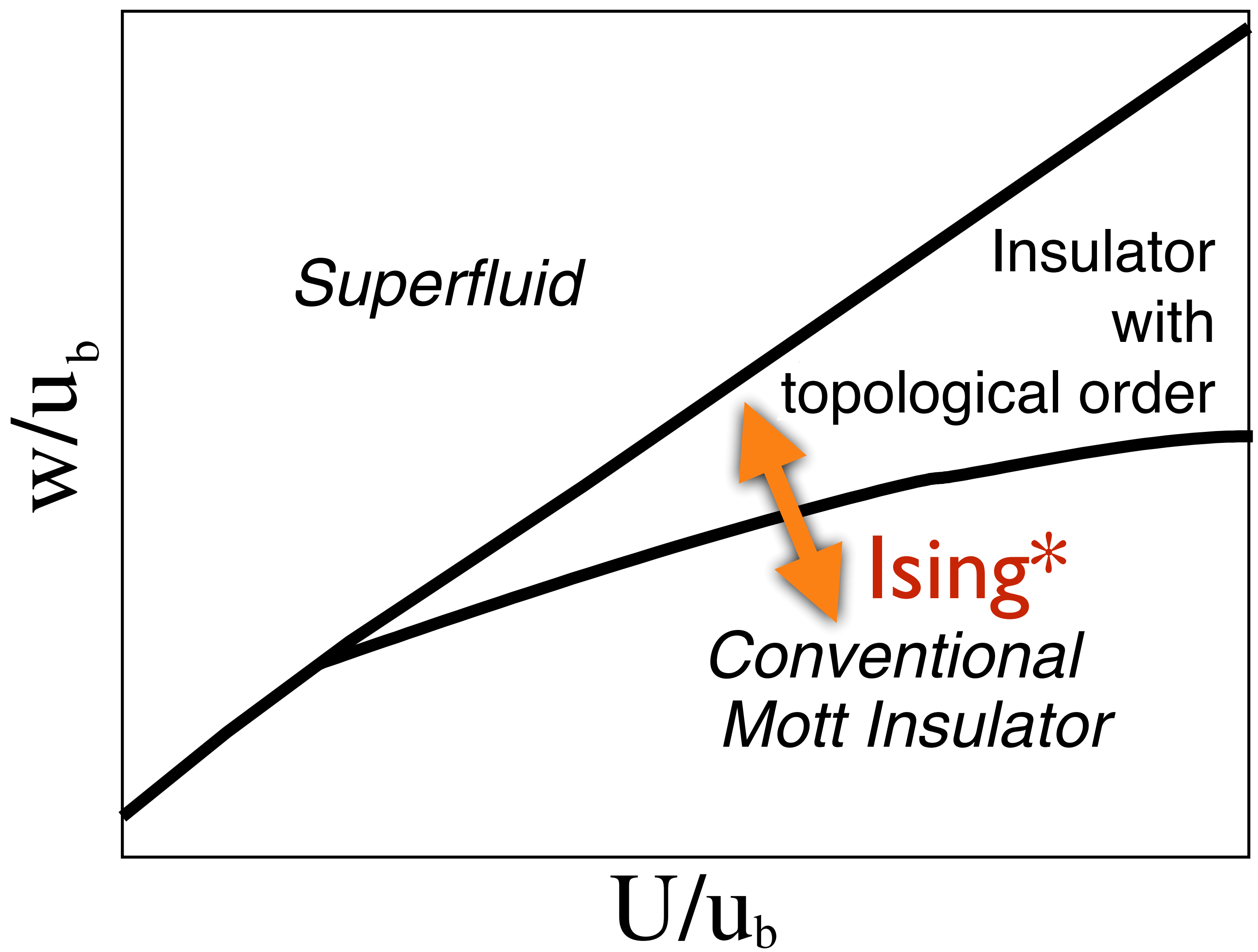
$$\mathcal{L} = |(\partial_\mu - A_\mu/2)h|^2 + m_h^2|h|^2 + u|h|^4$$

This is the O(2)* Wilson-Fisher theory. The $*$ refers to the fact that the spectrum of the theory only contains operators which are invariant under $h \to -h$: it is not possible to create a single half-boson, and they always appear in pairs. The critical theory therefore involves a fractionalized field, and *not* the order parameter: this is an example of a *deconfined critical point*.

Bosons at unit density on the square lattice

R. Jalabert and S. Sachdev Phys. Rev. B 44, 686 (1991); S. Sachdev and M. Vojta, Journal of the Physical Society of Japan **69,** Suppl. B, 1 (2000); O.I. Motrunich and T. Senthil, PRL **89**, 277004 (2002).





Bosons at unit density on the square lattice

Transition from trivial insulator to insulator with $\mathbb{Z}_2$ topological order

The half-charged boson $h \sim e^{i\theta}$ is gapped in both phases. So as a first attempt, we just write down the theory of the critical Higgs field $\Phi \sim e^{i\phi}$

$$\mathcal{L} = |(\partial_\mu - 2ia_\mu)\Phi|^2 + m_\Phi^2|\Phi|^2 + u|\Phi|^4 + \frac{1}{2e^2}(\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda)^2$$

However, this turns out to be incorrect: we cannot ignore the monopoles is the compact U(1) gauge field, a_μ.

Transition from trivial insulator to insulator with $\mathbb{Z}_2$ topological order

The correct theory can be obtained by performing a particle-vortex duality on both the half-charged boson $h \sim e^{i\theta}$ (to a double-vortex d) and the Higgs field $\Phi \sim e^{i\phi}$ (to the vortex v)

$$\begin{aligned}\mathcal{L} &= |(\partial_\mu - i c_\mu) d|^2 + m_d^2 |d|^2 + |(\partial_\mu - i b_\mu) v|^2 + m_v^2 |v|^2 \\ &+ \frac{i}{2\pi} \epsilon_{\mu\nu\lambda} b_\mu \partial_\nu A_\lambda + \frac{i}{2\pi} \epsilon_{\mu\nu\lambda} a_\mu \partial_\nu (c_\lambda - 2 b_\lambda) - K \sum_{\square} \cos(\epsilon_{\mu\nu\lambda} \Delta_\nu a_\lambda)\end{aligned}$$

After integrating over a_μ this becomes equivalent to the field theory

$$\begin{aligned}\mathcal{L} &= |(\partial_\mu - i 2 b_\mu) d|^2 + m_d^2 |d|^2 + |(\partial_\mu - i b_\mu) v|^2 + m_v^2 |v|^2 \\ &+ \frac{i}{2\pi} \epsilon_{\mu\nu\lambda} b_\mu \partial_\nu A_\lambda - y_m \, d^* \, v^2 + \text{c.c.},\end{aligned}$$

where y_m is the monopole fugacity.

Bosons at unit density on the square lattice

Transition from trivial insulator to insulator with $\mathbb{Z}_2$ topological order

The double vortex d is condensed in both phases: so we can replace d by a constant, and ignore the gapped b_μ gauge field. Because of the monopole fugacity term, the symmetry $v \to v e^{i\varphi}$ is broken. Only the *real part* of v becomes critical at the transition. Denoting $w \sim v + v^*$, the vison field, we have the critical theory

$$\mathcal{L} \;=\; (\partial_\mu w)^2 + m_w^2 w^2 + u w^4$$

This is the Ising* Wilson-Fisher critical theory. Again, the $*$ refers to the fact that all observable operators must be invariant under $w \to -w$.

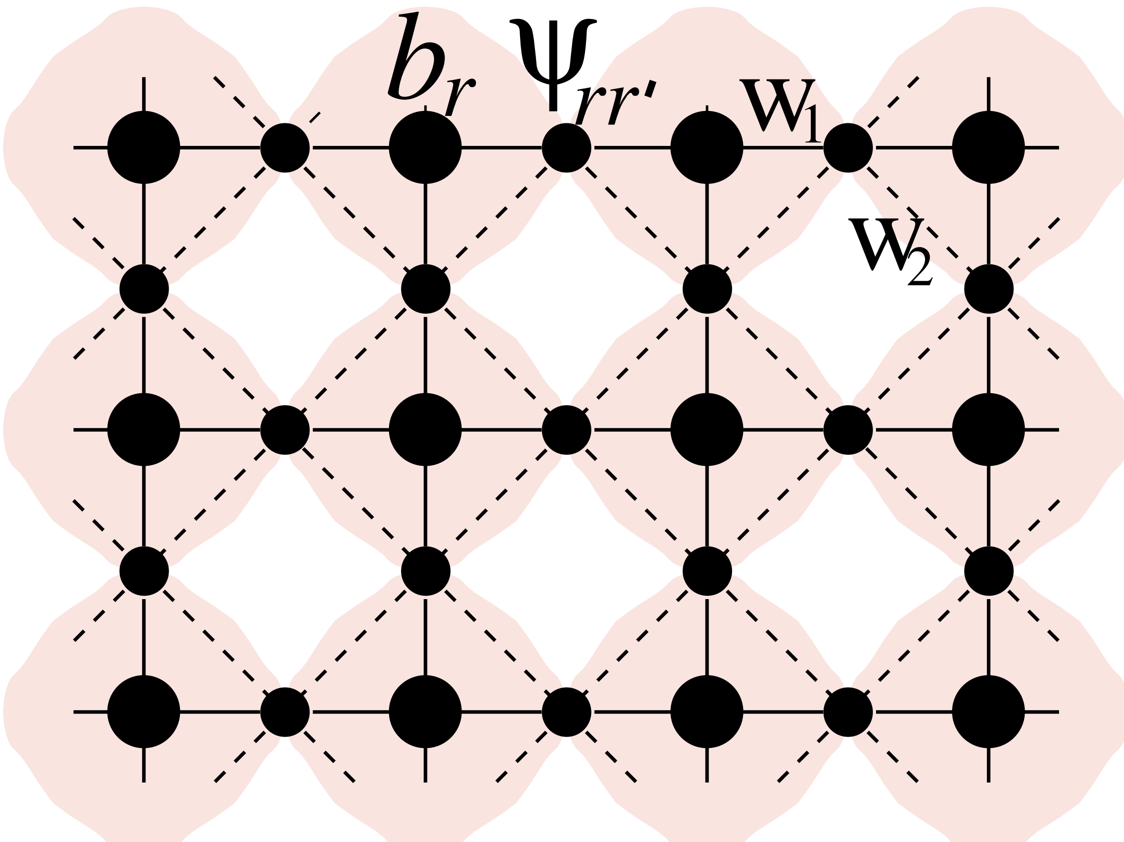
An alternative formulation in two dimensions preemptively accounts for the strong effects of monopoles. We take the strong-coupling limit, in which the Higgs field is locally condensed, to a $\mathbb{Z}_2$ gauge field where $e^{i\phi_r} = 1$ and $e^{ia_{r\mu}} = \pm 1$. Then we have the Hamiltonian of a $\mathcal{Z}_2$ gauge field coupled to a half-charged boson $e^{i\theta_r}$:

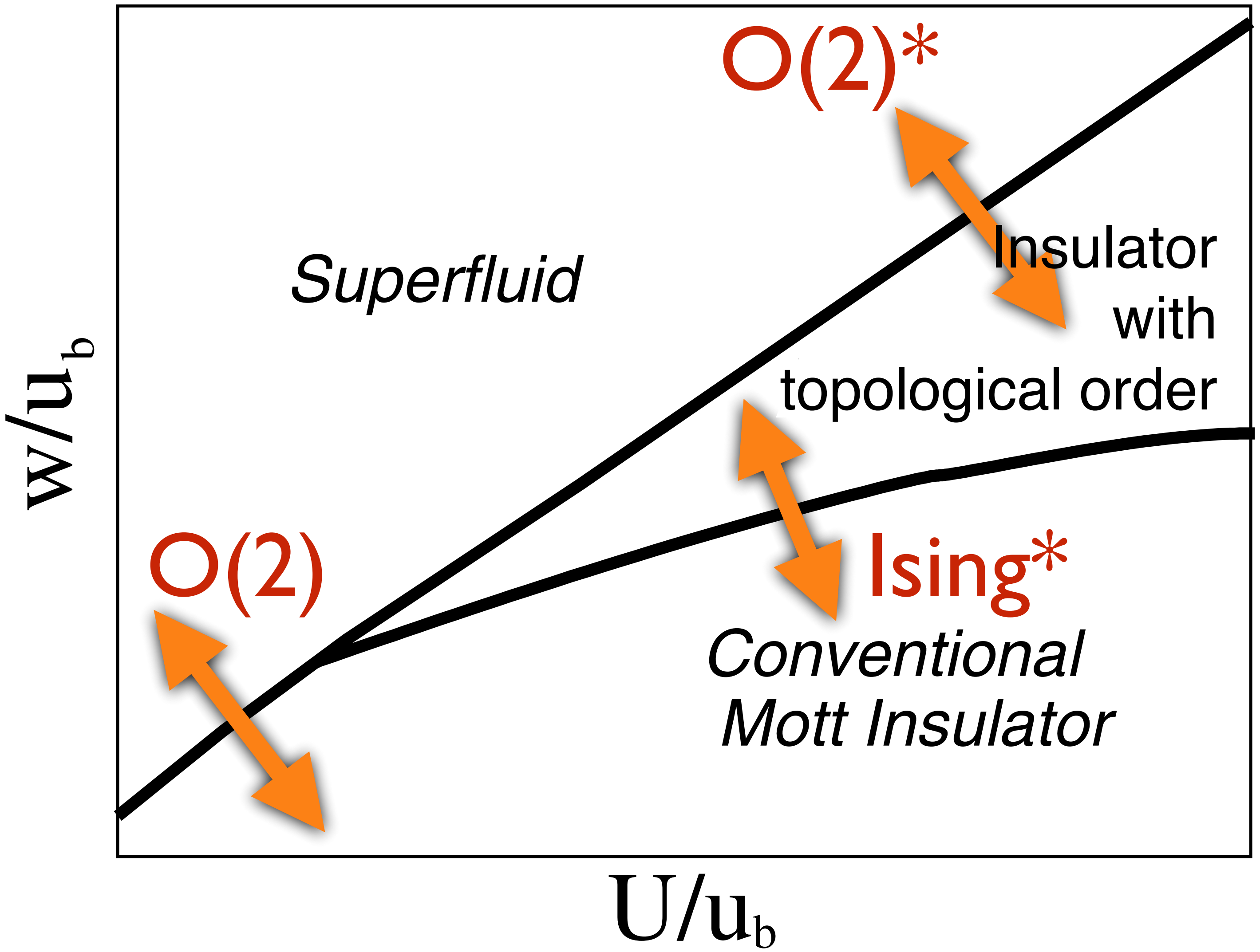
$$
\begin{aligned}
H &= -t \sum_{r,\alpha=x,y} \tau^z_{r\alpha} \cos(\Delta_\alpha \theta_r - A_{r\alpha}/2) \\
&\quad - K \sum_{\square} \tau^z \tau^z \tau^z \tau^z \\
&\quad - g \sum_{r,\alpha=x,y} \tau^x_{r\alpha}
\end{aligned}
$$

The $\mathbb{Z}_2$ gauge field is dual to the Ising* theory encountered earlier.

Bosons at unit density on the square lattice

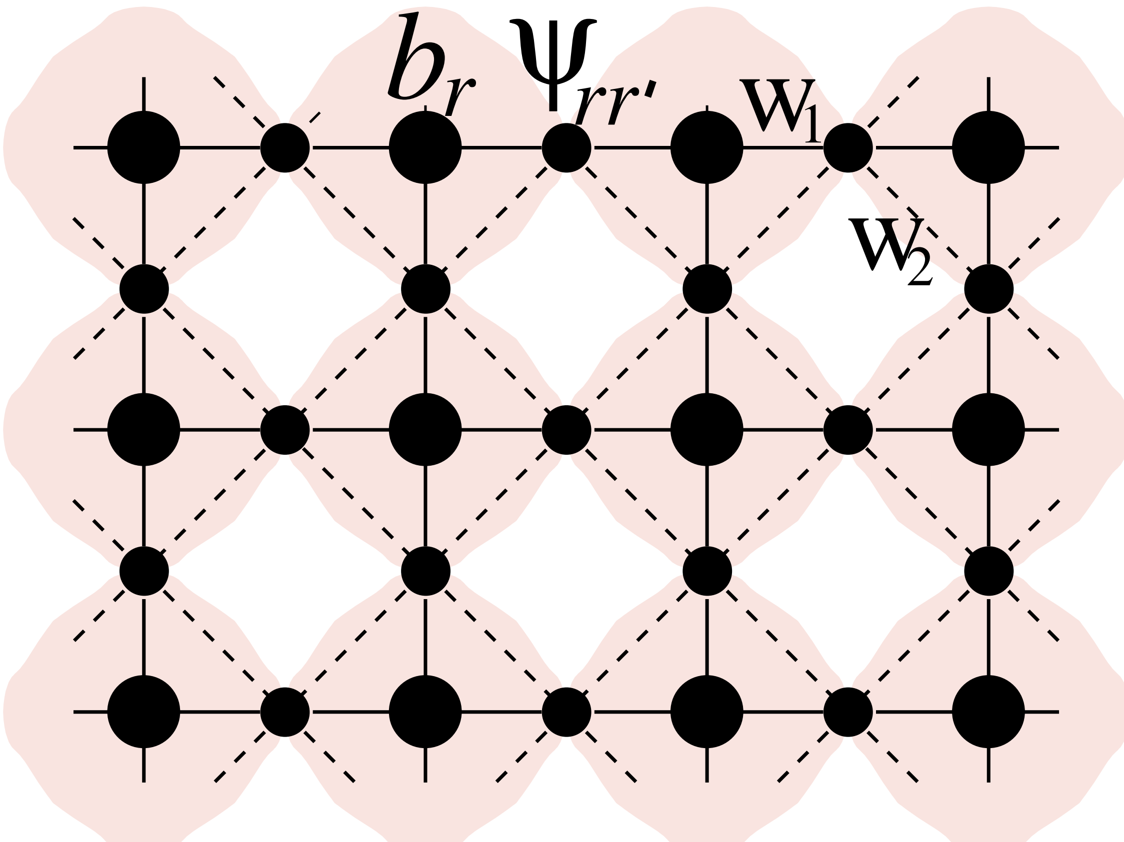
R. Jalabert and S. Sachdev PRB 44, 686 (1991); A.V. Chubukov, T. Senthil and S. Sachdev, PRL **72**, 2089 (1994); S. Sachdev and M. Vojta, Journal of the Physical Society of Japan **69,** Suppl. B, 1 (2000); O.I. Motrunich and T. Senthil, PRL **89**, 277004 (2002).

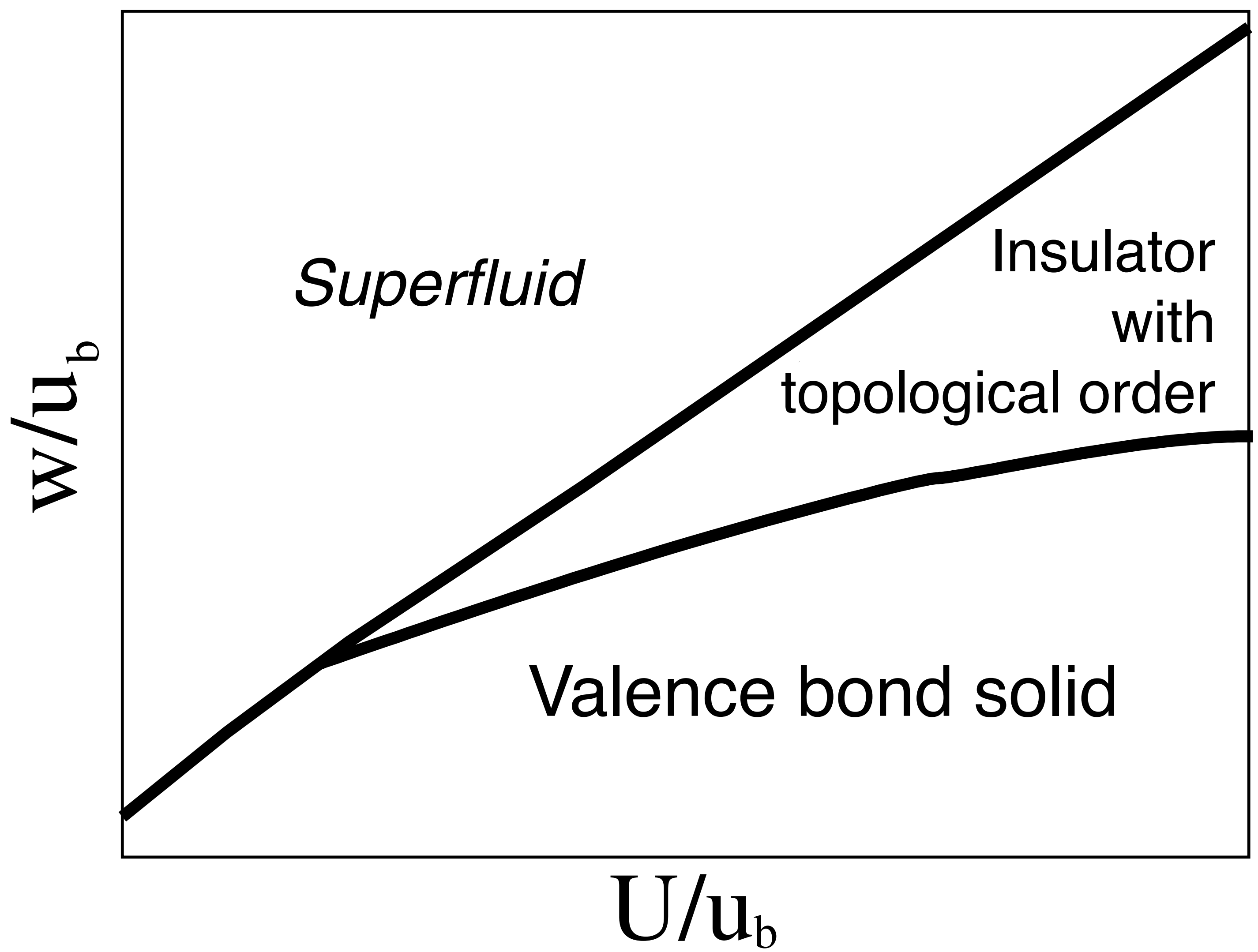




Bosons at unit density on the square lattice

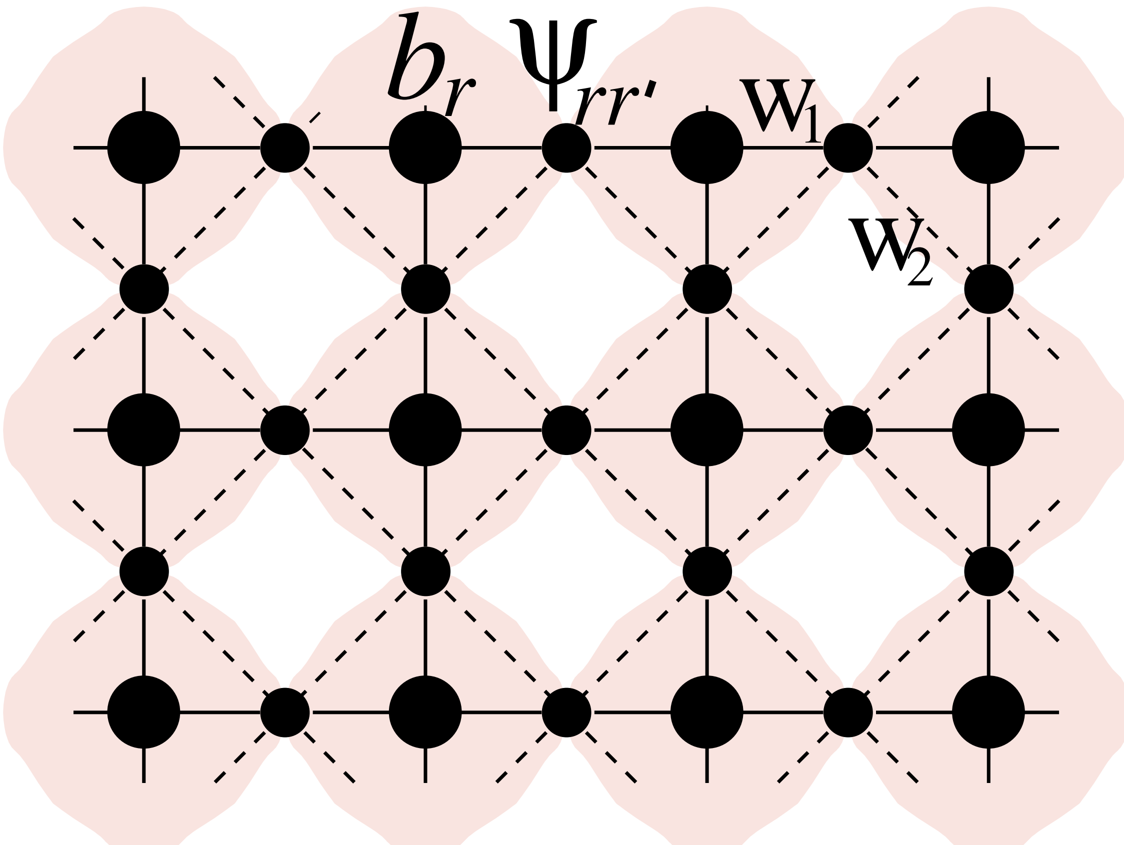
S. Sachdev and R. Jalabert, Modern Physics Letters B **4**, 1043 (1990); R. Jalabert and S. Sachdev Phys. Rev. B 44, 686 (1991); S. Sachdev and M. Vojta, Journal of the Physical Society of Japan **69,** Suppl. B, 1 (2000).

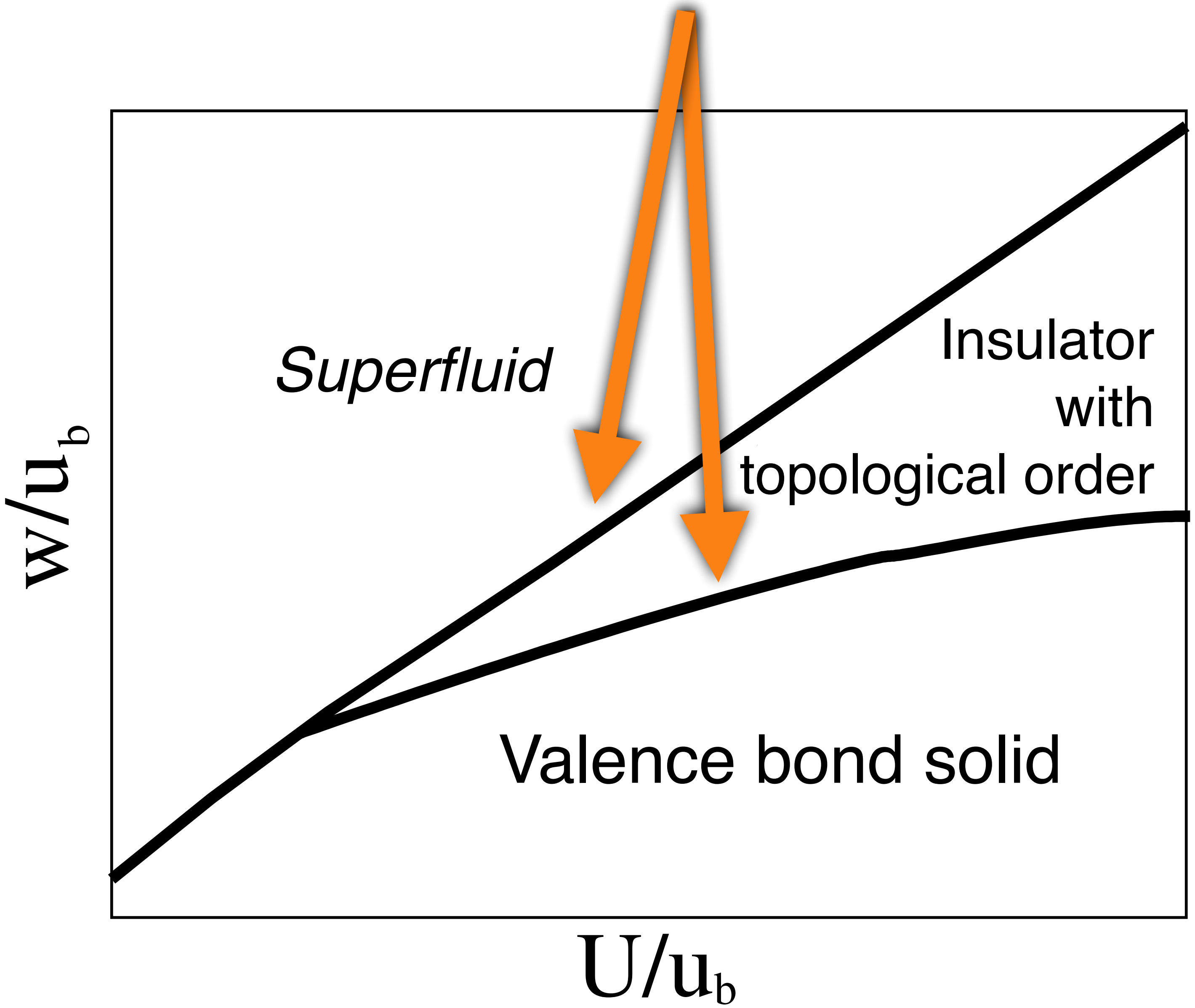




Bosons at half-integer density on the square lattice

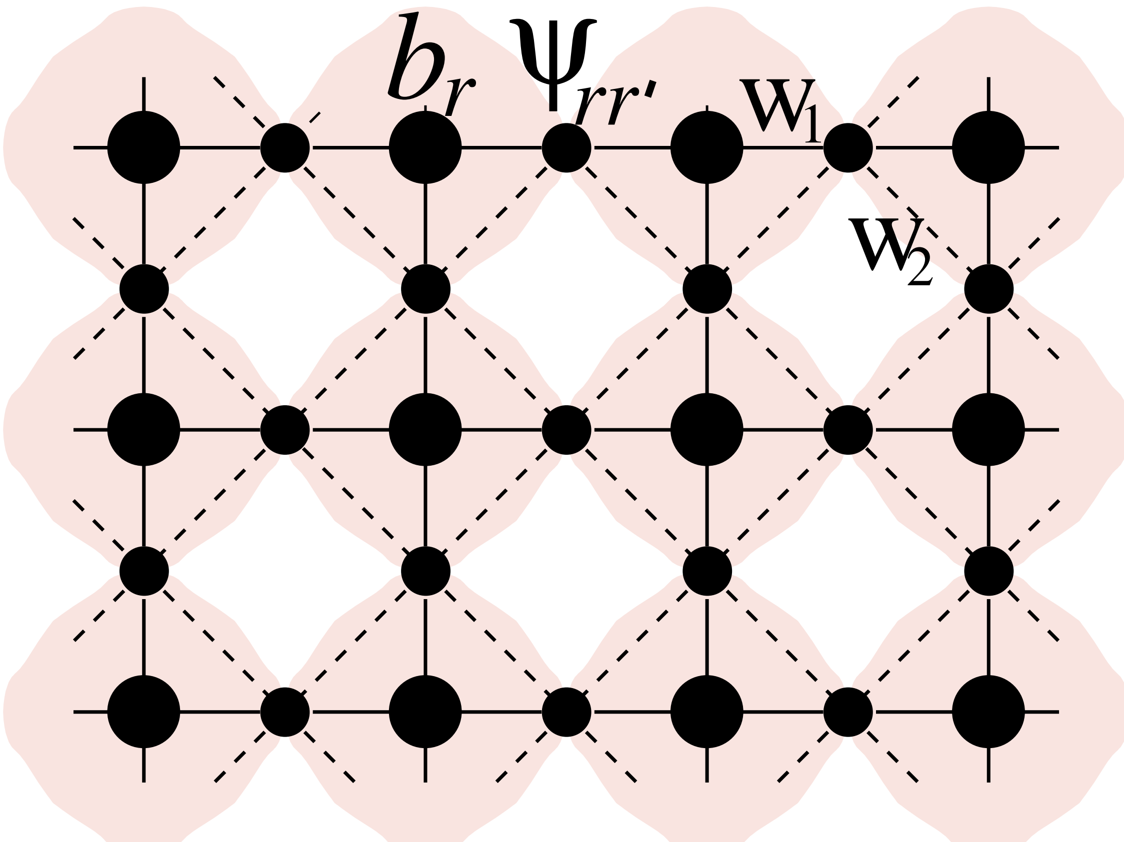
Topological phase transitions

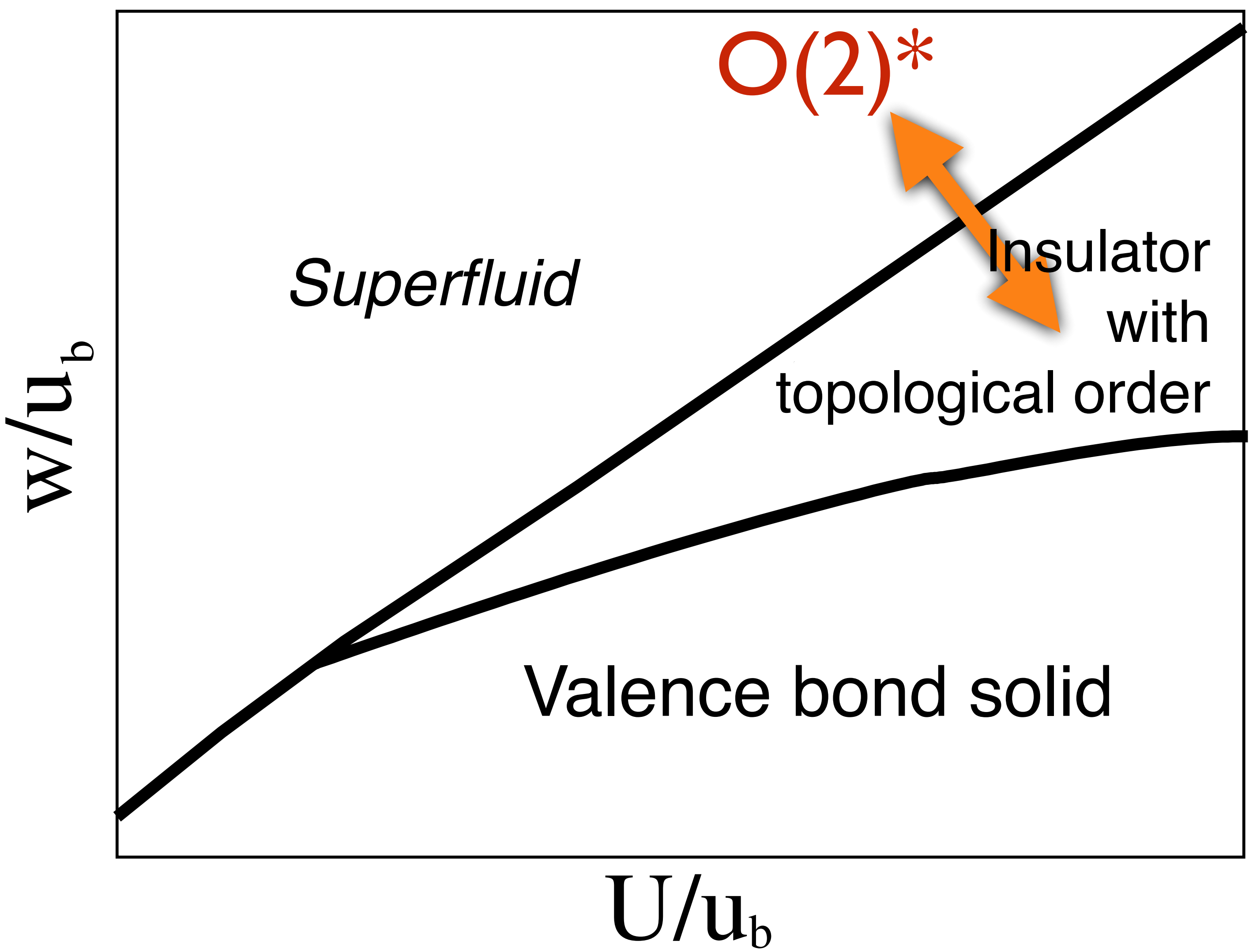




Bosons at half-integer density on the square lattice

A.V. Chubukov, T. Senthil and S. Sachdev, PRL **72**, 2089 (1994)

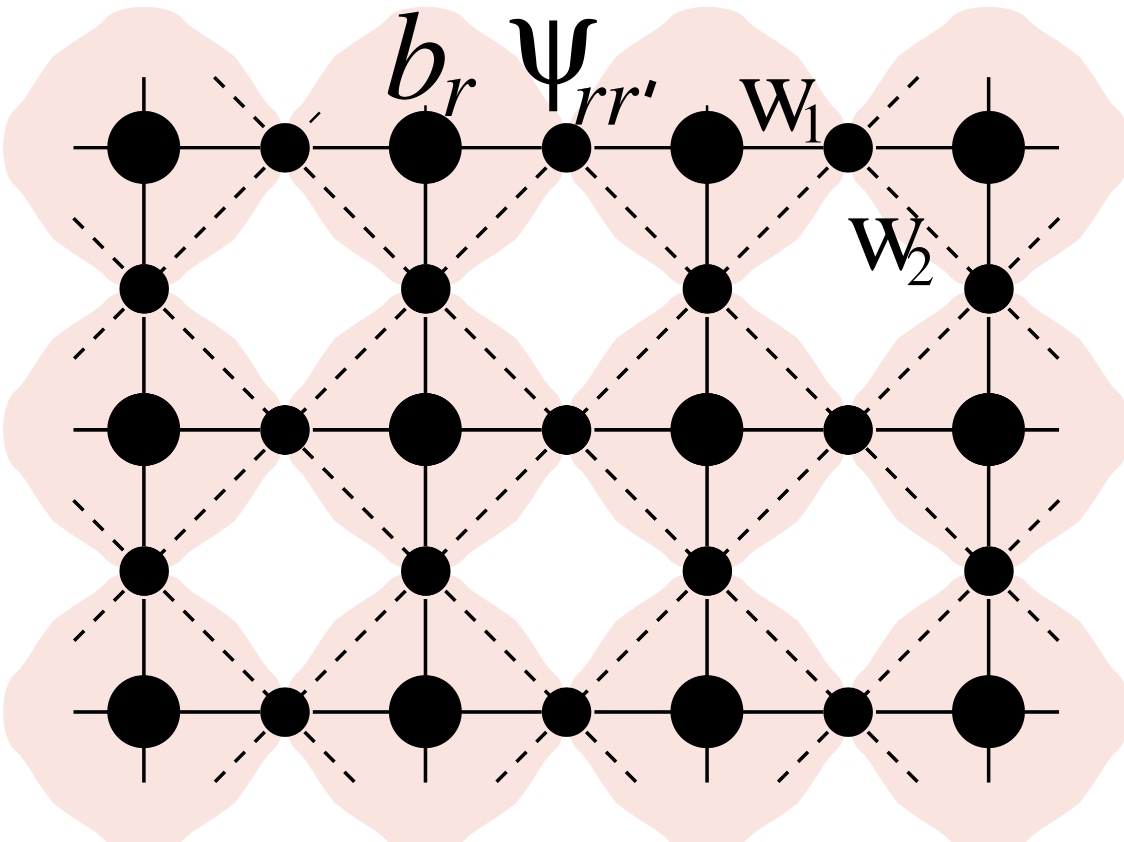


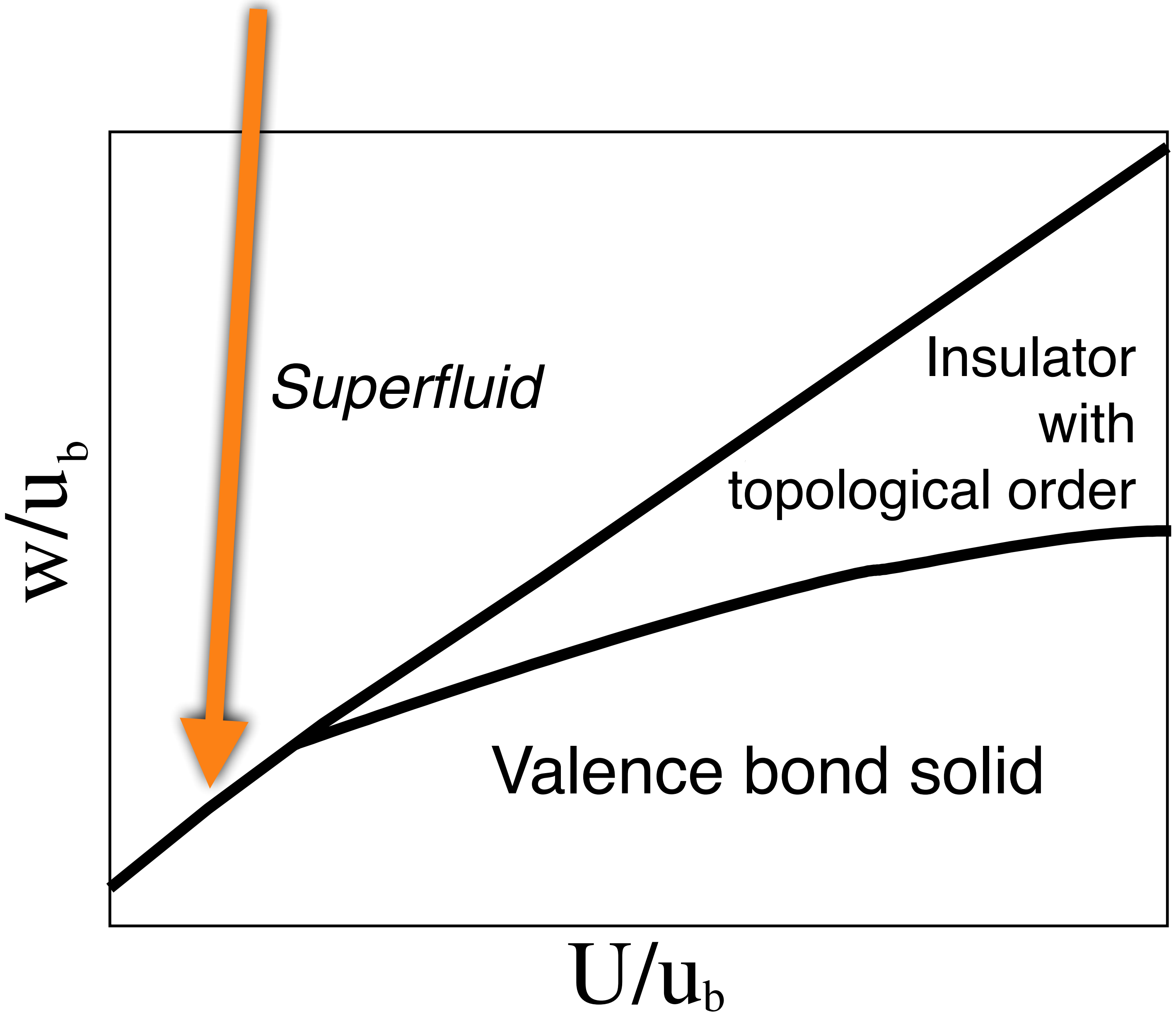


Bosons at half-integer density on the square lattice

T. Senthil, A. Vishwanath, L. Balents, S. Sachdev, and M. P. A. Fisher, Science **303**, 1490 (2004)

Deconfined criticality

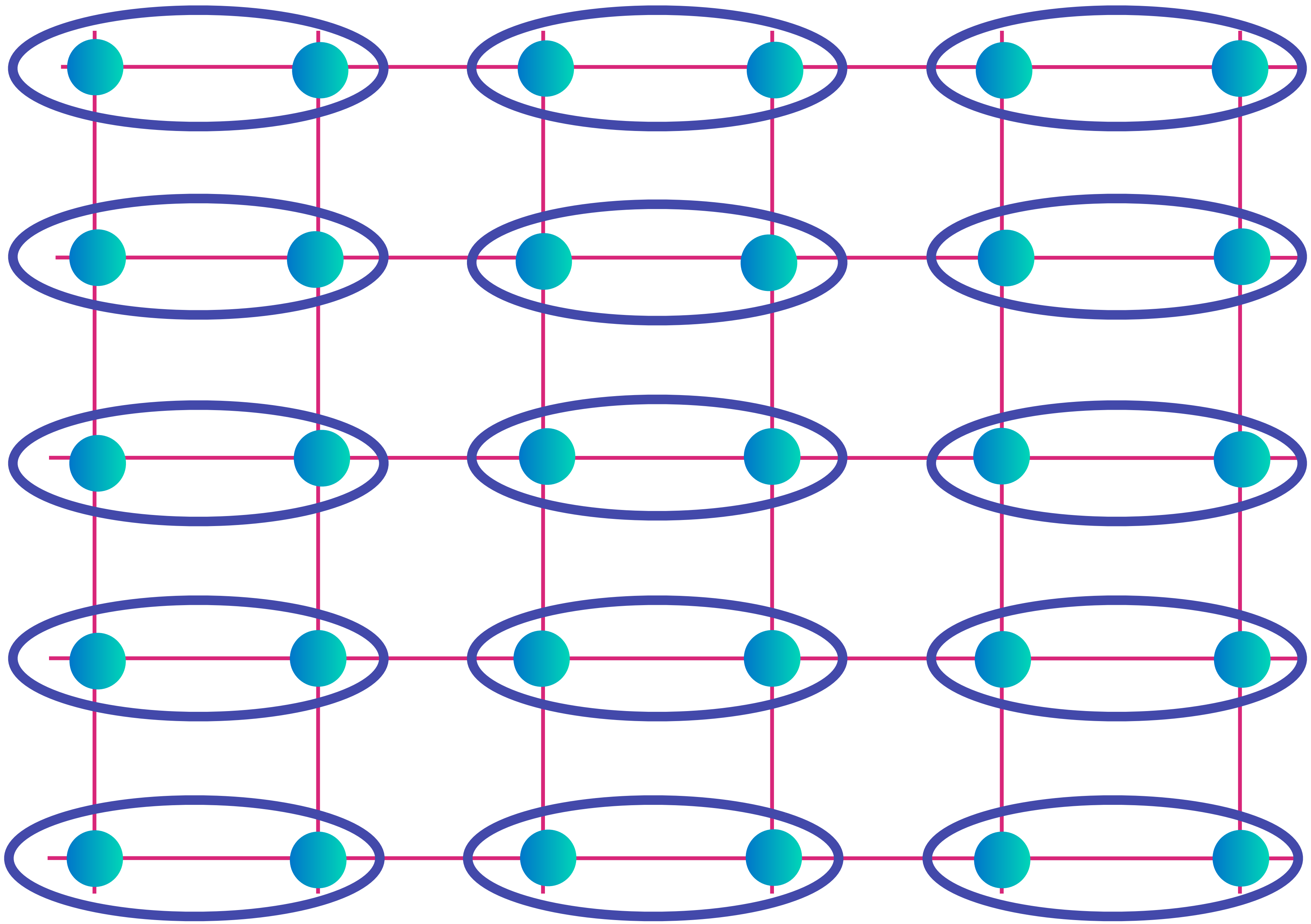




Bosons at half-integer density on the square lattice

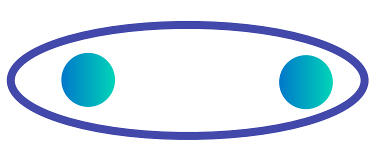
VBS states on the square lattice

Columnar VBS



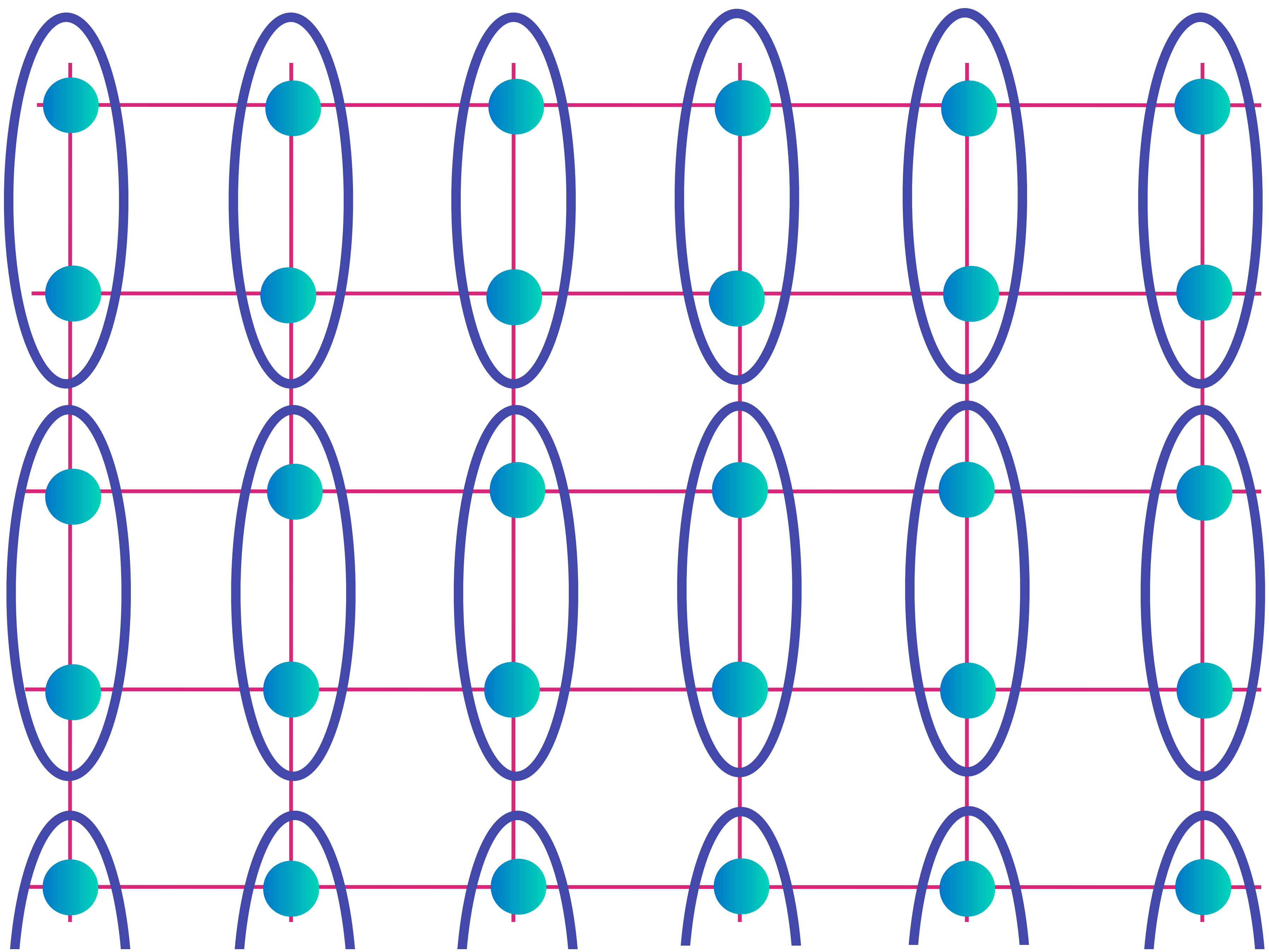
Equivalently, this can be interpreted as a model of bosons at 1/2 filling, with the mapping

$$|\uparrow\rangle \Rightarrow |0\rangle$$
$$|\downarrow\rangle \Rightarrow b^\dagger |0\rangle$$

 $= \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)/\sqrt{2}$

VBS states on the square lattice

Columnar VBS



Equivalently, this can be interpreted as a model of bosons at 1/2 filling, with the mapping
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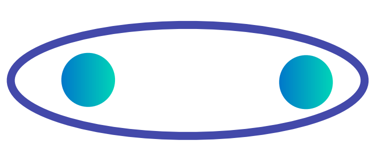 $= \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)/\sqrt{2}$

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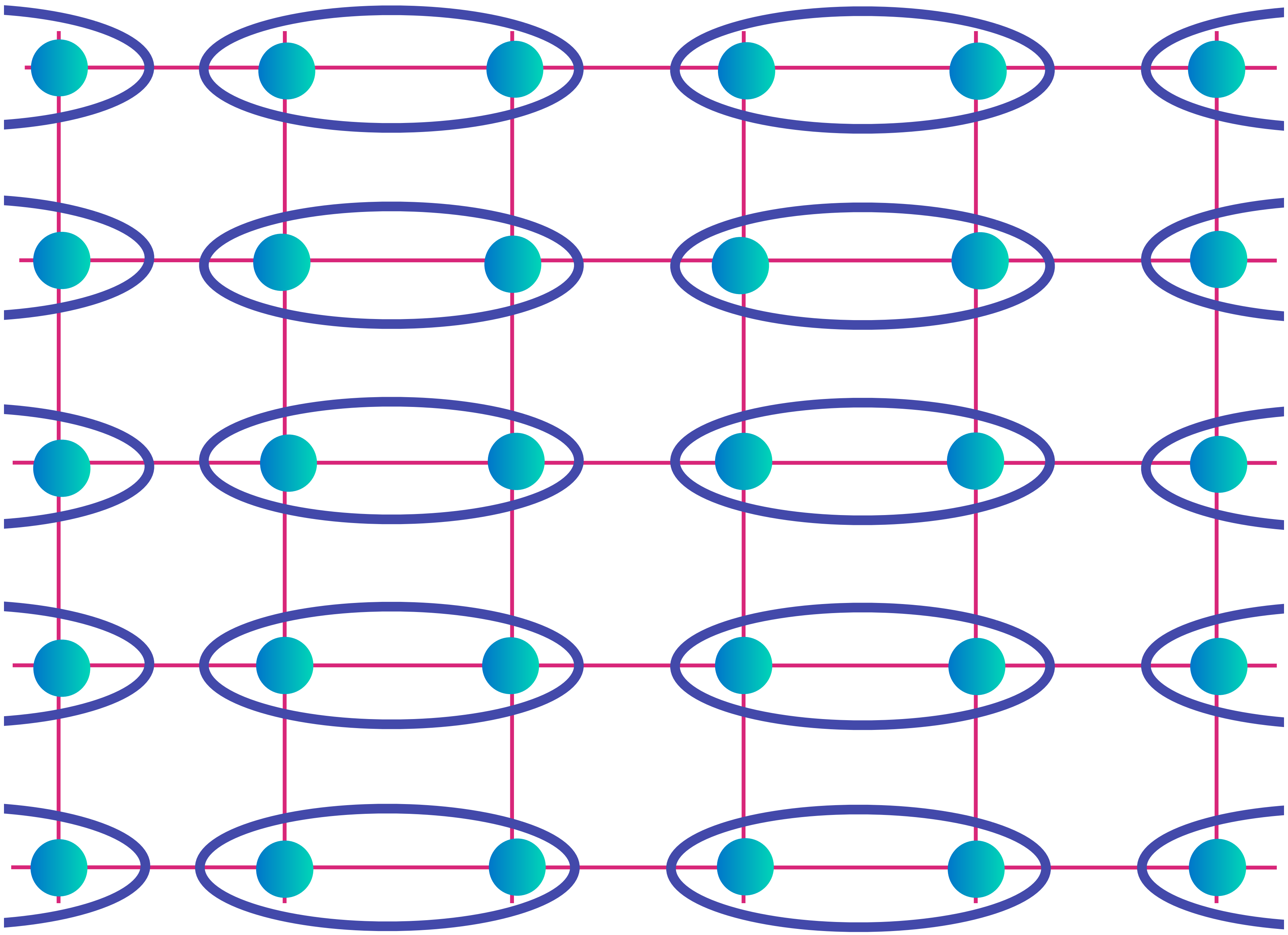

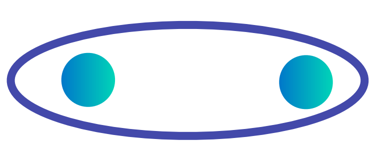

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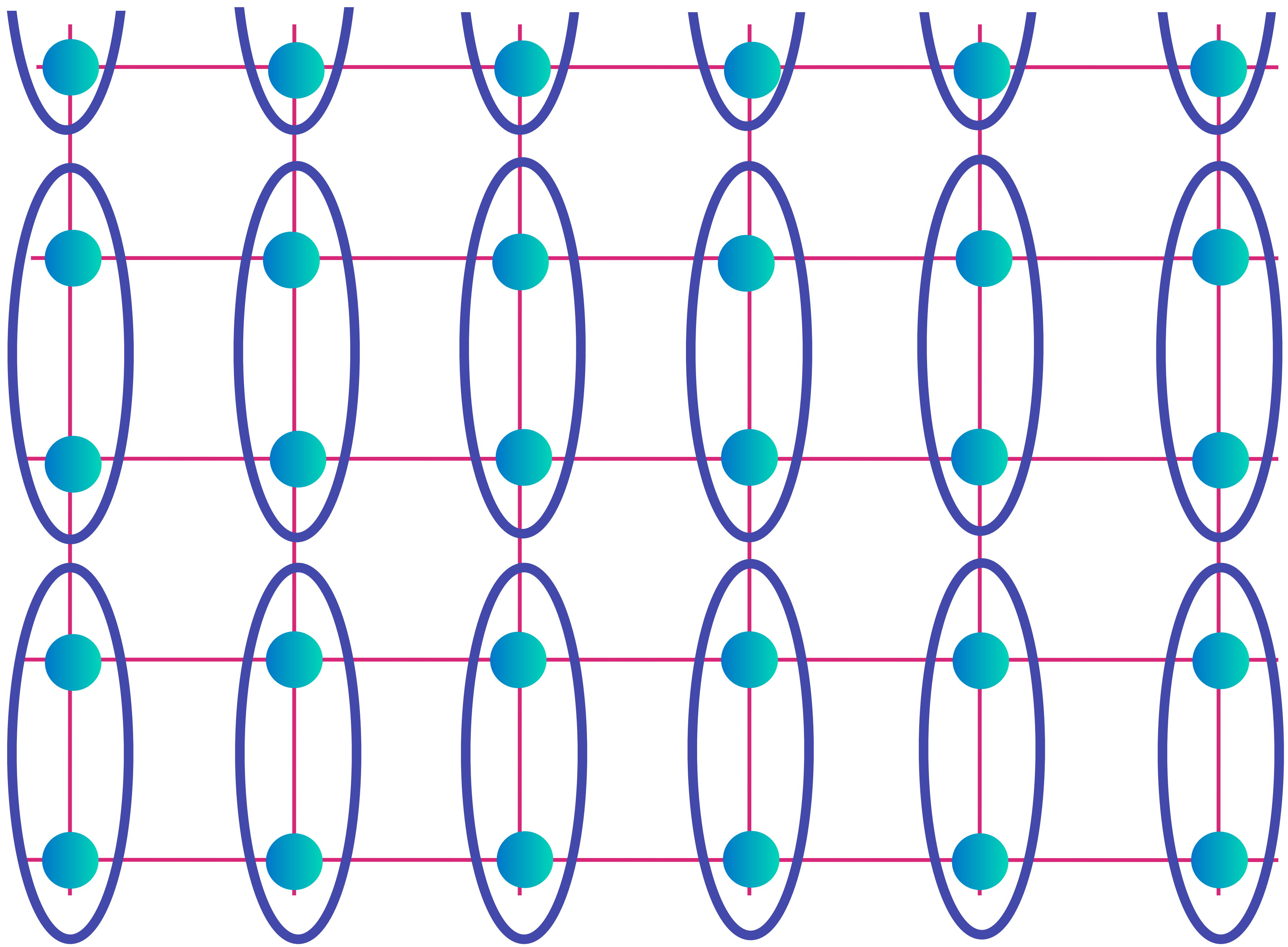

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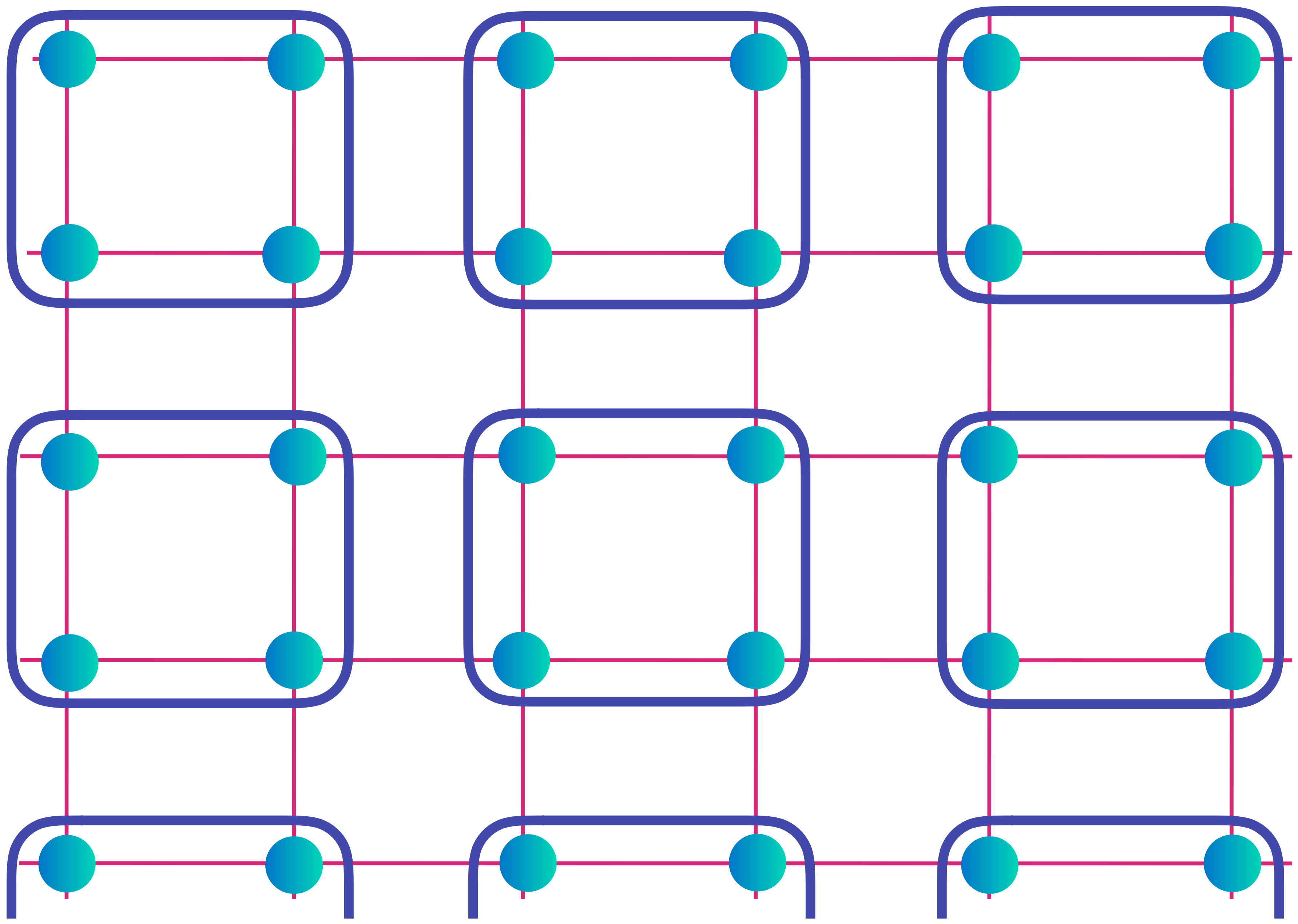
VBS states on the square lattice

Plaquette VBS



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VBS states on the square lattice

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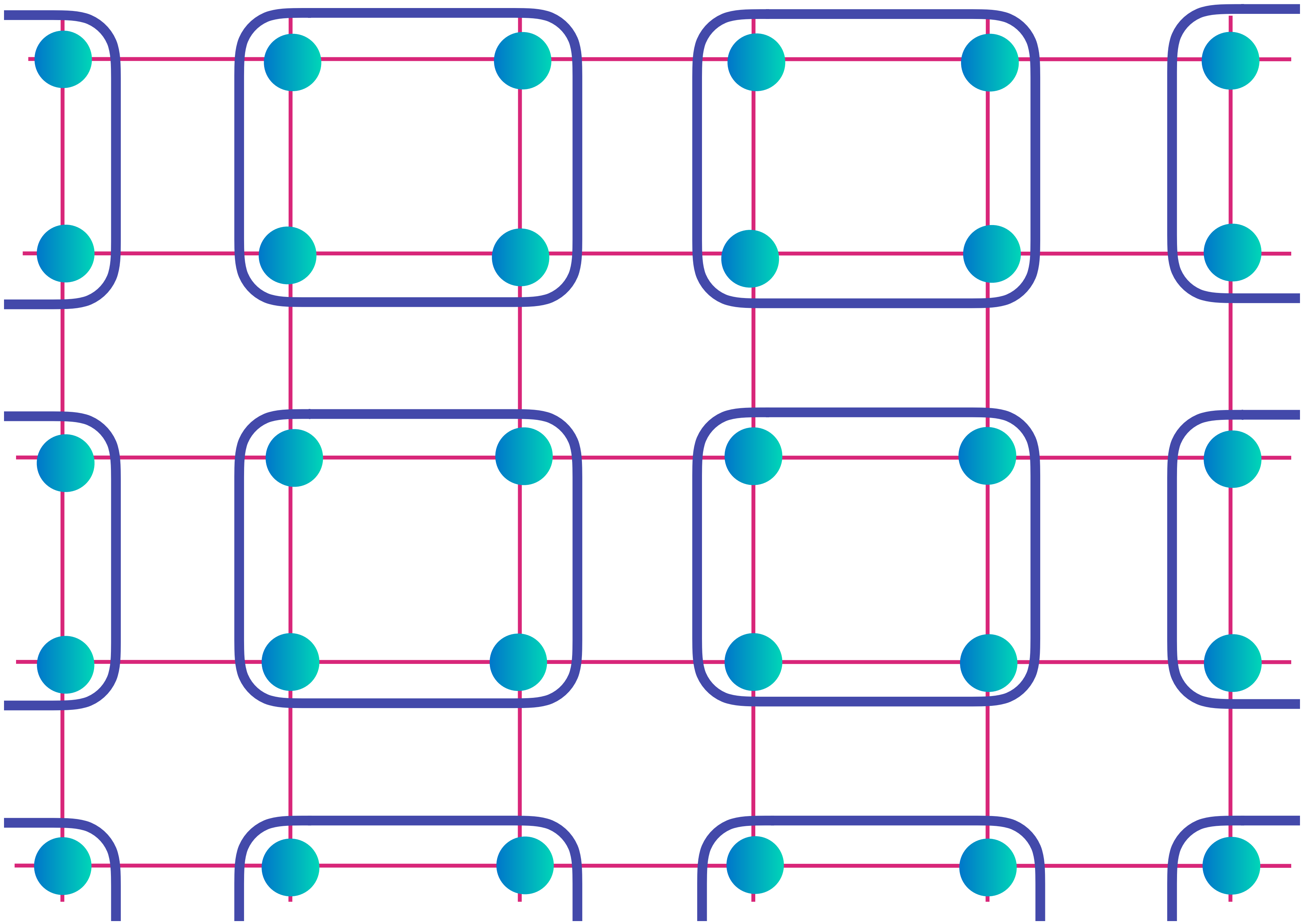

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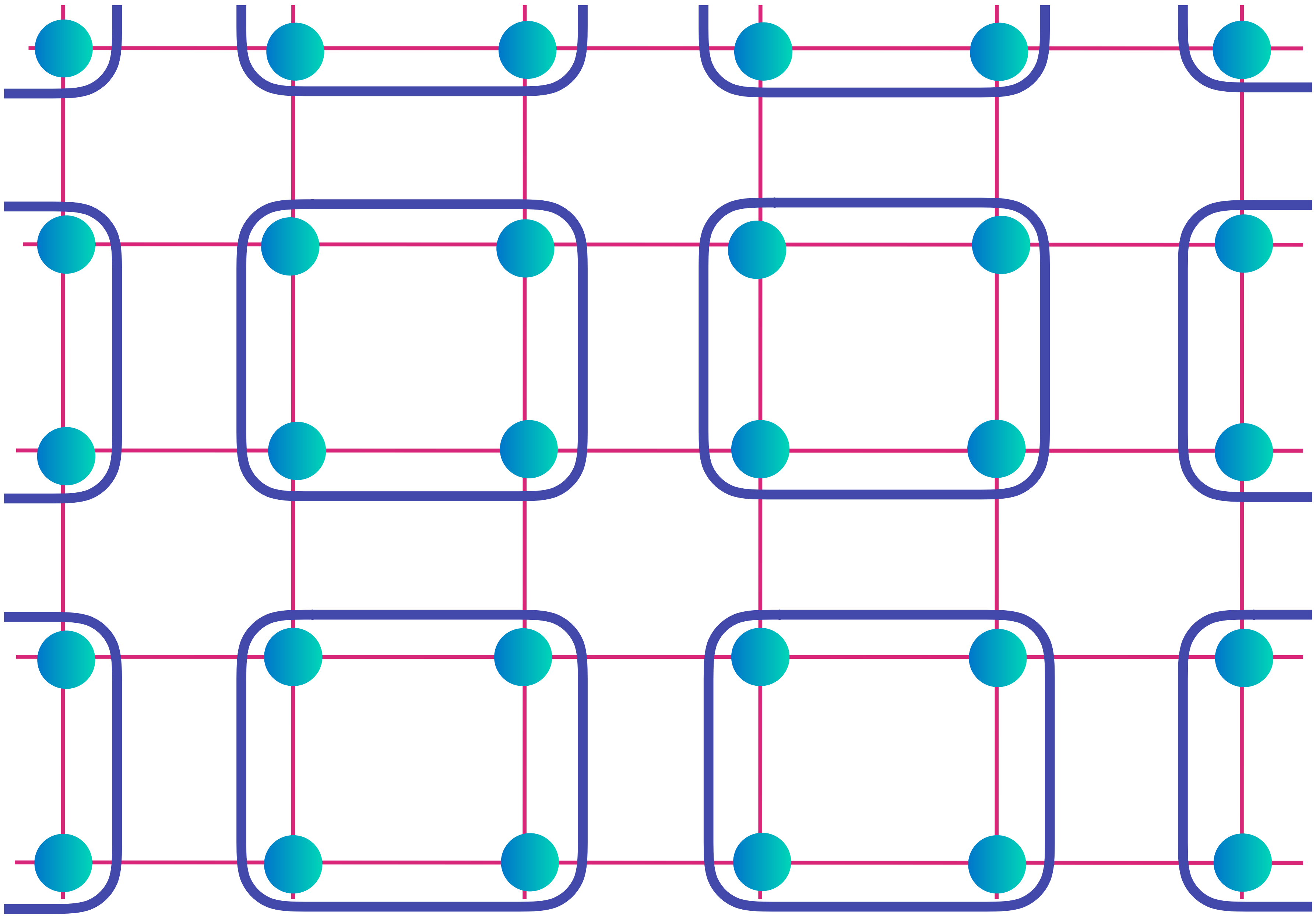

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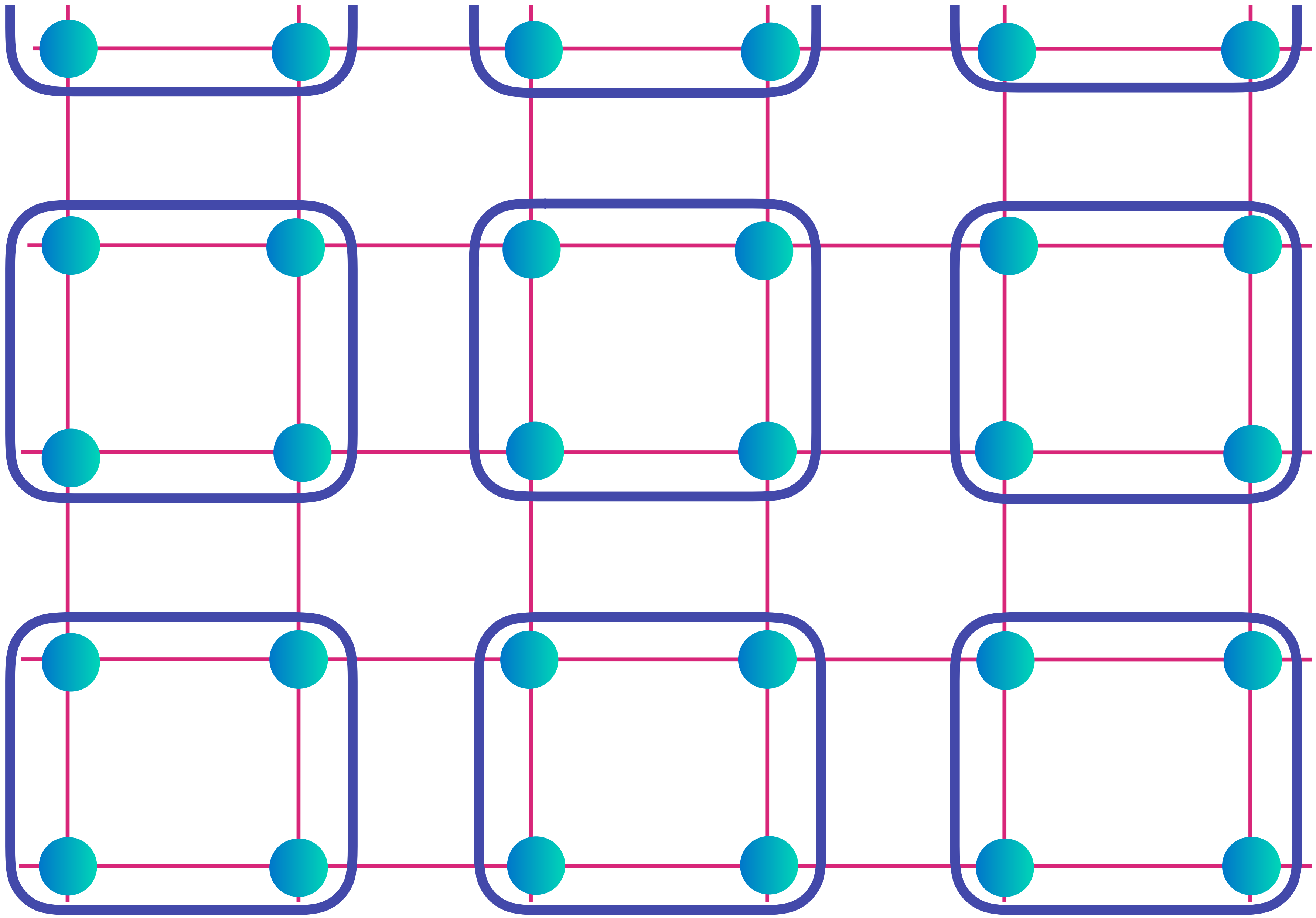

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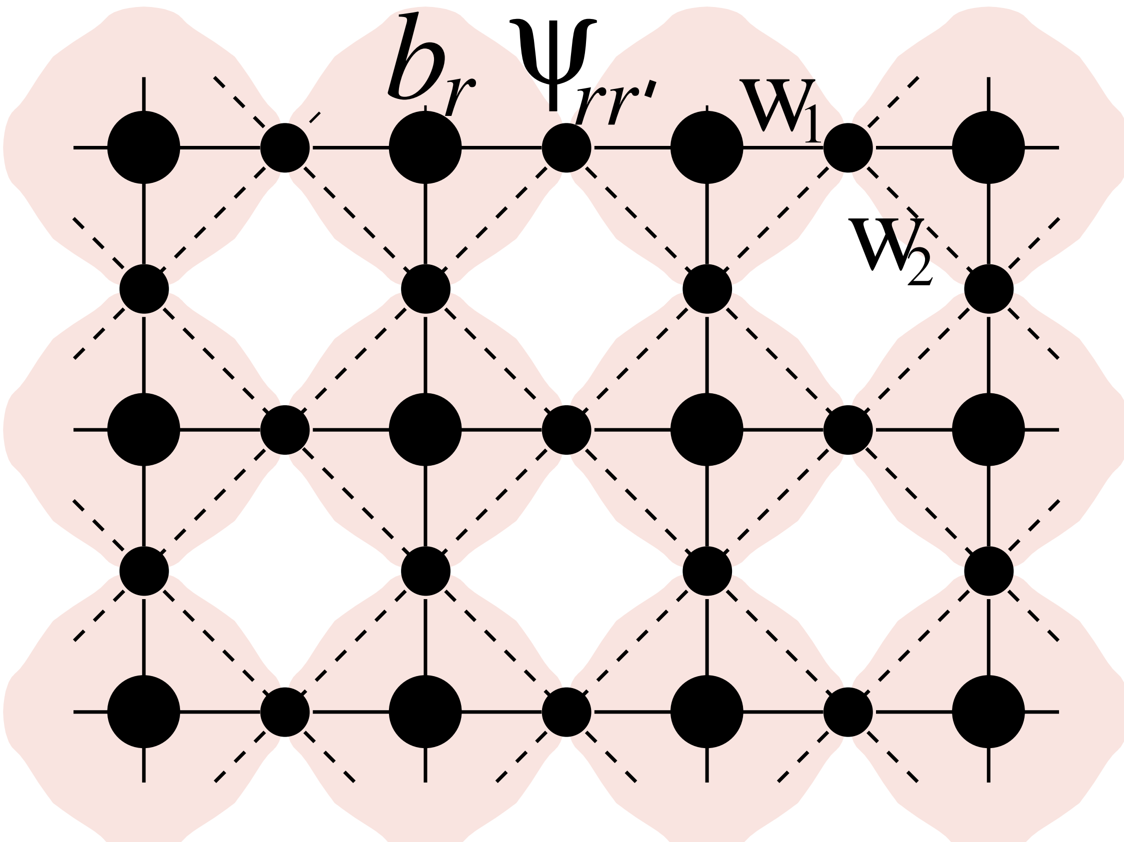
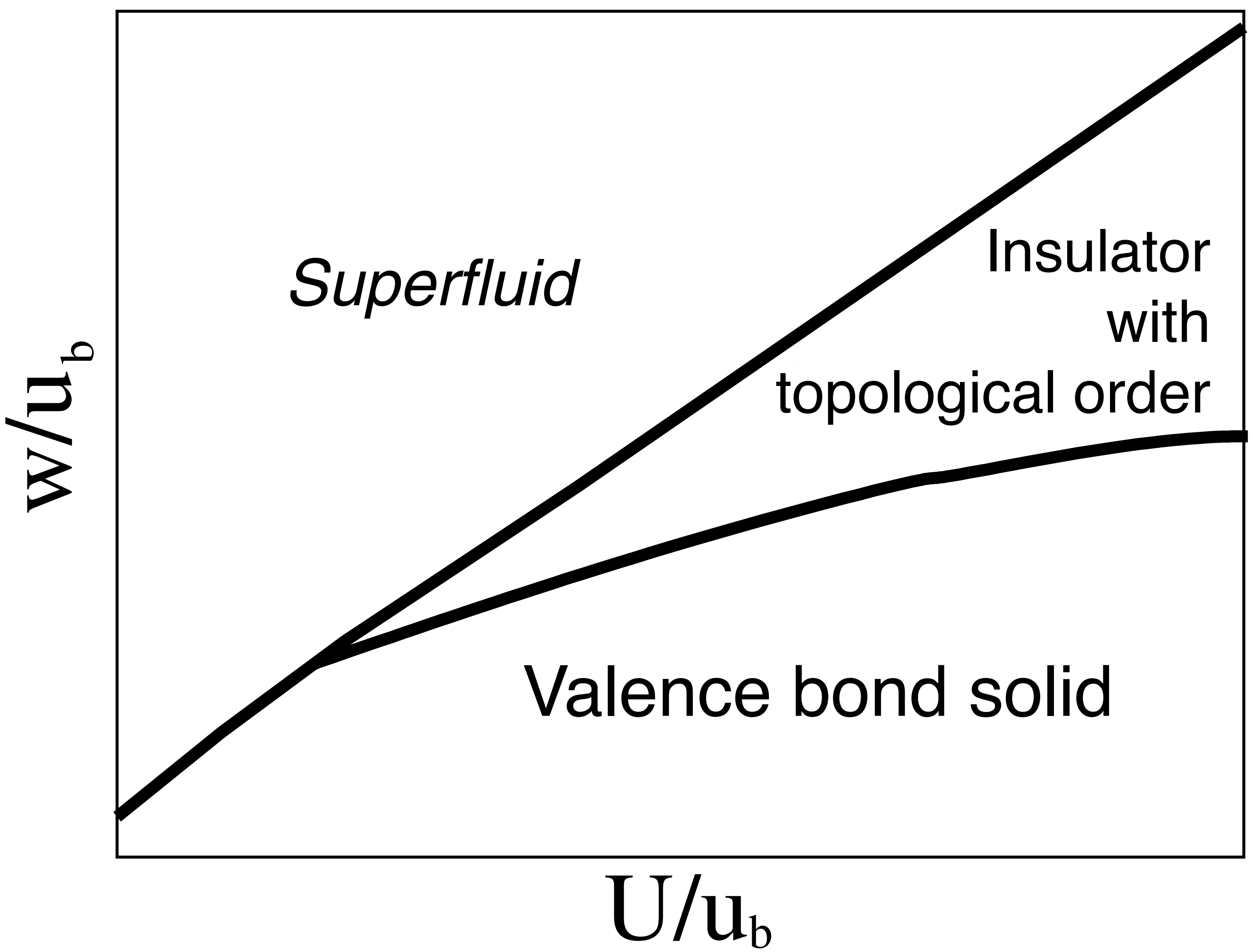
S. Sachdev and R. Jalabert, Modern Physics Letters B **4**, 1043 (1990); R. Jalabert and S. Sachdev Phys. Rev. B 44, 686 (1991); S. Sachdev and M. Vojta, Journal of the Physical Society of Japan **69,** Suppl. B, 1 (2000).





Bosons at half-integer density on the square lattice

S. Sachdev and R. Jalabert, Modern Physics Letters B **4**, 1043 (1990); R. Jalabert and S. Sachdev Phys. Rev. B 44, 686 (1991); S. Sachdev and M. Vojta, Journal of the Physical Society of Japan **69,** Suppl. B, 1 (2000).

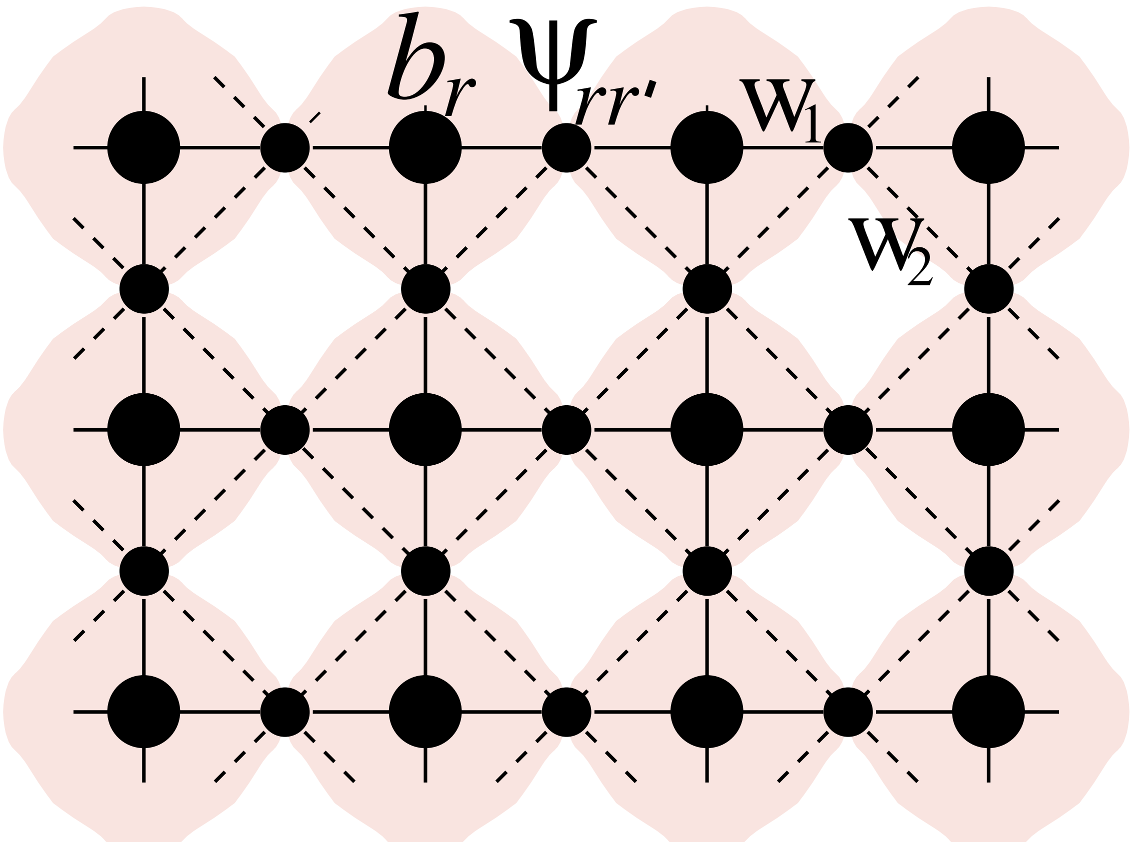
At half-integer density, we obtain an additional Berry phase term ($\varepsilon_r = \pm 1$ on the two sub lattices)

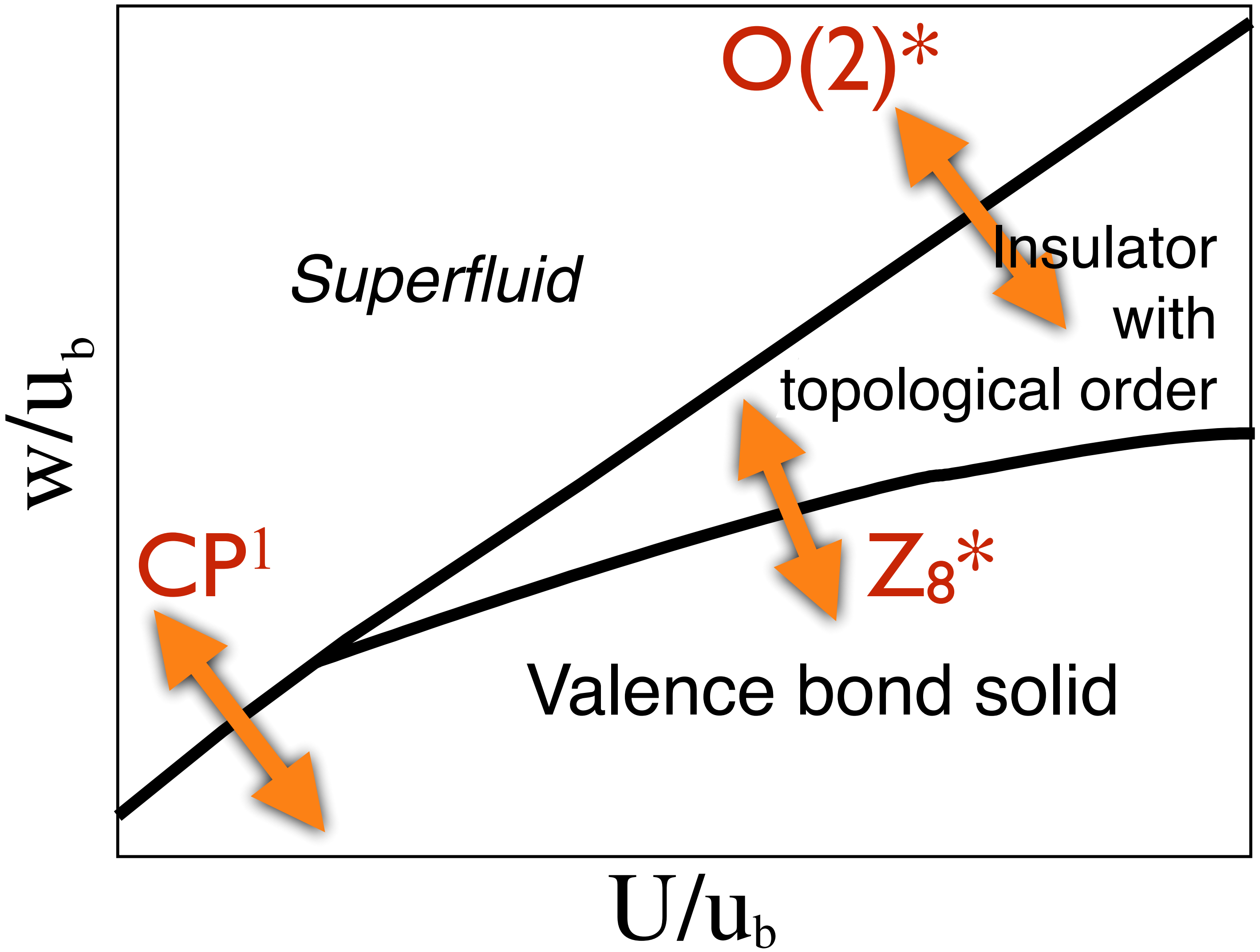
$$
\begin{aligned}
S &= i \sum_r \varepsilon_r a_{r\tau} \\
&\quad - t \sum_r \cos(\Delta_\mu \theta_r - a_{r\mu} - A_{r\mu}/2) \\
&\quad - J \sum_r \cos(\Delta_\mu \phi_r - 2a_{r\mu}) \\
&\quad - K \sum_{\square} \cos(\epsilon_{\mu\nu\lambda} \Delta_\nu a_\lambda)
\end{aligned}
$$

The Berry phases prohibit a "trivial" phase with no broken symmetry and no topological order: instead we obtain a phase with valence bond solid order and broken translational symmetry. Also, the $\mathbb{Z}_2$ spin liquid is now a "symmetry enriched topological" (SET) state.

Bosons at half-integer density on the square lattice

S. Sachdev and R. Jalabert, Modern Physics Letters B **4**, 1043 (1990); R. Jalabert and S. Sachdev PRB **44**, 686 (1991); A.V. Chubukov, T. Senthil and S. Sachdev, PRL **72**, 2089 (1994); S. Sachdev and M. Vojta, Journal of the Physical Society of Japan **69,** Suppl. B, 1 (2000); T. Senthil, A. Vishwanath, L. Balents, S. Sachdev, and M. P.A. Fisher, Science **303**, 1490 (2004)





Bosons at half-integer density on the square lattice

“Anomaly” constraints on phase diagram

Consider a general lattice Hamiltonian with 2 global symmetries: translation by a lattice spacing, $\hat{T}_x$, and a global U(1) symmetry, $\mathcal{U}$, associated in our case with conservation of boson number. The global U(1) symmetry is generated by

$$\mathcal{U} = \exp\left(i\alpha \sum_i \hat{n}_i\right)$$

where α is the rotation angle, and $\hat{n}_i$ is the boson number on site i. There two symmetries clearly commute

$$\hat{T}_x \mathcal{U} = \mathcal{U} \hat{T}_x$$

Now place the system on a $L_x \times L_y$ torus, let us consider a spatially-dependent rotation angle ($i.e.$ we gauge the U(1) symmetry)

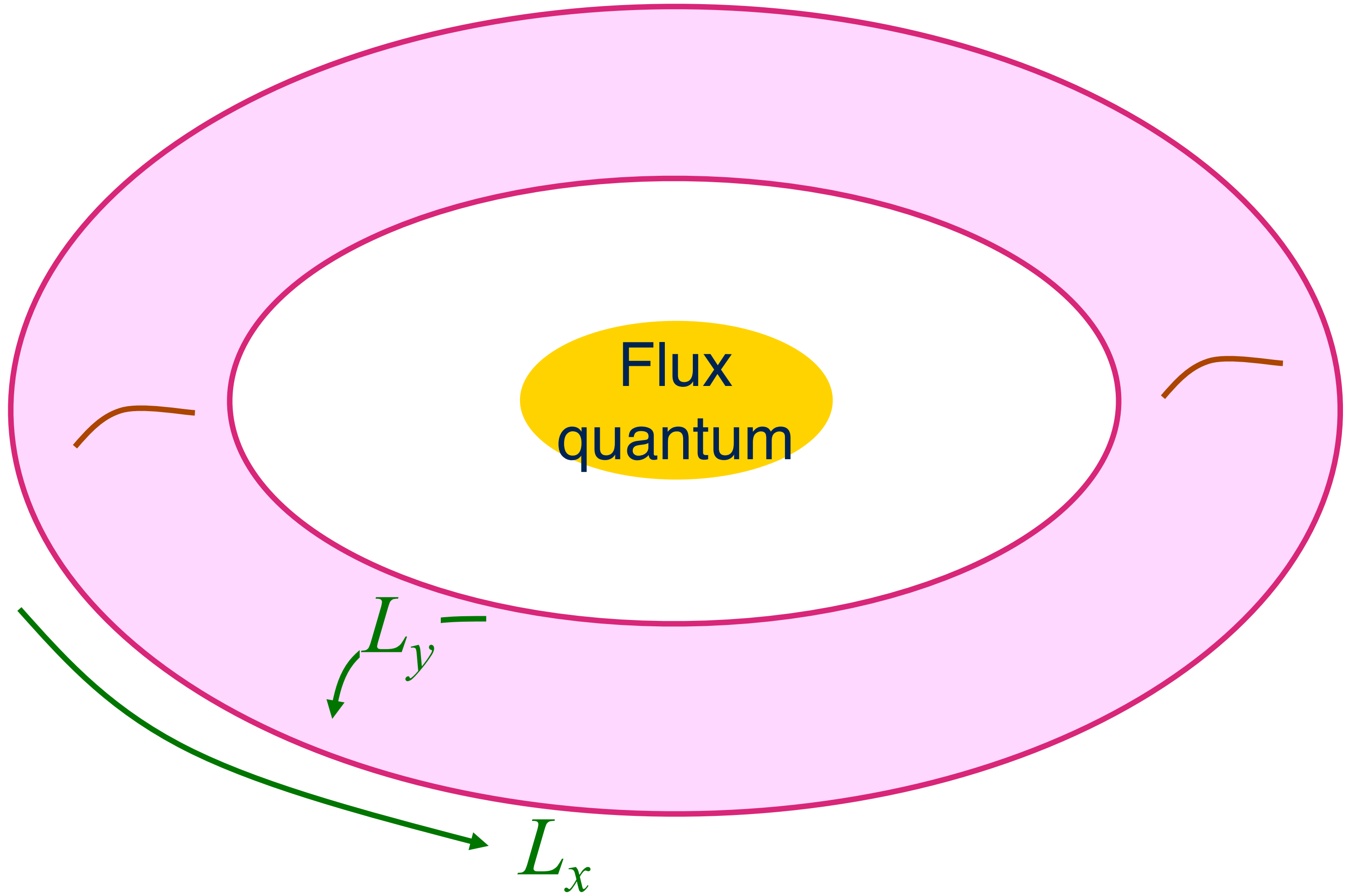
$$\mathcal{U}_G = \exp\left(i \frac{2\pi}{L_x} \sum_i x_i \hat{n}_i\right).$$

It is now easy to show that $\hat{T}_x$ and $\mathcal{U}_G$ do not commute, and

$$\hat{T}_x \mathcal{U}_G = \exp\left(-i 2\pi \frac{N}{L_x}\right) \mathcal{U}_G \hat{T}_x$$

where N is the total number of bosons. This anomaly is an obstruction to gauging the U(1)$\times\hat{T}_x$ global symmetry.

"Anomaly" constraints on phase diagram



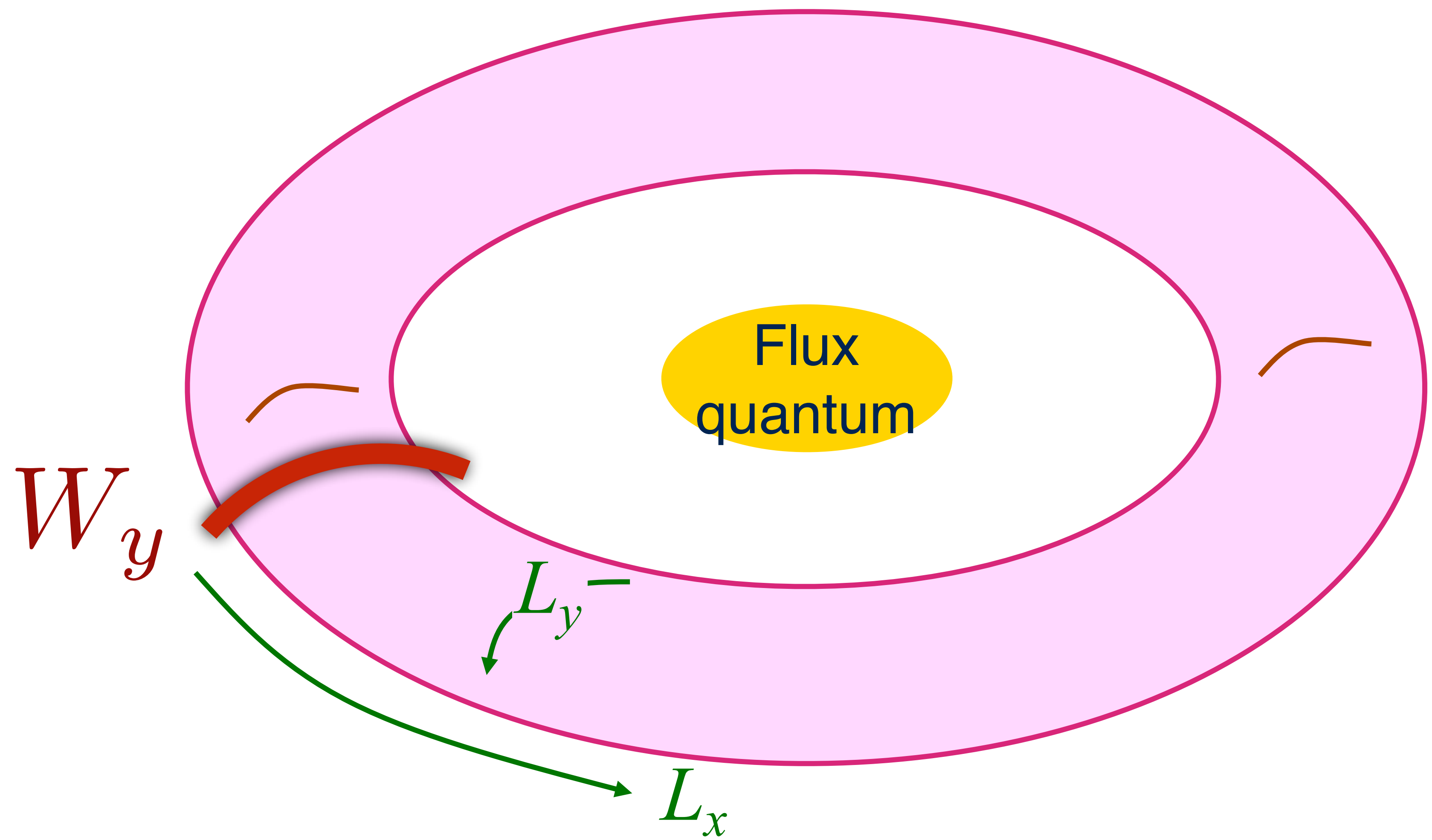
The operator $\mathcal{U}_G$ is equivalent to the adiabatic insertion of one flux quantum of the external gauge field, A_μ, which couples to the boson number. Specifically, we have $A_x = f(t)/L_x$, where $f(t)$ increases slowly from 0 to 2π. However, in the action of the TQFT we have the term

$$\exp\left(\frac{i}{2\pi}\int d^3r\,\epsilon_{\mu\nu\lambda}b_\mu\partial_\nu A_\lambda\right) = \exp\left(\frac{i}{2\pi}\int dydt\,b_y\frac{df}{dt}\right) = \exp\left(i\int dy\,b_y\right) \equiv W_y$$

Here b_μ is the gauge field that couples to the vison, and W_y is the 'branch-cut' operator. So we have the basic result that in an insulator with $\mathbb{Z}_2$ topological order, when acting on the low-energy sector

$$\mathcal{U}_G = W_y$$

"Anomaly" constraints on phase diagram



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“Anomaly” constraints on phase diagram

Combining our results we have for bosons at density ν

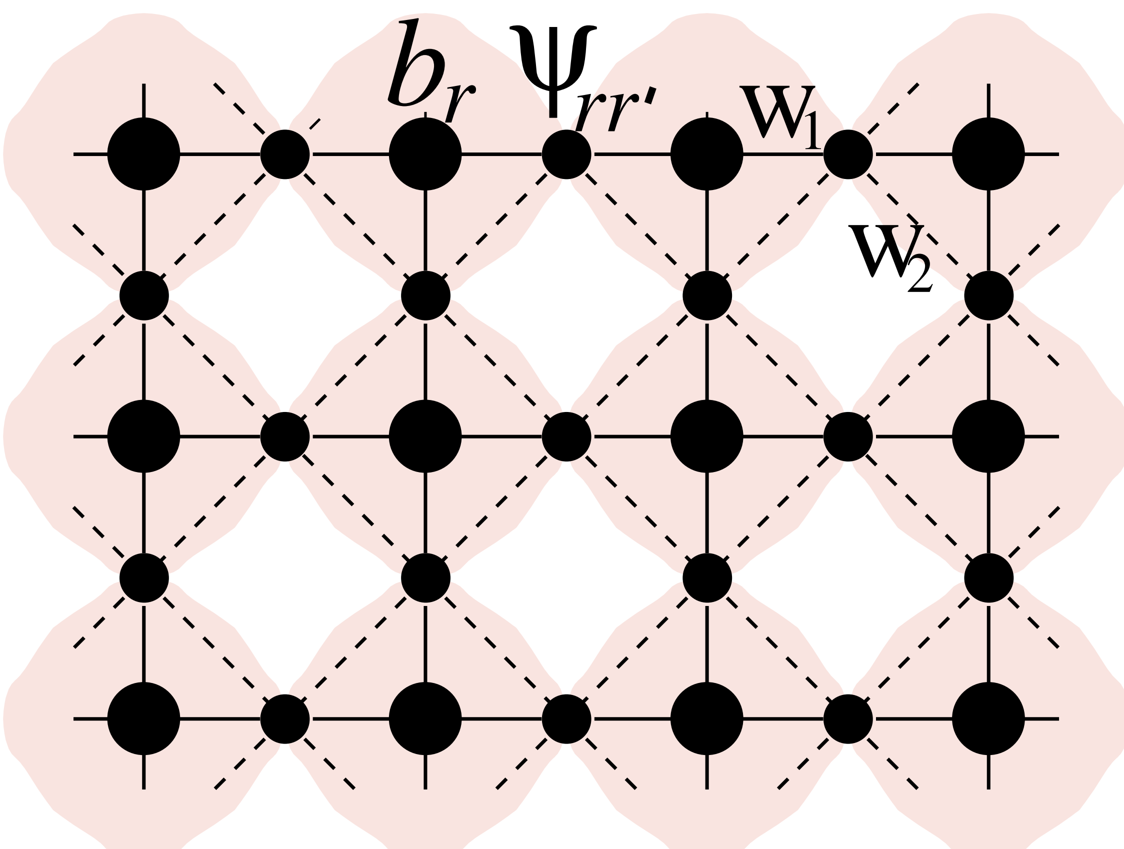
$$
\begin{aligned}
\hat{T}_x W_y &= \exp\left(2\pi i \nu L_y\right) W_y T_x \\
\hat{T}_y W_x &= \exp\left(2\pi i \nu L_x\right) W_x T_y
\end{aligned}
$$

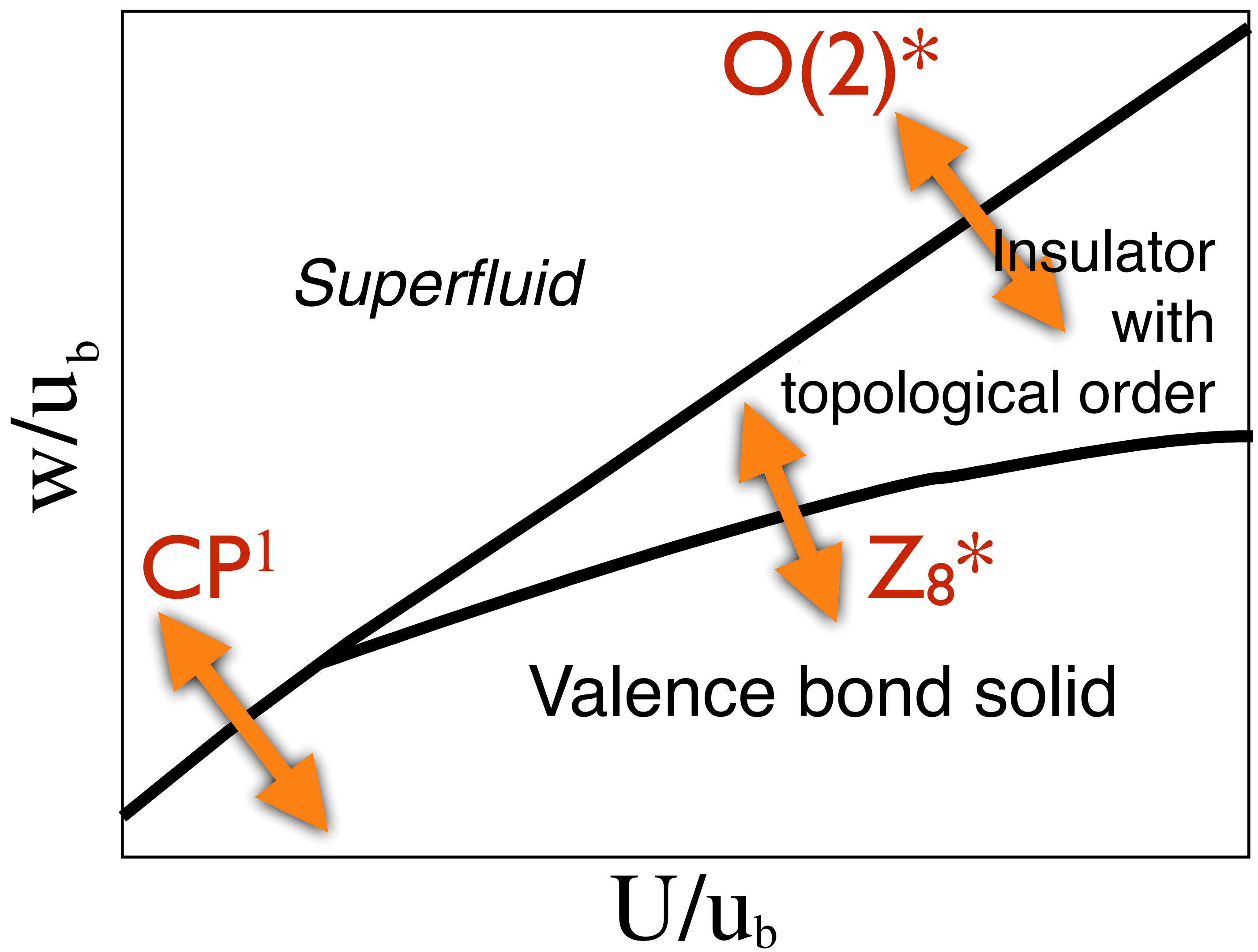
As the Wilson loop operator transports a vison around the torus, in a translationally invariant system, we can write these relations as a constraint on the translation operator acting on single vison states

$$
\hat{T}_x \hat{T}_y = \exp\left(2\pi i \nu\right) \hat{T}_y \hat{T}_x .
$$

So the vison sees each boson as a 2π flux quantum. At $\nu = 1/2$, this implies that each vison state is at least doubly-degenerate, and it transforms non-trivially under lattice translations.

S. Sachdev and R. Jalabert, Modern Physics Letters B **4**, 1043 (1990); R. Jalabert and S. Sachdev PRB **44**, 686 (1991); A.V. Chubukov, T. Senthil and S. Sachdev, PRL **72**, 2089 (1994); S. Sachdev and M. Vojta, Journal of the Physical Society of Japan **69,** Suppl. B, 1 (2000); T. Senthil, A. Vishwanath, L. Balents, S. Sachdev, and M. P.A. Fisher, Science **303**, 1490 (2004)





Bosons at half-integer density on the square lattice

Transition from VBS insulator to insulator with $\mathbb{Z}_2$ topological order

Our previous incorrect attempt for this transition at $\nu = 1$ turns out to correct at $\nu = 1/2$ (up to irrelevant terms at the critical point)! The half-charged boson $h \sim e^{i\theta}$ is gapped in both phases. So we just write down the theory of the critical Higgs field $\Phi \sim e^{i\phi}$

$$\mathcal{L} = |(\partial_\mu - 2ia_\mu)\Phi|^2 + m_\Phi^2|\Phi|^2 + u|\Phi|^4 + \frac{1}{2e^2}(\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda)^2$$

The multiple vison species suppress monopoles in the compact U(1) gauge field, a_μ. This theory is dual at an O(2)* Wilson-Fisher theory, and that is the critical theory of the $\mathbb{Z}_8^*$ transition.

Bosons at half-integer density on the square lattice

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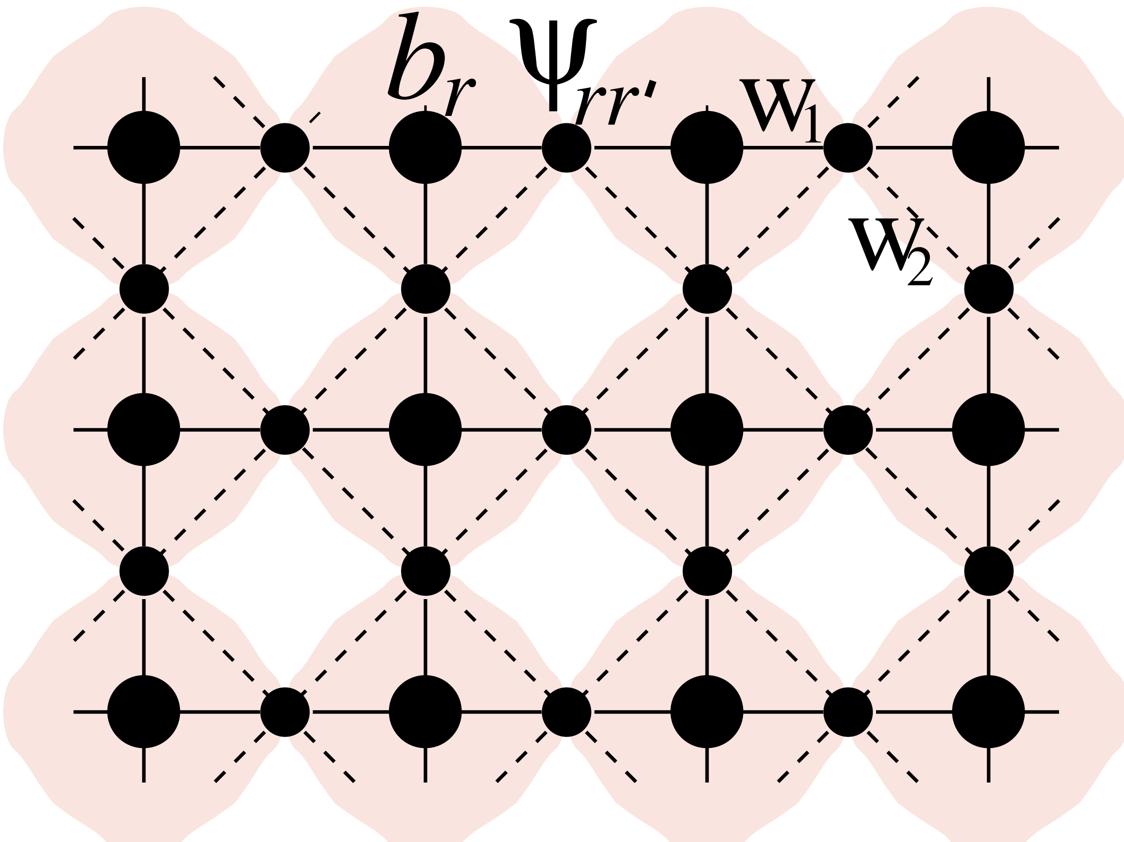
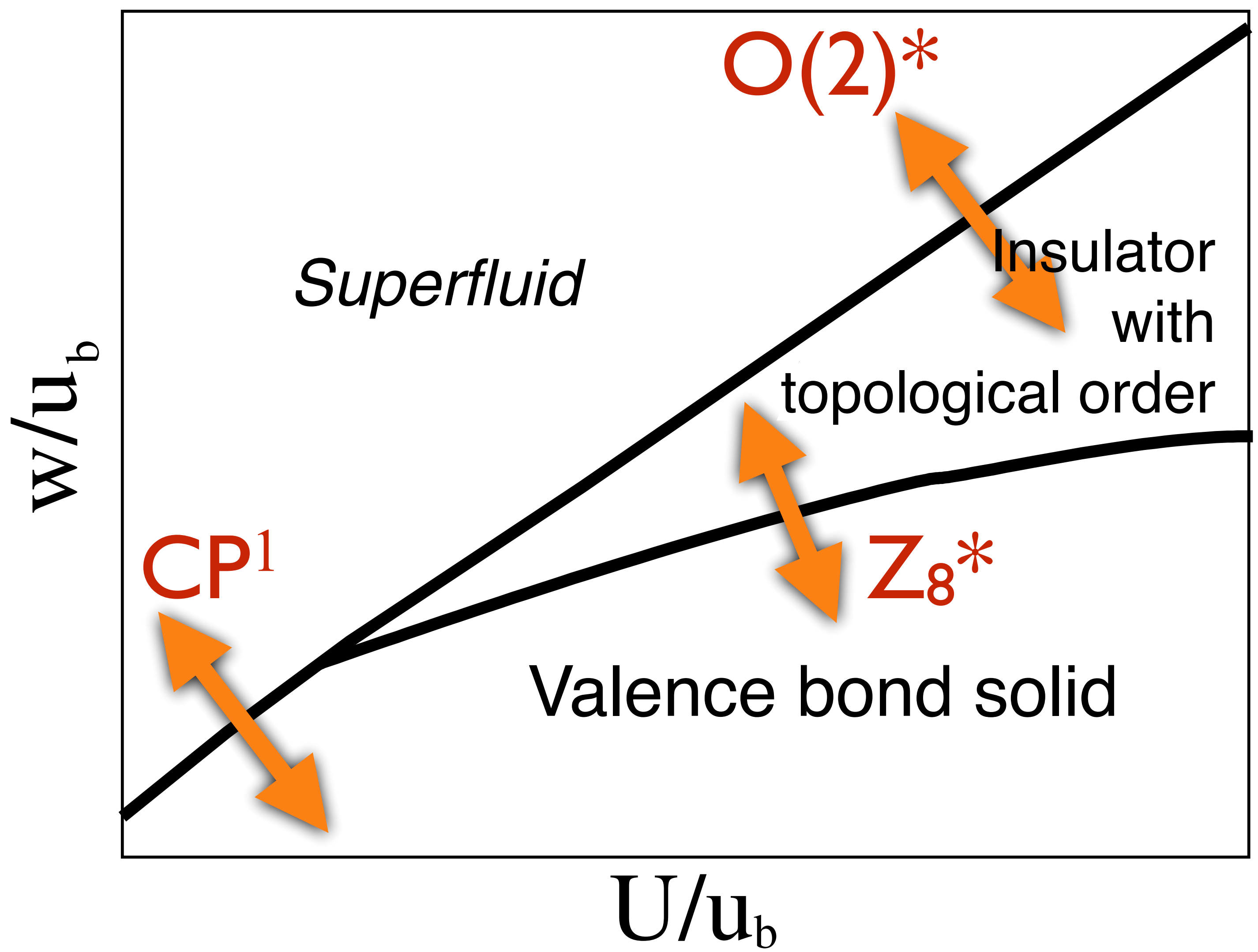
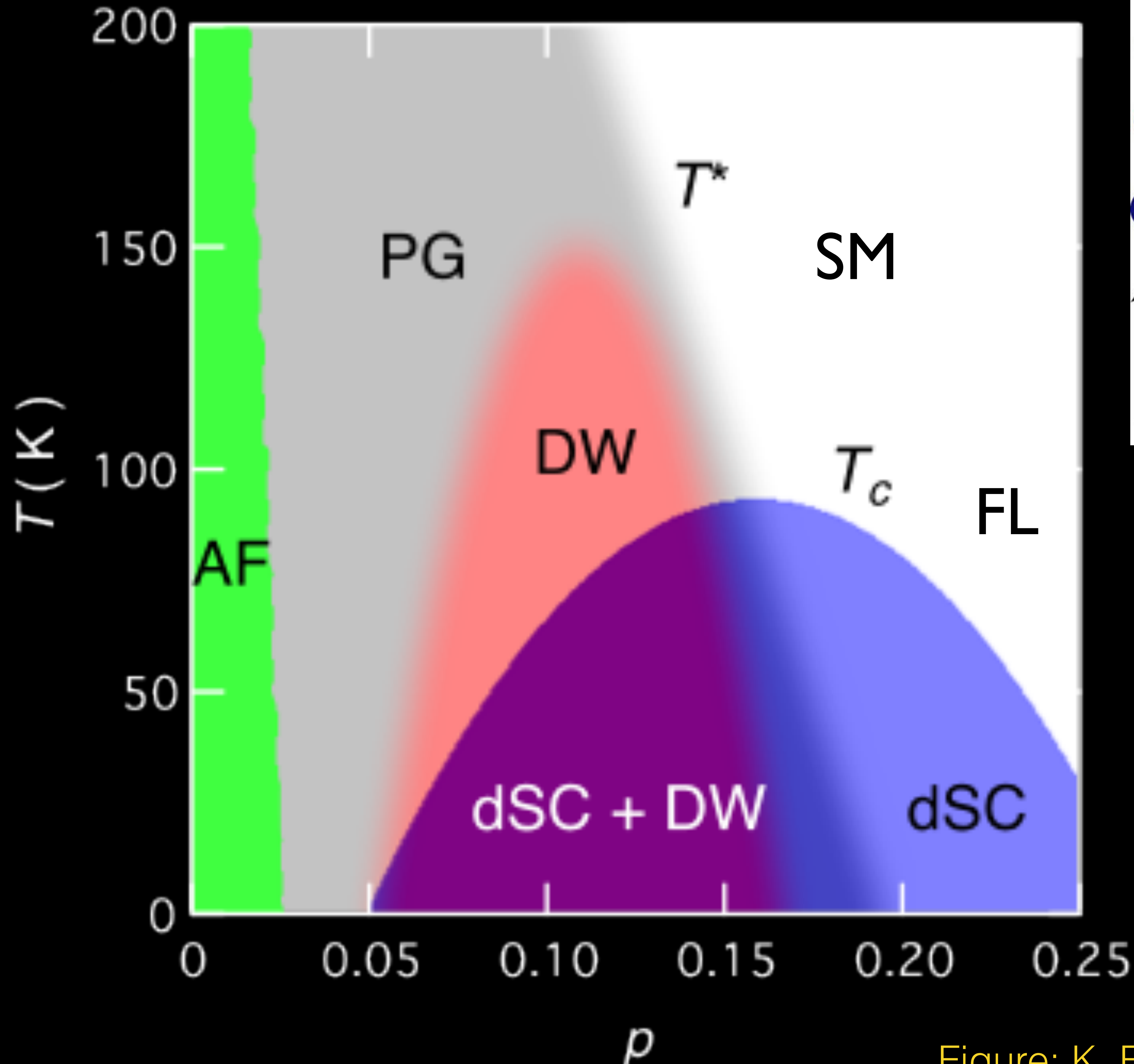
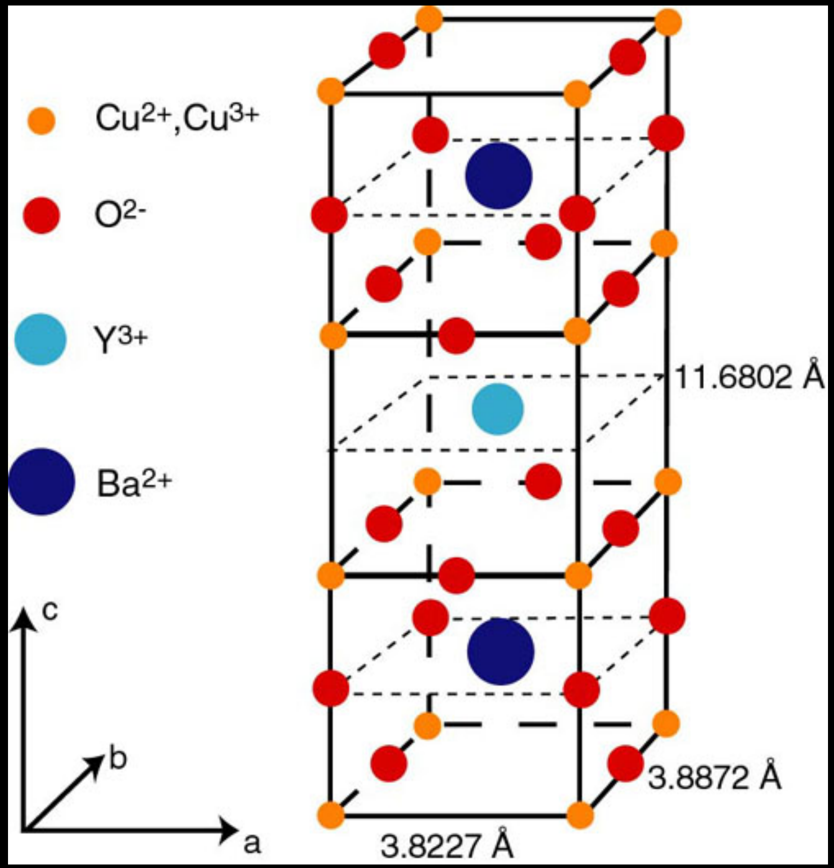




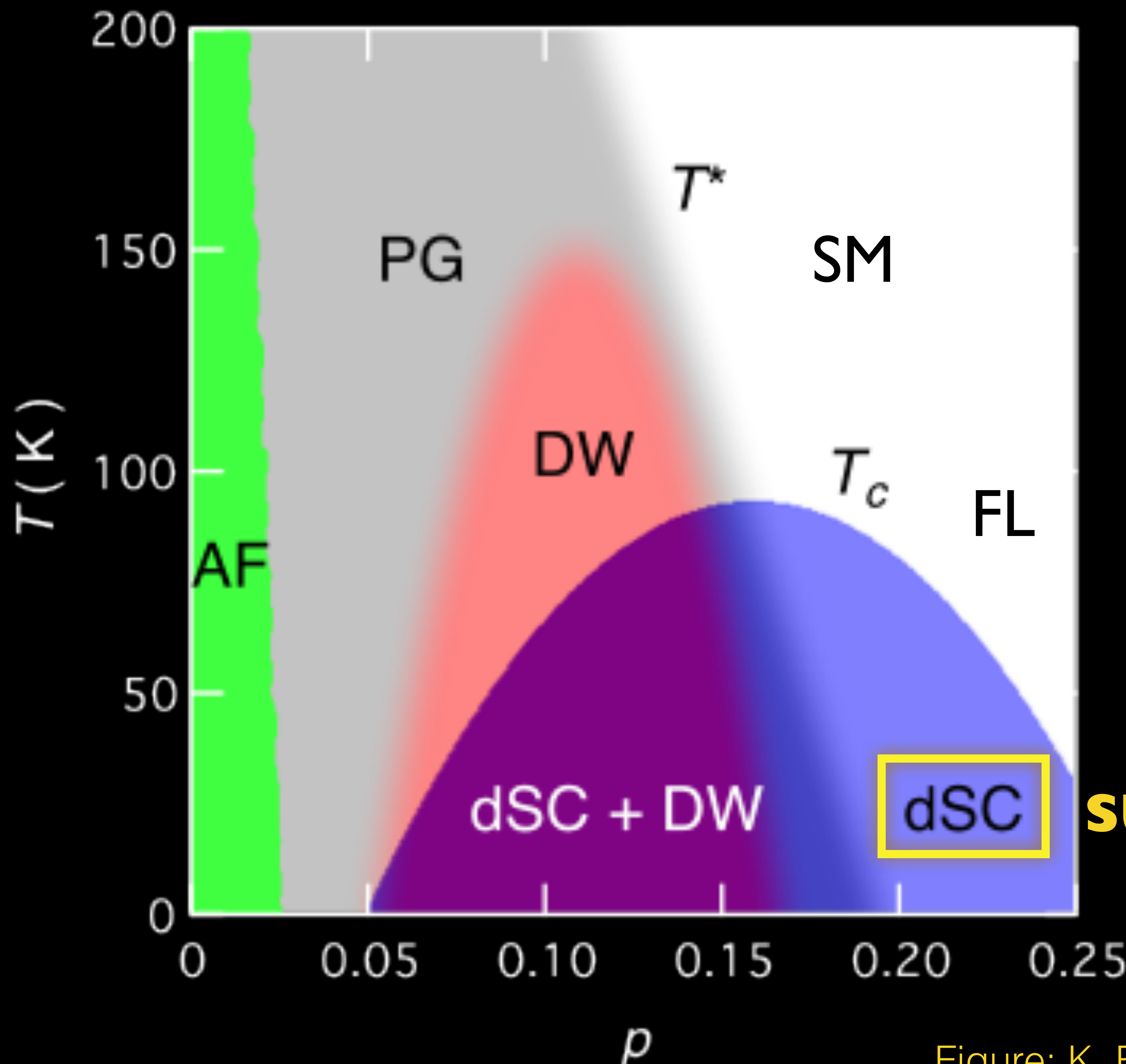
Bosons at half-integer density on the square lattice

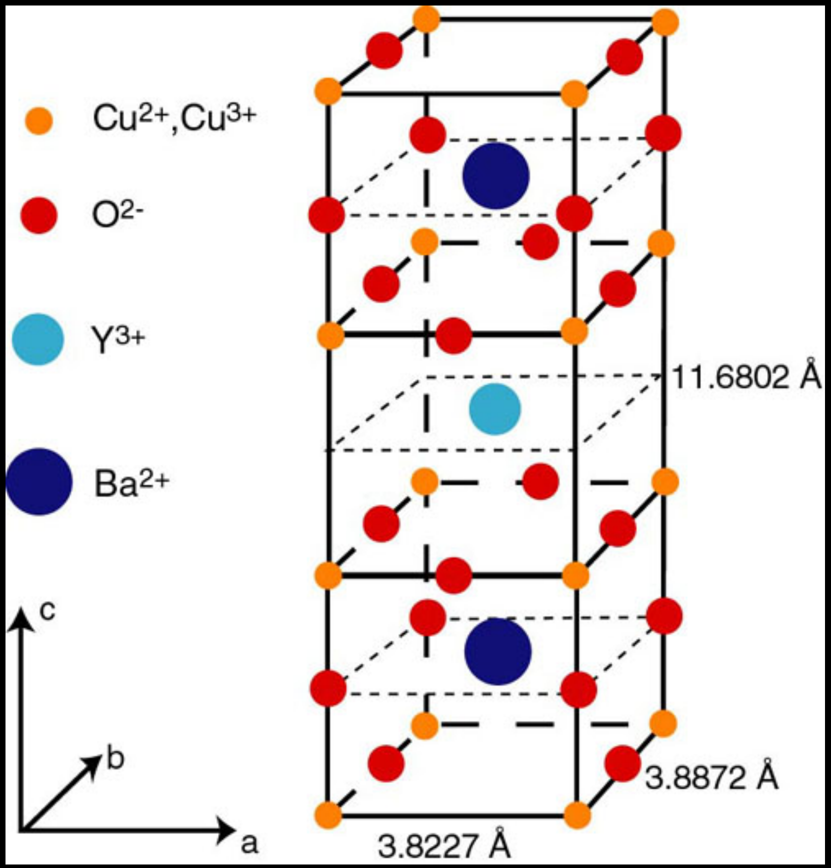




$YBa_2Cu_3O_{6+x}$

Figure: K. Fujita and J. C. Seamus Davis





$YBa_2Cu_3O_{6+x}$

d-wave superconductor

Figure: K. Fujita and J. C. Seamus Davis

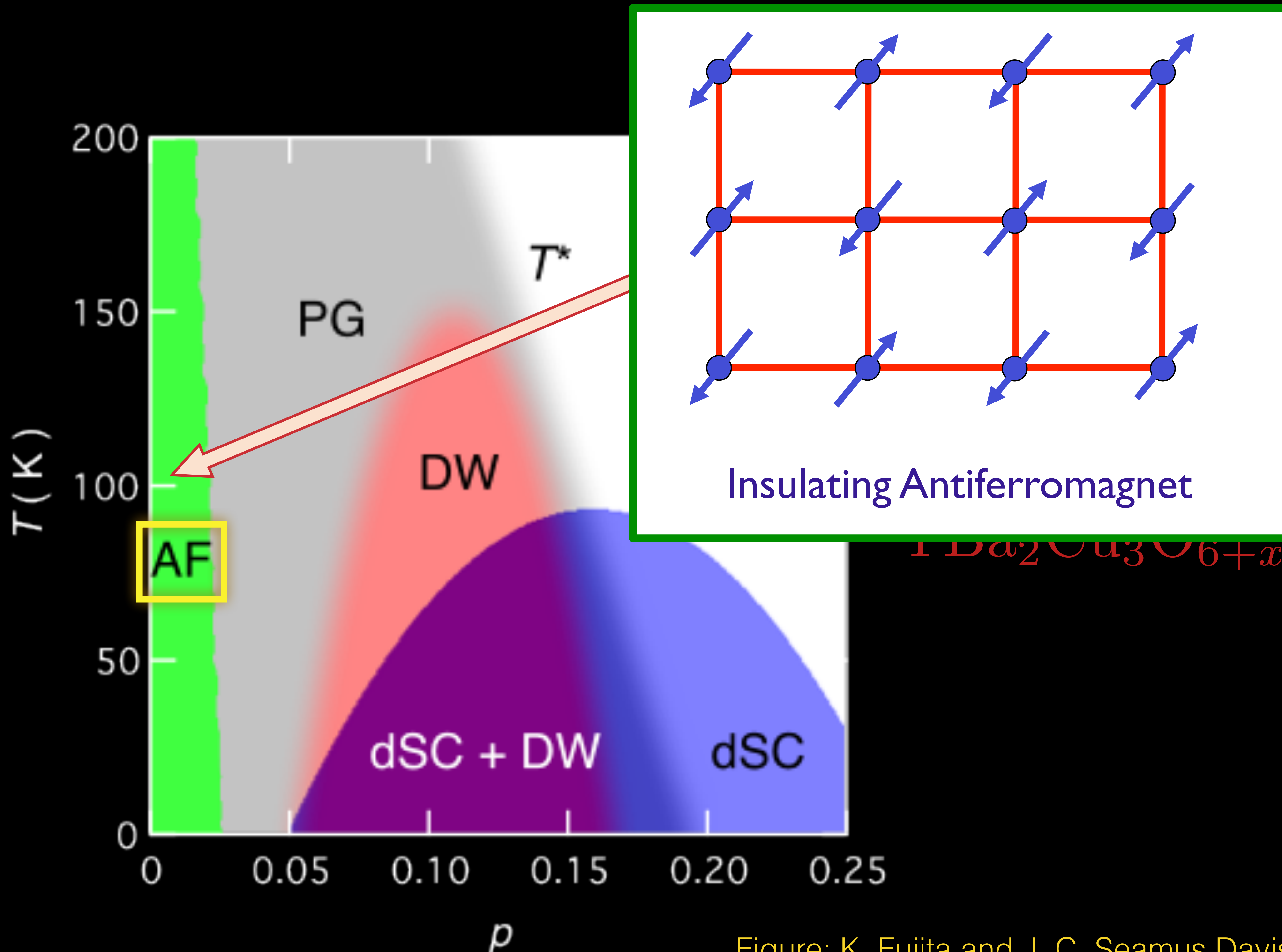
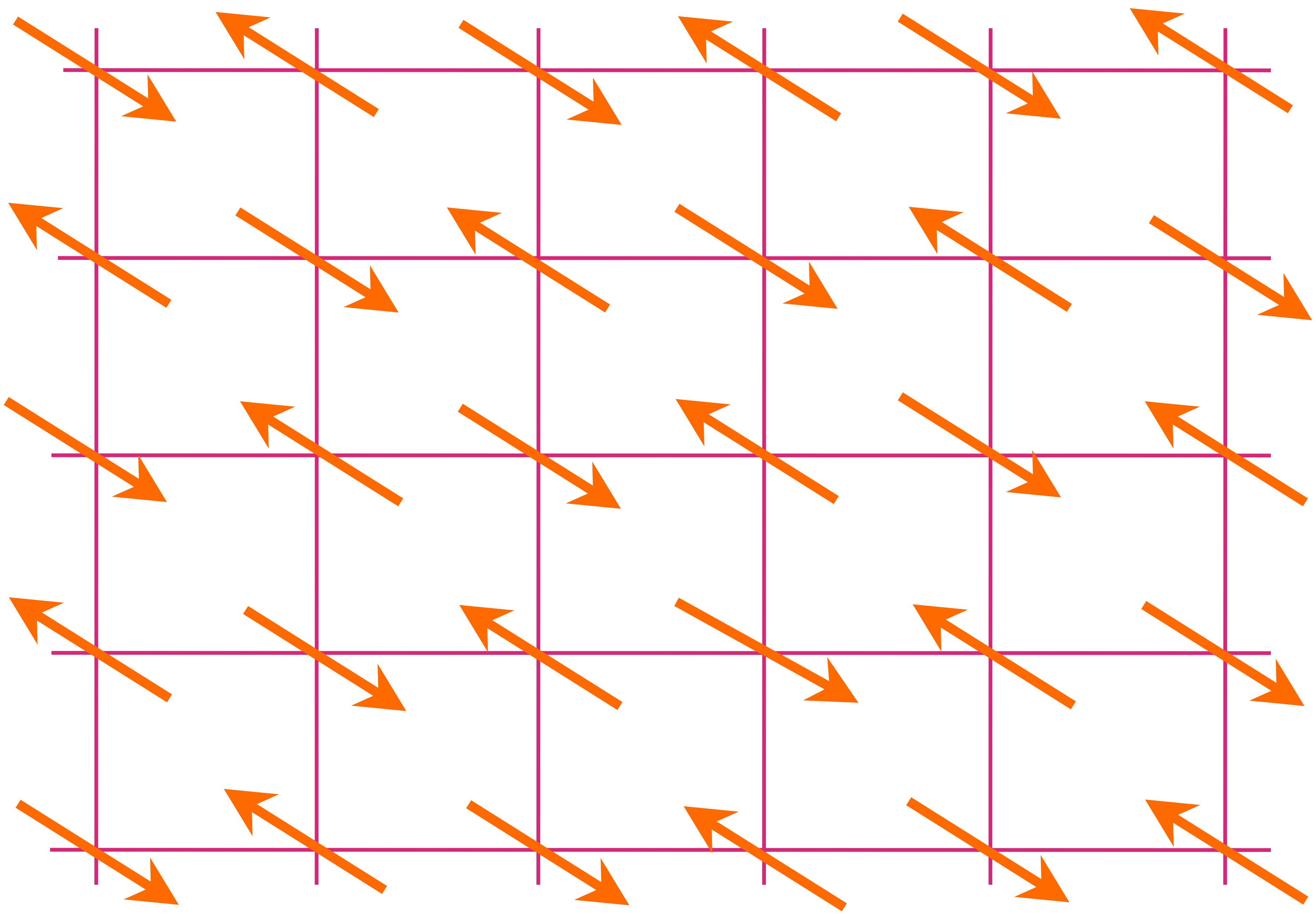
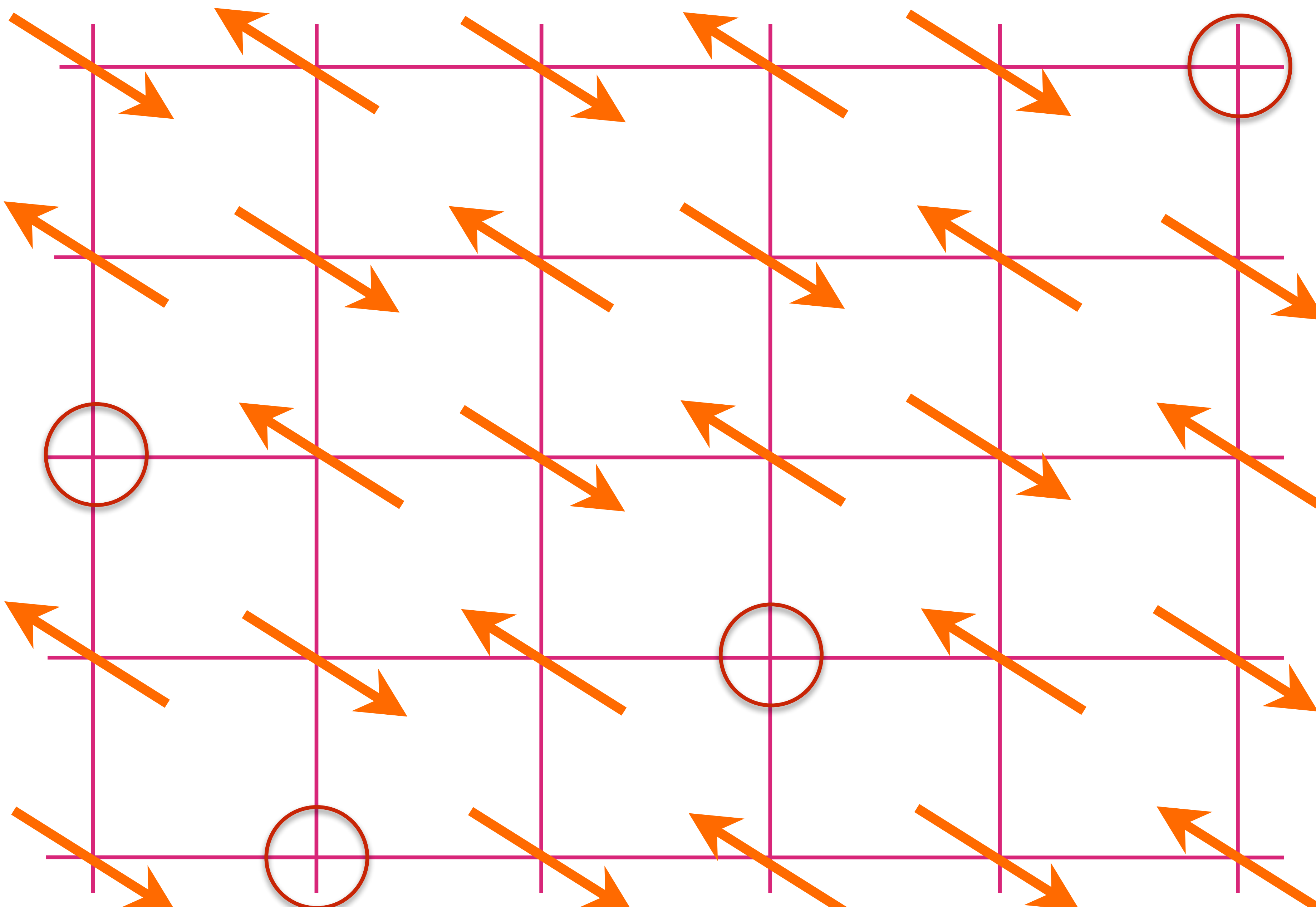


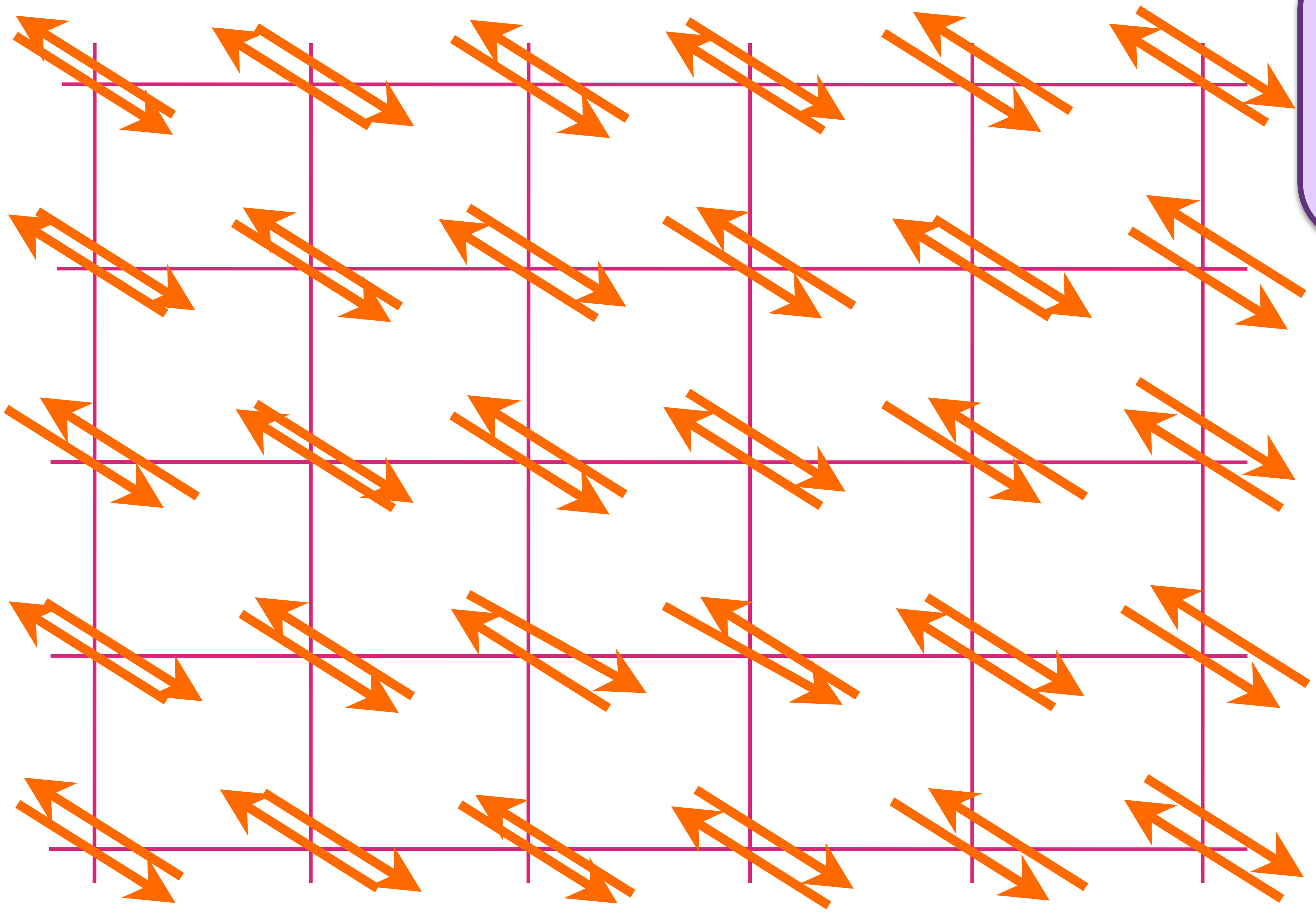
Figure: K. Fujita and J. C. Seamus Davis



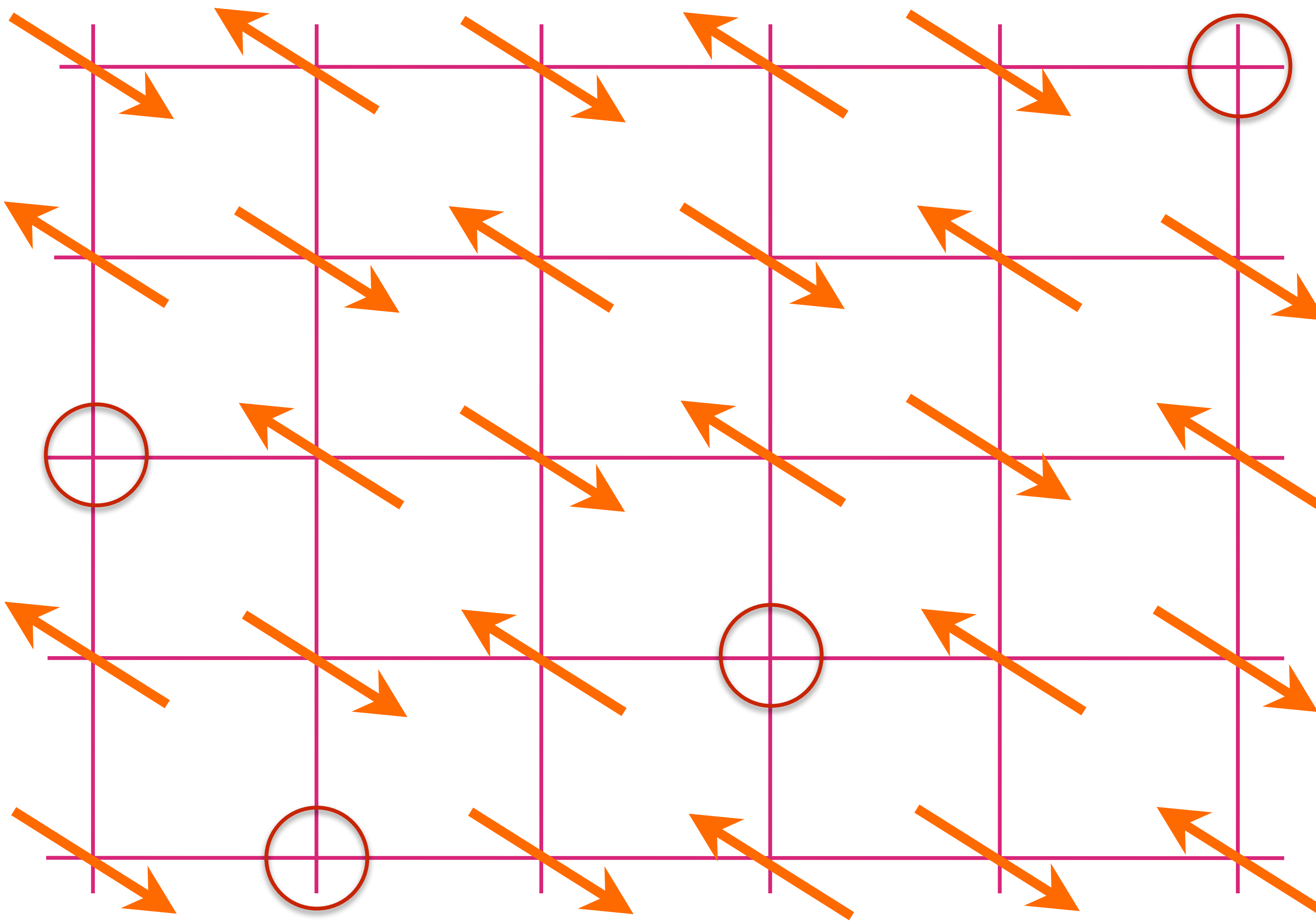
“Undoped” insulating anti-ferromagnet



Anti-
ferromagnet
with p mobile
holes
per square

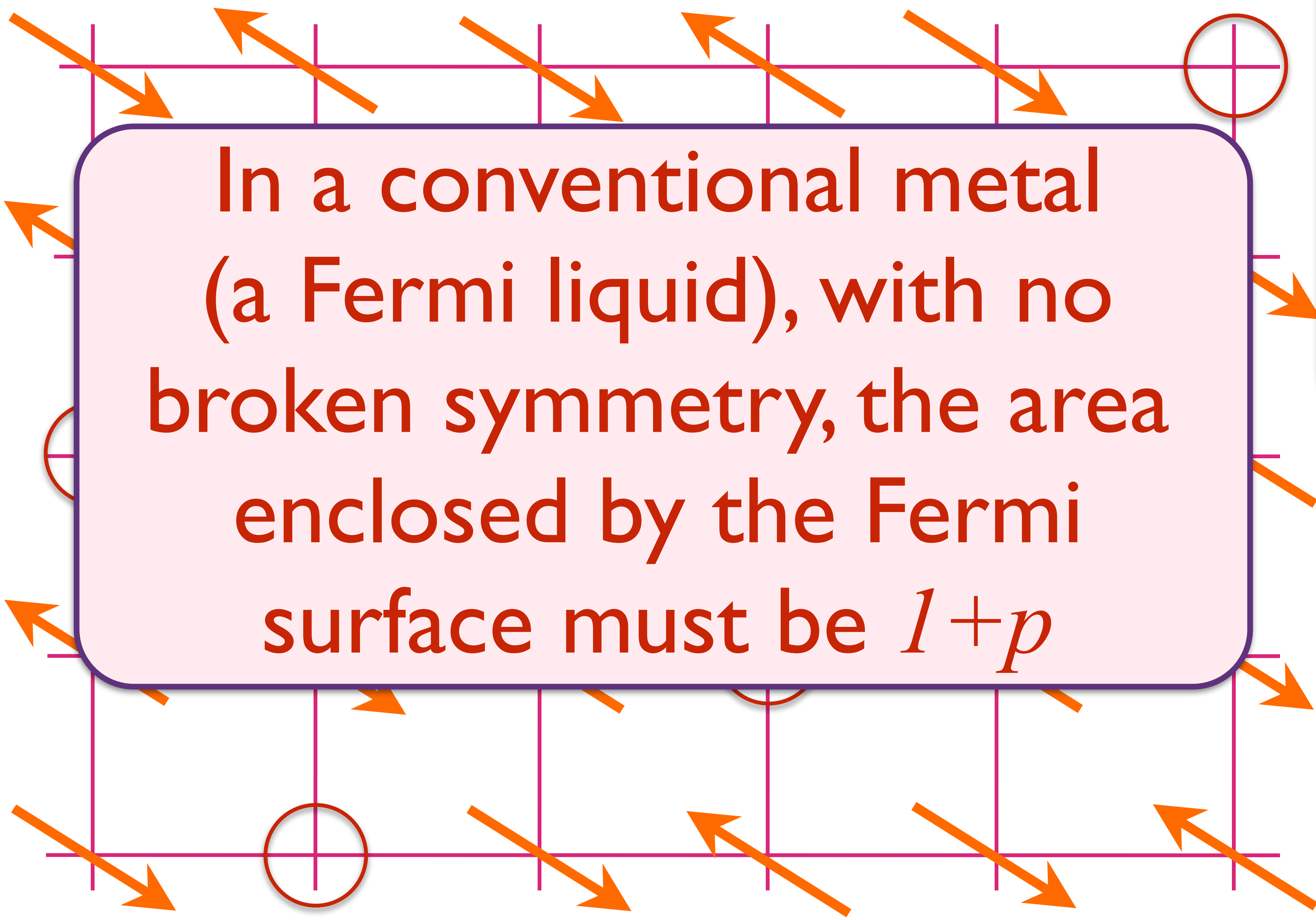


Filled
Band



Anti-
ferromagnet
with p mobile
holes
per square

But relative to
the band
insulator, there
are $1+p$ holes
per square



In a conventional metal (a Fermi liquid), with no broken symmetry, the area enclosed by the Fermi surface must be $1+p$

Anti-ferromagnet with p mobile holes per square

But relative to the band insulator, there are $1+p$ holes per square

The Hubbard Model

$$H = -\sum_{i<j} t_{ij} c_{i\alpha}^\dagger c_{j\alpha} + U \sum_i \left(n_{i\uparrow} - \frac{1}{2} \right) \left(n_{i\downarrow} - \frac{1}{2} \right) - \mu \sum_i c_{i\alpha}^\dagger c_{i\alpha}$$

$t_{ij} \rightarrow$ "hopping". $U \rightarrow$ local repulsion, $\mu \rightarrow$ chemical potential

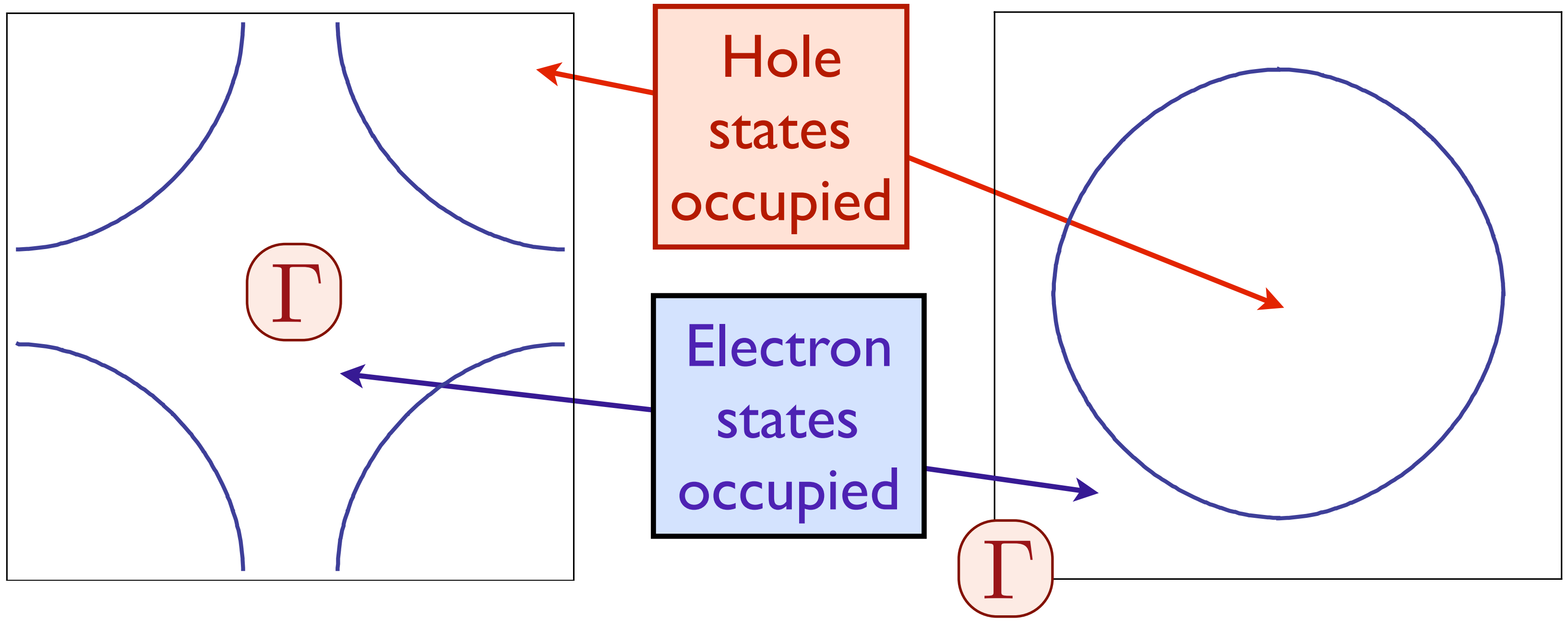
Spin index $\alpha = \uparrow, \downarrow$

$$n_{i\alpha} = c_{i\alpha}^\dagger c_{i\alpha}$$

$$c_{i\alpha}^\dagger c_{j\beta} + c_{j\beta} c_{i\alpha}^\dagger = \delta_{ij} \delta_{\alpha\beta}$$
$$c_{i\alpha} c_{j\beta} + c_{j\beta} c_{i\alpha} = 0$$

Will study on the square lattice

Fermi surfaces in electron- and hole-doped cuprates



Effective Hamiltonian for quasiparticles:

$$H_0 = -\sum_{i<j} t_{ij} c_{i\alpha}^\dagger c_{j\alpha} \equiv \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} c_{\mathbf{k}\alpha}^\dagger c_{\mathbf{k}\alpha}$$

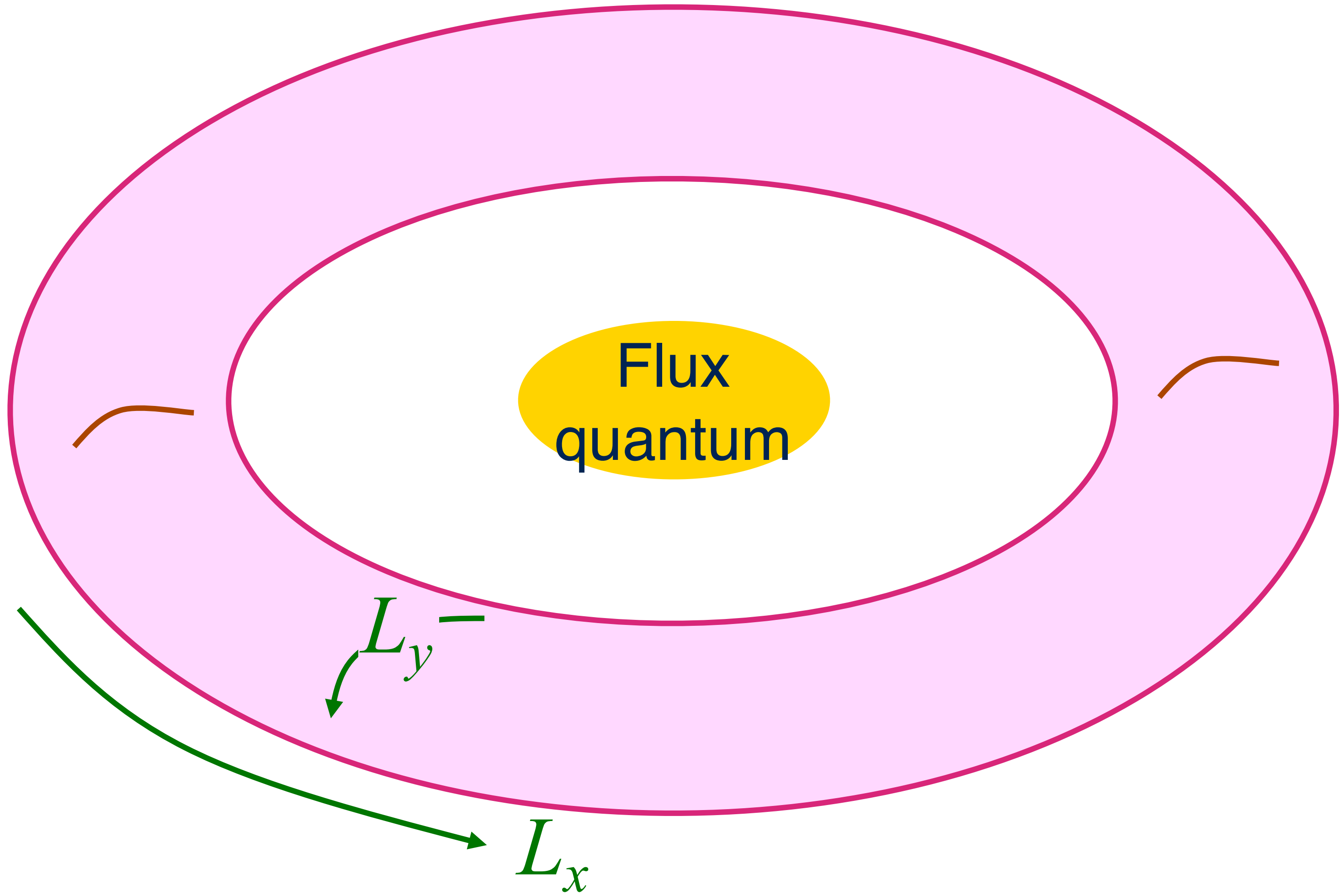
with t_{ij} non-zero for first, second and third neighbor, leads to satisfactory agreement with experiments. The area of the occupied electron states, $\mathcal{A}_e$, from Luttinger's theory is

$$\mathcal{A}_e = \begin{cases} 2\pi^2(1-x) & \text{for hole-doping } x \\ 2\pi^2(1+p) & \text{for electron-doping } p \end{cases}$$

The area of the occupied hole states, $\mathcal{A}_h$, which form a closed Fermi surface and so appear in quantum oscillation experiments is $\mathcal{A}_h = 4\pi^2 - \mathcal{A}_e$.

"Anomaly" constraints on the Fermi surface size

M. Oshikawa, PRL **84**, 3370 (2000)
A. Paramekanti and A. Vishwanath,
PRB **70**, 245118 (2004)



The anomaly argument can be restated as a computation of the change in crystal momentum, P, of the many-body state due to piercing by a flux quantum

$$\boxed{P_{xf} - P_{xi} = \frac{2\pi N}{L_x} (\mathrm{mod}\ 2\pi) = 2\pi \nu L_y (\mathrm{mod}\ 2\pi)}$$

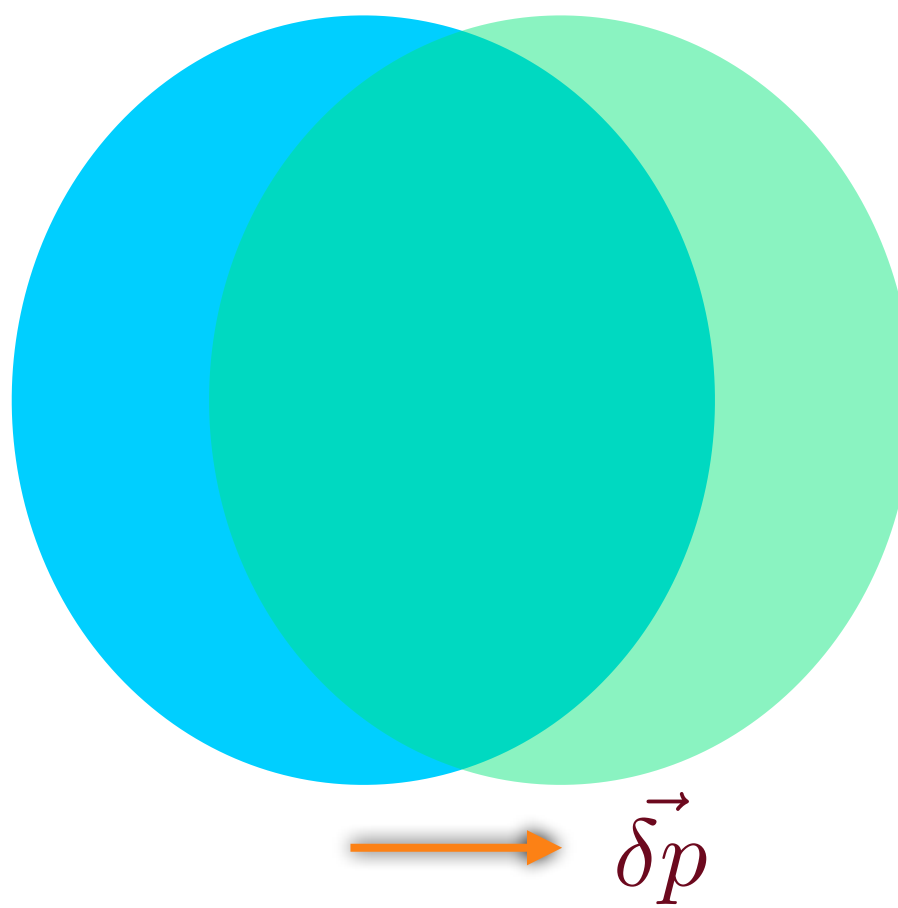
where $\nu = N/(L_x L_y)$ is the density of fermions.

"Anomaly" constraints on the Fermi surface size

$$\boxed{\Delta P_x = 2\pi\nu L_y (\mathrm{mod}\ 2\pi) \quad , \quad \Delta P_y = 2\pi\nu L_x (\mathrm{mod}\ 2\pi)}$$

Now we compute the momentum balance *assuming* that the only low energy excitations are quasiparticles near the Fermi surface, and these react like free particles to a sufficiently slow flux insertion. So each quasiparticle picks up a momentum $\vec{\delta p} \equiv (2\pi/L_x, 0)$, and then we can write (with δn_p the quasiparticle density excited by the flux insertion)

$$\Delta P_x = \sum_p \delta n_p p_x .$$



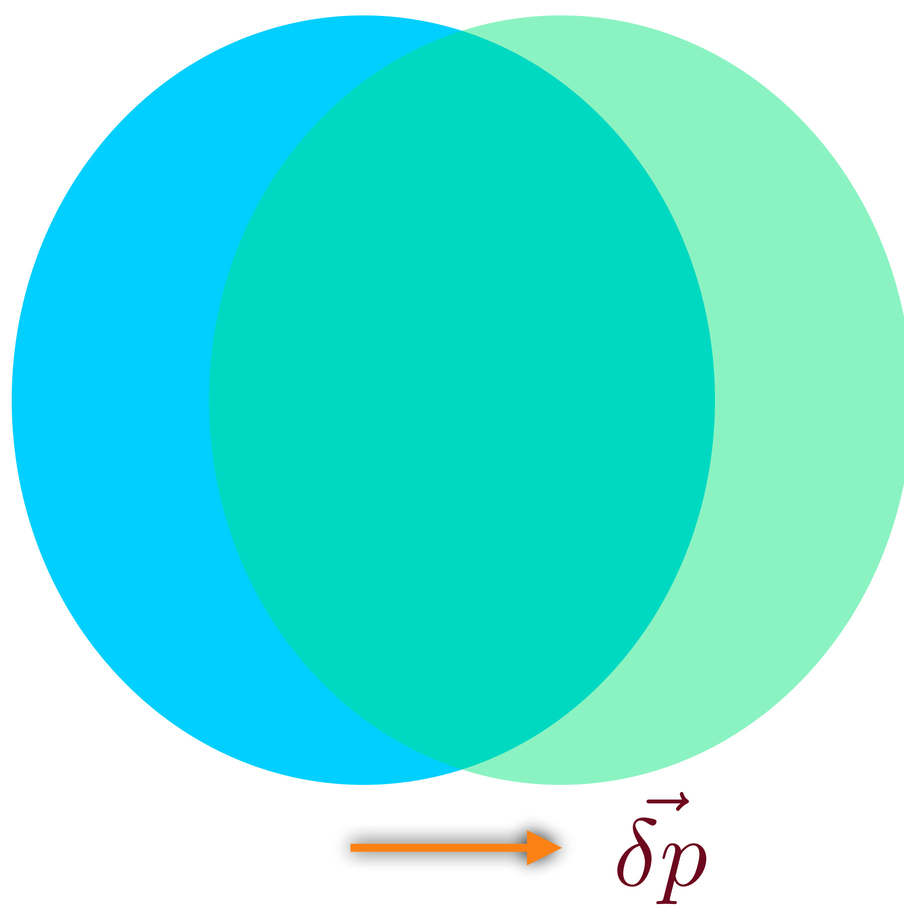
"Anomaly" constraints on the Fermi surface size

$$\boxed{\Delta P_x = 2\pi\nu L_y (\mathrm{mod}\ 2\pi) \quad , \quad \Delta P_y = 2\pi\nu L_x (\mathrm{mod}\ 2\pi)}$$

Now $\delta n_p = \pm 1$ on a shell of thickness $\vec{\delta p} \cdot d\vec{S}_p$ on the Fermi surface (where $\vec{S}_p$ is an area element on the Fermi surface). So we can write the above as a surface integral

$$\begin{aligned} \Delta P_x &= \oint_{\mathrm{FS}} p_x \left(\frac{L_x L_y}{4\pi^2}\right) \vec{\delta p} \cdot d\vec{S}_p \\ &= (\vec{\delta p} \cdot \hat{x}) \int_{\mathrm{FV}} \left(\frac{L_x L_y}{4\pi^2}\right) dV \end{aligned}$$

by the divergence theorem. So



“Anomaly” constraints on the Fermi surface size

$$\boxed{\Delta P_x = 2\pi\nu L_y (\text{mod } 2\pi) \quad , \quad \Delta P_y = 2\pi\nu L_x (\text{mod } 2\pi)}$$

$$\Delta P_x = \left(\frac{2\pi}{L_x}\right)\frac{L_x L_y}{4\pi^2} V_{\text{FS}} \quad , \quad \Delta P_y = \left(\frac{2\pi}{L_y}\right)\frac{L_x L_y}{4\pi^2} V_{\text{FS}}$$

where V_{FS} is the volume of the Fermi surface. So, although the quasiparticles are only defined near the Fermi surface, by using Gauss’s Law on the momentum acquired by quasiparticles near the Fermi surface, we have converted the answer to an integral over the volume enclosed by the Fermi surface.
Now we equate these values to those obtained above, and obtain

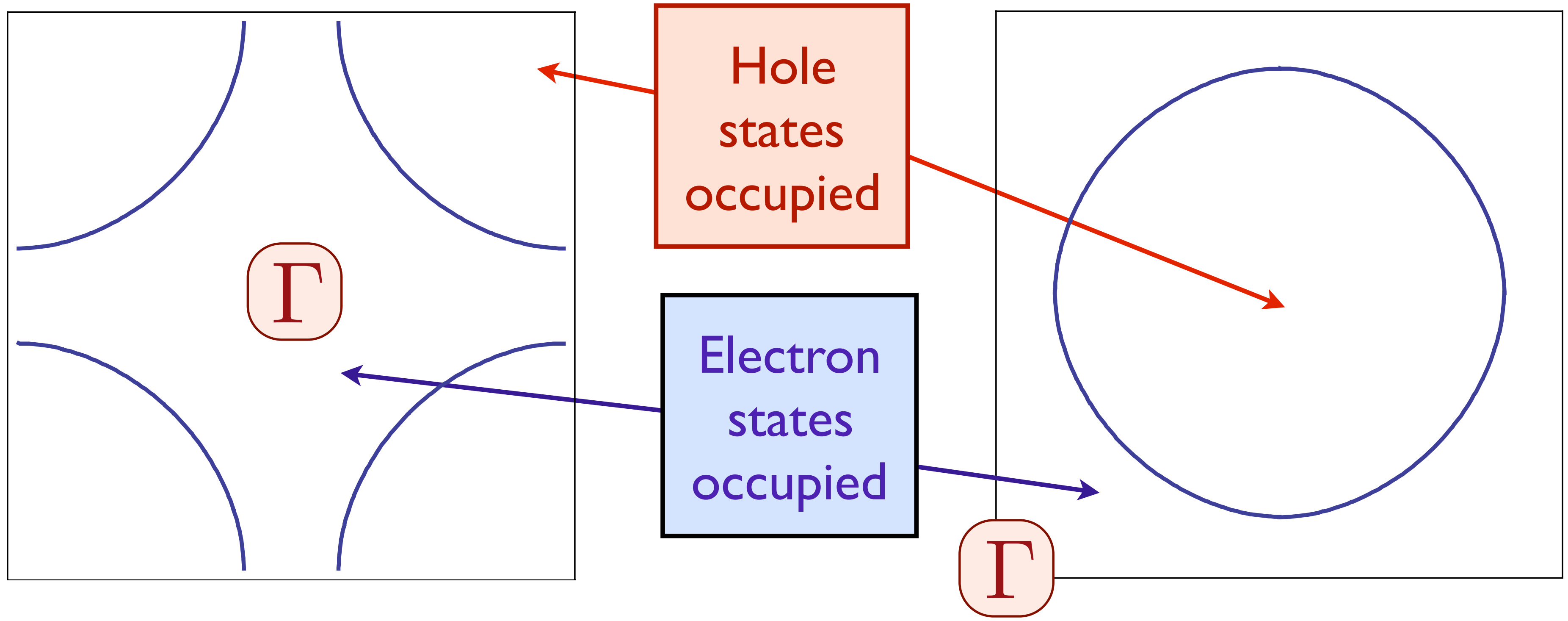
$$N - L_x L_y \frac{V_{\text{FS}}}{4\pi^2} = L_x m_x \quad , \quad N - L_x L_y \frac{V_{\text{FS}}}{4\pi^2} = L_y m_y$$

for some integers m_x, m_y. Now choose L_x, L_y mutually prime integers; then $m_x L_x = m_y L_y$ implies that $m_x L_x = m_y L_y = p L_x L_y$ for some integer p. Then we obtain

$$\nu = \frac{N}{L_x L_y} = \frac{V_{\text{FS}}}{4\pi^2} + p.$$

This is Luttinger’s theorem.

Fermi surface+antiferromagnetism



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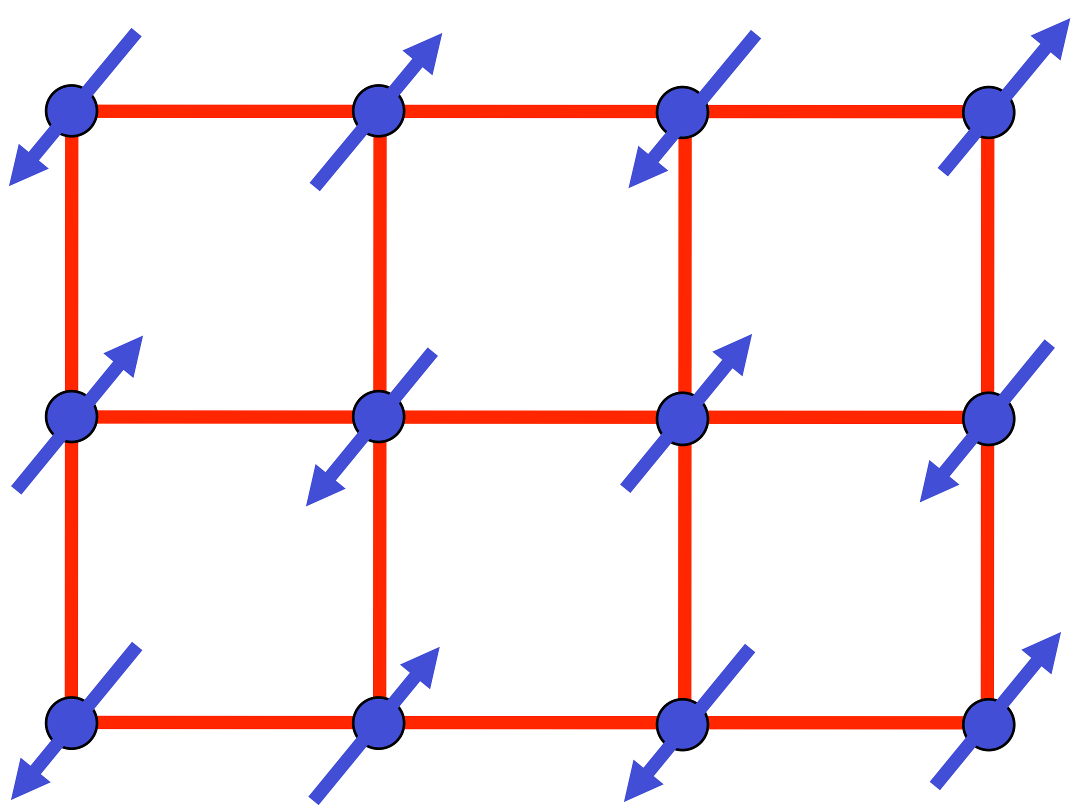


The electron spin polarization obeys

$$\left\langle \vec{S}(\mathbf{r},\tau) \right\rangle = \vec{\varphi}(\mathbf{r},\tau) e^{i\mathbf{K}\cdot\mathbf{r}}$$

where $\mathbf{K}$ is the ordering wavevector.

Fermi surface+antiferromagnetism

We use the operator equation (valid on each site i):

$$U\left(n_{\uparrow}-\frac{1}{2}\right)\left(n_{\downarrow}-\frac{1}{2}\right)=-\frac{2U}{3}\vec{S}^{2}+\frac{U}{4} \tag{1}$$

Then we decouple the interaction via

$$\exp\left(\frac{2U}{3}\sum_{i}\int d\tau \vec{S}_{i}^{2}\right)=\int \mathcal{D}\vec{J}_{i}(\tau)\exp\left(-\sum_{i}\int d\tau\left[\frac{3}{8U}\vec{J}_{i}^{2}-\vec{J}_{i}\vec{S}_{i}\right]\right) \tag{2}$$

We now integrate out the fermions, and look for the saddle point of the resulting effective action for $\vec{J}_i$. At the saddle-point we find that the lowest energy is achieved when the vector has opposite orientations on the A and B sublattices. Anticipating this, we look for a continuum limit in terms of a field $\vec{\varphi}_i$ where

$$\vec{J}_{i}=\vec{\varphi}_{i}\,e^{i\mathbf{K}\cdot\mathbf{r}_{i}} \tag{3}$$

Fermi surface+antiferromagnetism

In this manner, we obtain the "spin-fermion" model

$$\mathcal{Z} = \int \mathcal{D}c_\alpha \mathcal{D}\vec{\varphi} \exp\left(-\mathcal{S}\right)$$

$$\mathcal{S} = \int d\tau \sum_{\mathbf{k}} c^\dagger_{\mathbf{k}\alpha} \left(\frac{\partial}{\partial\tau} - \varepsilon_{\mathbf{k}} \right) c_{\mathbf{k}\alpha}$$

$$- \lambda \int d\tau \sum_i c^\dagger_{i\alpha} \vec{\varphi}_i \cdot \vec{\sigma}_{\alpha\beta} c_{i\beta} e^{i\mathbf{K}\cdot\mathbf{r}_i}$$

$$+ \int d\tau d^2 r \left[\frac{1}{2} \left(\boldsymbol{\nabla}_r \vec{\varphi}\right)^2 + \frac{1}{2} \left(\partial_\tau \vec{\varphi}\right)^2 + \frac{s}{2} \vec{\varphi}^2 + \frac{u}{4} \vec{\varphi}^4 \right]$$

Fermi surface+antiferromagnetism

In the Hamiltonian form (ignoring, for now, the time dependence of $\vec{\varphi}$), the coupling between $\vec{\varphi}$ and the electrons takes the form

$$H_{\rm sdw} = \lambda \sum_{\mathbf{k},\mathbf{q},\alpha,\beta} \vec{\varphi}_{\mathbf{q}} \cdot c^{\dagger}_{\mathbf{k}+\mathbf{q},\alpha} \vec{\sigma}_{\alpha\beta} c_{\mathbf{k}+\mathbf{K},\beta}$$

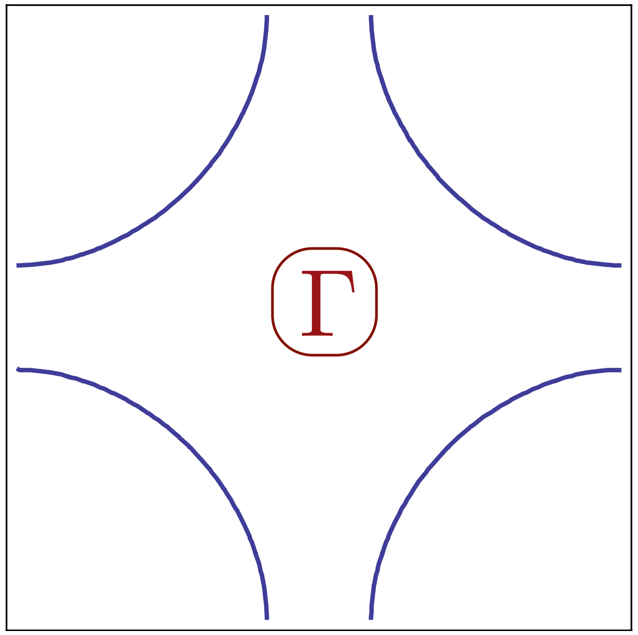
where $\vec{\sigma}$ are the Pauli matrices, the boson momentum $\mathbf{q}$ is small, while the fermion momenum $\mathbf{k}$ extends over the entire Brillouin zone. In the antiferromagnetically ordered state, we may take $\vec{\varphi} \propto (0,0,1)$, and the electron dispersions obtained by diagonalizing $H_0 + H_{\rm sdw}$ are

$$E_{\mathbf{k}\pm} = \frac{\varepsilon_{\mathbf{k}} + \varepsilon_{\mathbf{k}+\mathbf{K}}}{2} \pm \sqrt{\left(\frac{\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}+\mathbf{K}}}{2}\right)^2 + \lambda^2 |\vec{\varphi}|^2}$$

This leads to the Fermi surfaces shown in the following slides as a function of increasing $|\vec{\varphi}|$.

Square lattice Hubbard model at p=0

Increasing SDW order

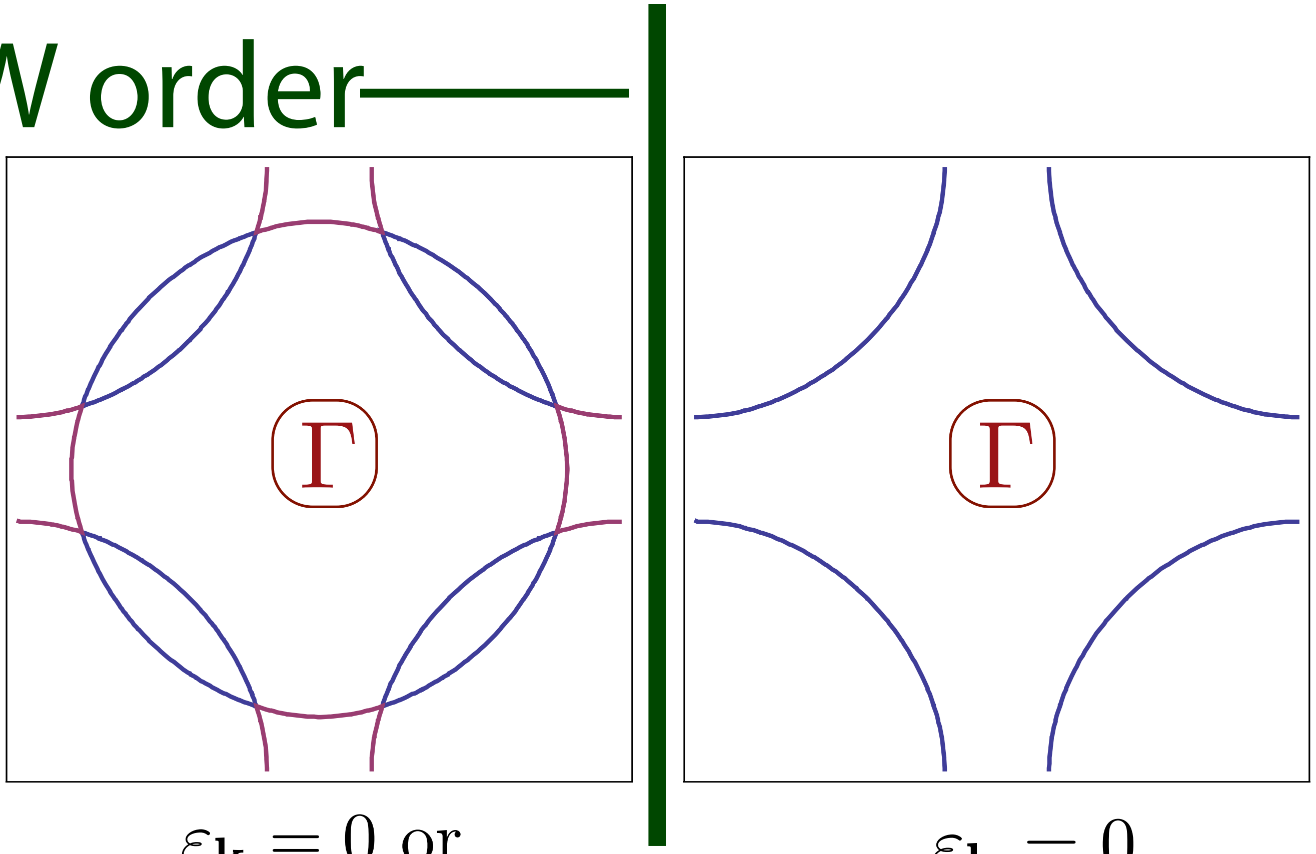


$\varepsilon_{\mathbf{k}} = 0$

S. Sachdev, A.V. Chubukov, and A. Sokol, *Phys. Rev. B* **51**, 14874 (1995).
A.V. Chubukov and D. K. Morr, *Physics Reports* **288**, 355 (1997).

Square lattice Hubbard model at p=0

Increasing SDW order



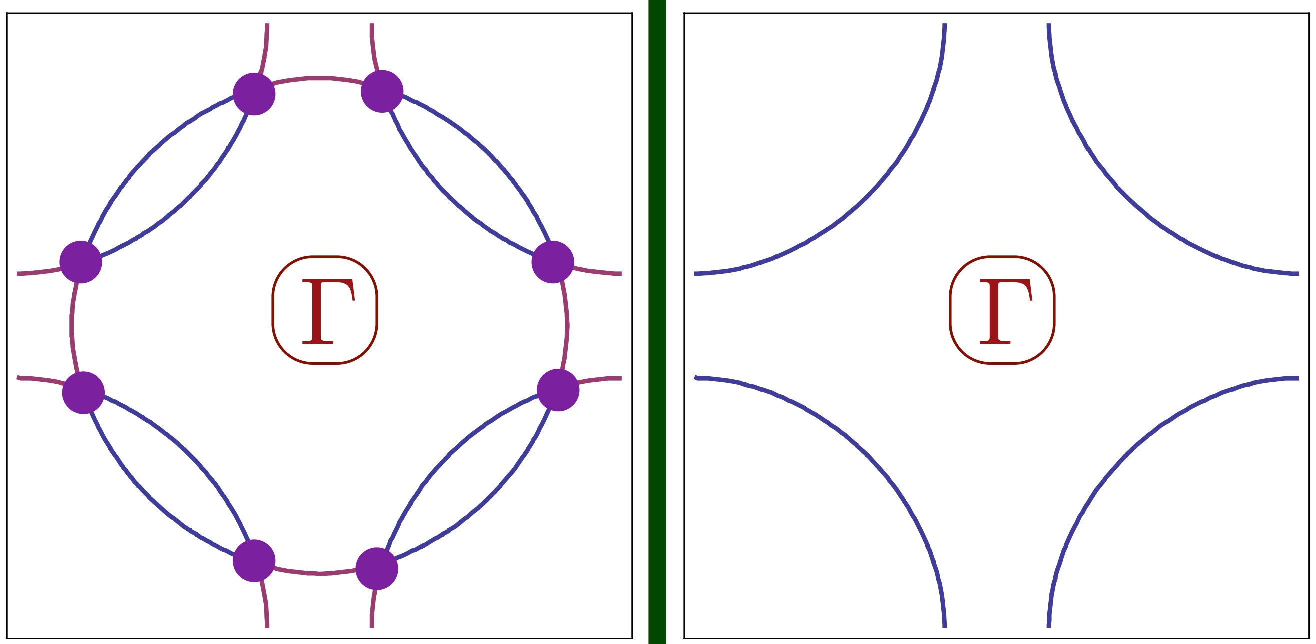
$\varepsilon_{\mathbf{k}} = 0$ or
$\varepsilon_{\mathbf{k}+\mathbf{K}} = 0$

$\varepsilon_{\mathbf{k}} = 0$

S. Sachdev, A.V. Chubukov, and A. Sokol, *Phys. Rev. B* **51**, 14874 (1995).
A.V. Chubukov and D. K. Morr, *Physics Reports* **288**, 355 (1997).

Square lattice Hubbard model at p=0

Increasing SDW order

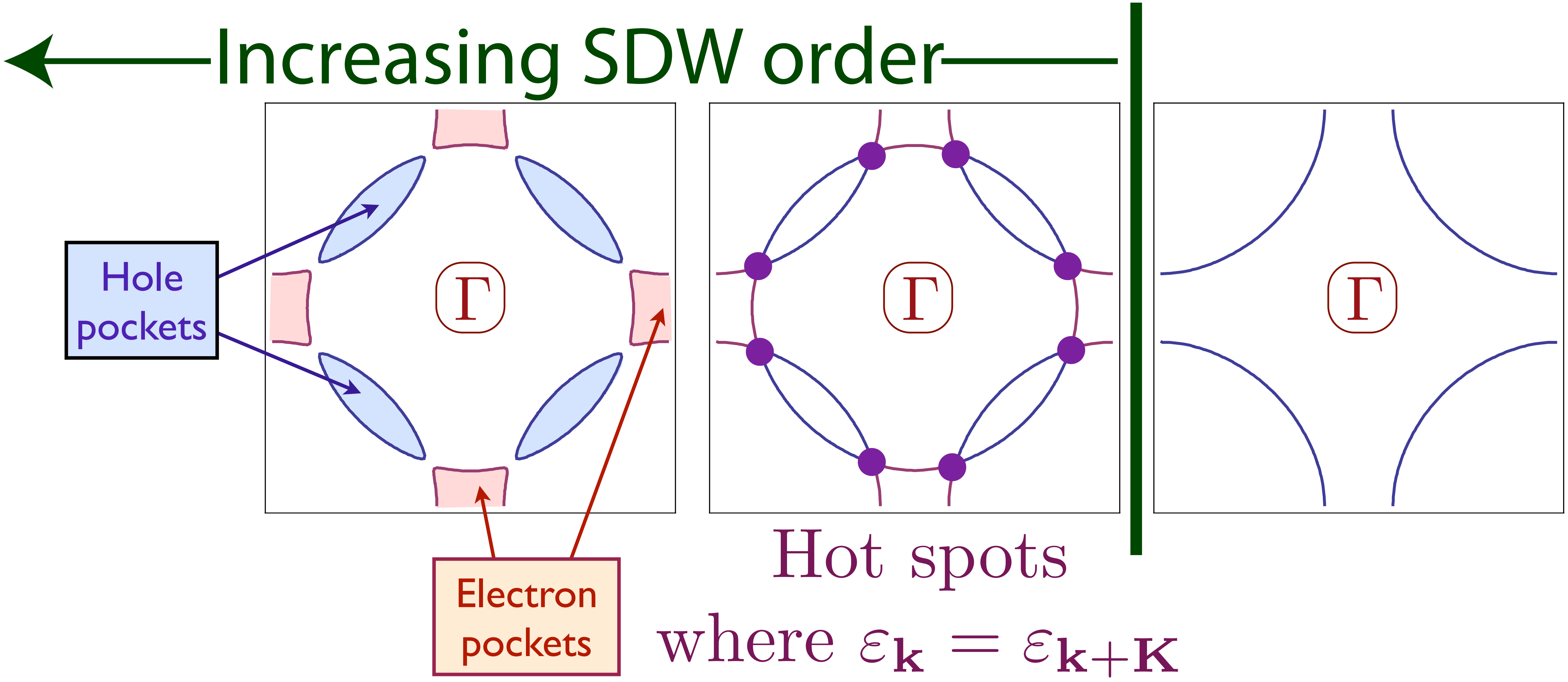


Hot spots

where $\varepsilon_{\mathbf{k}} = \varepsilon_{\mathbf{k}+\mathbf{K}}$

S. Sachdev, A.V. Chubukov, and A. Sokol, *Phys. Rev. B* **51**, 14874 (1995).
A.V. Chubukov and D. K. Morr, *Physics Reports* **288**, 355 (1997).

Square lattice Hubbard model at $p=0$



Fermi surface breaks up at hot spots into electron and hole "pockets"

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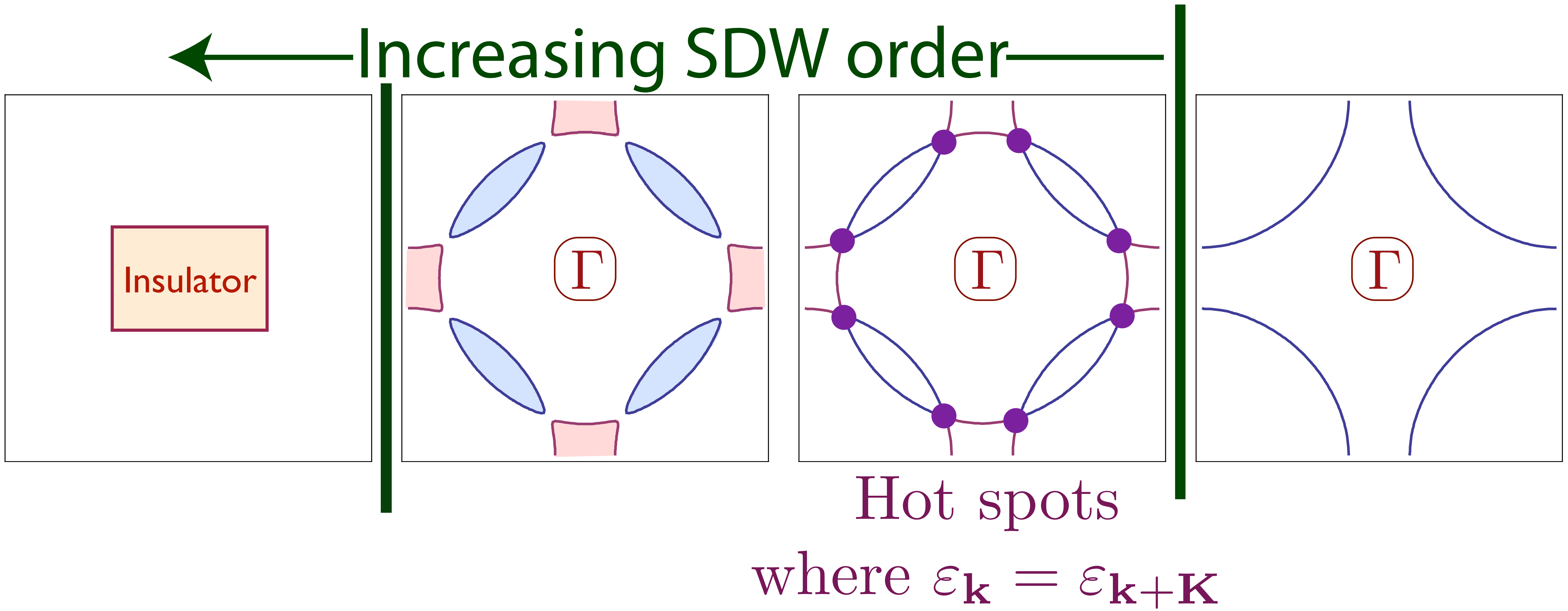
Square lattice Hubbard model at p=0



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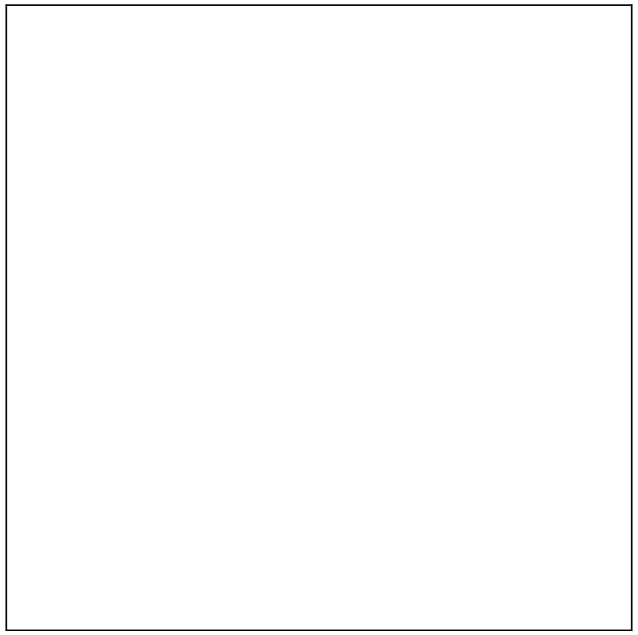
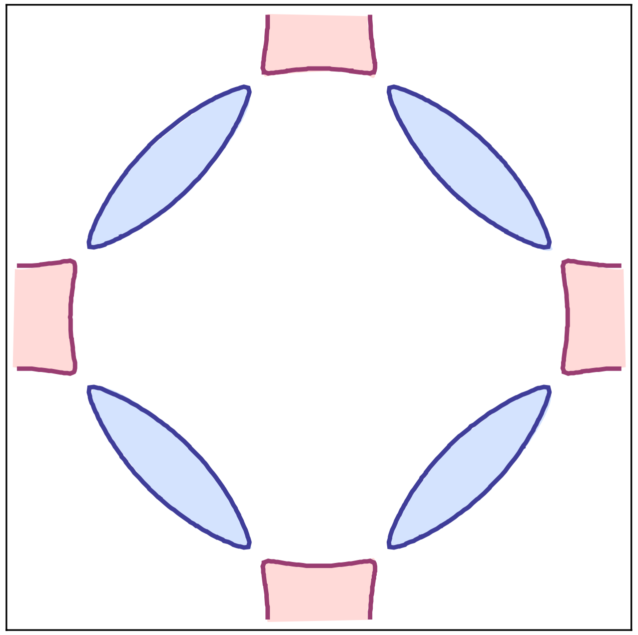
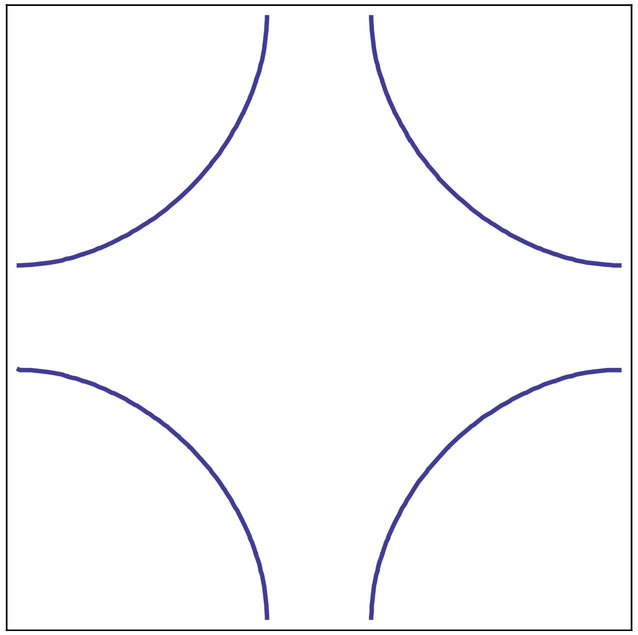
Square lattice Hubbard model at $p=0$

$\langle \vec{\varphi} \rangle \neq 0$ and large	$\langle \vec{\varphi} \rangle \neq 0$ and small	$\langle \vec{\varphi} \rangle = 0$
		
Insulator	Metal with electron and hole pockets	Metal with "large" Fermi surface

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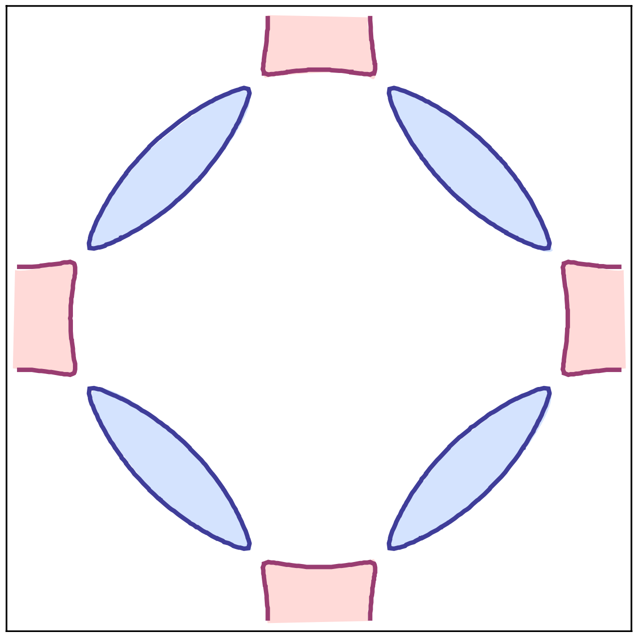
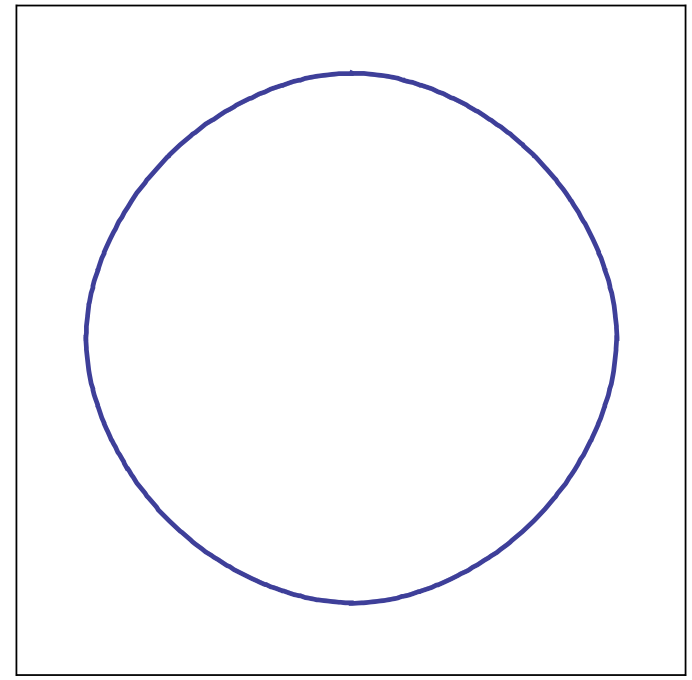
Spin density wave order, topological order, and Fermi surface reconstruction

Quantum phase transition with Fermi surface reconstruction



$\langle \vec{\varphi} \rangle \neq 0$

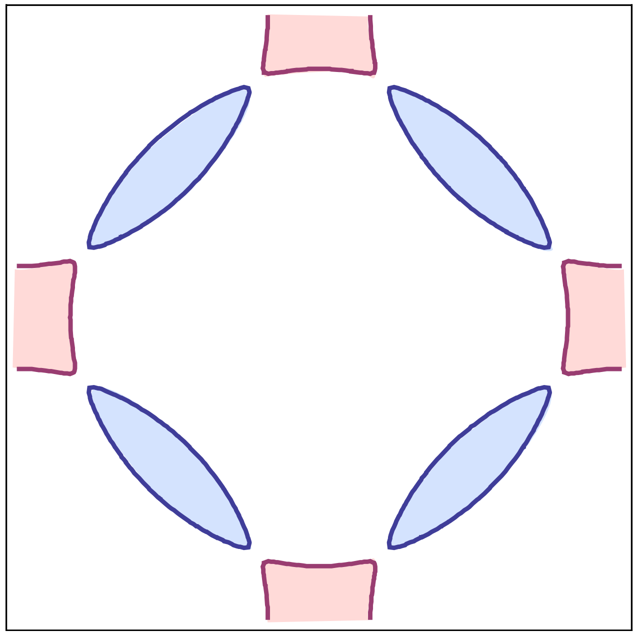
Metal with electron and hole pockets



$\langle \vec{\varphi} \rangle = 0$

Metal with “large” Fermi surface

Separating onset of SDW order and Fermi surface reconstruction



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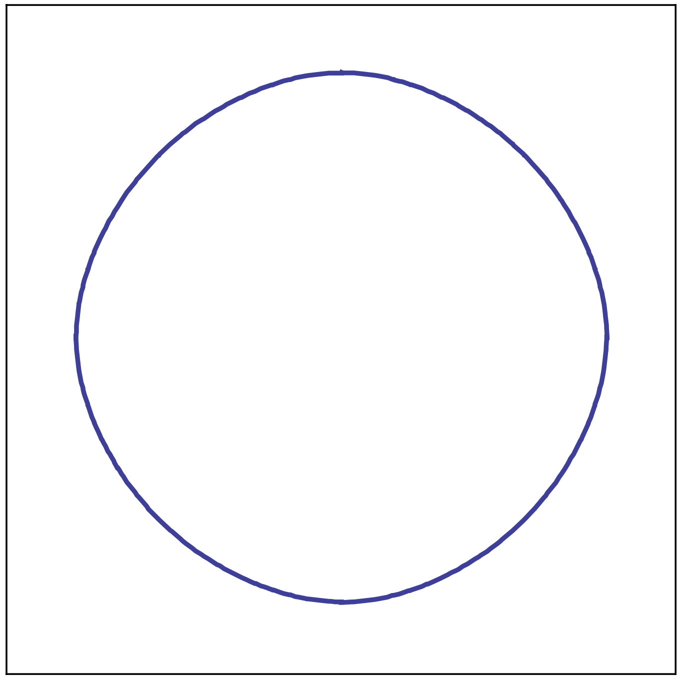
Metal with electron and hole pockets



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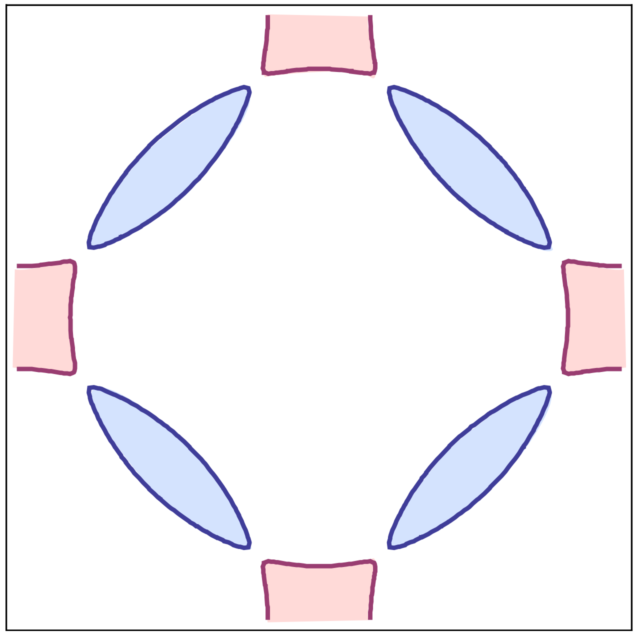
Separating onset of SDW order and Fermi surface reconstruction

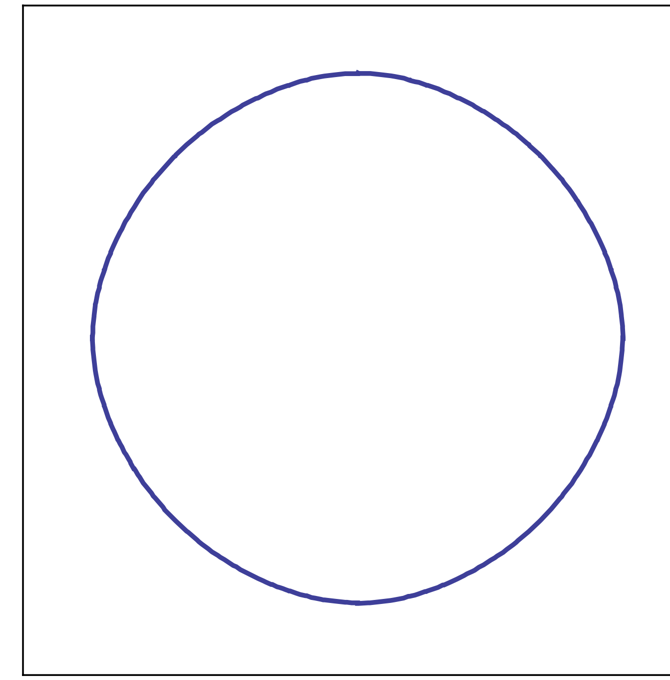


$\langle \vec{\varphi} \rangle \neq 0$

Metal with electron and hole pockets

Electron and/or hole Fermi pockets form in "local" SDW order, but quantum fluctuations destroy long-range SDW order

$\langle \vec{\varphi} \rangle = 0$

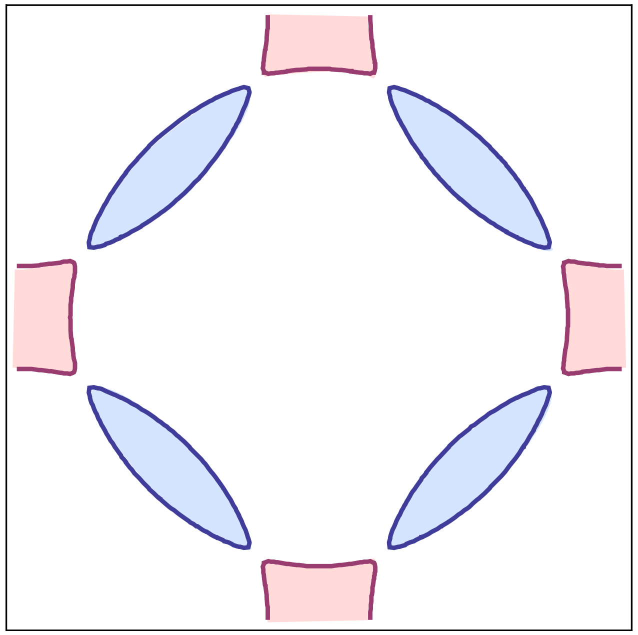


$\langle \vec{\varphi} \rangle = 0$

Metal with "large" Fermi surface

T. Senthil, S. Sachdev, and M. Vojta, *Phys. Rev. Lett.* **90**, 216403 (2003)

Separating onset of SDW order and Fermi surface reconstruction



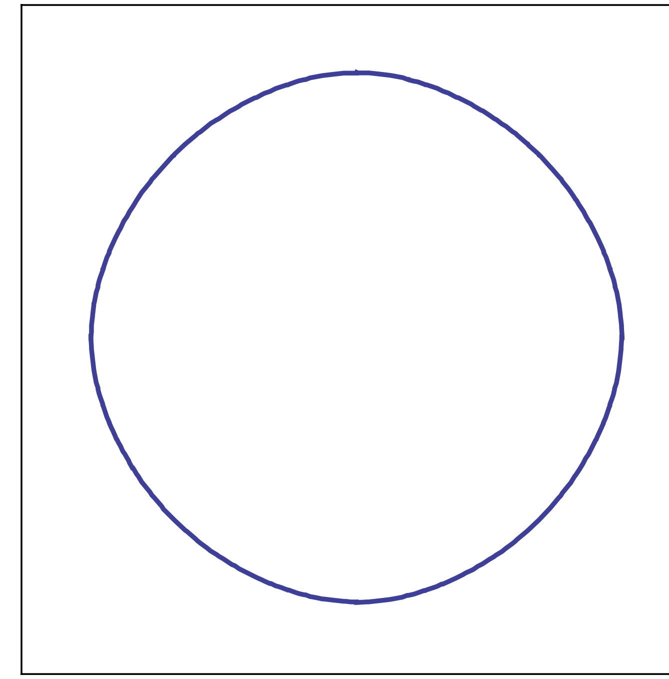
$\langle \vec{\varphi} \rangle \neq 0$

Metal with electron and hole pockets

Electron and/or hole Fermi pockets form in "local" SDW order, but quantum fluctuations destroy long-range SDW order

$\langle \vec{\varphi} \rangle = 0$

Algebraic Charge liquid (ACL) or Fractionalized Fermi liquid (FL*) phase with no symmetry breaking and pocket Fermi surfaces



$\langle \vec{\varphi} \rangle = 0$

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T. Senthil, S. Sachdev, and M. Vojta, *Phys. Rev. Lett.* **90**, 216403 (2003)

Spin density wave order, topological order, and Fermi surface reconstruction

Subir Sachdev,[1,2] Erez Berg,[3] Shubhayu Chatterjee,[1] and Yoni Schattner[3]

arXiv:1606.07813

Hertz theory for XY SDW order

The Hertz theory for the onset of SDW order can be described by the following Hamiltonian

$$H_{\rm sdw} = H_c + H_\theta + H_Y, \tag{1.1}$$

where H_c describes electrons (of density $(1-p)$) hopping on the sites of a square lattice

$$H_c = -\sum_{i,j} \left(t_{ij} + \mu\delta_{ij}\right) c_{i\alpha}^\dagger c_{j\alpha} \tag{1.2}$$

with $c_{i\alpha}$ the electron annihilation operator on site i with spin $\alpha =\uparrow,\downarrow$. We represent the SDW order by a lattice XY rotor model, described by an angle θ_i, and its canonically conjugate number operator N_i, obeying

$$H_\theta = -\sum_{i<j} J_{ij} \cos(\theta_i - \theta_j) + 4\Delta \sum_i N_i^2 \quad ; \quad [\theta_i, N_j] = i\delta_{ij}, \tag{1.3}$$

where J_{ij} positive exchange constants, and Δ is proportional to the bare spin-wave gap

Hertz theory for XY SDW order

Finally, there is a 'Yukawa' coupling between the XY order parameter, $e^{i\theta}$, and the fermions

$$H_Y = -\lambda \sum_i \eta_i \left[e^{-i\theta_i} c_{i\uparrow}^\dagger c_{i\downarrow} + e^{i\theta_i} c_{i\downarrow}^\dagger c_{i\uparrow} \right], \tag{1.4}$$

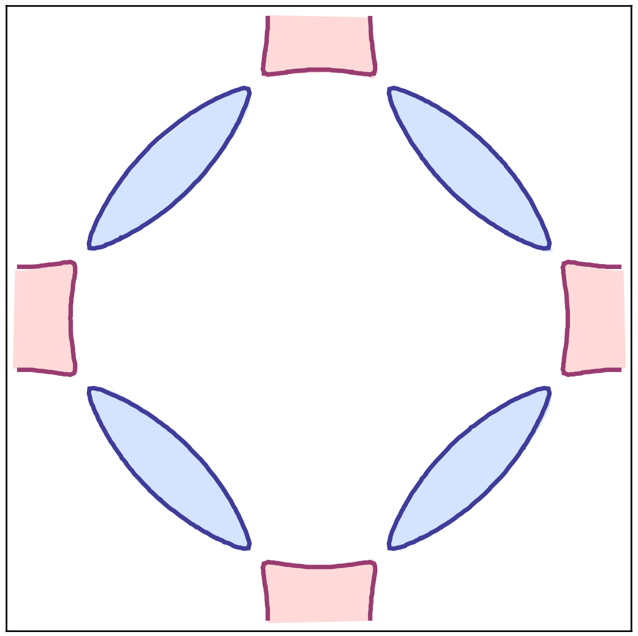
where

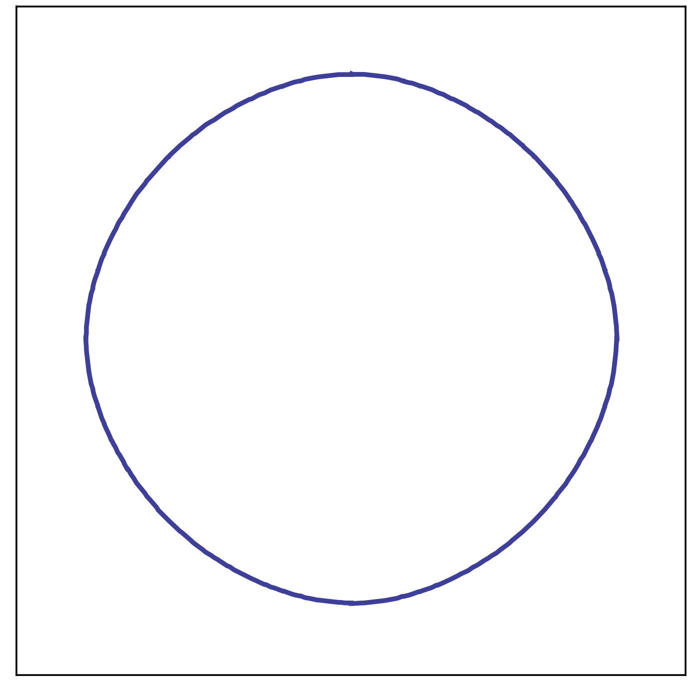
$$\eta_i \equiv (-1)^{x_i + y_i} \tag{1.5}$$

is the staggering factor representing the opposite spin orientations on the two sublattices. Note that the Yukawa coupling, and the remaining Hamiltonian, commute with the total spin along the z direction

$$S_z = \sum_i \left(N_i + \frac{1}{2} c_{i\uparrow}^\dagger c_{i\uparrow} - \frac{1}{2} c_{i\downarrow}^\dagger c_{i\downarrow} \right). \tag{1.6}$$

Quantum phase transition with Fermi surface reconstruction





$\langle \vec{\varphi} \rangle \neq 0$

$\langle \vec{\varphi} \rangle = 0$

Metal with electron and hole pockets

Metal with "large" Fermi surface

Z_2 gauge theory for fractionalized XY SDW order coupled to electrons

$$\mathcal{H}_1 = H_c + H_{\theta,\mathbb{Z}_2} + H_Y$$

$$H_c = -\sum_{i,j} (t_{ij} + \mu\delta_{ij})\, c_{i\alpha}^\dagger c_{j\alpha}$$

$$H_Y = -\lambda \sum_i \eta_i \left[e^{-i\theta_i} c_{i\uparrow}^\dagger c_{i\downarrow} + e^{i\theta_i} c_{i\downarrow}^\dagger c_{i\uparrow} \right]$$

$$H_{\theta,\mathbb{Z}_2} = -\sum_{i<j} J_{ij}\mu_{ij}^z \cos\left((\theta_i - \theta_j)/2\right) + 4\Delta \sum_i N_i^2 - g\sum_{\langle ij\rangle} \mu_{ij}^x - K \sum_{\square} \left[\prod_{\square} \mu_{ij}^z \right],$$

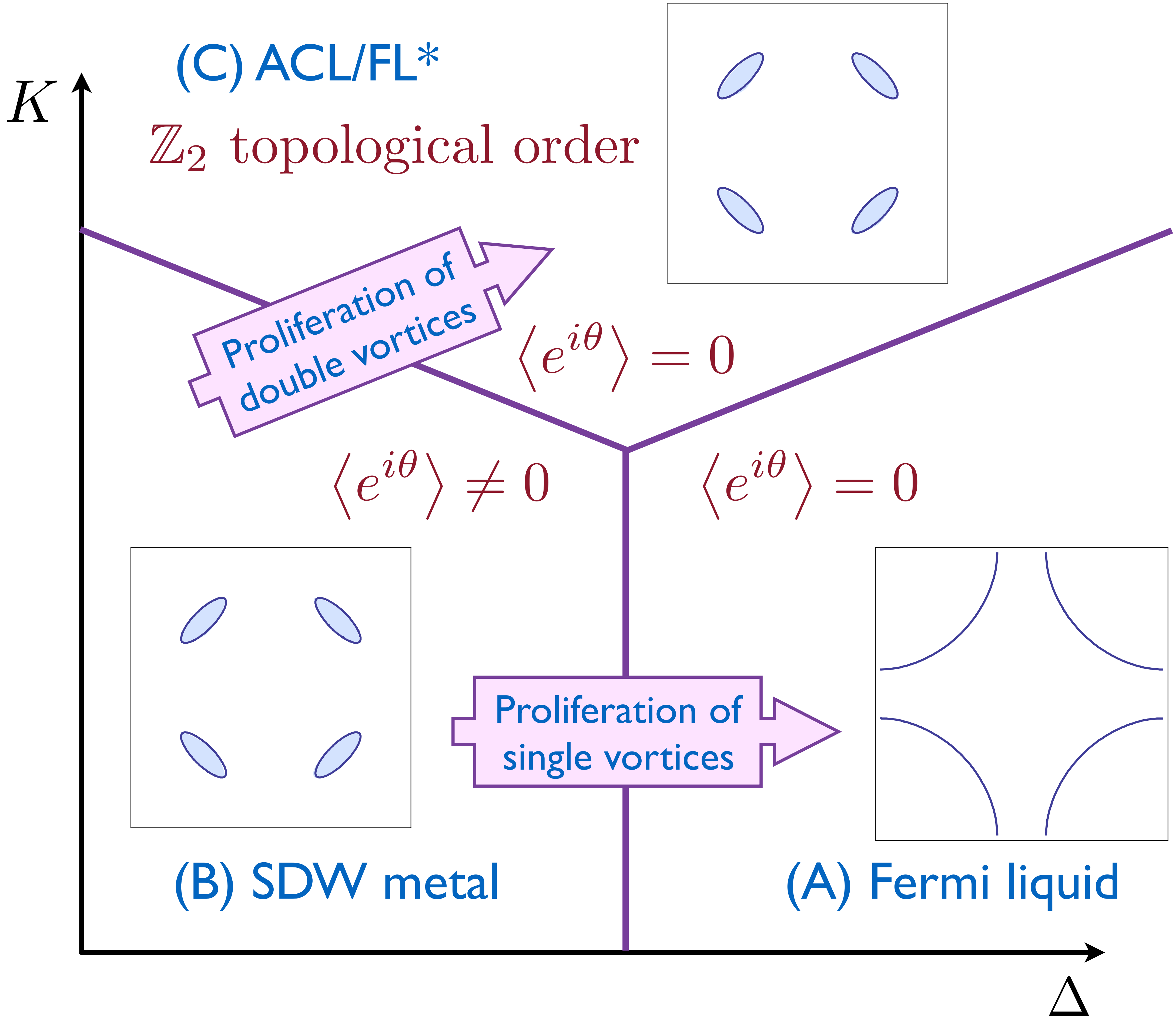
Consider the phase with $\mathbb{Z}_2$ topological order. In this state it is useful to perform a rotation about the z axis in spin space by introducing the fermion operators

$$\psi_+ = e^{i\theta/2} c_\uparrow \quad , \quad \psi_- = e^{-i\theta/2} c_\downarrow .$$

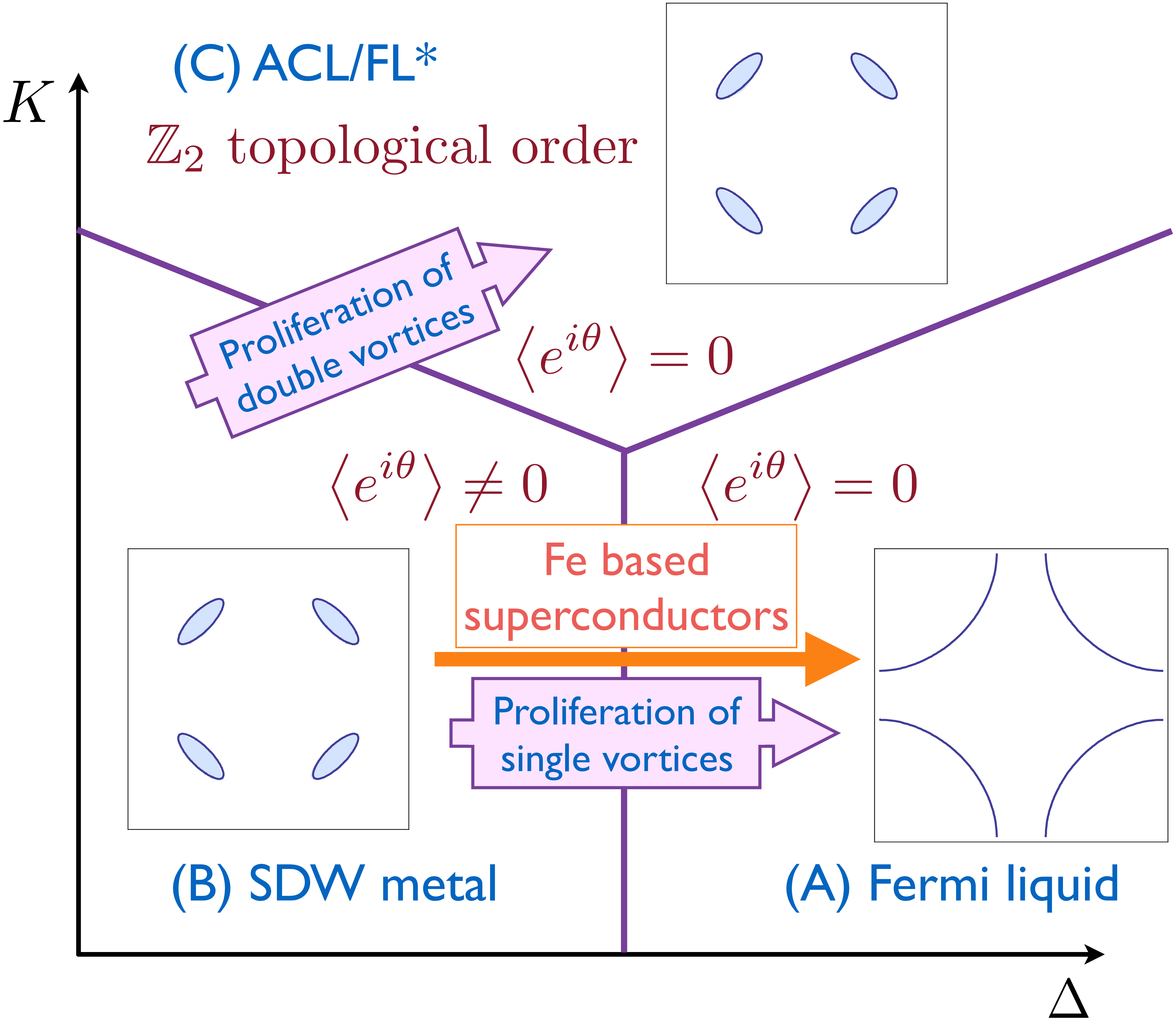
Then the Yukawa coupling, H_Y, takes a simple form independent of the orientation of the XY order:

$$H_Y = -\lambda \sum_i \eta_i \left[\psi_{i+}^\dagger \psi_{i-} + \psi_{i-}^\dagger \psi_{i+} \right] .$$

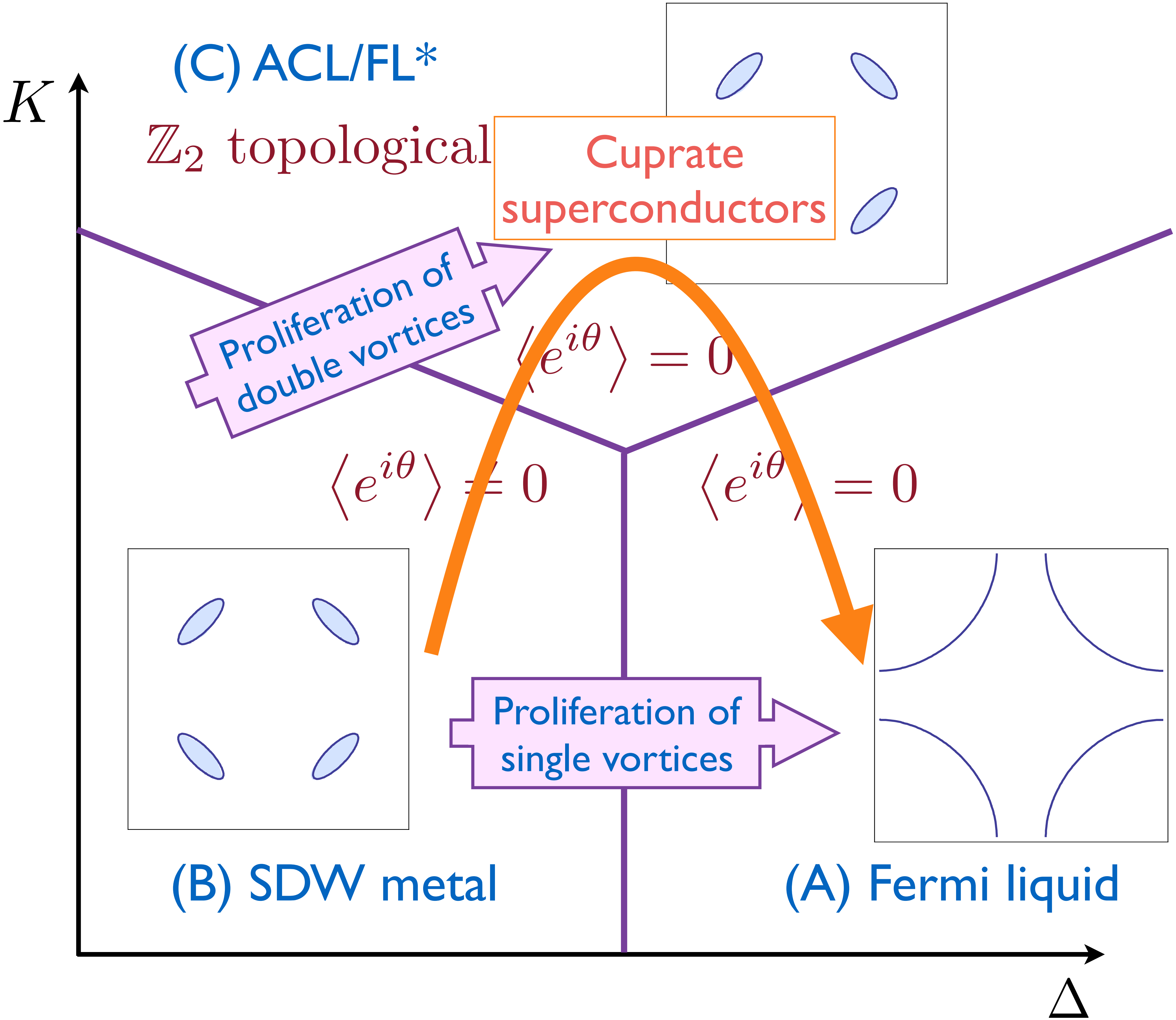
In other words, the $\psi_\pm$ fermions move in the presence of a spacetime-independent XY order, even though the actual orientation of the XY order rotates from point to point. Moreover, from the electron hopping term in H_c, we can obtain an effective hopping $Z_{ij} t_{ij} (\psi_{i+}^\dagger \psi_{j+} + \psi_{i-}^\dagger \psi_{j-})$ where $Z_{ij} = \langle e^{\pm i(\theta_i - \theta_j)/2} \rangle$ is a renormalization factor of order unity. So it appears we can realize a situation in which the $\psi_\pm$ fermions are approximately free, and their observation of constant XY order implies that they will form small pocket Fermi surfaces (or be fully gapped at $p = 0$).



(C) ACL/FL*
$\mathbb{Z}_2$ topological order
K
Proliferation of double vortices
$\langle e^{i\theta} \rangle = 0$
$\langle e^{i\theta} \rangle \neq 0$
$\langle e^{i\theta} \rangle = 0$
Proliferation of single vortices
(B) SDW metal
(A) Fermi liquid
Δ

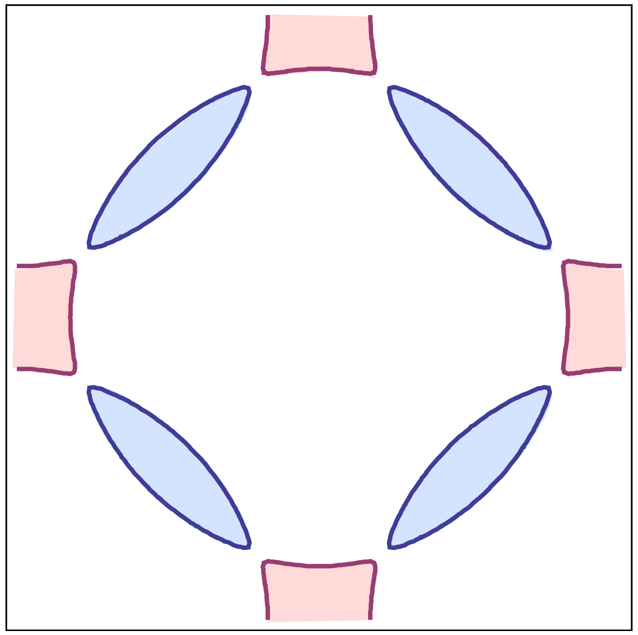


(C) ACL/FL*
$\mathbb{Z}_2$ topological order
K
Proliferation of double vortices
$\langle e^{i\theta} \rangle = 0$
$\langle e^{i\theta} \rangle \neq 0$
$\langle e^{i\theta} \rangle = 0$
Fe based superconductors
Proliferation of single vortices
(B) SDW metal
(A) Fermi liquid
Δ



(C) ACL/FL*
$\mathbb{Z}_2$ topological
Cuprate superconductors
Proliferation of double vortices
$\langle e^{i\theta} \rangle = 0$
$\langle e^{i\theta} \rangle \neq 0$
$\langle e^{i\theta} \rangle = 0$
Proliferation of single vortices
(B) SDW metal
(A) Fermi liquid
K
Δ

Separating onset of SDW order and Fermi surface reconstruction



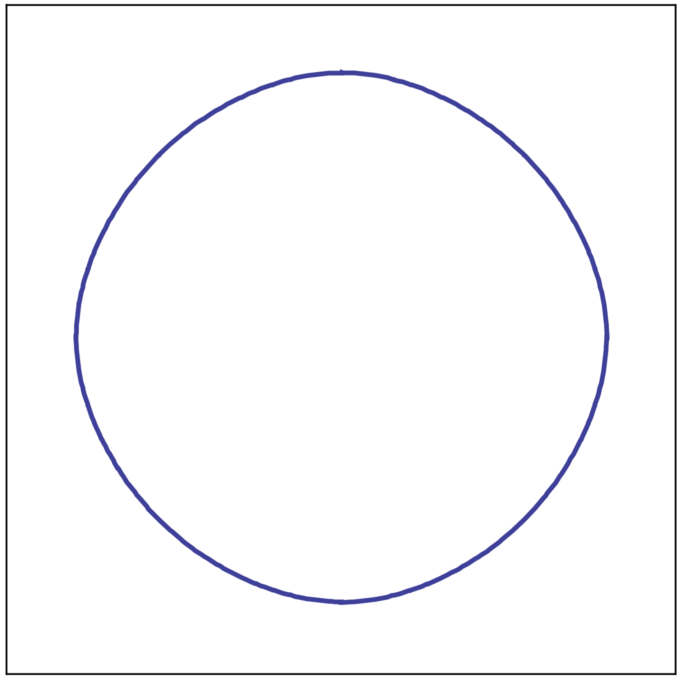
$\langle \vec{\varphi} \rangle \neq 0$

Metal with electron and hole pockets

Electron and/or hole Fermi pockets form in "local" SDW order, but quantum fluctuations destroy long-range SDW order

$\langle \vec{\varphi} \rangle = 0$

Algebraic Charge liquid (ACL) or Fractionalized Fermi liquid (FL*) phase with no symmetry breaking and pocket Fermi surfaces



$\langle \vec{\varphi} \rangle = 0$

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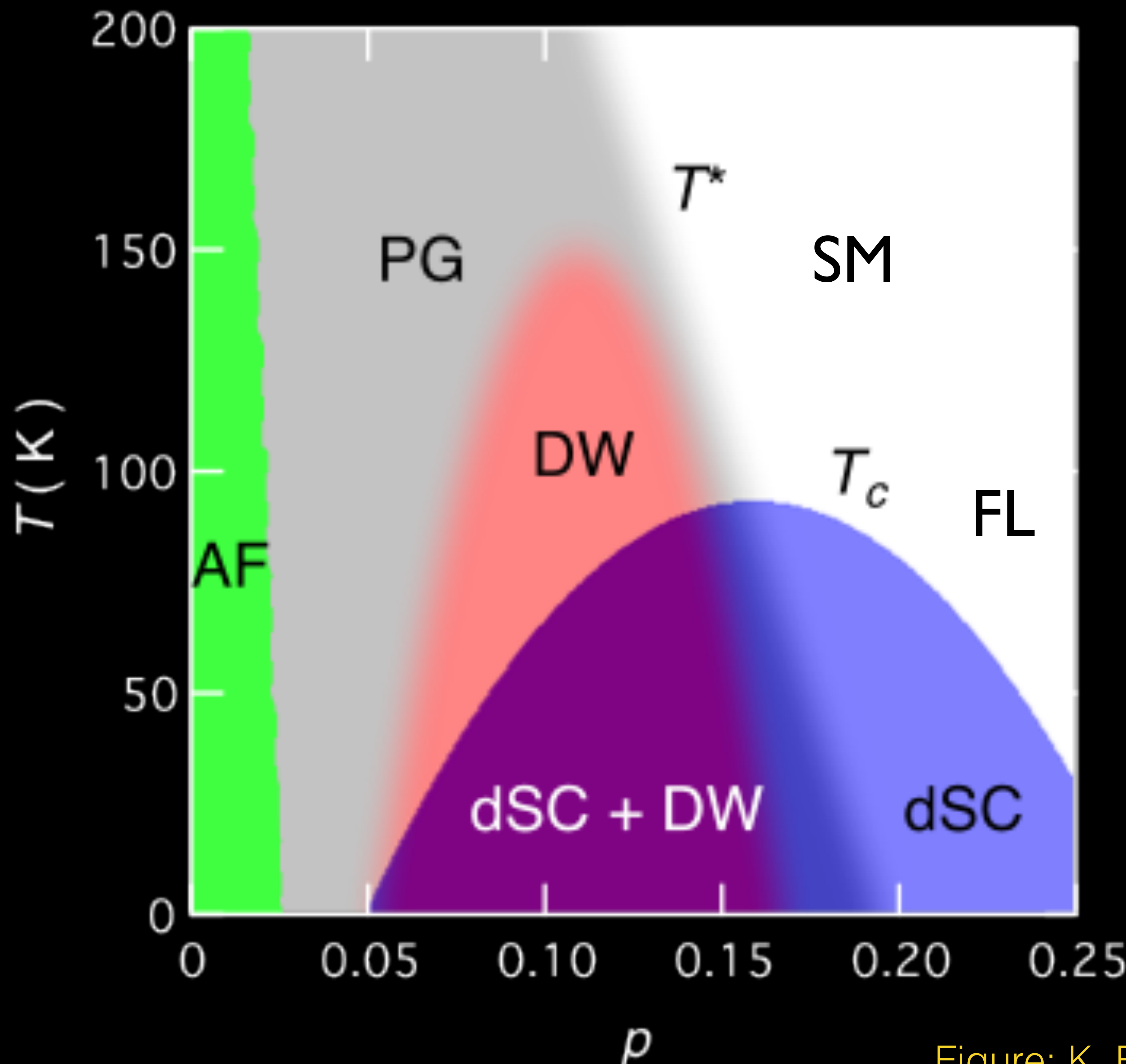
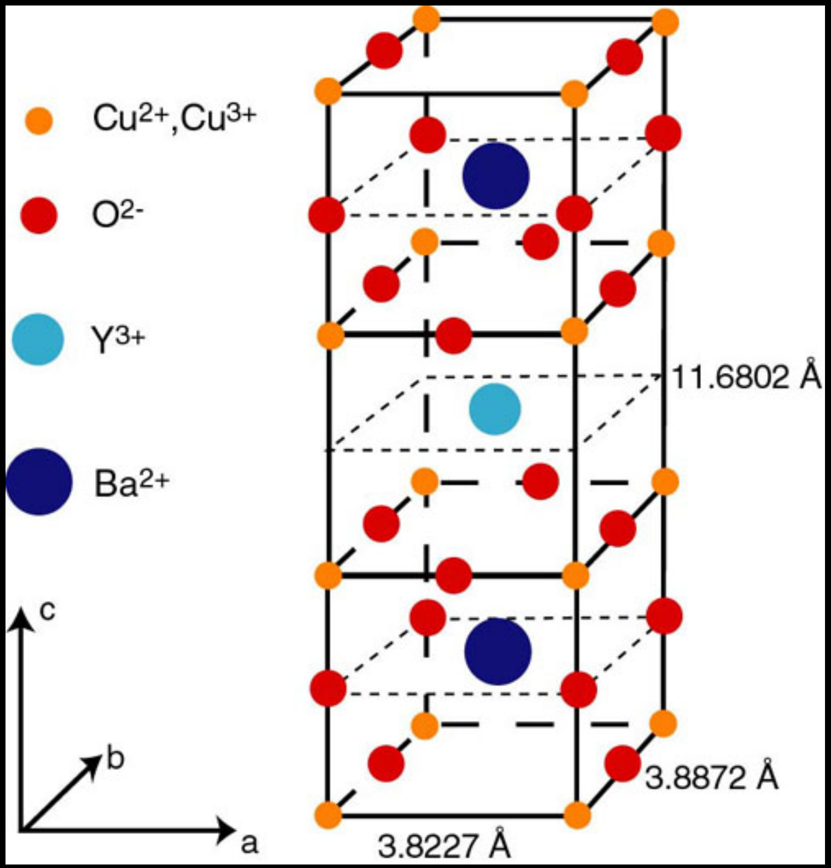
T. Senthil, S. Sachdev, and M. Vojta, *Phys. Rev. Lett.* **90**, 216403 (2003)





$YBa_2Cu_3O_{6+x}$

Figure: K. Fujita and J. C. Seamus Davis

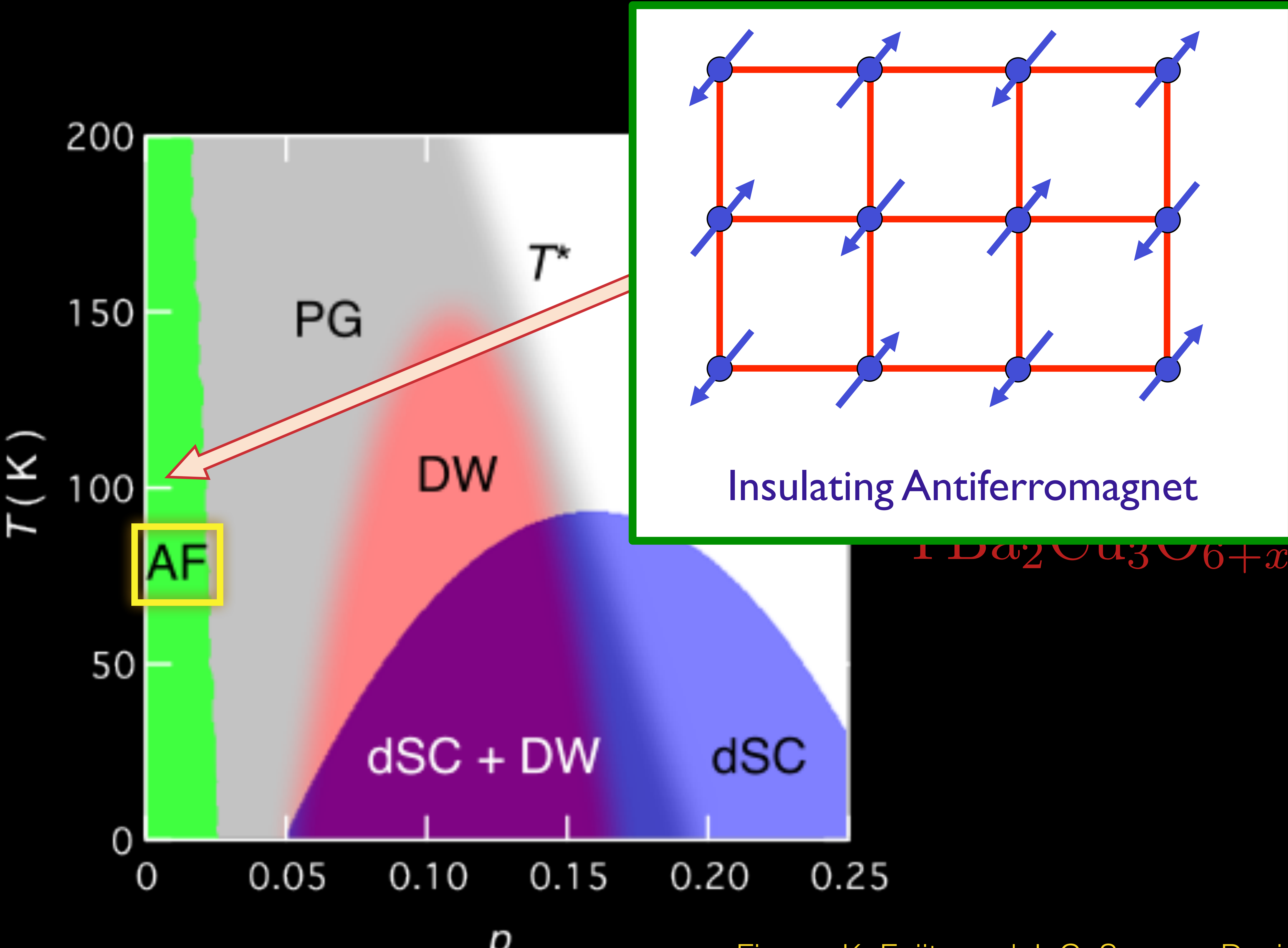
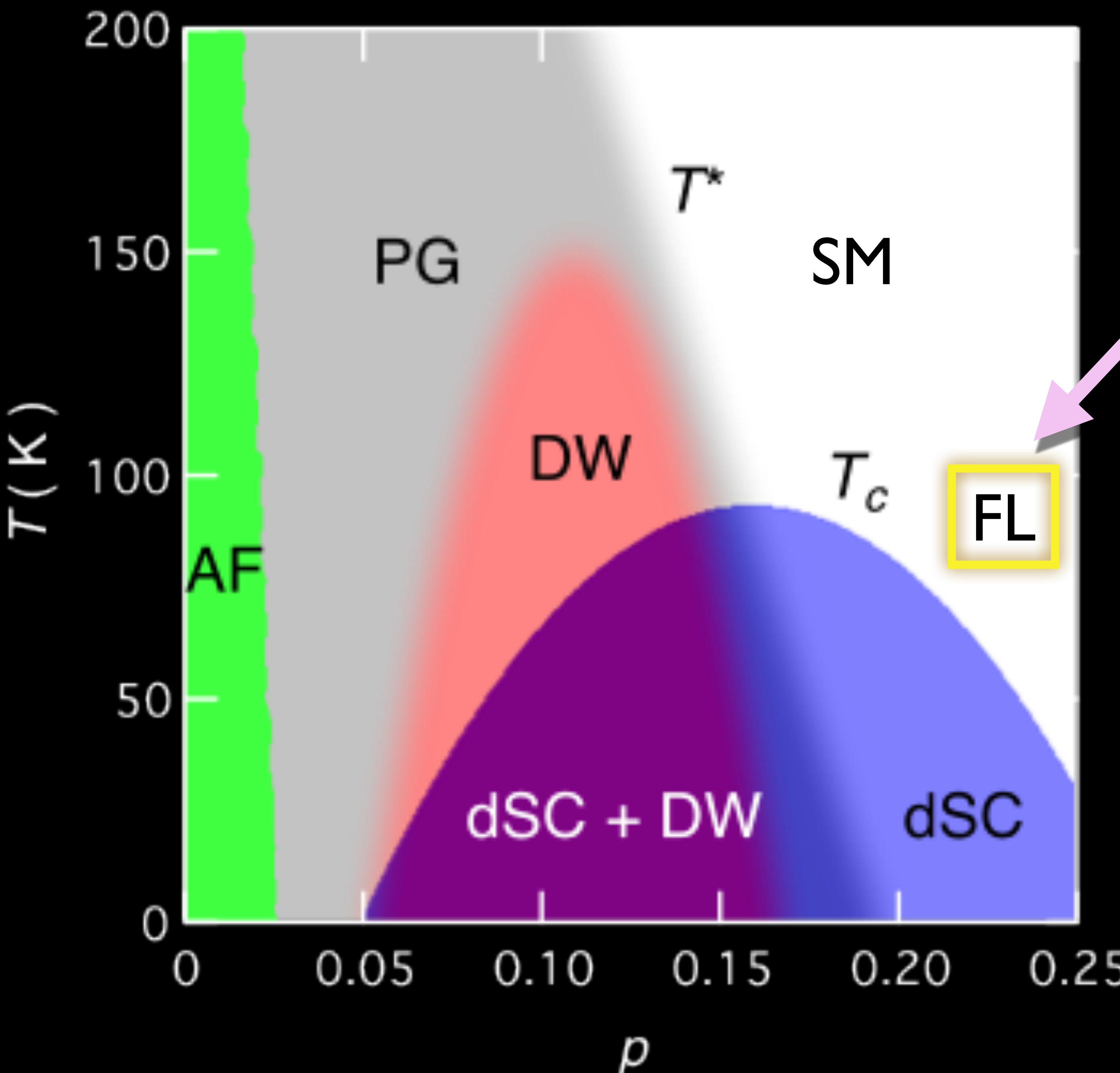
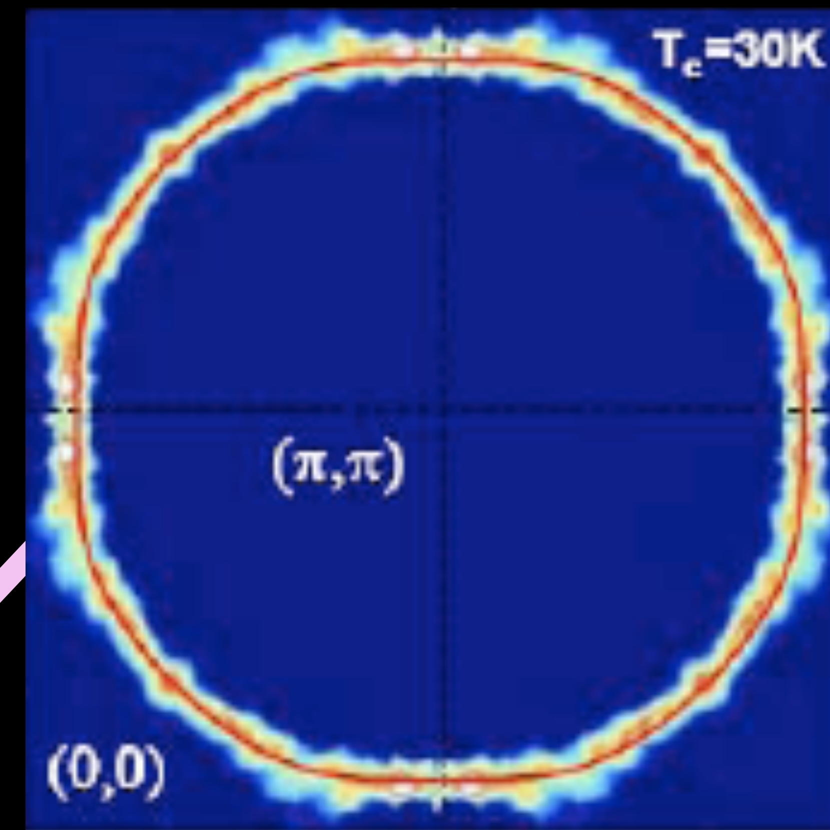


Figure: K. Fujita and J. C. Seamus Davis

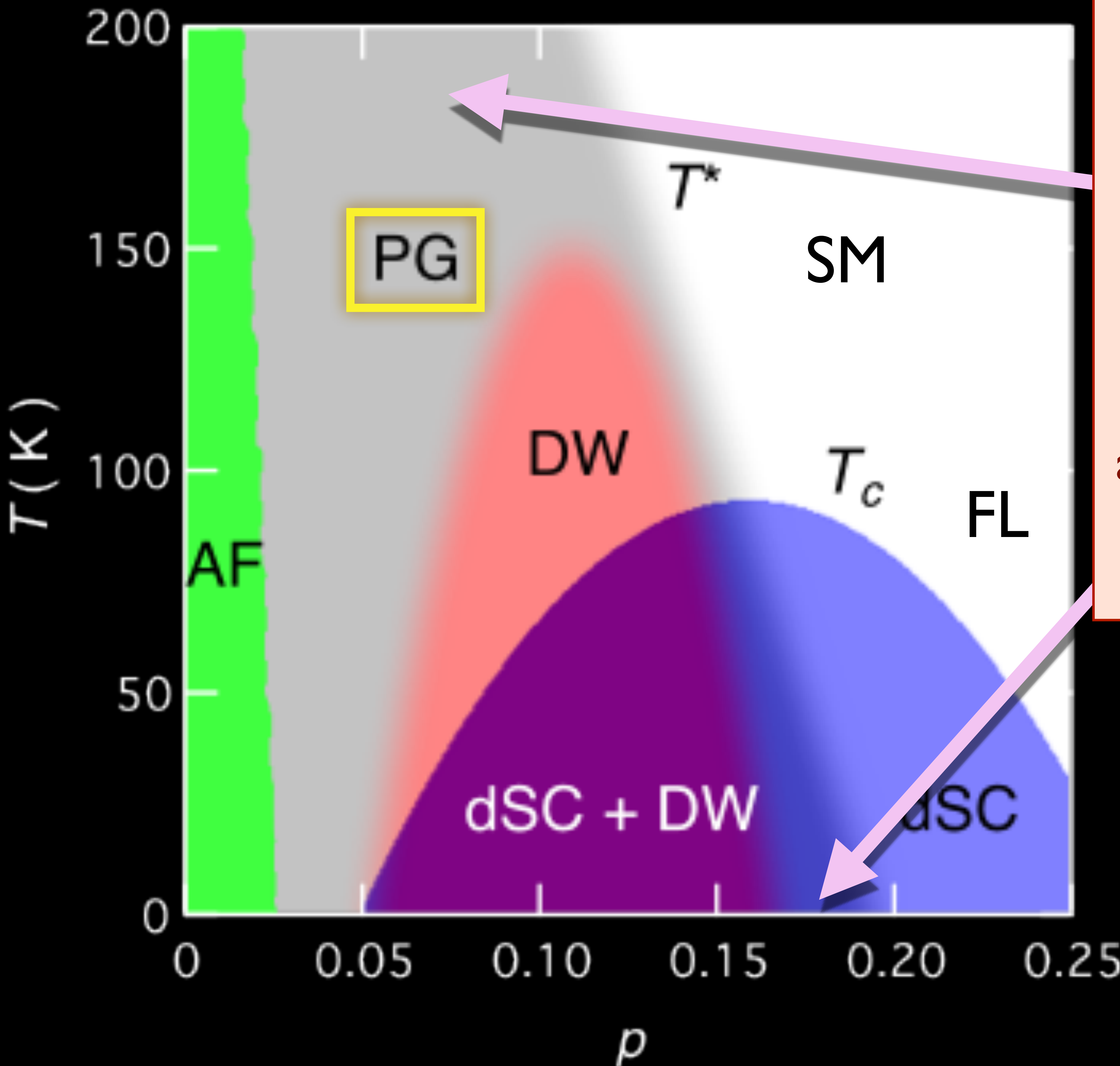
M. Platé, J. D. F. Mottershead, I. S. Elfimov, D. C. Peets, Ruixing Liang, D. A. Bonn, W. N. Hardy, S. Chiuzbaian, M. Falub, M. Shi, L. Patthey, and A. Damascelli, Phys. Rev. Lett. **95**, 077001 (2005)





A conventional metal:
the Fermi liquid with Fermi surface of size $1+p$

S. Badoux, W. Tabis, F. Laliberté, G. Grissonnanche, B. Vignolle, D. Vignolles, J. Béard, D.A. Bonn, W.N. Hardy, R. Liang, N. Doiron-Leyraud, L. Taillefer, and C. Proust, Nature **531**, 210 (2016).

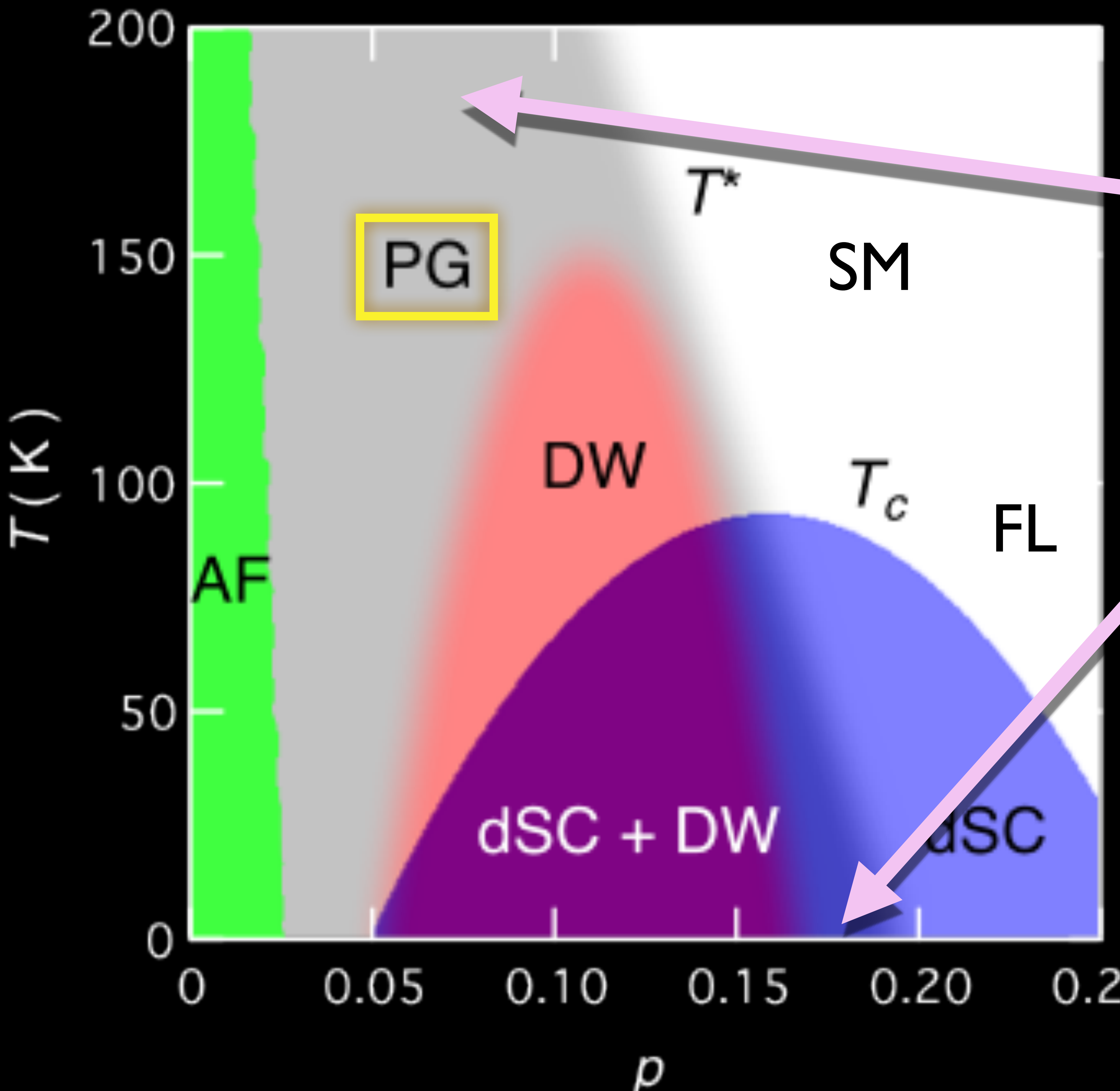


<u>Pseudogap metal</u>

at low p

Many indications that this metal behaves like a Fermi liquid, but with Fermi surface size p and *not* $1+p$.

T. Senthil, M. Vojta and S. Sachdev, PRB **69**, 035111 (2004)



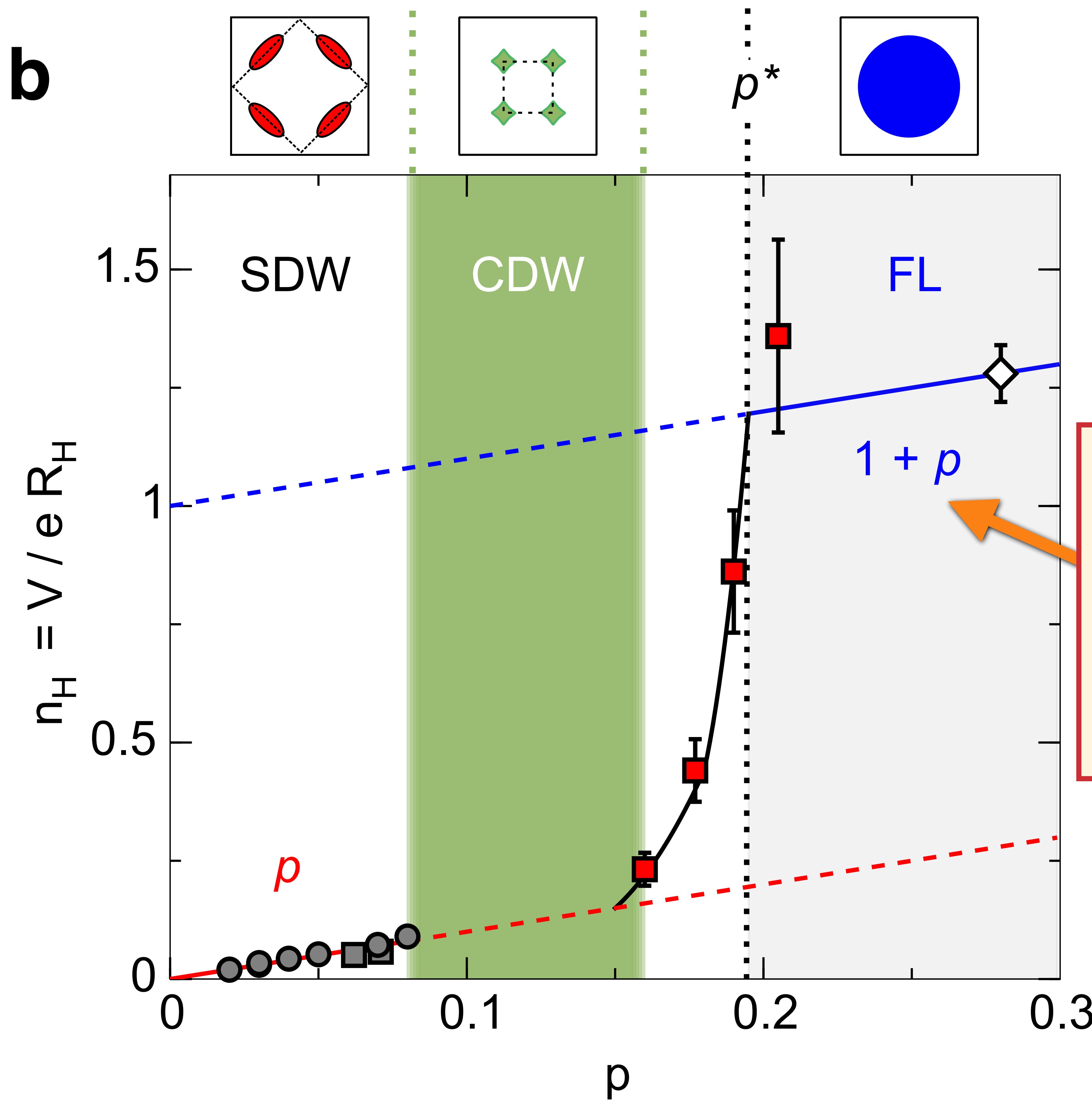
Pseudogap metal

at low p

Many indications that this metal behaves like a Fermi liquid, but with Fermi surface size p and *not* $1+p$.

If present at $T=0$, a metal with a size p Fermi surface (and translational symmetry preserved) *must* have *topological order*

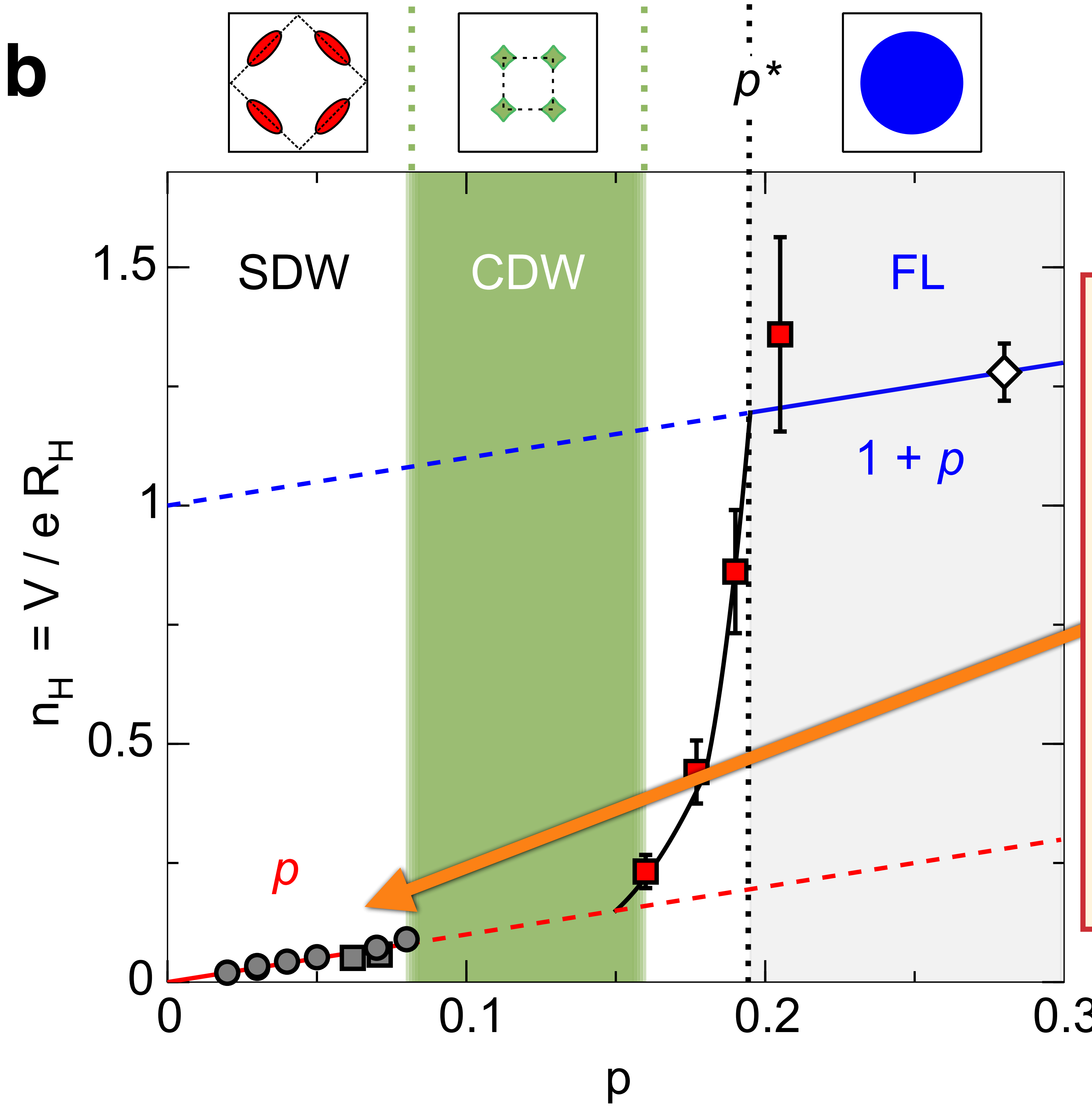
Hall effect measurements in YBCO



Fermi liquid (FL) with carrier density $1+p$

Badoux, Proust, Taillefer et al., Nature **531**, 210 (2016)

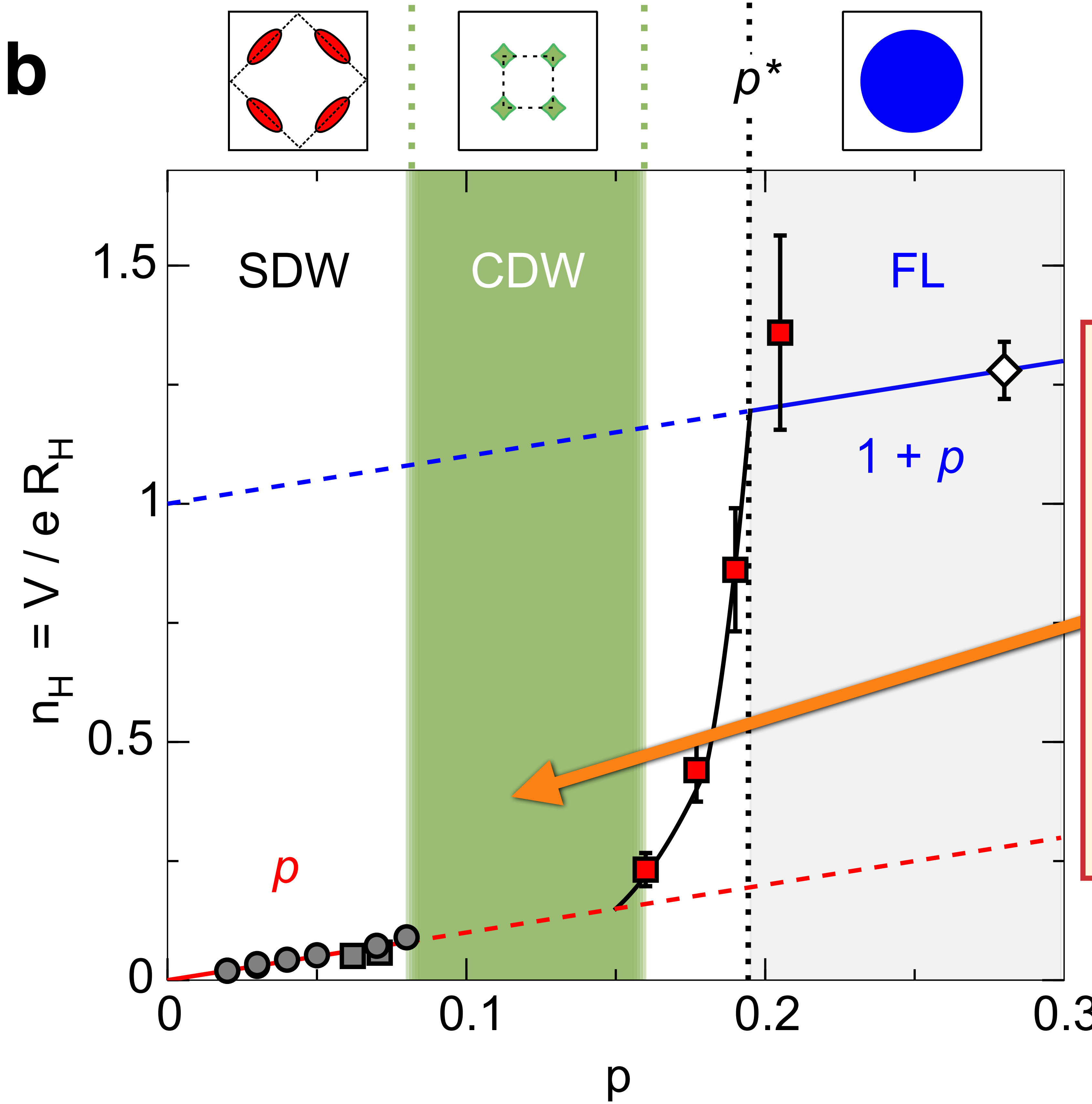
Hall effect measurements in YBCO



Spin density wave (SDW) breaks translational invariance, and the Fermi liquid then has carrier density p

Badoux, Proust, Taillefer et al., Nature **531**, 210 (2016)

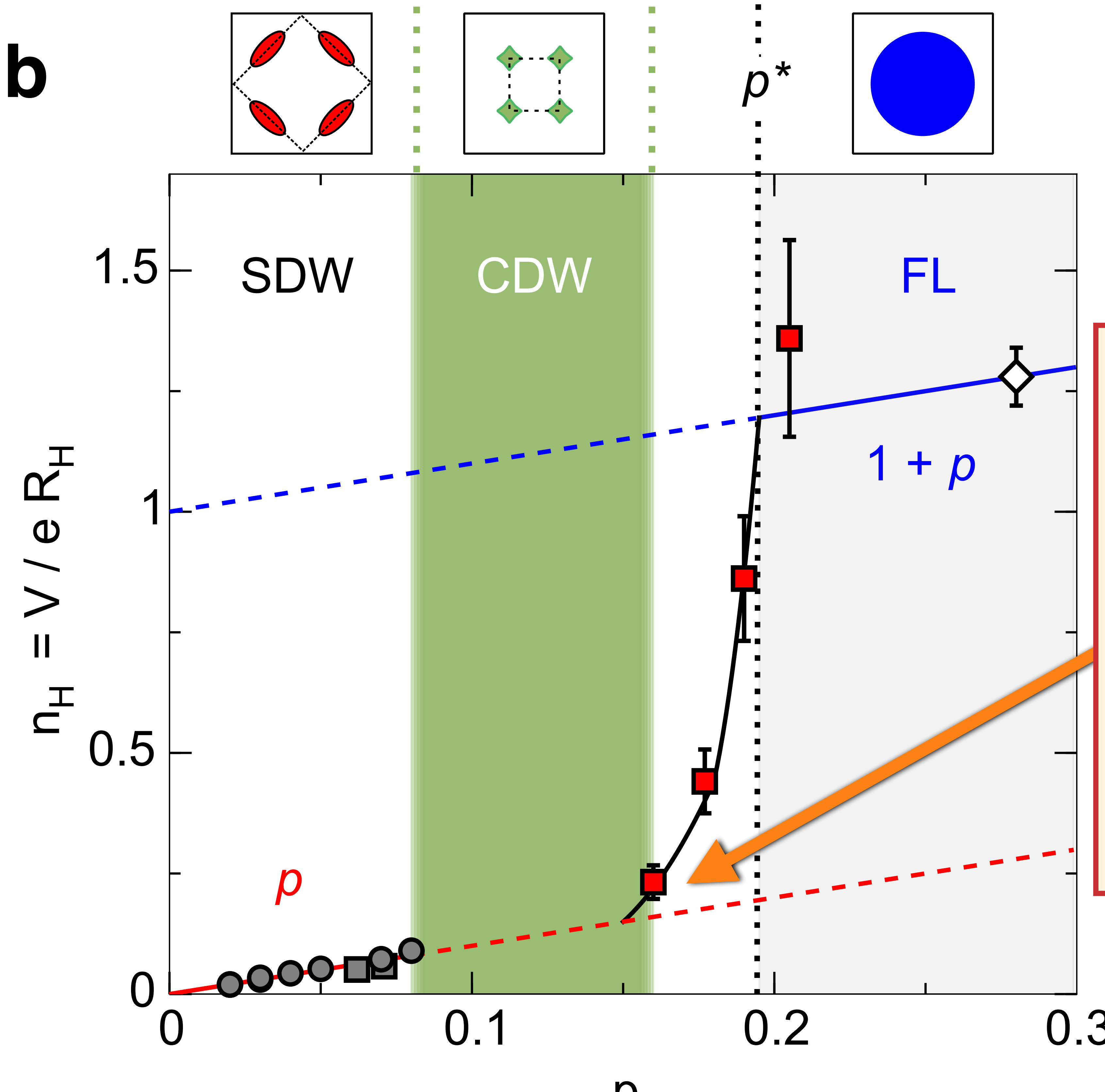
Hall effect measurements in YBCO



Charge density wave (CDW) leads to complex Fermi surface reconstruction and negative Hall resistance

Badoux, Proust, Taillefer et al., Nature **531**, 210 (2016)

Hall effect measurements in YBCO



Evidence for a metal with topological order: Fermi surface of size p !

Badoux, Proust, Taillefer et al., Nature **531**, 210 (2016)