The SYK model: a window into black holes and non-Fermi liquids

The Dual Mysteries of Gauge Theories and Gravity
IIT Madras
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Talk online: sachdev.physics.harvard.edu

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HARVARD

1. SYK models

2. Time reparameterization soft mode

3. Charged black holes

4. Critical Fermi surfaces: large *N* theory

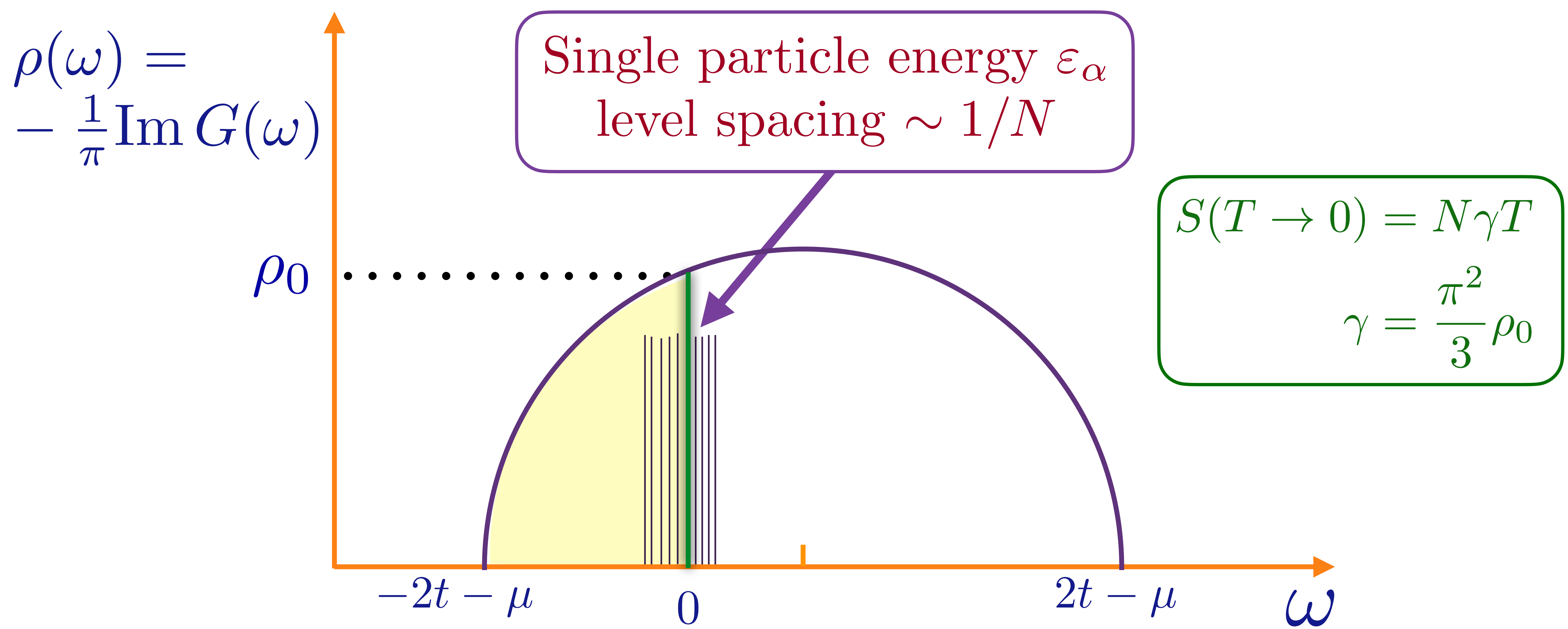
A simple model of a metal with quasiparticles

$$H = \frac{1}{(N)^{1/2}} \sum_{i,j=1}^{N} t_{ij} c_i^\dagger c_j - \mu \sum_i c_i^\dagger c_i$$

$$c_i c_j + c_j c_i = 0 \quad , \quad c_i c_j^\dagger + c_j^\dagger c_i = \delta_{ij}$$

$$\frac{1}{N} \sum_i c_i^\dagger c_i = \mathcal{Q}$$

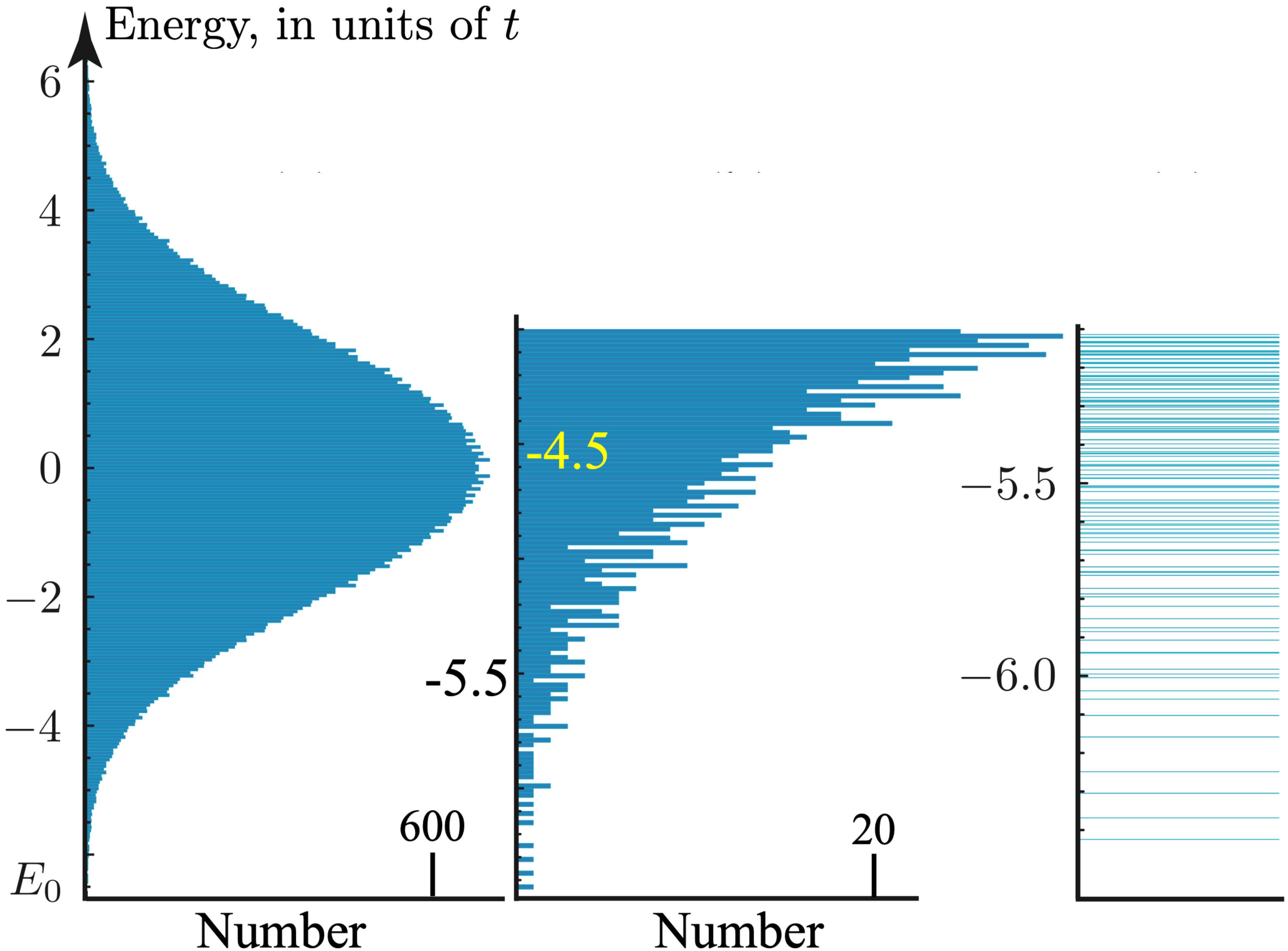
t_{ij} are independent random variables with $\overline{t_{ij}} = 0$ and $\overline{|t_{ij}|^2} = t^2$



Many-body density of states

$$D(E) = \sum_i \delta(E - E_i); \quad E_0 + E_i \Rightarrow \text{Many body eigenvalue}$$

For random matrix model: $E_0 + E_i = \sum_\alpha n_\alpha \varepsilon_\alpha$ $n_\alpha = 0, 1,$ occupation number



Random matrix model

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$$D(E) \sim e^{S(E)}$$
$$= e^{\sqrt{2N\gamma E}}$$
$$S(T \to 0) = N\gamma T$$

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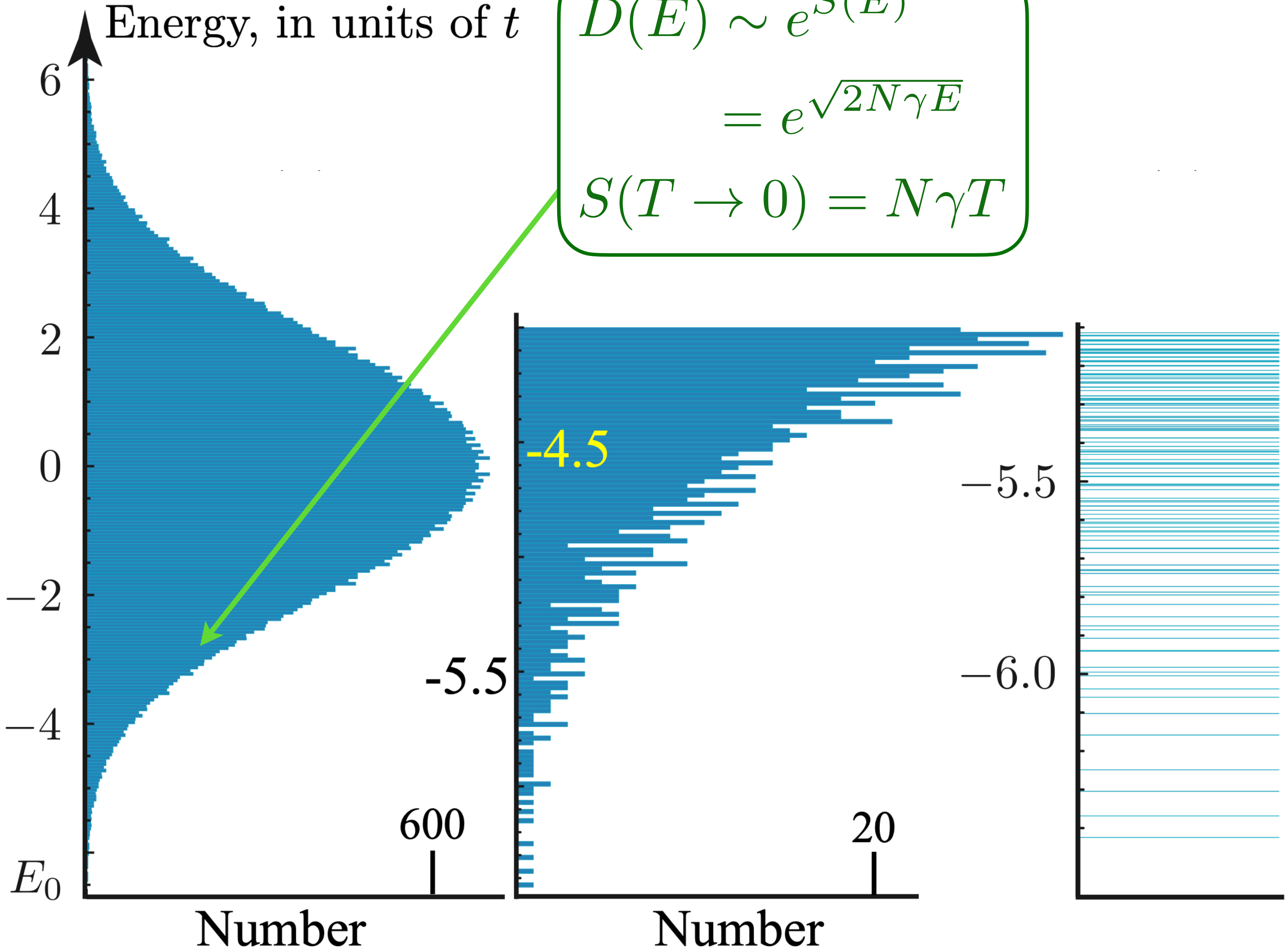

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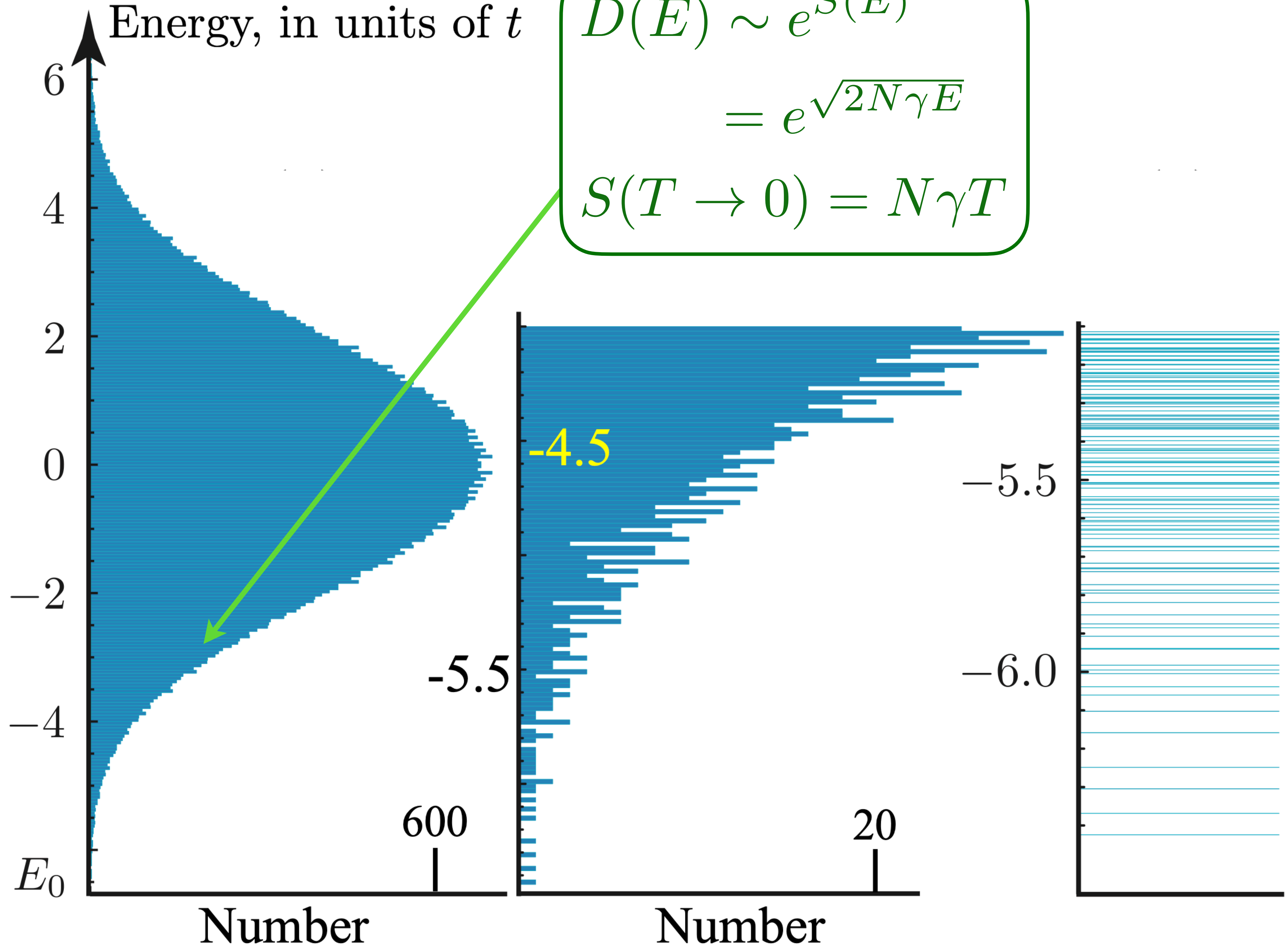


$$D(E) \sim N$$

Random matrix model

The Sachdev-Ye-Kitaev (SYK) model

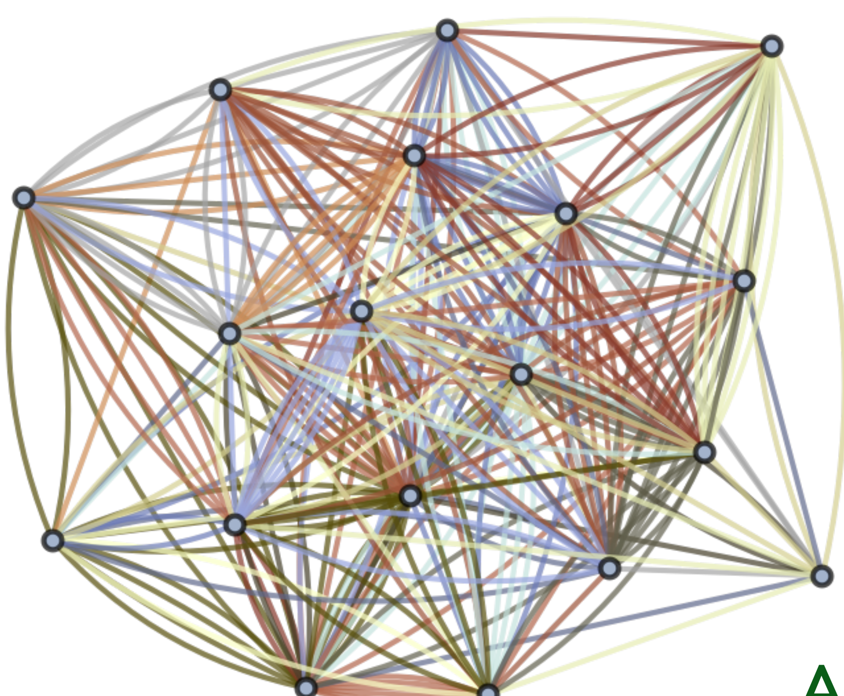
(See also: the "2-Body Random Ensemble" in nuclear physics; did not obtain the large *N* limit;
T.A. Brody, J. Flores, J.B. French, P.A. Mello, A. Pandey, and S.S.M. Wong, Rev. Mod. Phys. **53**, 385 (1981))

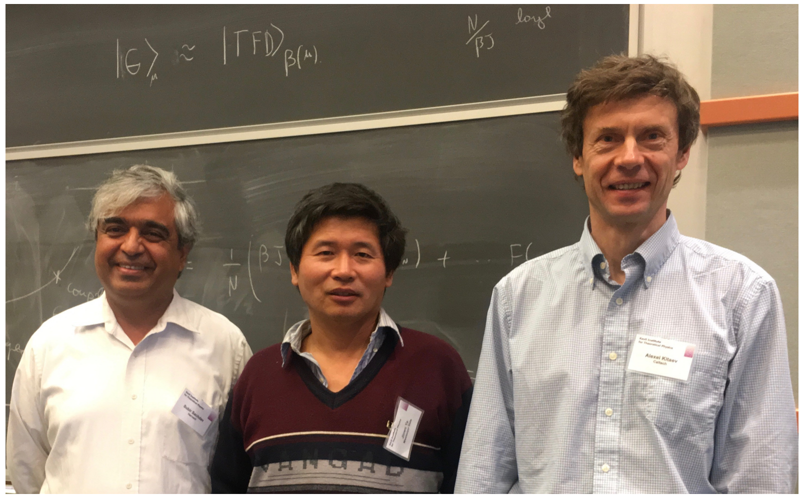
$$H = \frac{1}{(2N)^{3/2}} \sum_{\alpha,\beta,\gamma,\delta=1}^{N} U_{\alpha\beta;\gamma\delta}\, c^\dagger_\alpha c^\dagger_\beta c_\gamma c_\delta - \mu \sum_\alpha c^\dagger_\alpha c_\alpha$$

$$c_\alpha c_\beta + c_\beta c_\alpha = 0 \quad , \quad c_\alpha c^\dagger_\beta + c^\dagger_\beta c_\alpha = \delta_{\alpha\beta}$$

$$\mathcal{Q} = \frac{1}{N} \sum_\alpha c^\dagger_\alpha c_\alpha$$

$U_{\alpha\beta;\gamma\delta}$ are independent random variables with $\overline{U_{\alpha\beta;\gamma\delta}} = 0$ and $\overline{|U_{\alpha\beta;\gamma\delta}|^2} = U^2$
$N \to \infty$ yields critical strange metal.





S. Sachdev and J. Ye, PRL **70**, 3339 (1993)

A. Kitaev, unpublished; S. Sachdev, PRX **5**, 041025 (2015)

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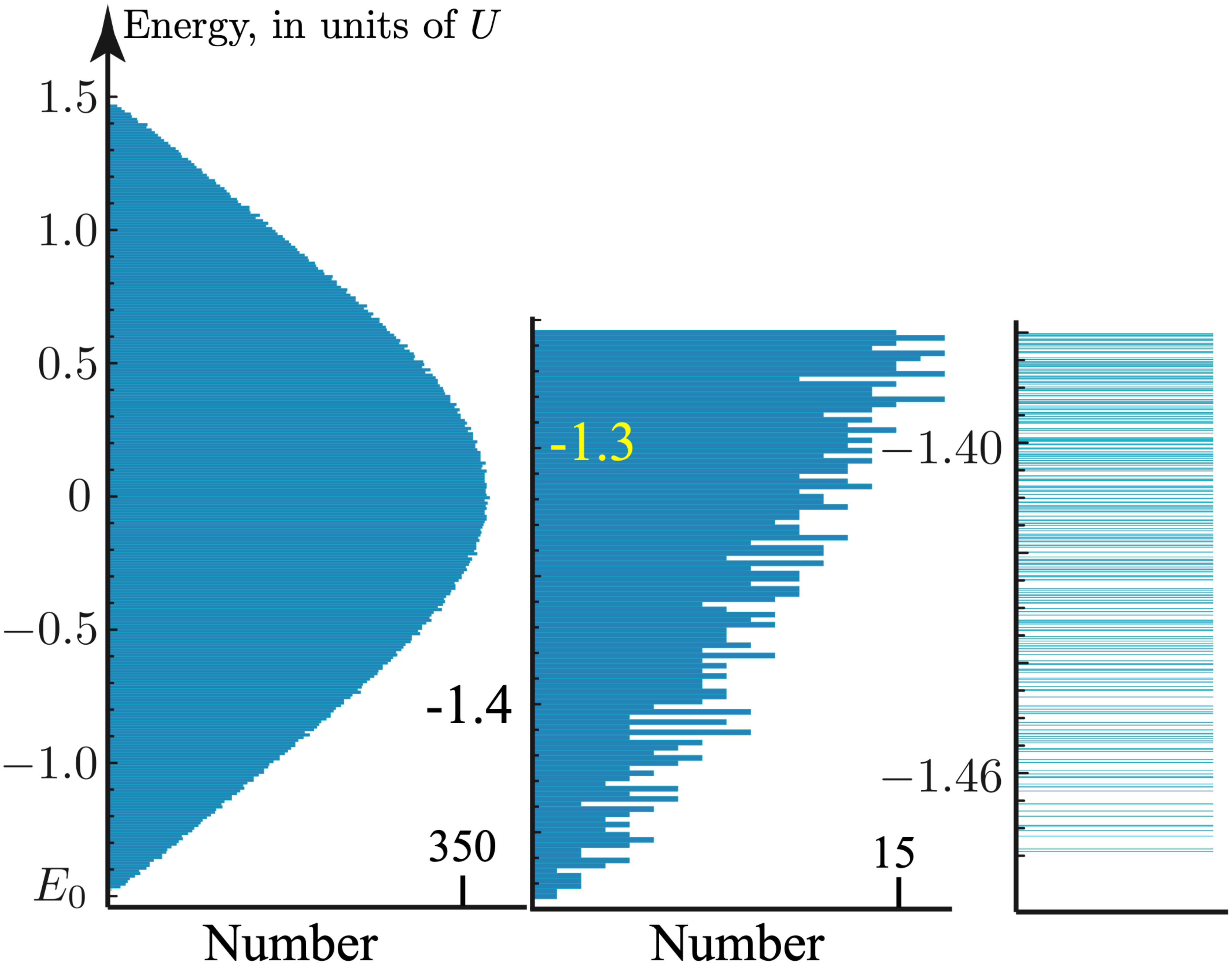


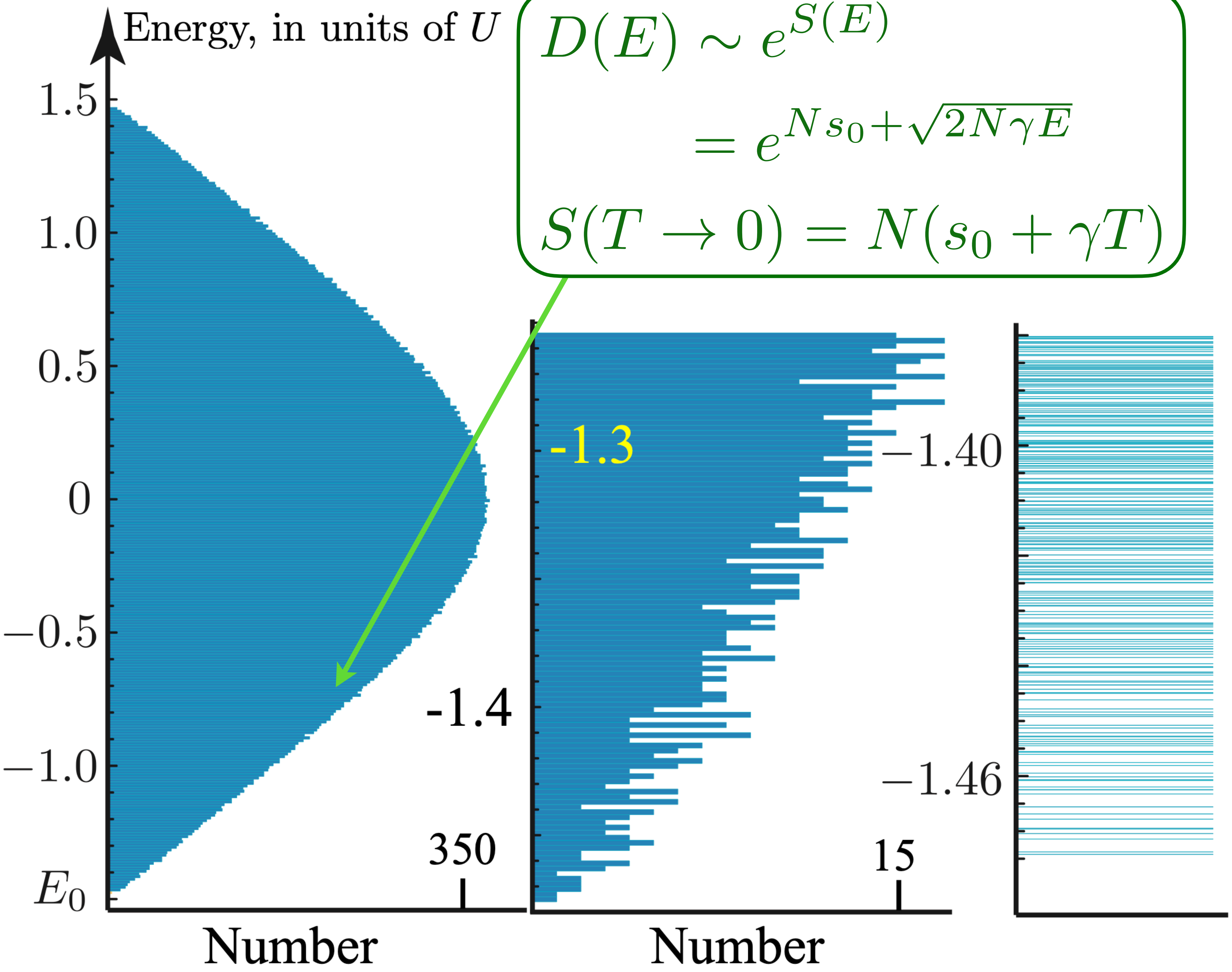
Complex SYK model

Many-body density of states

$$D(E) = \sum_i \delta(E - E_i); \quad E_0 + E_i \Rightarrow \text{Many body eigenvalue}$$

$$D(E) \sim e^{S(E)} = e^{N s_0 + \sqrt{2N\gamma E}}$$

$$S(T \to 0) = N(s_0 + \gamma T)$$



$$s_0 = 0.464848\ldots$$

A. Georges, O. Parcollet, and S. Sachdev, PRB **63**, 134406 (2001)

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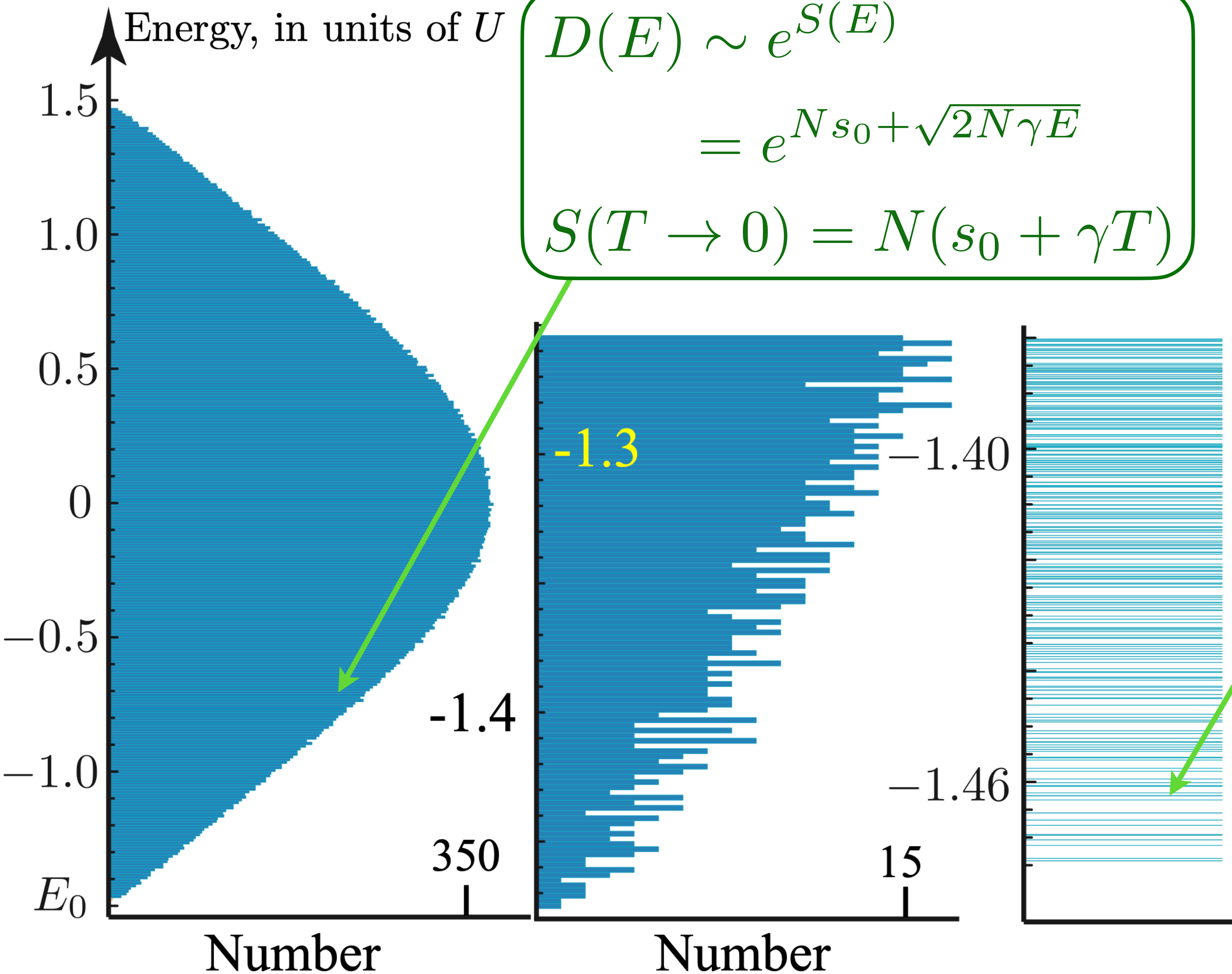

$$D(E) \sim e^{S(E)} = e^{N s_0 + \sqrt{2N\gamma E}}$$

$$S(T \to 0) = N(s_0 + \gamma T)$$

$$D(E) \sim 2\, e^{N s_0} \sqrt{2N\gamma E}$$

No quasiparticle decomposition of many-body states

$$s_0 = 0.464848\ldots$$

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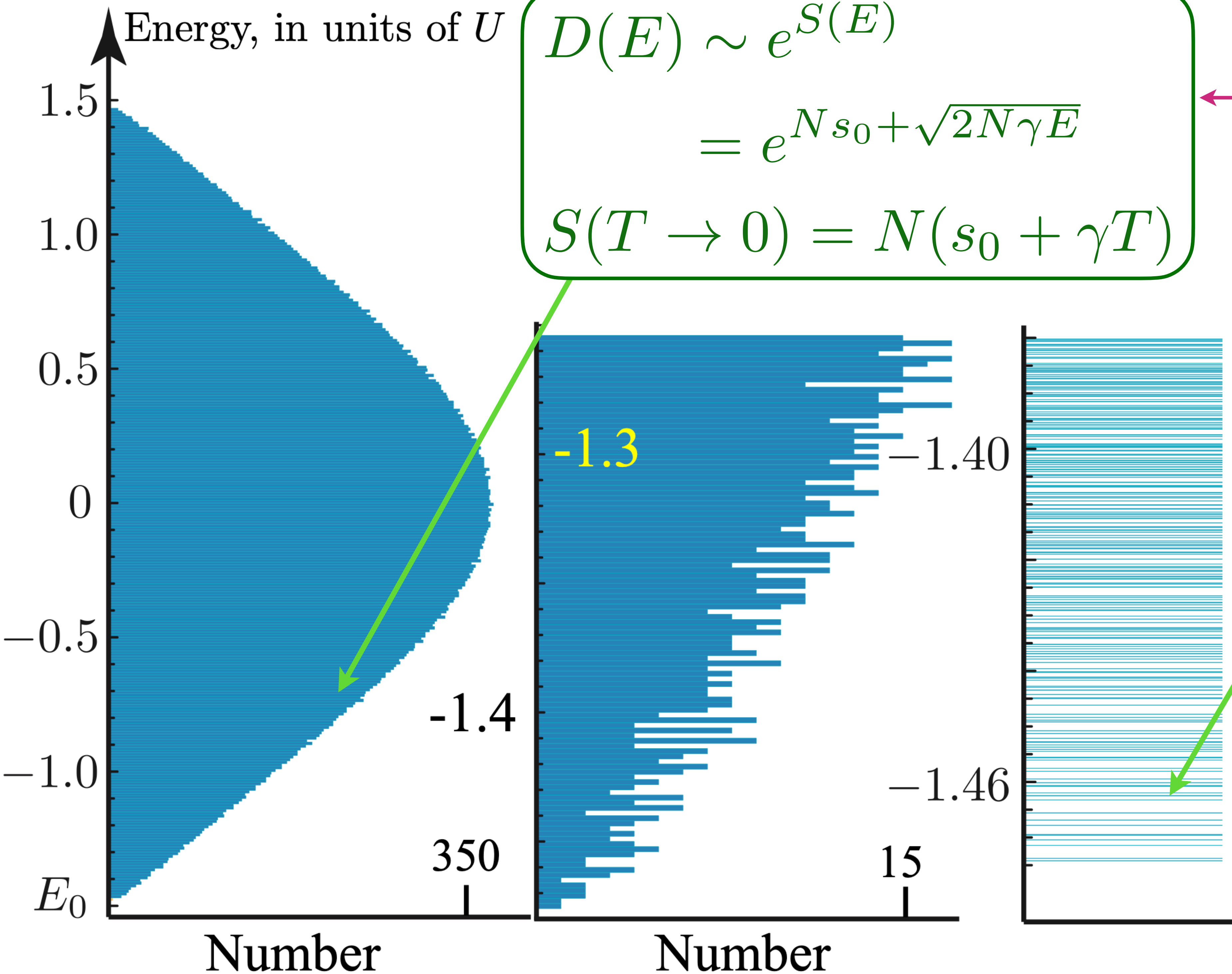

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$$D(E) \sim 2\, e^{N s_0} \sinh(\sqrt{2N\gamma E})$$
$$S(T) = N(s_0 + \gamma T) - \frac{3}{2} \ln\left(\frac{U}{T}\right)$$

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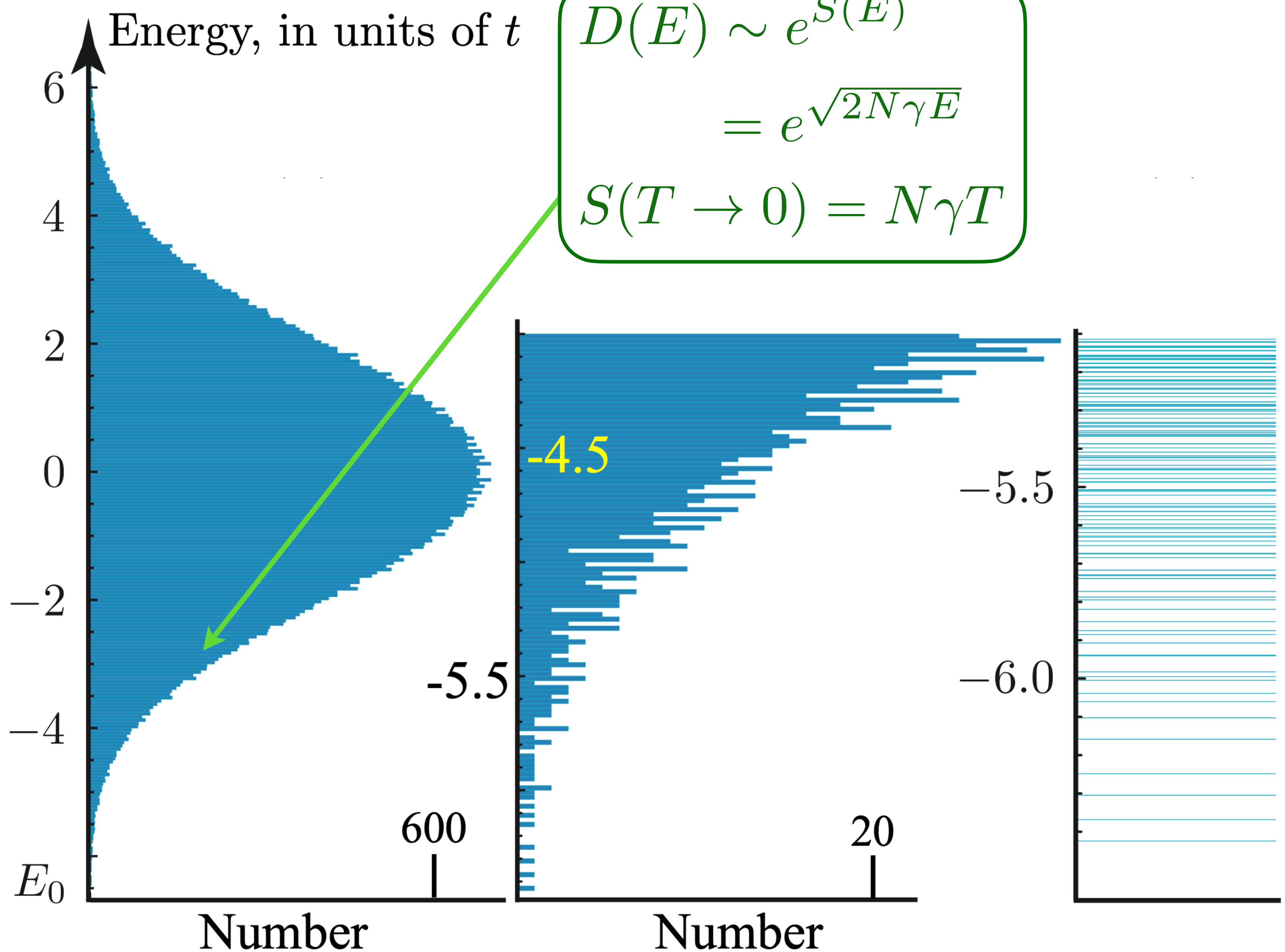

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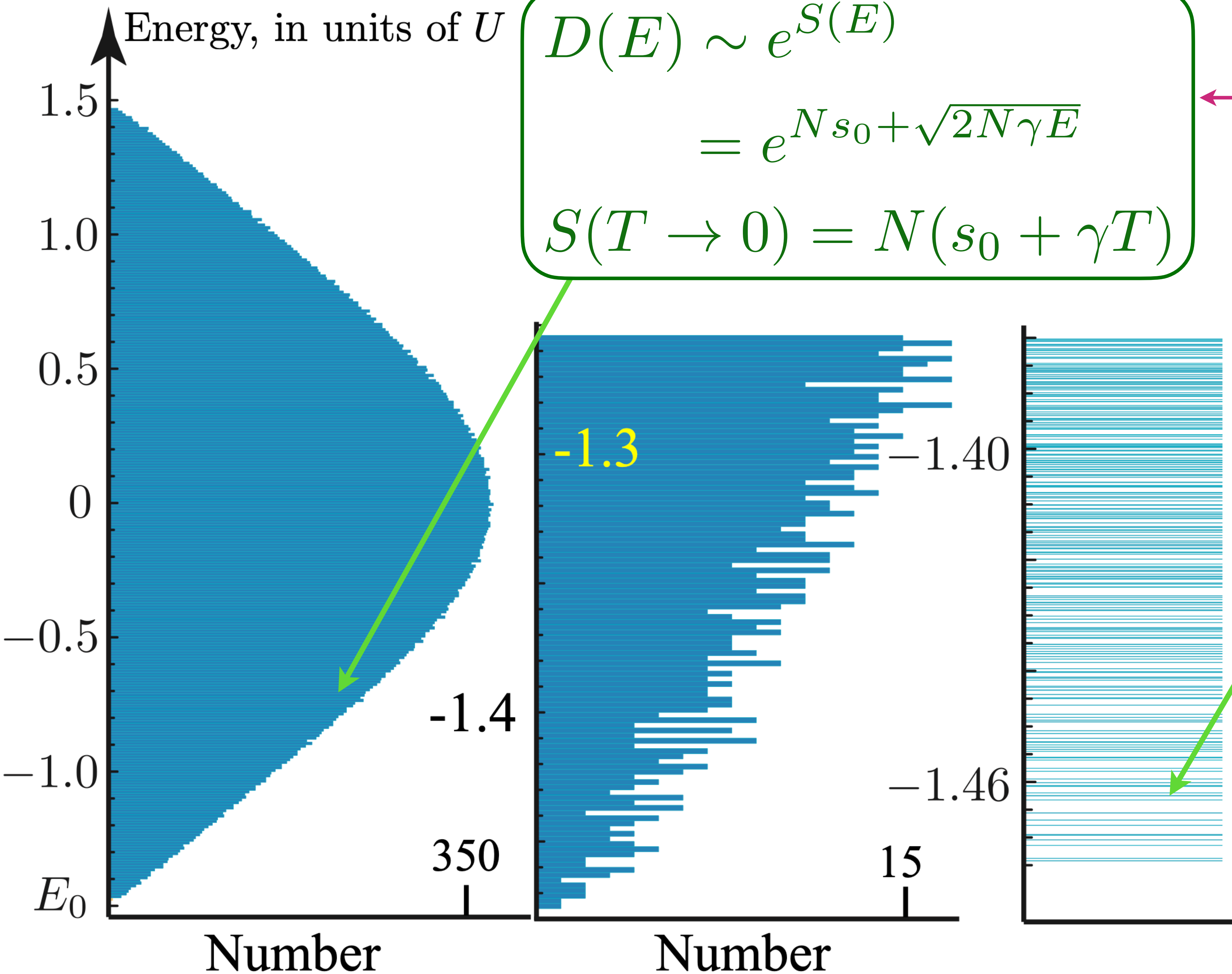

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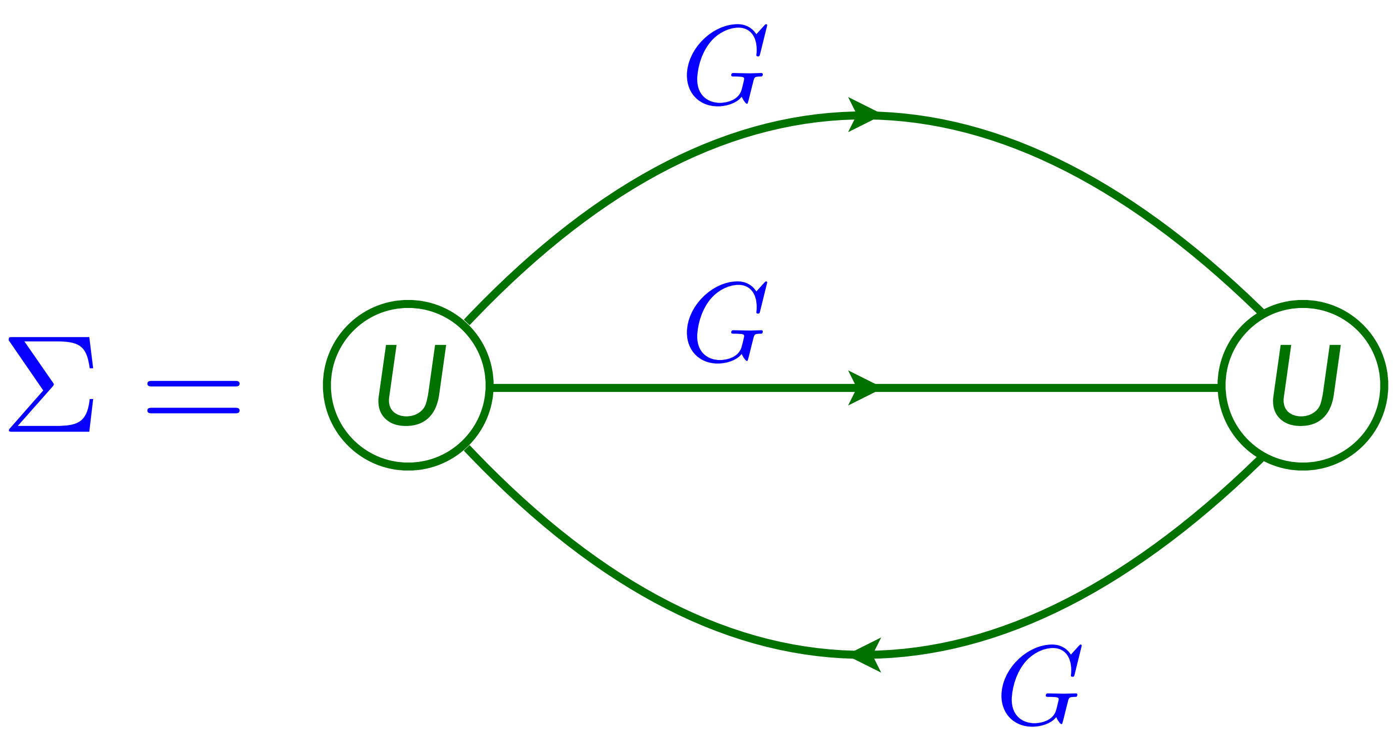
Complex SYK model

The complex SYK model

Feynman graph expansion in $U_{\alpha\beta;\gamma\delta}$, and graph-by-graph average, yields exact equations in the large N limit:

$$G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = -U^2 G^2(\tau) G(-\tau)$$
$$G(\tau = 0^-) = \mathcal{Q}.$$



S. Sachdev and J. Ye, PRL **70**, 3339 (1993)

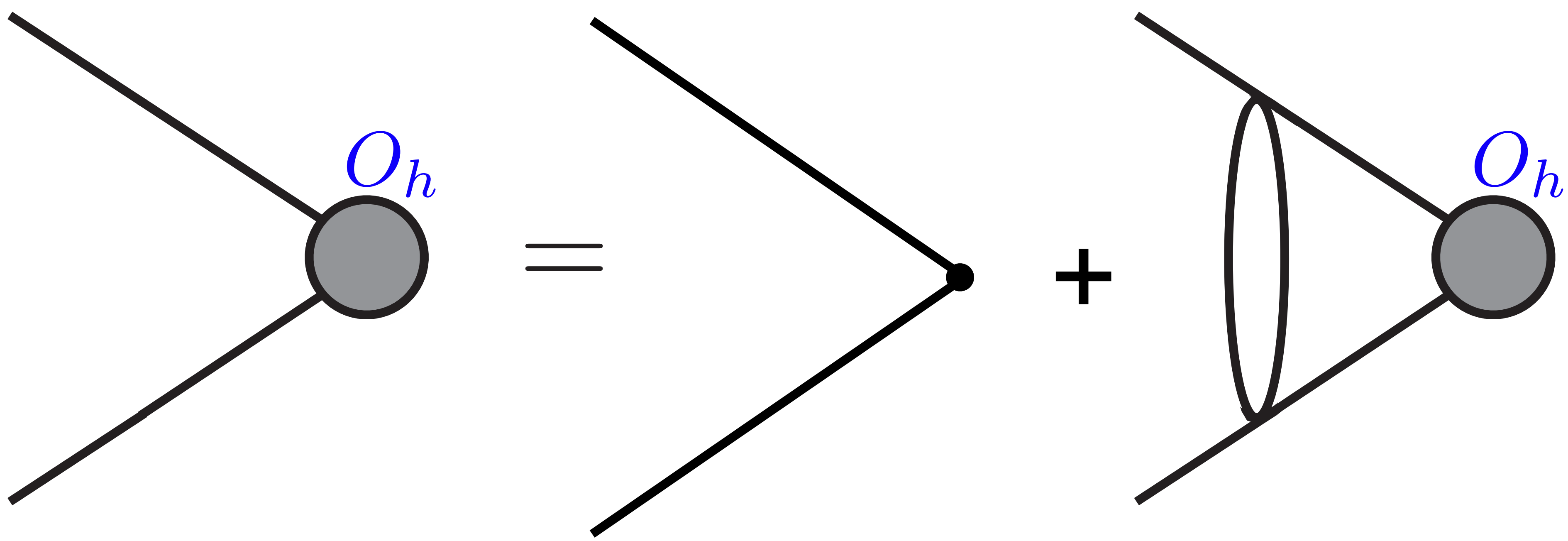
Conformal solution at $\mu = 0$, $G(\tau) \sim \frac{\mathrm{sgn}(\tau)}{\sqrt{|\tau|}}$.

The SYK model

Conformal Perturbation theory

$$S = S_{\rm CFT} + \sum_h g_h \int_0^\beta d\tau O_h(\tau)$$

where $G_{\rm CFT} \sim {\rm sgn}(\tau)/\sqrt{|\tau|}$ and $\langle O_h(\tau) O_h(0) \rangle \sim 1/|\tau|^{2h}$



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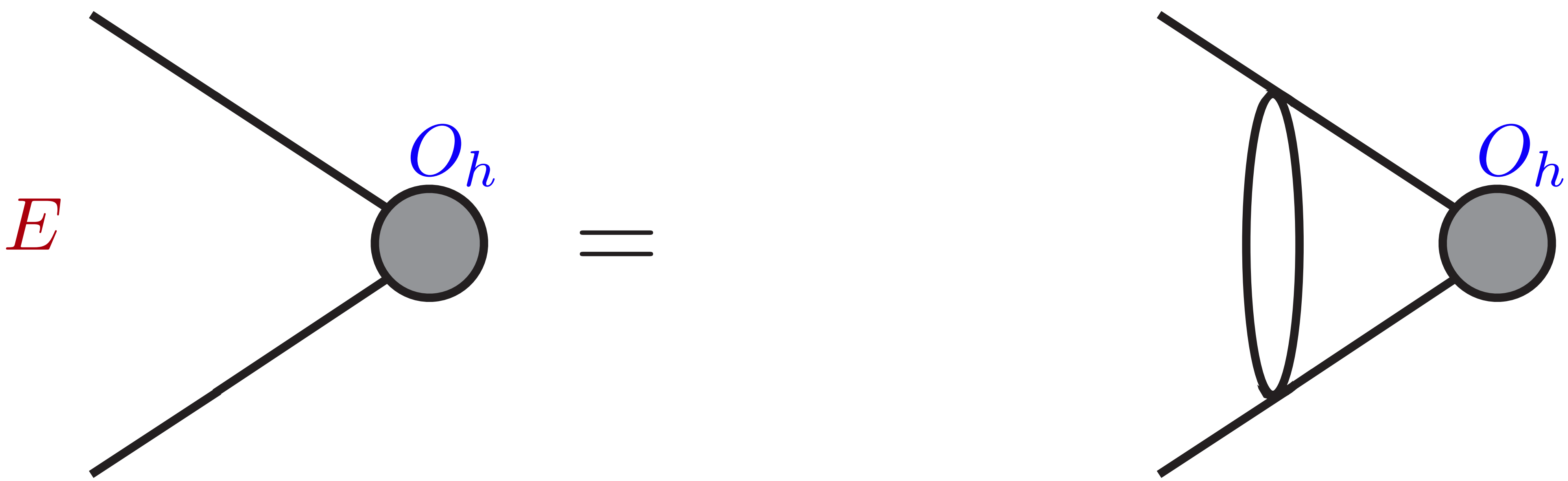


Solution of eigenvalue equation with $E=1$ yields a tower of operators O_h with scaling dimensions h. Smallest non-trivial value is $h=2$, and O_2 is the 'boundary graviton'.

$$G(\tau) \sim \frac{\text{sgn}(\tau)}{\sqrt{|\tau|}} \left(1 + \frac{c g_h}{|\tau|} + \dots \right)$$

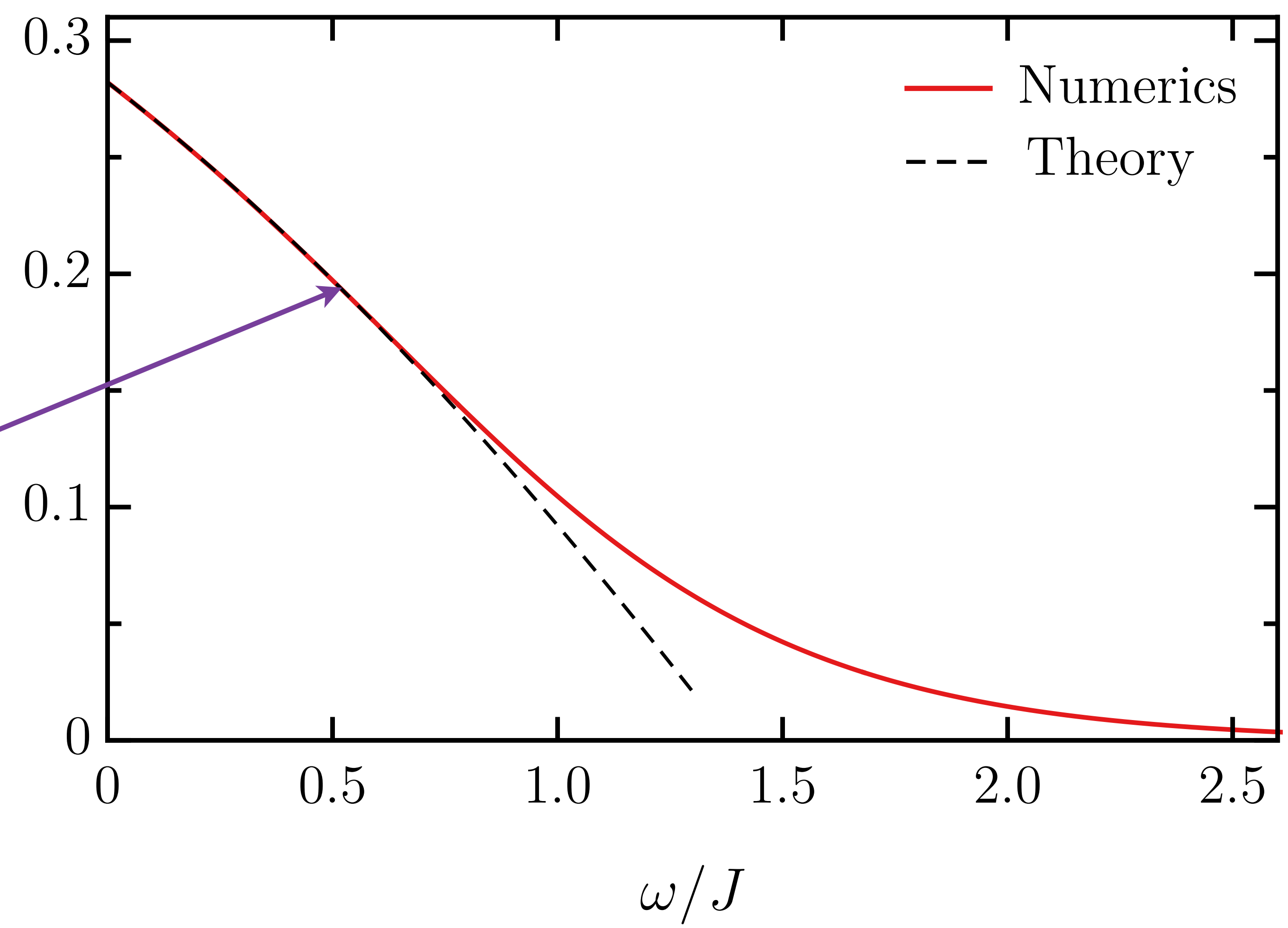
Corrections to the dynamic spin susceptibility of SYK model

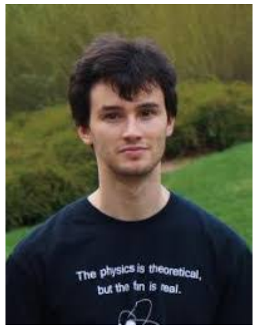
$$\chi_L(\tau) = G(\tau)G(-\tau)$$

$$\mathrm{Im}\chi_L(\omega) \sim \mathrm{sgn}(\omega)\left[1 - \mathcal{C}\gamma|\omega| - \frac{7}{16}(\mathcal{C}\gamma)^2|\omega|^2 - \mathcal{C}'|\omega|^{2.77354\ldots} + \frac{37}{48}(\mathcal{C}\gamma)^3|\omega|^3 - \ldots\right]$$

Numerical solution of SYK equations (SY, PRL 1993), compared with conformal perturbation theory. $\mathcal{C}$ is a known number, and γ is the co-efficient of the action for the 'boundary graviton' in holographic dual.

Correction from the boundary graviton



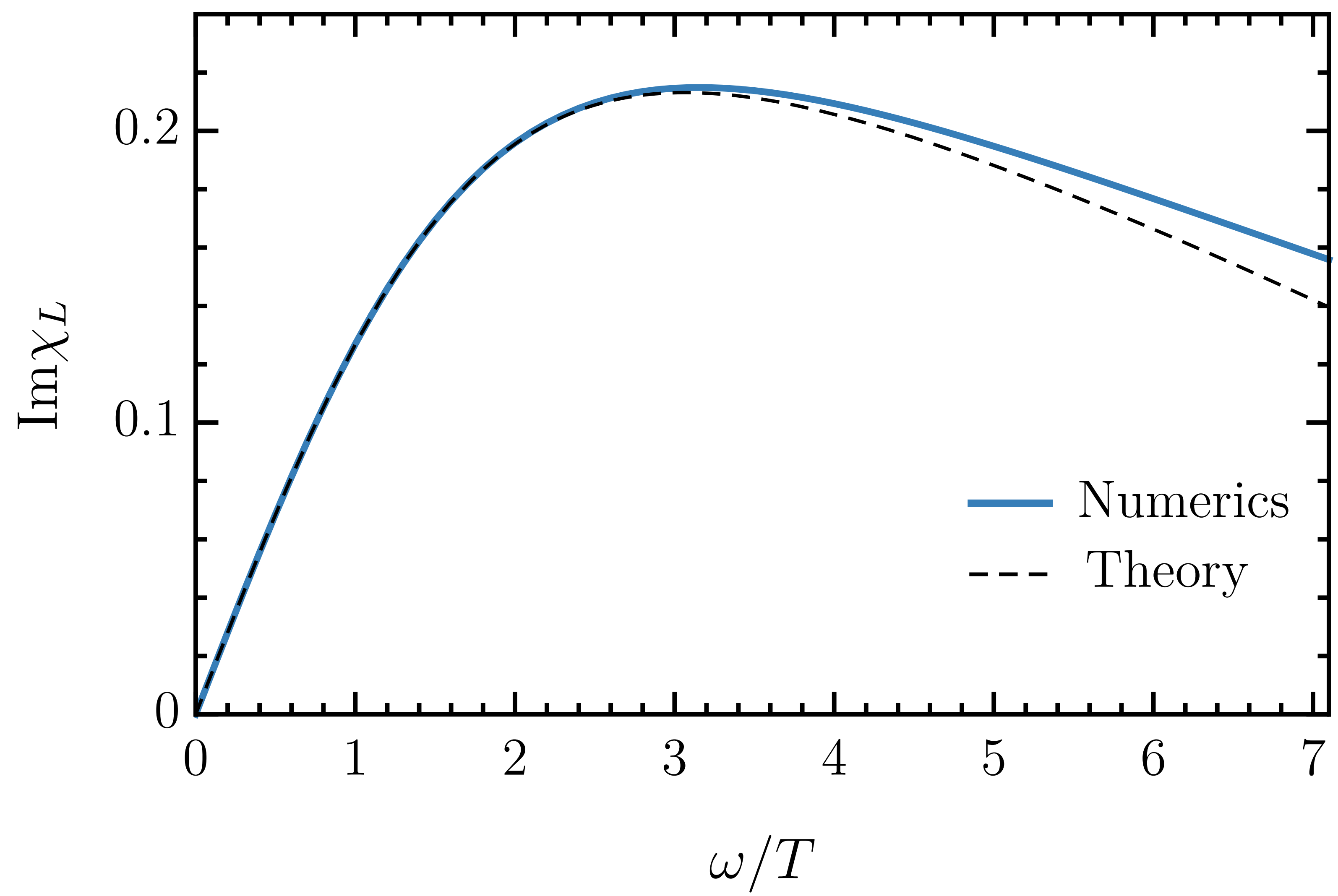


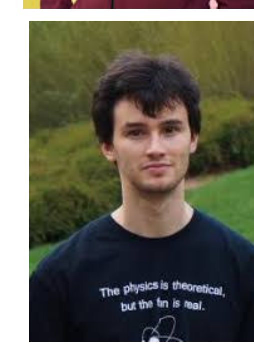
Maria Tikhanovskaya, Haoyu Guo, S. Sachdev, G. Tarnopolsky, arXiv: 2010.09742, 2012.14449

Corrections to the dynamic spin susceptibility of SYK model

$$\chi_L(\tau) = G(\tau)G(-\tau)$$

$$\chi_L(\omega) \sim \tanh\left(\frac{\hbar\omega}{2k_BT}\right)\left[1 - \mathcal{C}\gamma\,\omega\tanh\left(\frac{\hbar\omega}{2k_BT}\right) - \ldots\right]$$





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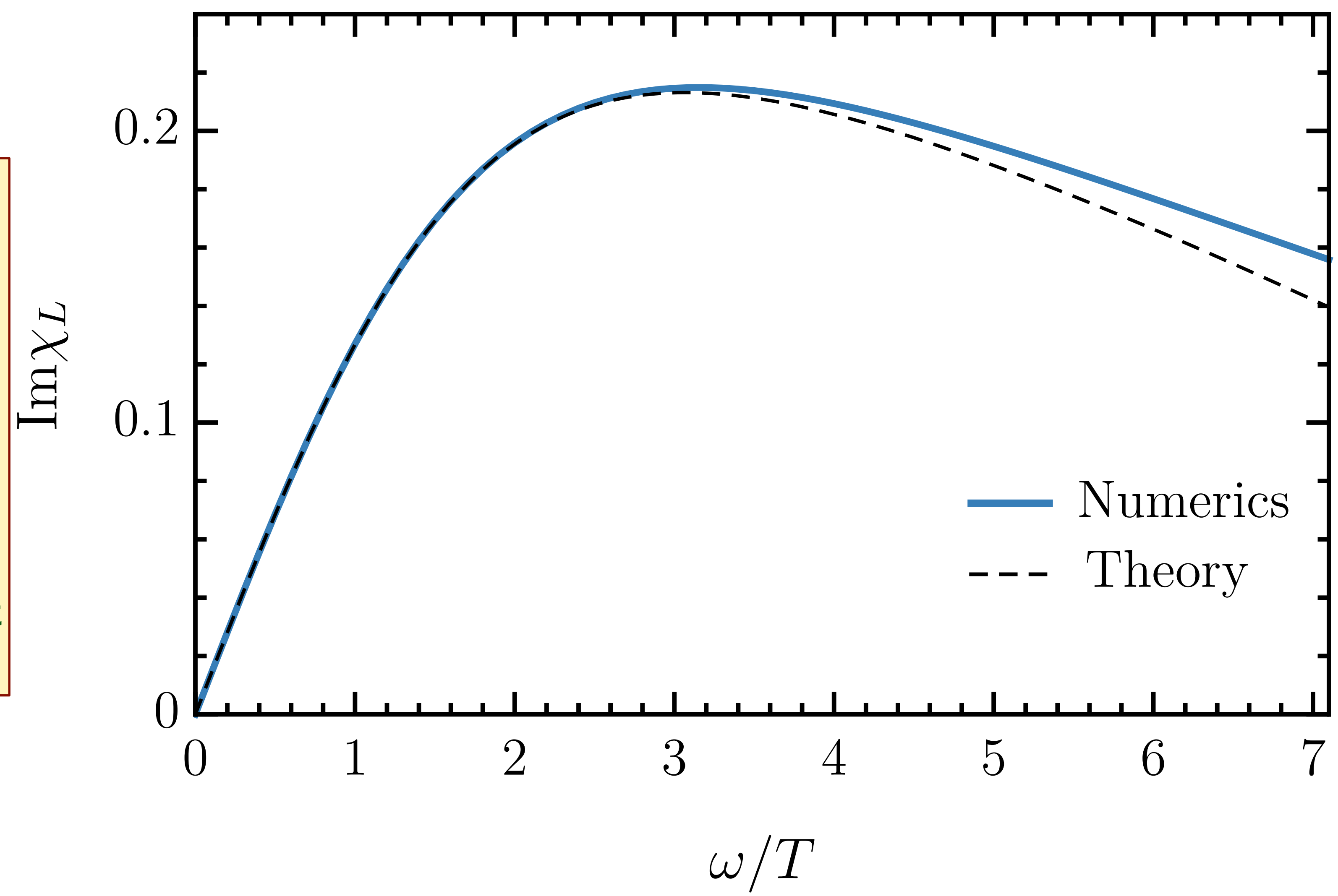
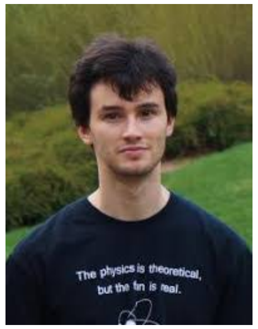
Conformally (SL(2,R)) invariant form with 'Planckian' dissipative time $\sim \hbar/(k_B T)$, *independent* of U.

A. Georges and O. Parcollet PRB **59**, 5341 (1999)





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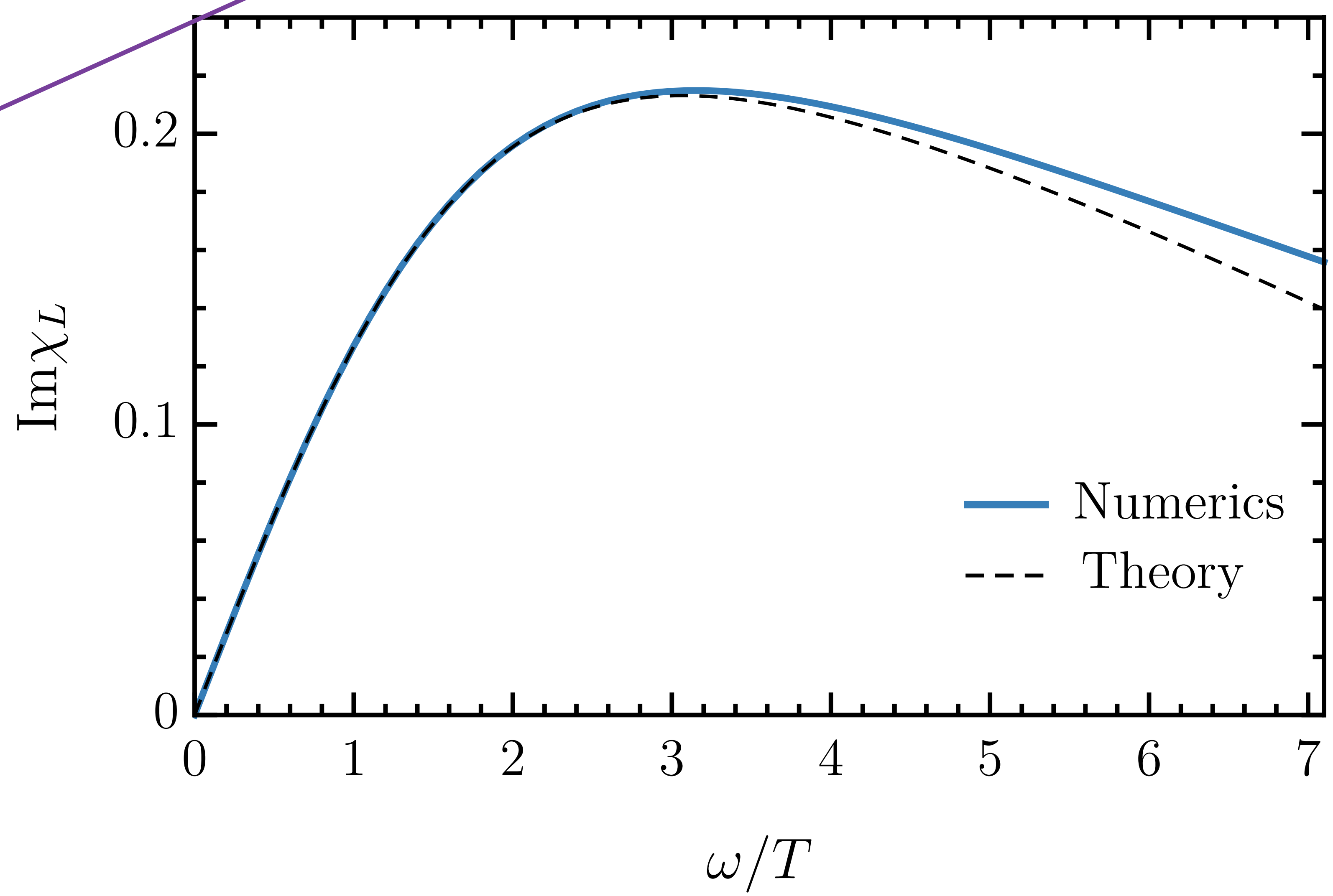
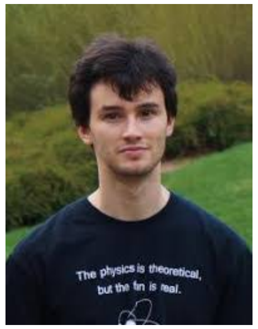
Correction from the boundary graviton





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Time reparameterization symmetry and 2D gravity

After introducing replicas $a = 1 \ldots n$, and integrating out the disorder, the partition function can be written as

$$Z = \int \mathcal{D}c_{\alpha a}(\tau) \exp\left[- \sum_{ia} \int_0^\beta d\tau \, c_{\alpha a}^\dagger \left(\frac{\partial}{\partial\tau} - \mu \right) c_{\alpha a} \right.$$
$$\left. - \frac{U^2}{4N^3} \sum_{ab} \int_0^\beta d\tau d\tau' \left| \sum_i c_{\alpha a}^\dagger(\tau) c_{\alpha b}(\tau') \right|^4 \right] .$$

For simplicity, we neglect the replica indices, and introduce the identity

$$1 = \int \mathcal{D}G(\tau_1, \tau_2) \mathcal{D}\Sigma(\tau_1, \tau_2) \exp\left[-N \int_0^\beta d\tau_1 d\tau_2 \Sigma(\tau_1, \tau_2) \left(G(\tau_2, \tau_1) \right.\right.$$
$$\left.\left. + \frac{1}{N} \sum_i c_\alpha(\tau_2) c_\alpha^\dagger(\tau_1) \right) \right] .$$

Time reparameterization symmetry and 2D gravity

Then the partition function can be written as a path integral with an action S analogous to a Luttinger-Ward functional

$$Z = \int \mathcal{D}G(\tau_1, \tau_2)\mathcal{D}\Sigma(\tau_1, \tau_2) \exp(-NS)$$

$$S = \ln \det \left[\delta(\tau_1 - \tau_2)(\partial_{\tau_1} + \mu) - \Sigma(\tau_1, \tau_2)\right] + \int d\tau_1 d\tau_2 \left[\Sigma(\tau_1, \tau_2)G(\tau_2, \tau_1) + (U^2/2)G^2(\tau_2, \tau_1)G^2(\tau_1, \tau_2)\right]$$

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$$+ \int d\tau_1 d\tau_2 \left[\Sigma(\tau_1, \tau_2)G(\tau_2, \tau_1) + (U^2/2)G^2(\tau_2, \tau_1)G^2(\tau_1, \tau_2)\right]$$

At frequencies $\ll U$, the time derivative in the determinant is less important, and without it the path integral is invariant under the reparametrization and gauge transformations

A. Georges and O. Parcollet
PRB **59**, 5341 (1999)
A. Kitaev, 2015
S. Sachdev, PRX **5,** 041025 (2015)

$$\tau = f(\sigma)$$

$$G(\tau_1, \tau_2) = [f'(\sigma_1)f'(\sigma_2)]^{-1/4} \frac{g(\sigma_1)}{g(\sigma_2)} \widetilde{G}(\sigma_1, \sigma_2)$$

$$\Sigma(\tau_1, \tau_2) = [f'(\sigma_1)f'(\sigma_2)]^{-3/4} \frac{g(\sigma_1)}{g(\sigma_2)} \widetilde{\Sigma}(\sigma_1, \sigma_2)$$

where $f(\sigma)$ and $g(\sigma)$ are arbitrary functions.

Time reparameterization symmetry and 2D gravity

Reparametrization and phase zero modes

We can write the path integral for the SYK model as

$$\mathcal{Z} = \int \mathcal{D}G(\tau_1, \tau_2)\mathcal{D}\Sigma(\tau_1, \tau_2) e^{-NS[G,\Sigma]}$$

for a known action $S[G, \Sigma]$. We find the saddle point, G_s, Σ_s, and only focus on the "Nambu-Goldstone" modes associated with breaking reparameterization and U(1) gauge symmetries by writing

$$G(\tau_1, \tau_2) = [f'(\tau_1)f'(\tau_2)]^{1/4} G_s(f(\tau_1) - f(\tau_2)) e^{i\phi(\tau_1) - i\phi(\tau_2)}$$

(and similarly for Σ). Then the path integral is approximated by

$$\mathcal{Z} = \int \mathcal{D}f(\tau)\mathcal{D}\phi(\tau) e^{-E_0/T + NS(E_0) - NS_{\text{eff}}[f,\phi]},$$

where $E_0 \propto N$ is the ground state energy.

J. Maldacena and D. Stanford, arXiv:1604.07818;
R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, arXiv.1612.00849;
S. Sachdev, PRX **5,** 041025 (2015); J. Maldacena, D. Stanford, and Zhenbin Yang, arXiv:1606.01857;
K. Jensen, arXiv:1605.06098; J. Engelsoy, T.G. Mertens, and H. Verlinde, arXiv:1606.03438

Time reparameterization symmetry and 2D gravity

Symmetry arguments, and explicit computations, show that the effective action is

$$S_{\text{eff}}[f,\phi] = \frac{NK}{2}\int_0^{1/T} d\tau(\partial_\tau\phi + i(2\pi\mathcal{E}T)\partial_\tau f)^2 - \frac{N\gamma}{4\pi^2}\int_0^{1/T} d\tau\,\{\tan(\pi T f(\tau)),\tau\},$$

where $f(\tau)$ is a monotonic map from $[0,1/T]$ to $[0,1/T]$, the couplings K, γ, and $\mathcal{E}$ can be related to thermodynamic derivatives and we have used the Schwarzian:

$$\{g,\tau\} \equiv \frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2 .$$

Specifically, an argument constraining the effective at $T=0$ is

$$S_{\text{eff}}\left[f(\tau) = \frac{a\tau+b}{c\tau+d}, \phi(\tau)=0\right] = 0,$$

and this is origin of the Schwarzian.

The same effective action is obtained for the boundary graviton of 2D gravity on AdS_2.

J. Maldacena and D. Stanford, arXiv:1604.07818;
R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, PRB **95**, 155131 (2017);
A. Gaikwad, L.K. Joshi, G. Mandal, and S.R. Wadia, arXiv:1802.07746

- Exact evaluation of the path integral over $f(\tau)$ and $\phi(\tau)$ leads to the many-body density of states

$$D(E) \sim 2e^{S_0} \sinh(\sqrt{2N\gamma E})$$

- Saddle-point shift leads to a correction to the Green's function:

$$G(\tau) \sim \frac{\mathrm{sgn}(\tau)}{\sqrt{|\tau|}} \left(1 + \frac{\alpha_G}{|\tau|} + \ldots\right)$$

  From this, we can compute the susceptibility $\chi(\tau) \sim G(\tau)G(-\tau)$

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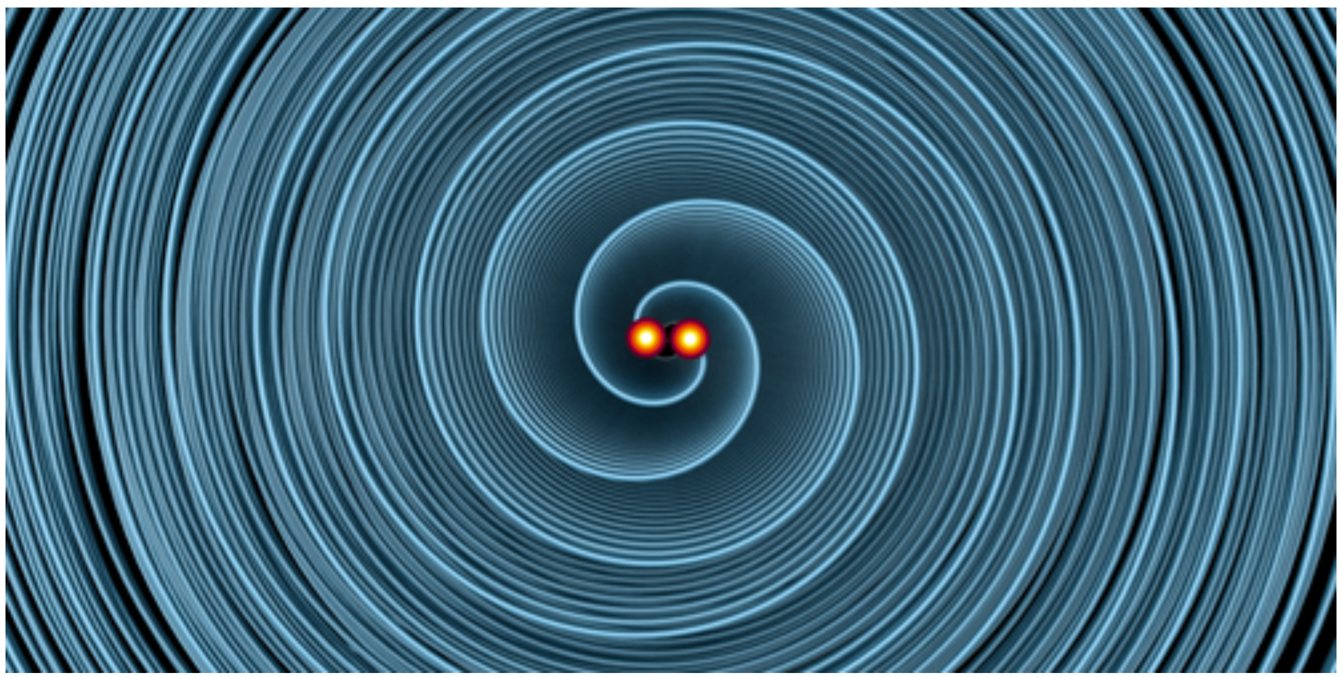
Black Holes Obey Information-Emission Limits

April 22, 2021 • *Physics* 14, s47 –Christopher Crockett

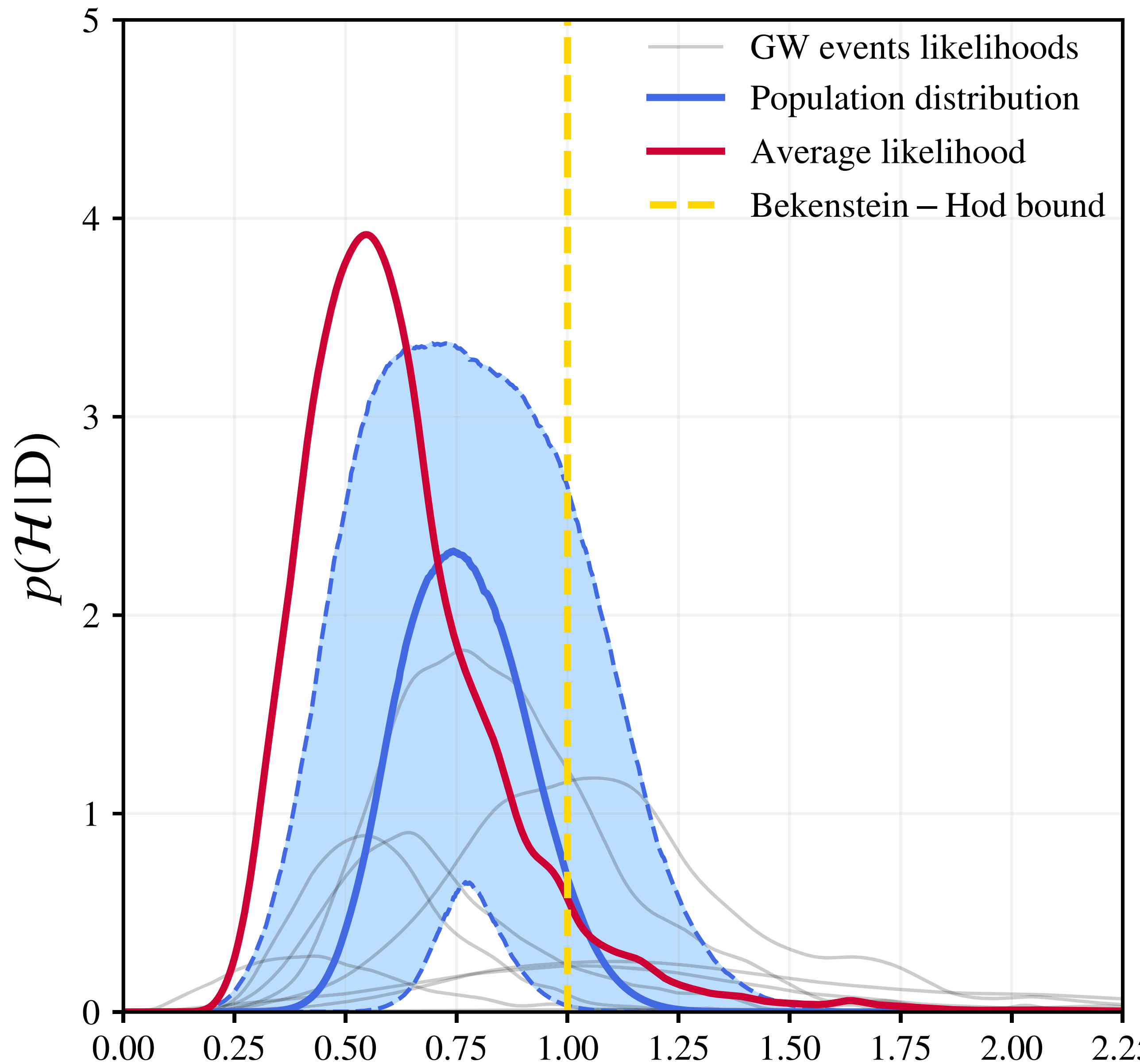
G. Carullo, D. Laghi, J. Veitch, W. Del Pozzo, Phys. Rev. Lett. **126**, 161102 (2021)

An analysis of the gravitational waves emitted from black hole mergers confirms that black holes are the fastest known information dissipaters.



Gravity wave observations of 8 different black holes show a relaxation time

$$\tau \sim \frac{\hbar}{k_B T}$$



$$\mathcal{H} = \frac{1}{\pi}\frac{\hbar/\tau}{k_B T}$$

Thermodynamics of quantum black holes:

$$\int \mathcal{D}g_{\mu\nu} \exp\left(-\frac{1}{\hbar}\, \mathcal{S}^{(3+1)}_{\text{Einstein gravity}}[g_{\mu\nu}]\right)$$

Metric of spacetime

In general, this summation is not well defined, because to the uncontrollably large number of spacetime configurations.

Thermodynamics of quantum black holes:

$$\int \mathcal{D}g_{\mu\nu} \exp\left(-\frac{1}{\hbar}\, \mathcal{S}^{(3+1)}_{\text{Einstein gravity}}[g_{\mu\nu}]\right)$$

$$= \exp\Big(S_{BH}\Big) \times \left(\quad ....????... \quad \right)$$

Metric of spacetime

Gibbons, Hawking (1977)

$$S_{BH} = \frac{A(T)c^3}{4G\hbar}$$

$(\hbar/(k_B T)$ is the length of the Euclidean time circle)
$A(T)$ is the area of the black hole horizon at a temperature T.
Interpretation: Black hole entropy is entanglement entropy across the horizon.

Thermodynamics of quantum black holes with charge $\mathcal{Q}$:

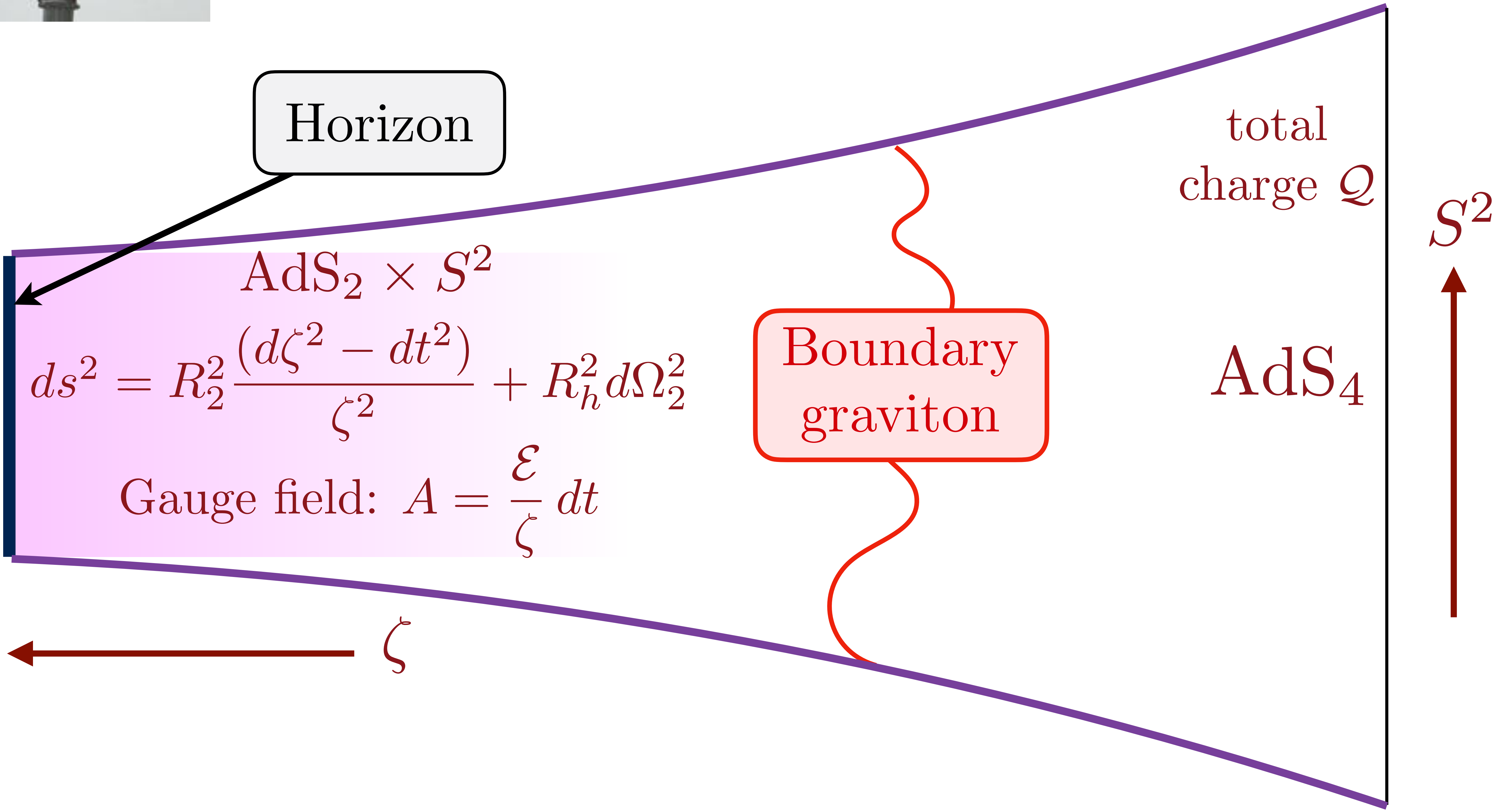
$$\int \mathcal{D}g_{\mu\nu}\mathcal{D}A_\mu \exp\left(-\frac{1}{\hbar}\,\mathcal{S}^{(3+1)}_{\text{Einstein gravity+Maxwell EM}}[g_{\mu\nu}, A_\mu]\right)$$
$$= \exp\Big(S_{BH}\Big) \times \Bigg(\quad ....????... \quad \Bigg)$$

Gibbons, Hawking (1977)

$$S_{BH}(T \to 0, \mathcal{Q}) = \frac{A(T)c^3}{4G\hbar} = \frac{A_0 c^3}{4G\hbar}\left(1 + \frac{2(\pi A_0)^{1/2} T}{\hbar c}\right)$$

A_0 is the area of the charged black hole horizon at $T = 0$, $\mathcal{Q}$ is the black hole charge.

Reissner-Nordstrom black hole of Einstein-Maxwell theory



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$$S(T\to 0, \mathcal{Q}) = S_{BH} - \frac{3}{4}\ln\left(\frac{\hbar c^5}{GT^2}\right)$$

$$S_{BH} = \frac{A(T)c^3}{4G\hbar} = \frac{A_0 c^3}{4G\hbar}\left(1 + \frac{2(\pi A_0)^{1/2}T}{\hbar c}\right)$$

A_0 is the area of the charged black hole horizon at $T=0$, $\mathcal{Q}$ is the black hole charge. The $\ln T$ term is the contribution of the boundary graviton.

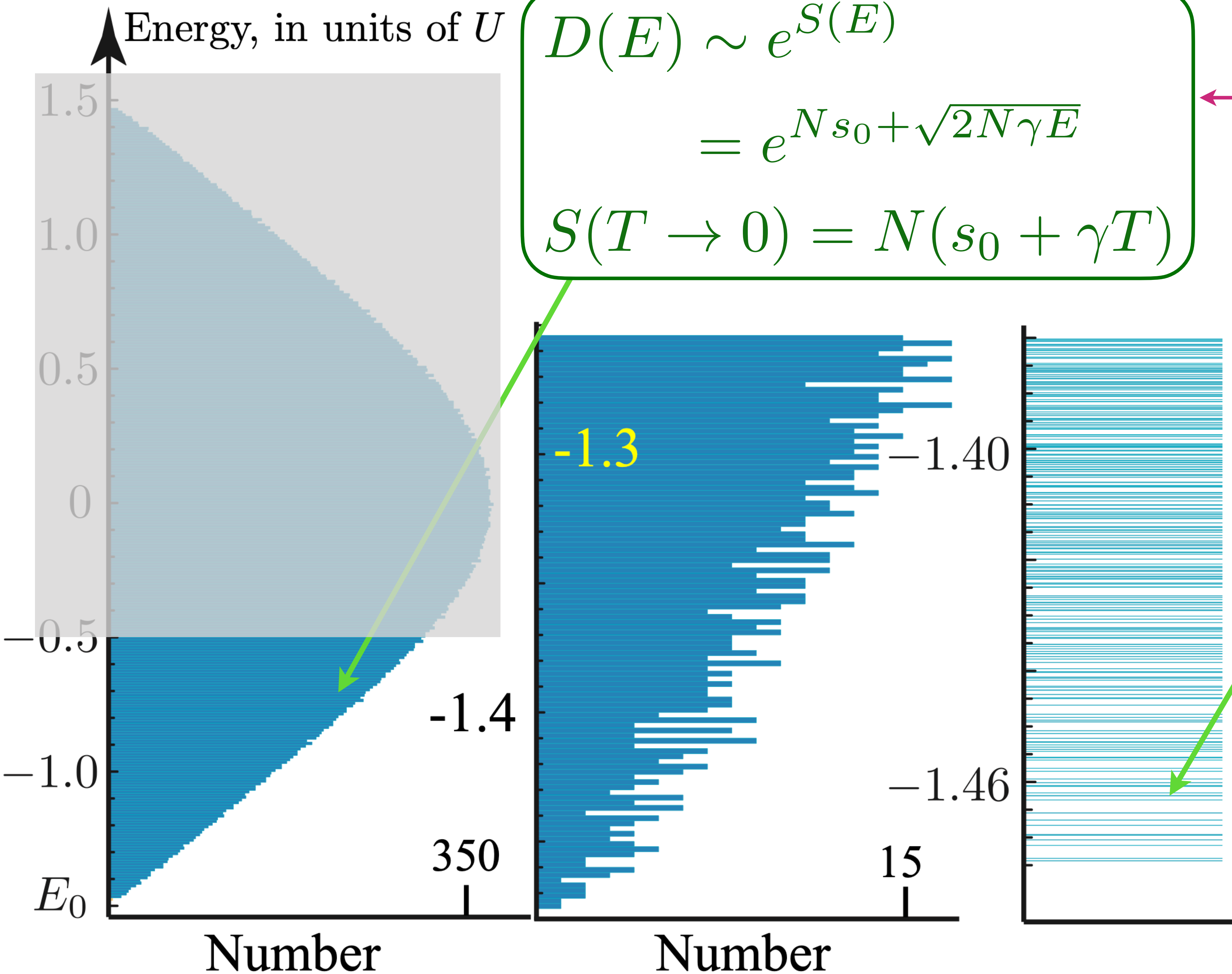
Sachdev (2010); Kitaev (2015); Sachdev (2015); Bagrets, Altland, Kamenev (2016); Maldacena, Stanford, Yang (2016); Moitra, Trivedi, Vishal (2018) ; Gaikwad, Joshi, Mandal, Wadia (2018); Iliesiu, Turaci (2020)

Many-body density of states

$$D(E) = \sum_i \delta(E - E_i); \quad E_0 + E_i \Rightarrow \text{Many body eigenvalue}$$

$$D(E) \sim e^{S(E)}$$
$$= e^{N s_0 + \sqrt{2N\gamma E}}$$
$$S(T \to 0) = N(s_0 + \gamma T)$$

$$D(E) \sim 2\, e^{N s_0} \sinh(\sqrt{2N\gamma E})$$
$$S(T) = N(s_0 + \gamma T) - \frac{3}{2} \ln\left(\frac{U}{T}\right)$$



$$D(E) \sim 2\, e^{N s_0} \sqrt{2N\gamma E}$$

No quasiparticle decomposition of many-body states

$$s_0 = 0.464848\ldots$$

A. Georges, O. Parcollet, and S. Sachdev, PRB **63**, 134406 (2001)

Complex SYK model

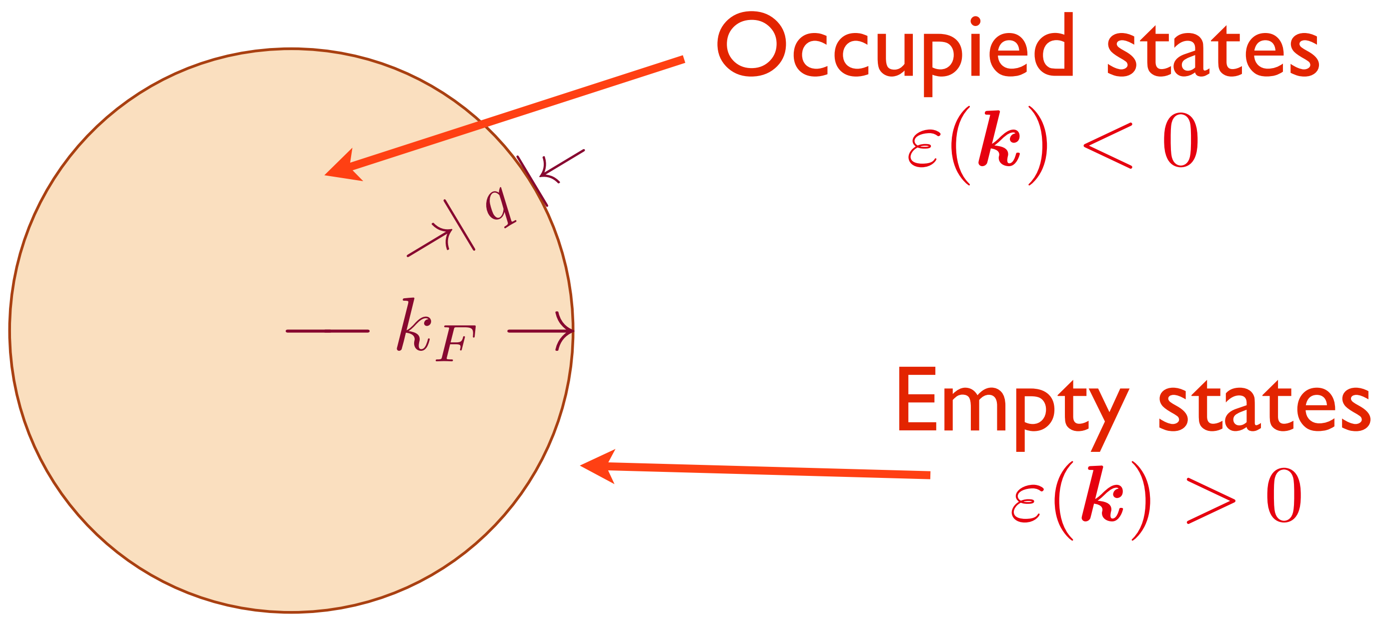
1. SYK models

2. Time reparameterization soft mode

3. Charged black holes

4. Critical Fermi surfaces: large N theory

FL Fermi liquid

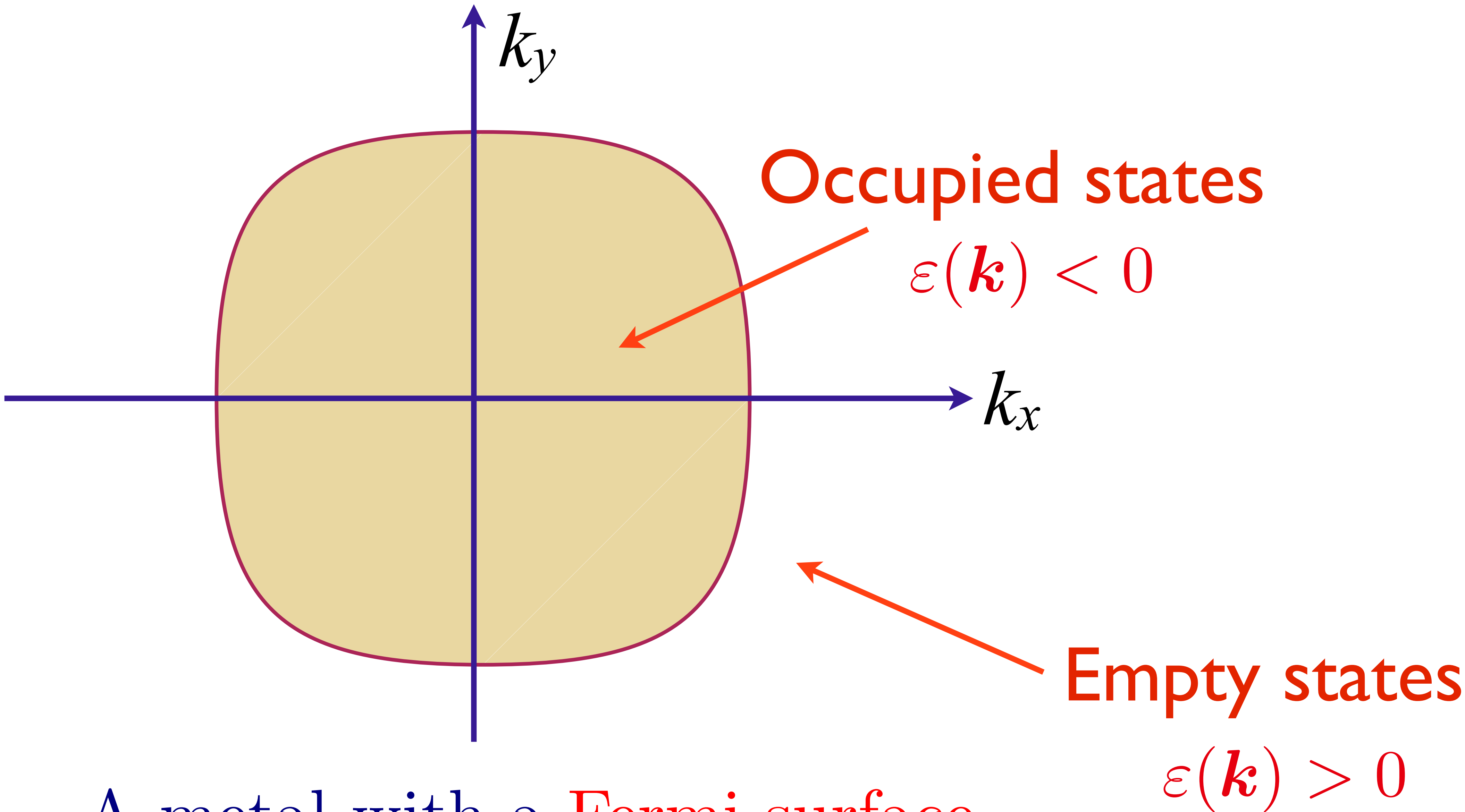


- $k_F^d \sim \mathcal{Q}$, the fermion density

- Sharp fermionic excitations near Fermi surface with $\omega \sim |q|^z$, and $z = 1$.

- Entropy density $S \sim T^{(d-\theta)/z}$ with violation of hyperscaling exponent $\theta = d - 1$.

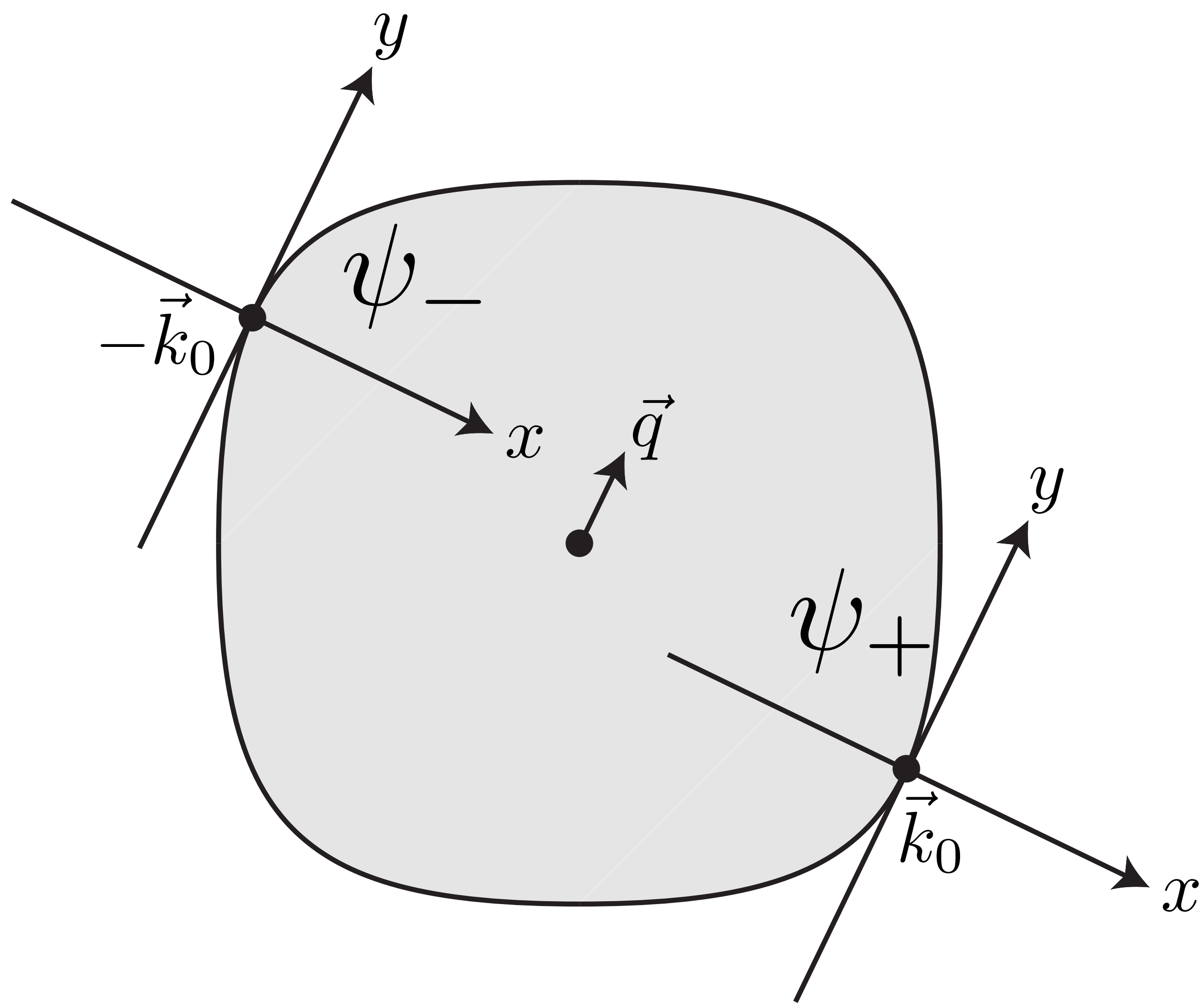
Fermi surface coupled to a gauge field



A metal with a Fermi surface
minimally coupled to a gauge field $\boldsymbol{A}$

$$\mathcal{L} = c_{\boldsymbol{k}}^{\dagger} \left(\frac{\partial}{\partial \tau} + \varepsilon(-i\boldsymbol{\nabla} - g\boldsymbol{A}) - \mu \right) c_{\boldsymbol{k}} + \frac{1}{2}(\boldsymbol{\nabla} \times \boldsymbol{A})^2$$

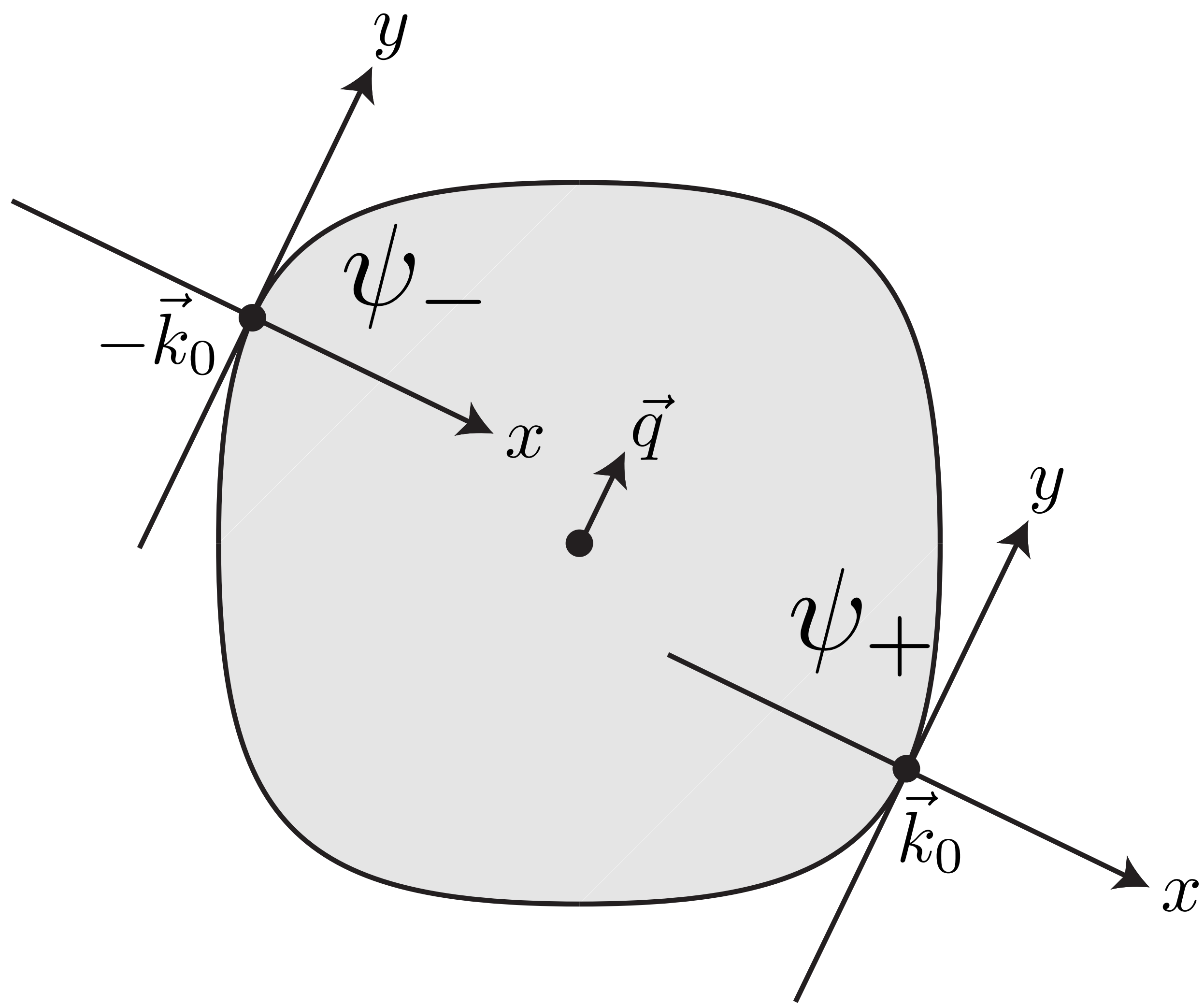
Fermi surface coupled to a gauge field



- Gauge fluctuation at wavevector $\boldsymbol{q}$ couples most efficiently to fermions near $\pm\boldsymbol{k}_0$.

- Expand fermion kinetic energy at wavevectors about $\pm\boldsymbol{k}_0$. In Landau gauge $\boldsymbol{A} = (a, 0)$.

Fermi surface coupled to a gauge field



$$\mathcal{L}[\psi_\pm, a] =$$

$$\psi_+^\dagger \left(\partial_\tau - i\partial_x - \partial_y^2\right) \psi_+ + \psi_-^\dagger \left(\partial_\tau + i\partial_x - \partial_y^2\right) \psi_-$$

$$-g\, a \left(\psi_+^\dagger \psi_+ - \psi_-^\dagger \psi_-\right) + \frac{1}{2} \left(\partial_y a\right)^2$$

M. A. Metlitski and S. Sachdev,
Phys. Rev. B **82**, 075127 (2010)

Fermi surface coupled to a gauge field

$$\mathcal{L} = \psi_+^\dagger \left(\partial_\tau - i\partial_x - \partial_y^2\right)\psi_+ + \psi_-^\dagger \left(\partial_\tau + i\partial_x - \partial_y^2\right)\psi_- - g\, a \left(\psi_+^\dagger \psi_+ - \psi_-^\dagger \psi_-\right) + \frac{1}{2}\left(\partial_y a\right)^2$$

Simple scaling argument for $z = 3/2$.

Under the rescaling $x \to x/s$, $y \to y/s^{1/2}$, and $\tau \to \tau/s^z$, we find invariance provided

$$\begin{aligned} a &\to a\, s^{(2z+1)/4} \\ \psi &\to \psi\, s^{(2z+1)/4} \\ g &\to g\, s^{(3-2z)/4} \end{aligned}$$

So the action is invariant provided $z = 3/2$.

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$$\mathcal{L} = \psi_+^\dagger \left(\cancel{\partial_\tau} - i\partial_x - \partial_y^2\right)\psi_+ + \psi_-^\dagger \left(\cancel{\partial_\tau} + i\partial_x - \partial_y^2\right)\psi_- - g\, a \left(\psi_+^\dagger \psi_+ - \psi_-^\dagger \psi_-\right) + \frac{1}{2}\left(\partial_y a\right)^2$$

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Because the bare time derivatives are irrelevant, the critical theory has an emergent time reparameterization symmetry:

$$\begin{aligned}\tau &\to f(\tau) \\ d\tau &\to f'(\tau)\, d\tau \\ x &\to [f'(\tau)]^{1/z}\, x \\ y &\to [f'(\tau)]^{1/(2z)}\, y \\ a &\to [f'(\tau)]^{-(2z+1)/(4z)}\, a \\ \psi &\to [f'(\tau)]^{-(2z+1)/(4z)}\, \psi\end{aligned}$$

Fermi surface coupled to a gauge field

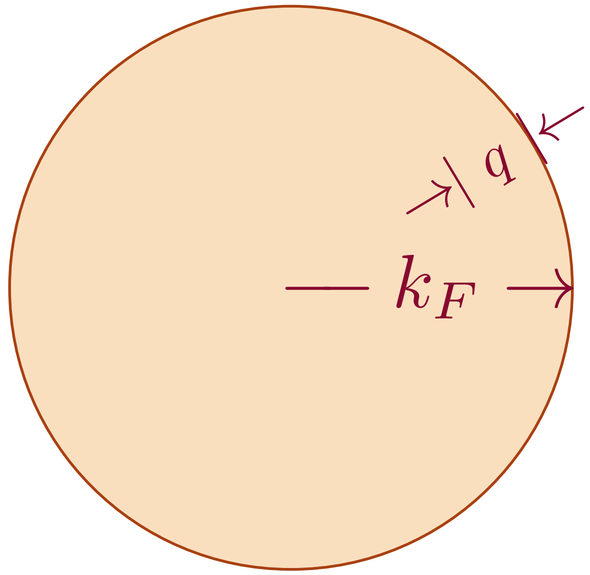
$$\mathcal{L} = \psi_+^\dagger \left(\not{\partial_\tau} - i\partial_x - \partial_y^2\right)\psi_+ + \psi_-^\dagger \left(\not{\partial_\tau} + i\partial_x - \partial_y^2\right)\psi_- \\ - g\,a\left(\psi_+^\dagger\psi_+ - \psi_-^\dagger\psi_-\right) + \frac{1}{2}\left(\partial_y a\right)^2$$

Various uncontrolled computations show

$$G(\boldsymbol{k}, i\omega) = \frac{1}{i\omega - v_F(|\boldsymbol{k}| - k_F) + i\#\, g^{4/3}\mathrm{sgn}(\omega)|\omega|^{2/3}}$$

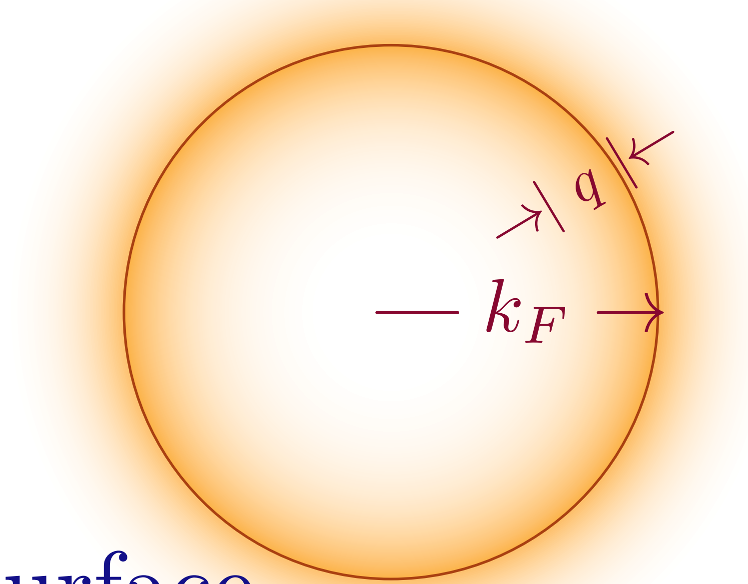
Needed: systematic theory, which can also address possible pairing and density-wave instabilities, and the role of time-reparameterizations

FL Fermi liquid



- $k_F^d \sim \mathcal{Q}$, the fermion density
- Sharp fermionic excitations near Fermi surface with $\omega \sim |q|^z$, and $z = 1$.
- Entropy density $S \sim T^{(d-\theta)/z}$ with violation of hyperscaling exponent $\theta = d - 1$.

NFL

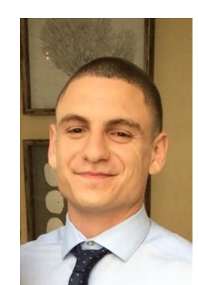


- Fermi surface with $k_F^d \sim \mathcal{Q}$.
- Diffuse fermionic excitations with $z = 3/2$ to three loops.
- $S \sim T^{(d-\theta)/z}$ with $\theta = d - 1$.

Large N theory of a critical Fermi surface

Main idea:

Introduce N flavors of fermions and bosons, and examine an *ensemble* of theories with different Yukawa couplings. In the large N limit, every member of the ensemble is expected to have the same critical properties, and so it is easier to study the average theory.



Ilya Esterlis, Haoyu Guo, Aavishkar Patel, S.S. arXiv: 2103.08615

Large N theory of a critical Fermi surface

N flavors of fermions $\psi_{\pm\alpha}$,
M flavors of a boson a_α, and
a "Yukawa coupling" $g_{\alpha\beta\gamma}$ which is a random function in flavor space. Note: there is *no spatial randomness*. Take the large N limit with M/N fixed.

$$\mathcal{L} = \psi^\dagger_{+\alpha}\left(\partial_\tau - i\partial_x - \partial_y^2\right)\psi_{+\alpha} + \psi^\dagger_{-\alpha}\left(\partial_\tau + i\partial_x - \partial_y^2\right)\psi_{-\alpha} - \frac{g_{\alpha\beta\gamma}}{N} a_\alpha\left(\eta_{+\alpha}\psi^\dagger_{+\beta}\psi_{+\gamma} + \eta_{-\alpha}\psi^\dagger_{-\beta}\psi_{-\gamma}\right) + \frac{1}{2}\left(\partial_y a_\alpha\right)^2$$

$\eta_{\pm\alpha} = \pm 1$ depending upon nature of a_α: gauge field, Higgs field, order parameter ....

$$\overline{g_{\alpha\beta\gamma}} = 0 \quad , \quad \overline{|g_{\alpha\beta\gamma}|^2} = g^2$$

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Large N theory of a critical Fermi surface

We can now proceed just as in the SYK model: we obtain a theory for Green's functions which are bilocal in both space and time. Using the spacetime coordinate $X \equiv (\tau, x, y)$, we can write the averaged partition function

$$\overline{\mathcal{Z}}_{\psi\phi} = \int \mathcal{D}G(X_1, X_2)\mathcal{D}\Sigma(X_1, X_2)\mathcal{D}D(X_1, X_2) \times \mathcal{D}\Pi(X_1, X_2) \exp\left[-NI(G, \Sigma, D, \Pi)\right] .$$

The G-Σ-D-Π action is now

$$I(G, \Sigma, D, \Pi) = \frac{g^2}{2}\mathrm{Tr}\left(G \cdot [GD]\right) - \mathrm{Tr}(G \cdot \Sigma) + \frac{1}{2}\mathrm{Tr}(D \cdot \Pi) - \ln\det\left[\left(\partial_{\tau_1} - i\partial_{x_1} - \partial_{y_1}^2\right)\delta(X_1 - X_2) + \Sigma(X_1, X_2)\right] + \frac{1}{2}\ln\det\left[\left(-K\partial_{y_1}^2\right)\delta(X_1 - X_2) - \Pi(X_1, X_2)\right] .$$

where we have introduced notation

$$\mathrm{Tr}\left(f \cdot g\right) \equiv \int dX_1 dX_2\, f(X_2, X_1) g(X_1, X_2) .$$

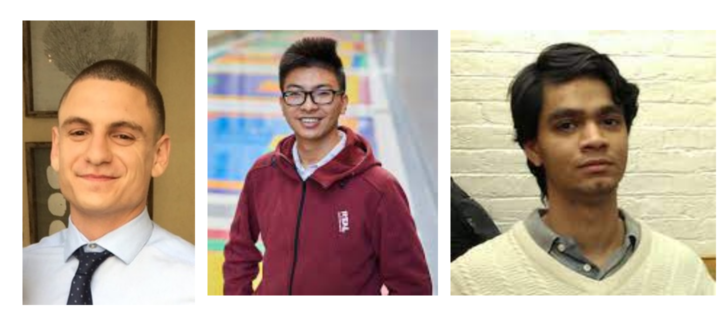
Ilya Esterlis, Haoyu Guo, Aavishkar Patel, S.S. arXiv: 2103.08615

Large N theory of a critical Fermi surface

Saddle-point equations

$$G(\boldsymbol{k}, i\omega) = \frac{1}{i\omega - k_x - k_y^2 - \Sigma(\boldsymbol{k}, i\omega)}, \quad D(\boldsymbol{k}, i\omega) = \frac{1}{k_y^2 - \Pi(\boldsymbol{k}, i\omega)}$$

$$\Sigma(\boldsymbol{r}, \tau) = g^2 D(\boldsymbol{r}, \tau) G(\boldsymbol{r}, \tau), \quad \Pi(\boldsymbol{r}, \tau) = -g^2 G(-\boldsymbol{r}, -\tau) G(\boldsymbol{r}, \tau)$$



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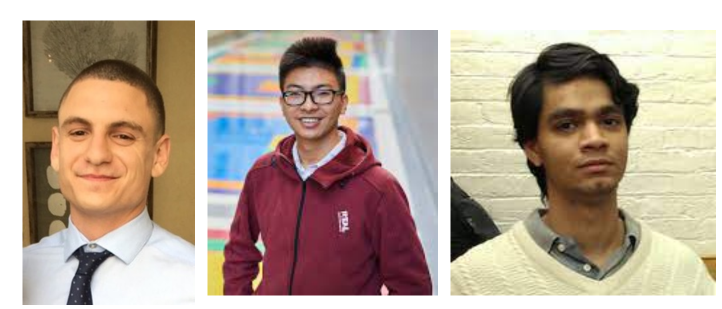
Exact solution at low energies:

$$\Sigma(\boldsymbol{k}, \omega) = g^{4/3} T^{2/3} \Phi\left(\frac{\omega}{T}\right),$$

where $\Phi(z)$ is a universal scaling function, obtained by analytical continuation from imaginary Matsubara frequencies $\omega_n = (2n-1)\pi T$

$$\Phi\left(\frac{i\omega_n}{T}\right) = -i\,\mathrm{sgn}(\omega_n) \frac{2^{5/3}}{3\sqrt{3}} H_{1/3}\left(\frac{|\omega_n| - \pi T}{2\pi T}\right)$$

$$H_r(n \in \mathbb{Z}^+) = \sum_{j=1}^{n} \frac{1}{j^r}$$



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Large N theory of a critical Fermi surface

- There is no time-reparameterization soft mode, and no associated violation of scaling: because of non-trivial action of time reparameterization on spatial co-ordinates.

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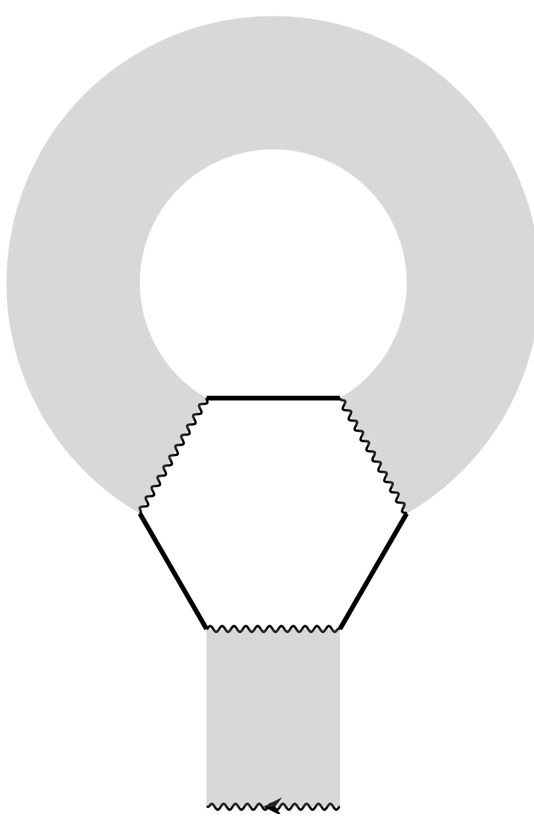
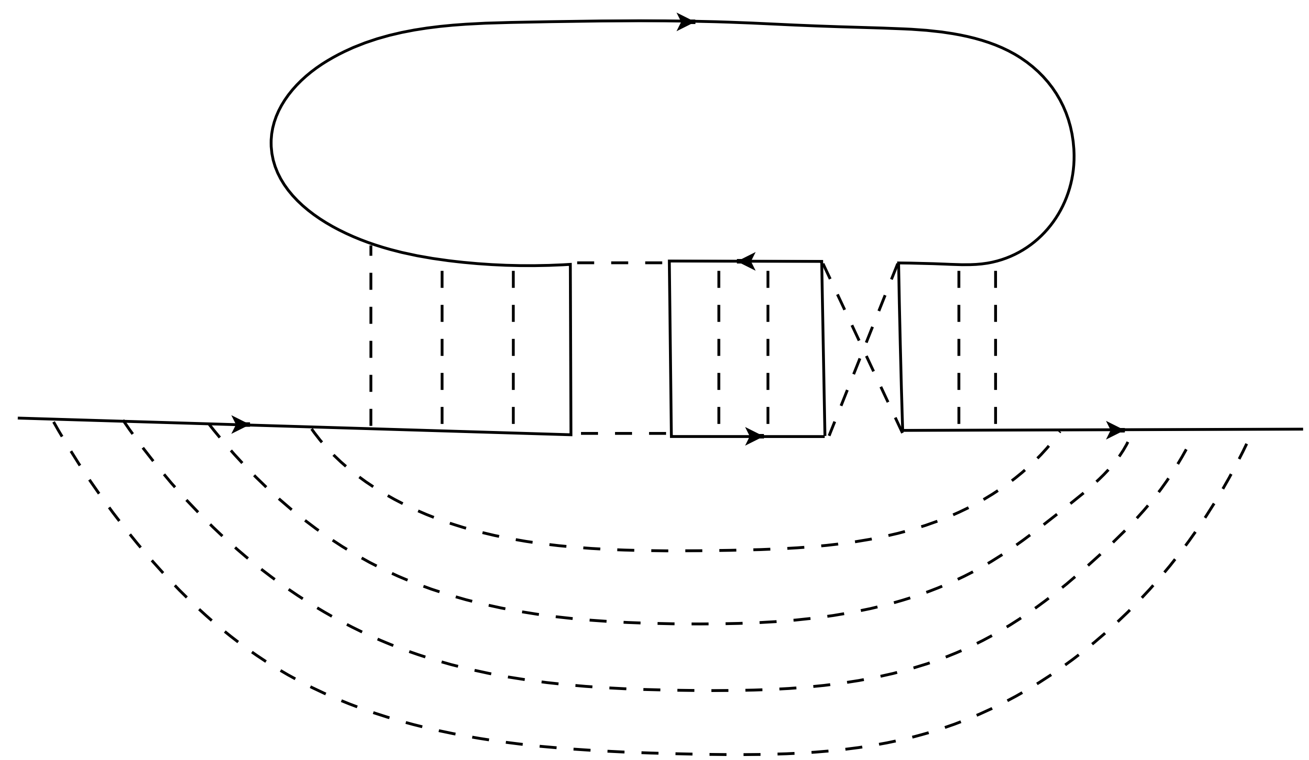
- Dynamic scaling exponent z, and fermion anomalous dimension can (in principle) have corrections at higher orders in $1/N$: these can be systematically computed in theory of fields bilocal in spacetime.





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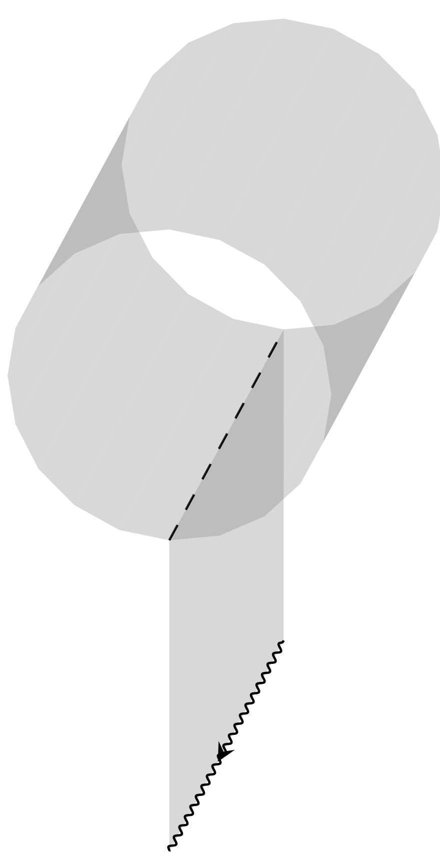
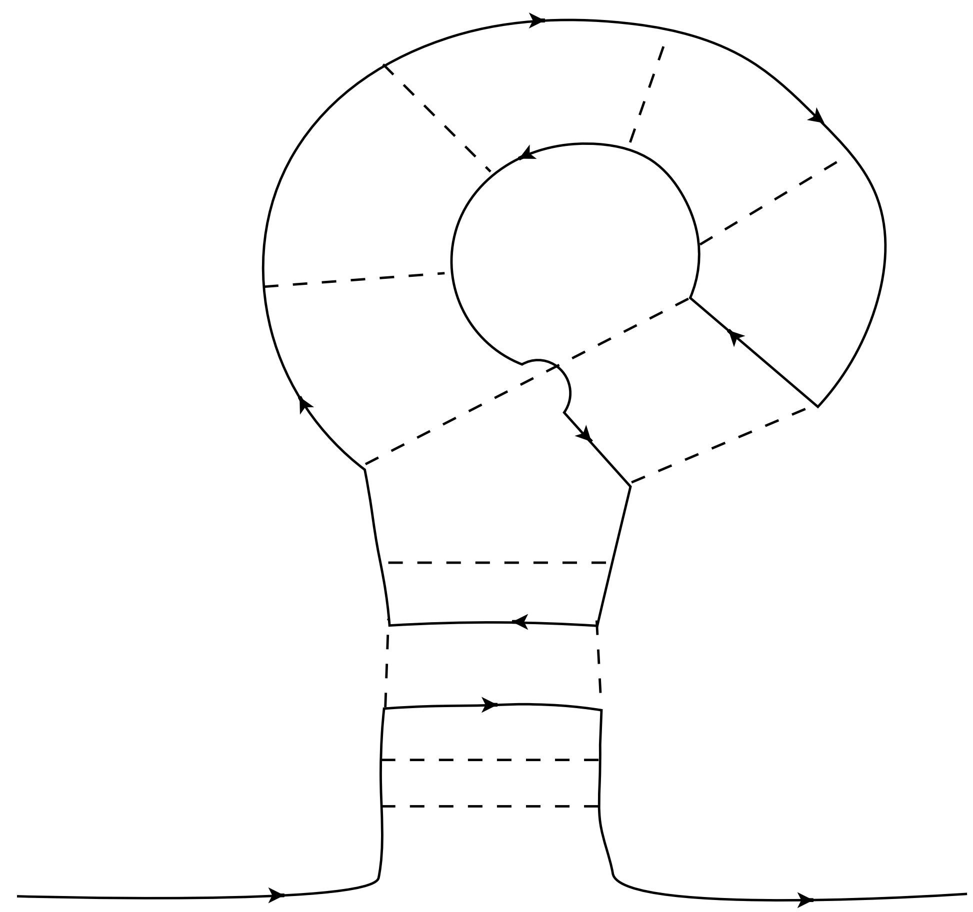
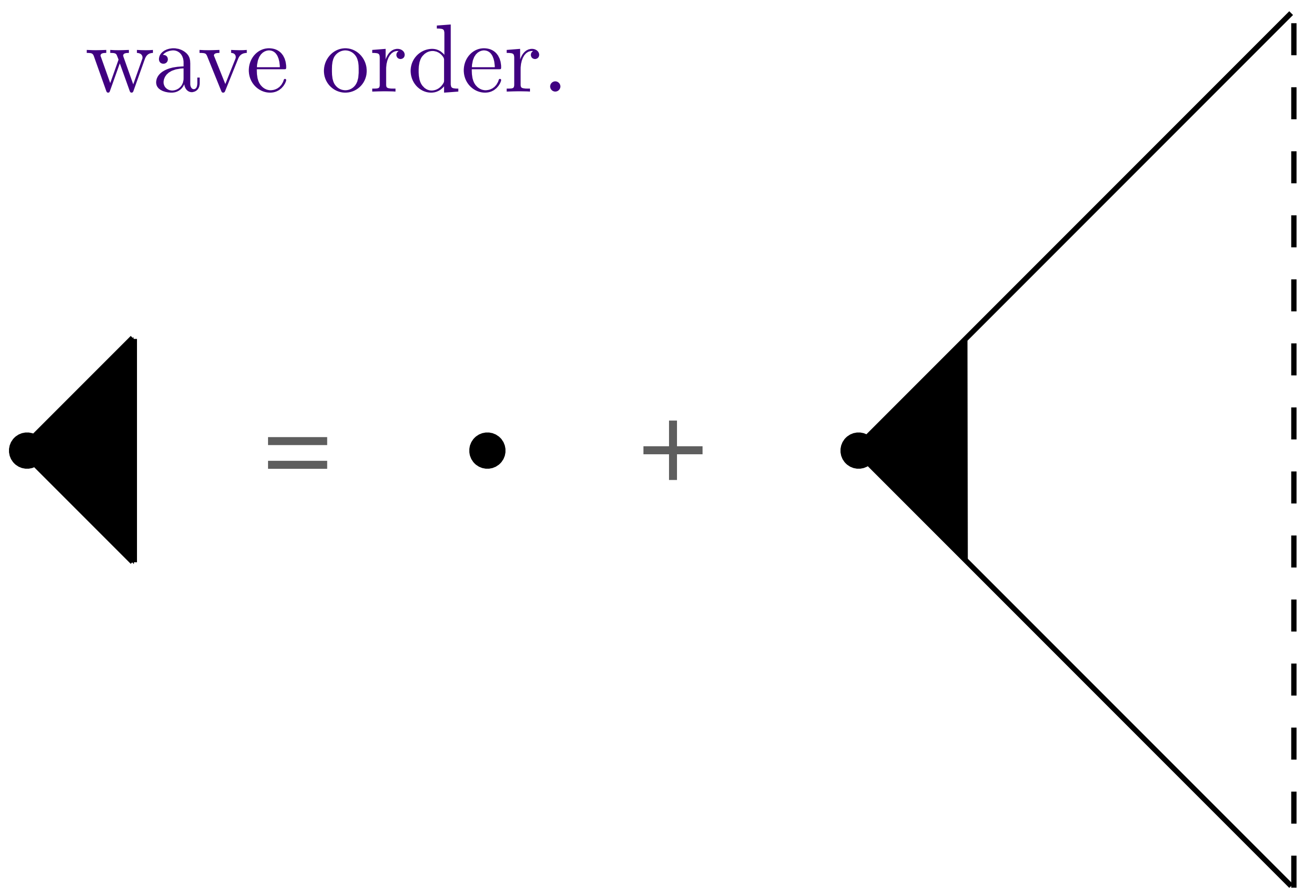




Large N theory of a critical Fermi surface

- Can determine scaling dimensions of Cooper pair and $2k_F$ density wave operators in $N = \infty$ theory: these are universal functions of M/N, $\eta_{\pm\alpha} = \pm 1$ (and ratios of diamagnetic susceptibilities with multiple gauge fields). Complex scaling dimension implies an instability to superconductivity/density wave order.



After dropping the bare term, search for an eigenvector

$$\Phi_\Psi(i\Omega_m) = \frac{1}{|\Omega_m|^\alpha}$$

with eigenvalue $E = 1$.

$$E\Phi_\Psi(i\Omega_m) = -\sum_a \frac{M_a \zeta_a g^2}{N} \frac{T}{3\sqrt{3}} \sum_{\omega_n \neq \Omega_m} \frac{\Phi_\Psi(i\omega_n)}{|\omega_n + i\Sigma(i\omega_n)|} \frac{(4\pi)^{1/3}}{(gK_a)^{2/3}|\omega_n - \Omega_m|^{1/3}},$$

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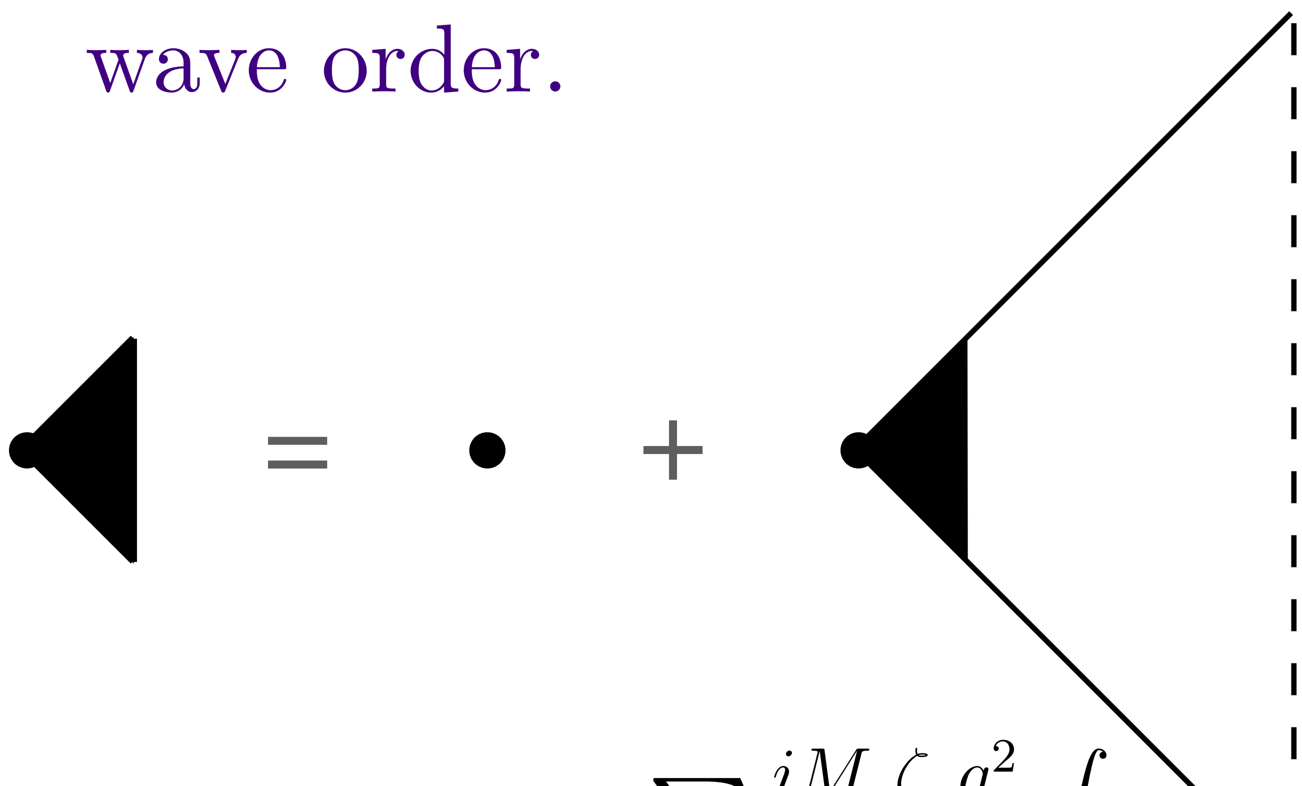

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$$\Phi_{2k_F}(q_y, i\Omega_m) = \frac{1}{|q_y|^\alpha} \Psi\left(\frac{\Omega_m}{|q_y|^3}\right)$$

with eigenvalue $E = 1$.

$$E\Phi_{2k_F}(q_y, i\Omega_m) = -\sum_a \frac{iM_a \zeta_a g^2}{2N} \int_{k_y,\omega_n} \frac{\mathrm{sgn}(\omega_n)}{k_y^2 - \frac{ig^{4/3}}{2^{1/3}\pi^{2/3}\sqrt{3}} \mathrm{sgn}(\omega_n)|\omega_n|^{2/3} \left(\frac{M_1}{K_1^{2/3}N} + \frac{M_2}{K_2^{2/3}N}\right)}$$
$$\times \frac{|k_y - q_y|}{K_a|k_y - q_y|^3 + \frac{g^2}{4\pi}|\omega_n - \Omega_m|} \Phi_{2k_F}(k_y, i\omega_n).$$

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(Talk at ICTS-TIFR on Wed)

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- Boundary graviton leads to:
  - Universal $-3/2 \ln(1/T)$ correction to Bekenstein-Hawking entropy of low T charged black holes in Einstein gravity.
- Systematic large N theory for a non-Fermi liquid is obtained by averaging an ensemble of theories. The is describes by a G-Σ theory with fields which are bi-local in spacetime.