Maximal quantum chaos of the critical Fermi surface

Probing Complex Quantum Dynamics through
Out-of-time-ordered Correlators
MPIPKS, Dresden
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Subir Sachdev

Talk online: sachdev.physics.harvard.edu

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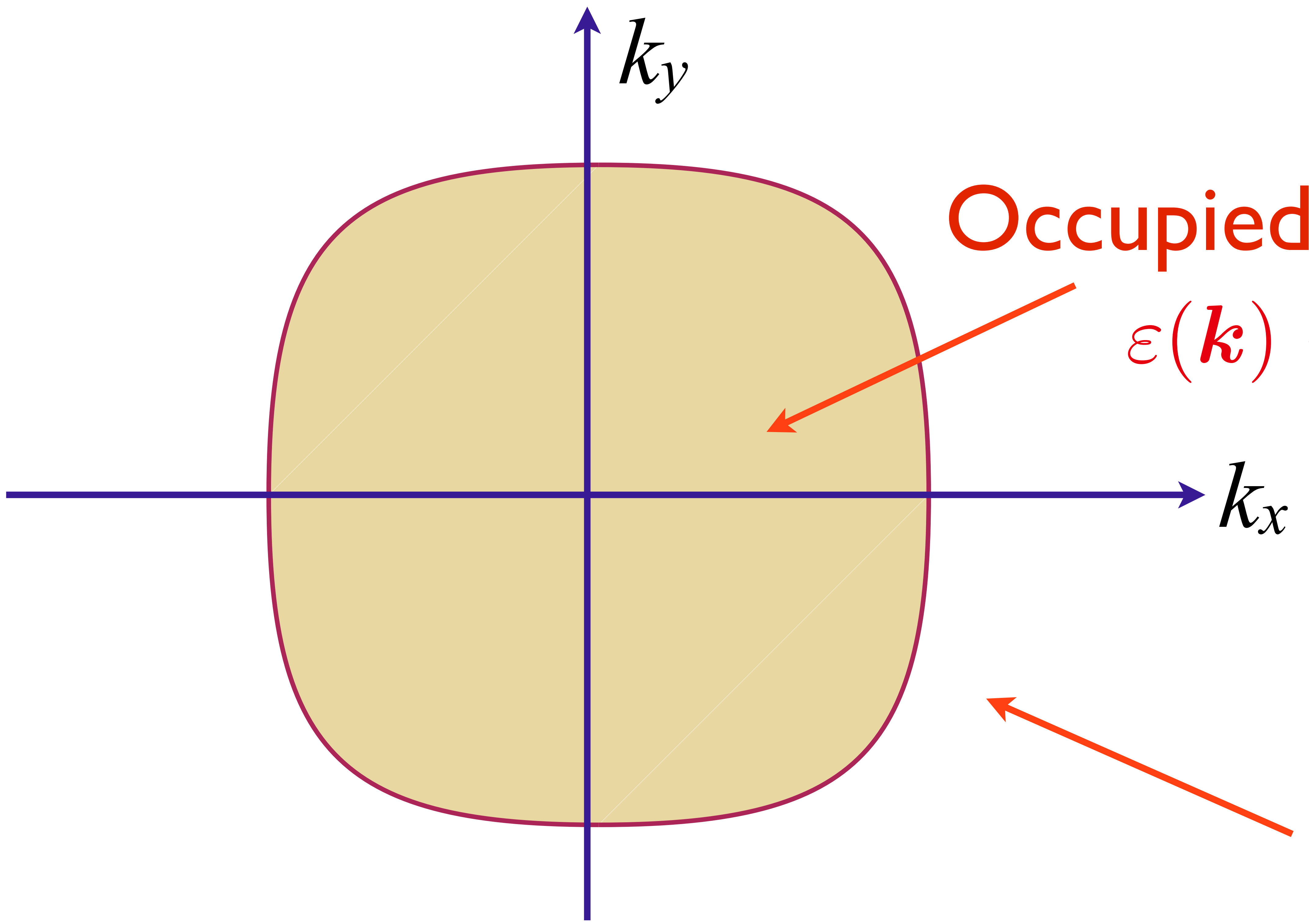
Maria Tikhanovskaya
Harvard

Aavishkar Patel
Berkeley

1. Critical Fermi surfaces: large *N* theory
2. Gu-Kitaev theory of OTOCs with spatio-temporal chaos
3. Maximal quantum chaos of the critical Fermi surface

Fermi surface coupled to a gauge field



Occupied states

$\varepsilon(\boldsymbol{k}) < 0$

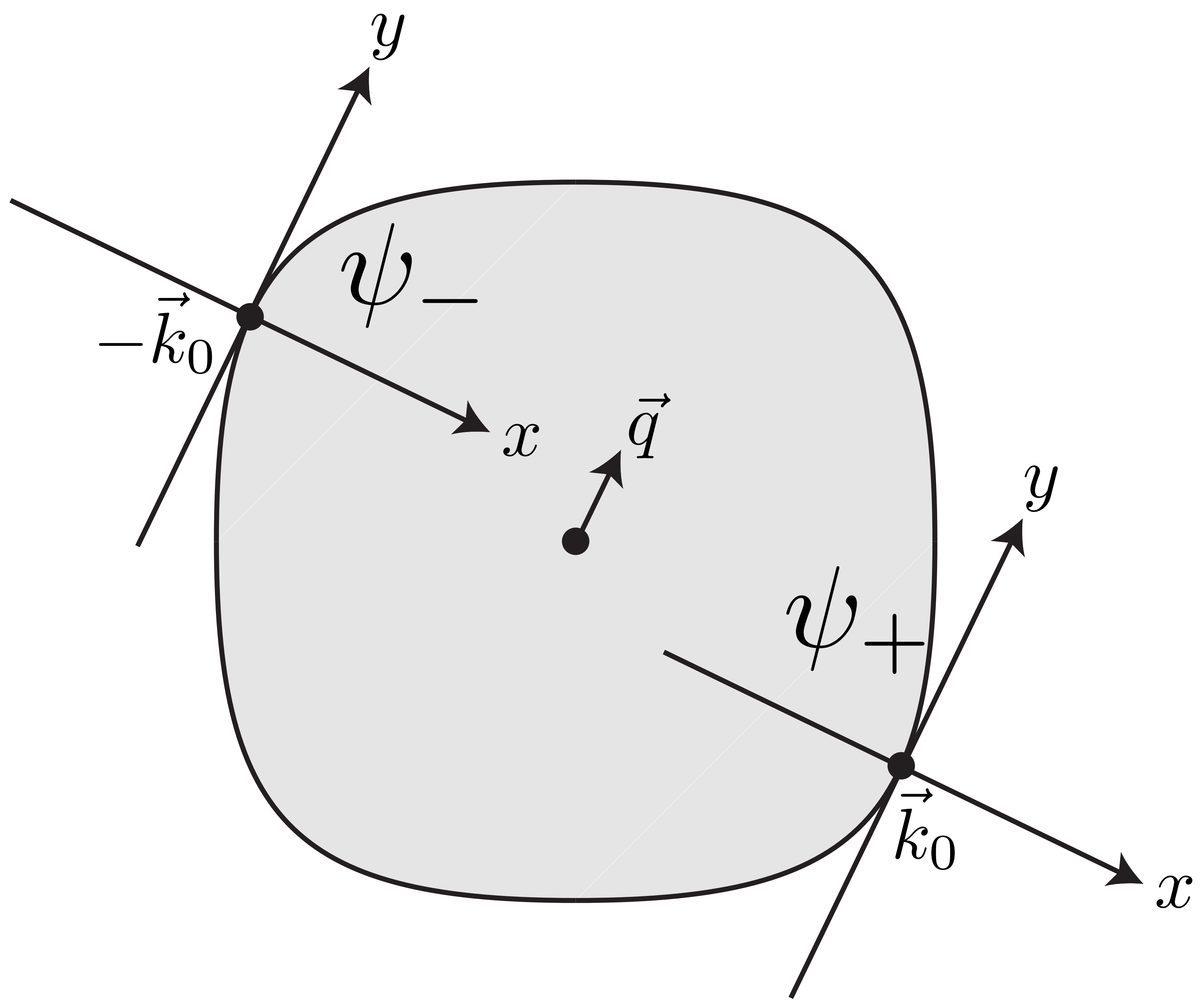
Empty states

$\varepsilon(\boldsymbol{k}) > 0$

A metal with a Fermi surface
minimally coupled to a gauge field $\boldsymbol{A}$

$$\mathcal{L} = c_{\boldsymbol{k}}^{\dagger} \left(\frac{\partial}{\partial \tau} + \varepsilon(-i\boldsymbol{\nabla} - g\boldsymbol{A}) - \mu \right) c_{\boldsymbol{k}} + \frac{1}{2}(\boldsymbol{\nabla} \times \boldsymbol{A})^2$$

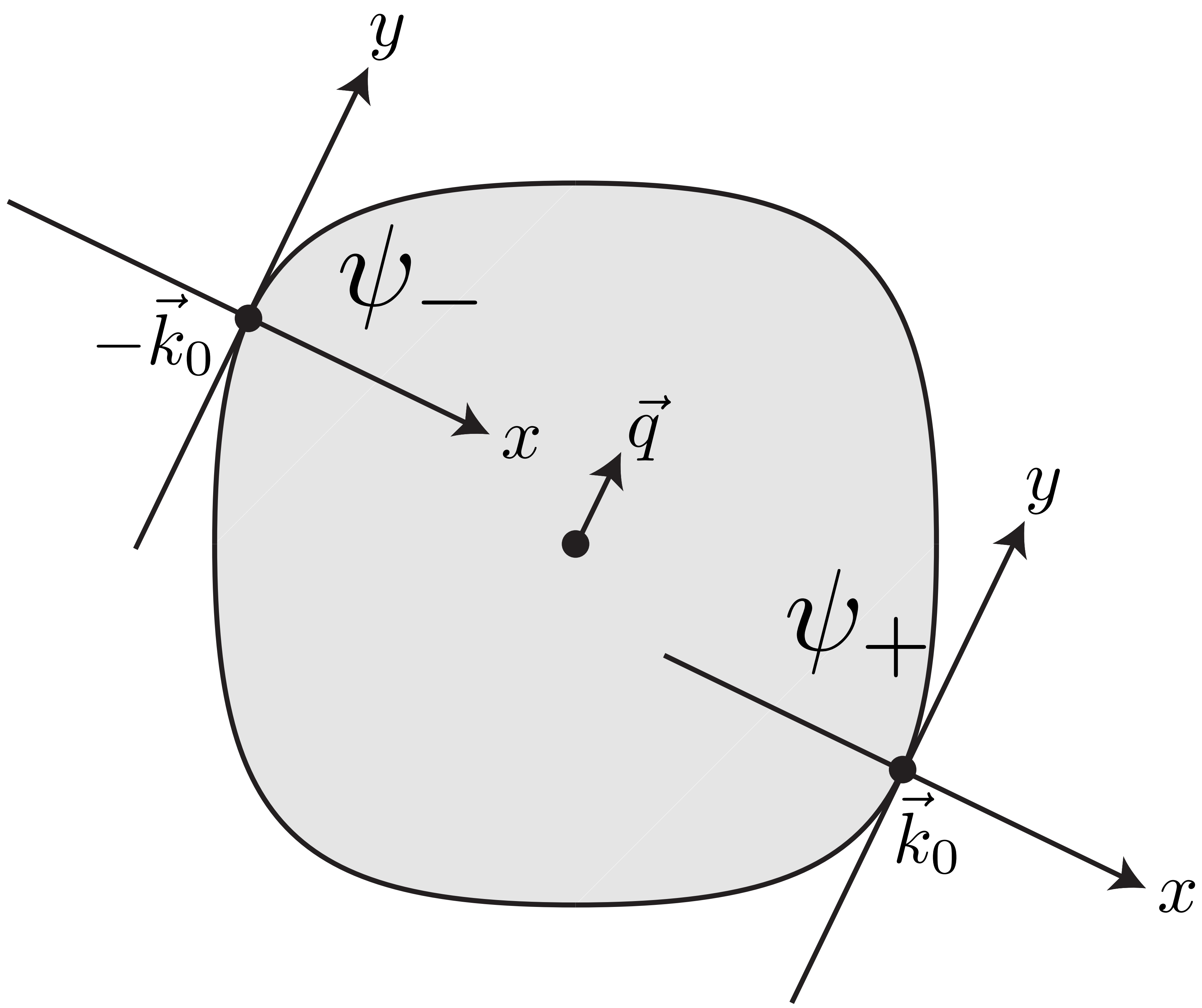
Fermi surface coupled to a gauge field



- Gauge fluctuation at wavevector $\boldsymbol{q}$ couples most efficiently to fermions near $\pm\boldsymbol{k}_0$.

- Expand fermion kinetic energy at wavevectors about $\pm\boldsymbol{k}_0$. In Landau gauge $\boldsymbol{A} = (a, 0)$.

Fermi surface coupled to a gauge field



$$\mathcal{L}[\psi_\pm, a] =$$

$$\psi_+^\dagger \left(\partial_\tau - i\partial_x - \partial_y^2\right)\psi_+ + \psi_-^\dagger \left(\partial_\tau + i\partial_x - \partial_y^2\right)\psi_-$$

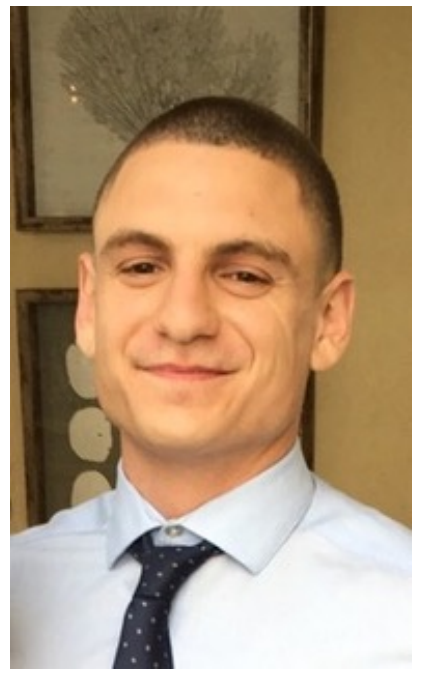
$$-g\, a \left(\psi_+^\dagger \psi_+ - \psi_-^\dagger \psi_-\right) + \frac{1}{2}\left(\partial_y a\right)^2$$

M. A. Metlitski and S. Sachdev,
Phys. Rev. B **82**, 075127 (2010)

Large N theory of a critical Fermi surface

Main idea:

Introduce N flavors of fermions and bosons, and examine an *ensemble* of theories with different Yukawa couplings. In the large N limit, every member of the ensemble is expected to have the same critical properties, and so it is easier to study the average theory.



Ilya Esterlis, Haoyu Guo, Aavishkar Patel, S.S. arXiv: 2103.08615

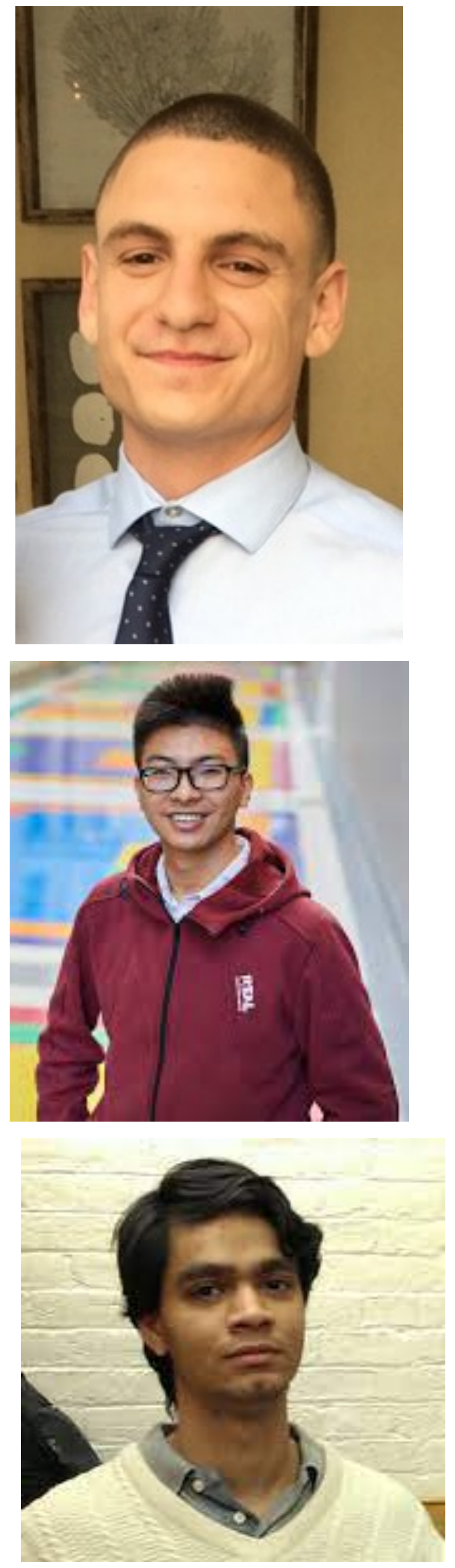
Large N theory of a critical Fermi surface

N flavors of fermions $\psi_{\pm\alpha}$,
M flavors of a boson a_α, and
a "Yukawa coupling" $g_{\alpha\beta\gamma}$ which is a random function in flavor space. Note: there is *no spatial randomness*. Take the large N limit with M/N fixed.

$$\begin{aligned}\mathcal{L} \;=\;& \psi^\dagger_{+\alpha}\left(\partial_\tau - i\partial_x - \partial_y^2\right)\psi_{+\alpha} + \psi^\dagger_{-\alpha}\left(\partial_\tau + i\partial_x - \partial_y^2\right)\psi_{-\alpha} \\ & - \frac{g_{\alpha\beta\gamma}}{N} a_\alpha \left(\eta_{+\alpha}\psi^\dagger_{+\beta}\psi_{+\gamma} + \eta_{-\alpha}\psi^\dagger_{-\beta}\psi_{-\gamma}\right) + \frac{1}{2}\left(\partial_y a_\alpha\right)^2\end{aligned}$$

$\eta_{\pm\alpha} = \pm 1$ depending upon nature of a_α: gauge field, Higgs field, order parameter ....

$$\overline{g_{\alpha\beta\gamma}} = 0 \quad , \quad \overline{|g_{\alpha\beta\gamma}|^2} = g^2$$



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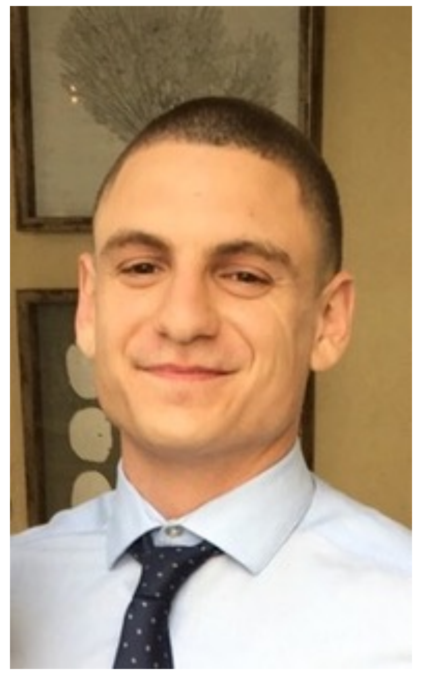
Large *N* theory of a critical Fermi surface

We can now proceed just as in the SYK model: we obtain a theory for Green's functions which are bilocal in both space and time. Using the spacetime coordinate $X \equiv (\tau, x, y)$, we can write the averaged partition function

$$\overline{\mathcal{Z}}_{\psi\phi} = \int \mathcal{D}G(X_1, X_2)\mathcal{D}\Sigma(X_1, X_2)\mathcal{D}D(X_1, X_2) \\ \times \mathcal{D}\Pi(X_1, X_2) \exp\left[-N I(G, \Sigma, D, \Pi)\right] .$$



The G-Σ-D-Π action is now

$$\begin{aligned} I(G, \Sigma, D, \Pi) &= \frac{g^2}{2}\mathrm{Tr}\left(G \cdot [GD]\right) - \mathrm{Tr}(G \cdot \Sigma) + \frac{1}{2}\mathrm{Tr}(D \cdot \Pi) \\ &- \ln\det\left[\left(\partial_{\tau_1} - i\partial_{x_1} - \partial_{y_1}^2\right)\delta(X_1 - X_2) + \Sigma(X_1, X_2)\right] \\ &+ \frac{1}{2}\ln\det\left[\left(-K\partial_{y_1}^2\right)\delta(X_1 - X_2) - \Pi(X_1, X_2)\right] . \end{aligned}$$

where we have introduced notation

$$\mathrm{Tr}\left(f \cdot g\right) \equiv \int dX_1 dX_2\, f(X_2, X_1) g(X_1, X_2) .$$

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Large *N* theory of a critical Fermi surface

Saddle-point equations

$$G(\boldsymbol{k}, i\omega) = \frac{1}{i\omega - k_x - k_y^2 - \Sigma(\boldsymbol{k}, i\omega)}, \quad D(\boldsymbol{k}, i\omega) = \frac{1}{k_y^2 - \Pi(\boldsymbol{k}, i\omega)}$$

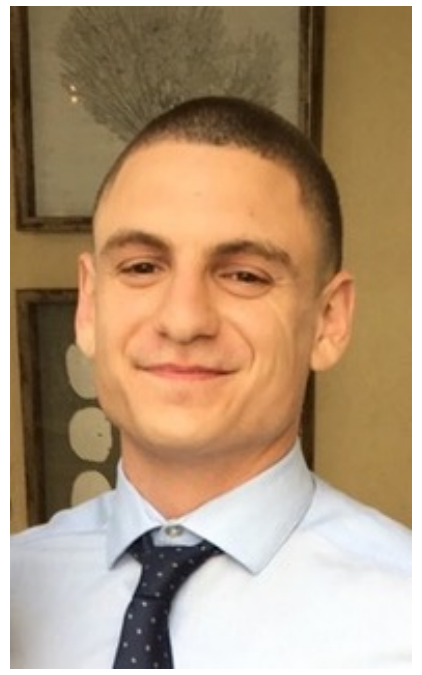
$$\Sigma(\boldsymbol{r}, \tau) = g^2 D(\boldsymbol{r}, \tau) G(\boldsymbol{r}, \tau), \quad \Pi(\boldsymbol{r}, \tau) = -g^2 G(-\boldsymbol{r}, -\tau) G(\boldsymbol{r}, \tau)$$

Exact solution at low energies:

$$\Sigma(\boldsymbol{k}, \omega) = g^{4/3} T^{2/3} \Phi\left(\frac{\omega}{T}\right),$$

where $\Phi(z)$ is a universal scaling function, obtained by analytical continuation from imaginary Matsubara frequencies $\omega_n = (2n-1)\pi T$

$$\begin{aligned} \Phi\left(\frac{i\omega_n}{T}\right) &= -i\,\mathrm{sgn}(\omega_n)\frac{2^{5/3}}{3\sqrt{3}} H_{1/3}\left(\frac{|\omega_n| - \pi T}{2\pi T}\right) \\ H_r(n \in \mathbb{Z}^+) &= \sum_{j=1}^{n} \frac{1}{j^r} \end{aligned}$$



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Slides
(mostly)
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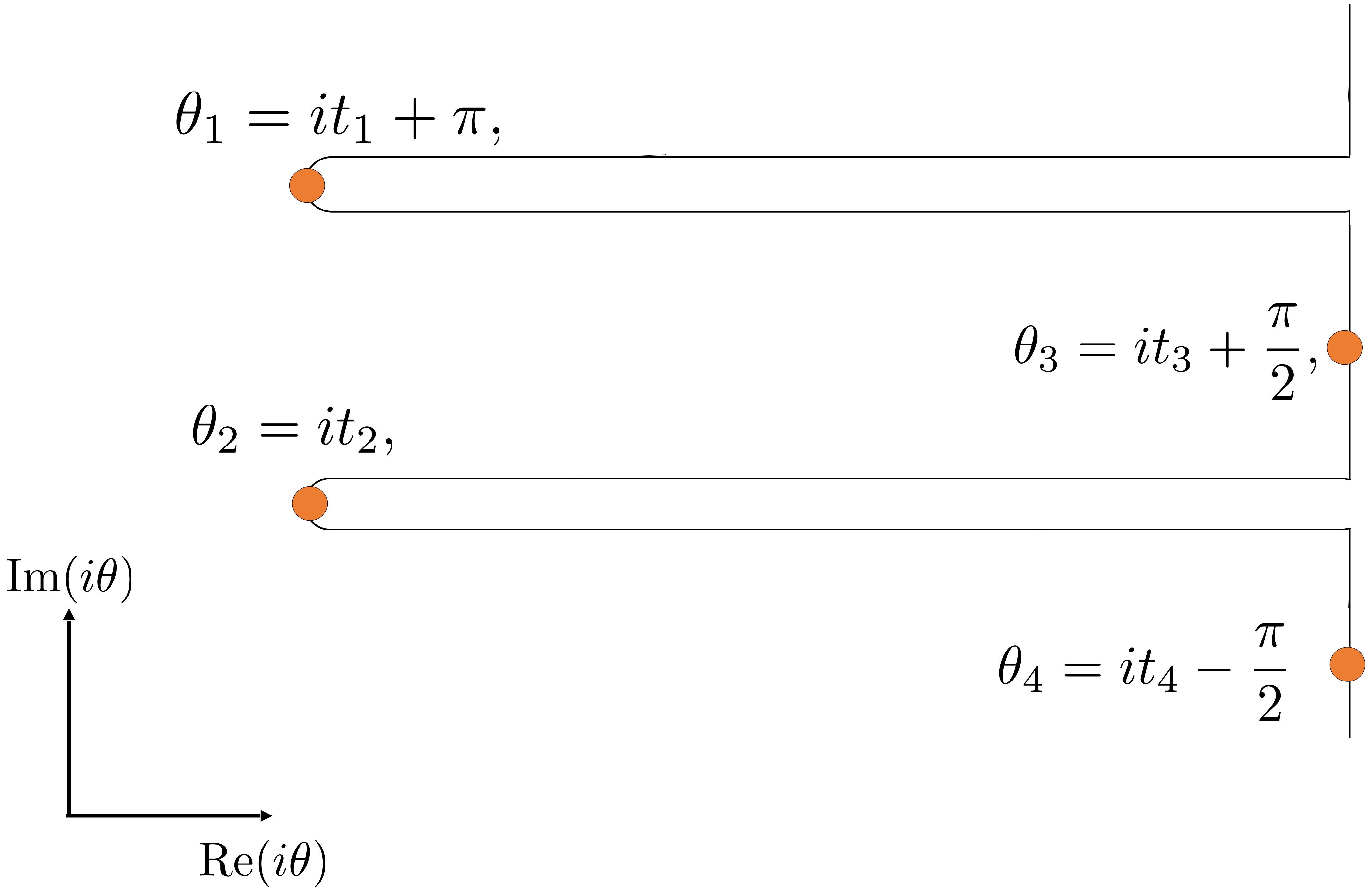
Out-of-time-order correlator (OTOC)

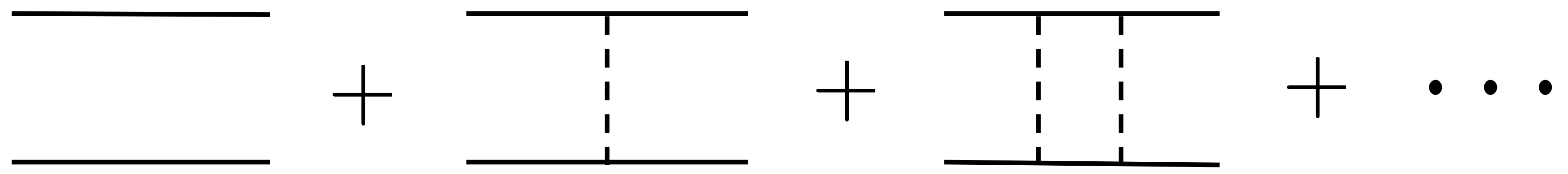
Here $\beta \equiv 1/T = 2\pi$.

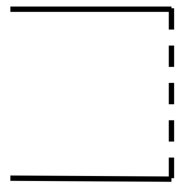
- Definition $\quad \mathrm{OTOC} = \langle X(\theta_1)Y(\theta_3)X(\theta_2)Y(\theta_4)\rangle$ (connected part)

$$\theta_1 = it_1 + \pi, \quad \theta_3 = it_3 + \frac{\pi}{2}, \quad \theta_2 = it_2, \quad \theta_4 = it_4 - \frac{\pi}{2}$$



Assume the leading contribution comes from ladders

$\mathrm{OTOC} \sim$  $+ \cdots$

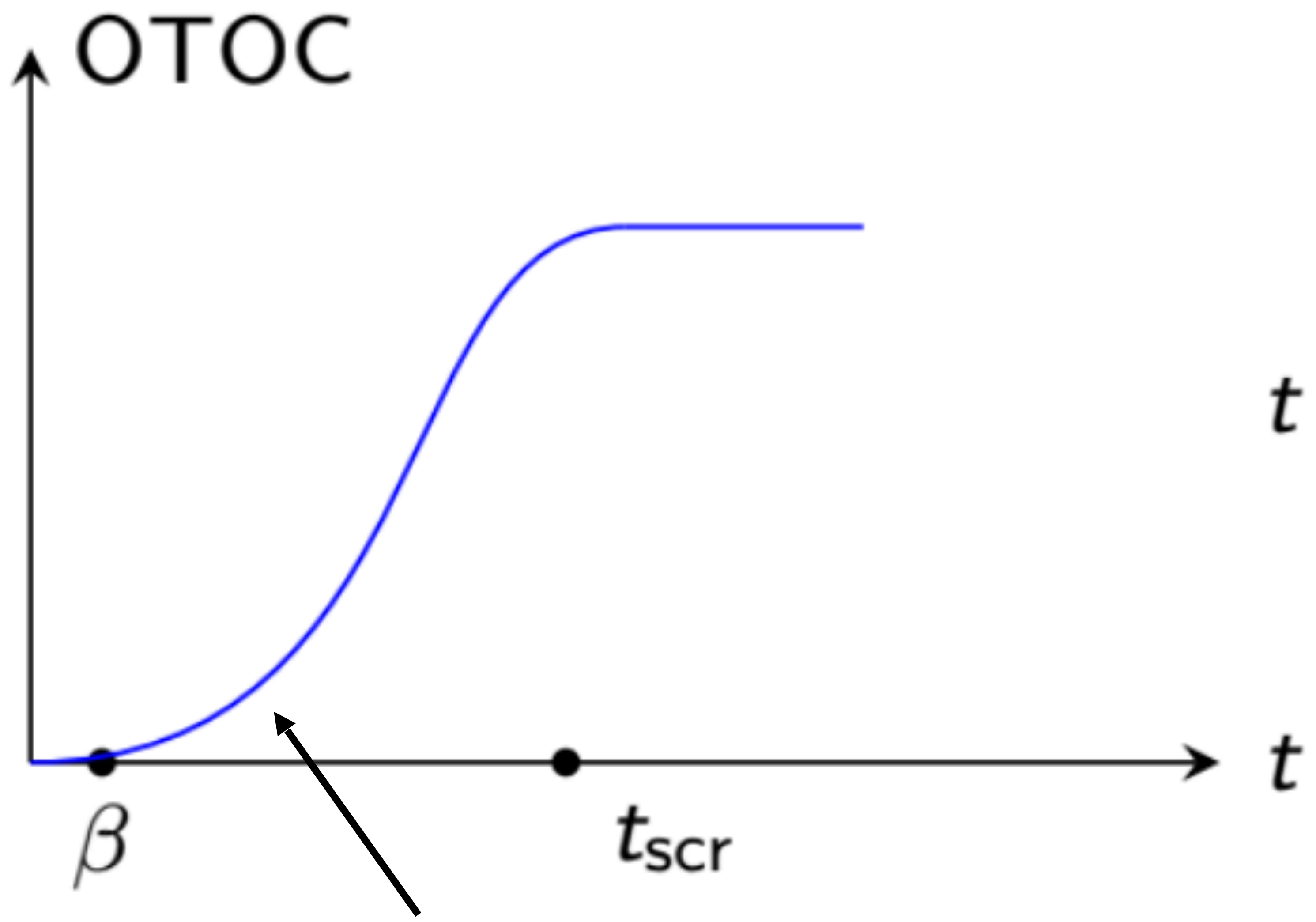
$\mathcal{F}_n =$  $\cdot \mathcal{F}_{n-1}, \quad \mathcal{F} = \sum_n \mathcal{F}_n \ \Rightarrow \mathcal{F} = \mathcal{F}_0 + K\mathcal{F}$

Eigenvalue equation

Structure of OTOC – early time regime

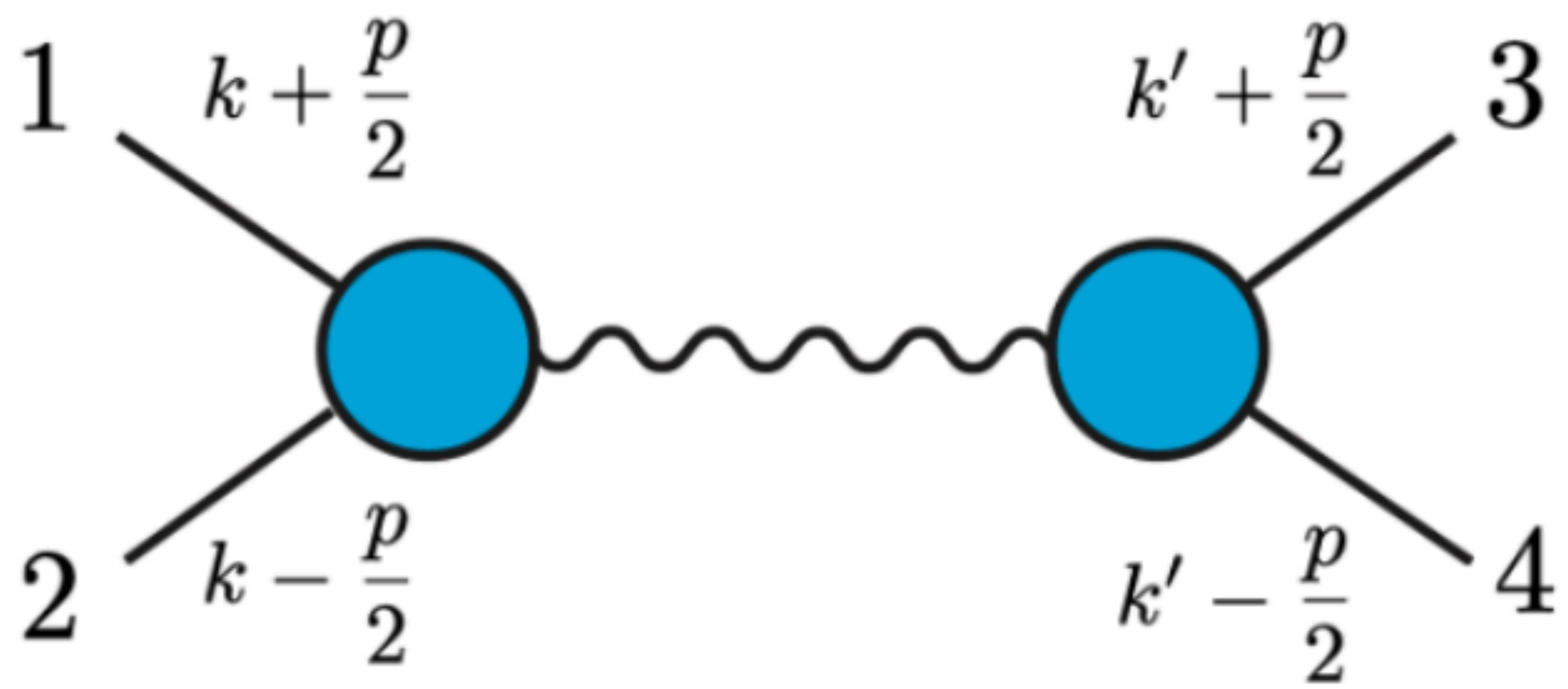
Consider a single mode ansatz for early times

$$\mathrm{OTOC}_p(t_1, t_2, t_3, t_4; k, k') \approx \frac{e^{\kappa(p)(t_1+t_2-t_3-t_4)/2}}{C(p)} \Upsilon_p^R(t_{12}, k)\Upsilon_p^A(t_{34}, k')$$



Exponential growth

$$t := \frac{t_1+t_2-t_3-t_4}{2}$$



Josephine Suh, Alexei Kitaev 2017

Yingfei Gu, Alexei Kitaev 2018

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Vertex functions

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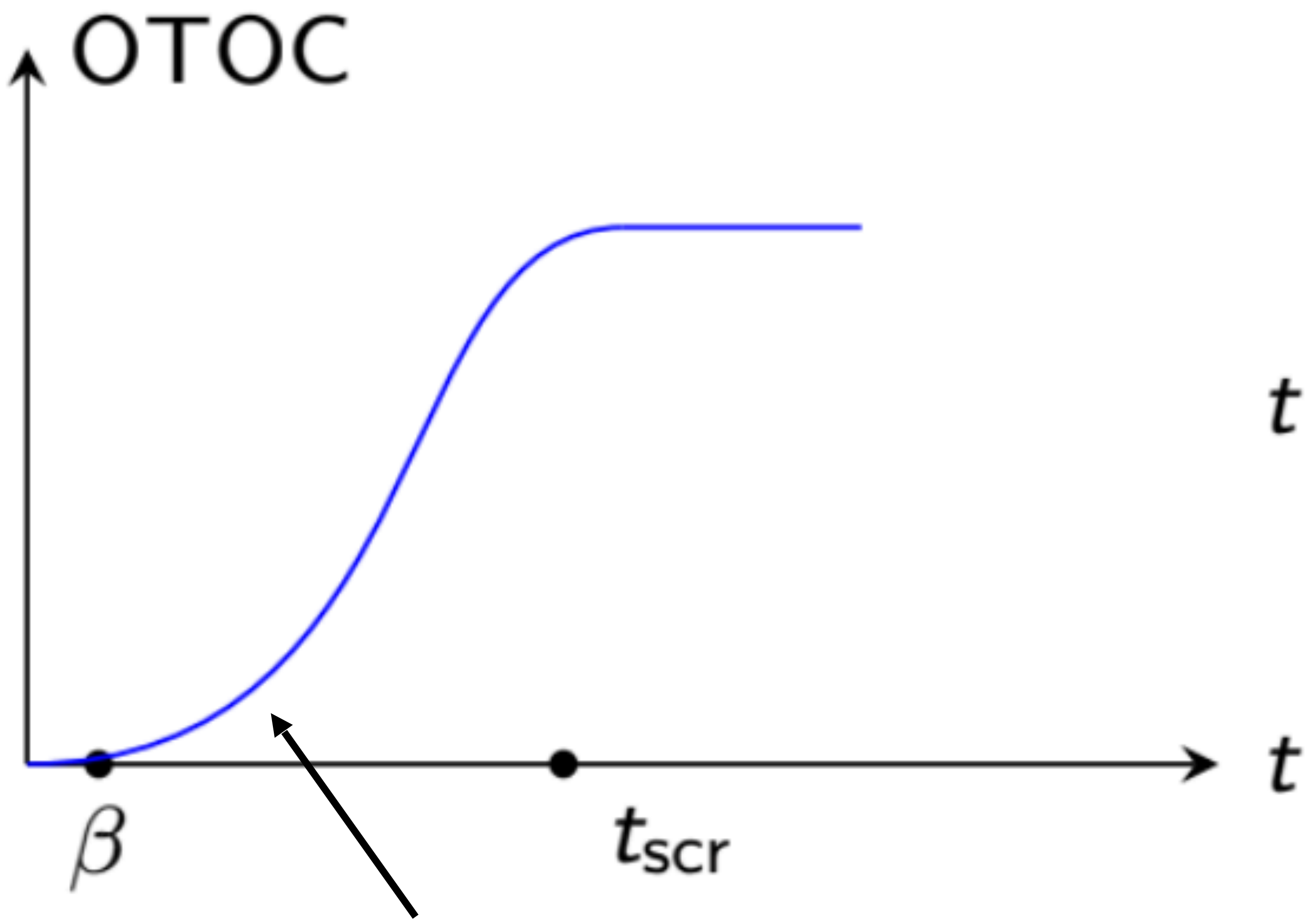

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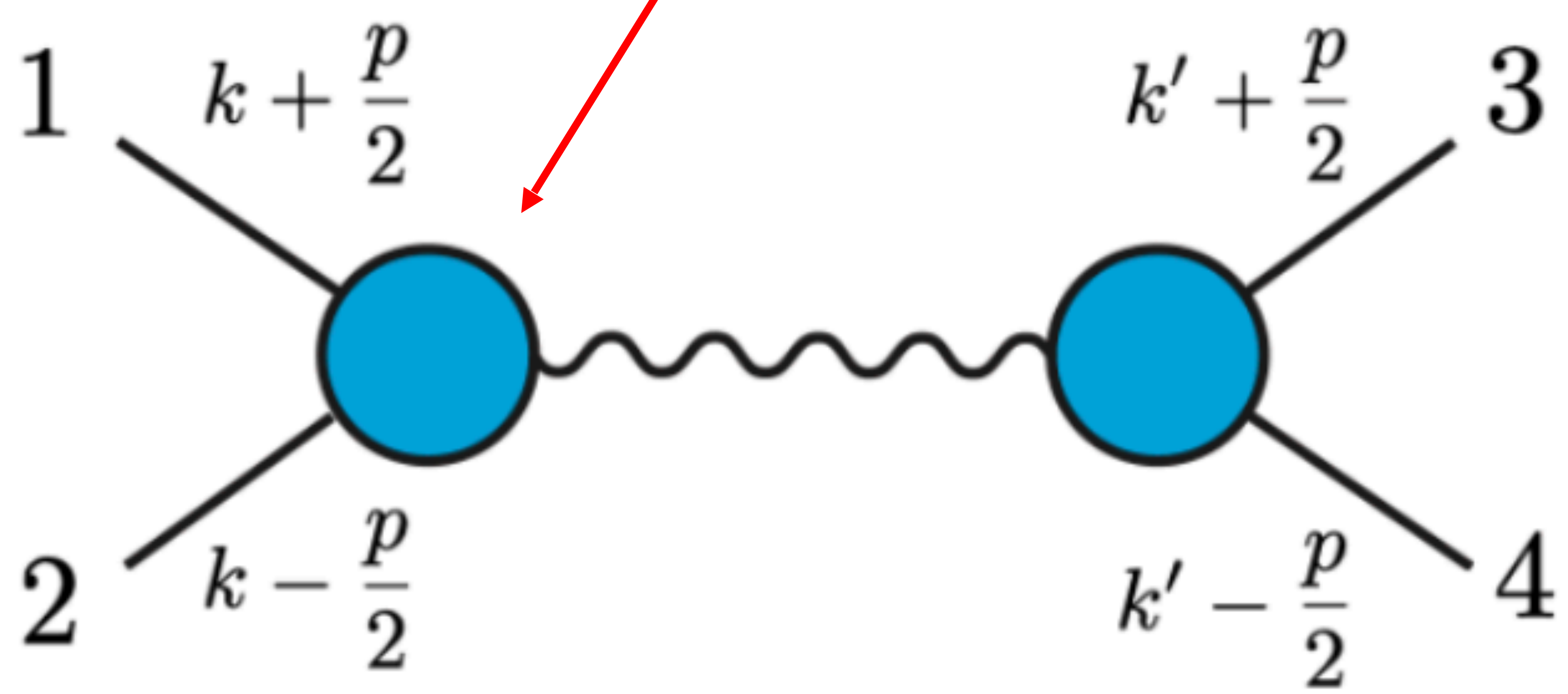

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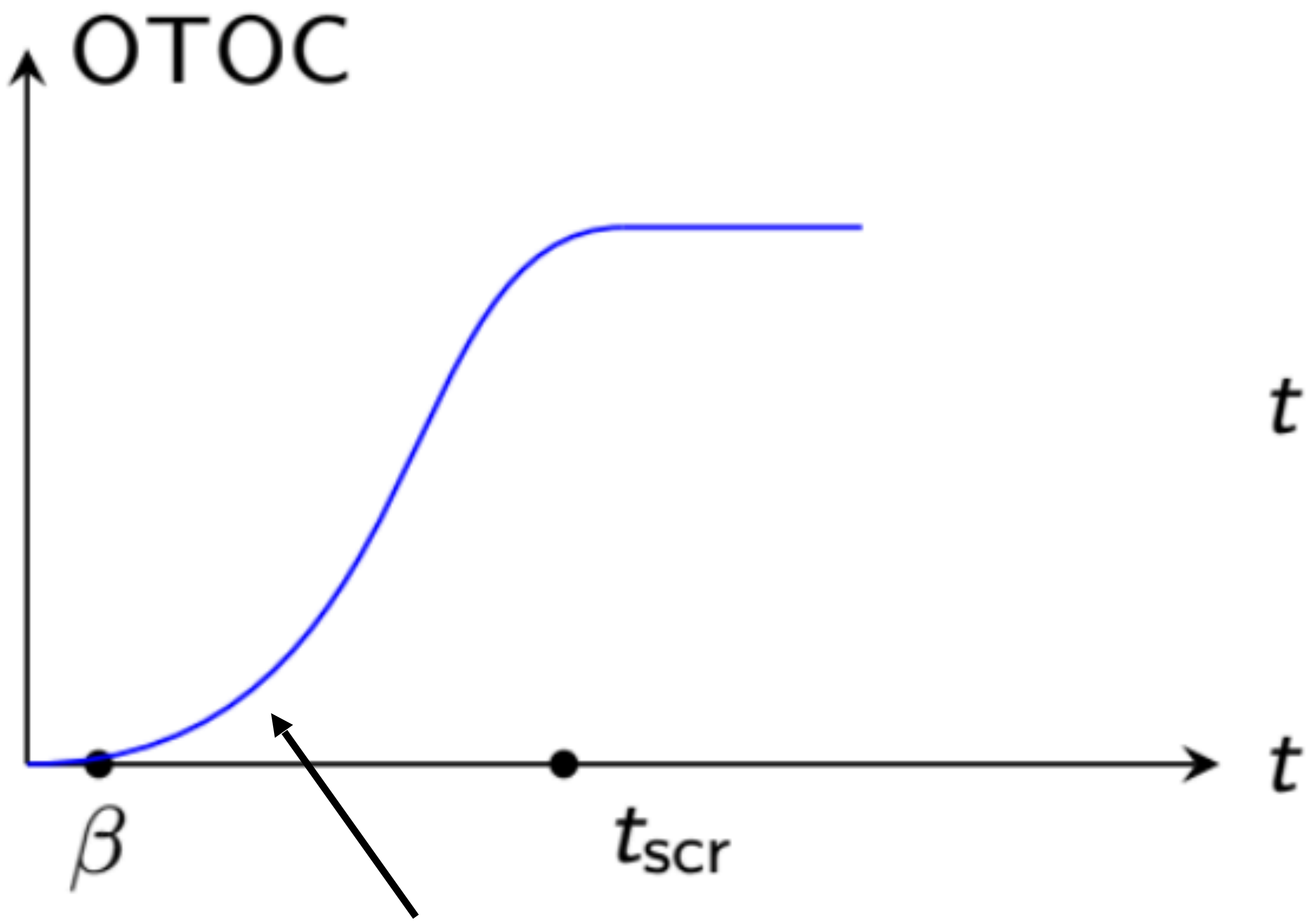

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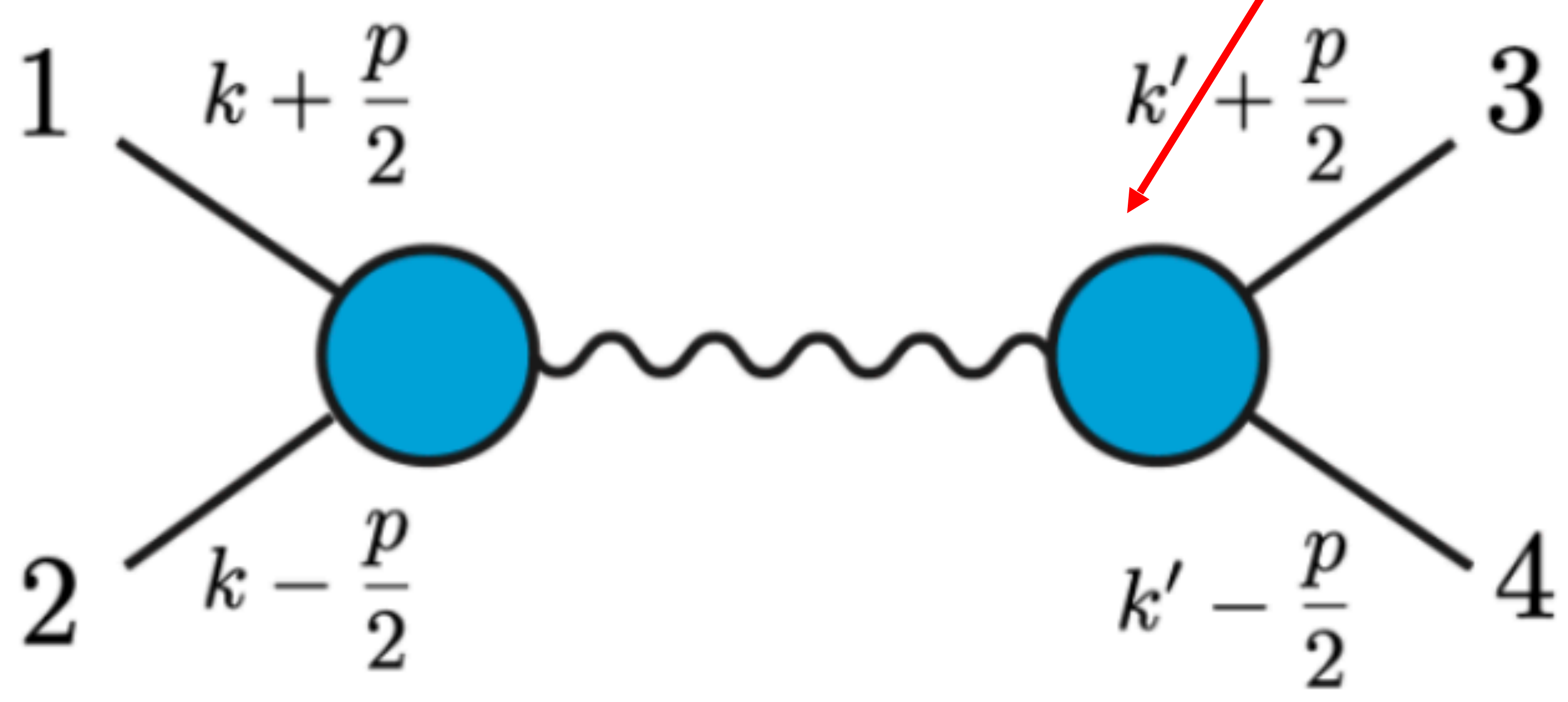

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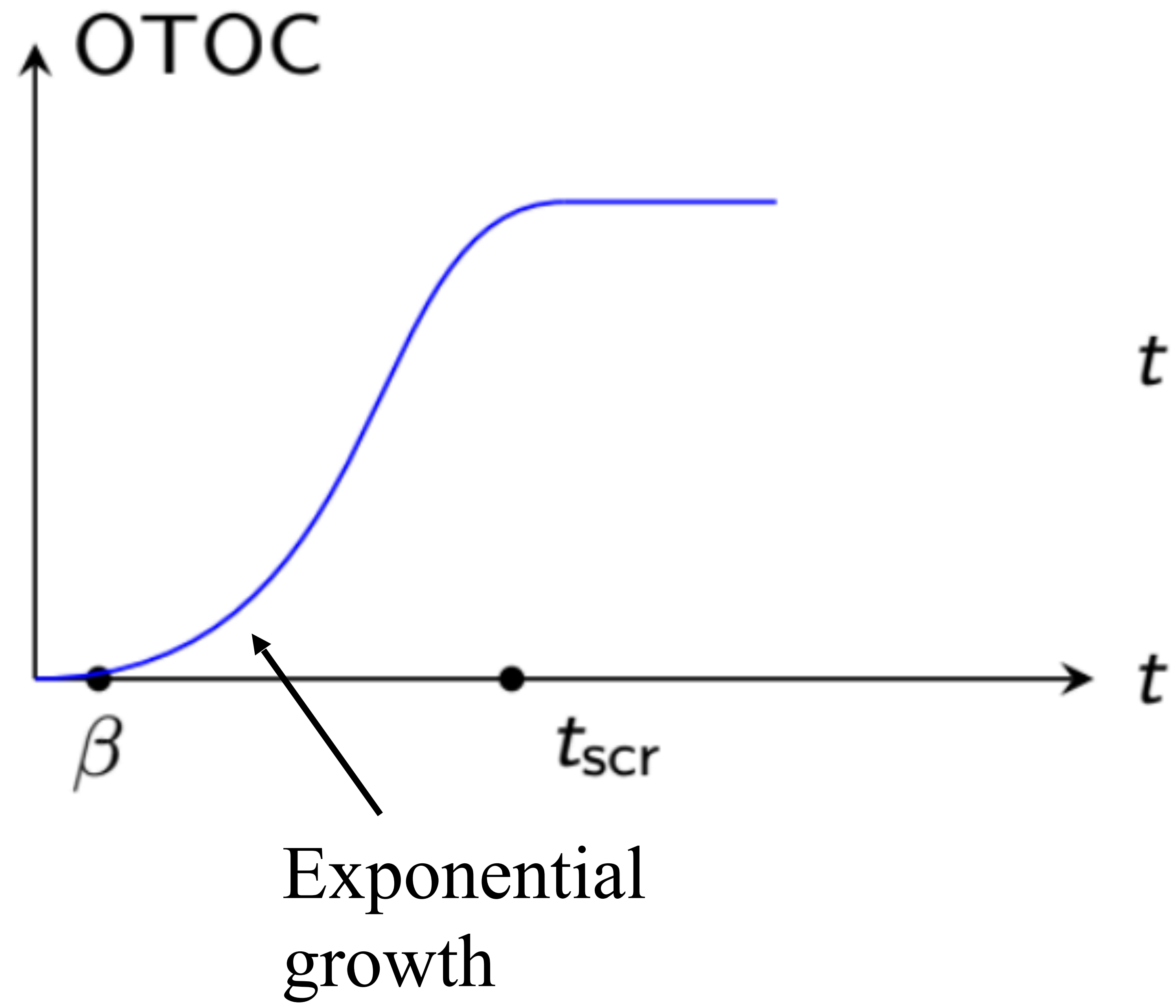
Structure of OTOC – early time regime

Consider a single mode ansatz for early times

"Scramblon"

$$\mathrm{OTOC}_p(t_1,t_2,t_3,t_4;k,k') \approx \frac{\boxed{e^{\kappa(p)(t_1+t_2-t_3-t_4)/2}}}{C(p)}\Upsilon_p^R(t_{12},k)\Upsilon_p^A(t_{34},k')$$



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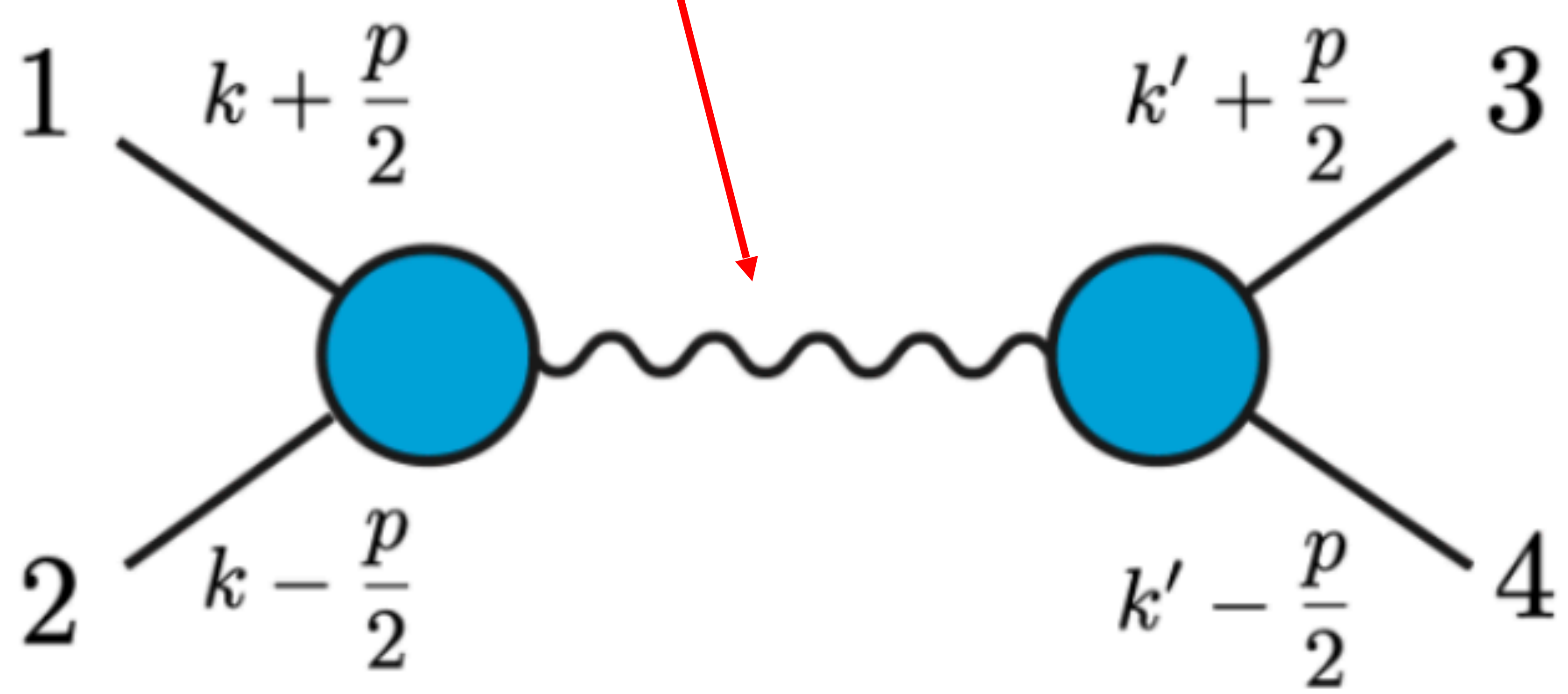

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Consider a single mode ansatz for early times

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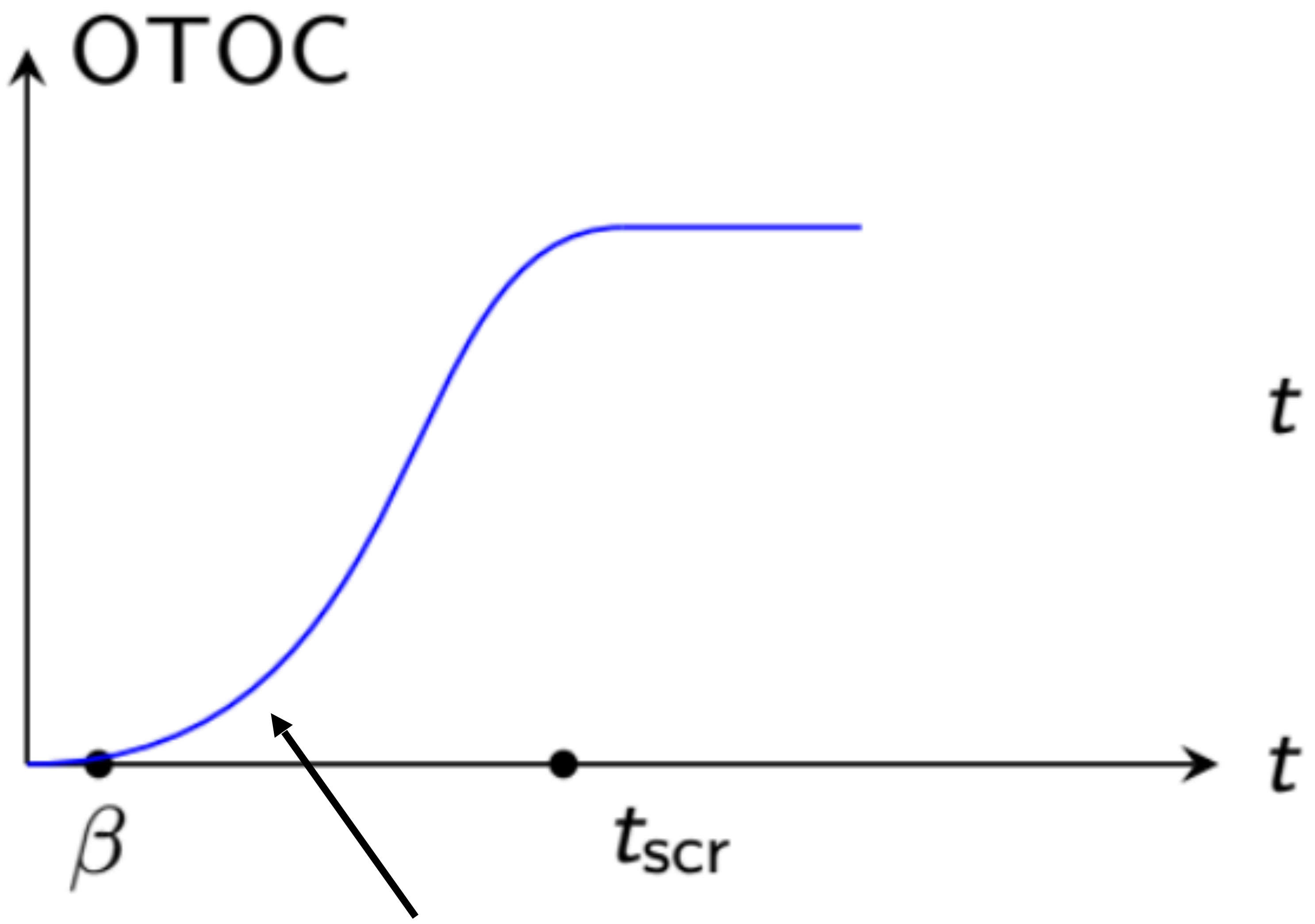
Function to be found



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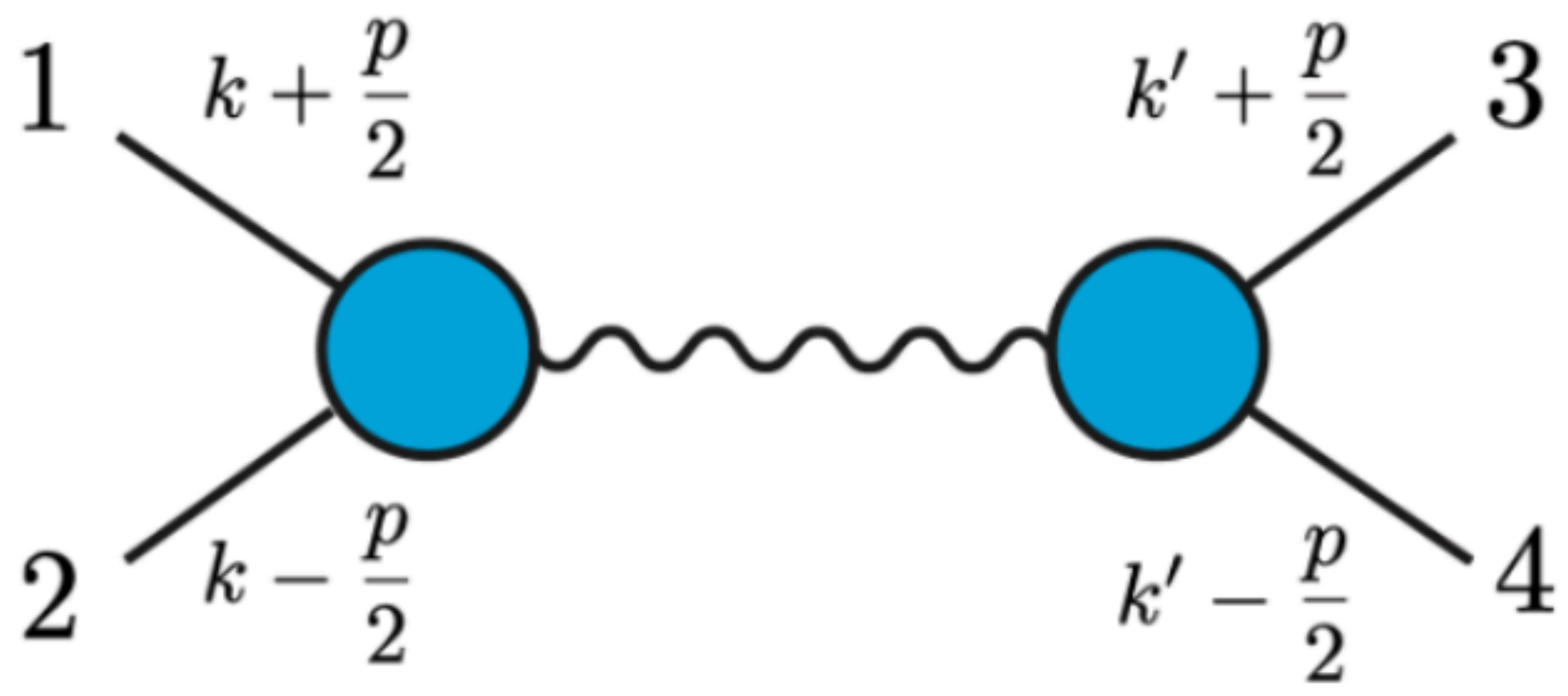

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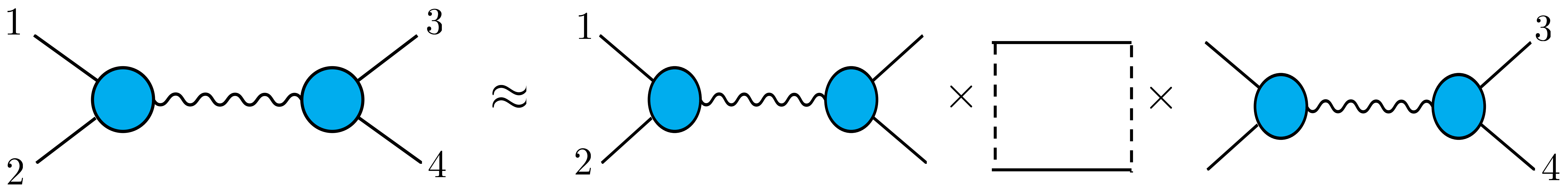
Structure of OTOC – ladder identity

- What is C(p)?
- A way to find it is to use the ladder identity from [Gu, Kitaev 2018]

1) Write down the self-consistency condition:

$$\mathrm{OTOC}_p(t_1,t_2,t_3,t_4;k,k') \approx \int_{t_5,t_6,t_7,t_8;q,q'} \mathrm{OTOC}_p^R(t_1,t_2,t_5,t_6;k,q) \times \underbrace{\begin{array}{cc} 5 & 7 \\ \bullet\!\!-\!\!-\!\!-\!\!\bullet & \\ 6 & 8 \end{array}}_{\mathrm{BOX}} \times \mathrm{OTOC}_p(t_7,t_8,t_3,t_4;q',k')$$



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Structure of OTOC – ladder identity

- What is C(p)?
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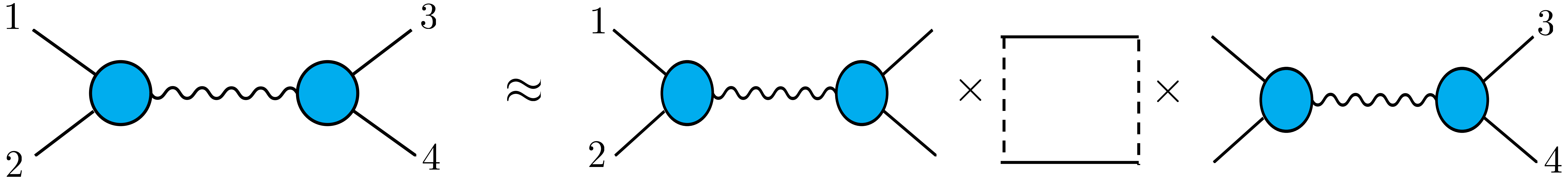
2) Apply the single-mode ansatz:



$$\frac{e^{\kappa(p)\frac{(t_1+t_2-t_3-t_4)}{2}}}{C(p)}\Upsilon_p^R(t_{12},k)\Upsilon_p^A(t_{34},k') = N\frac{2\cos\frac{\kappa(p)\pi}{2}}{C^2(p)}e^{\kappa(p)\frac{(t_1+t_2-t_3-t_4)}{2}}$$

$$\times\int_{t_5,t_6,t_7,t_8;q,q'} e^{\kappa(p)\frac{-(t_5+t_6)}{2}}\Upsilon_p^R(t_{12},k)\Upsilon_p^A(t_{56},q)\cdot \mathrm{BOX}(t_5,t_6,t_7,t_8;q,q')\cdot e^{\kappa(p)\frac{(t_7+t_8)}{2}}\Upsilon_p^R(t_{78},q')\Upsilon_p^A(t_{34},k')$$

Yingfei Gu, Alexei Kitaev 2018

Structure of OTOC – ladder identity

- What is C(p)?
- A way to find it is to use the ladder identity from [Gu, Kitaev 2018]

3) Find C(p):

$$C(p)= 2N\cos\frac{\kappa(p)\pi}{2}\int_{t_5,t_6,t_7,t_8;q,q'} e^{\kappa(p)(t_7+t_8-t_5-t_6)/2}\Upsilon_p^A(t_{56},q)\cdot \mathrm{BOX}(t_5,t_6,t_7,t_8;q,q')\cdot\Upsilon_p^R(t_{78},q')$$

Important for us: $\boxed{C(p)\sim N\cos\left(\frac{\kappa(p)\pi}{2}\right)}$

OTOC - early time behavior:

$$\mathrm{OTOC}_p(t_1,t_2,t_3,t_4;k,k')\sim\frac{e^{\kappa(p)(t_1+t_2-t_3-t_4)/2}}{\cos\frac{\kappa(p)\pi}{2}}\Upsilon_p^R(t_{12},k)\Upsilon_p^A(t_{34},k')$$

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Structure of OTOC: saddle point vs pole

Fourier transform: $\mathrm{OTOC}_{x,0}(t_1,t_2,t_3,t_4)=\int\frac{dp}{2\pi}e^{ipx}\mathrm{OTOC}_p(t_1,t_2,t_3,t_4)$

Early time: $\mathrm{OTOC}_{x,0}(t_1,t_2,t_3,t_4)\sim\frac{1}{N}\underbrace{\int_{-\infty}^{+\infty}\frac{dp}{2\pi}\frac{e^{\kappa(p)t+ipx}}{\cos\frac{\pi\kappa(p)}{2}}}_{u(x,t)}\int_{k,q}\Upsilon_p^R(t_{12},k)\Upsilon_p^A(t_{34},q)$

Estimate the integral:

1) Saddle point solution: $u_s(x,t)\sim\frac{e^{\kappa(p_s)t+ip_sx}}{\cos\frac{\pi\kappa(p_s)}{2}},\qquad p_s:\ \kappa'(p_s)t+ix=0$

2) Pole contribution: $u_1(x,t)\sim\frac{e^{t+ip_1x}}{\pi i\kappa'(p_1)},\qquad p_1:\ \kappa(p_1)=1$

Note: p_1 and p_s are pure imaginary: *i.e.* $p_1=i|p_1|$ and $p_s=i|p_s|$.

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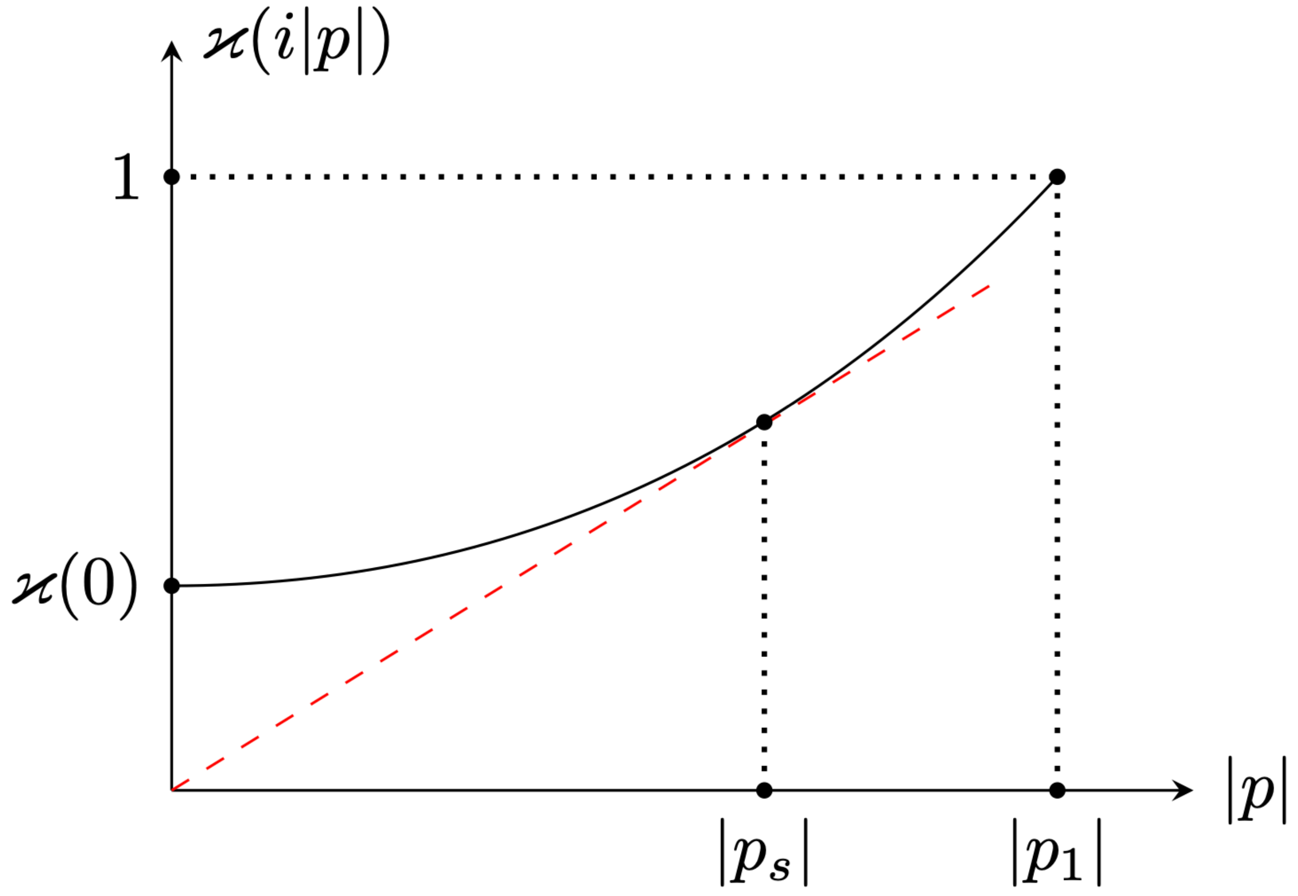
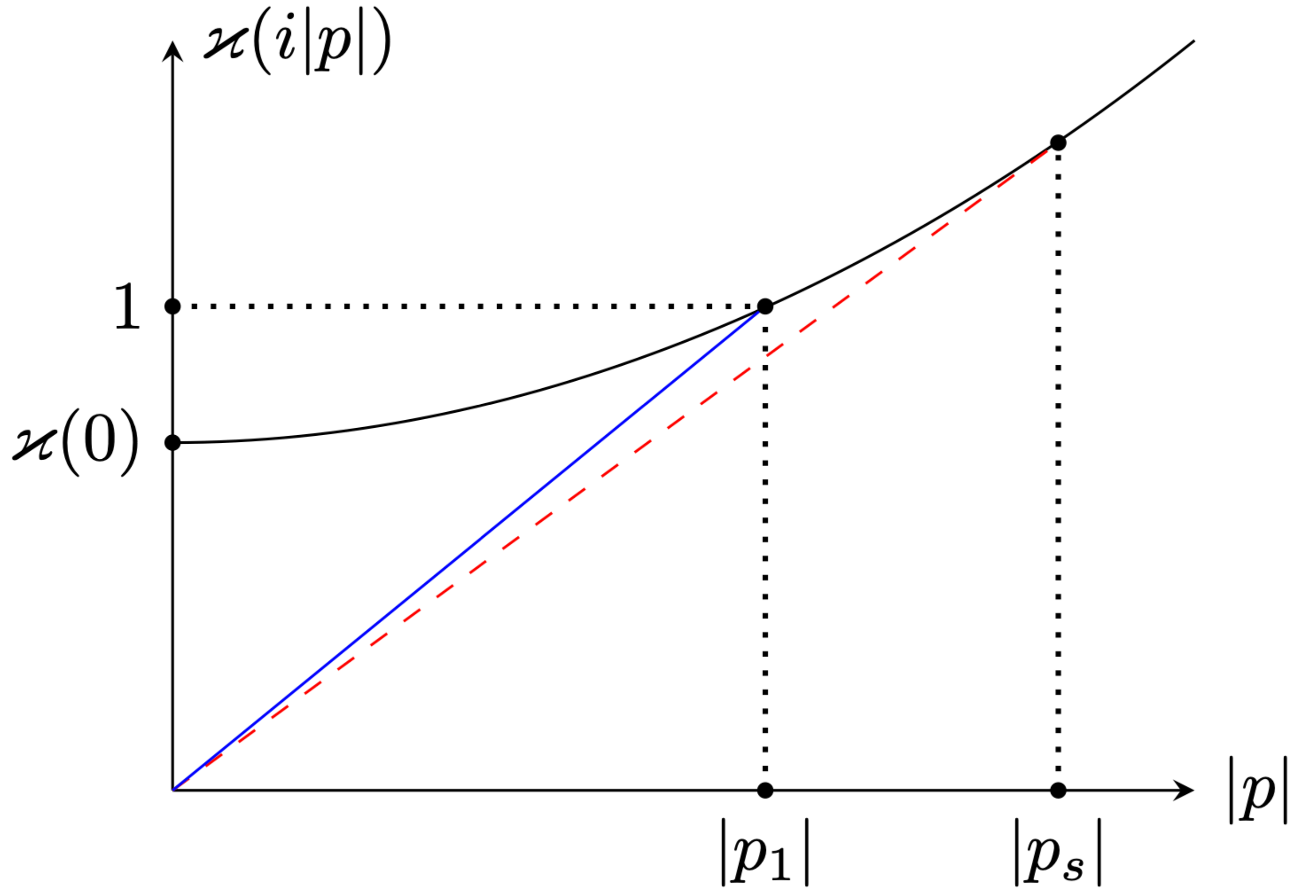
Maximal chaos requires a large butterfly velocity, or $\kappa(0)$ not too far from 1.



(a) $|p_s| < |p_1|$

Saddle point contribution –
weak quantum chaos



(b) $|p_s| > |p_1|$

Pole contribution –
maximal chaos

Note: p_1 and p_s are pure imaginary: *i.e.* $p_1 = i|p_1|$ and $p_s = i|p_s|$.

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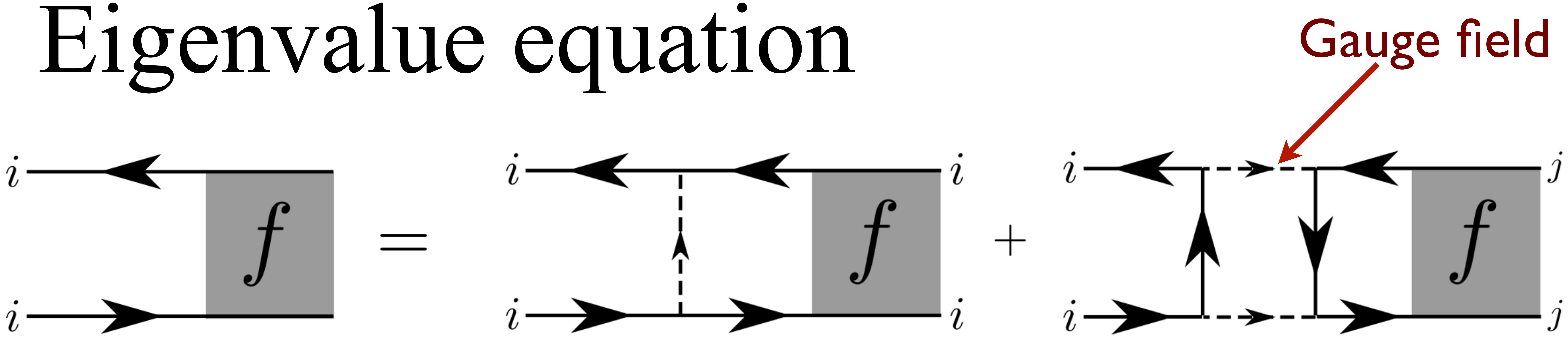
Eigenvalue equation

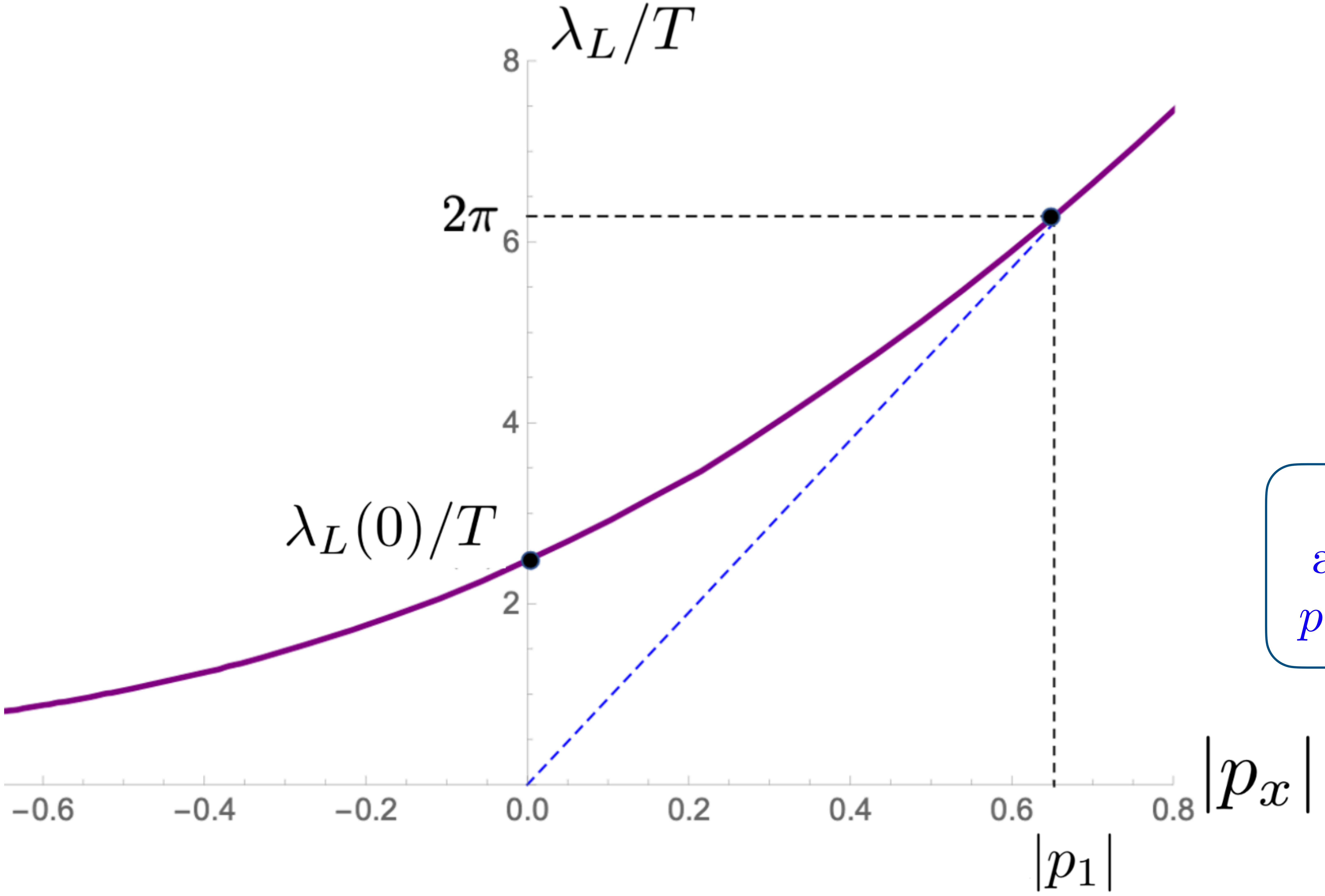


Invariance under adding a ladder

$$f(t) = \frac{1}{N^2}\theta(t)\sum_{i,j=1}^{N}\int d^2x\ \mathrm{Tr}\left[e^{-\beta H/2}\{\psi_i(x,t),\psi_j^\dagger(0)\}e^{-\beta H/2}\{\psi_i(x,t),\psi_j^\dagger(0)\}^\dagger\right] = \int d^2x\ f(t,x)$$

$$\begin{aligned}\left[T^{2/z}c_{f,z}\left(H_{1-2/z}\left(\frac{-ik_0-\pi T}{2\pi T}\right)+H_{1-2/z}\left(\frac{-i(\omega-k_0)-\pi T}{2\pi T}\right)\right)+2\mu(T)-p_x\right]\tilde{f}(k_0,\omega) \\ = g^2\int\frac{dk_0'dk_y'}{(2\pi)^2}\frac{c_b(k_0-k_0')|k_y'|}{(|k_y'|^z+m^2)^2+c_b^2(k_0-k_0')^2}\frac{\tilde{f}(k_0',\omega)}{\sinh\frac{k_0-k_0'}{2T}} \\ -\frac{ig^4}{8z\sin(\frac{2\pi}{z})c_b^{2-\frac{2}{z}}}\int\frac{dk_0'dk_{01}}{(2\pi)^2}\frac{(-ik_{01})^{\frac{2}{z}-1}-(i(k_{01}-\omega))^{\frac{2}{z}-1}}{2k_{01}-\omega}\frac{\tilde{f}(k_0',\omega)}{\cosh\frac{k_0-k_{01}}{2T}\cosh\frac{k_0'-k_{01}}{2T}}\end{aligned}$$

As in [Patel, Sachdev 2016]



Note: p_1 and p_x are pure imaginary: *i.e.* $p_1 = i|p_1|$ and $p_x = i|p_x|$.

We find maximal chaos with $\lambda_L = 2\pi T$, and butterfly velocity $v_1 = 9.67 g^{-4/3} T^{1/3}$

Outlook

- Large N theory for non-quasiparticle dynamics of a critical Fermi surface. Saddle point allows systematic computation of fluctuation corrections, and pairing and density-wave instabilities.

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- Critical Fermi surface has maximal chaos with $\lambda_L = 2\pi T$, and butterfly velocity $\sim T^{1/3}$.

Outlook

- Large N theory for non-quasiparticle dynamics of a critical Fermi surface. Saddle point allows systematic computation of fluctuation corrections, and pairing and density-wave instabilities.

- Critical Fermi surface has maximal chaos with $\lambda_L = 2\pi T$, and butterfly velocity $\sim T^{1/3}$.

- Well-defined regime of exponential growth of chaos can exist even in systems with short-range interactions and a small number of degrees of freedom on each site provided the butterfly velocity is large enough. (Gu, Kitaev, 2018; Keselman, Nie, Berg, 2020)

Outlook

- Large N theory for non-quasiparticle dynamics of a critical Fermi surface. Saddle point allows systematic computation of fluctuation corrections, and pairing and density-wave instabilities.

- Critical Fermi surface has maximal chaos with $\lambda_L = 2\pi T$, and butterfly velocity $\sim T^{1/3}$.

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- Hydrodynamic effective theory for 'strong' quantum chaos provides rationale for why it is always maximal, with $\lambda_L = 2\pi T$ (Blake, Liu, 2021).