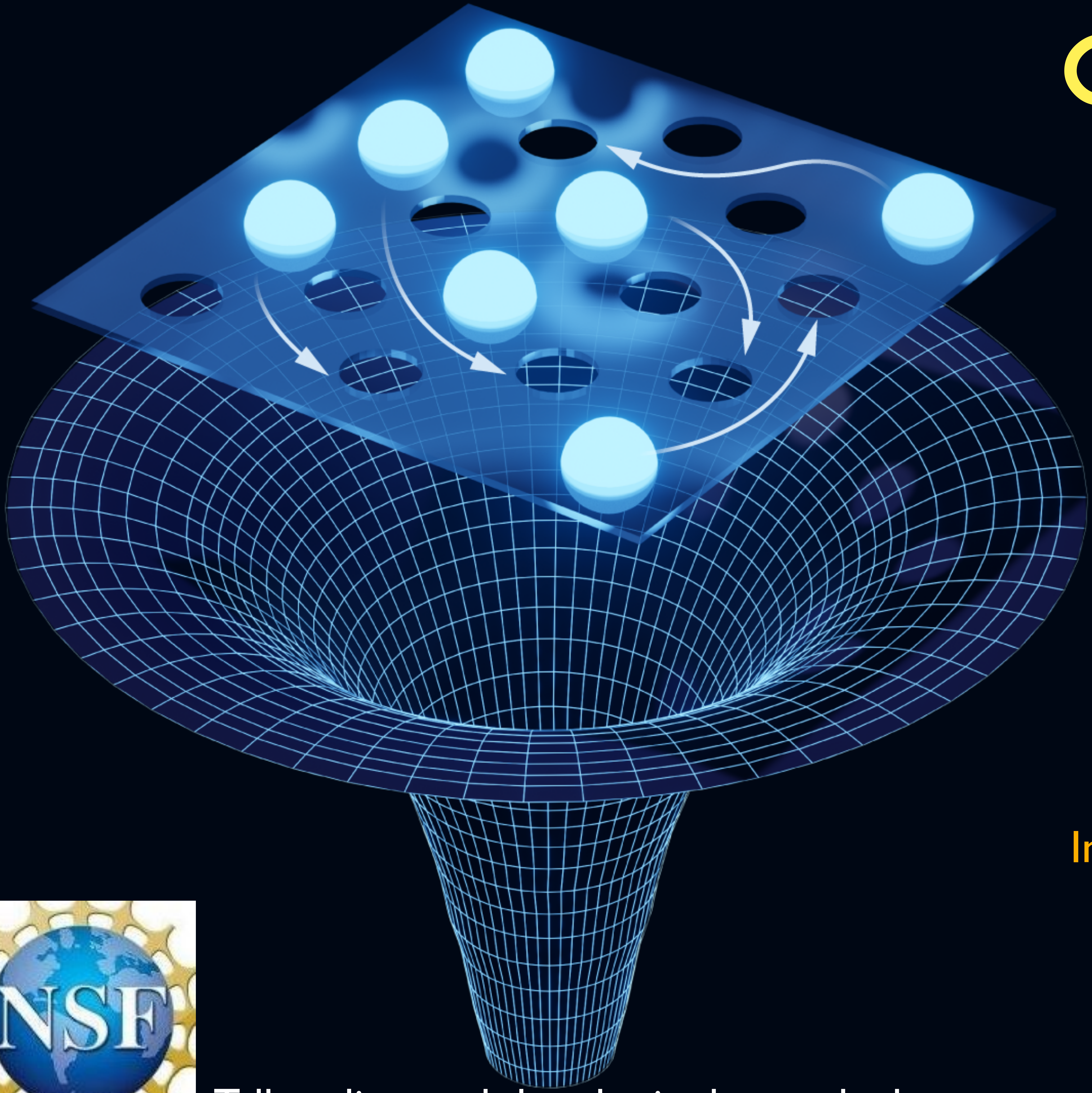


Quantum statistical mechanics of strange metals and black holes

Review articles:
arXiv:2304.13744, 2305.01001

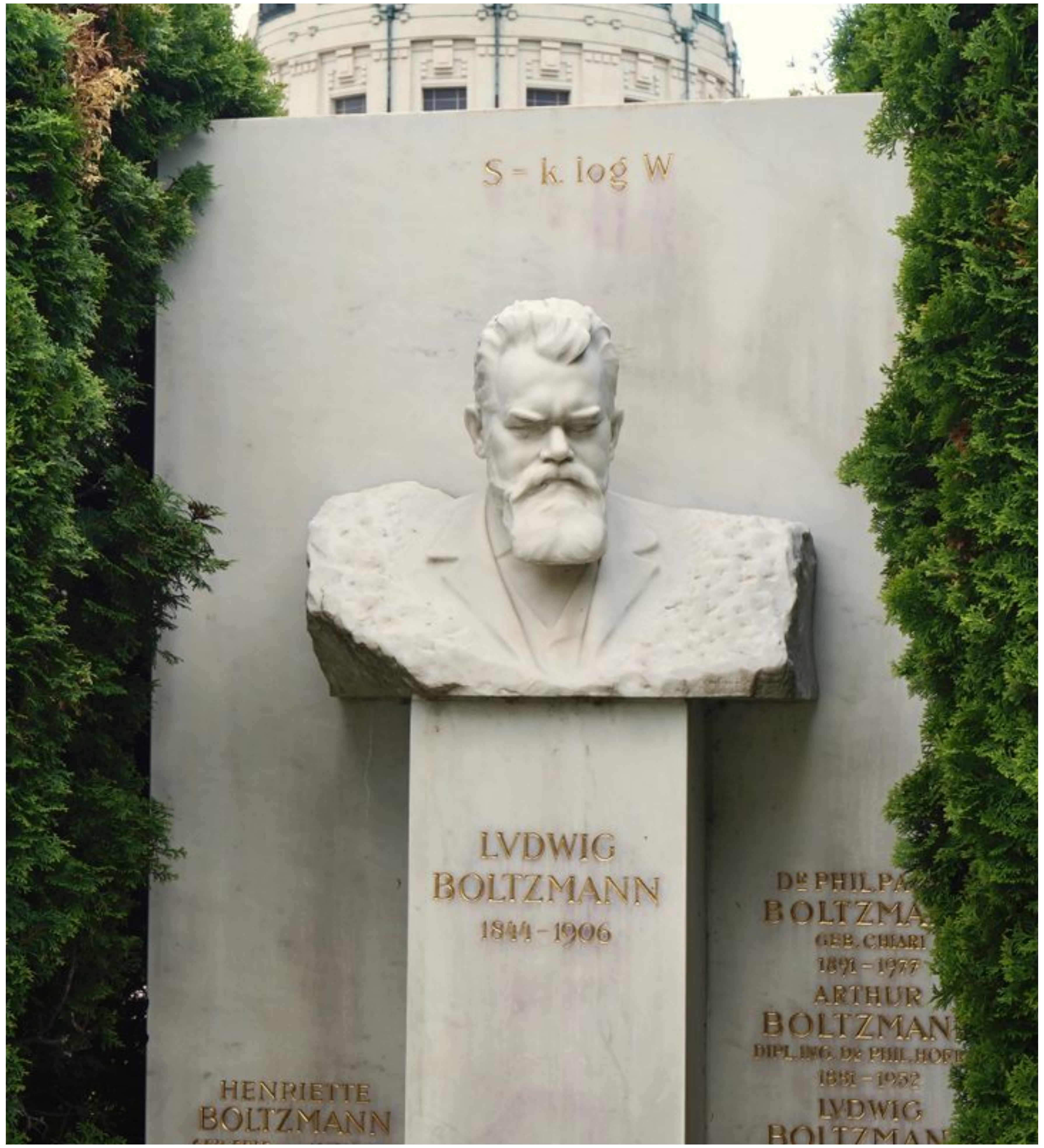
Indian Institute of Science
Bengaluru
December 29, 2023

Subir Sachdev

Talk online: sachdev.physics.harvard.edu

Foundations
by
Boltzmann

Statistical interpretation of entropy (1870)

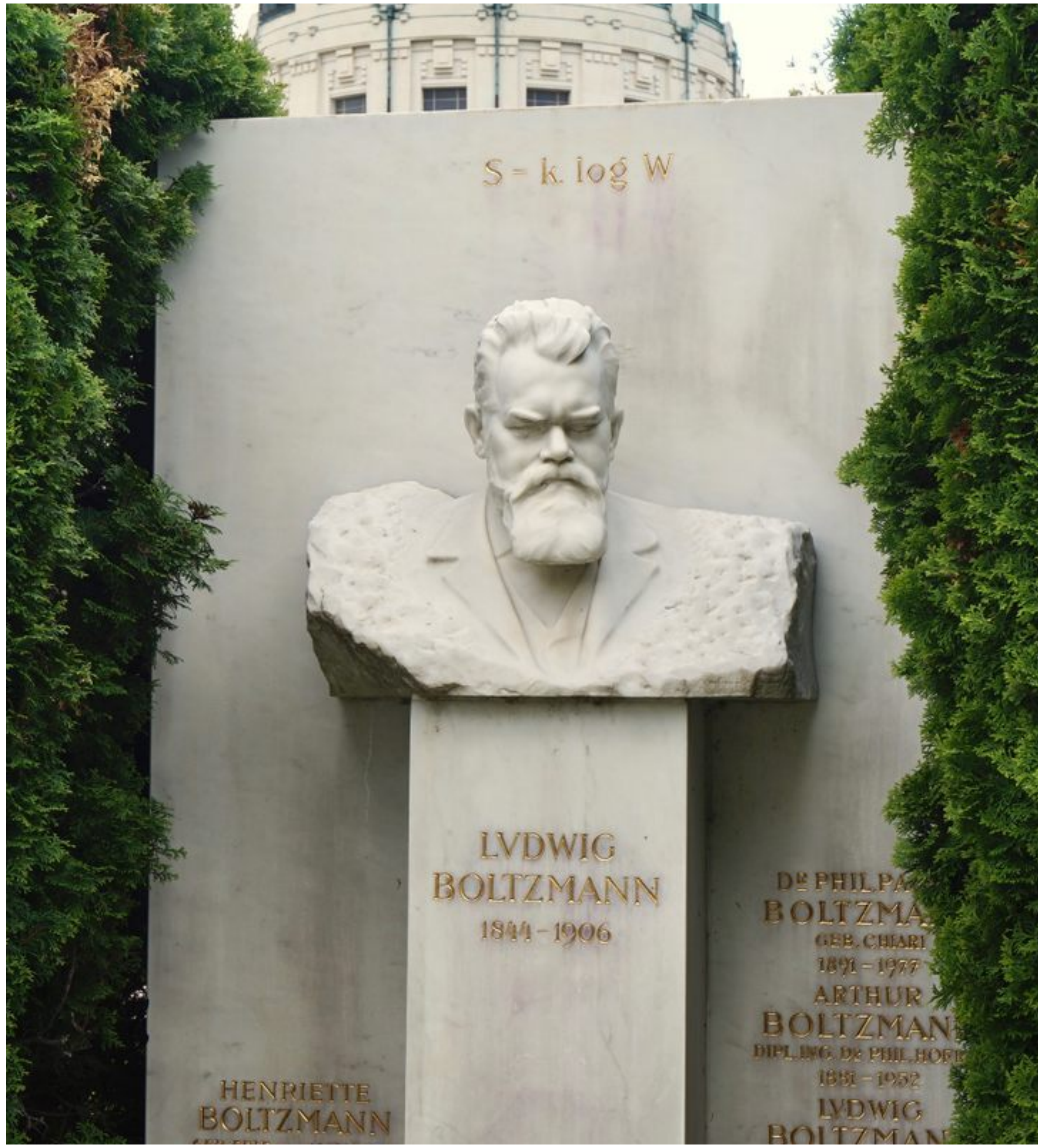


Ludwig Boltzmann
20 February 1844 - September 5, 1906
Vienna, Austria

$$S = k_B \log W$$

Density of quantum states $D(E) = \exp(S(E)/k_B)$

Boltzmann equation (1872)
Dilute classical gas



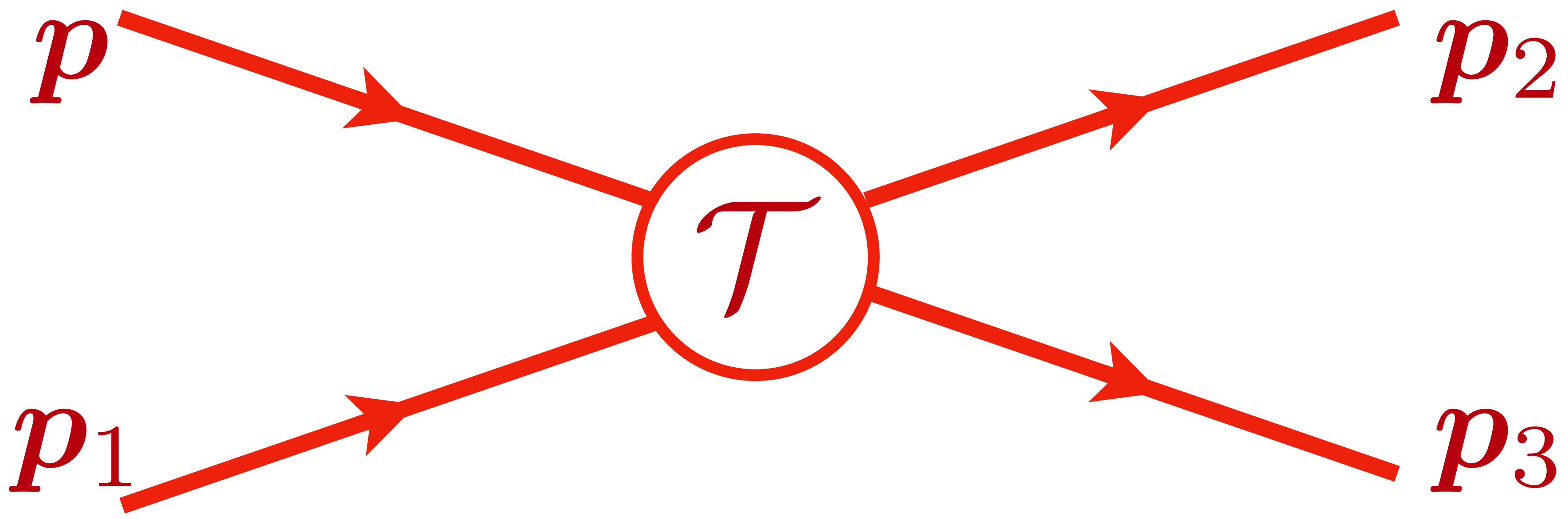
Ludwig Boltzmann
20 February 1844 - September 5, 1906
Vienna, Austria

Molecular chaos: successive collisions are statistically independent

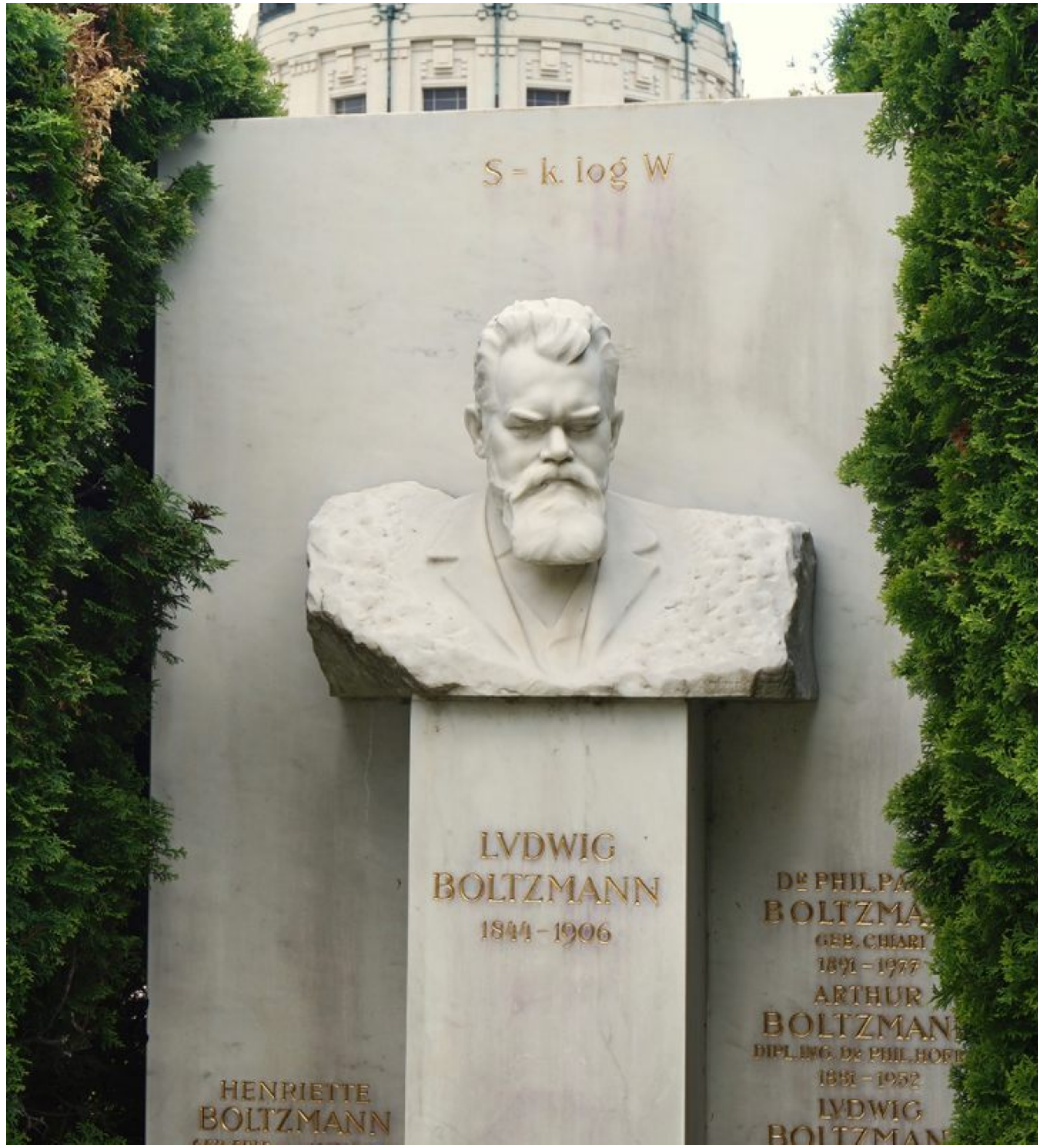
$$\frac{\partial f_{\boldsymbol{p}}}{\partial t} + \frac{\partial \varepsilon_{\boldsymbol{p}}}{\partial \boldsymbol{p}} \cdot \nabla_{\boldsymbol{r}} f_{\boldsymbol{p}} + \boldsymbol{F} \cdot \nabla_{\boldsymbol{p}} f_{\boldsymbol{p}} =$$

$$-2\pi \int_{\boldsymbol{p}_{1,2,3}} |\mathcal{T}|^2 \delta(\varepsilon_{\boldsymbol{p}} + \varepsilon_{\boldsymbol{p}_1} - \varepsilon_{\boldsymbol{p}_2} - \varepsilon_{\boldsymbol{p}_3}) \delta(\boldsymbol{p} + \boldsymbol{p}_1 - \boldsymbol{p}_2 - \boldsymbol{p}_3)$$

$$\times \left[f_{\boldsymbol{p}} f_{\boldsymbol{p}_1} - f_{\boldsymbol{p}_2} f_{\boldsymbol{p}_3} \right]$$



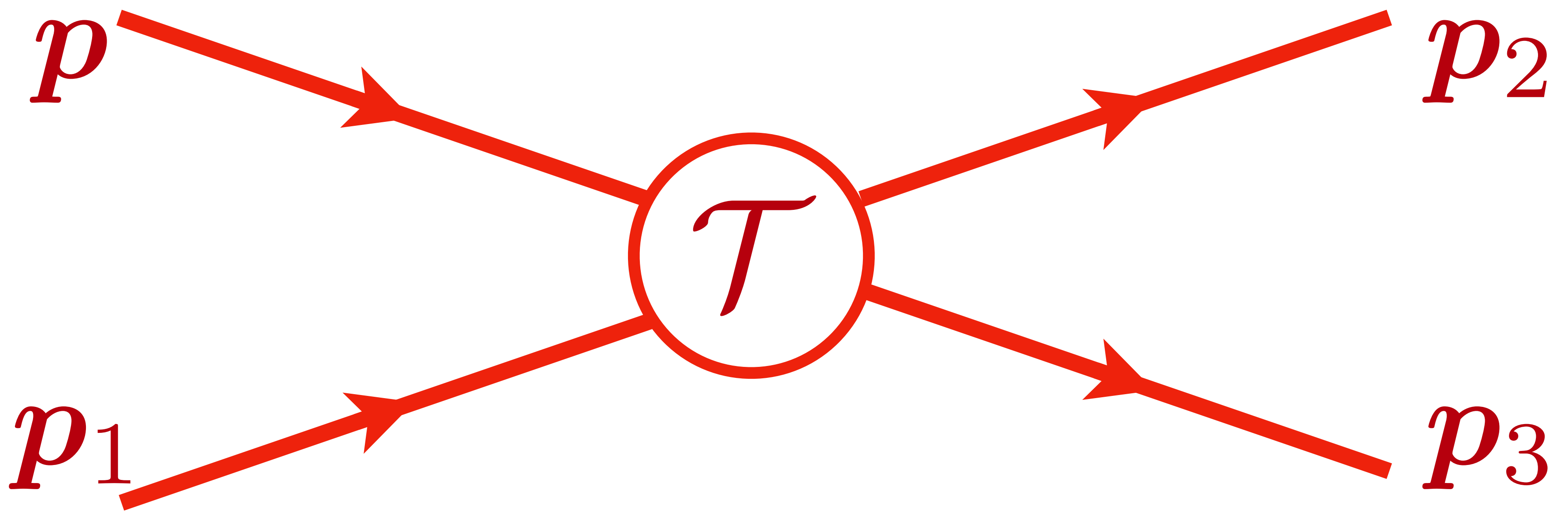
Quantum Boltzmann equation (Landau)
Dense gas of electrons



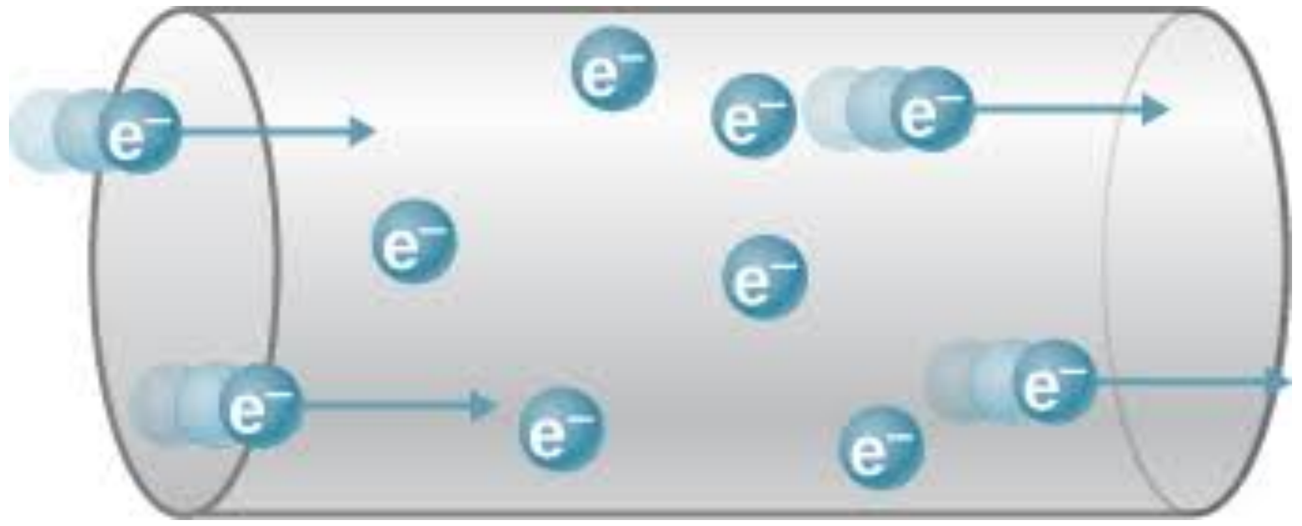
Ludwig Boltzmann
20 February 1844 - September 5, 1906
Vienna, Austria

Neglects quantum interference (entanglement)
between successive collisions

$$\frac{\partial f_{\boldsymbol{p}}}{\partial t} + \frac{\partial \varepsilon_{\boldsymbol{p}}}{\partial \boldsymbol{p}} \cdot \nabla_{\boldsymbol{r}} f_{\boldsymbol{p}} + \boldsymbol{F} \cdot \nabla_{\boldsymbol{p}} f_{\boldsymbol{p}} =$$
$$-2\pi \int_{\boldsymbol{p}_{1,2,3}} |\mathcal{T}|^2 \delta(\varepsilon_{\boldsymbol{p}} + \varepsilon_{\boldsymbol{p}_1} - \varepsilon_{\boldsymbol{p}_2} - \varepsilon_{\boldsymbol{p}_3}) \delta(\boldsymbol{p} + \boldsymbol{p}_1 - \boldsymbol{p}_2 - \boldsymbol{p}_3)$$
$$\times \left[f_{\boldsymbol{p}} f_{\boldsymbol{p}_1} (1 - f_{\boldsymbol{p}_2})(1 - f_{\boldsymbol{p}_3}) - f_{\boldsymbol{p}_2} f_{\boldsymbol{p}_3} (1 - f_{\boldsymbol{p}})(1 - f_{\boldsymbol{p}_1}) \right]$$



Current flow with electrons in ordinary metals

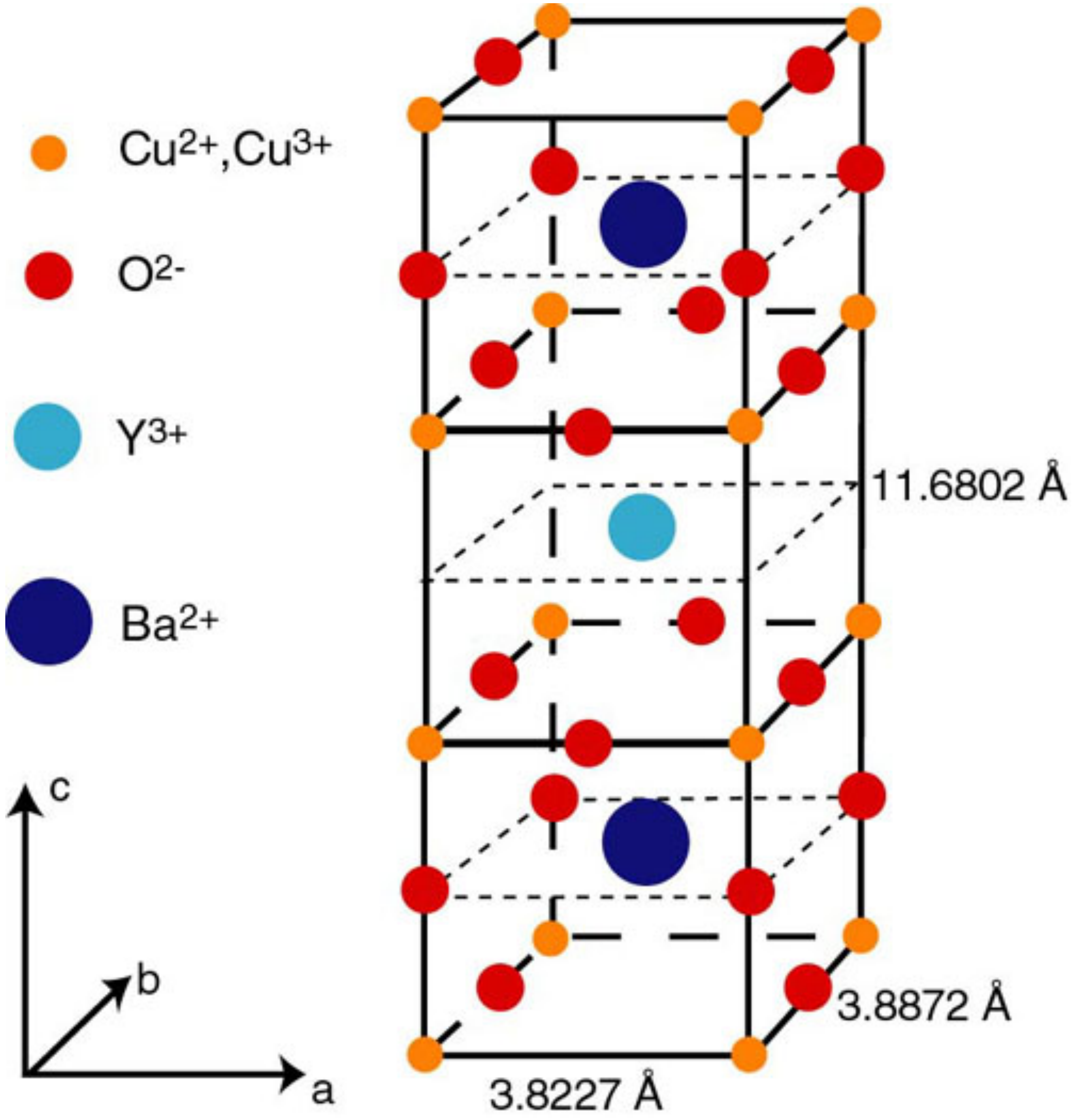


Flow of electrons described by Boltzmann equation $\Rightarrow$
typical scattering time $\tau \sim 1/T^2$, resistivity $\rho(T) = \rho(0) + AT^2$

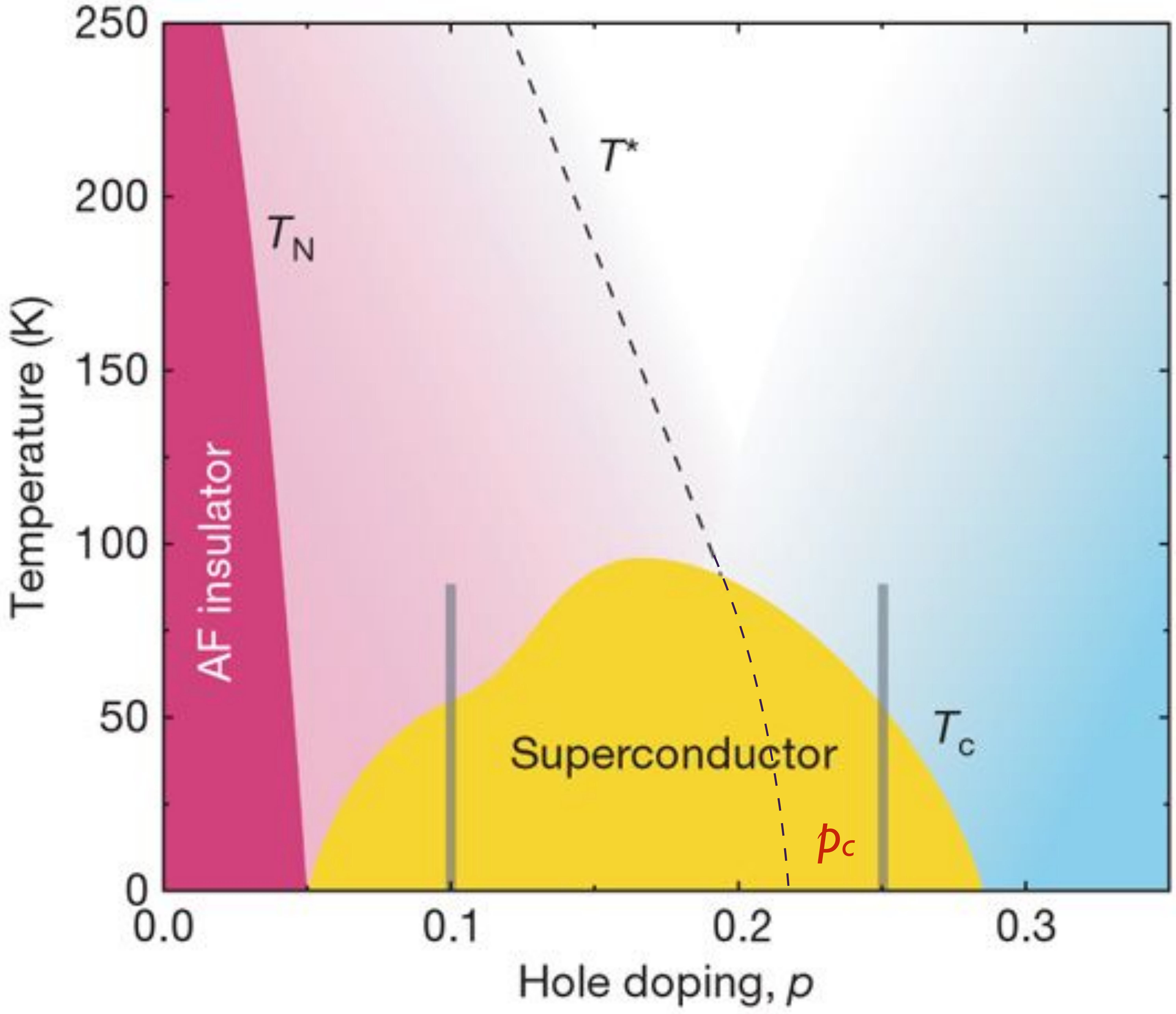
The time τ is much longer than a limiting 'Planckian time' $\dfrac{\hbar}{k_B T}$.

The long scattering time implies that individual electrons are well-defined.

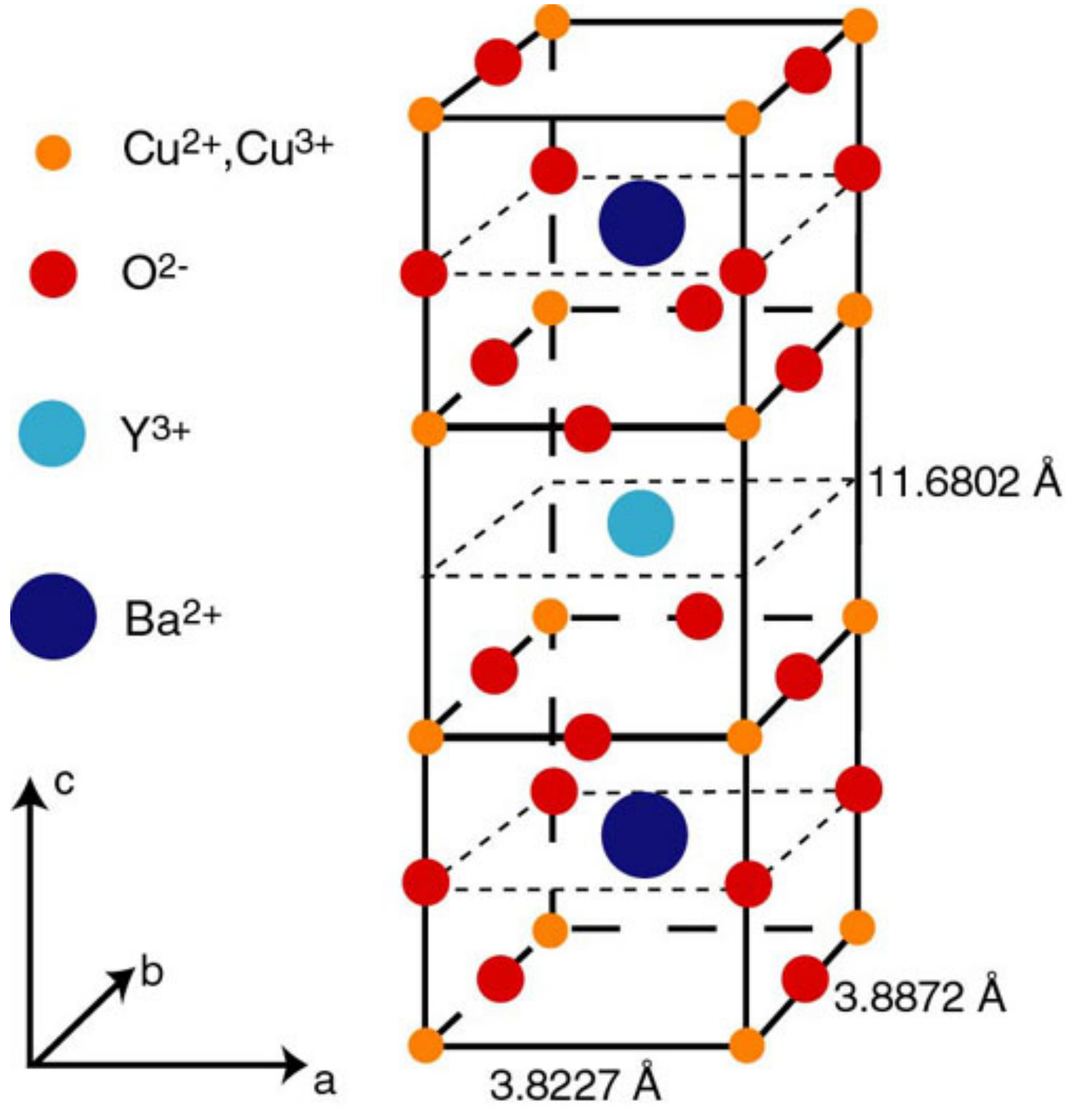
Cuprate high temperature superconductors



$YBa_2Cu_3O_{6+x}$

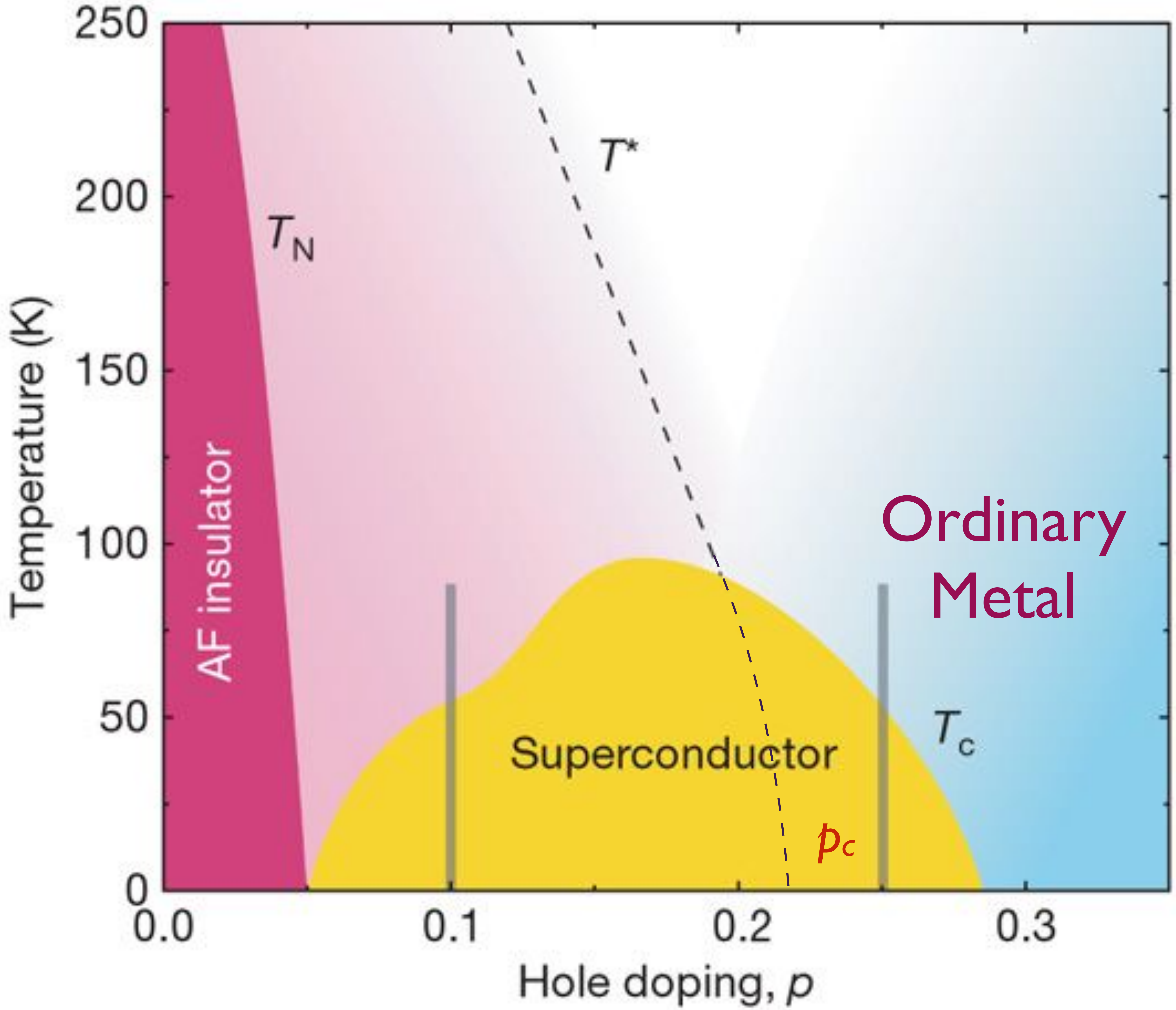


Temperature (K)
250
200
150
100
50
0
0.0
0.1
0.2
0.3
Hole doping, p
T_N
T^*
AF insulator
Superconductor
T_c
p_c



Cu2+,Cu3+
O2-
Y3+
Ba2+
c
b
a
11.6802 Å
3.8872 Å
3.8227 Å

$YBa_2Cu_3O_{6+x}$


250
200
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Temperature (K)
0.0
0.1
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Hole doping, p
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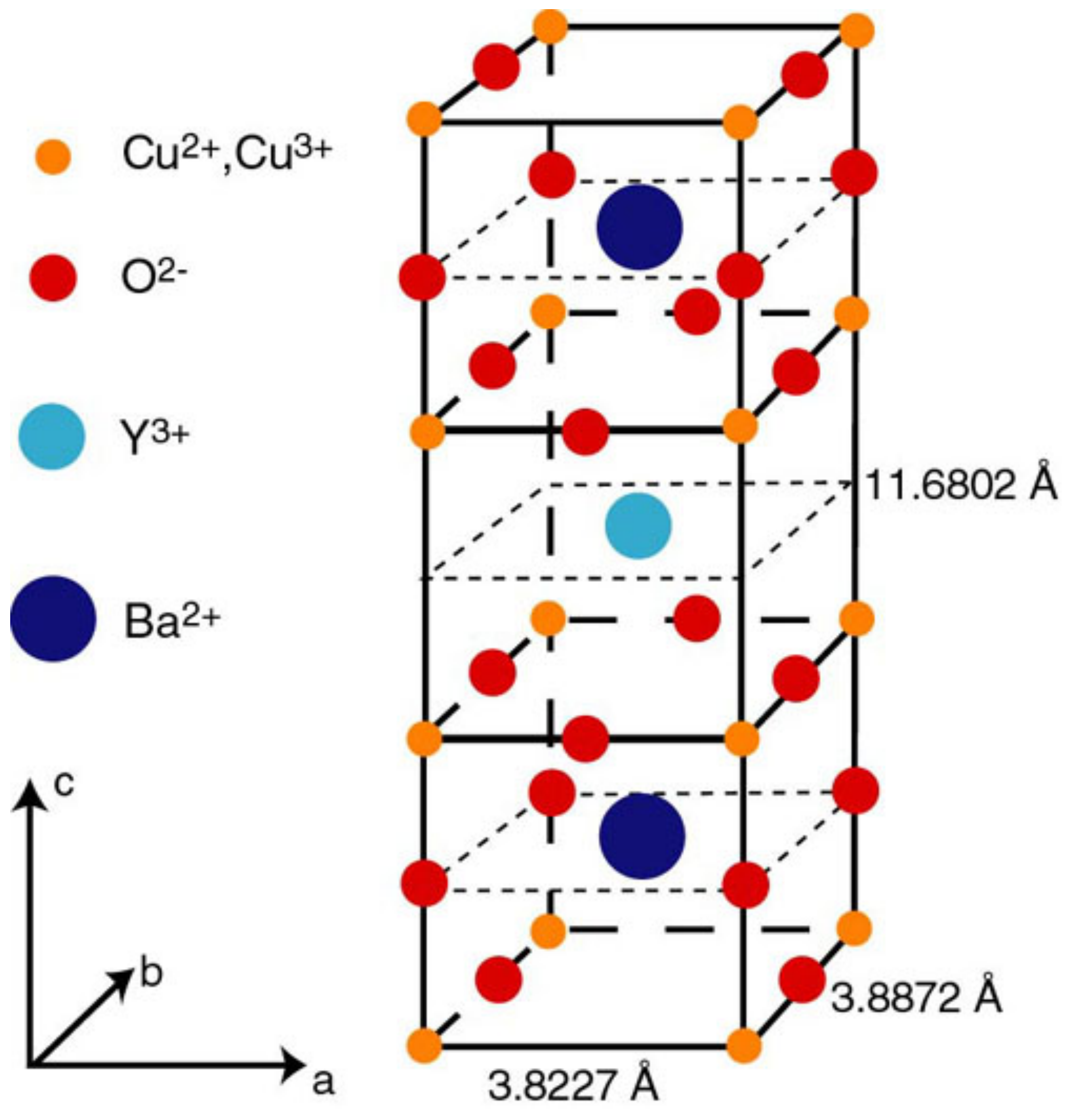
Cu^{2+},Cu^{3+}
O^{2-}
Y^{3+}
Ba^{2+}
c
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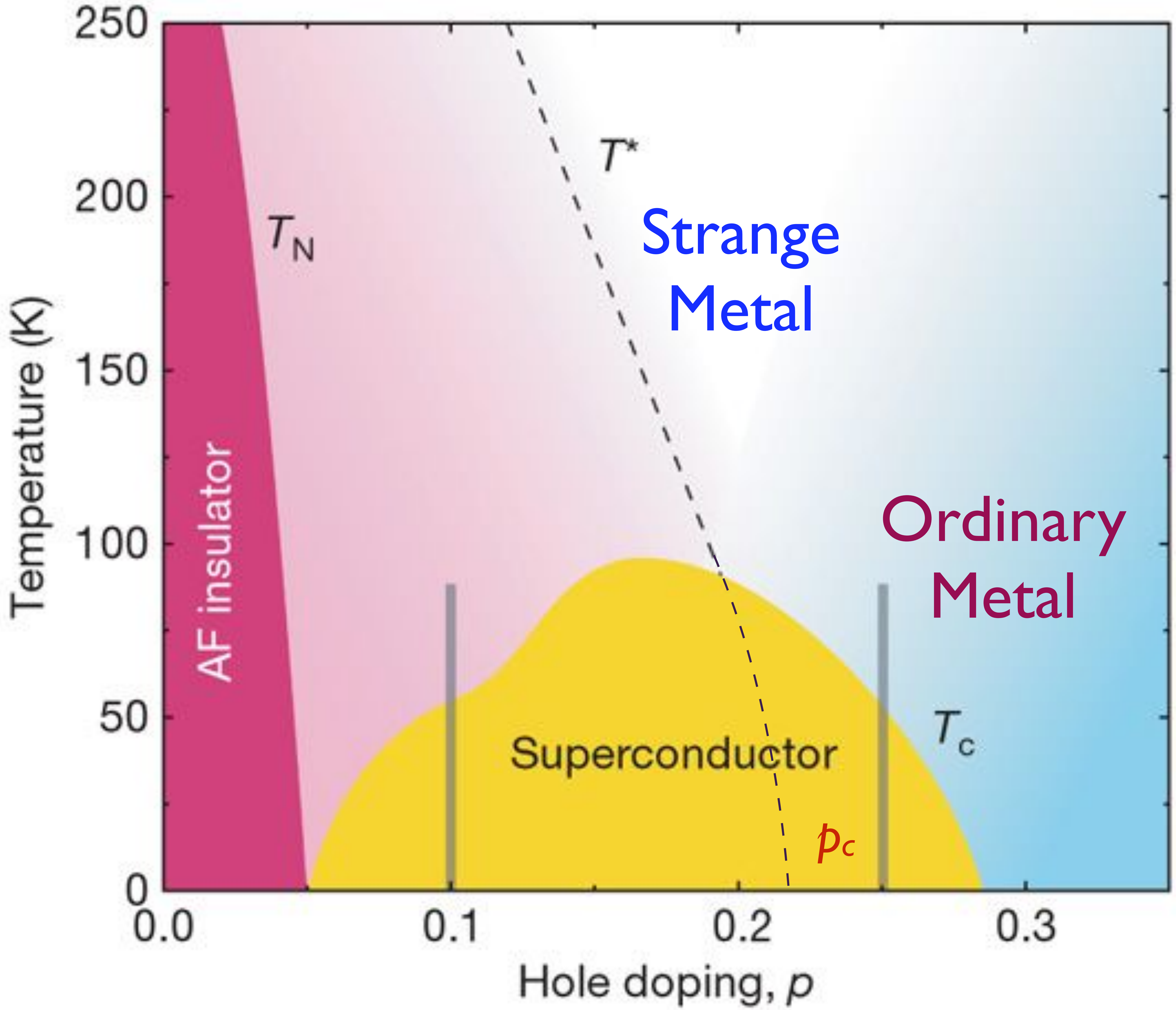
250
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Temperature (K)
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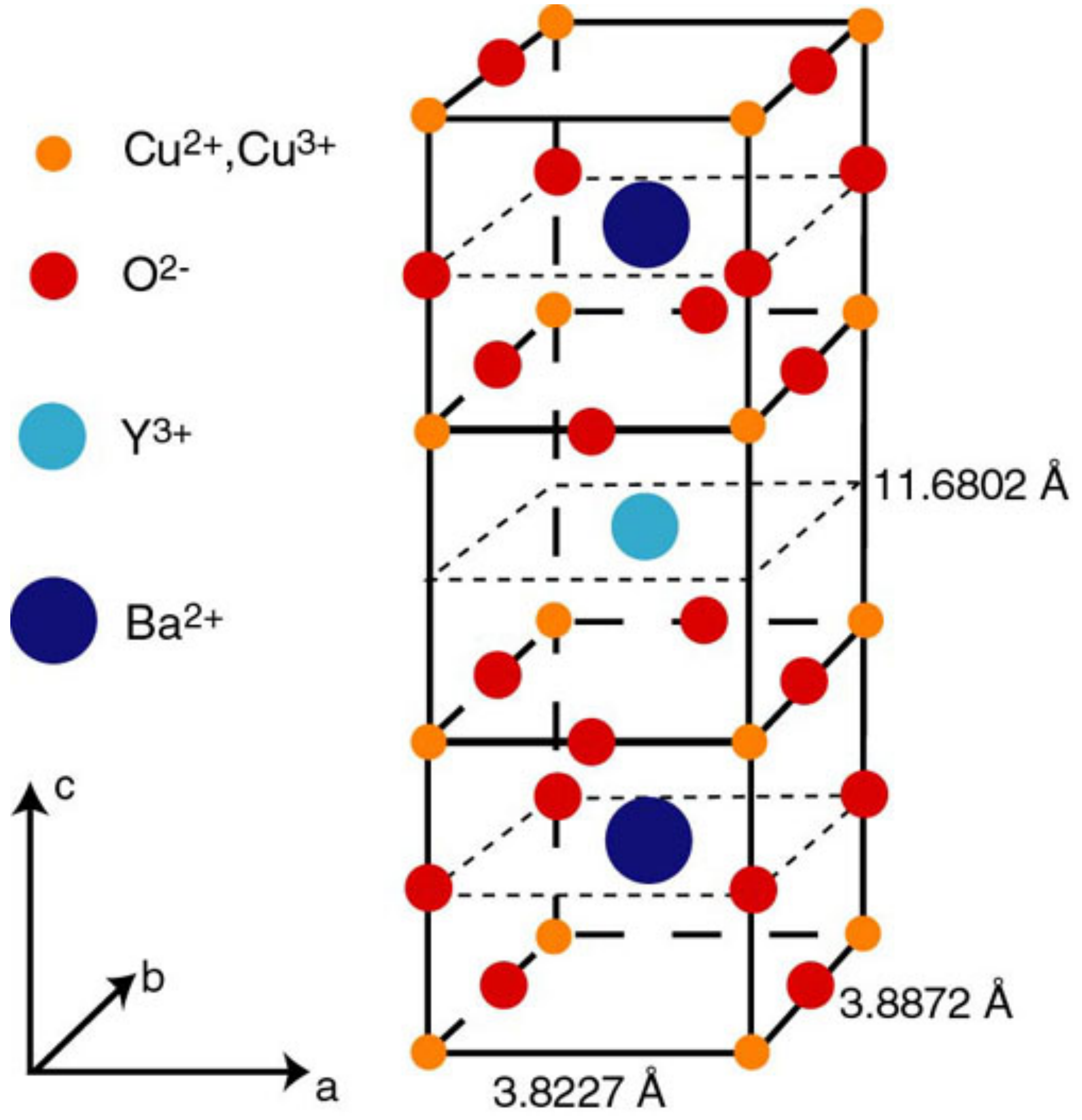
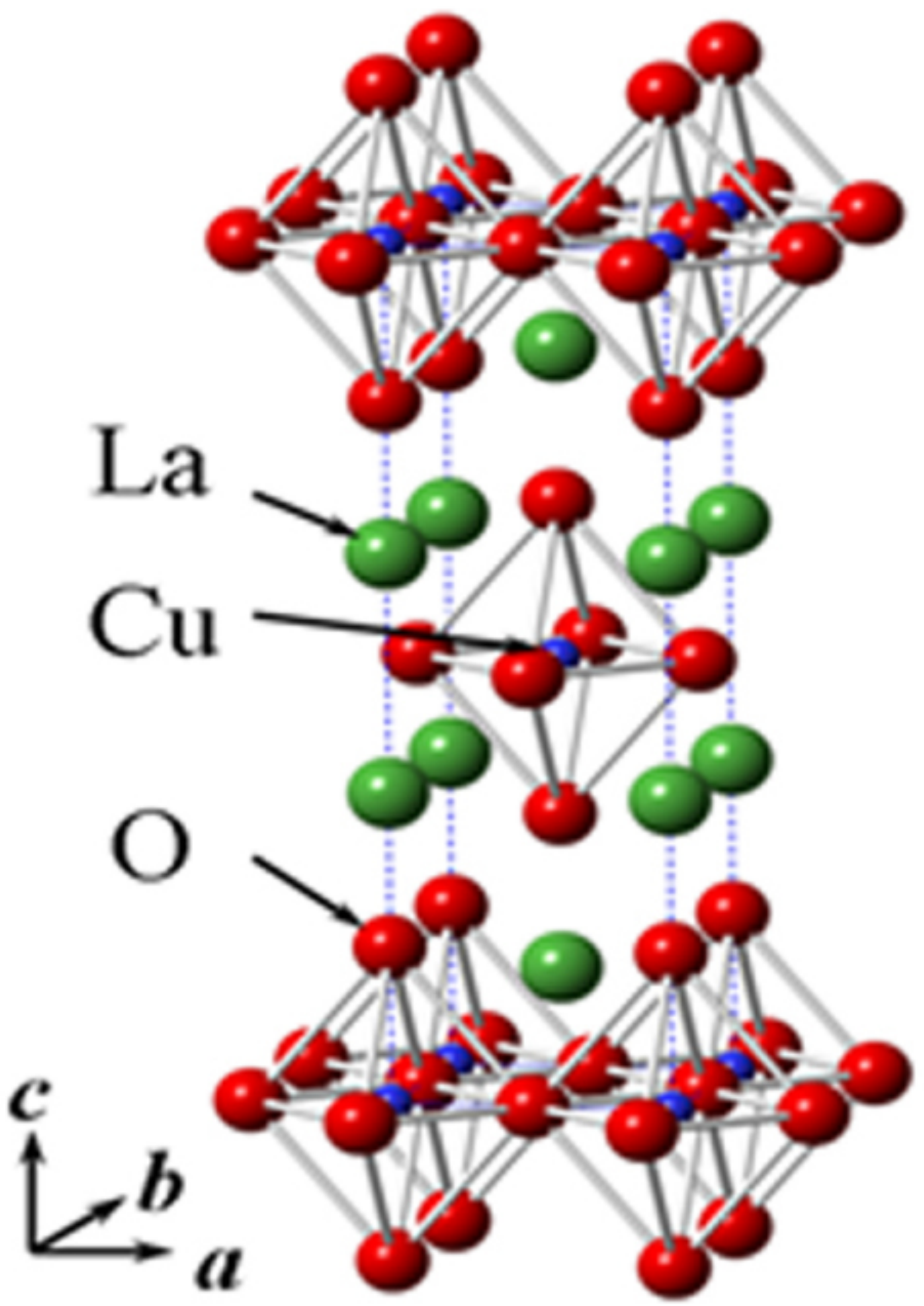

Cu^{2+},Cu^{3+}
O^{2-}
Y^{3+}
Ba^{2+}
c
b
a
11.6802 Å
3.8872 Å
3.8227 Å

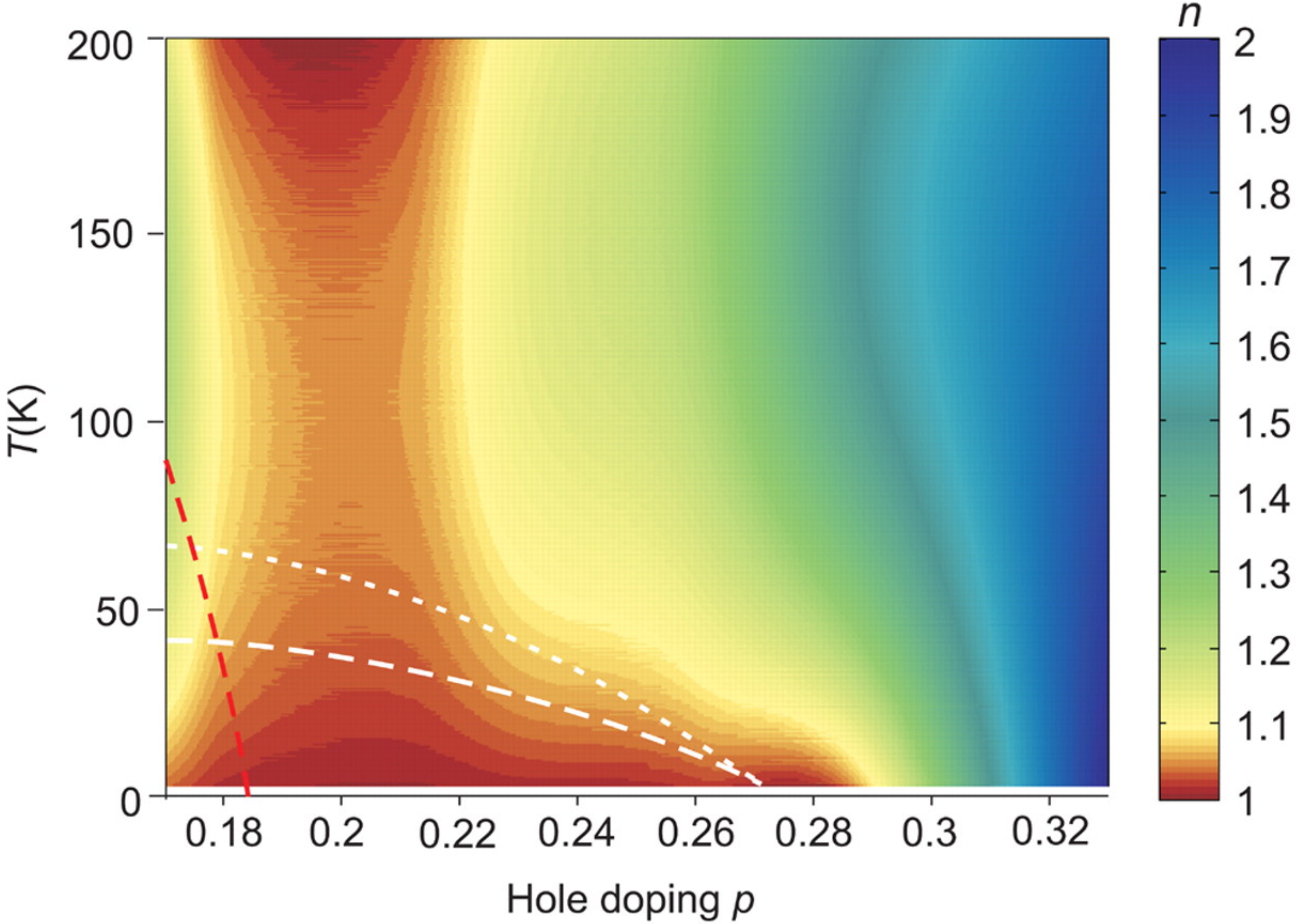
$YBa_2Cu_3O_{6+x}$

Anomalous Criticality in the Electrical Resistivity of $La_{2-x}Sr_xCuO_4$

R. A. Cooper,[1] Y. Wang,[1] B. Vignolle,[2] O. J. Lipscombe,[1] S. M. Hayden,[1] Y. Tanabe,[3] T. Adachi,[3]
Y. Koike,[3] M. Nohara,[4]* H. Takagi,[4] Cyril Proust,[2] N. E. Hussey[1]†

SCIENCE VOL 323 **603** 2009

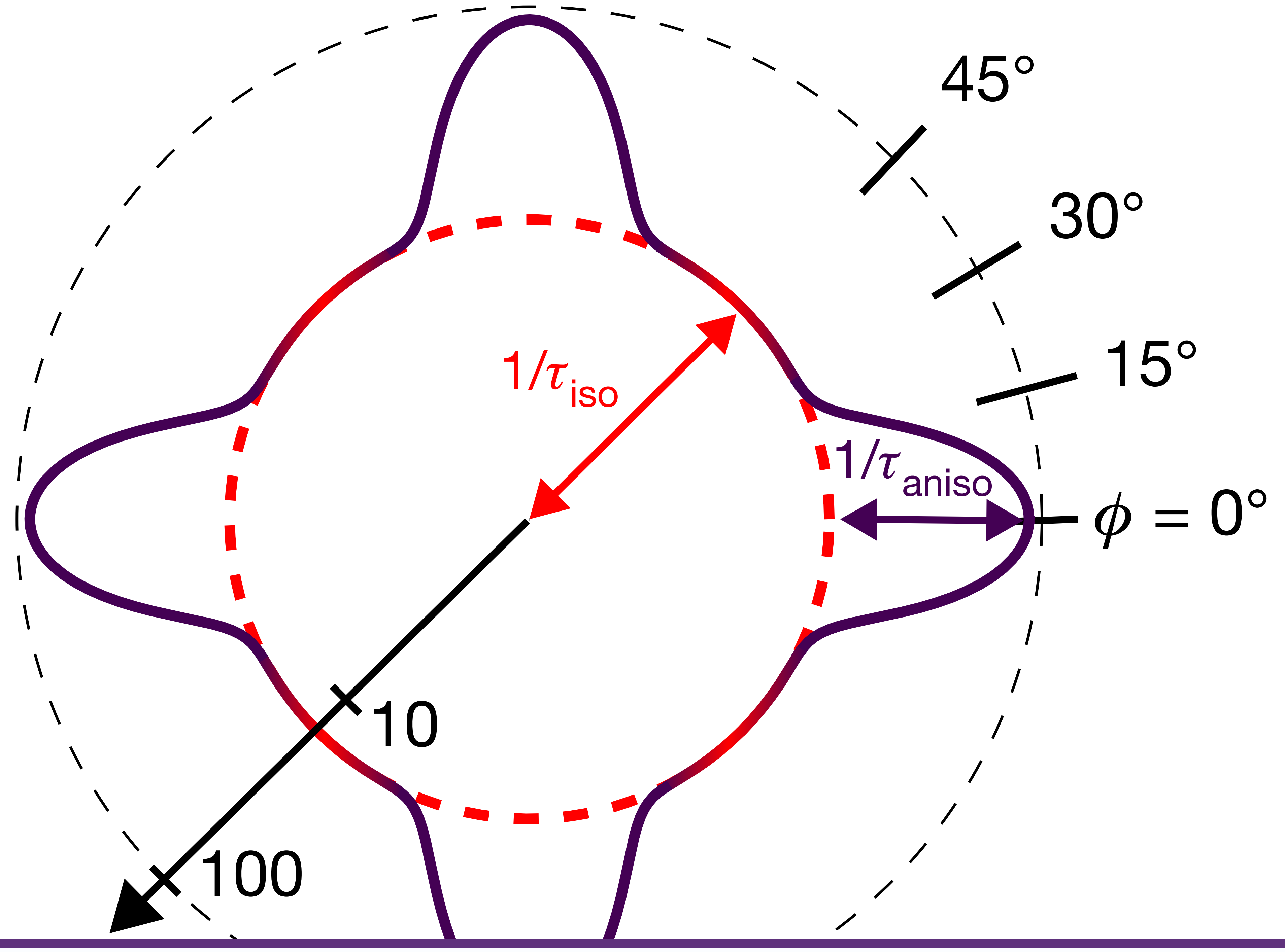


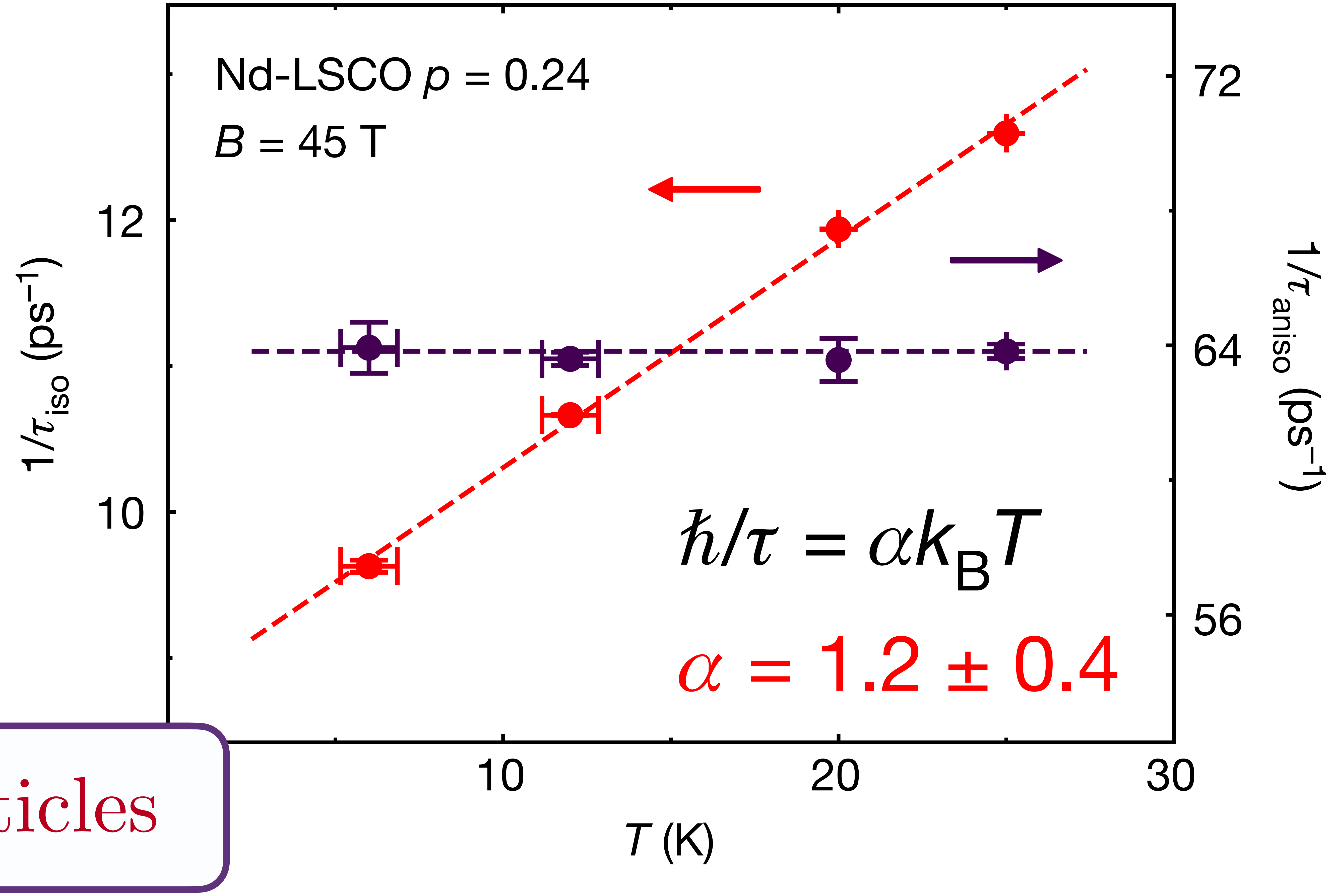


Linear-in temperature resistivity from an isotropic Planckian scattering rate

Nature **595**, 667-672 (2021)

G. Grissonnanche, Y. Fang, A. Legros, S. Verret, F. Laliberté, C. Collignon, J. Zhou, D. Graf, P. Goddard, L. Taillefer, B. J. Ramshaw





Current flow without quasiparticles

No Boltzmann-Landau quasiparticle description $\Rightarrow$
Many particle quantum entanglement
from quantum interference between “collisions”

Sachdev-Ye-Kitaev Model

The SYK model

Sachdev, Ye (1993); Kitaev (2015)

A solvable model of multi-particle quantum entanglement.

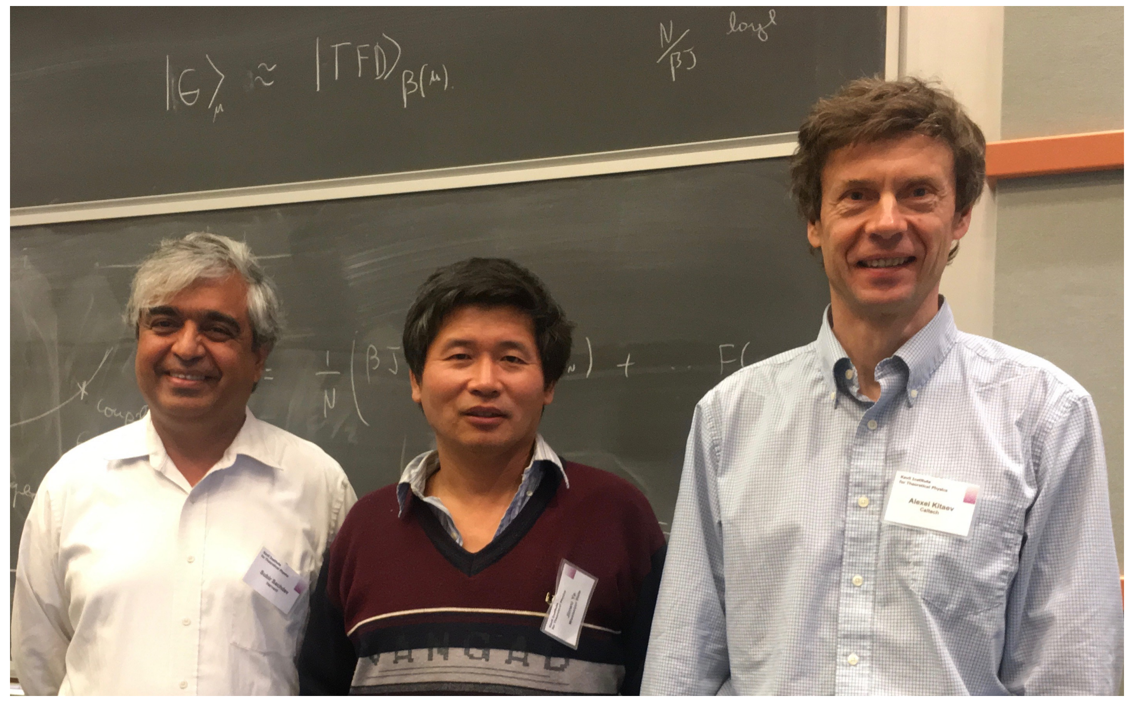
Yields a metal in which current is carried
not by individual electrons,
but by an entangled "quantum soup"

The SYK model

Sachdev, Ye (1993); Kitaev (2015)

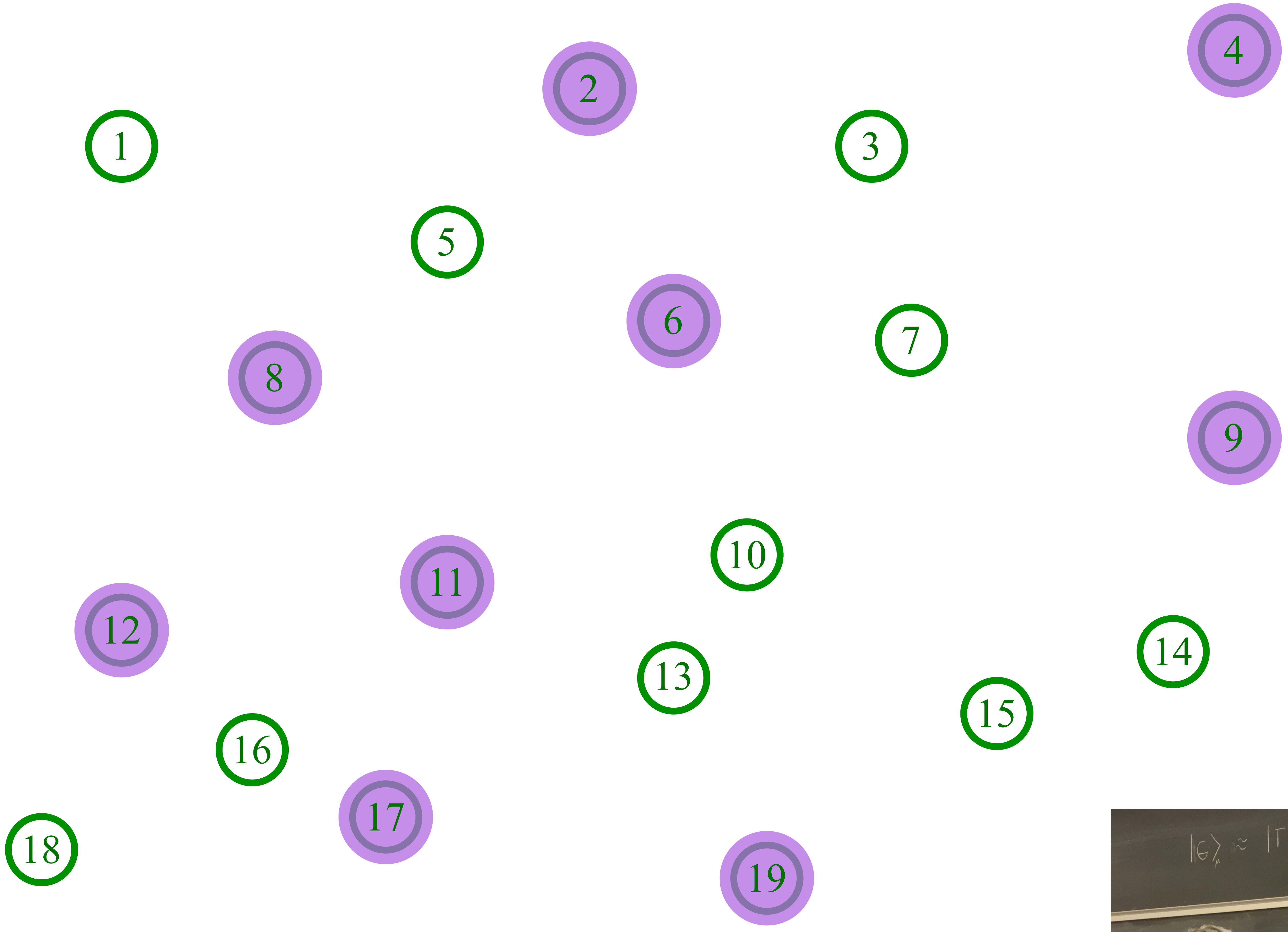
④
②
① ③
⑤
⑥ ⑦
⑧
⑨
⑩
⑪
⑫ ⑭
⑬
⑮
⑯
⑰
⑱ ⑲

Pick a set of random positions

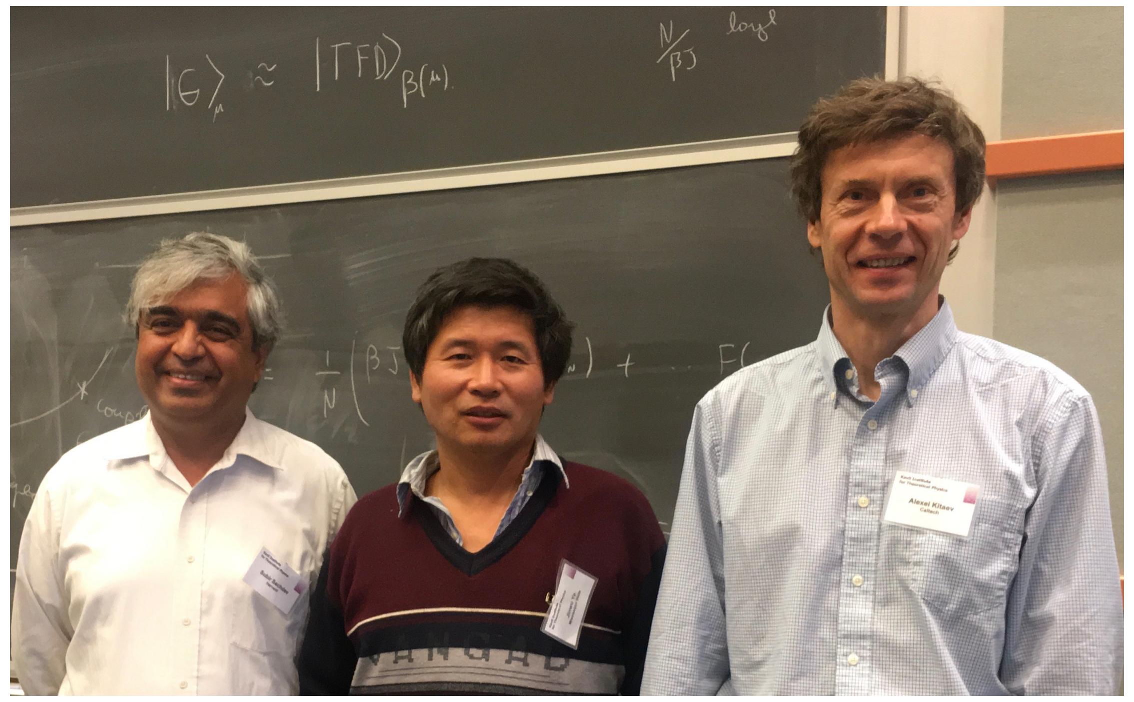


The SYK model

Sachdev, Ye (1993); Kitaev (2015)

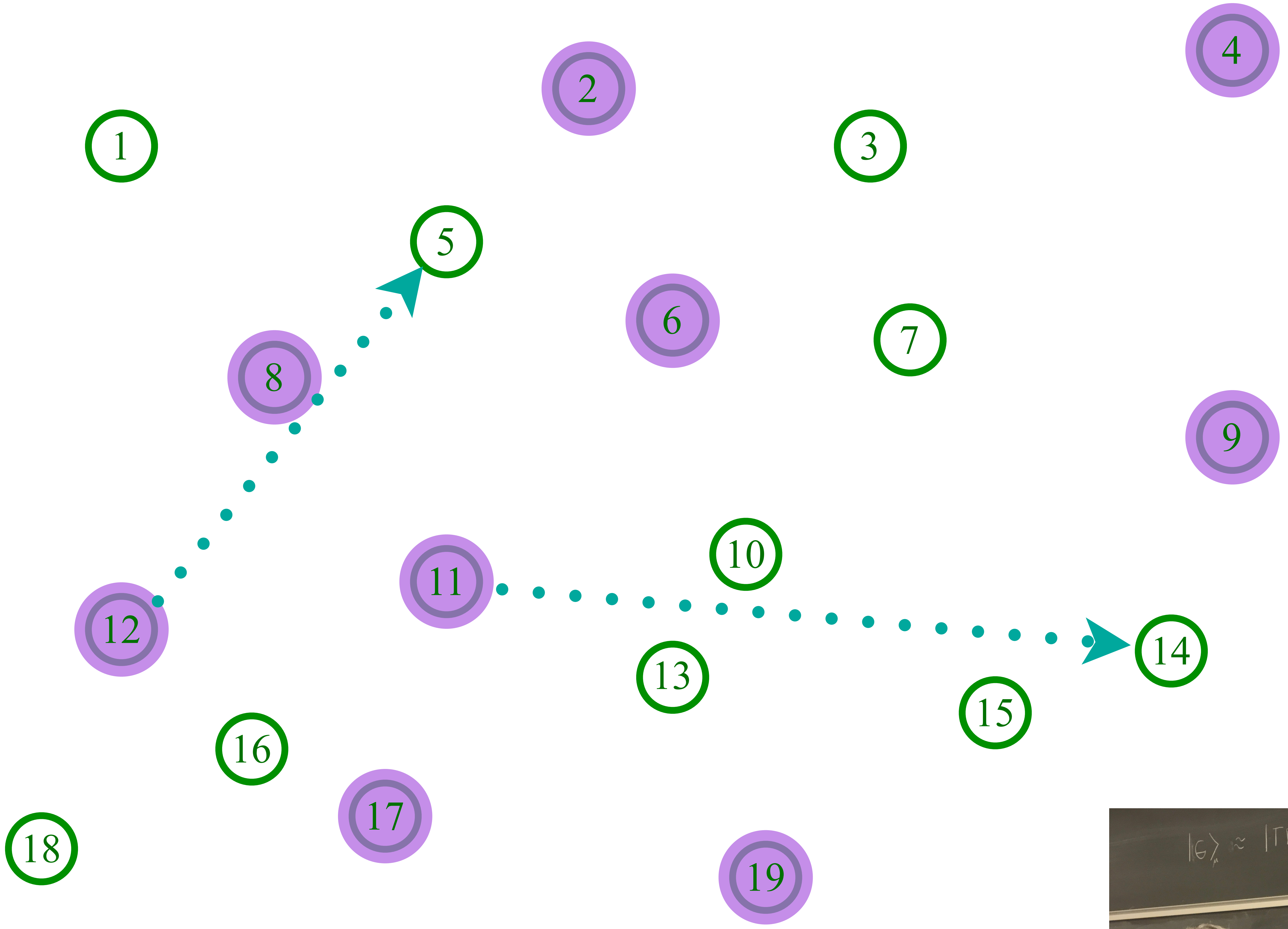


Place electrons randomly on some sites



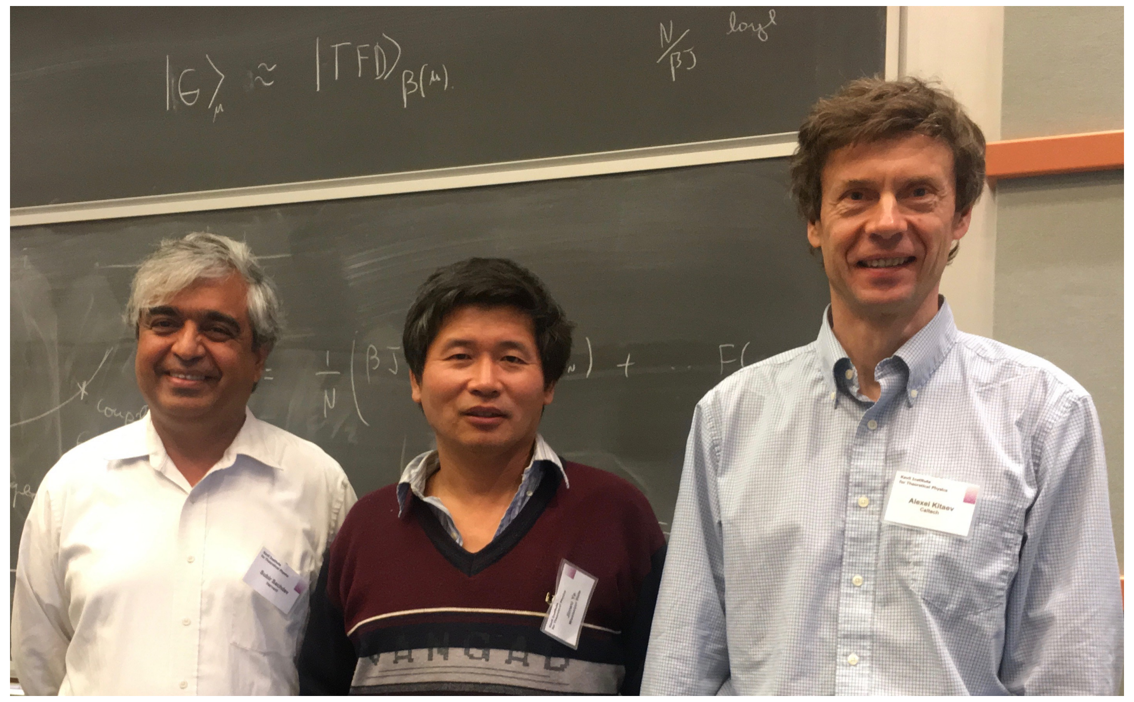
The SYK model

Sachdev, Ye (1993); Kitaev (2015)



$$U_{11,12;5,14}$$

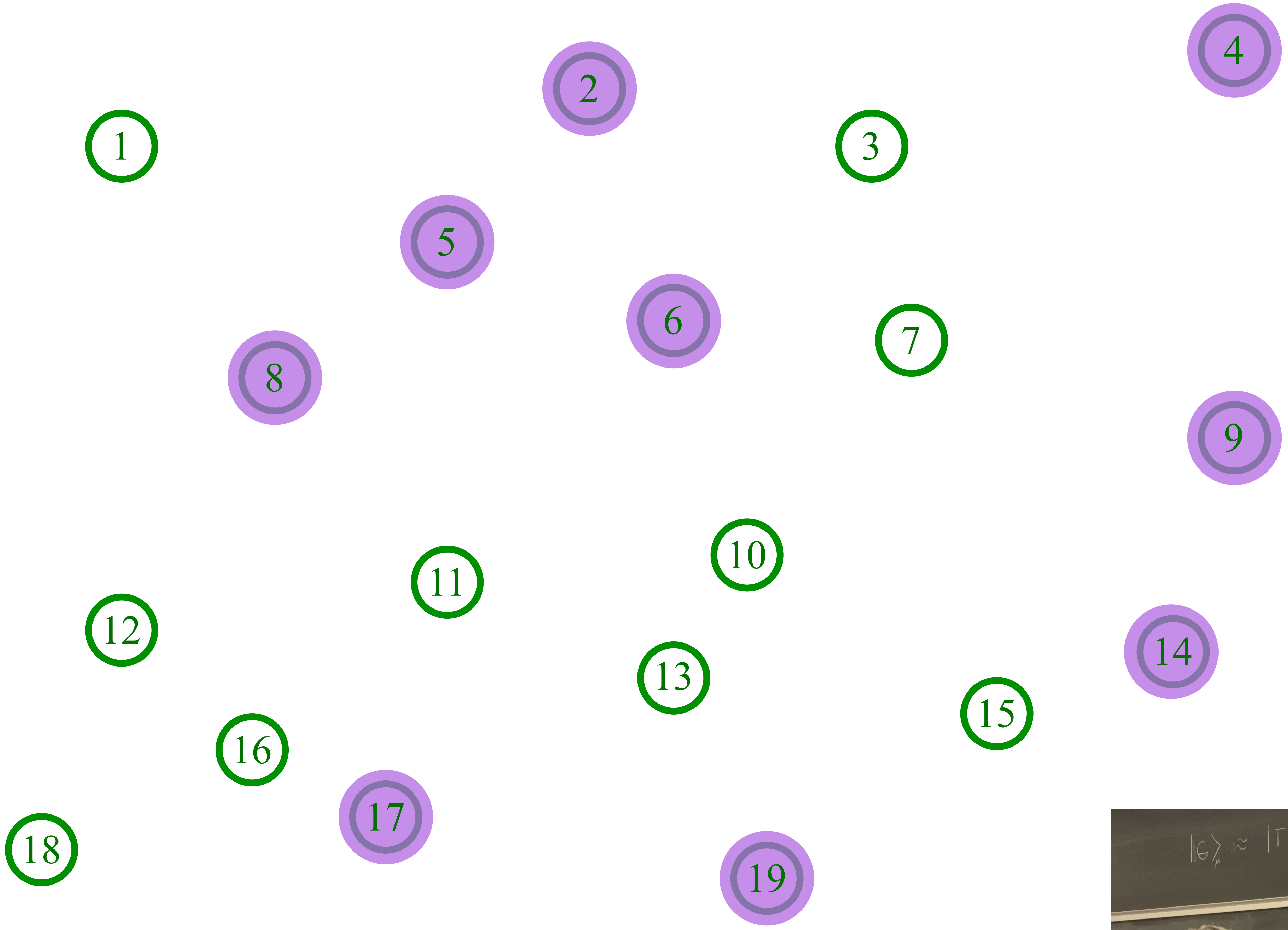
Place electrons randomly on some sites



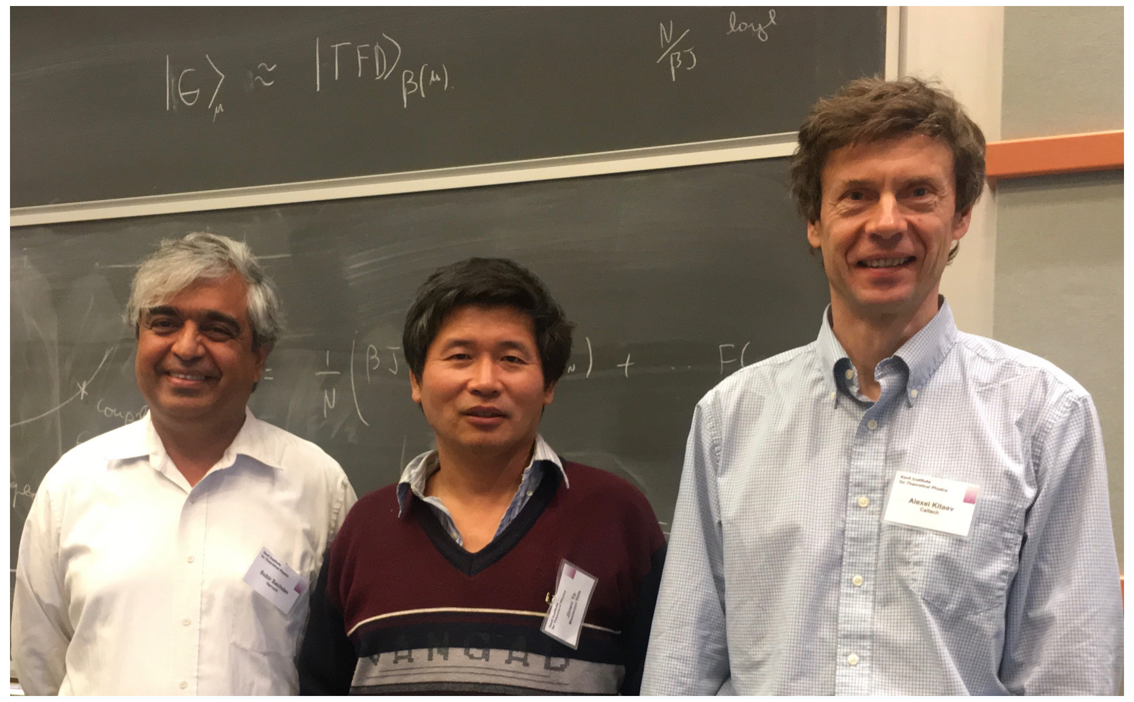
The SYK model

Sachdev, Ye (1993); Kitaev (2015)

$$U_{11,12;5,14}$$



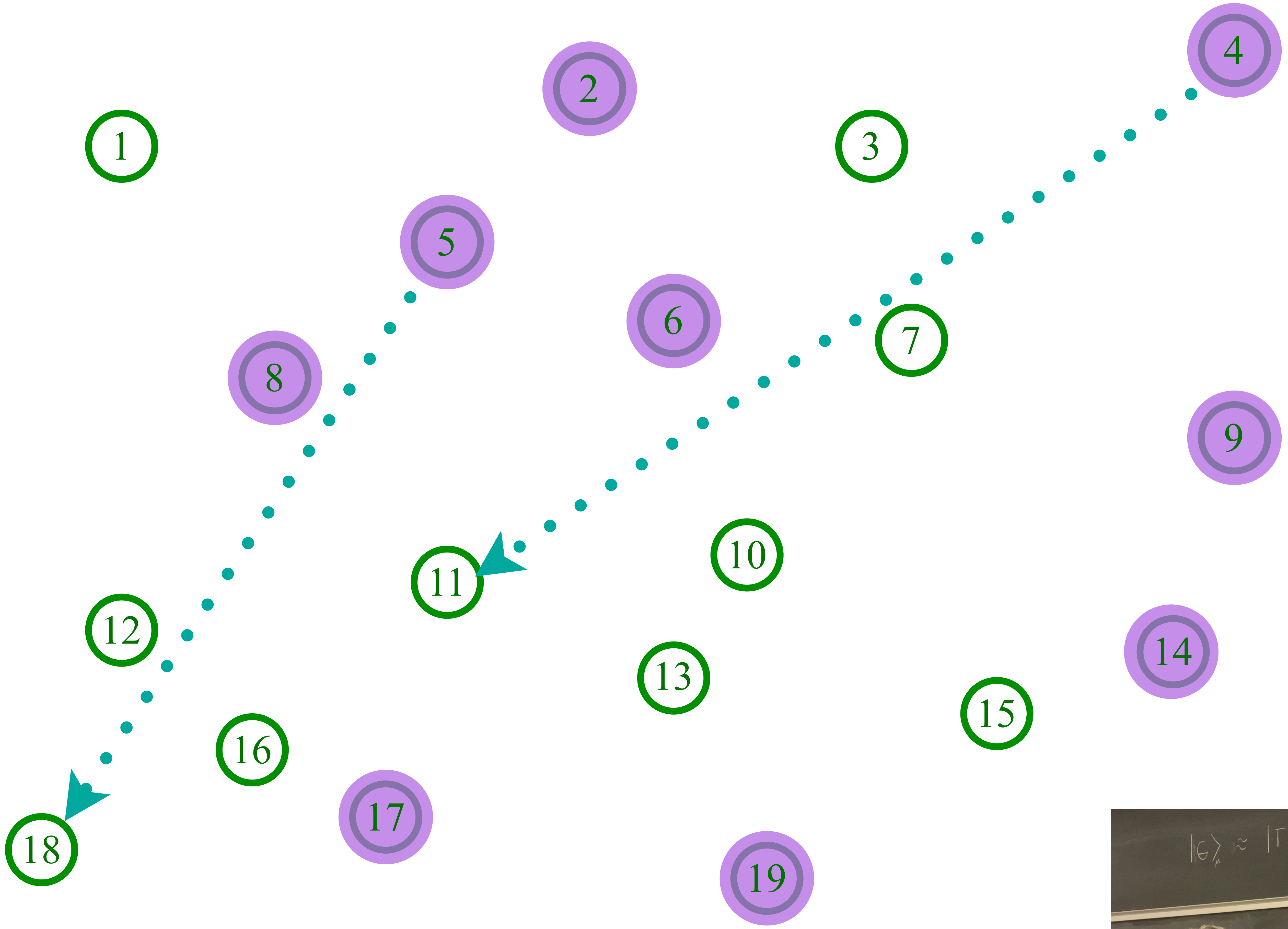
Entangle electrons pairwise randomly



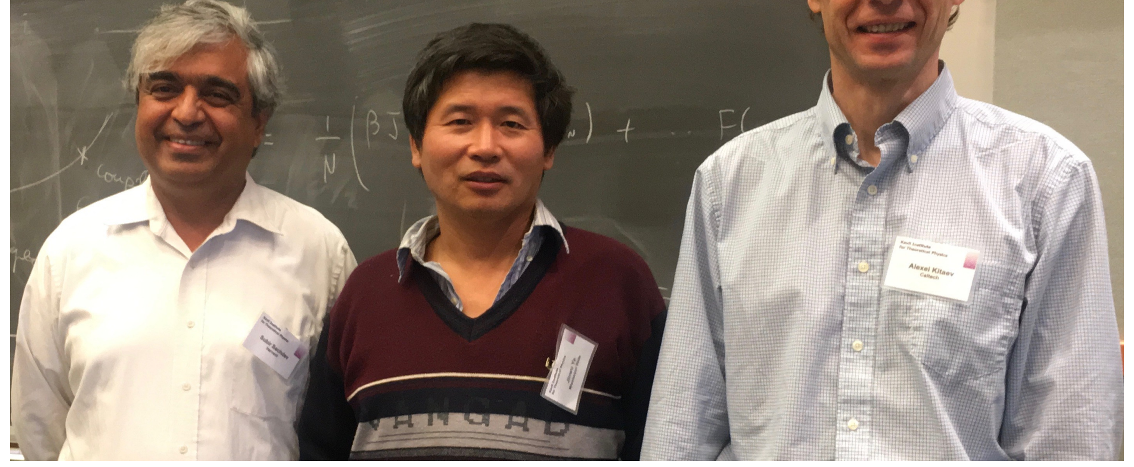
The SYK model

Sachdev, Ye (1993); Kitaev (2015)

$U_{4,5;11,18}$

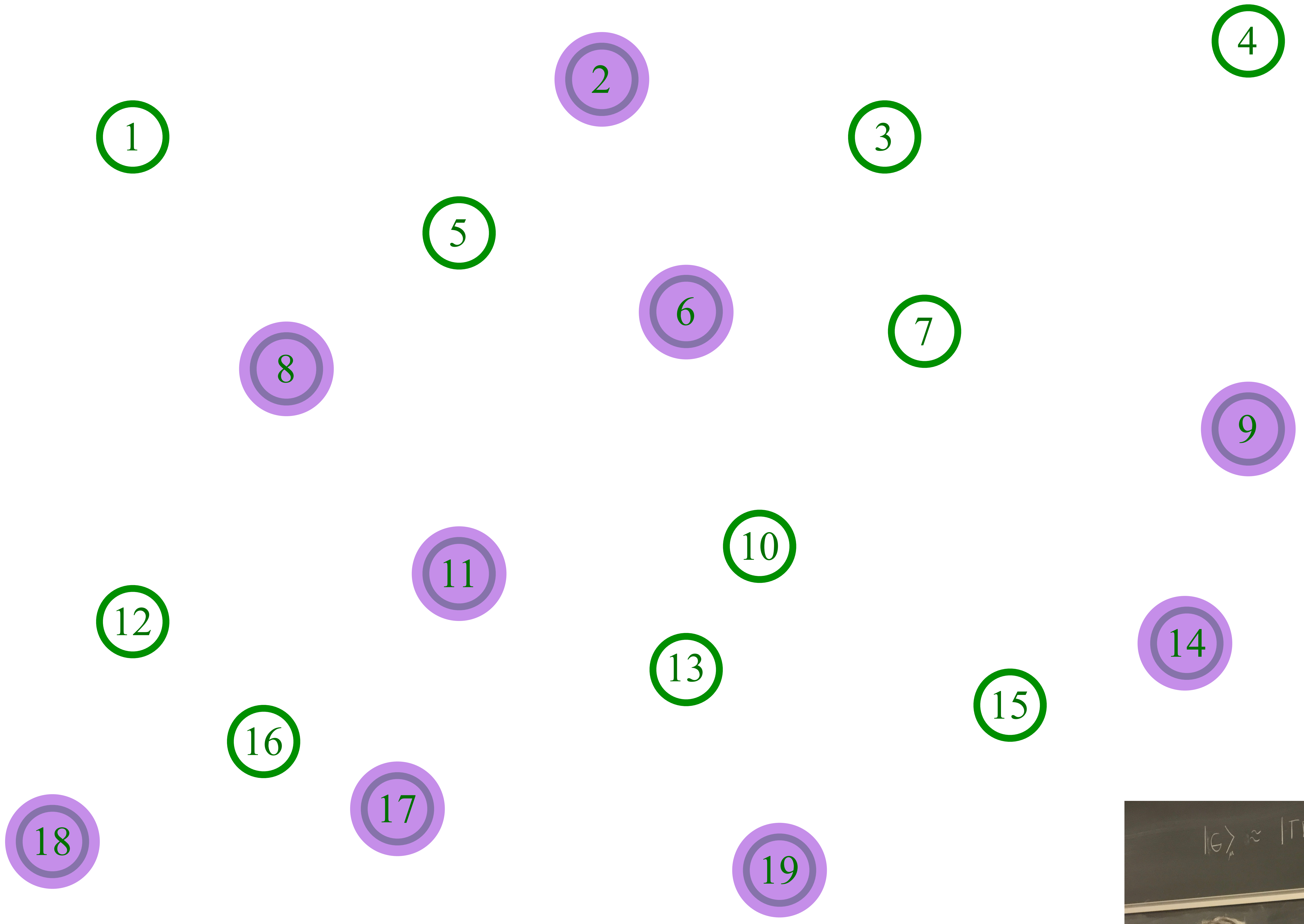


Entangle electrons pairwise randomly



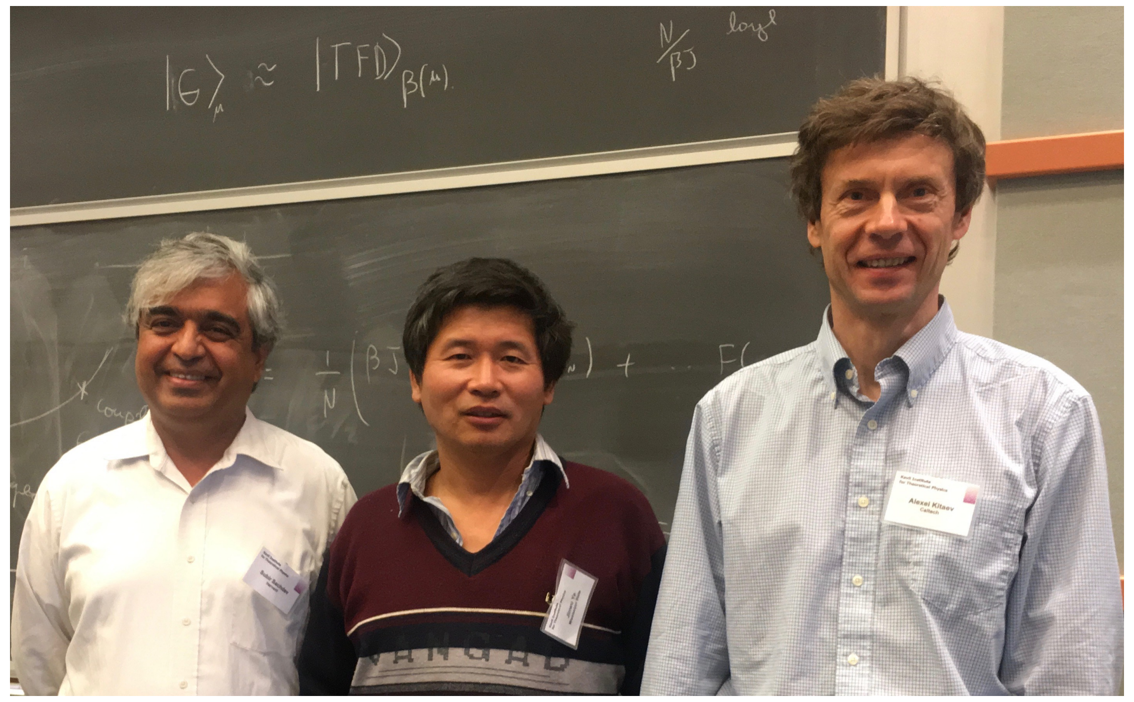
The SYK model

Sachdev, Ye (1993); Kitaev (2015)



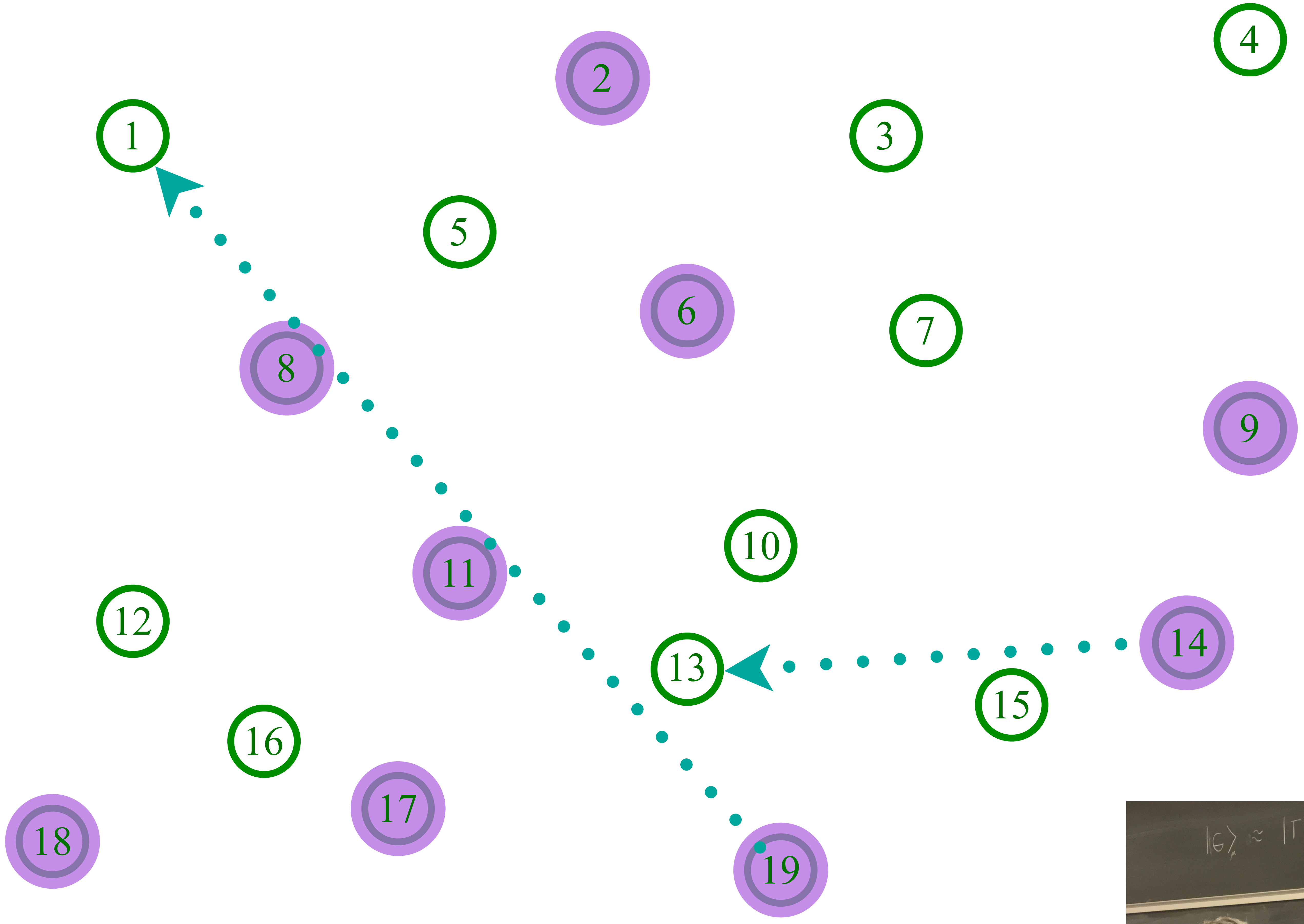
$$U_{4,5;11,18}$$

Entangle electrons pairwise randomly



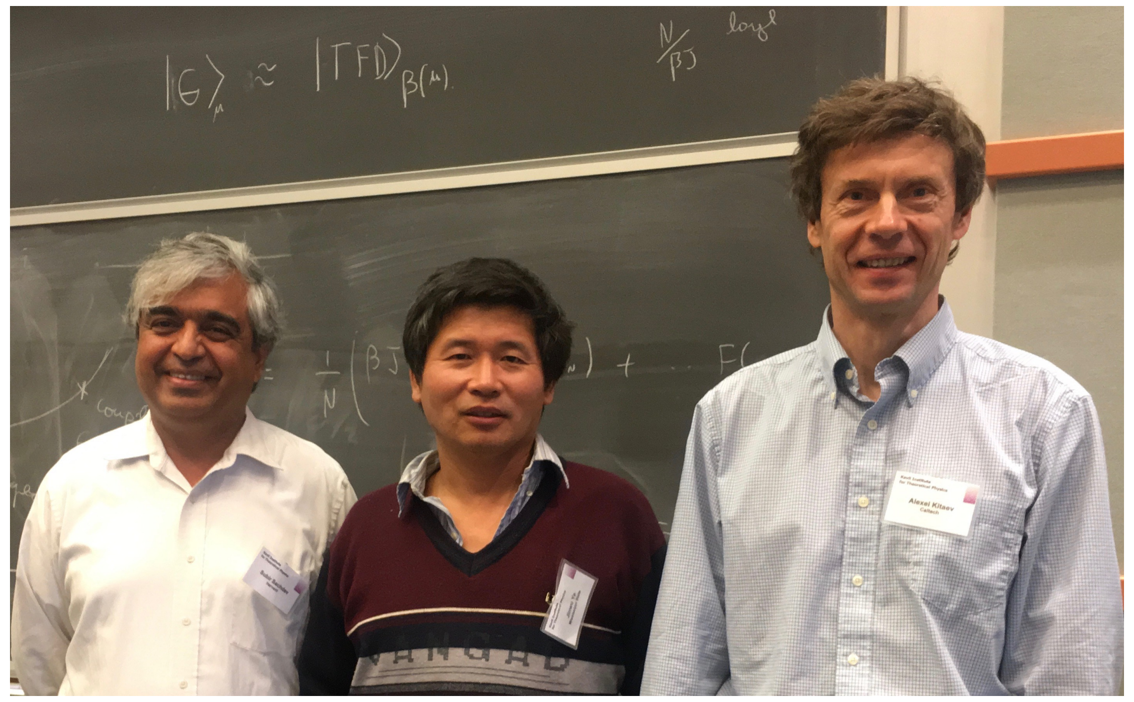
The SYK model

Sachdev, Ye (1993); Kitaev (2015)



$U_{14,19;1,13}$

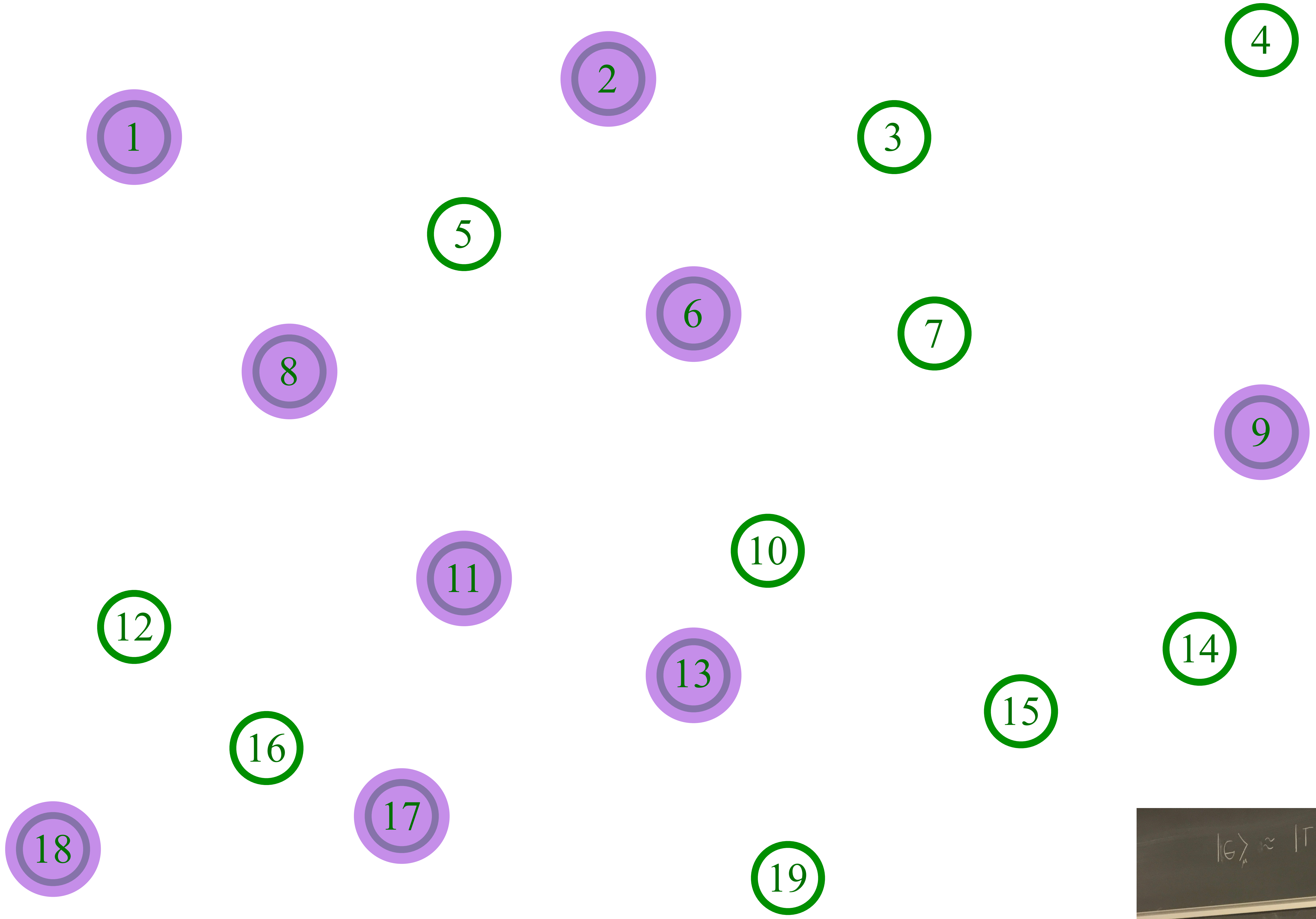
Entangle electrons pairwise randomly



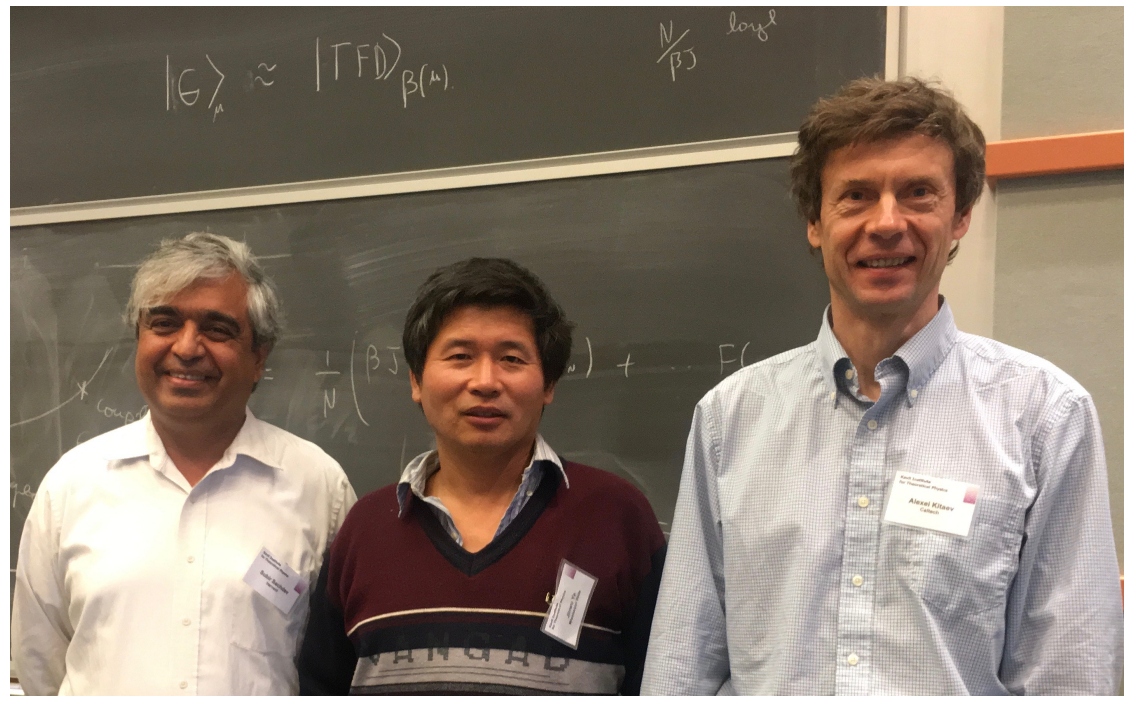
The SYK model

Sachdev, Ye (1993); Kitaev (2015)

$$U_{14,19;1,13}$$



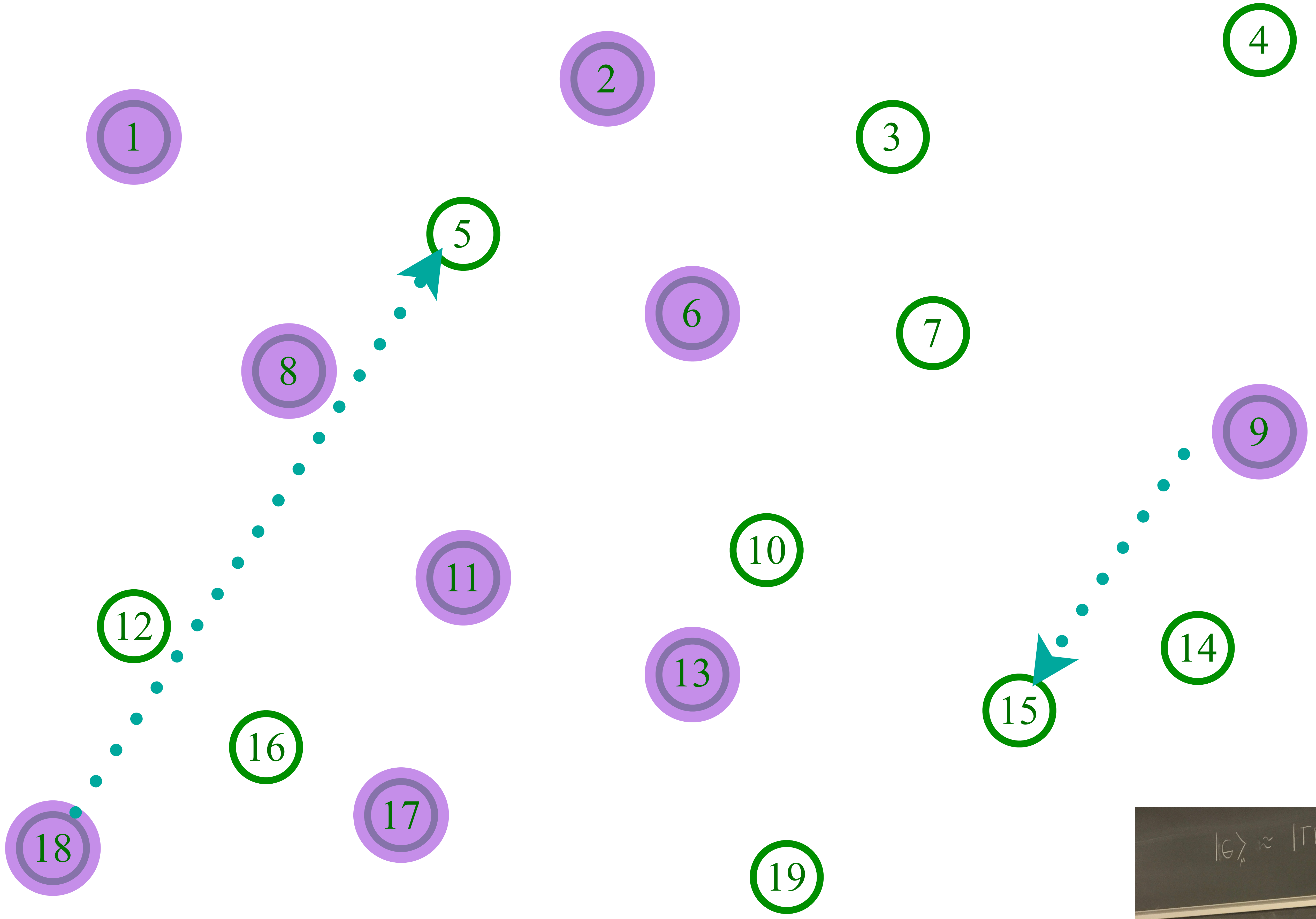
Entangle electrons pairwise randomly



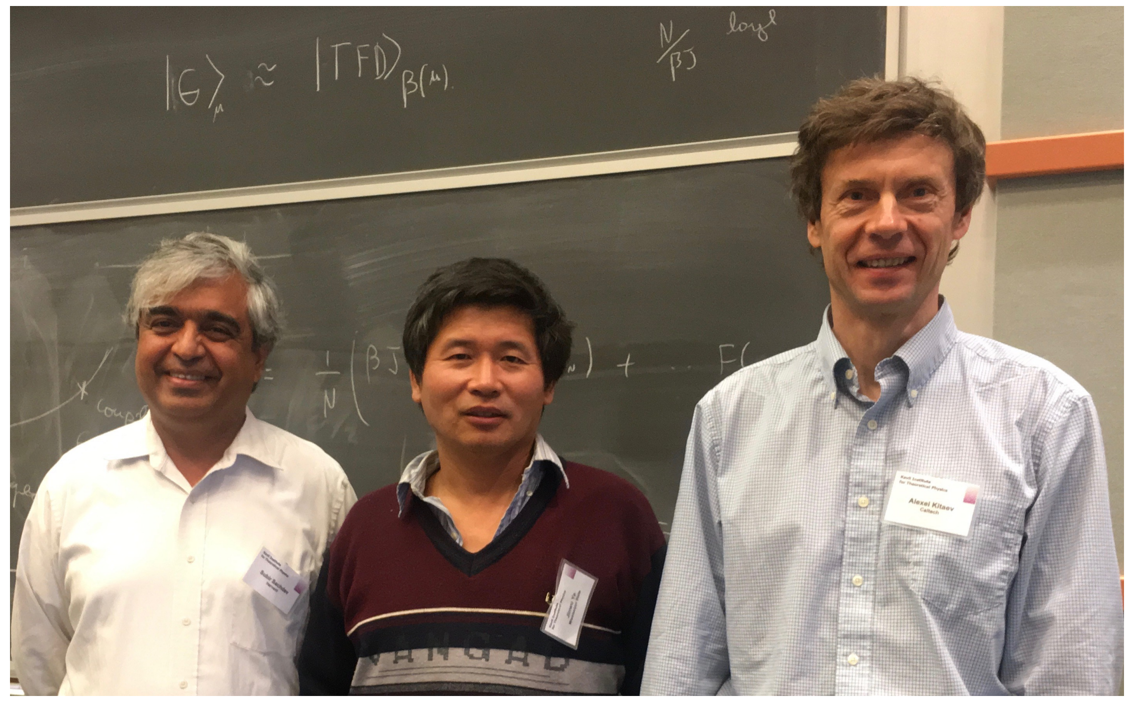
The SYK model

Sachdev, Ye (1993); Kitaev (2015)

$$U_{9,18;5,15}$$

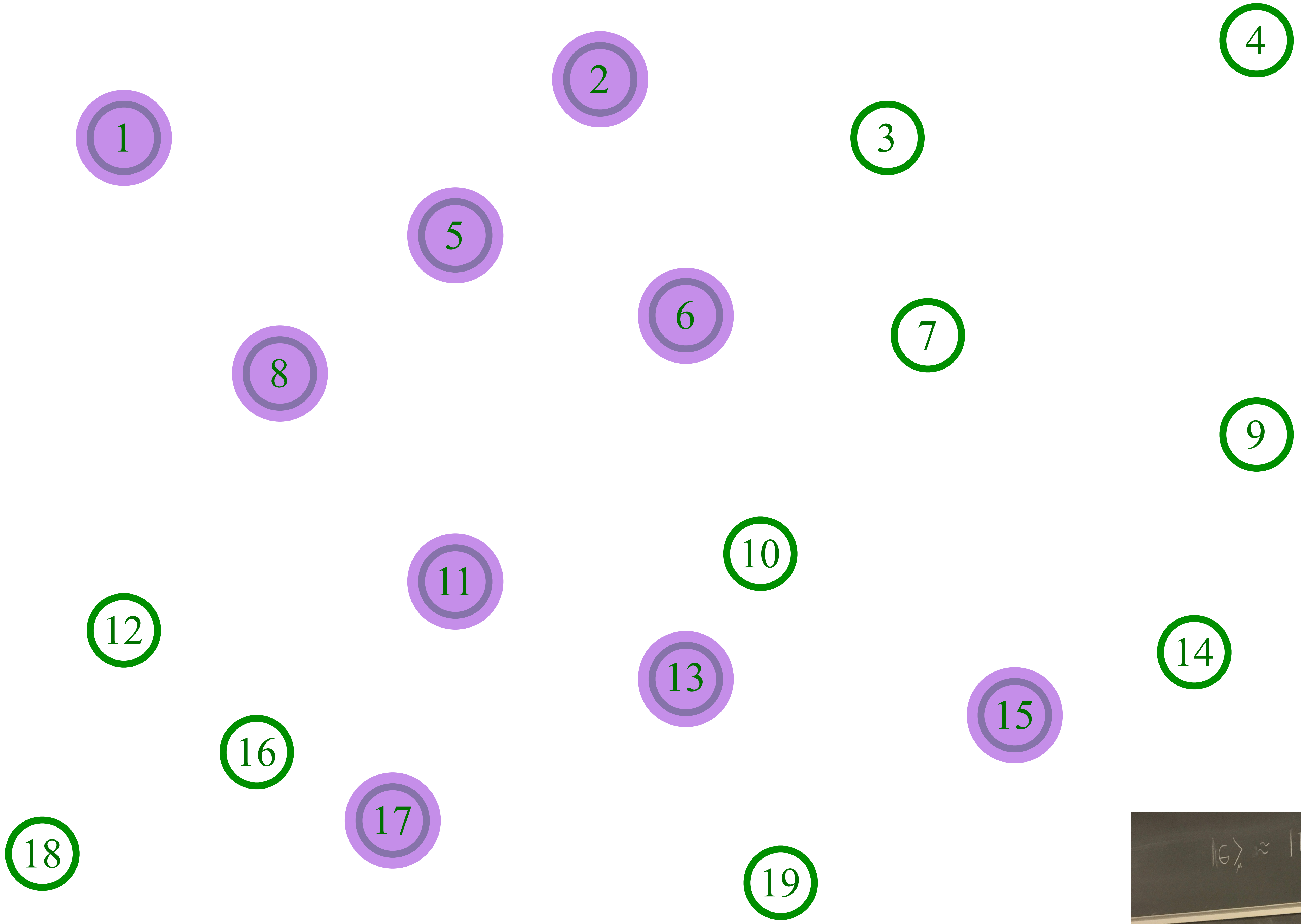


Entangle electrons pairwise randomly



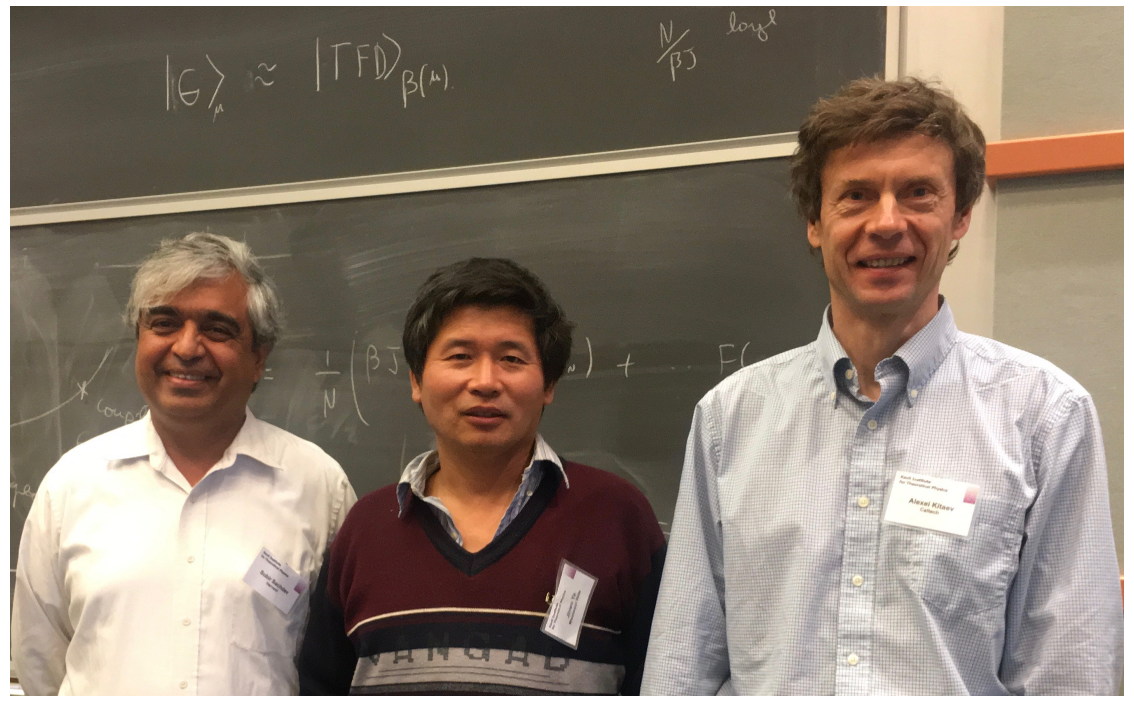
The SYK model

Sachdev, Ye (1993); Kitaev (2015)



$$U_{9,18;5,15}$$

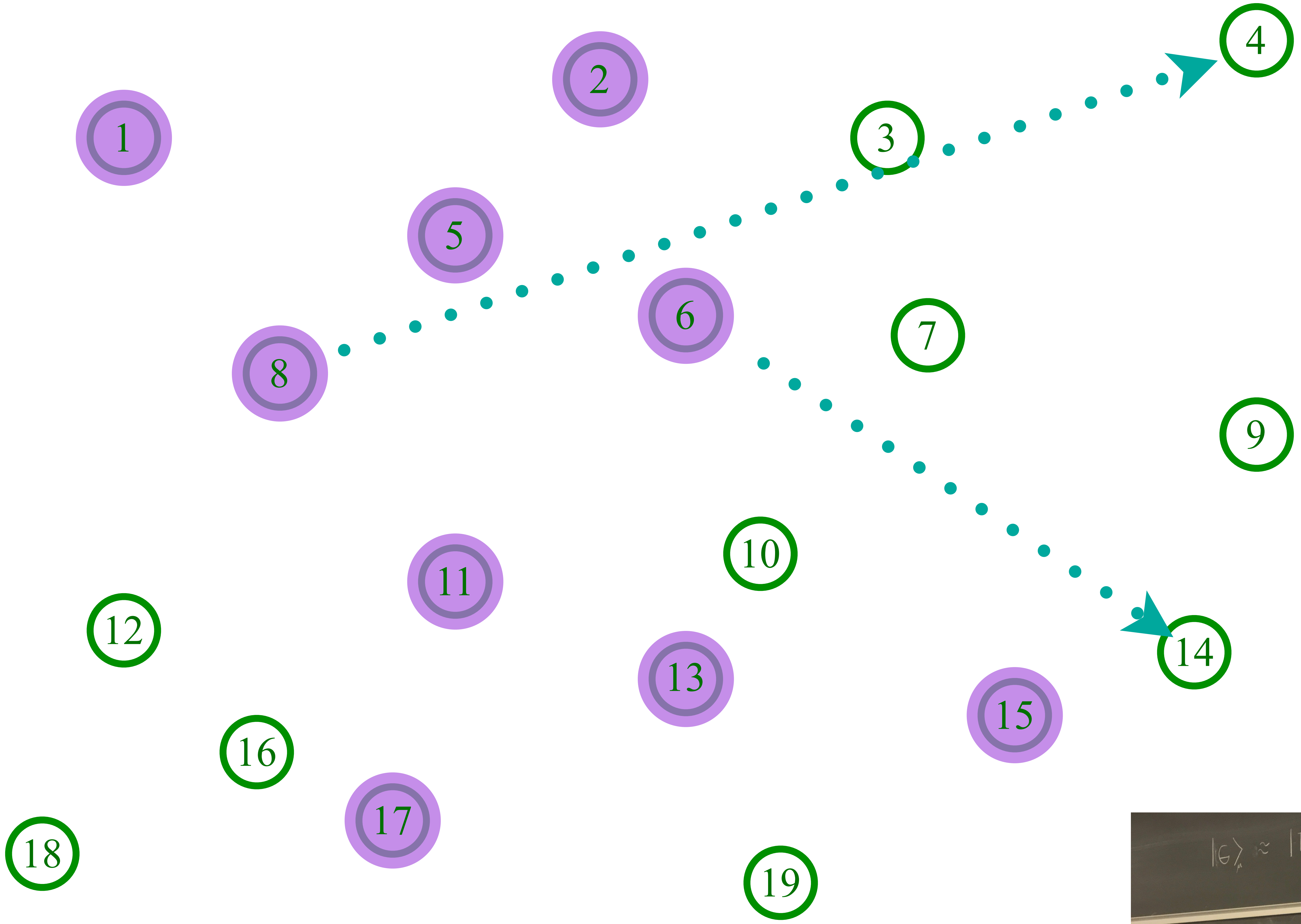
Entangle electrons pairwise randomly



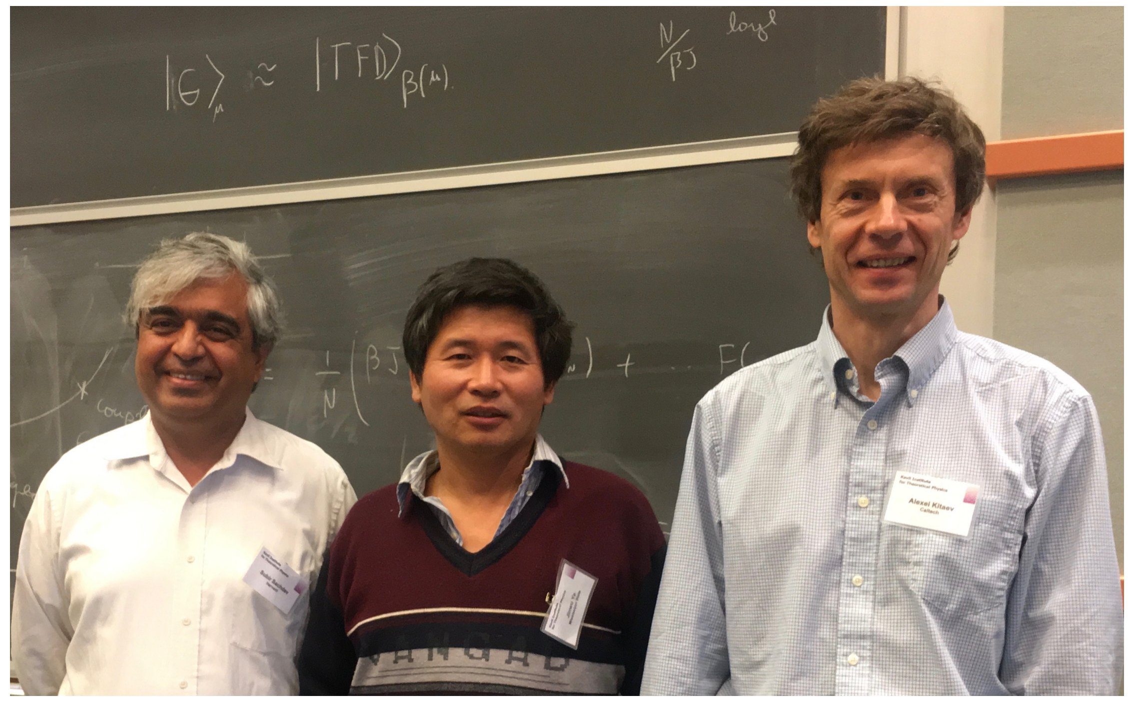
The SYK model

Sachdev, Ye (1993); Kitaev (2015)

$$U_{6,8;4,14}$$



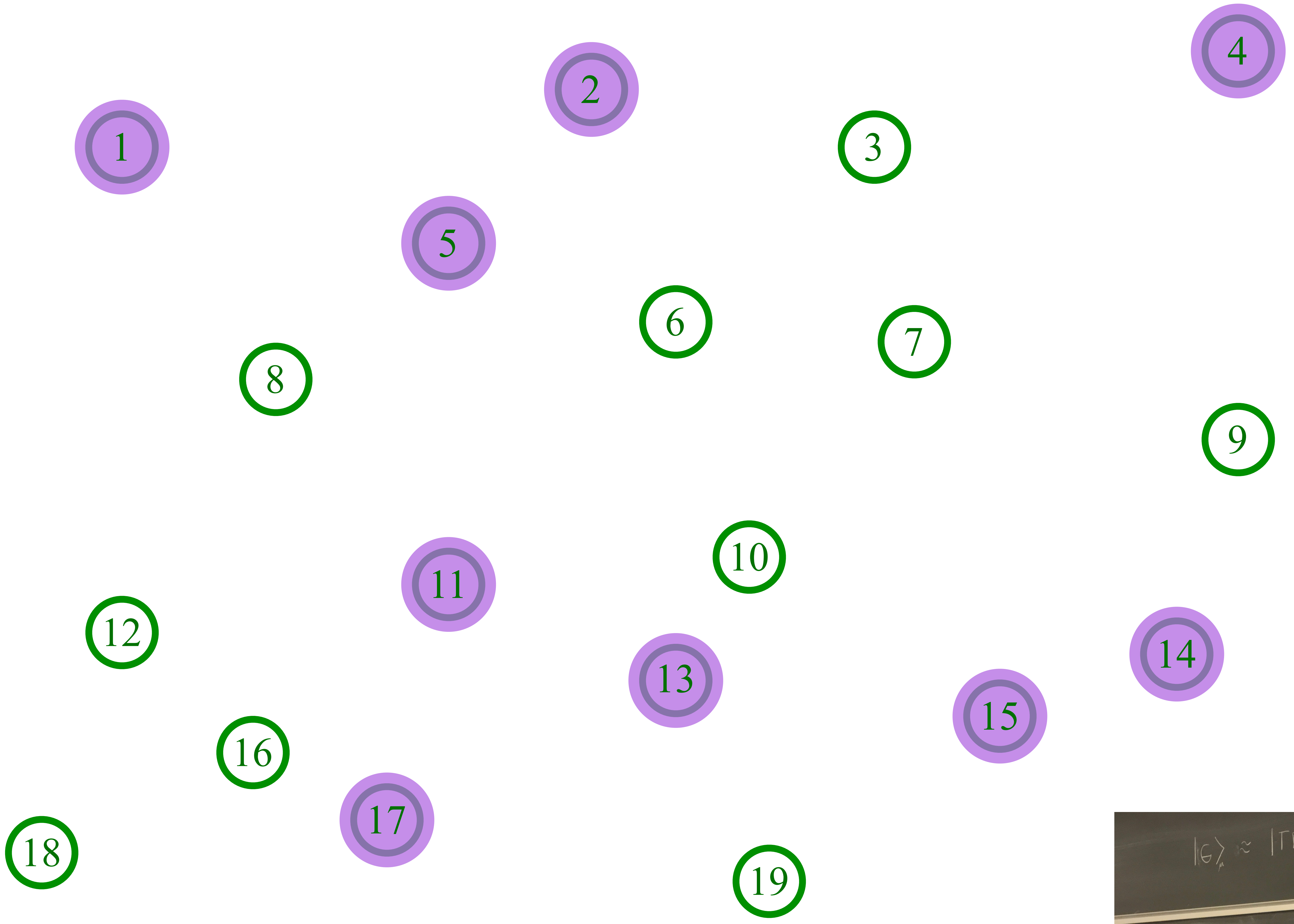
Entangle electrons pairwise randomly



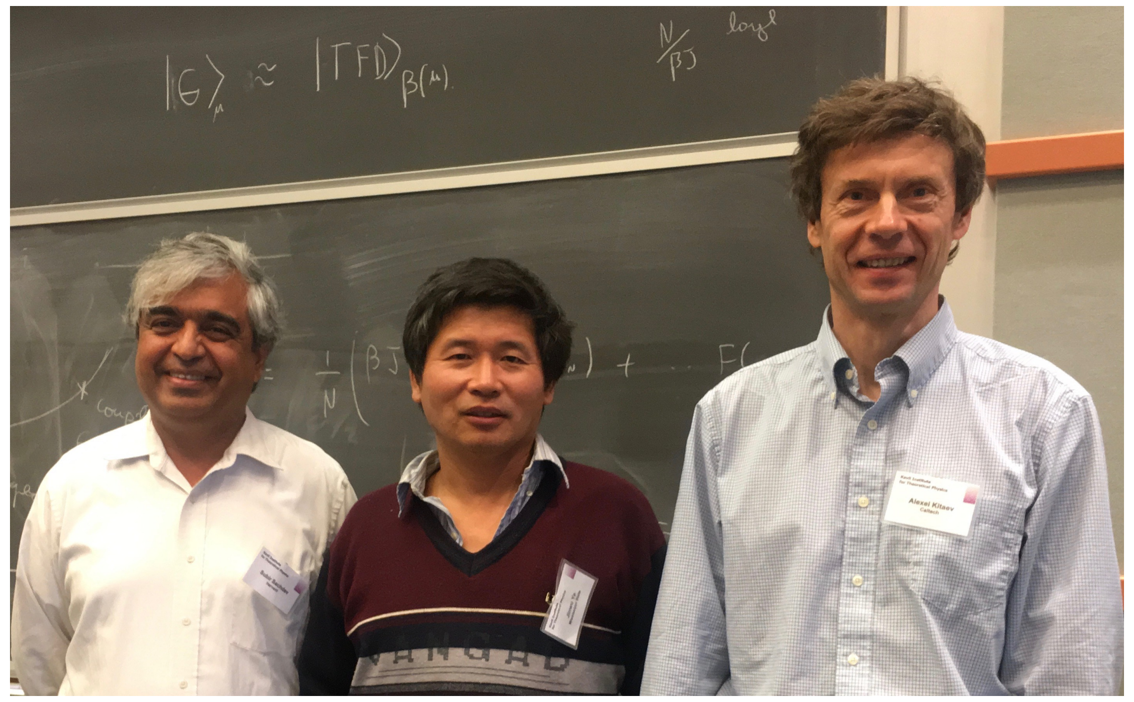
The SYK model

Sachdev, Ye (1993); Kitaev (2015)

$$U_{6,8;4,14}$$



Entangle electrons pairwise randomly



The SYK model

(See also: the "2-Body Random Ensemble" in nuclear physics; did not obtain the large *N* limit;
T.A. Brody, J. Flores, J.B. French, P.A. Mello, A. Pandey, and S.S.M. Wong, Rev. Mod. Phys. **53**, 385 (1981))

$$\mathcal{H} = \frac{1}{(2N)^{3/2}} \sum_{\alpha,\beta,\gamma,\delta=1}^{N} U_{\alpha\beta;\gamma\delta}\, c_\alpha^\dagger c_\beta^\dagger c_\gamma c_\delta - \mu \sum_\alpha c_\alpha^\dagger c_\alpha$$

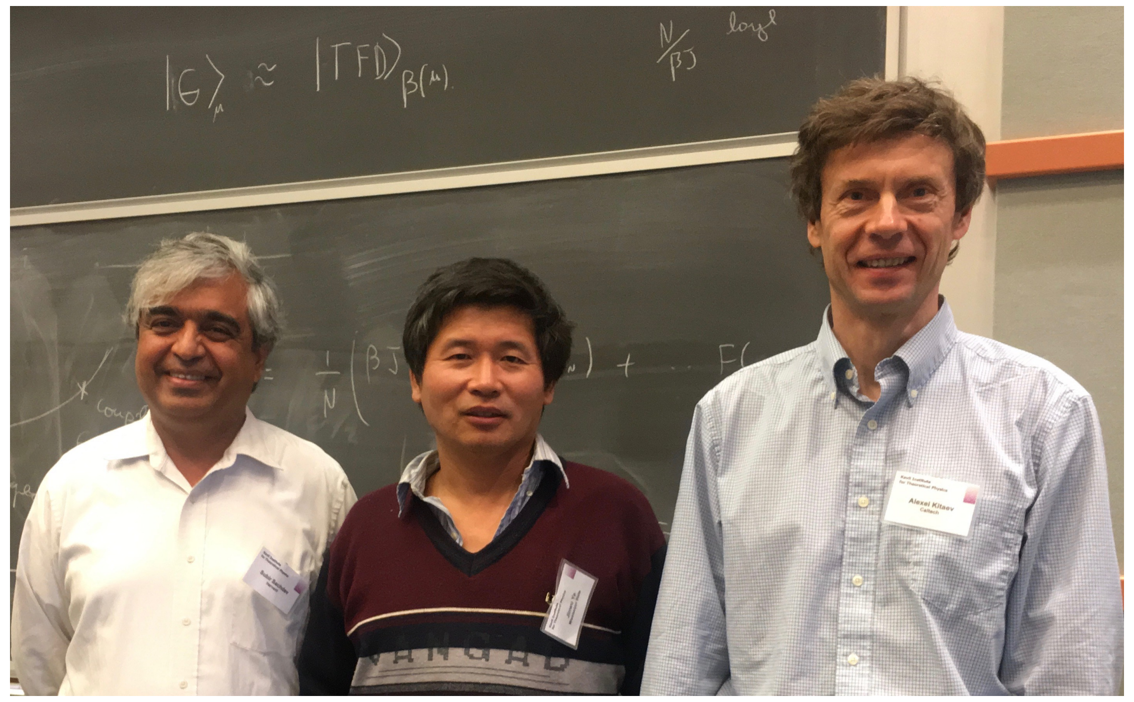
$$c_\alpha c_\beta + c_\beta c_\alpha = 0 \quad , \quad c_\alpha c_\beta^\dagger + c_\beta^\dagger c_\alpha = \delta_{\alpha\beta}$$

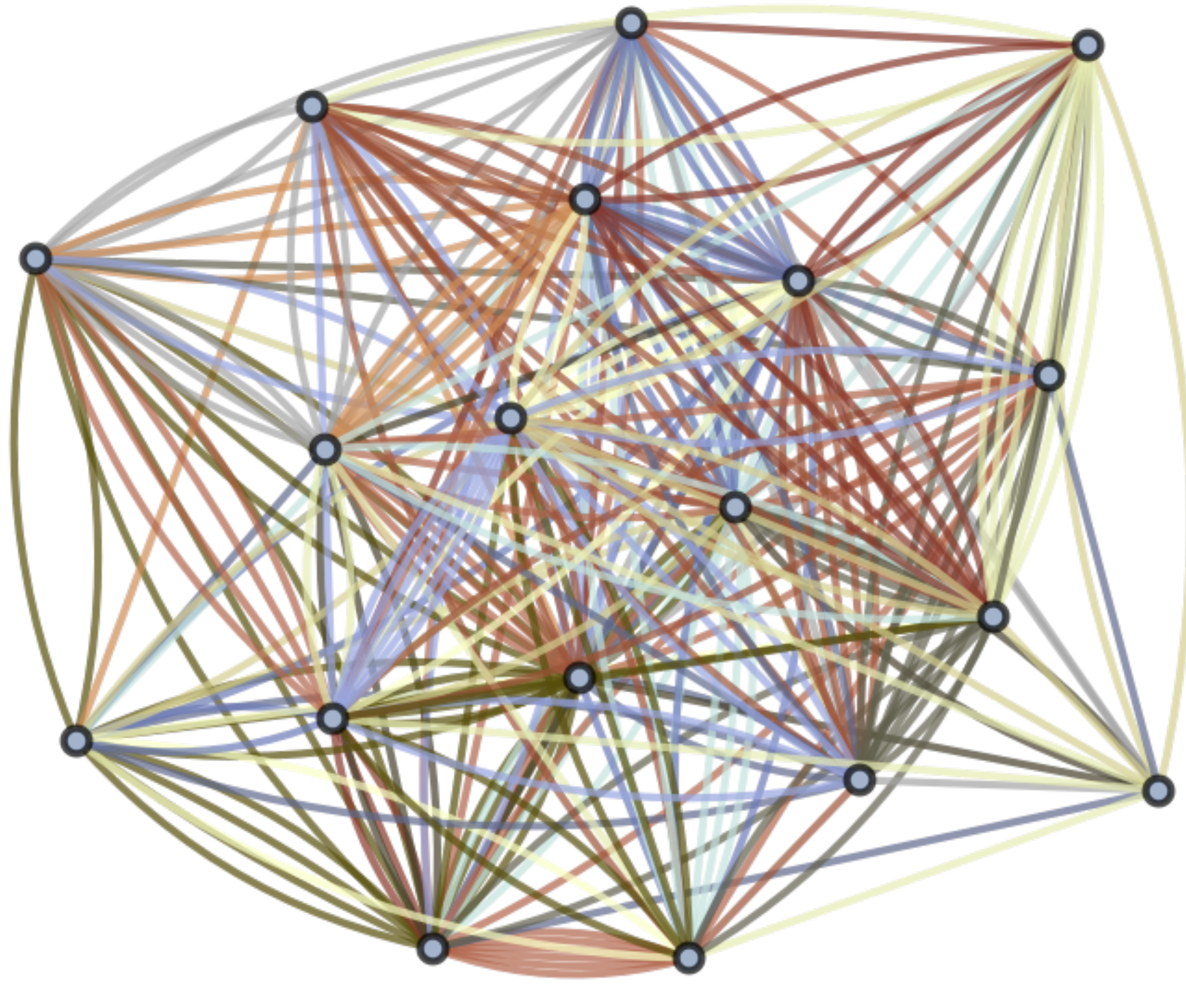
$$\mathcal{Q} = \frac{1}{N} \sum_\alpha c_\alpha^\dagger c_\alpha \, ; \quad [\mathcal{H}, \mathcal{Q}] = 0\, ; \quad 0 \leq \mathcal{Q} \leq 1$$

$U_{\alpha\beta;\gamma\delta}$ are independent random variables with $\overline{U_{\alpha\beta;\gamma\delta}} = 0$ and $\overline{|U_{\alpha\beta;\gamma\delta}|^2} = U^2$
$N \to \infty$ yields critical strange metal.

S. Sachdev and J. Ye, PRL **70**, 3339 (1993)

A. Kitaev, unpublished; S. Sachdev, PRX **5**, 041025 (2015)





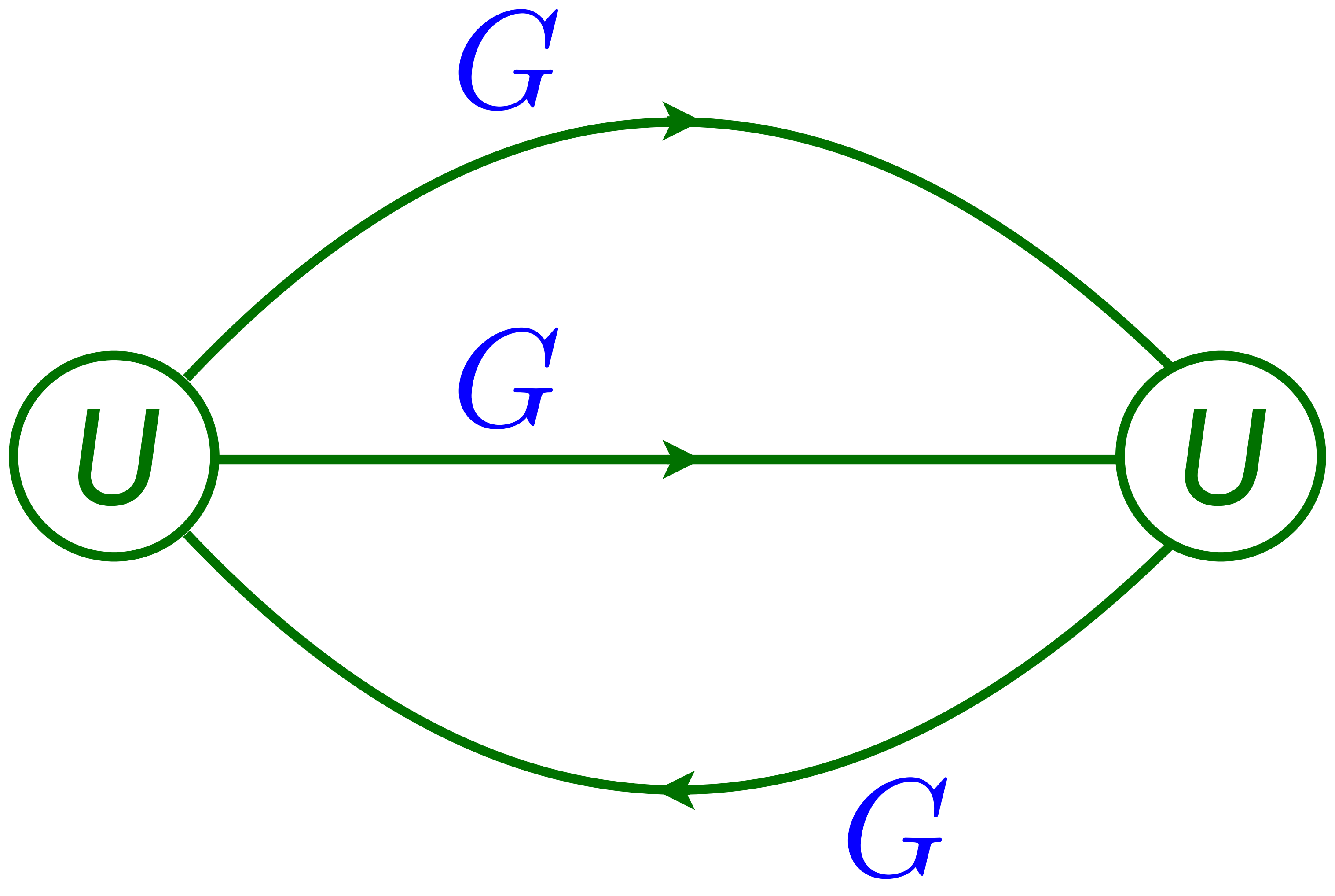
The SYK model

Feynman graph expansion in $U_{\alpha\beta;\gamma\delta}$, and graph-by-graph average, yields exact equations for the fermion Green's function $G(\tau) = -\sum_\alpha \left\langle c_\alpha(\tau) c_\alpha^\dagger(\tau) \right\rangle / N$ in the large N limit:

$$G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = -U^2 G^2(\tau) G(-\tau)$$

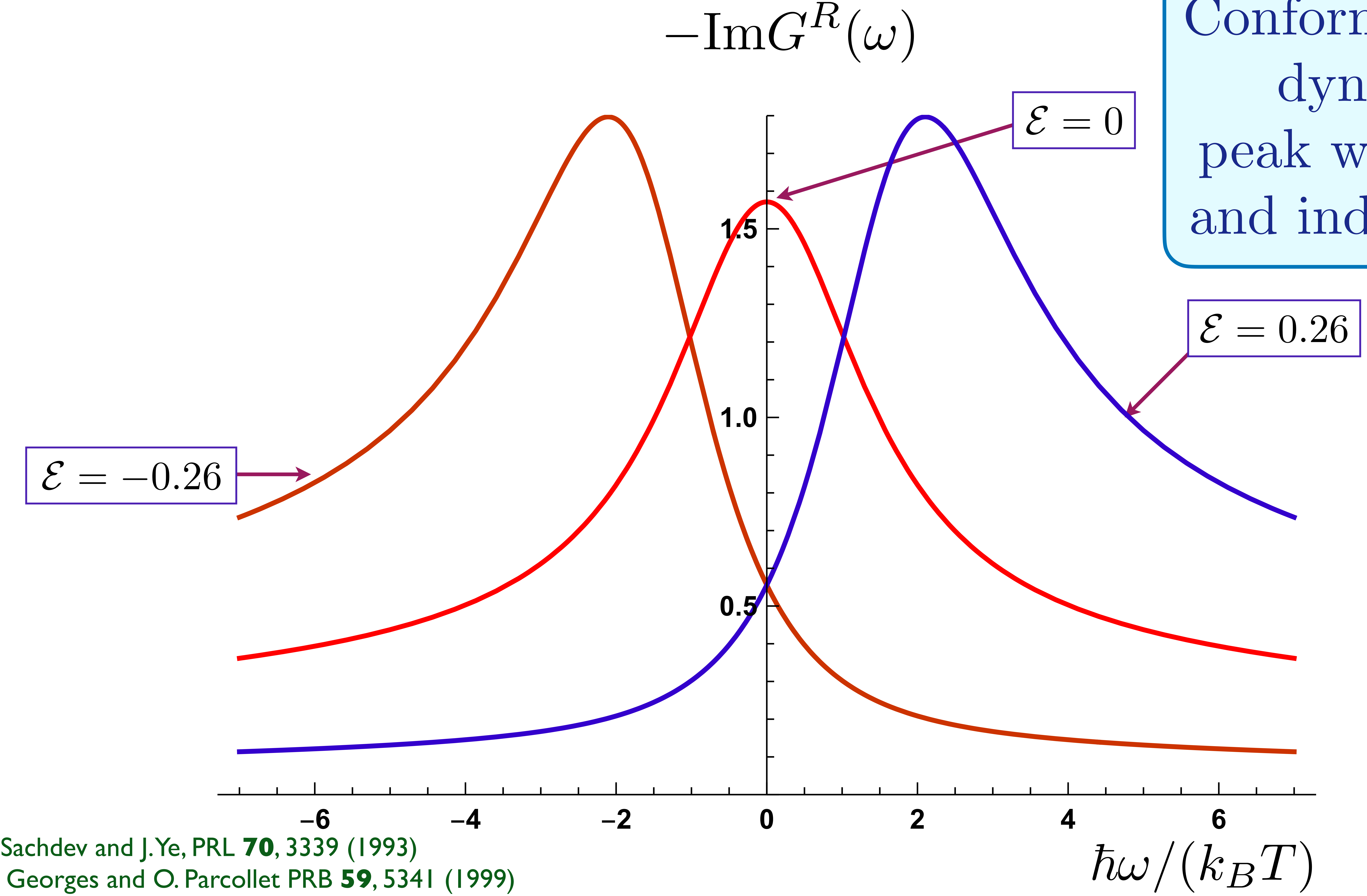
$$G(\tau = 0^-) = \mathcal{Q}.$$

$\Sigma =$



S. Sachdev and J. Ye,
PRL **70**, 3339 (1993)

The SYK model

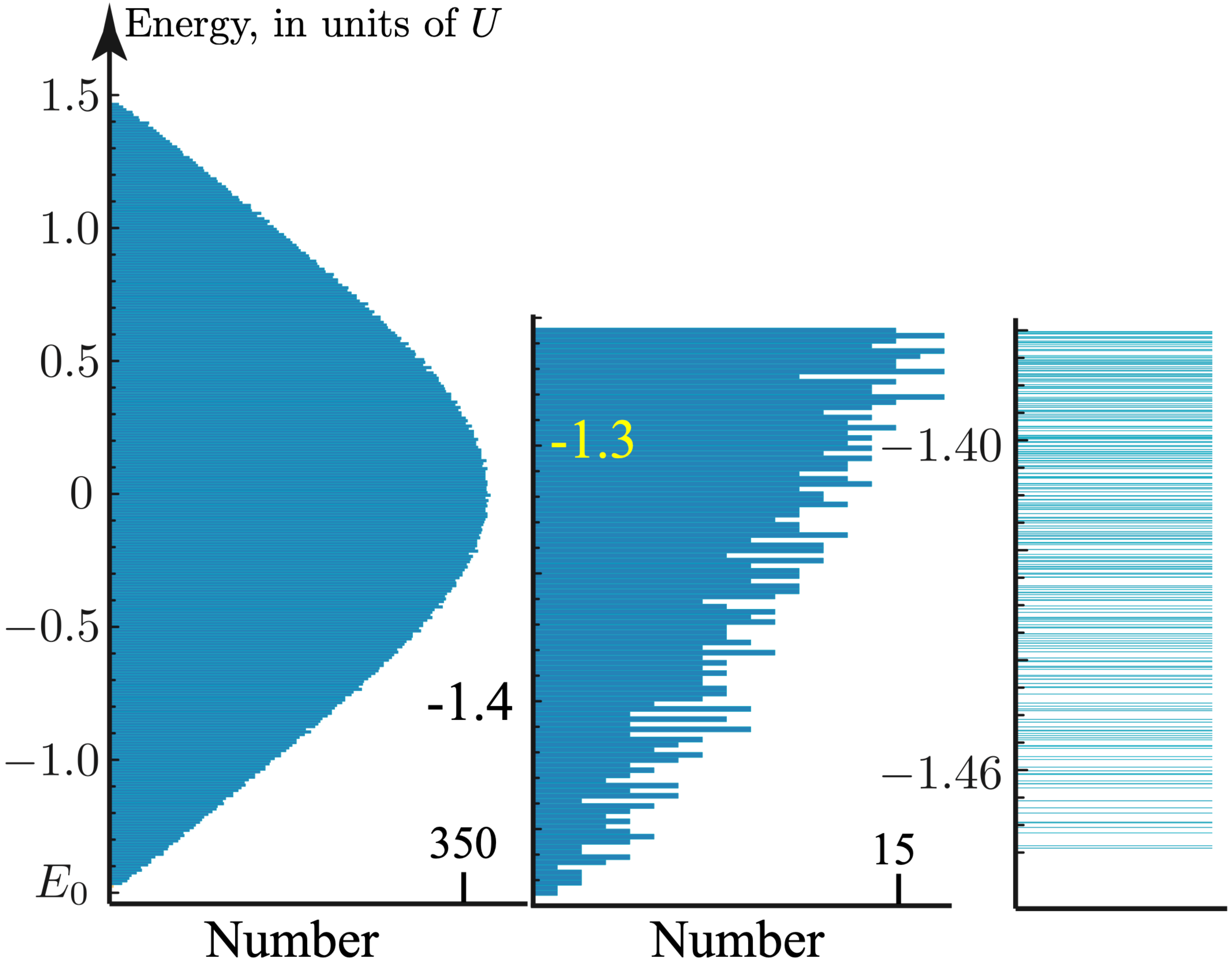


Conformal 'Planckian'
dynamics with
peak width $\sim k_B T/\hbar$
and independent of U

S. Sachdev and J. Ye, PRL **70**, 3339 (1993)
A. Georges and O. Parcollet PRB **59**, 5341 (1999)
S. Sachdev, PRX **5**, 041025 (2015)

Many-body density of states

$$D(E) = \sum_i \delta(E - E_i); \quad E_0 + E_i \Rightarrow \text{Many body eigenvalue}$$



Complex SYK model

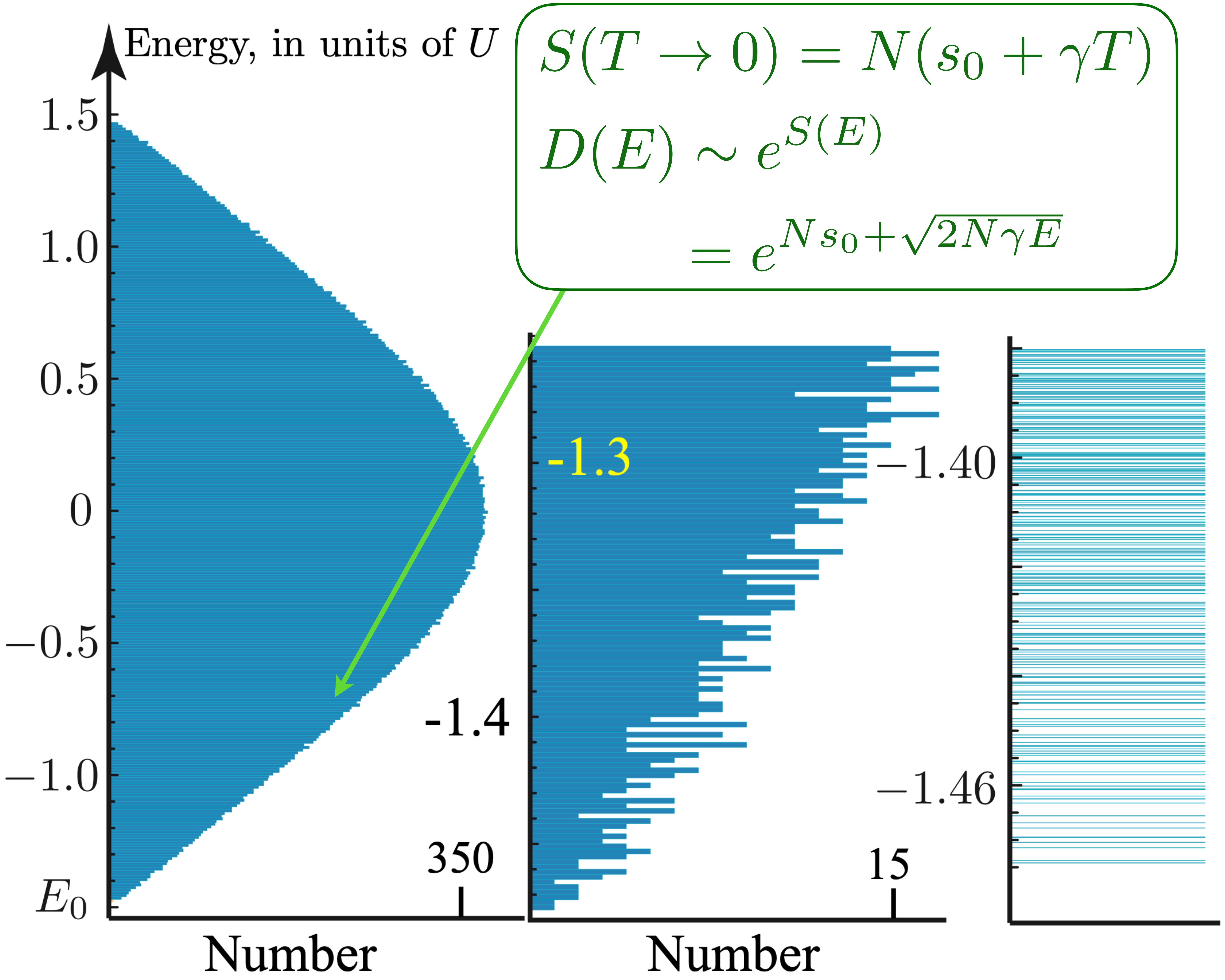
(Numerics: G. Tarnopolsky)

Many-body density of states

Boltzmann

$$D(E) = \sum_i \delta(E - E_i); \quad E_0 + E_i \Rightarrow \text{Many body eigenvalue}$$

At $\mathcal{Q} = 1/2$



$$S(T \to 0) = N(s_0 + \gamma T)$$

$$D(E) \sim e^{S(E)}$$

$$= e^{N s_0 + \sqrt{2N\gamma E}}$$

$$s_0 = \frac{\text{Catalan}}{\pi} + \frac{\ln 2}{4}$$

$$= 0.46484769917\ldots.$$

A. Georges, O. Parcollet, and S. Sachdev,
PRB **63**, 134406 (2001)

Complex SYK model

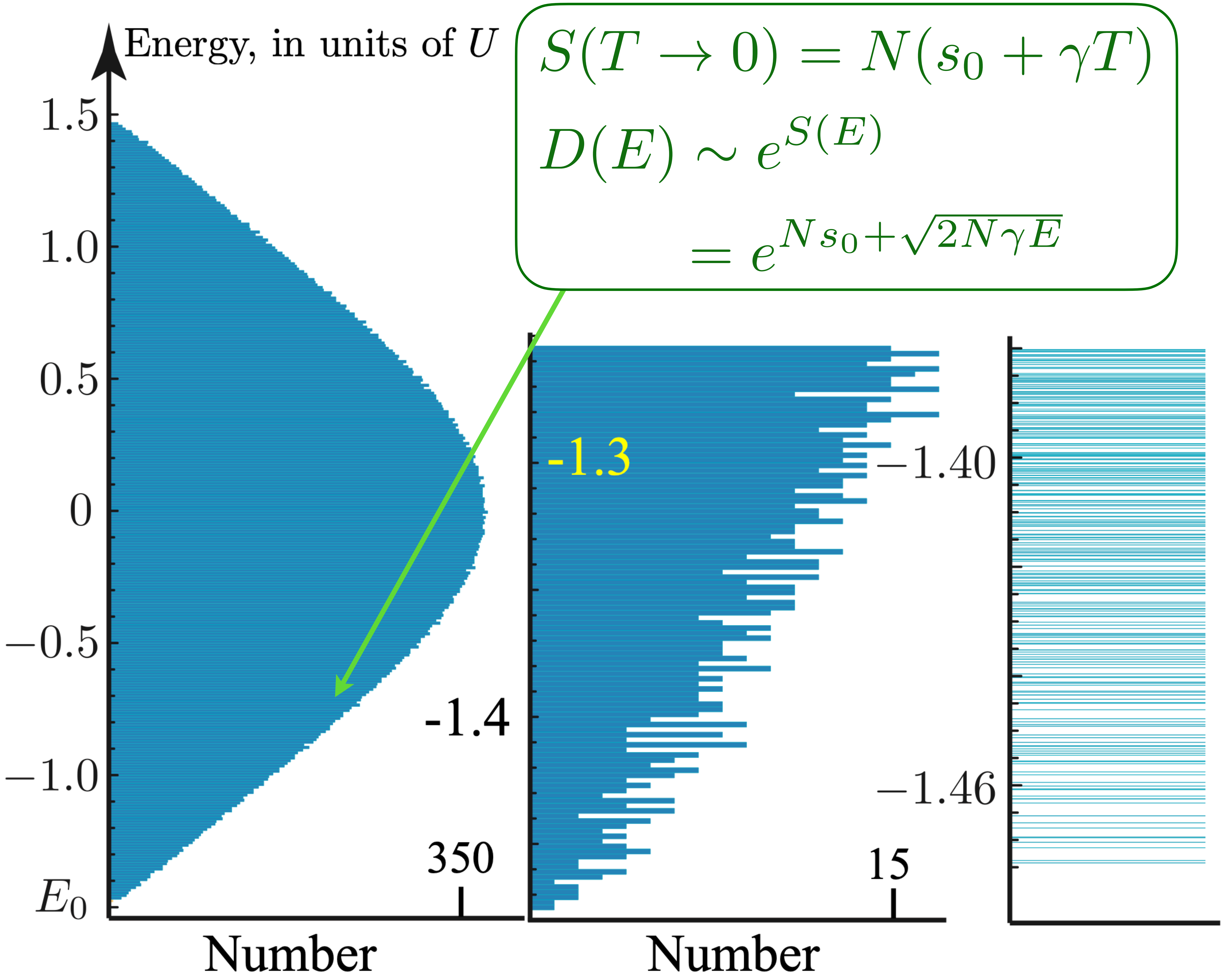
(Numerics: G. Tarnopolsky)

Many-body density of states

Boltzmann

$$D(E) = \sum_i \delta(E - E_i); \quad E_0 + E_i \Rightarrow \text{Many body eigenvalue}$$

At $\mathcal{Q} = 1/2$



$$S(T \to 0) = N(s_0 + \gamma T)$$

$$D(E) \sim e^{S(E)}$$

$$= e^{N s_0 + \sqrt{2 N \gamma E}}$$

$$s_0 = \frac{\text{Catalan}}{\pi} + \frac{\ln 2}{4}$$

$$= 0.46484769917\ldots.$$

A. Georges, O. Parcollet, and S. Sachdev,
PRB **63**, 134406 (2001)

Energy level spacing $\sim e^{-N s_0}$!

No quasiparticle decomposition: wavefunctions change chaotically from one state to the next.

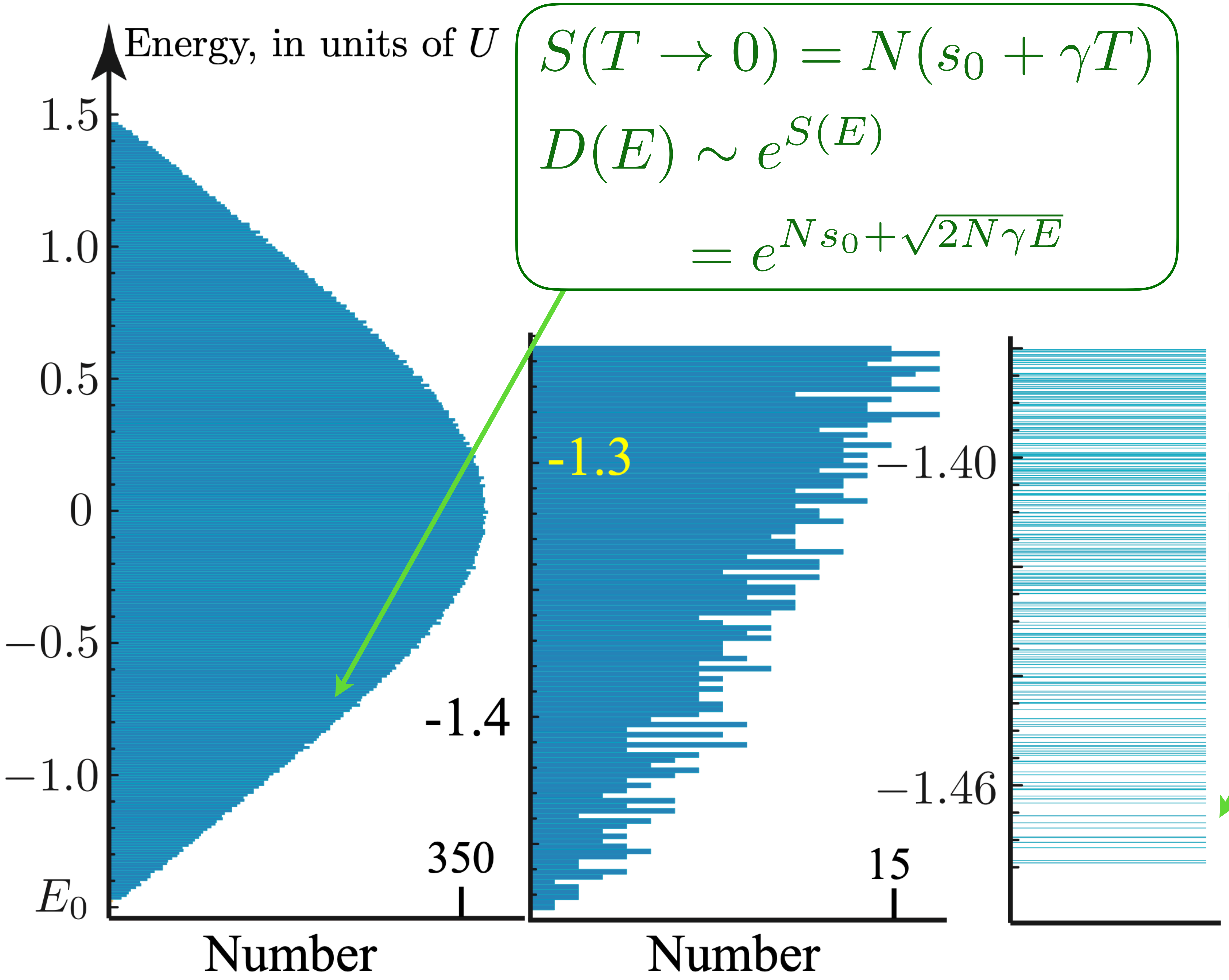
Complex SYK model

(Numerics: G. Tarnopolsky)

Many-body density of states

Beyond Boltzmann

$$D(E) = \sum_i \delta(E - E_i); \quad E_0 + E_i \Rightarrow \text{Many body eigenvalue}$$



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$$= 0.46484769917\ldots.$$

A. Georges, O. Parcollet, and S. Sachdev, PRB **63**, 134406 (2001)

$$D(E) \sim$$

$$N^{-1} \exp\left(N s_0\right) \sinh(\sqrt{2N\gamma E})$$

J. S. Cotler et al., JHEP 05 (2017) 118

Yingfei Gu, A. Kitaev, S. Sachdev, and G. Tarnopolsky, JHEP 02 (2020) 157

Complex SYK model

(Numerics: G. Tarnopolsky)

The SYK model

The (averaged) partition function can be written as path integral over the bilocal fermion Green's function $G(\tau_1,\tau_2) \sim \frac{1}{N}\sum_\alpha c_\alpha(\tau_1) c_\alpha^\dagger(\tau_2)$

$$\overline{\mathcal{Z}} = \int \mathcal{D}G(\tau_1,\tau_2)\exp\left(-N S_{\text{eff}}[G]\right)$$

The large N saddle point equation $\delta S_{\text{eff}}/\delta G = 0$ for $G(\tau_1,\tau_2) = G_s(\tau_1-\tau_2)$ is

$$G_s(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)} \quad , \quad \Sigma_s(\tau) = -U^2 G_s^2(\tau) G_s(-\tau)$$

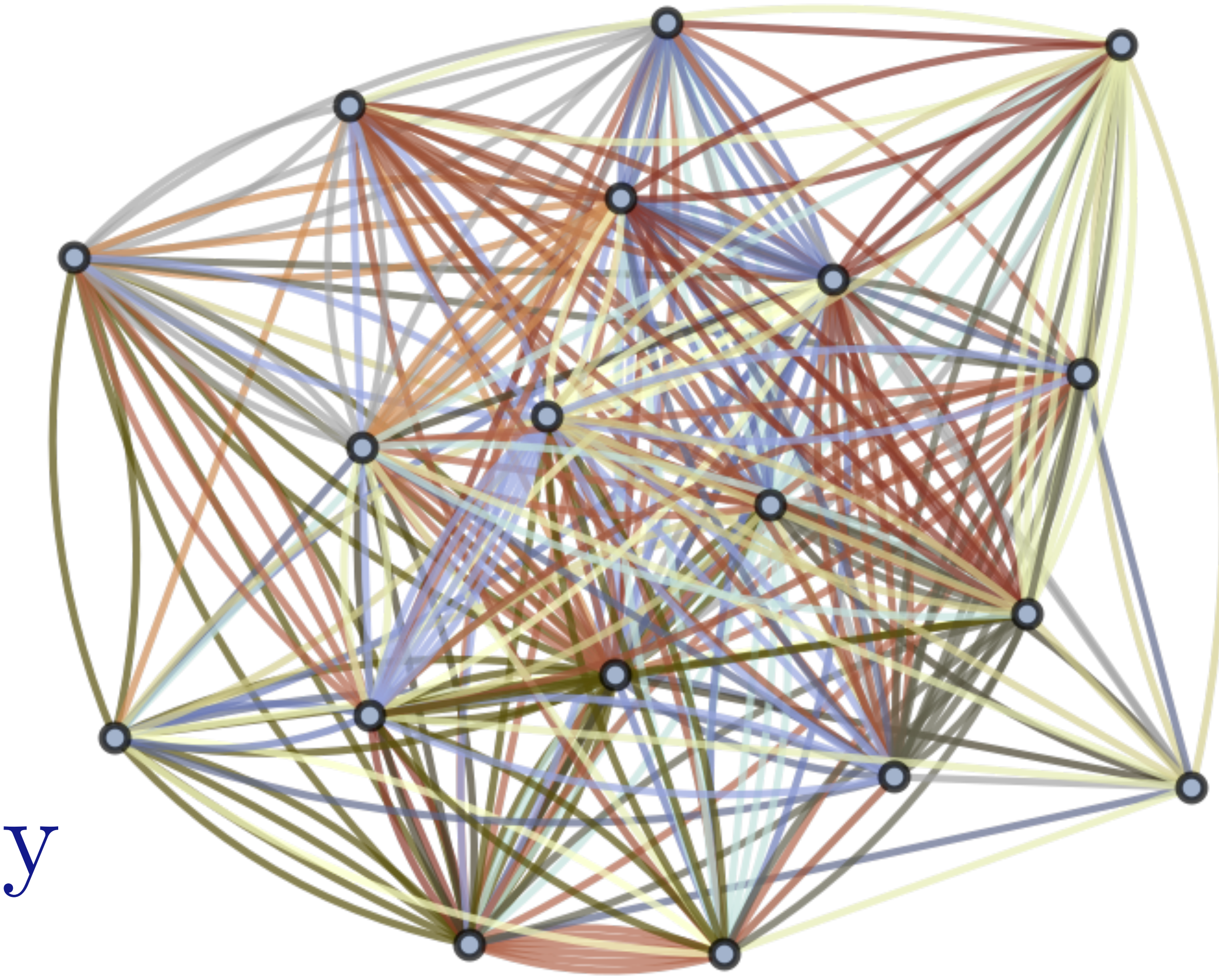
A. Georges and O. Parcollet
PRB **59**, 5341 (1999)
A. Kitaev, 2015
S. Sachdev, PRX **5,** 041025 (2015)

Time reparameterization symmetry:

At frequencies $\ll U$, the path integral for is invariant under time reparametrization $f(\sigma)$

$$\tau = f(\sigma)$$

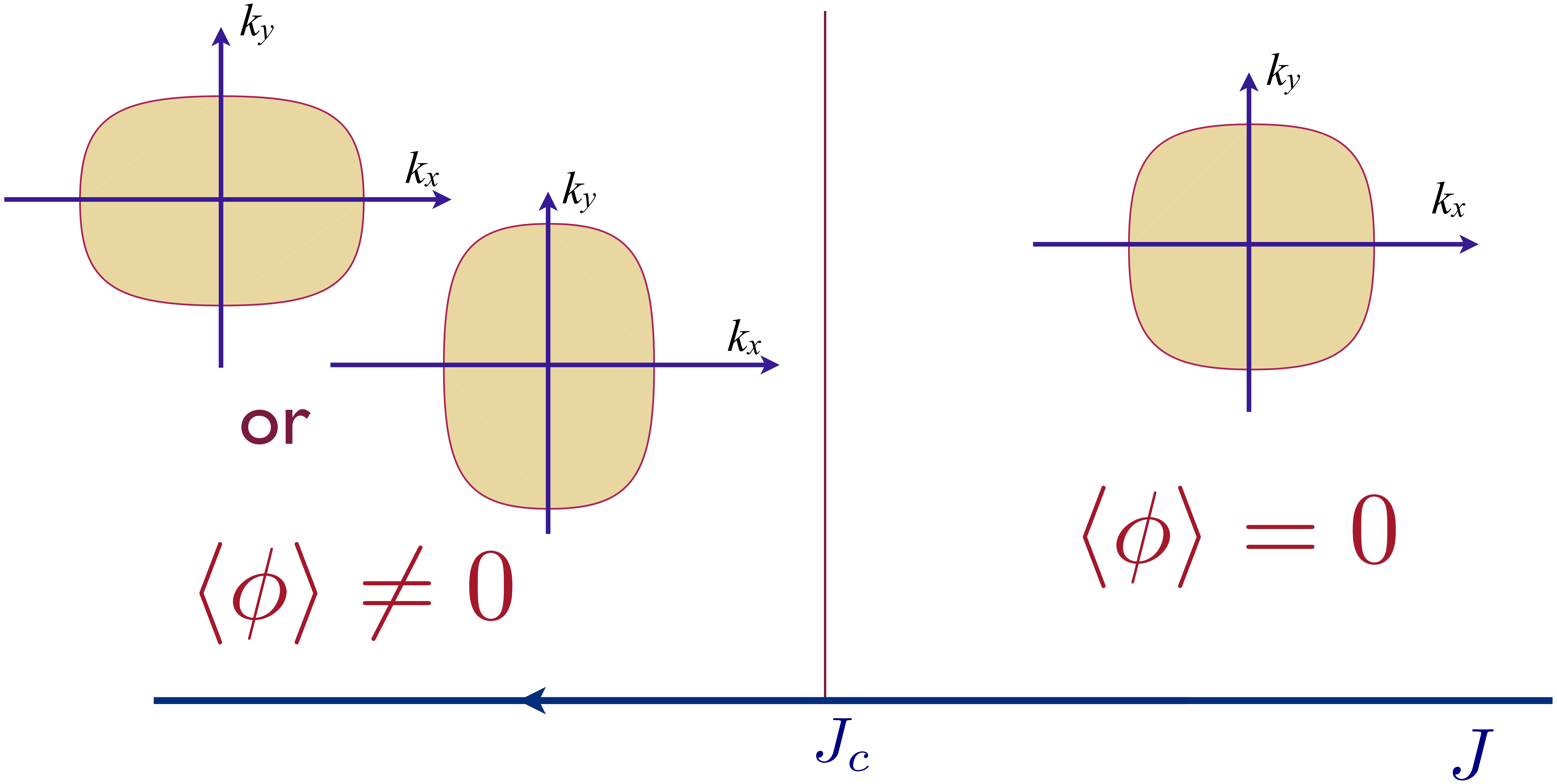
$$G(\tau_1,\tau_2) = \left[f'(\sigma_1)f'(\sigma_2)\right]^{-1/4}\widetilde{G}(\sigma_1,\sigma_2)$$



There is also an emergent U(1) gauge symmetry. Hints that the low energy theory is quantum gravity+electromagnetism!

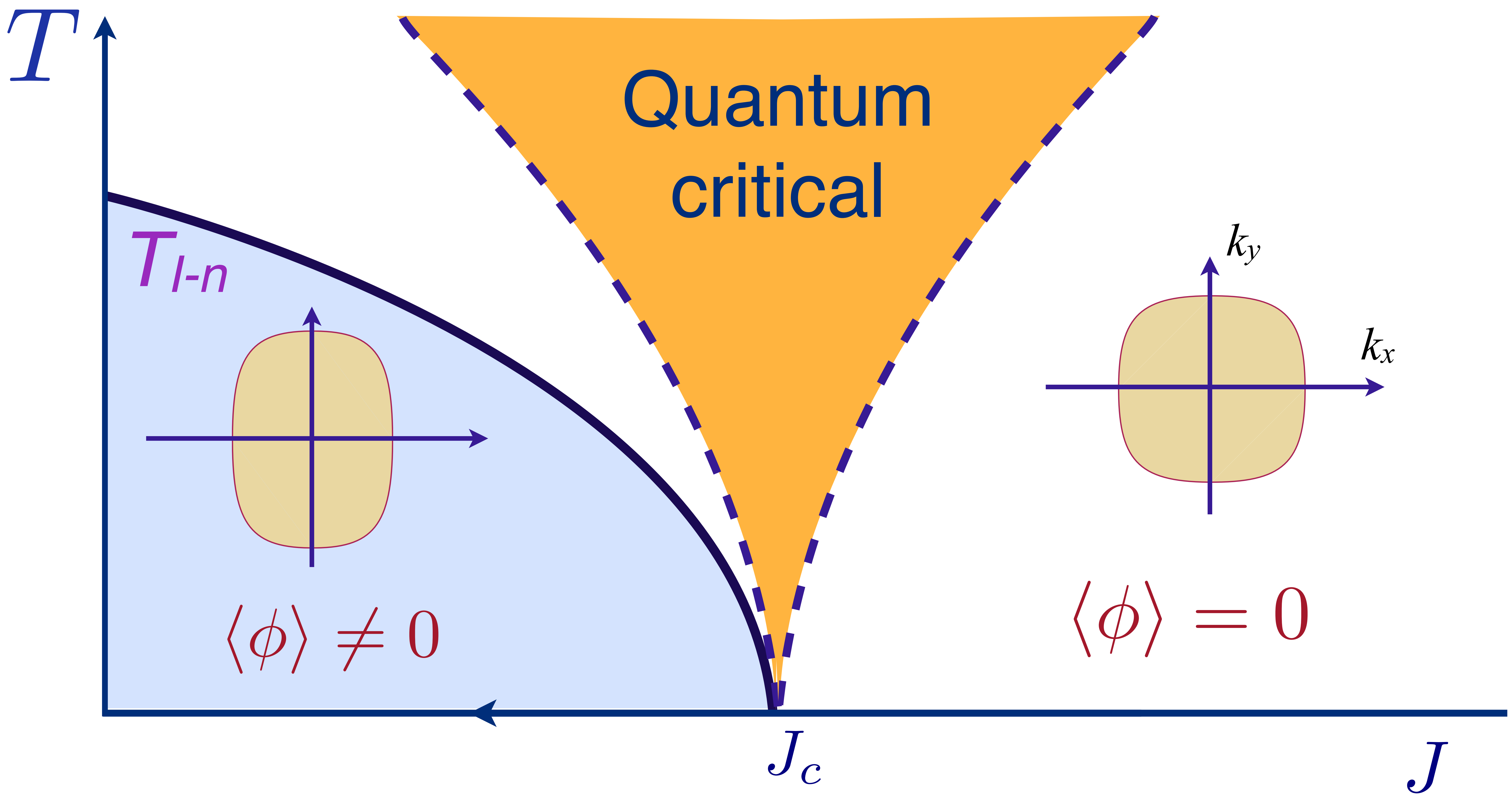
From the SYK model to a universal theory of strange metals

Quantum criticality of Ising-nematic ordering in a metal



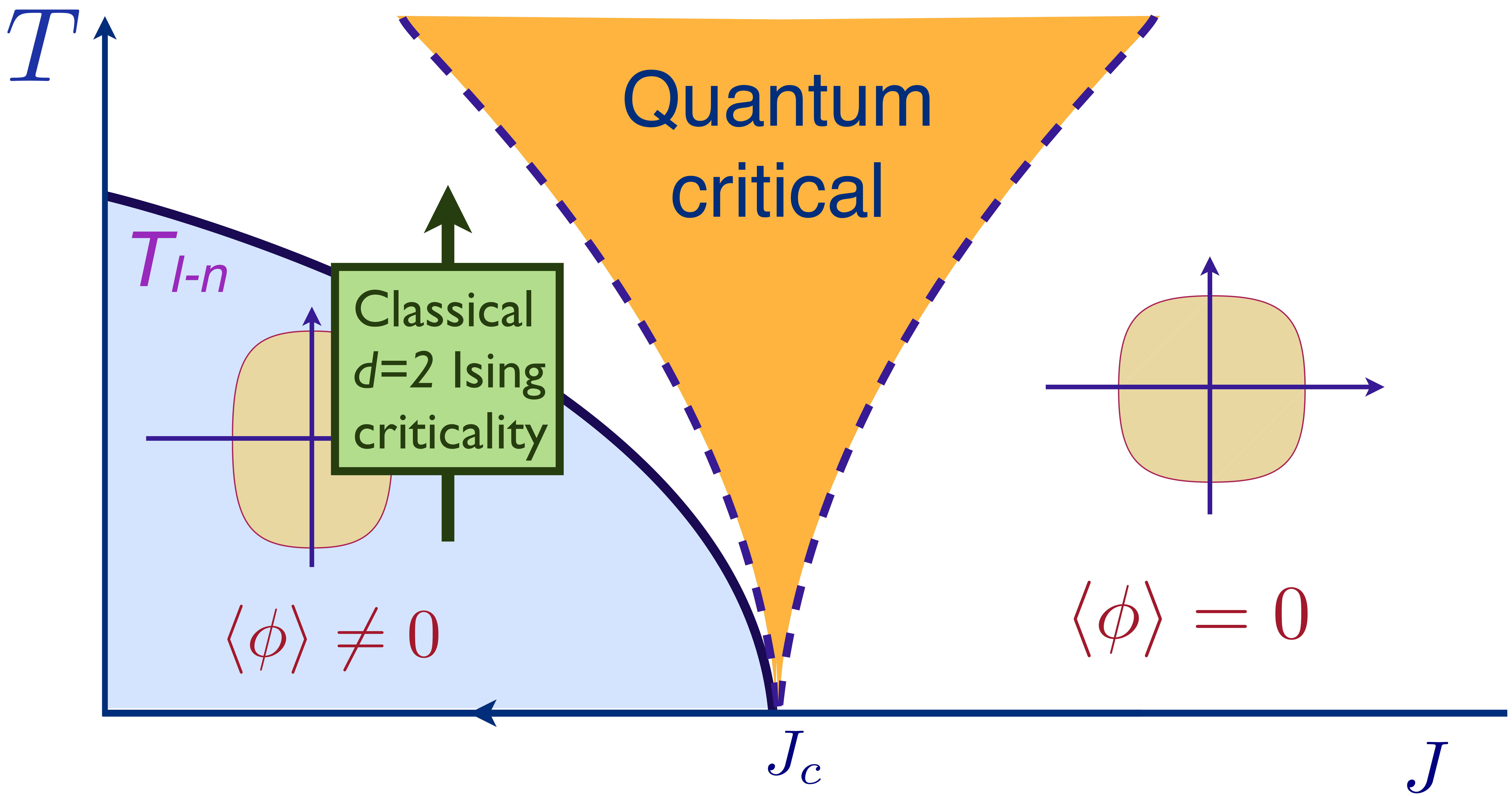
Pomeranchuk instability as a function of coupling J
(Similar considerations apply to the cuprates, with a Higgs boson Φ
describing a Fermi volume changing transition)

Quantum criticality of Ising-nematic ordering in a metal



Phase diagram as a function of T and J

Quantum criticality of Ising-nematic ordering in a metal



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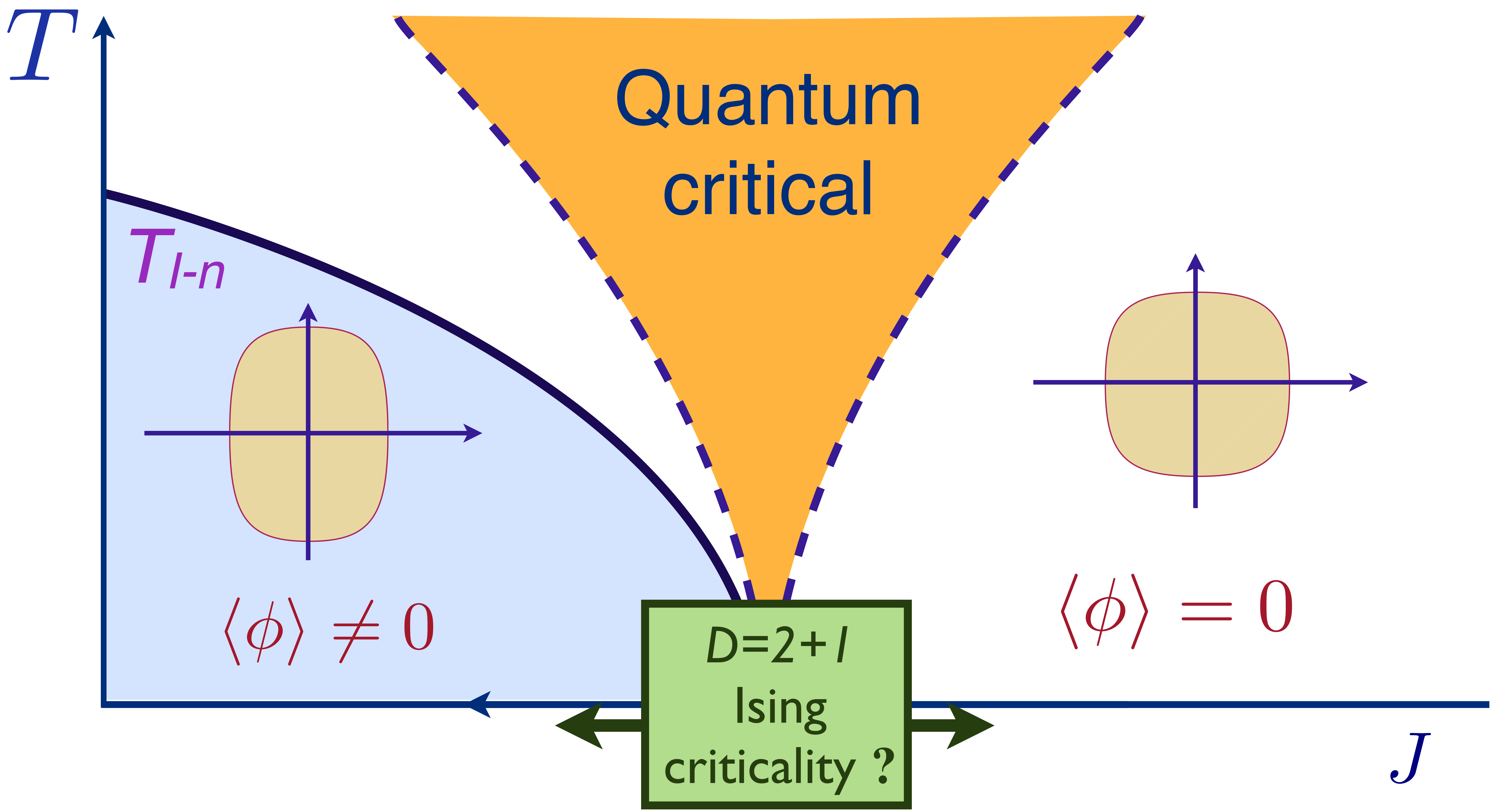

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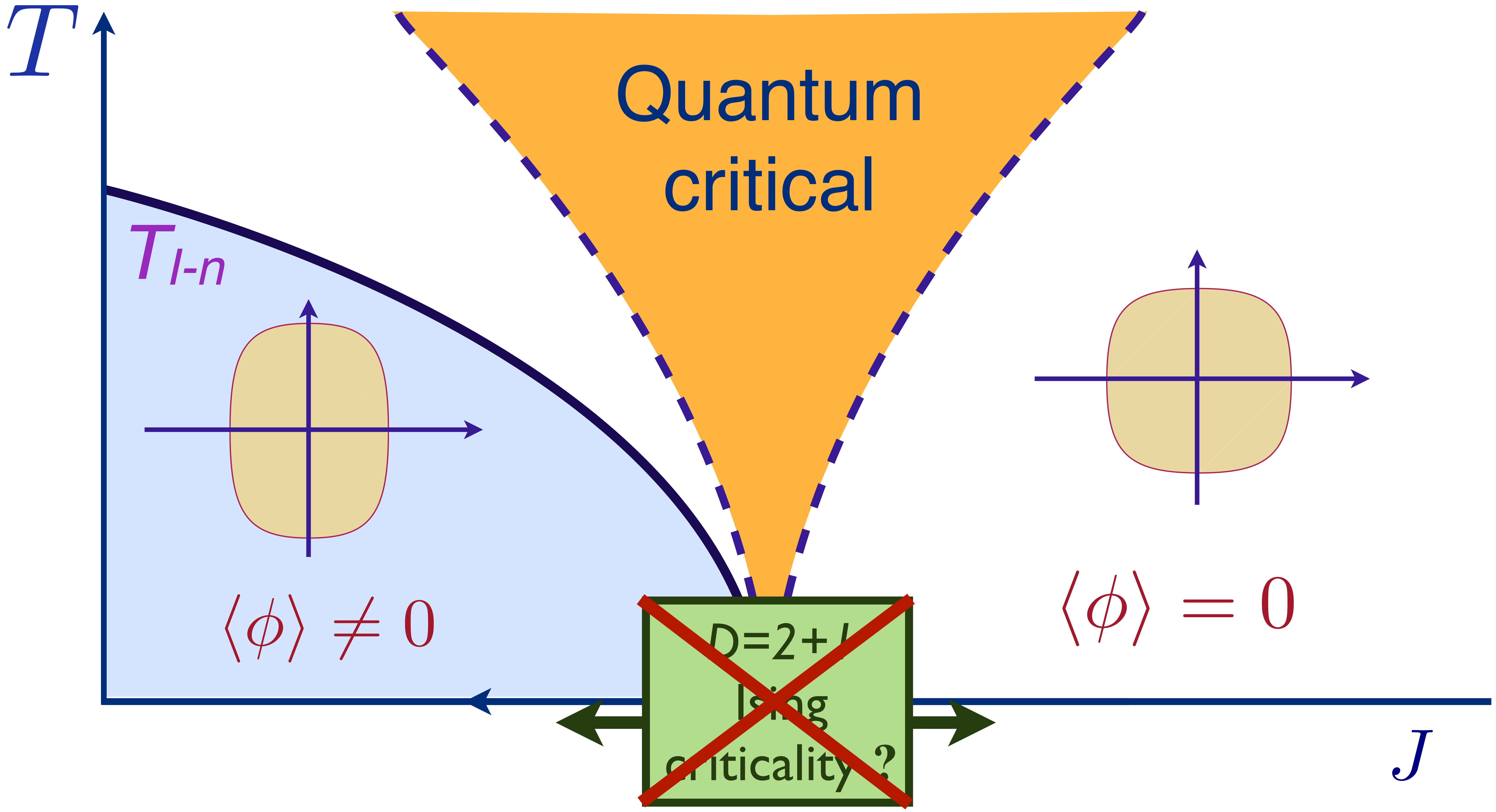


Phase diagram as a function of T and J

Critical boson

A critical boson ϕ
e.g. Ising-nematic order,
spin-density wave order,
Higgs boson for Fermi-volume changing transition

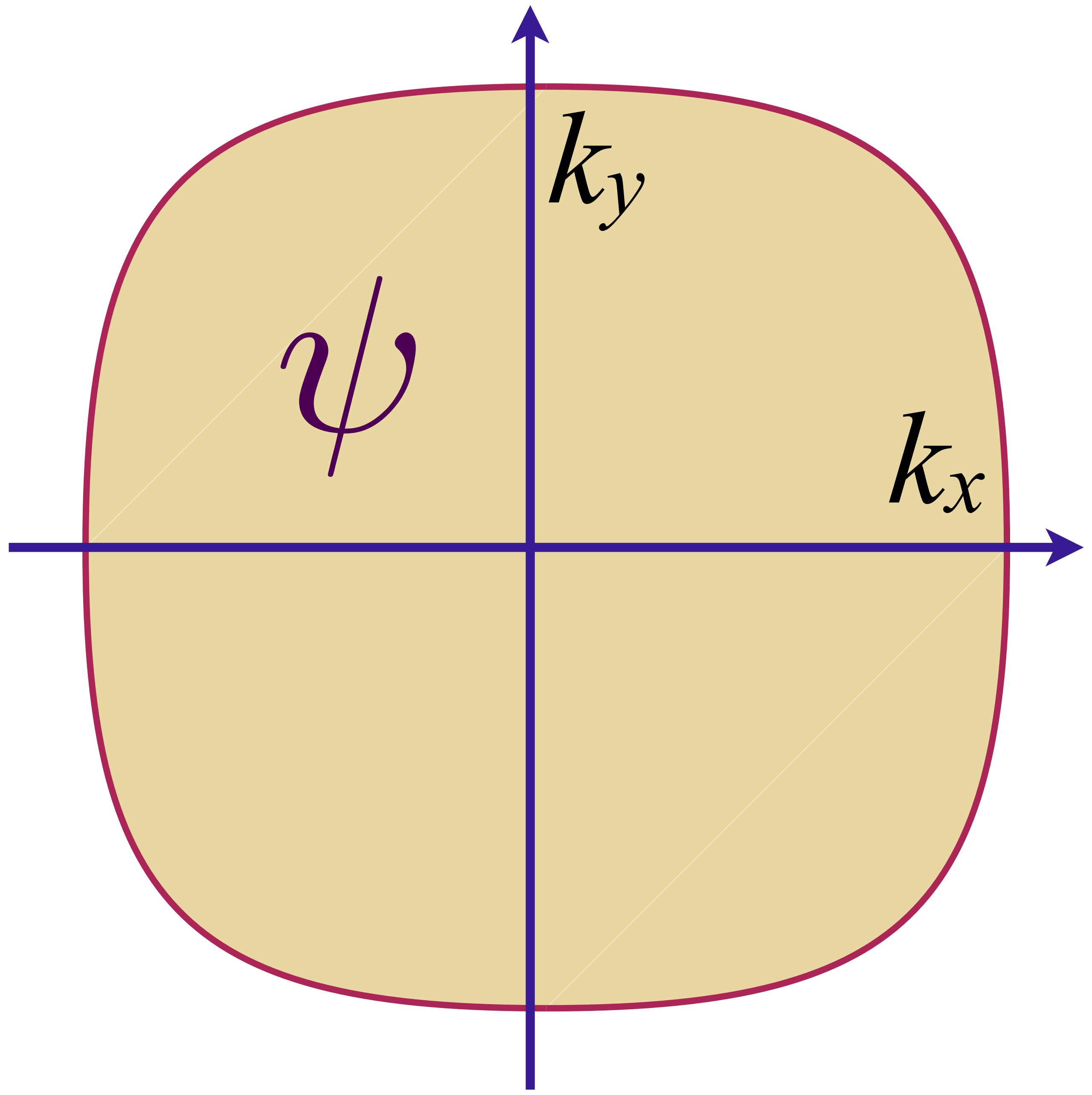
$$+s\left[\phi(\boldsymbol{r})\right]^2 + \qquad +g\,\psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})\,\phi(\boldsymbol{r})$$

$$+K\left[\nabla_{\boldsymbol{r}}\phi(\boldsymbol{r})\right]^2 + u\left[\phi(\boldsymbol{r})\right]^4$$

$$s \sim J_c - J$$

Critical boson + Fermi surface

$$\mathcal{L}_\psi = \psi_{\boldsymbol{k}}^\dagger \left(\frac{\partial}{\partial \tau} + \varepsilon(\boldsymbol{k}) \right) \psi_{\boldsymbol{k}}$$



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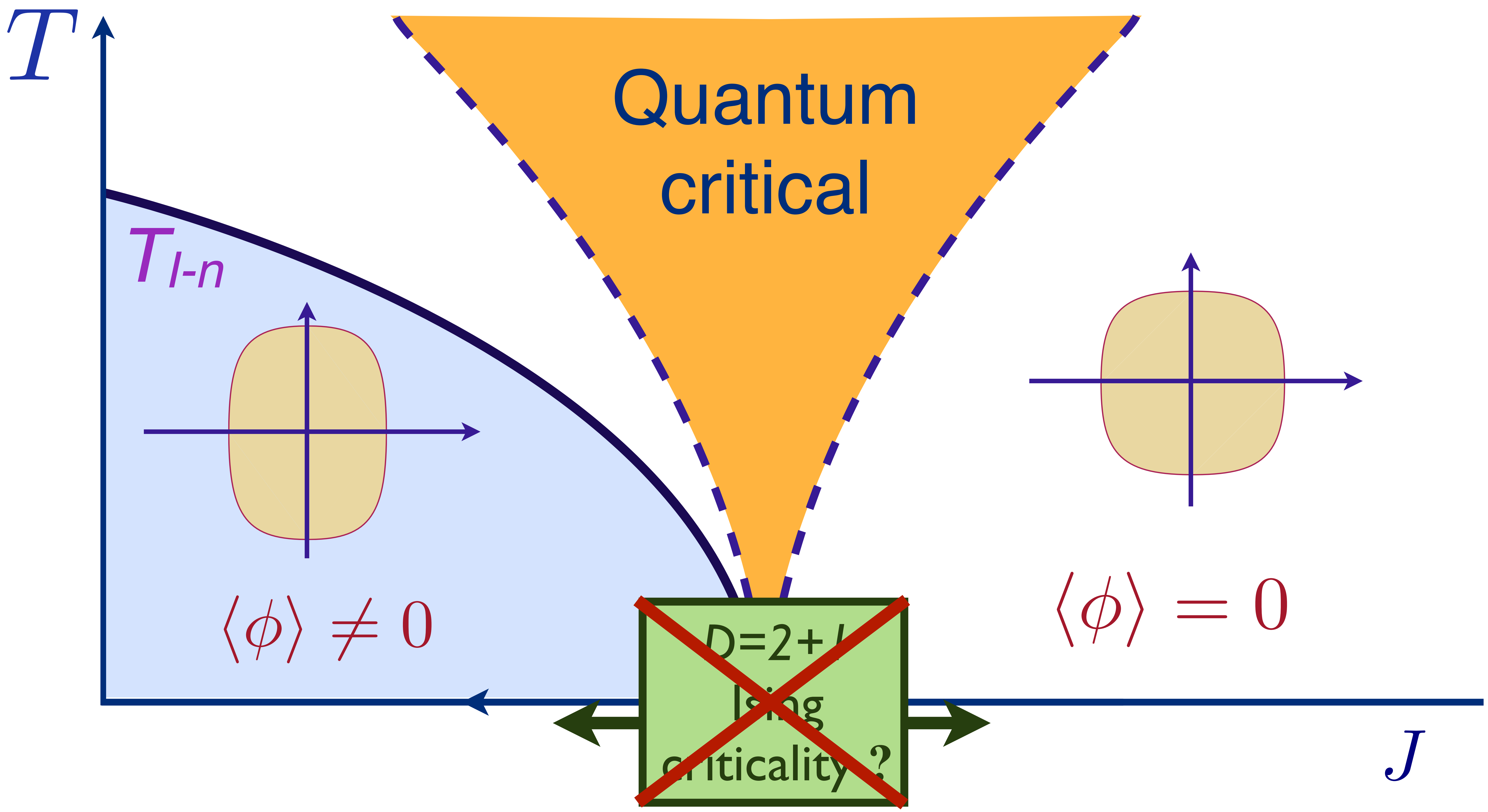
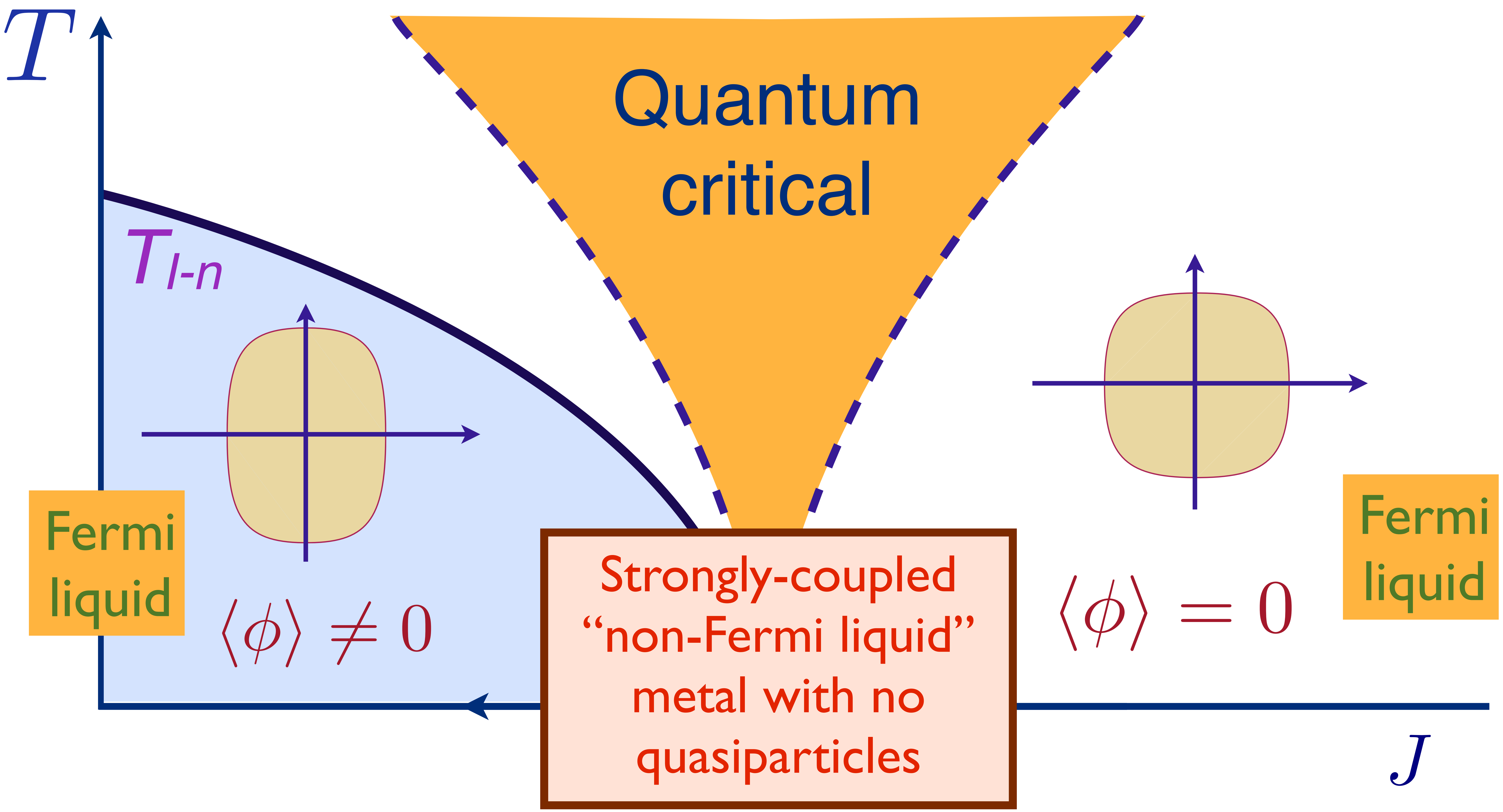


Phase diagram as a function of T and J

Quantum criticality of Ising-nematic ordering in a metal



Quantum critical

Fermi liquid

$\langle \phi \rangle \neq 0$

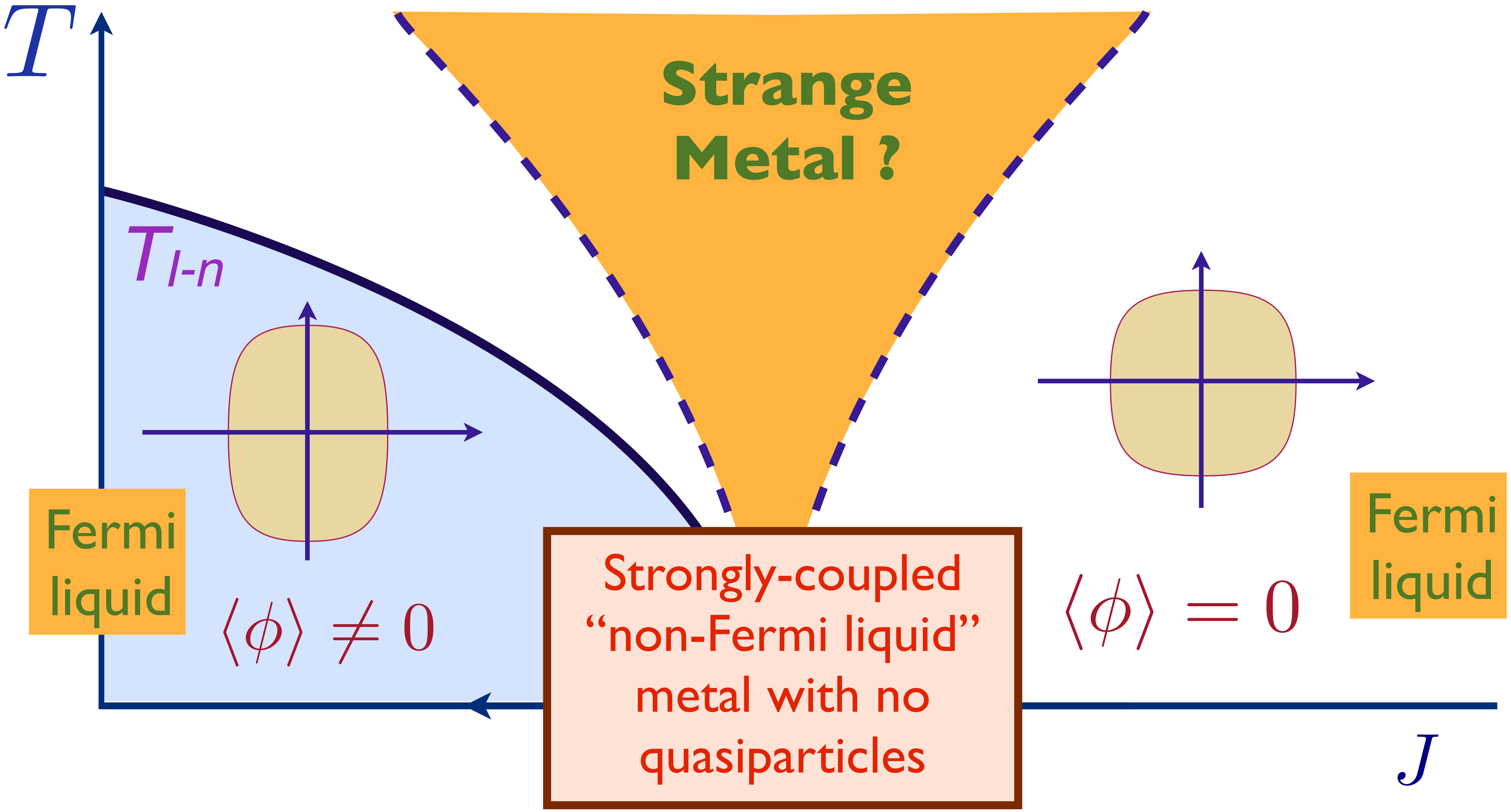
Strongly-coupled "non-Fermi liquid" metal with no quasiparticles

$\langle \phi \rangle = 0$

Fermi liquid

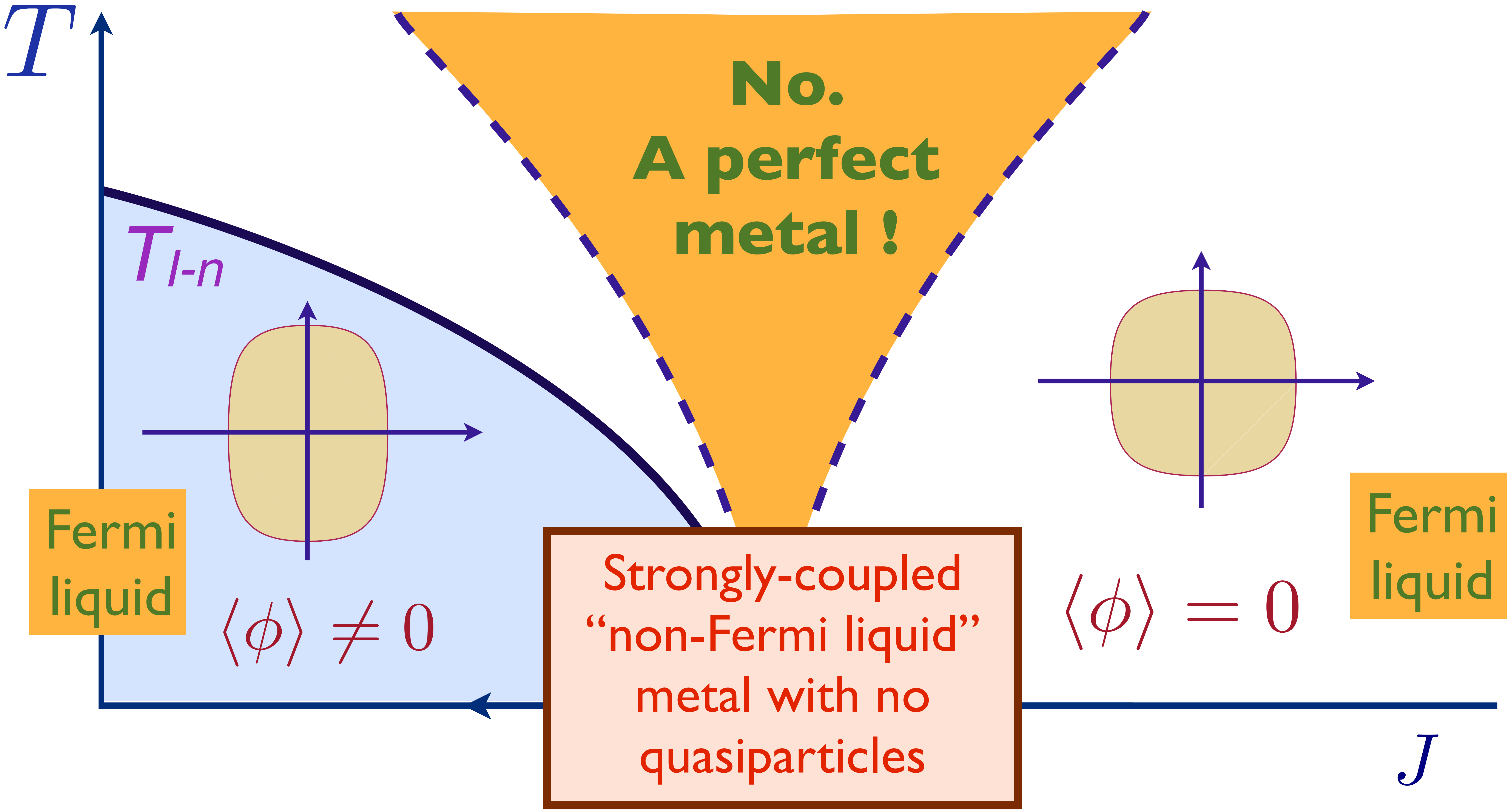
Phase diagram as a function of T and J

Quantum criticality of Ising-nematic ordering in a metal



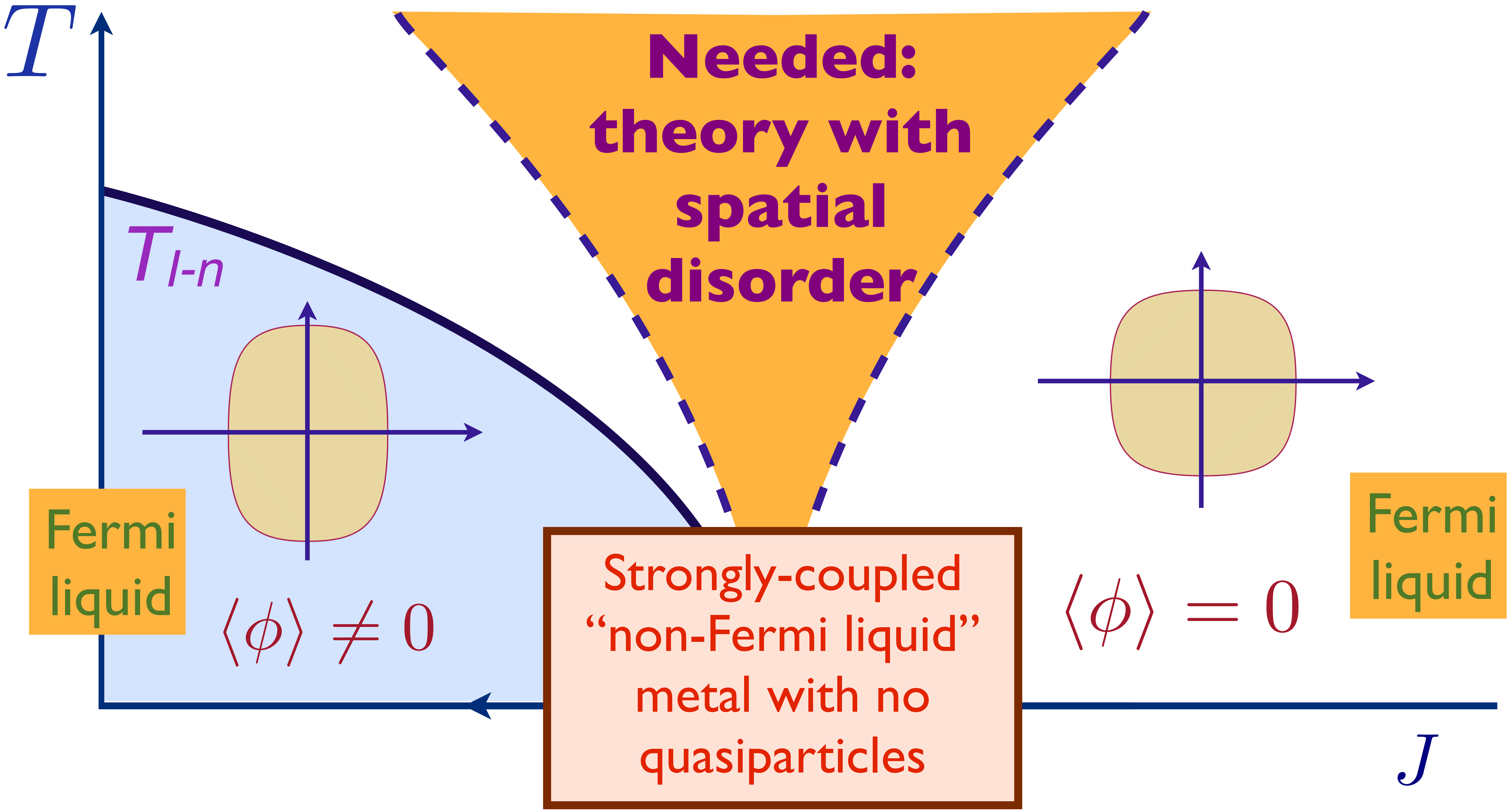
Phase diagram as a function of T and J

Quantum criticality of Ising-nematic ordering in a metal



Phase diagram as a function of T and J

Quantum criticality of Ising-nematic ordering in a metal



Needed: theory with spatial disorder

Fermi liquid

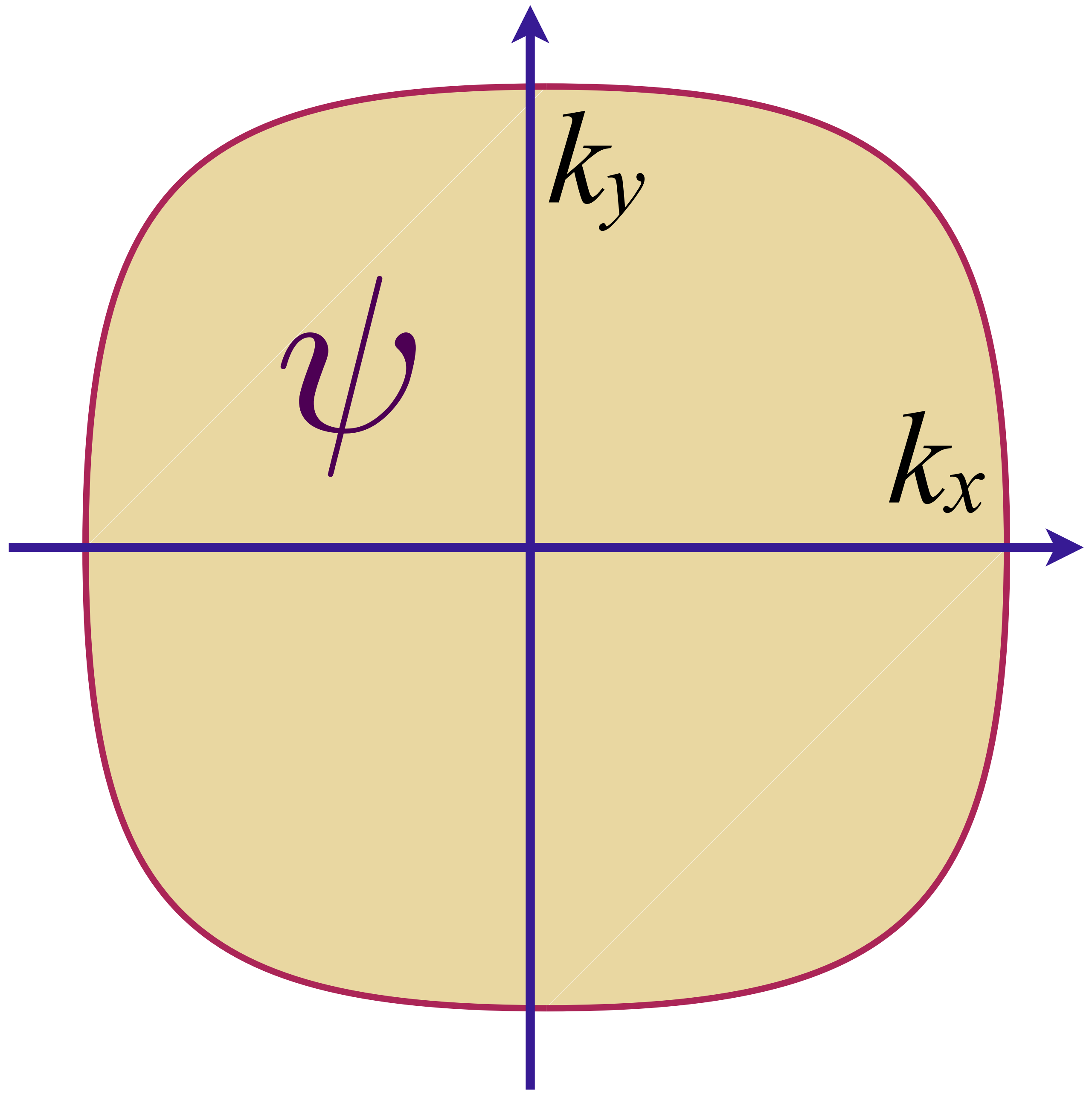
Fermi liquid

Strongly-coupled
"non-Fermi liquid"
metal with no
quasiparticles

Phase diagram as a function of T and J

Critical boson + Fermi surface

$$\mathcal{L}_\psi = \psi_{\boldsymbol{k}}^\dagger \left(\frac{\partial}{\partial \tau} + \varepsilon(\boldsymbol{k}) \right) \psi_{\boldsymbol{k}}$$



A critical boson ϕ
e.g. Ising-nematic order,
spin-density wave order,
Higgs boson for Fermi-volume changing transition

$$+s\,[\phi(\boldsymbol{r})]^2$$

$$+K\,[\nabla_{\boldsymbol{r}}\phi(\boldsymbol{r})]^2 + u\,[\phi(\boldsymbol{r})]^4$$

Critical boson + Fermi surface with potential and interaction disorder

$$\mathcal{L}_\psi = \psi^\dagger_{\boldsymbol{k}} \left(\frac{\partial}{\partial \tau} + \varepsilon(\boldsymbol{k}) \right) \psi_{\boldsymbol{k}}$$

A critical boson ϕ
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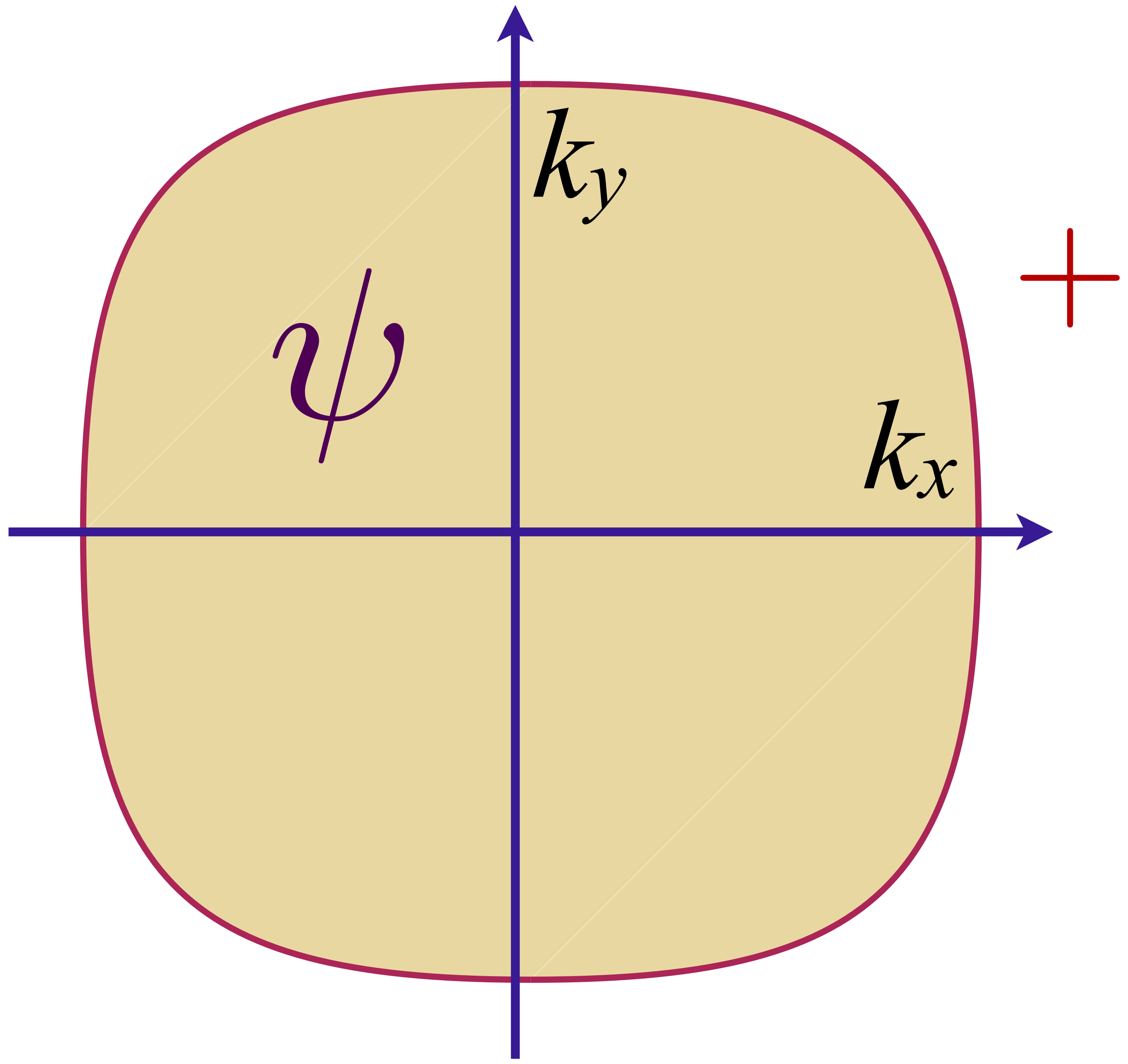


$$+[s + \delta s(\boldsymbol{r})] \, [\phi(\boldsymbol{r})]^2 + [g + g'(\boldsymbol{r})] \, \psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r}) \, \phi(\boldsymbol{r})$$

$$+K \, [\nabla_{\boldsymbol{r}} \phi(\boldsymbol{r})]^2 + u \, [\phi(\boldsymbol{r})]^4 \; + v(\boldsymbol{r})\psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})$$

Spatially random Yukawa coupling $g'(\boldsymbol{r})$ with $\overline{g'(\boldsymbol{r})} = 0$, $\overline{g'(\boldsymbol{r})g'(\boldsymbol{r}')} = g'^2 \delta(\boldsymbol{r} - \boldsymbol{r}')$

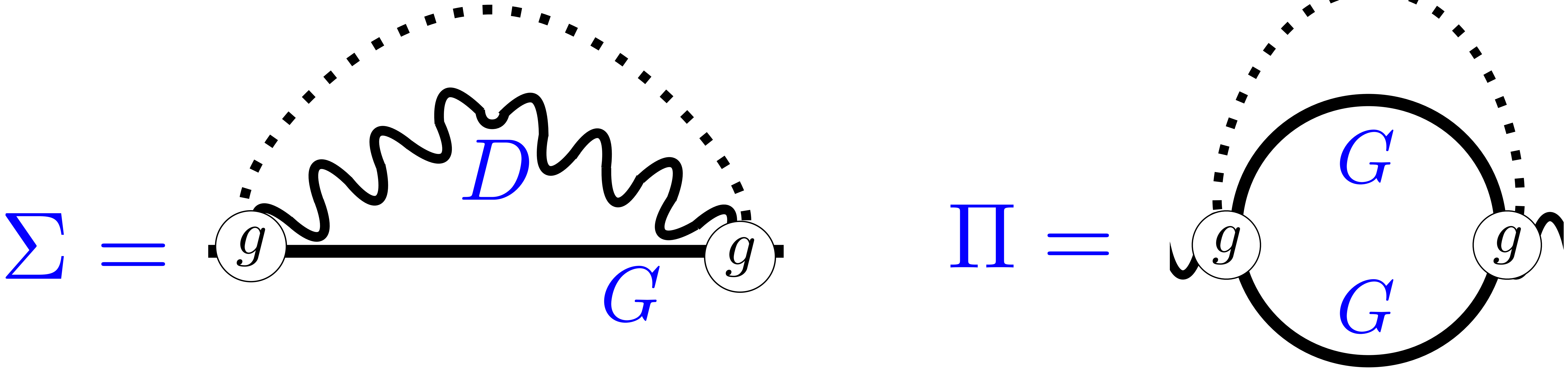
Spatially random mass $\delta s(\boldsymbol{r})$ with $\overline{\delta s(\boldsymbol{r})} = 0$, $\overline{\delta s(\boldsymbol{r}) \delta s(\boldsymbol{r}')} = \delta s^2 \delta(\boldsymbol{r} - \boldsymbol{r}')$

Spatially random potential $v(\boldsymbol{r})$ with $\overline{v(\boldsymbol{r})} = 0$, $\overline{v(\boldsymbol{r})v(\boldsymbol{r}')} = v^2 \delta(\boldsymbol{r} - \boldsymbol{r}')$

Yukawa-SYK model

$$H = \sum_{ij} t_{ij} \psi_i^\dagger \psi_j + \sum_{\ell} \frac{1}{2} \left(\pi_\ell^2 + \omega_\ell^2 \phi_\ell^2 \right) + \sum_{ij\ell} g_{ij\ell} \psi_i^\dagger \psi_j \phi_\ell$$

Leads to fully self-consistent Migdal-Eliashberg equations
$\Sigma_\psi \sim g^2 G_\psi G_\phi$, $\Sigma_\phi \sim g^2 G_\psi G_\psi$ in a SYK-like large N limit
and properties very similar to the SYK model



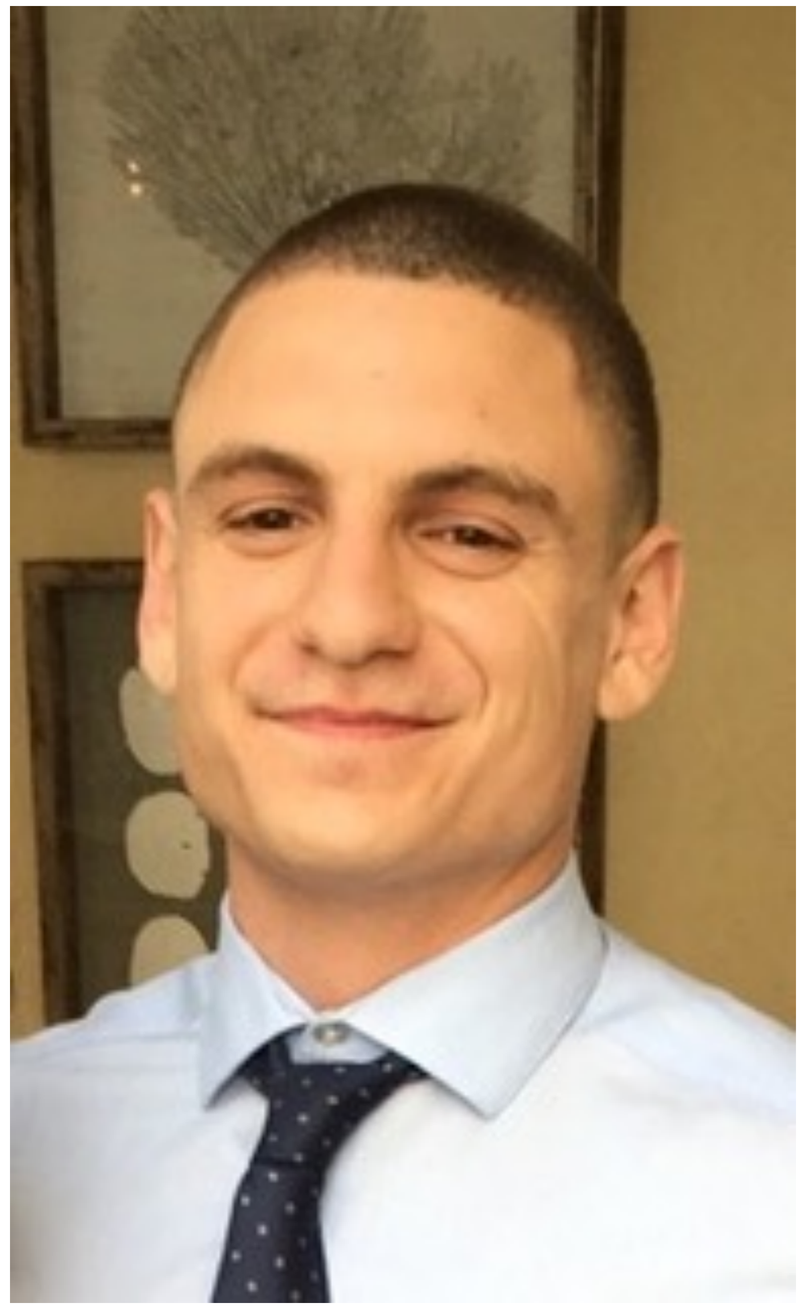
W. Fu, D. Gaiotto, J. Maldacena, and S. Sachdev, PRD **95**, 026009 (2017); J. Murugan, D. Stanford, and E. Witten, JHEP 08, 146 (2017); A. A. Patel and S. Sachdev, PRB **98**, 125134 (2018); E. Marcus and S. Vandoren, JHEP 01, 166 (2018); Yuxuan Wang, PRL **124**, 017002 (2020); I. Esterlis and J. Schmalian, PRB **100**, 115132 (2019); Yuxuan Wang and A. V. Chubukov, PRR **2**, 033084 (2020).

Aavishkar Patel

Flatiron Institute, NYC

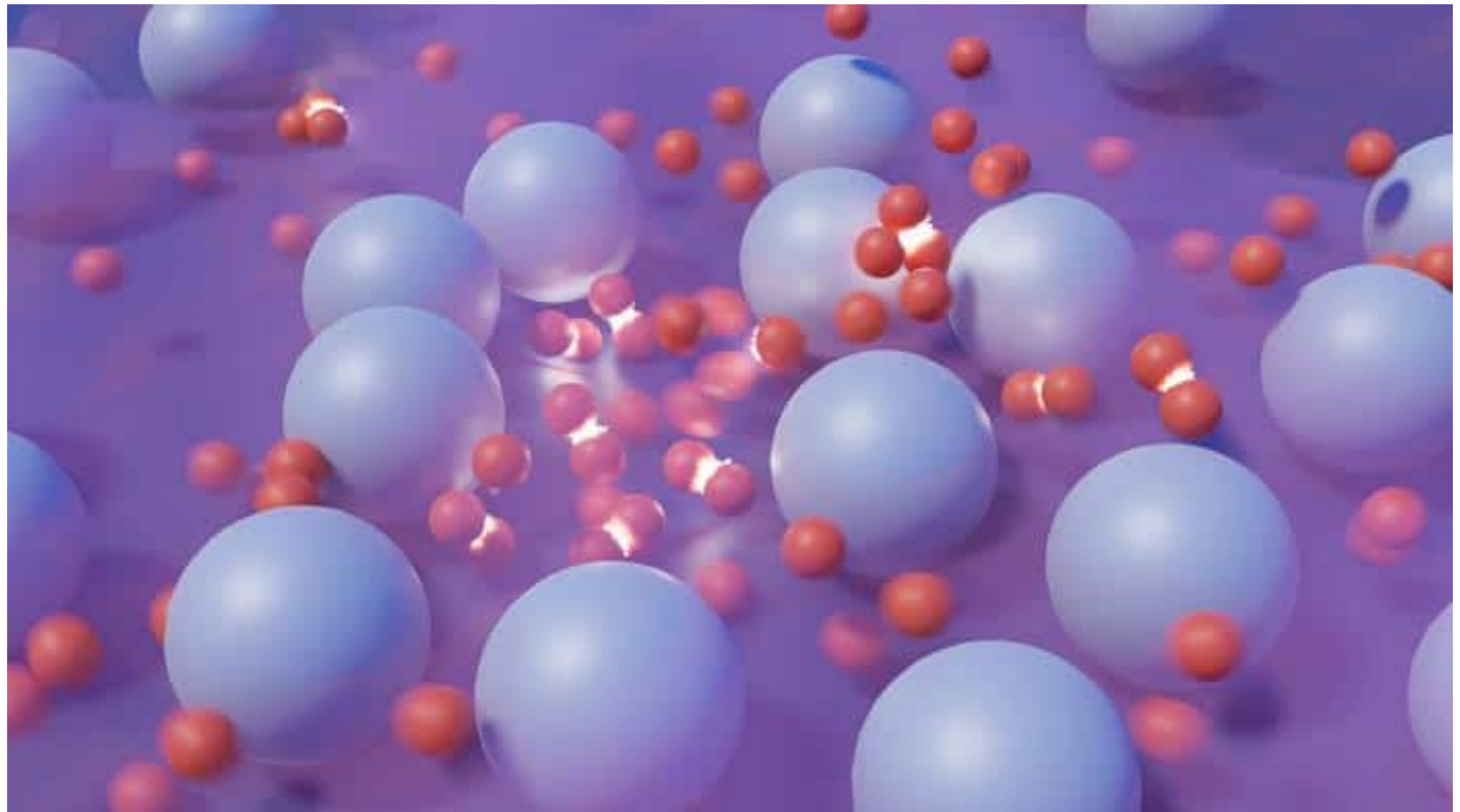
Haoyu Guo

Cornell



Ilya Esterlis

Wisconsin

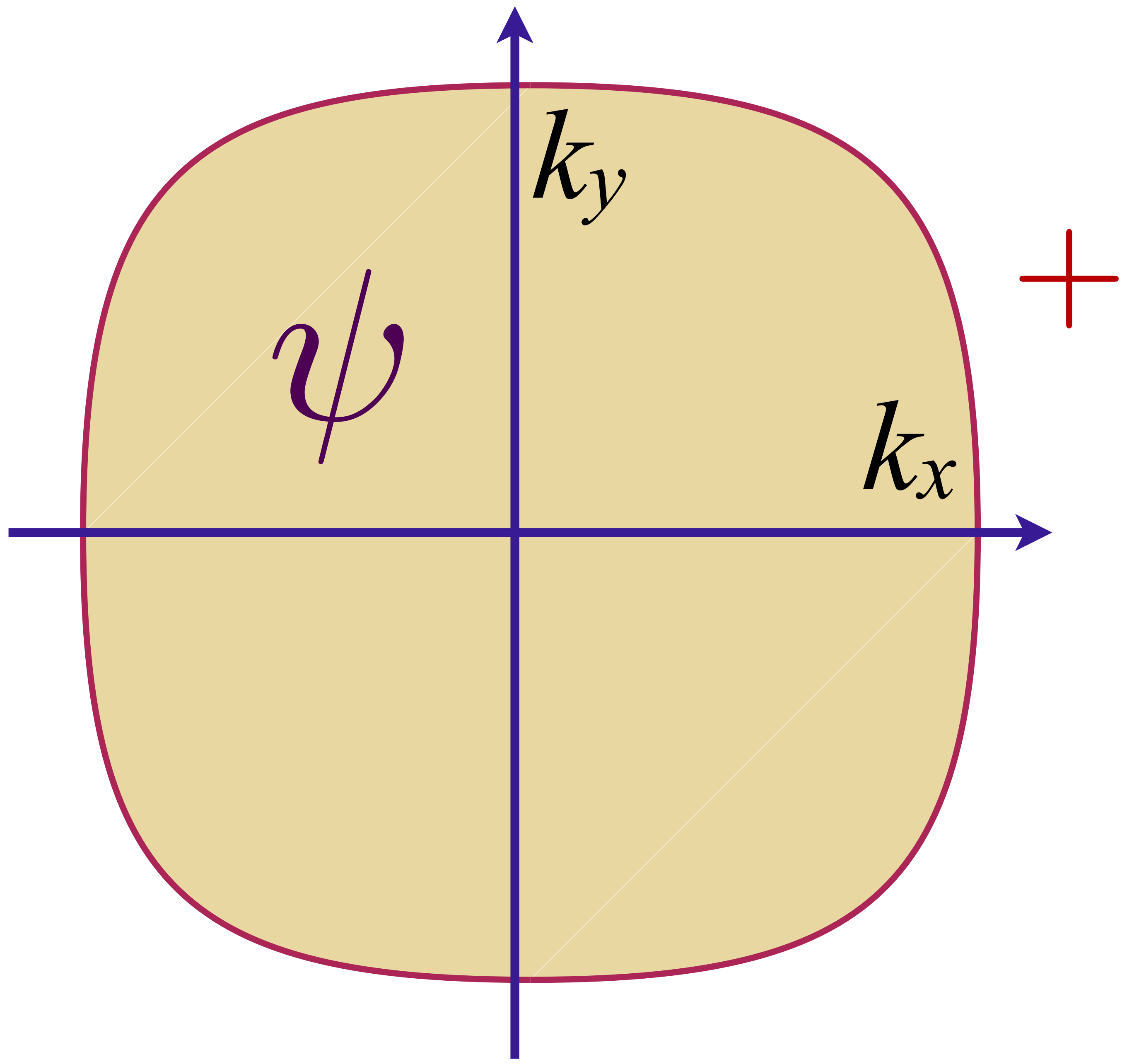


Universal theory of strange metals from spatially random interactions,
Aavishkar A. Patel, Haoyu Guo,
Ilya Esterlis, and S. Sachdev,
Science **381**, 790 (2023)

Critical boson + Fermi surface with potential and interaction disorder

$$\mathcal{L}_\psi = \psi^\dagger_{\boldsymbol{k}} \left(\frac{\partial}{\partial \tau} + \varepsilon(\boldsymbol{k}) \right) \psi_{\boldsymbol{k}}$$

A critical boson ϕ
e.g. Ising-nematic order,
spin-density wave order,
Higgs boson for Fermi-volume changing transition



$$+[s + \delta s(\boldsymbol{r})]\,[\phi(\boldsymbol{r})]^2 + [g + g'(\boldsymbol{r})]\,\psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})\,\phi(\boldsymbol{r})$$

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Critical boson + Fermi surface with potential and interaction disorder

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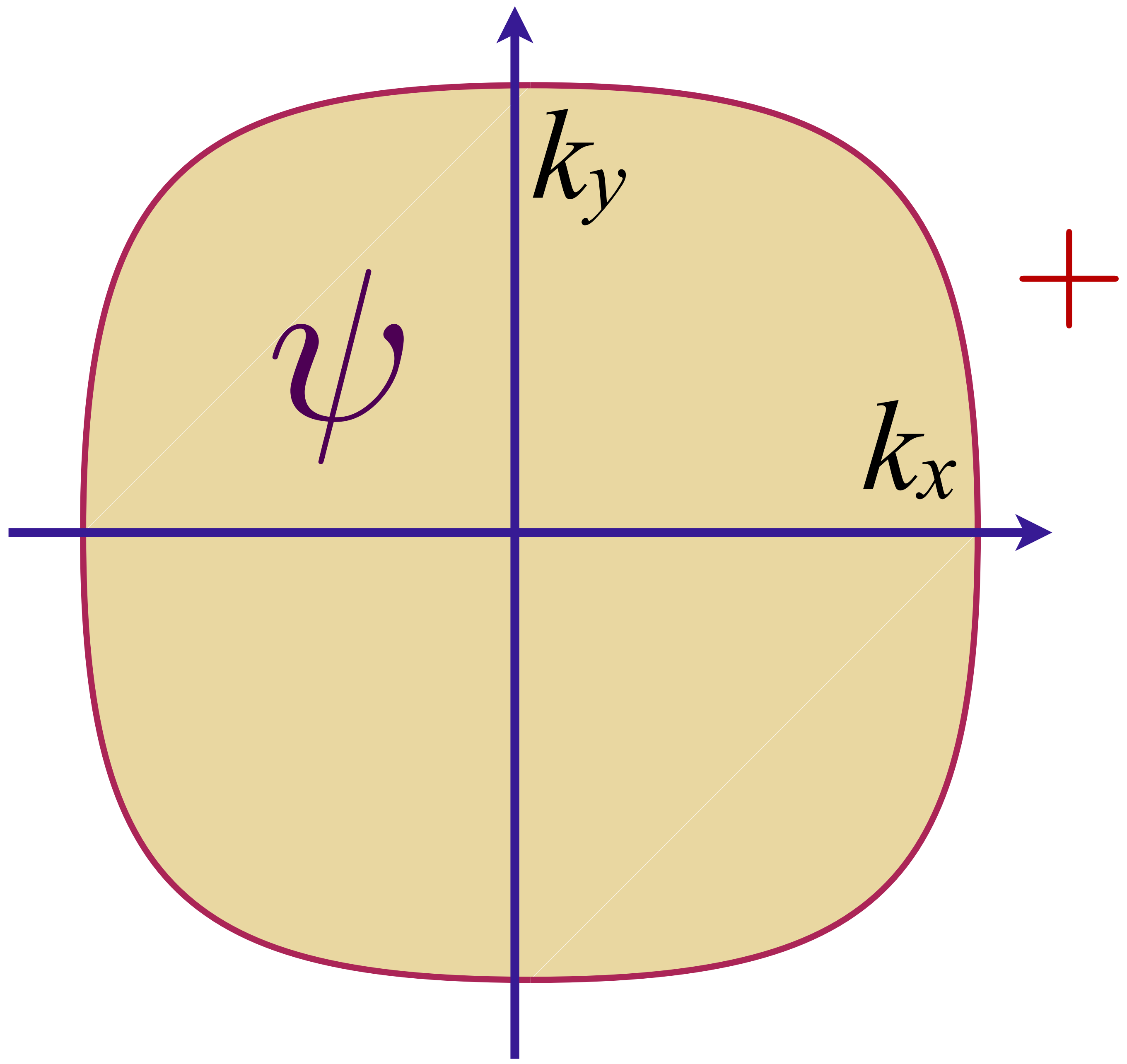

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Rescale $\phi(\boldsymbol{r})$ to absorb random mass $\delta s(\boldsymbol{r})$ into
the random Yukawa coupling $g'(\boldsymbol{r})$
and analyze with self-averaging as in Yukawa-SYK model.

Fermi surface + critical boson with potential and interaction disorder

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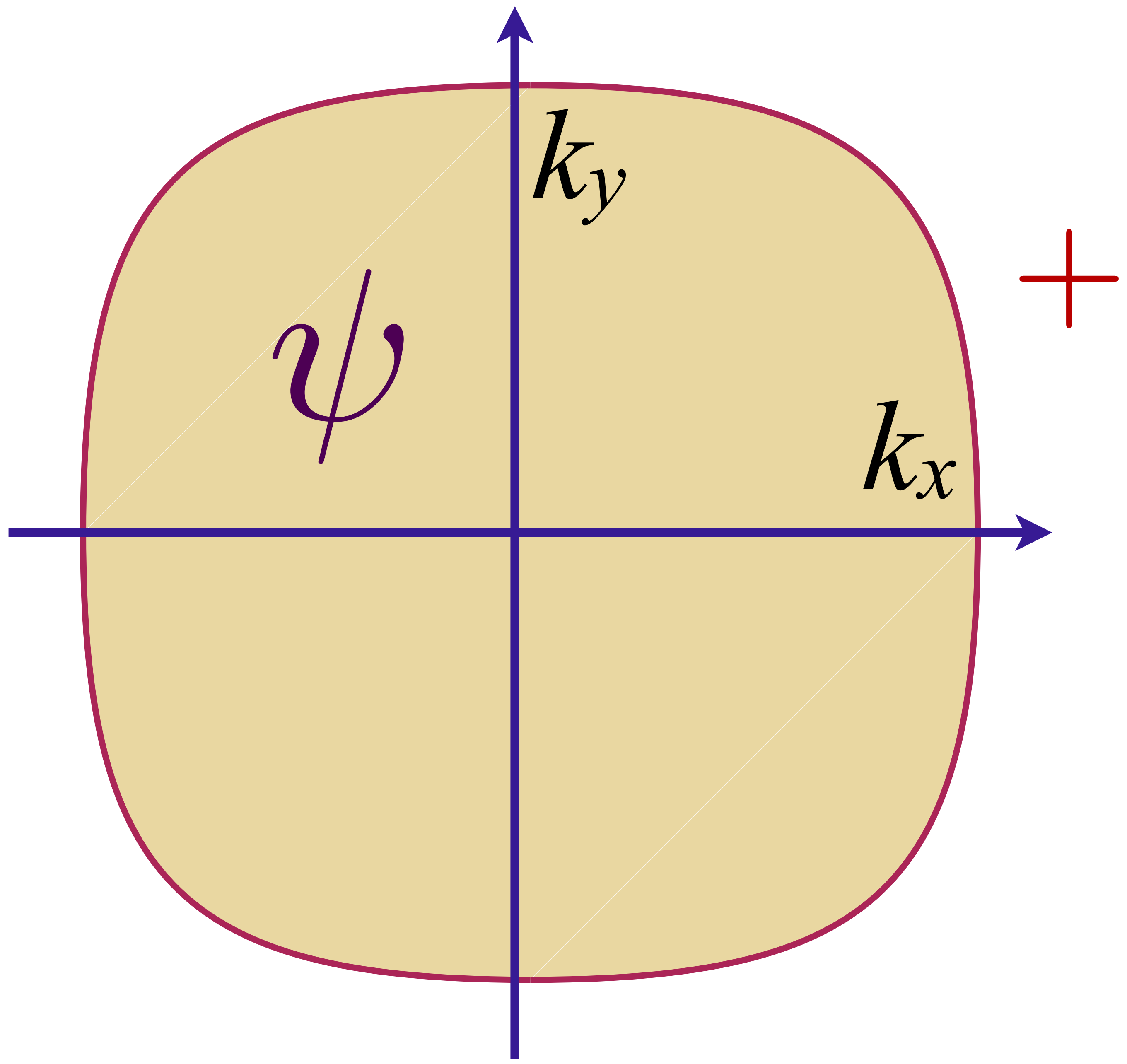

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Rescale $\phi(\boldsymbol{r})$ to absorb random mass $\delta s(\boldsymbol{r})$ into
the random Yukawa coupling $g'(\boldsymbol{r})$
and analyze with self-averaging as in Yukawa-SYK model.

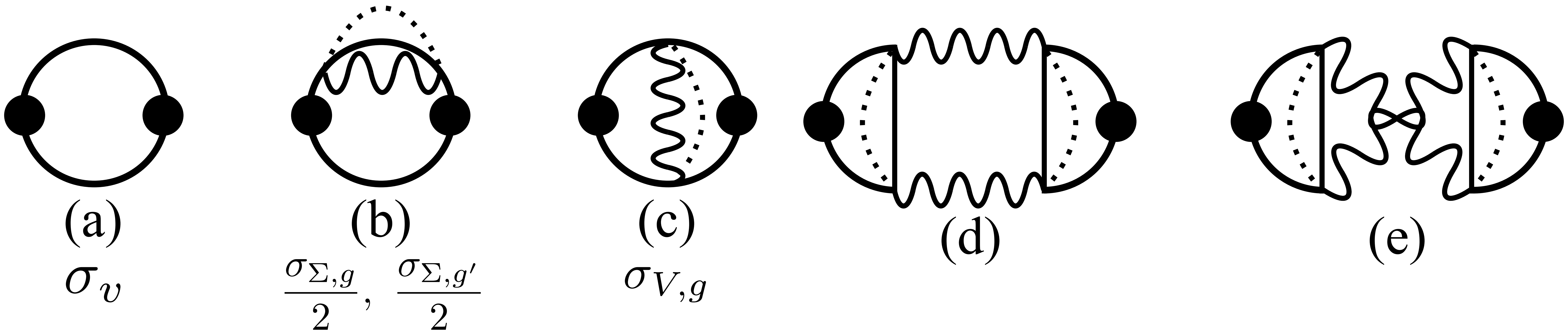
Should be applicable as long as eigenmodes of $\phi(\boldsymbol{r})$ are extended.

Critical boson + Fermi surface with potential and interaction disorder

SYK-type self-consistent equations

$$\Sigma(\tau,\mathbf{r}) = g^2 D(\tau,\mathbf{r}) G(\tau,\mathbf{r}) + v^2 G(\tau,\mathbf{r})\delta^2(\mathbf{r}) + {g'}^2 G(\tau,\mathbf{r}) D(\tau,\mathbf{r})\delta^2(\mathbf{r}),$$
$$\Pi(\tau,\mathbf{r}) = -g^2 G(-\tau,-\mathbf{r}) G(\tau,\mathbf{r}) - {g'}^2 G(-\tau,\mathbf{r}) G(\tau,\mathbf{r})\delta^2(\mathbf{r}),$$
$$G(i\omega,\mathbf{k}) = \frac{1}{i\omega - \varepsilon(\mathbf{k}) + \mu - \Sigma(i\omega,\mathbf{k})},$$
$$D(i\Omega,\mathbf{q}) = \frac{1}{\Omega^2 + \mathbf{q}^2 + m_b^2 - \Pi(i\Omega,\mathbf{q})}.$$

Conductivity:



+ all ladders and bubbles…..

Fermi surface + critical boson with potential and interaction disorder

$$\text{Conductivity: } \sigma(\omega) \sim \frac{1}{\dfrac{1}{\tau_{\text{trans}}(\omega)} - i\omega \dfrac{m^*_{\text{trans}}(\omega)}{m}}$$

$$\frac{1}{\tau_{\text{trans}}(\omega)} \sim v^2 + g'^2 |\omega| \quad ; \quad \frac{m^*_{\text{trans}}(\omega)}{m} \sim \frac{2g'^2}{\pi} \ln(\Lambda/\omega)$$

$$\text{Electron Green's function: } G(\omega) \sim \frac{1}{\omega \dfrac{m^*(\omega)}{m} - \varepsilon(\boldsymbol{k}) + i \left(\dfrac{1}{\tau_e} + \dfrac{1}{\tau_{\text{in}}(\omega)} \right) \text{sgn}(\omega)}$$

$$\frac{1}{\tau_e} \sim v^2 \quad ; \quad \frac{1}{\tau_{\text{in}}(\omega)} \sim \left(\frac{g^2}{v^2} + g'^2 \right) |\omega| \quad ; \quad \frac{m^*(\omega)}{m} \sim \frac{2}{\pi} \left(\frac{g^2}{v^2} + g'^2 \right) \ln(\Lambda/\omega)$$

Residual resistivity is determined by v^2; Linear-in-T resistivity determined by g'^2; Transport insensitive to g; Marginal Fermi liquid self energy and $T\ln(1/T)$ specific heat.

Aavishkar A. Patel, Haoyu Guo, Ilya Esterlis, S. Sachdev, *Science* **381**, 790 (2023)

Observable properties:

1. Resistivity $\rho(T) \sim v^2 + g'^2 T$.
2. Optical conductivity

$$\sigma(\omega) = \frac{K}{\dfrac{1}{\tau_{\rm trans}(\omega)} - i\omega \dfrac{m^*_{\rm trans}(\omega)}{m}} \quad ; \quad \frac{1}{\tau_{\rm trans}(\omega)} \sim |\omega| \Phi_\sigma \left(\frac{\hbar\omega}{k_B T} \right)$$

B. Michon……A. Georges, Nat. Commun. **14**, 3033 (2023)

3. Photoemission: nearly marginal Fermi liquid electron spectral density:

$$\mathrm{Im}\Sigma(\omega) \sim |\omega|^{2\alpha} \Phi_\Sigma \left(\frac{\hbar\omega}{k_B T} \right) \quad \text{with } \alpha \approx 1/2 \quad ; \quad \frac{1}{\tau_{\rm in}(\omega)} \sim |\omega| \Phi_\Sigma \left(\frac{\hbar\omega}{k_B T} \right)$$

T.J. Reber….D. Dessau, Nature Communications **10**, 5737 (2019)

4. Specific heat $\sim T \ln(1/T)$ as $T \to 0$.

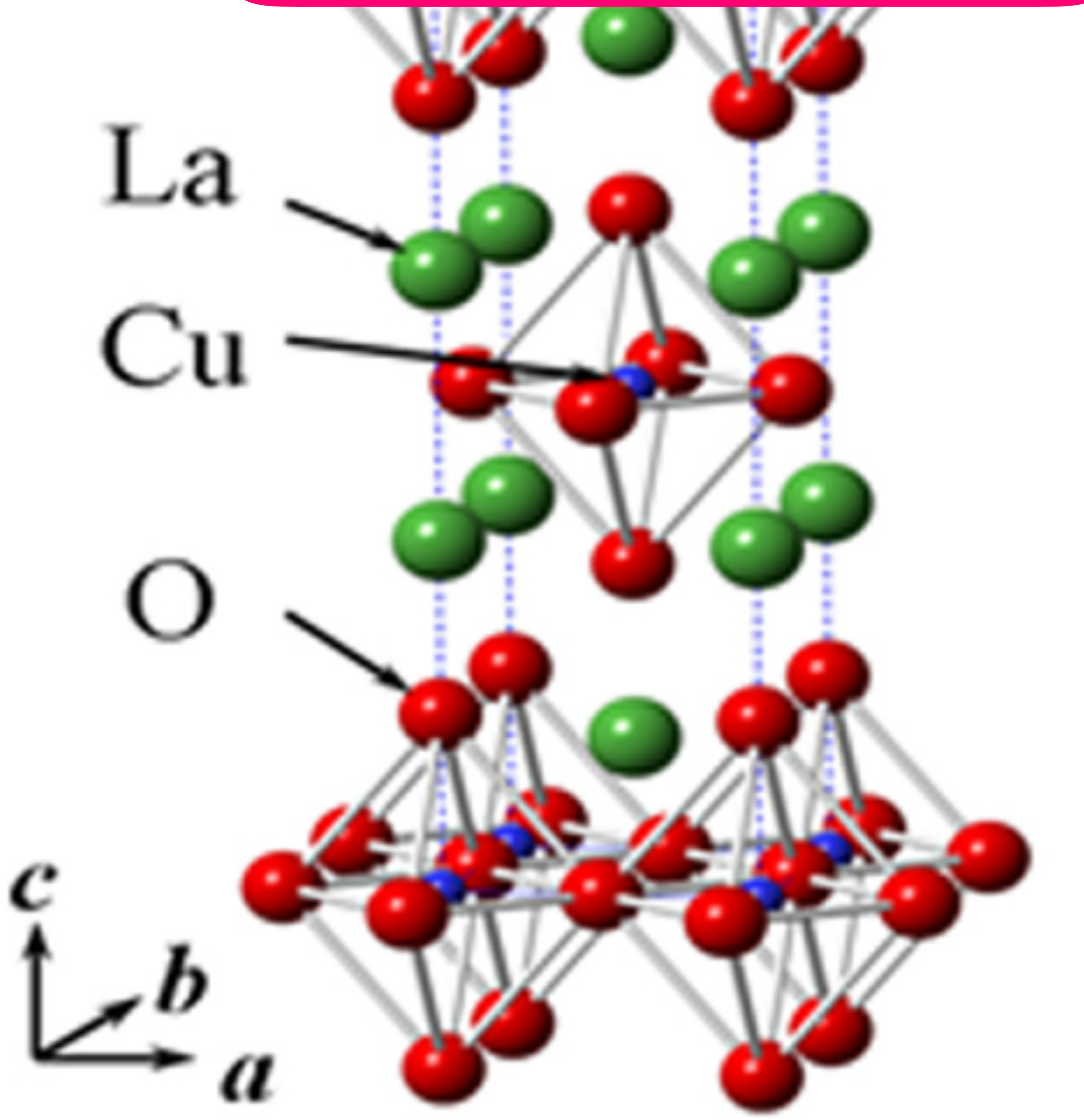
S.A. Hartnoll and A.P. MacKenzie, RMP (2022)

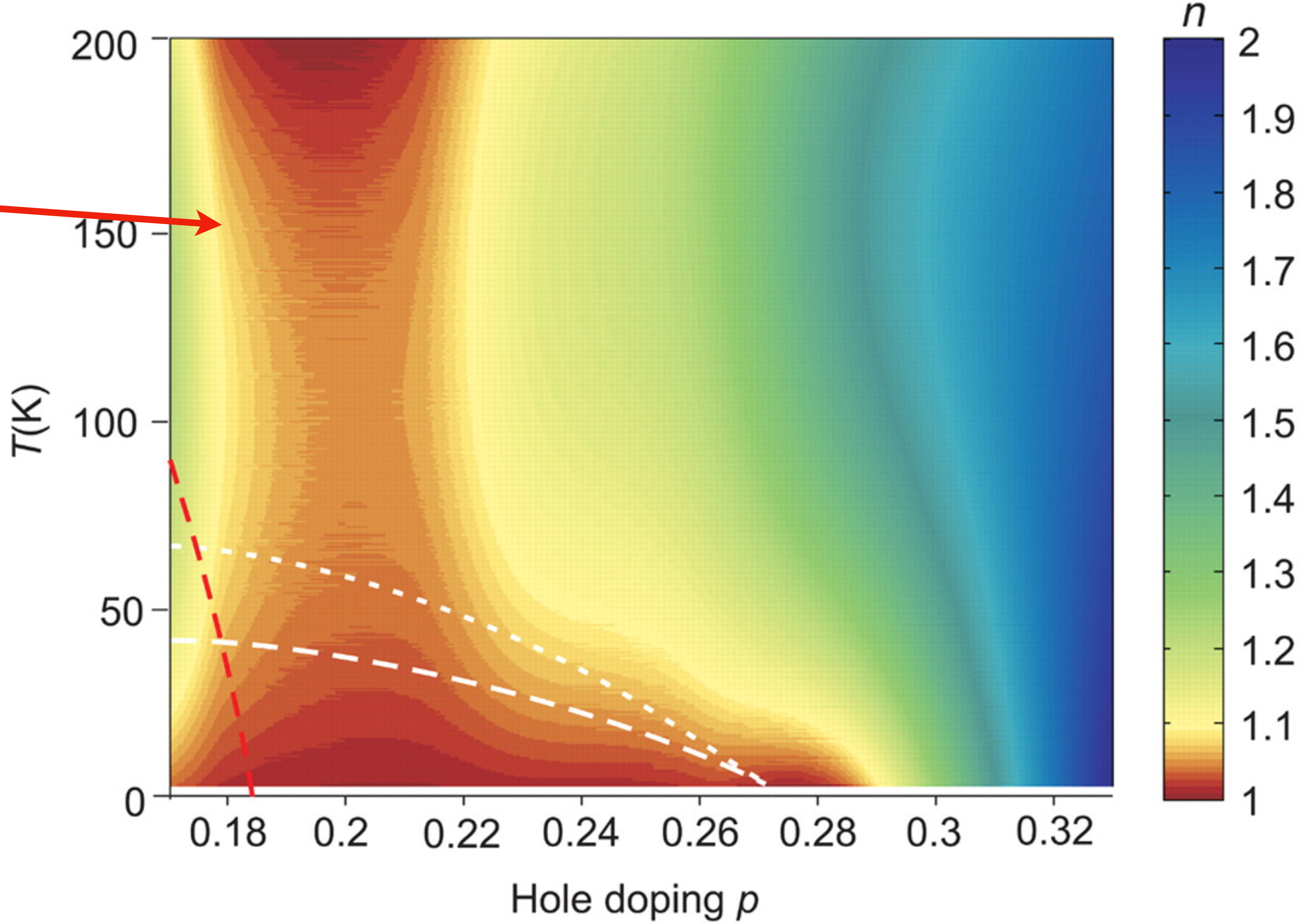
Anomalous Criticality in the Electrical Resistivity of $La_{2-x}Sr_xCuO_4$

R. A. Cooper,[1] Y. Wang,[1] B. Vignolle,[2] O. J. Lipscombe,[1] S. M. Hayden,[1] Y. Tanabe,[3] T. Adachi,[3]
Y. Koike,[3] M. Nohara,[4*] H. Takagi,[4] Cyril Proust,[2] N. E. Hussey[1†]

SCIENCE VOL 323 **603** 2009

Yukawa-SYK
in $d = 2$





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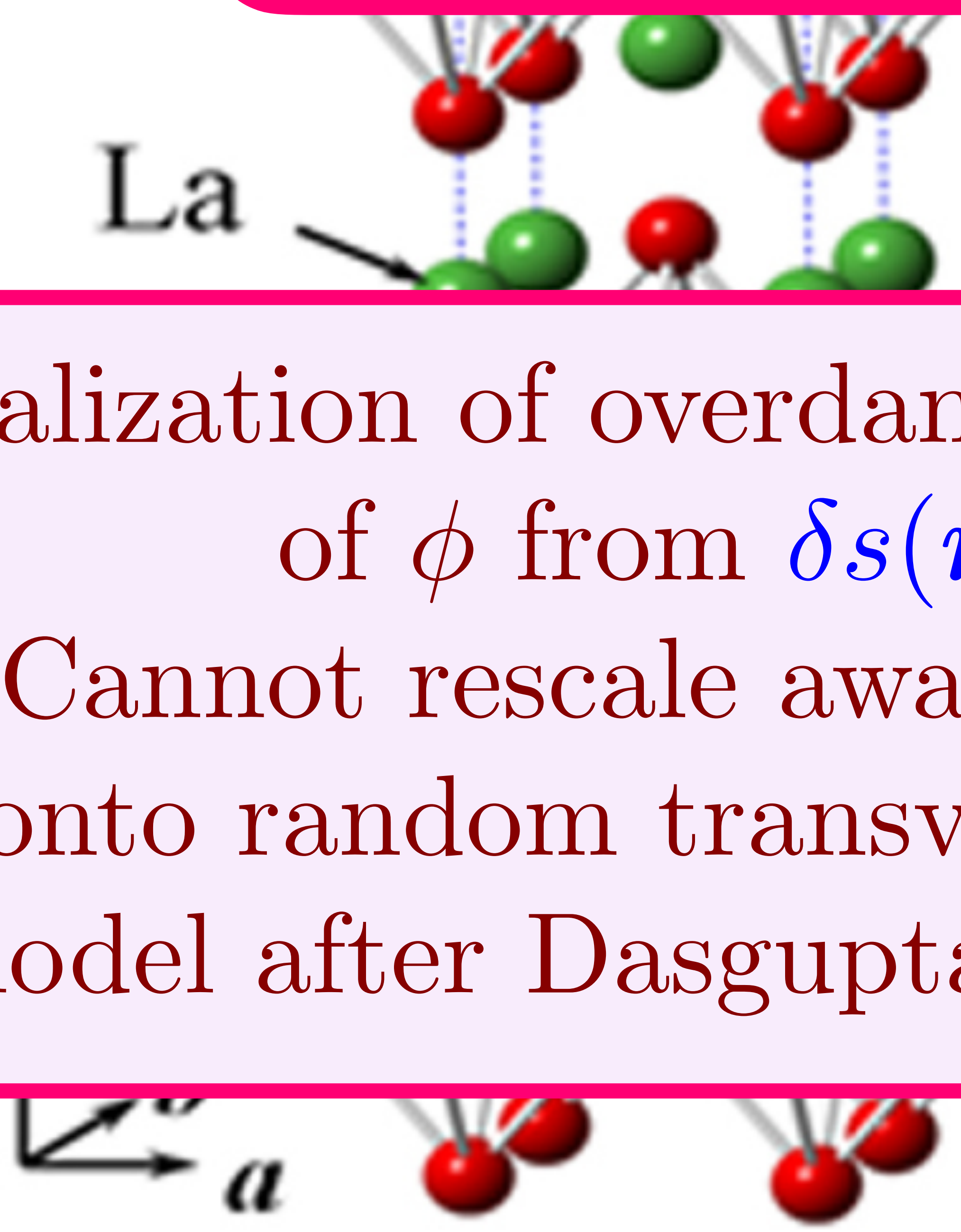
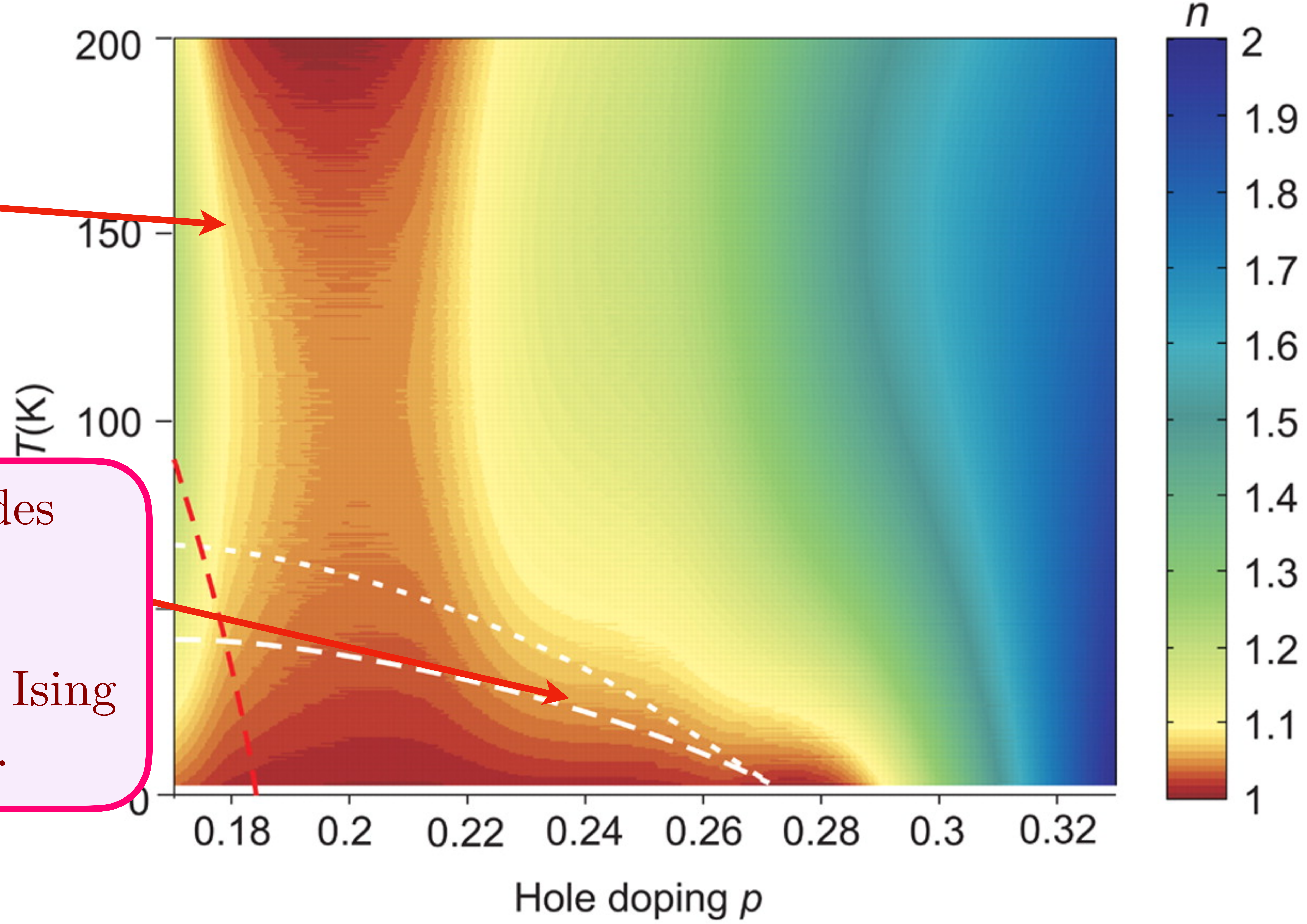
Aavishkar A. Patel, Peter Lunts,
S. Sachdev, *arXiv:2312.06751*

Yukawa-SYK
in $d = 2$





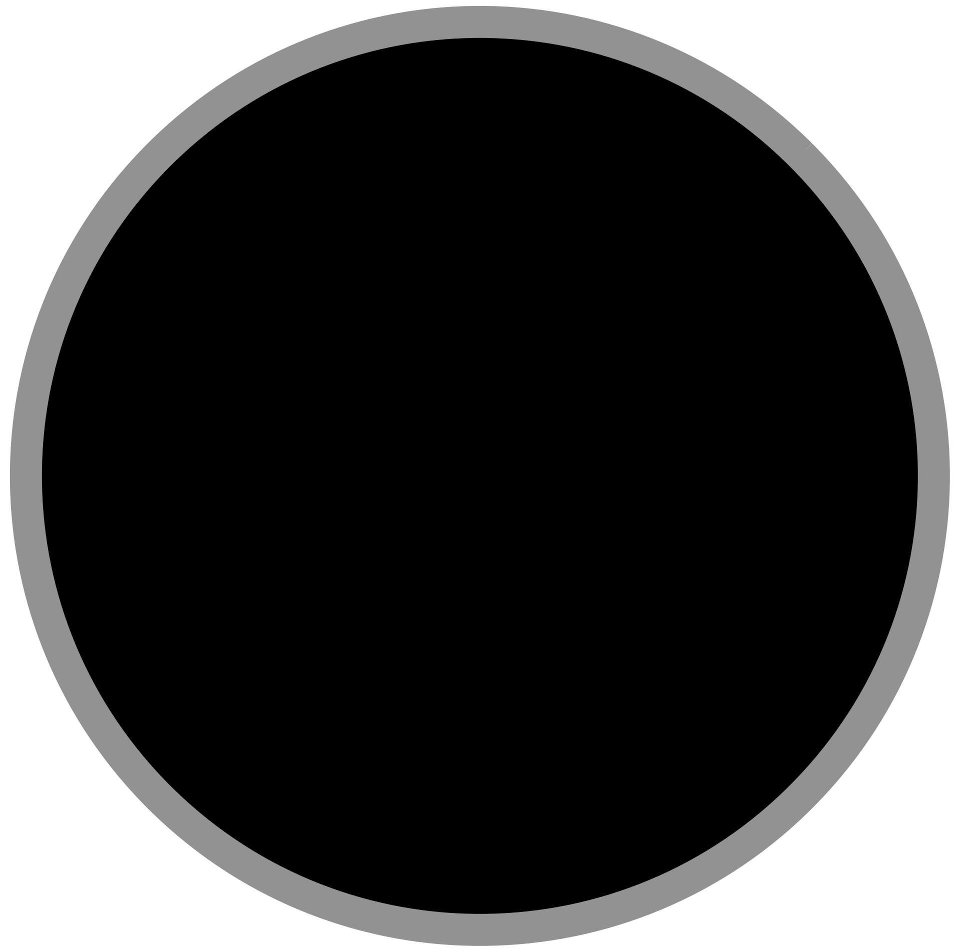
Localization of overdamped modes
of ϕ from $\delta s(\boldsymbol{r})$.
Cannot rescale away $\delta s(\boldsymbol{r})$.
Maps onto random transverse field Ising
model after Dasgupta-Ma RG.

Quantum black holes

Black Holes

Objects so dense that light is gravitationally bound to them.

$$\text{Horizon radius } R = \frac{2GM}{c^2}$$

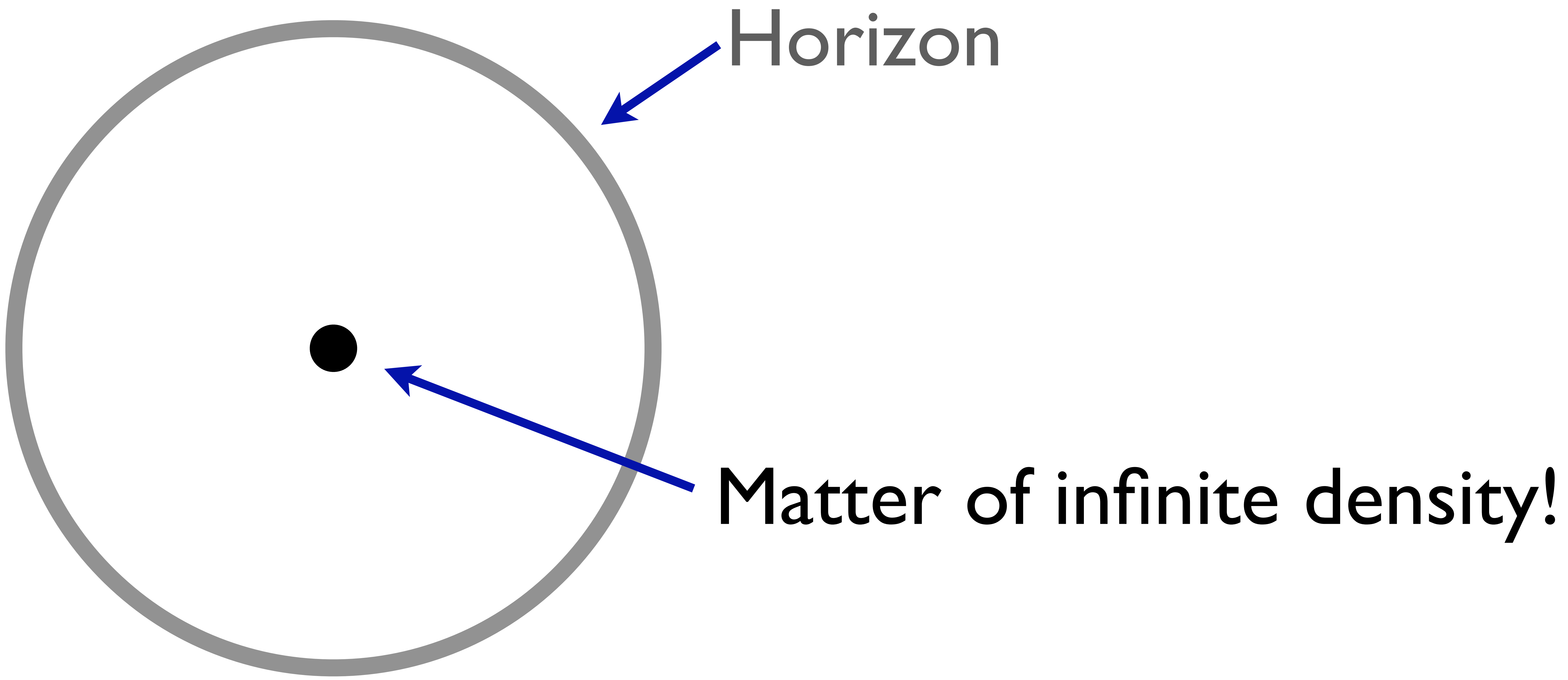


G Newton's constant, c velocity of light, M mass of black hole

For $M =$ earth's mass, $R \approx 9\,mm$!

What is inside a black hole ???

In Einstein’s theory, all the matter in a black hole collapses to a singularity at the center of the black hole.



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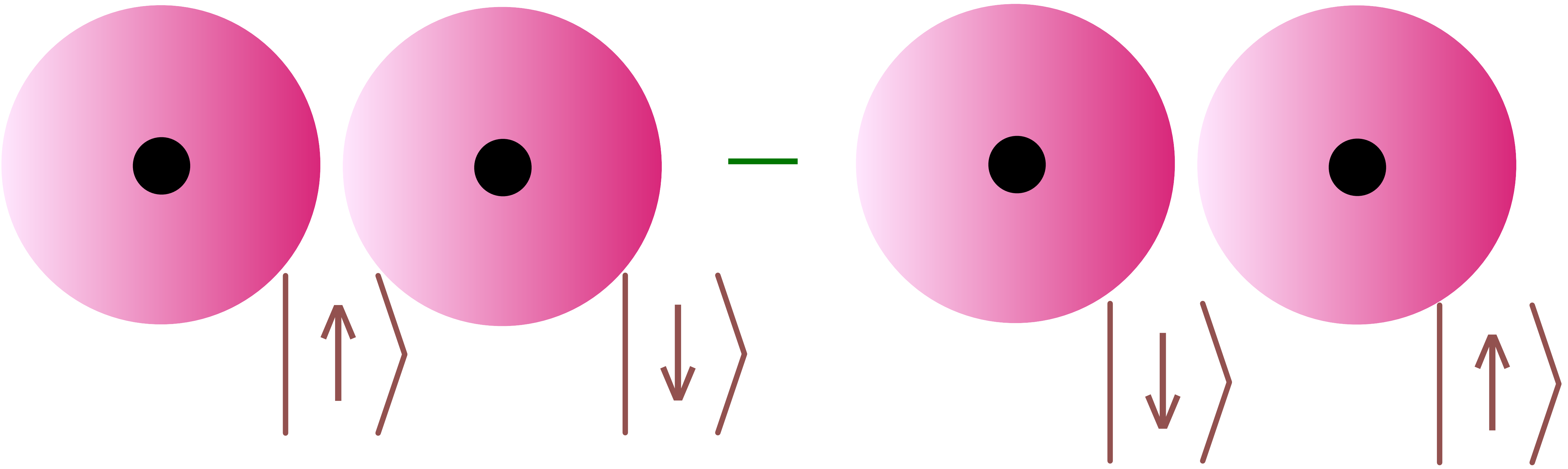
Quantum Entanglement across a black hole horizon



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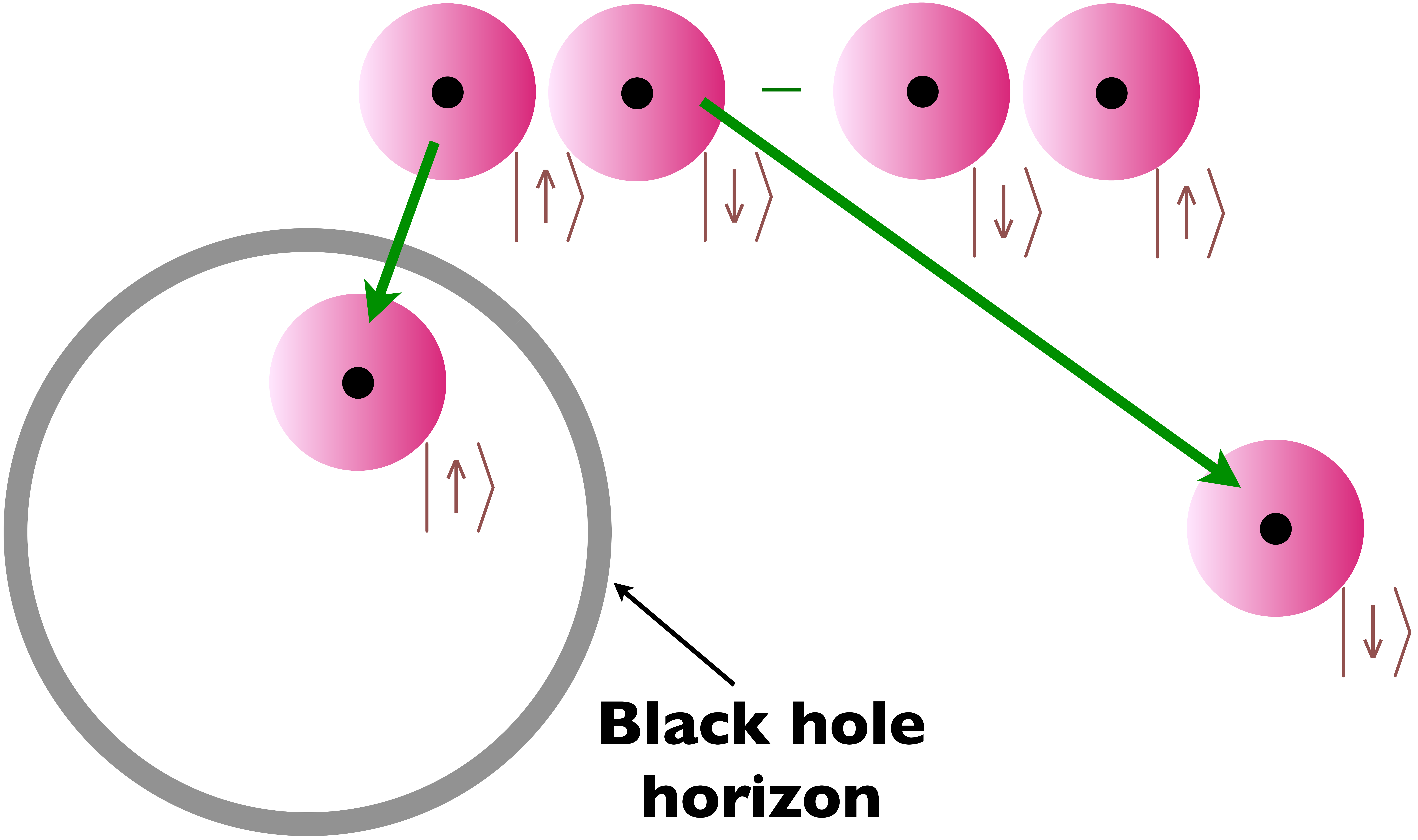

Black hole horizon

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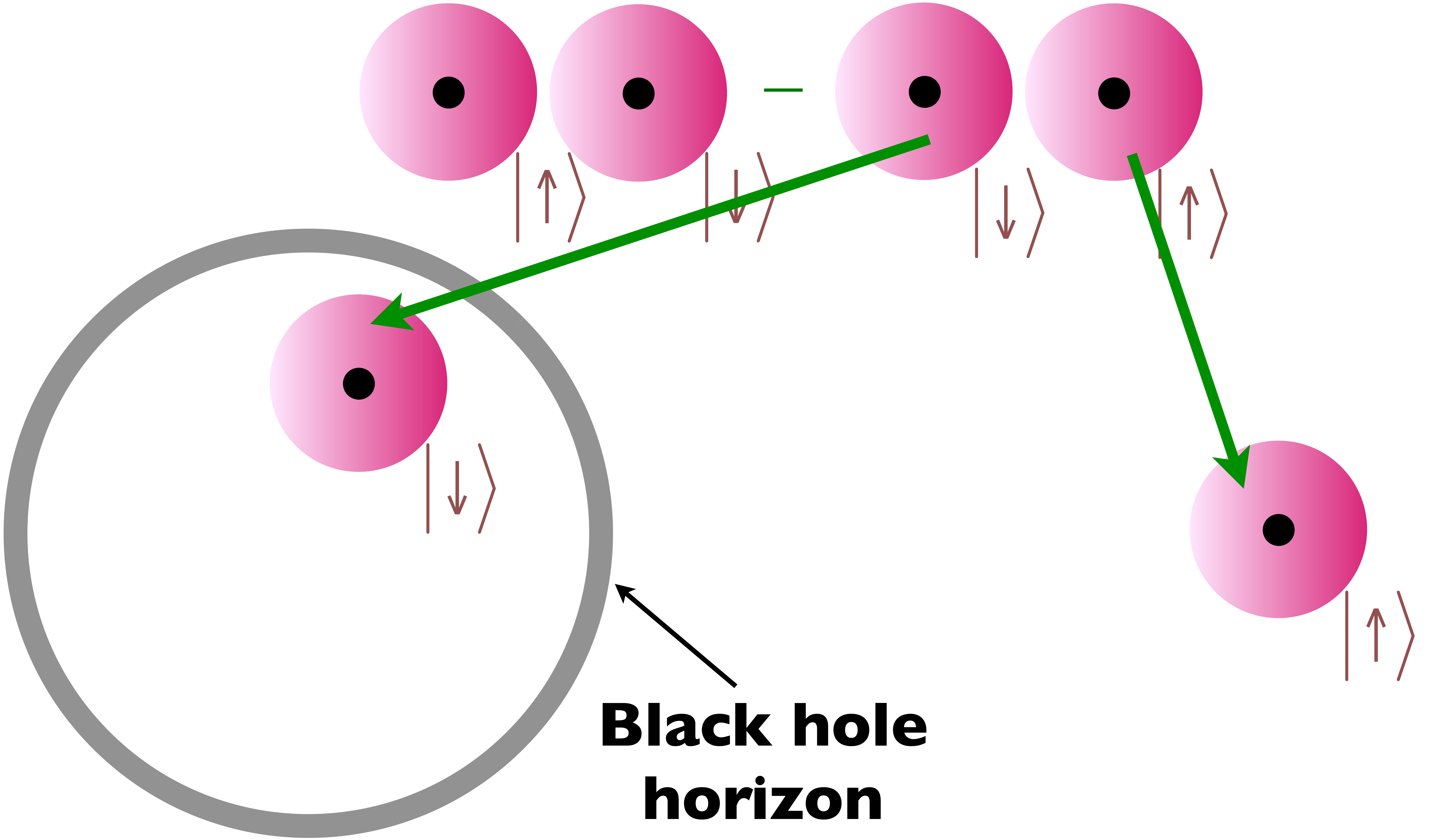
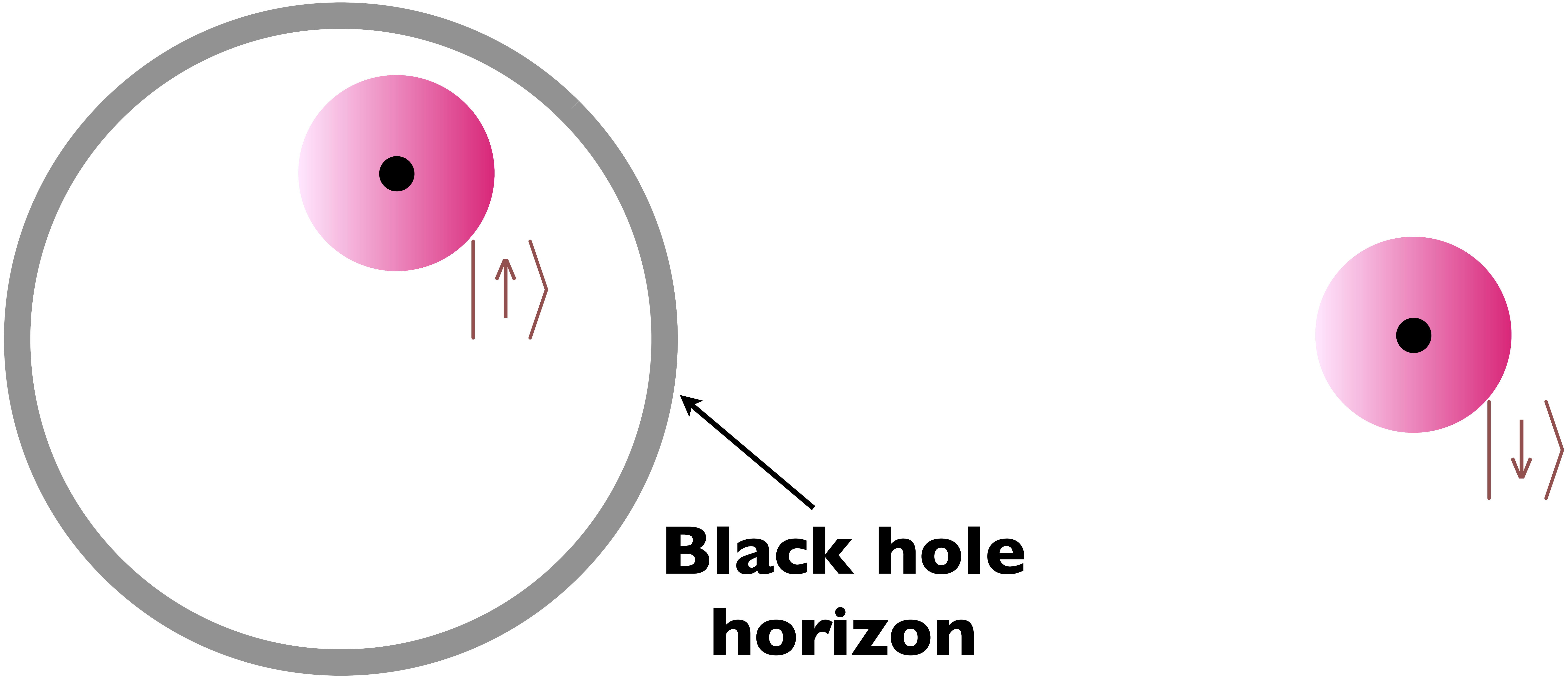


Black hole horizon

Quantum Entanglement across a black hole horizon

Hawking (1975): Black holes have a temperature and an entropy!

To an outside observer, the state of the electron inside the black hole cannot be known, and so the outside electron is in a random state.



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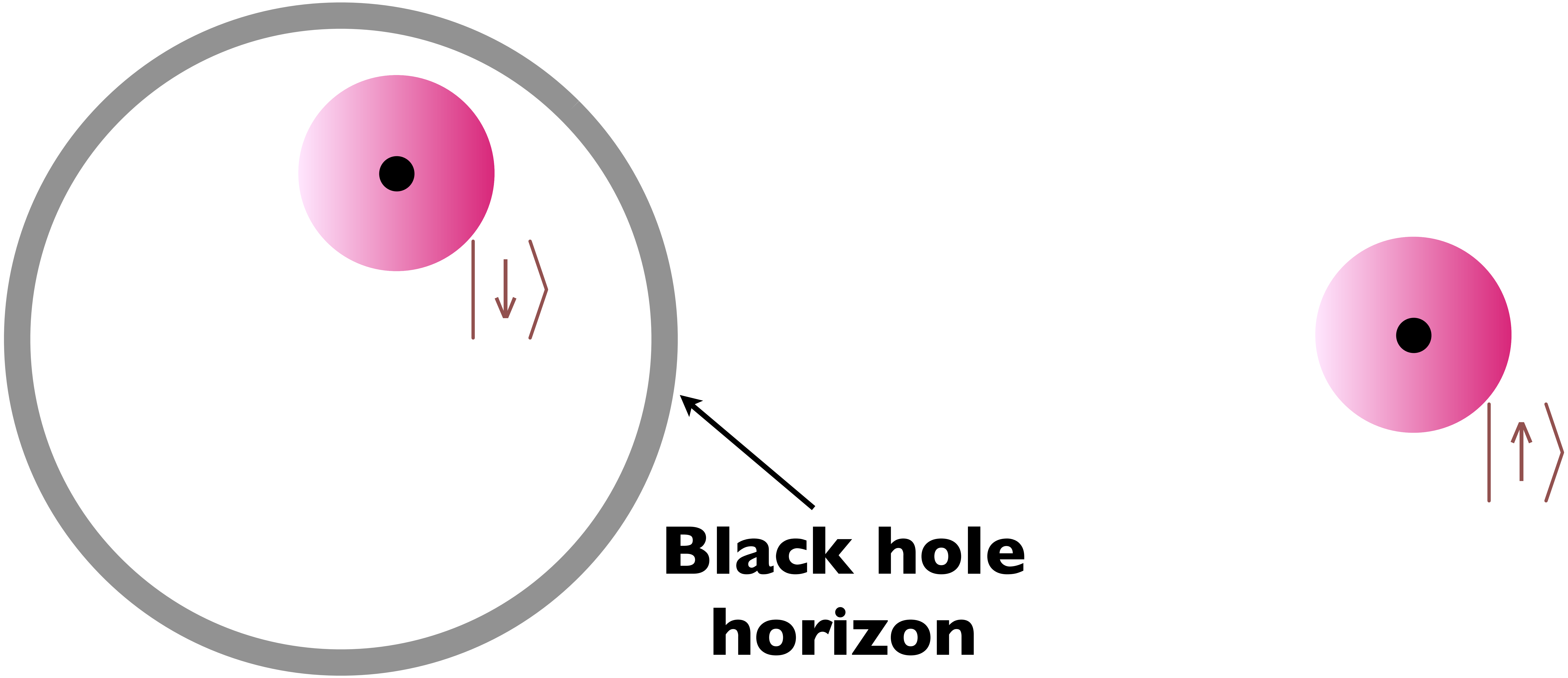
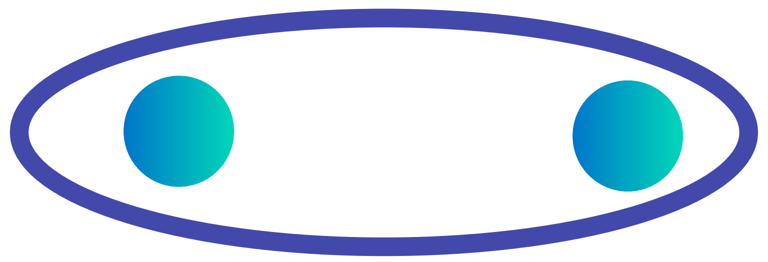
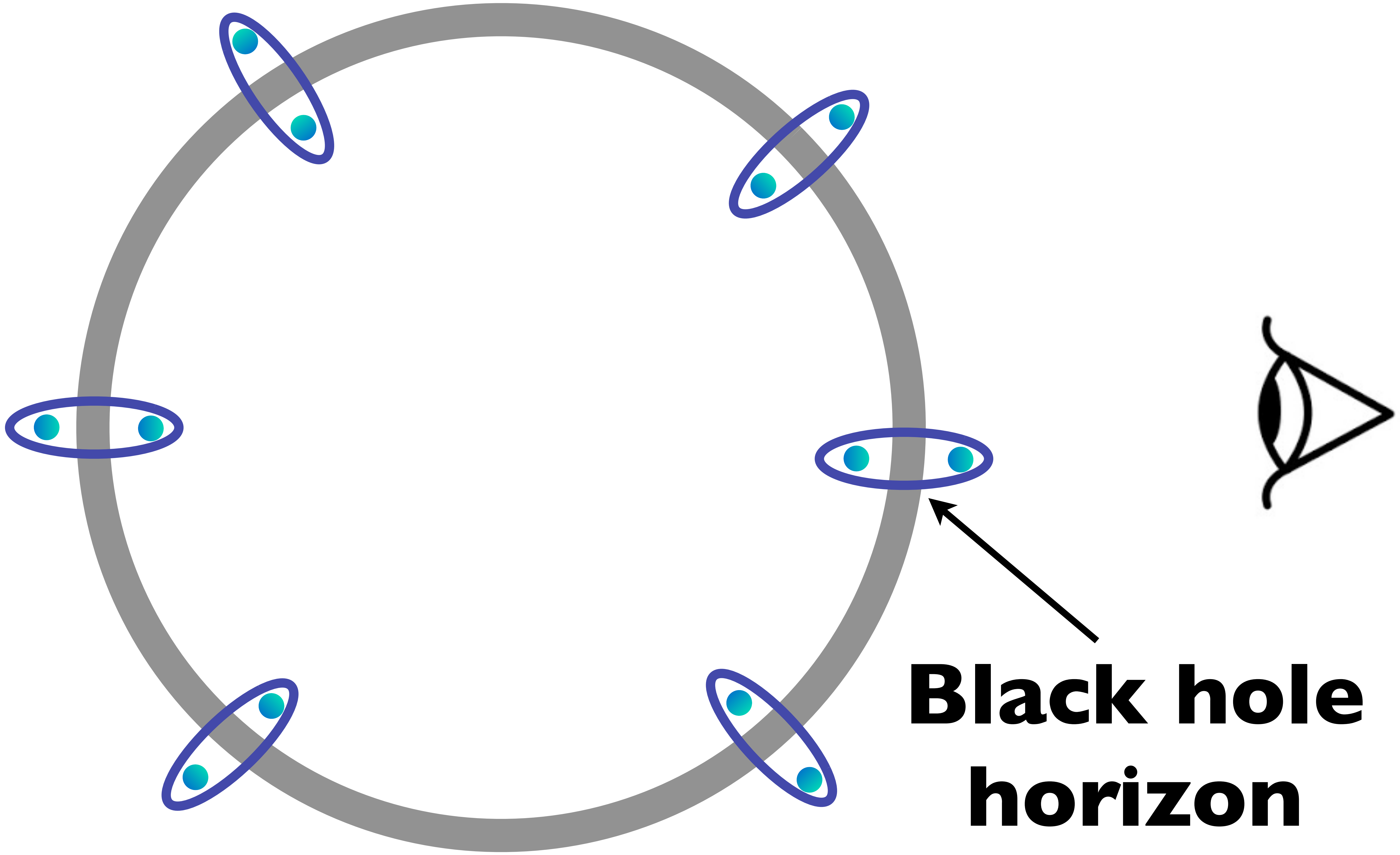


Quantum Entanglement across a black hole horizon

Quantum entanglement
on the surface



$= |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$



By computations *outside* the black hole,
Hawking obtained

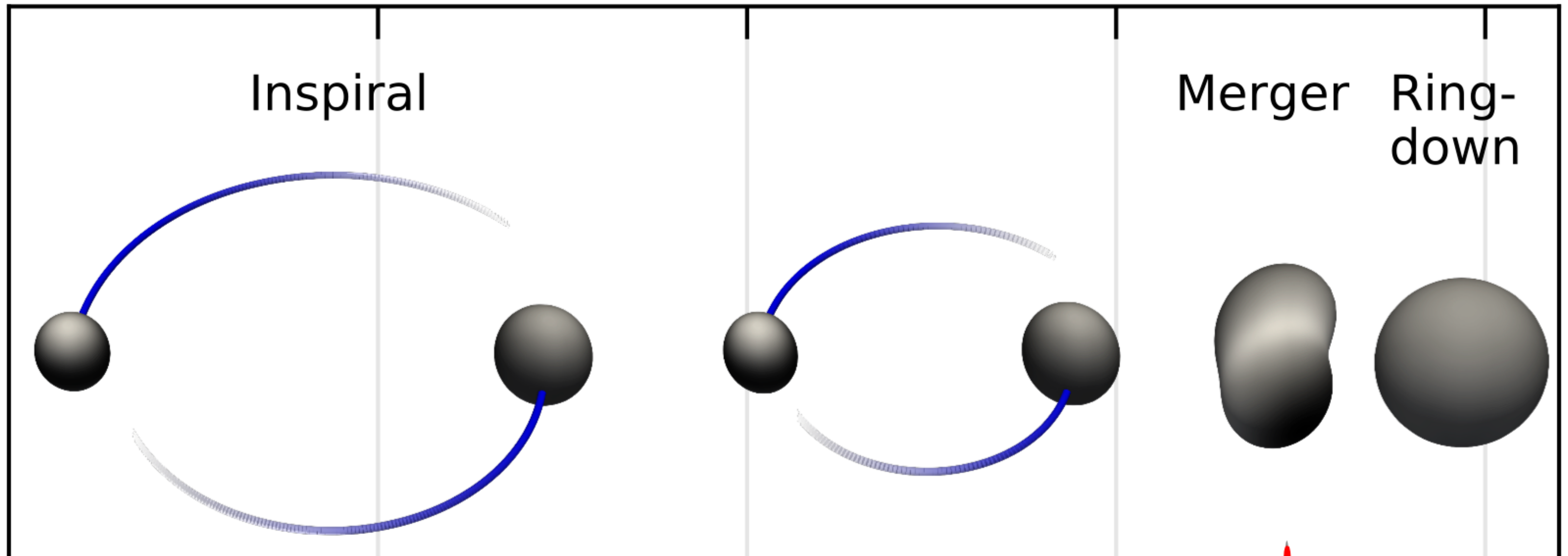
$$S = \frac{Ac^3}{4G\hbar}$$

where A is area of the black hole horizon.

All other systems have entropy proportional to their volume.

Quantum black holes

- Black holes have an entropy and a temperature, $T_H = \hbar c^3/(8\pi G M k_B)$.
- The entropy is proportional to their surface area. $S = A k_B c^3/(4G\hbar)$.
- They relax to thermal equilibrium in a time $\sim 8\pi G M/c^3 = \hbar/(k_B T_H)$ which is Planckian!



J. D. Bekenstein, PRD **7**, 2333 (1973)
S. W. Hawking, Nature **248**, 30 (1974)
C.V. Vishveshwara, Nature **227**, 936 (1970)

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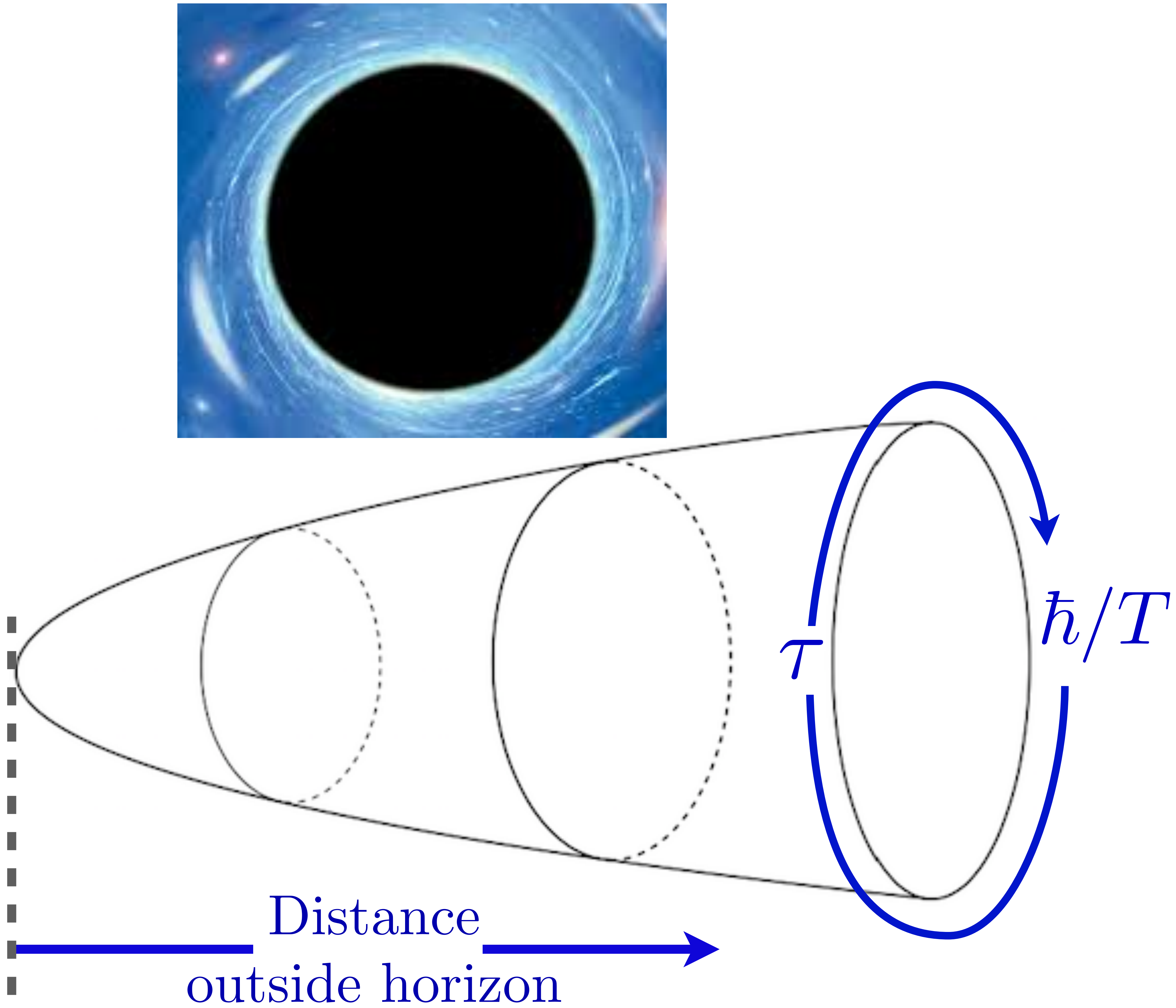
The many-body quantum system describing black holes should have no quasiparticle excitations!

From the SYK model to a quantum theory of charged black holes

Thermodynamics of quantum black holes with charge $\mathcal{Q}$:

$$\mathcal{Z}(\mathcal{Q},T) = \int \mathcal{D}g_{\mu\nu}\mathcal{D}A_{\mu}\, \exp\left(-\frac{1}{\hbar}\, I^{(3+1)}_{\text{Einstein gravity+Maxwell EM}}[g_{\mu\nu},A_{\mu}]\right)$$



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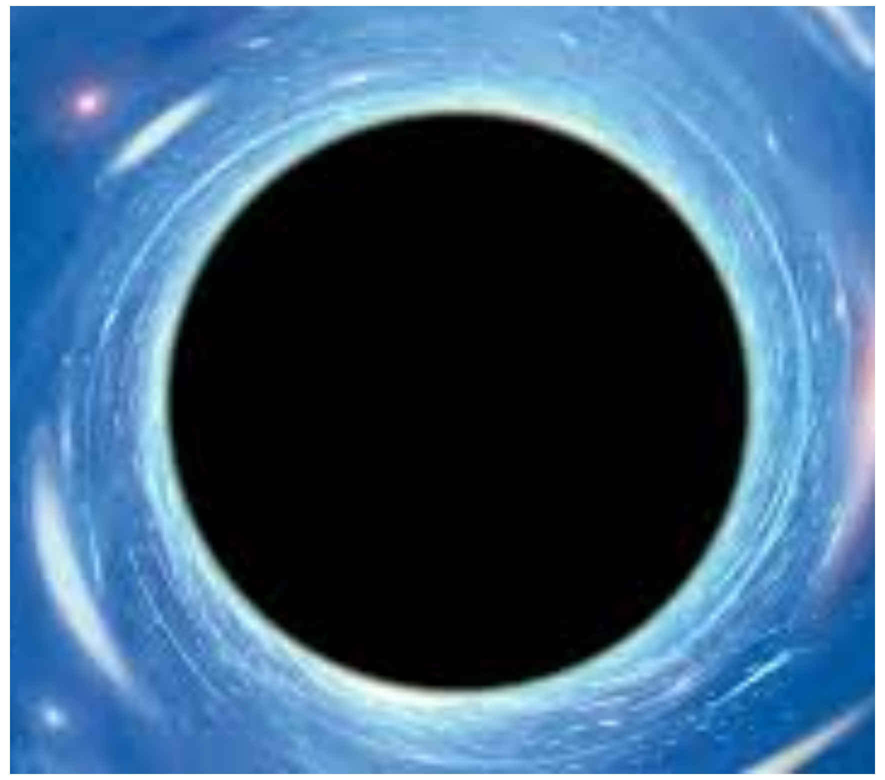
$$= \exp\Big(S_{BH}\Big) \times \Bigg(\quad ....????... \quad \Bigg)$$

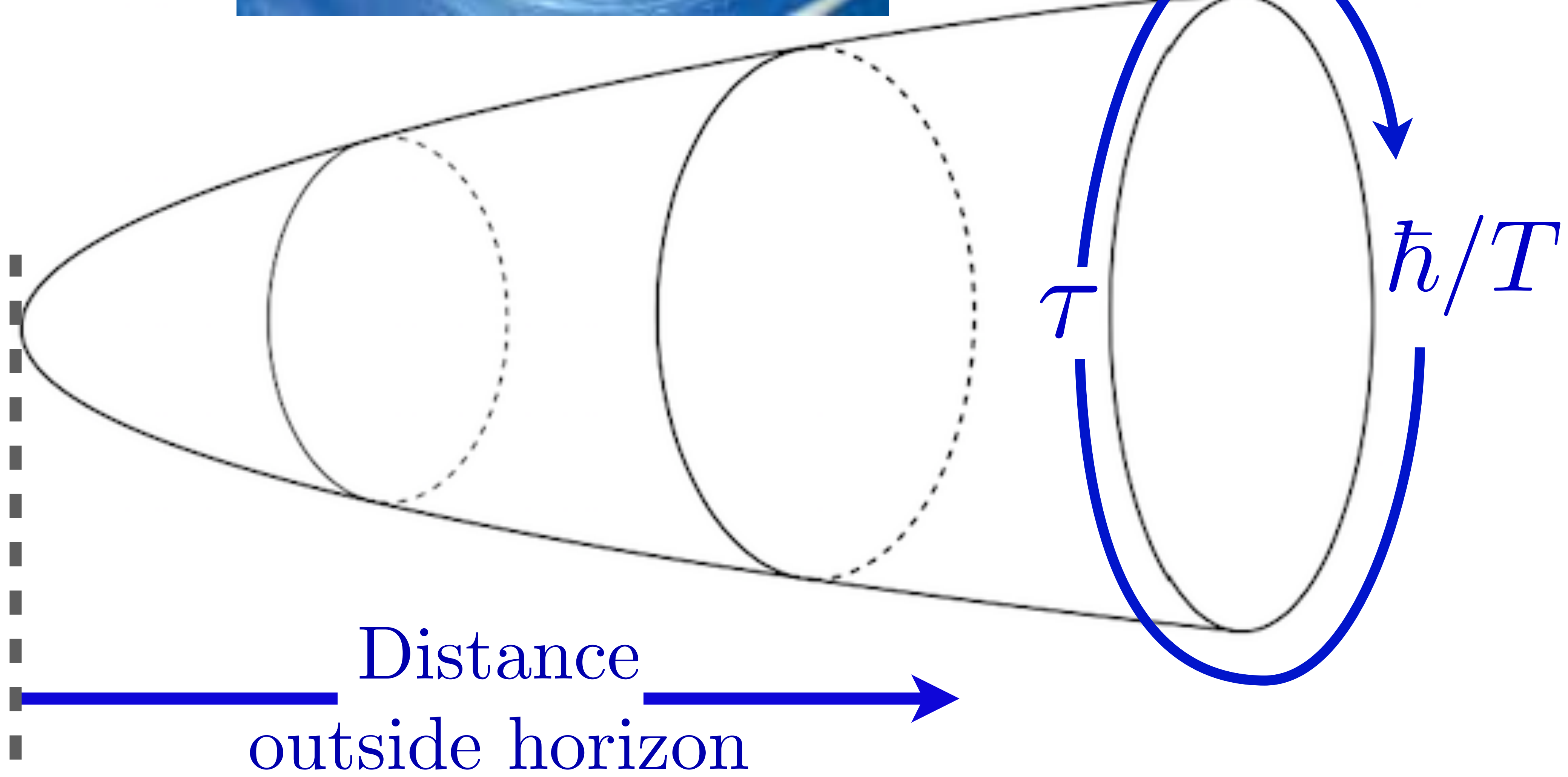
Gibbons, Hawking (1977)
Chambin, Emparan, Johnson, Myers (1999)

$$S_{BH}(T \to 0, \mathcal{Q}) = \frac{A(T)c^3}{4G\hbar} = \frac{A_0 c^3}{4G\hbar}\left(1 + \frac{2(\pi A_0)^{1/2} T}{\hbar c}\right)$$



$A_0 = 2G\mathcal{Q}^2/c^4$ is the area of the charged black hole horizon at $T = 0$.

Obtained from the saddle-point of the gravity path integral in the imaginary time spacetime outside the black hole.



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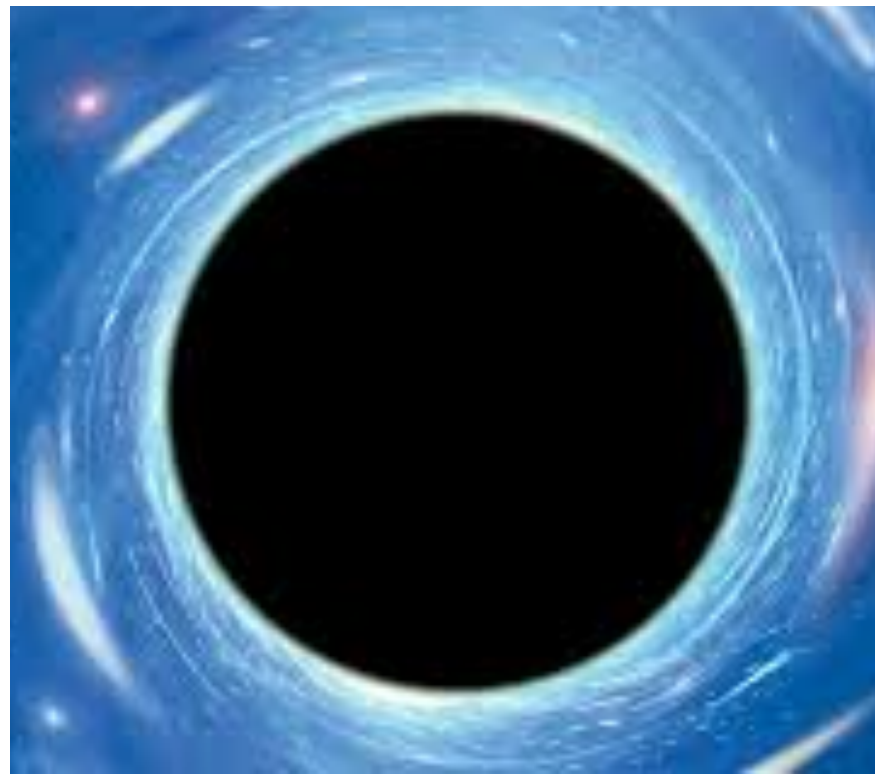

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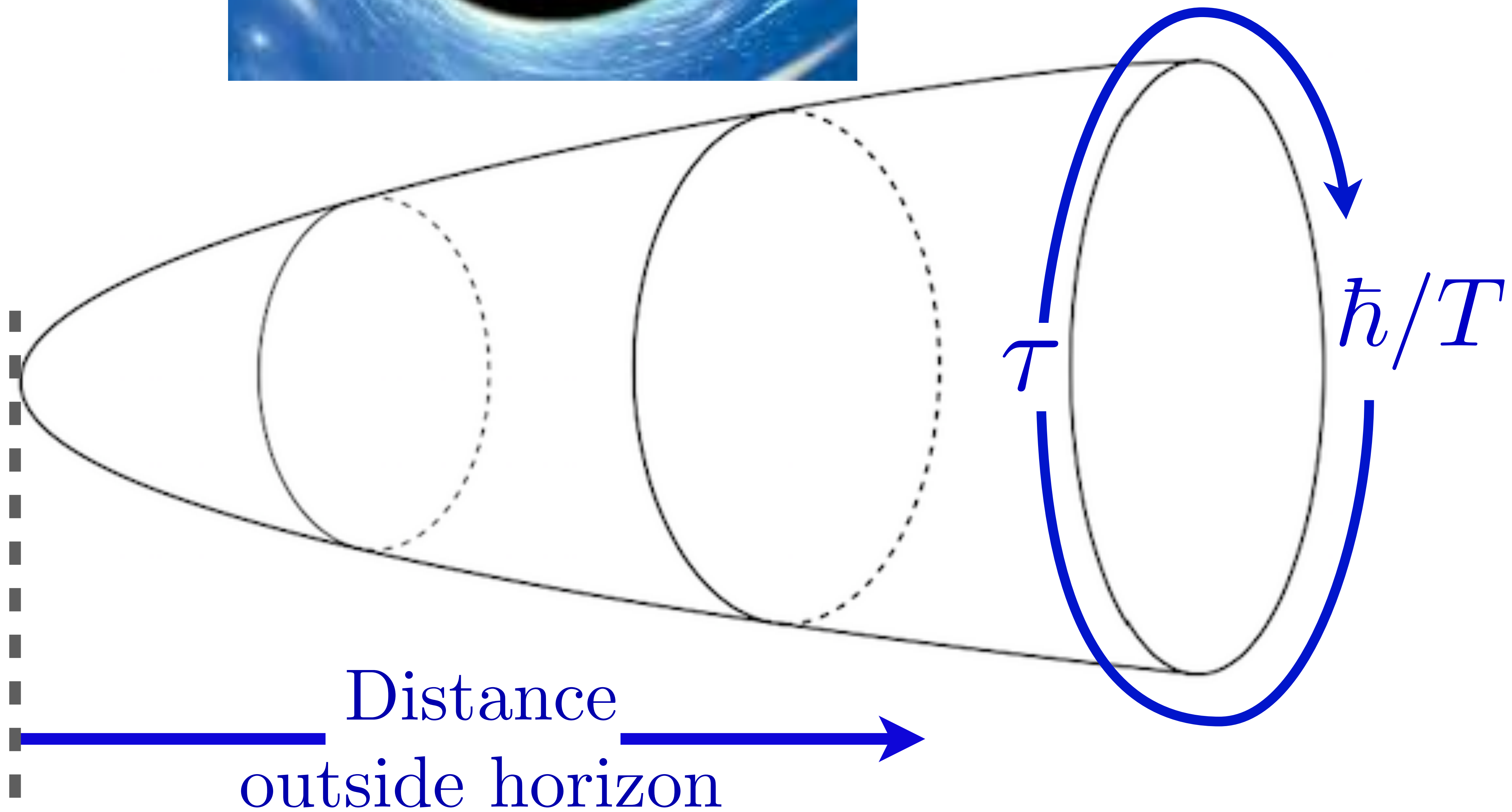

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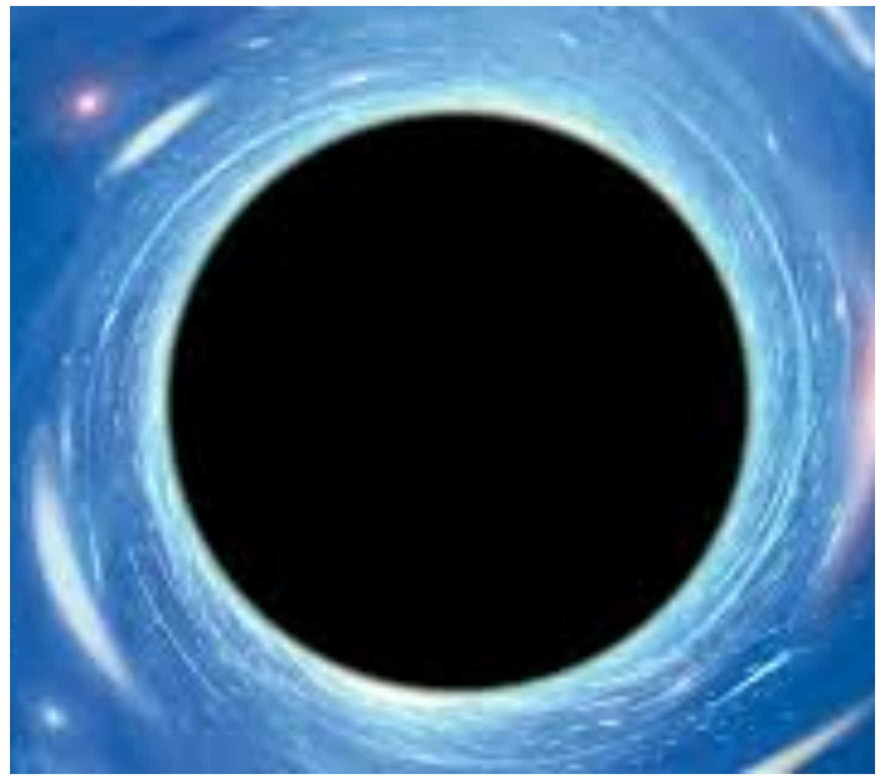

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PHYSICAL REVIEW LETTERS **105**, 151602 (2010)

Holographic Metals and the Fractionalized Fermi Liquid

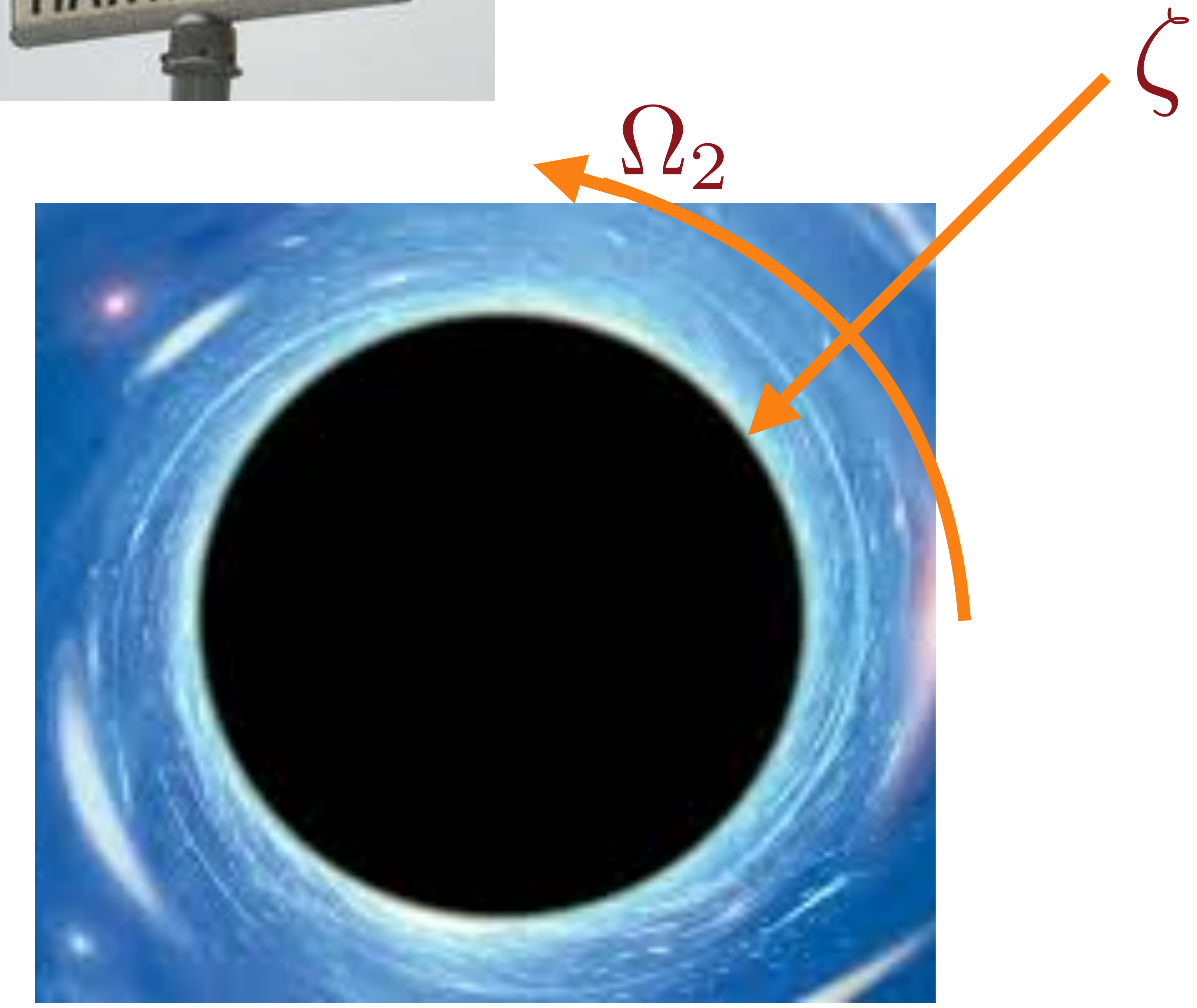
Subir Sachdev

Department of Physics, Harvard University, Cambridge, Massachusetts 02138, USA

(Received 23 June 2010; published 4 October 2010)

We show that there is a close correspondence between the physical properties of holographic metals near charged black holes in anti–de Sitter (AdS) space, and the fractionalized Fermi liquid phase of the lattice Anderson model. The latter phase has a "small" Fermi surface of conduction electrons, along with a spin liquid of local moments. This correspondence implies that certain mean-field gapless spin liquids are states of matter at nonzero density realizing the near-horizon, $\mathrm{AdS}_2 \times \mathrm{R}^2$ physics of Reissner-Nordström black holes.

Reissner-Nordstrom black hole of Einstein-Maxwell theory



Dimensional reduction from 3+1 dimensions to 1+1 dimensions (AdS_2) at low energies!

Faulkner, Liu, McGreevy, Vegh 2009; Cubrovic, Zaanen, Schalm 2009; Sachdev 2010

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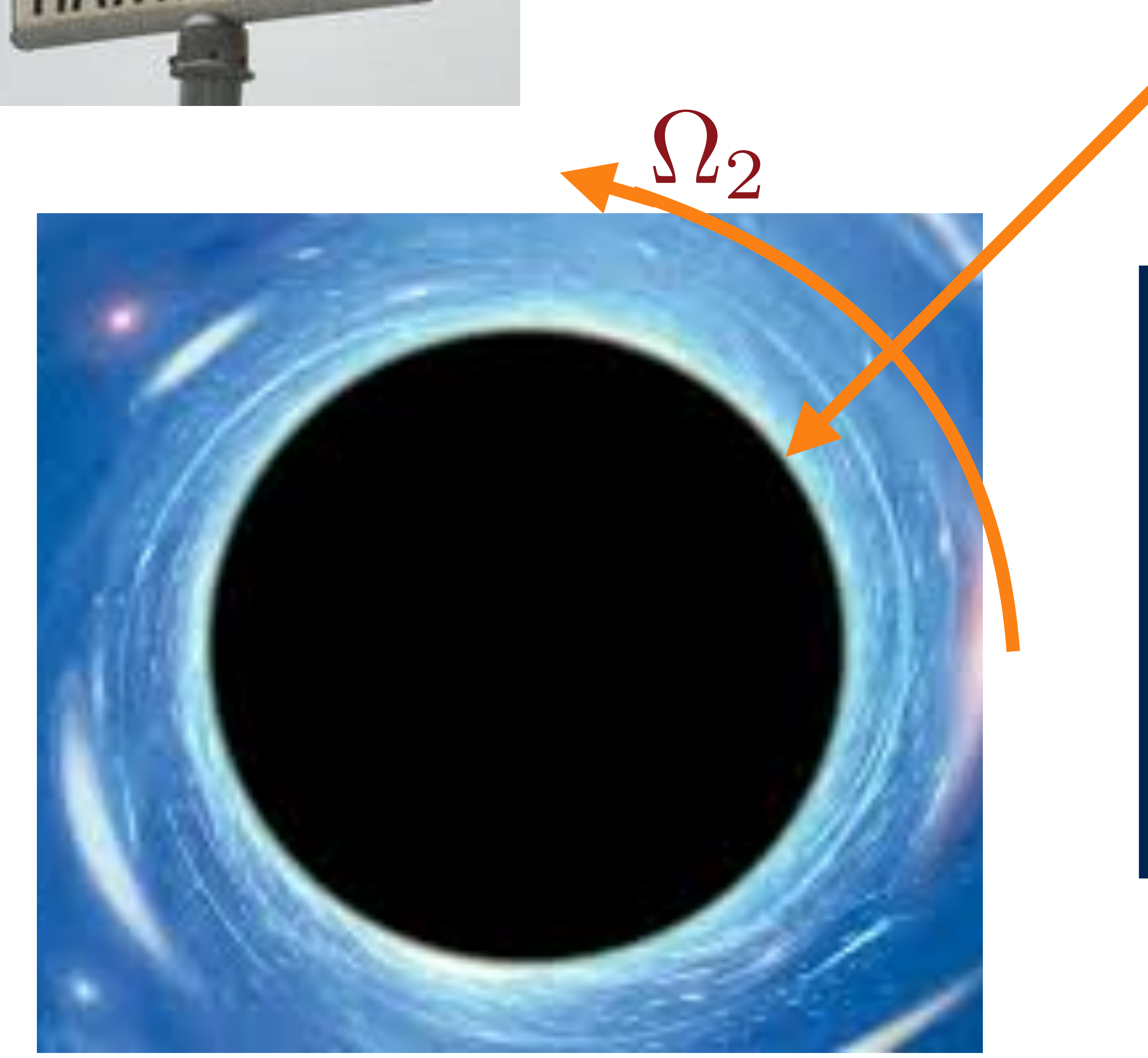

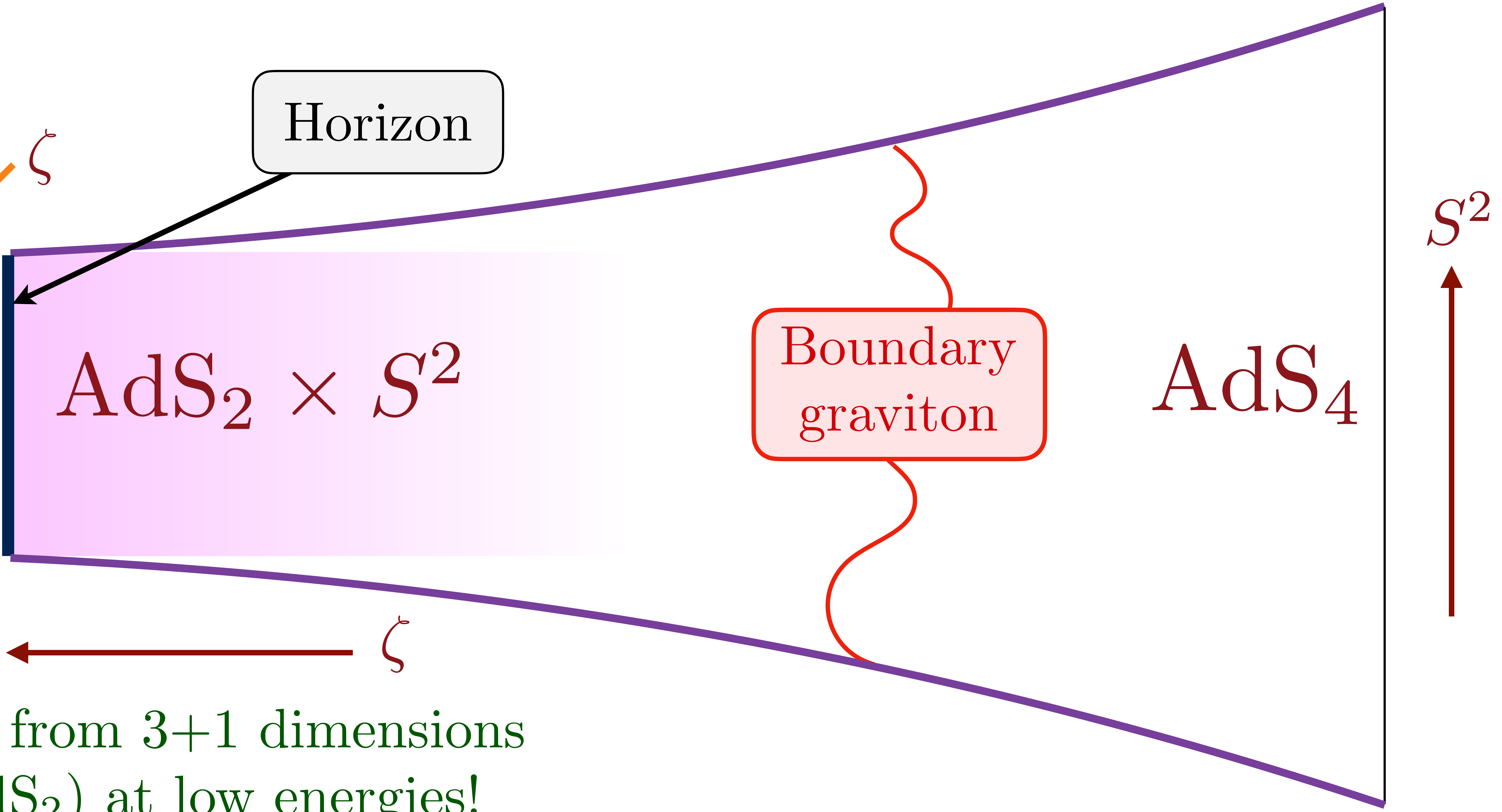

Dimensional reduction from 3+1 dimensions to 1+1 dimensions (AdS_2) at low energies!

The isometry group of AdS_2 is the 0+1 dimensional conformal group SL(2,R).

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$$\approx \exp\left(\frac{A_0 c^3}{4\hbar G}\right)\int \mathcal{D}g_{\mu\nu}\mathcal{D}A_\mu \exp\left(-\frac{1}{\hbar}\, I^{(1+1)}_{\text{JT gravity of AdS}_2\text{+boundary graviton}}[g_{\mu\nu},A_\mu]\right)$$

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Maldacena, Stanford, Yang (2016)

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$$= \int \mathcal{D}f(\tau)\mathcal{D}\phi(\tau) \exp\left(-\frac{1}{\hbar} I_{\rm SYK}[\text{time reparameterizations } f(\tau), \text{ phase rotations } \phi(\tau)]\right)$$

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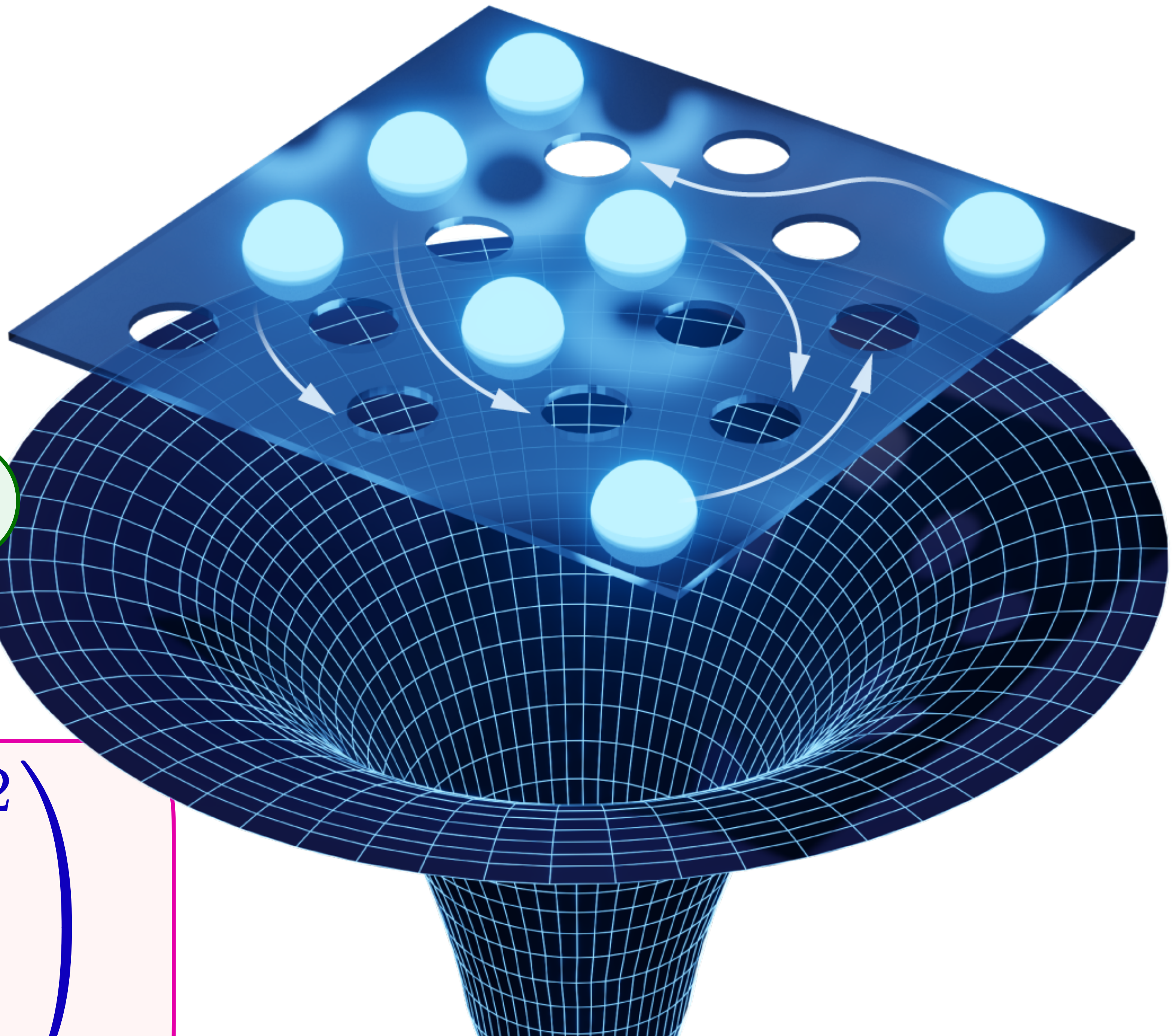
Sachdev (2010); Kitaev (2015); Sachdev (2015); Bagrets, Altland, Kamenev (2016); Maldacena, Stanford, Yang (2016); Moitra, Trivedi, Vishal (2018) ; Gaikwad, Joshi, Mandal, Wadia (2018); Sachdev (2019); Iliesiu, Turaci (2020)

Quantum simulation of charged black holes by the SYK model

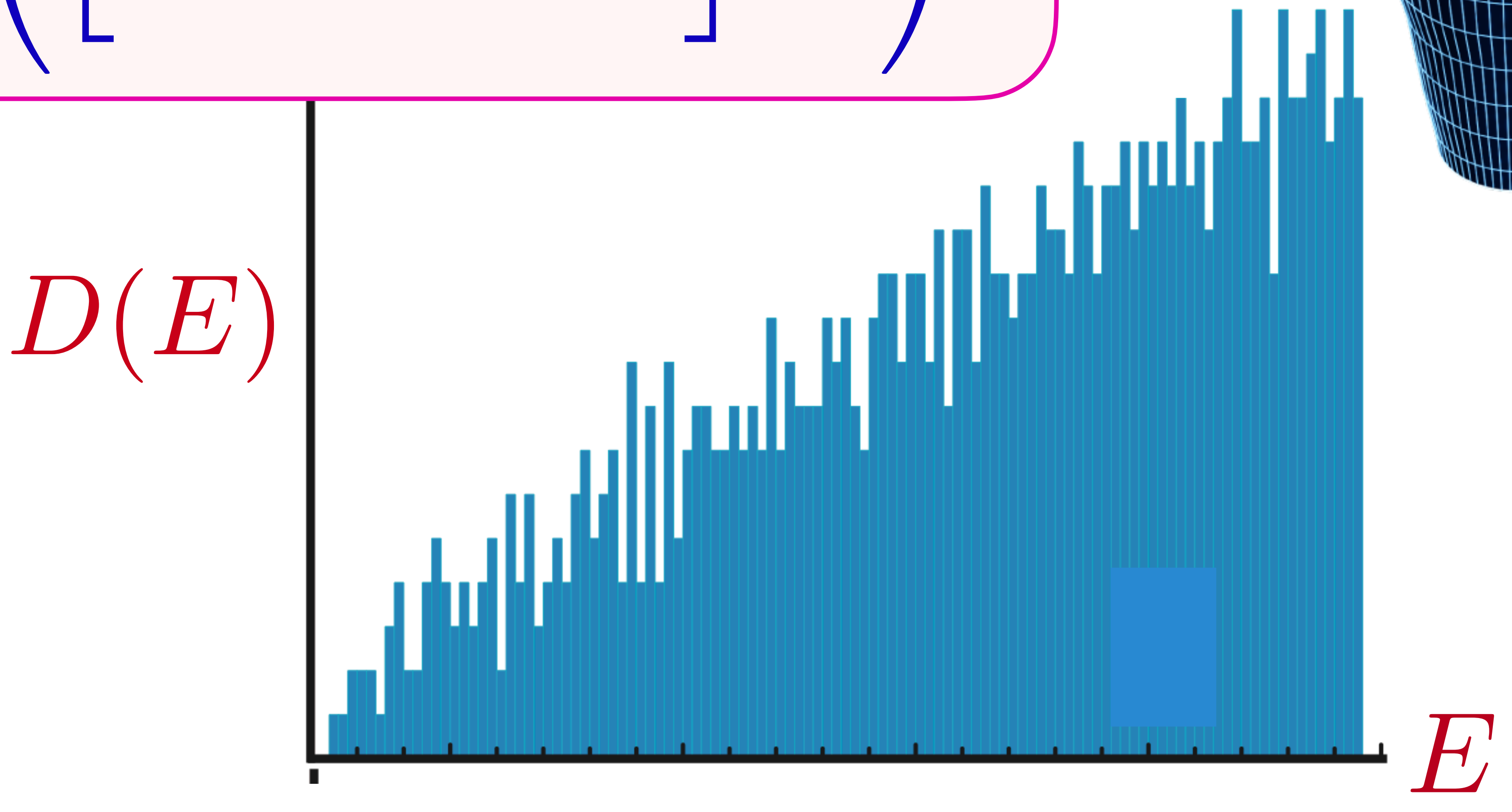


- For generic charged black holes in 3+1 dimensions, the SYK model yields, in terms of $A_0 = 2GQ^2/c^4$ the horizon area at $T = 0$:

Iliesiu, Murthy, Turiaci (2022) Bekenstein-Hawking Developments from the SYK model

$$D(E) \sim \left(\frac{A_0 c^3}{\hbar G}\right)^{-347/90} \exp\left(\frac{A_0 c^3}{4\hbar G}\right) \sinh\left(\left[\frac{\sqrt{\pi} A_0^{3/2} c^2}{\hbar^2 G} E\right]^{1/2}\right)$$

There is no degeneracy, but an exponentially small level spacing down to the ground state.



D. Chowdhury, A. Georges, O. Parcollet, and S. S., Rev. Mod. Phys. **94**, 035004 (2022)

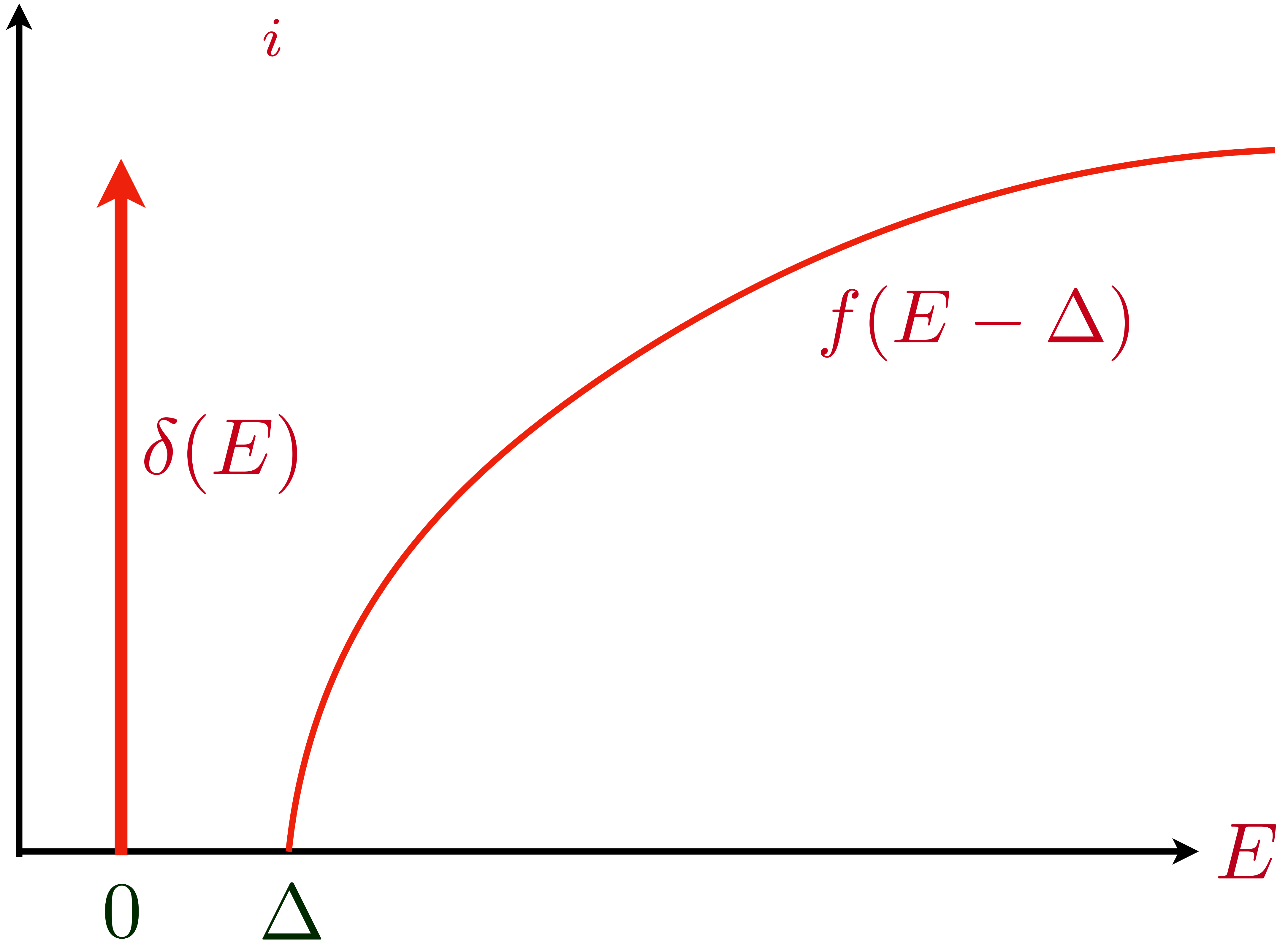
String theory of charged black holes

- With sufficient low energy supersymmetry, string theory yields:

$$D(E) = \exp\left(\frac{A_0 c^3}{4\hbar G}\right)\delta(E) + \theta(E-\Delta) f(E-\Delta) + \ldots$$

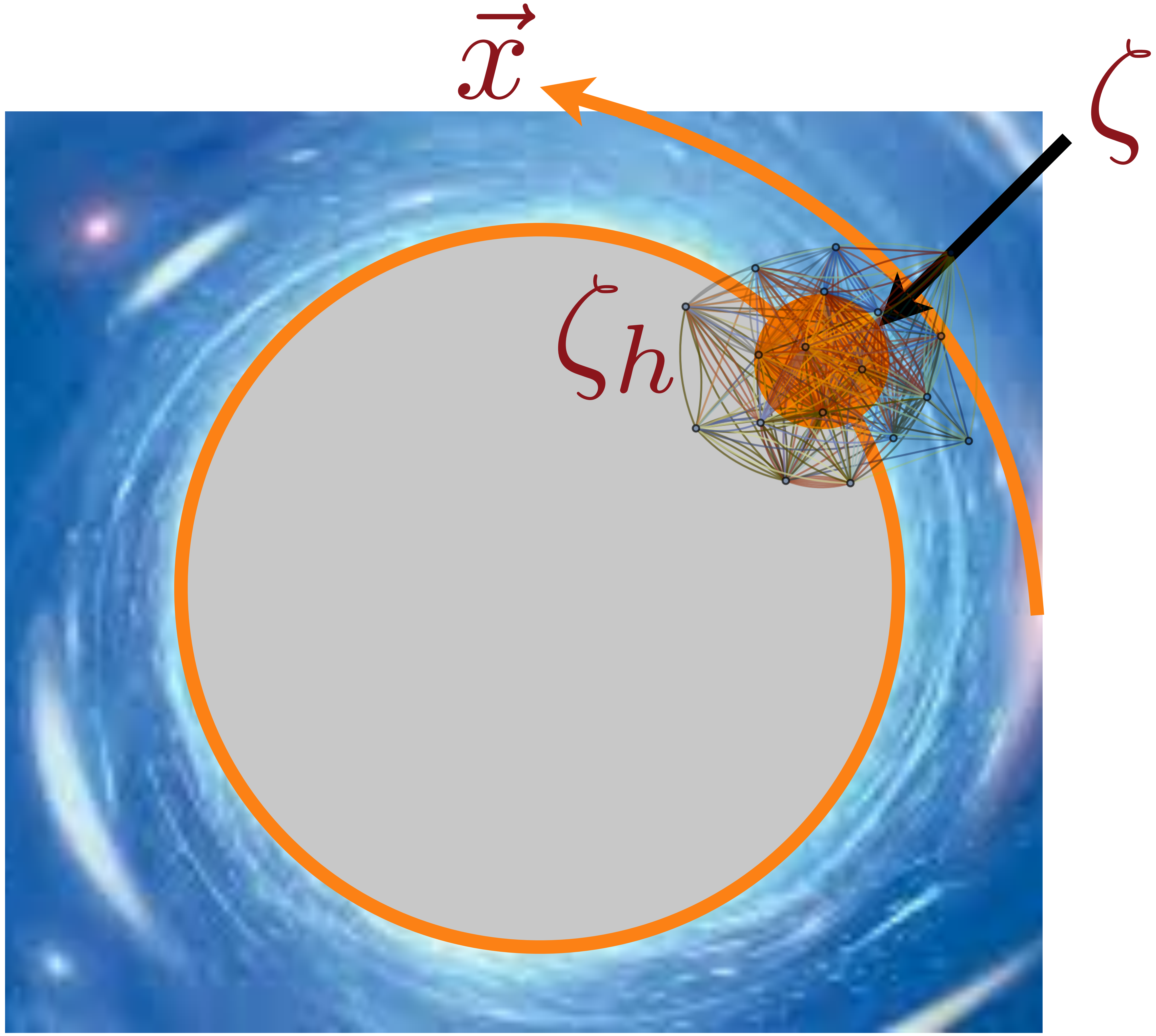
There are exponentially many degenerate BPS ground states, and an energy gap Δ above the ground state.

$$D(E) = \sum_i \delta(E - E_i)$$



M. Heydeman, L.V. Iliesiu, G. J. Turiaci, and W. Zhao, 2020
L.V. Iliesiu, S. Murthy, G. J. Turiaci, 2022

Quantum simulation of charged black holes by the SYK model



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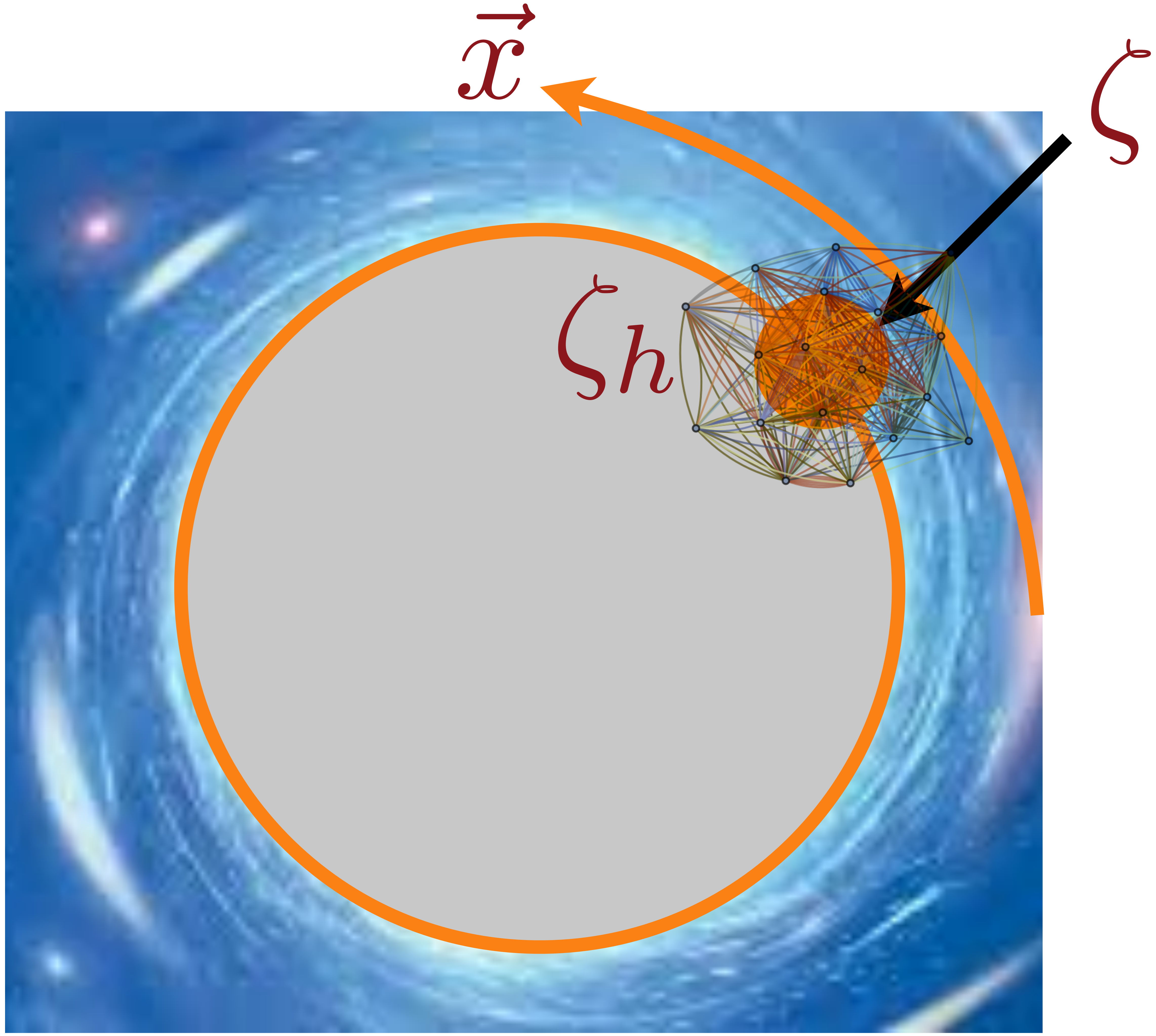

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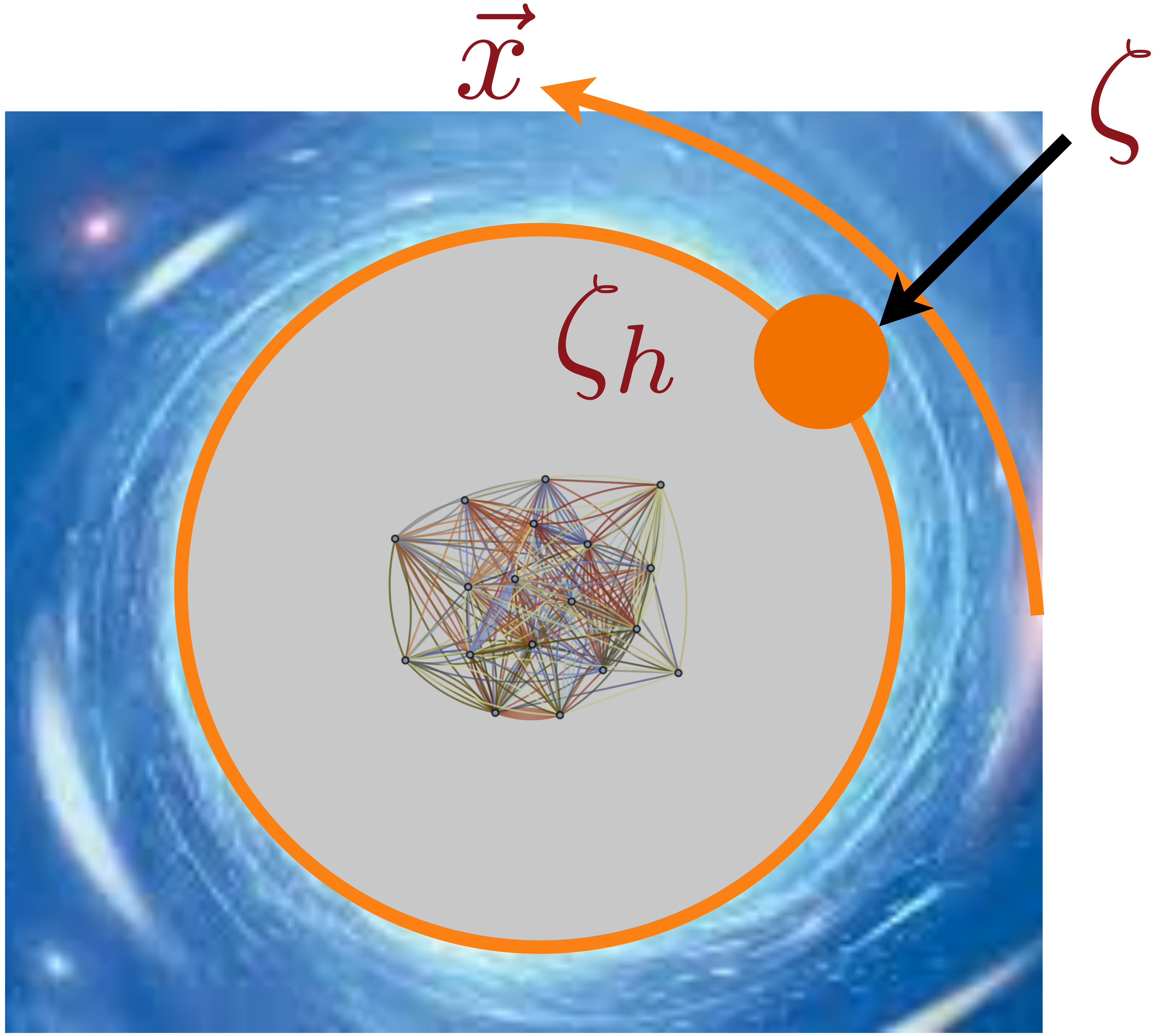

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The many faces of multi-particle entanglement

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- Higher temperature superconductivity (?)