

# Planckian Dynamics

$$\tau_r \geq C \frac{\hbar}{k_B T} \quad (\text{SS, } \textit{Quantum Phase Transitions}, 1999)$$

1. Quantum critical dynamics of the Ising model in 2 spatial dimensions
2. Quantum critical transport at the superfluid-insulator transition in 2 spatial dimensions

Thermal Seminars  
DESY  
September 2, 2025

Subir Sachdev

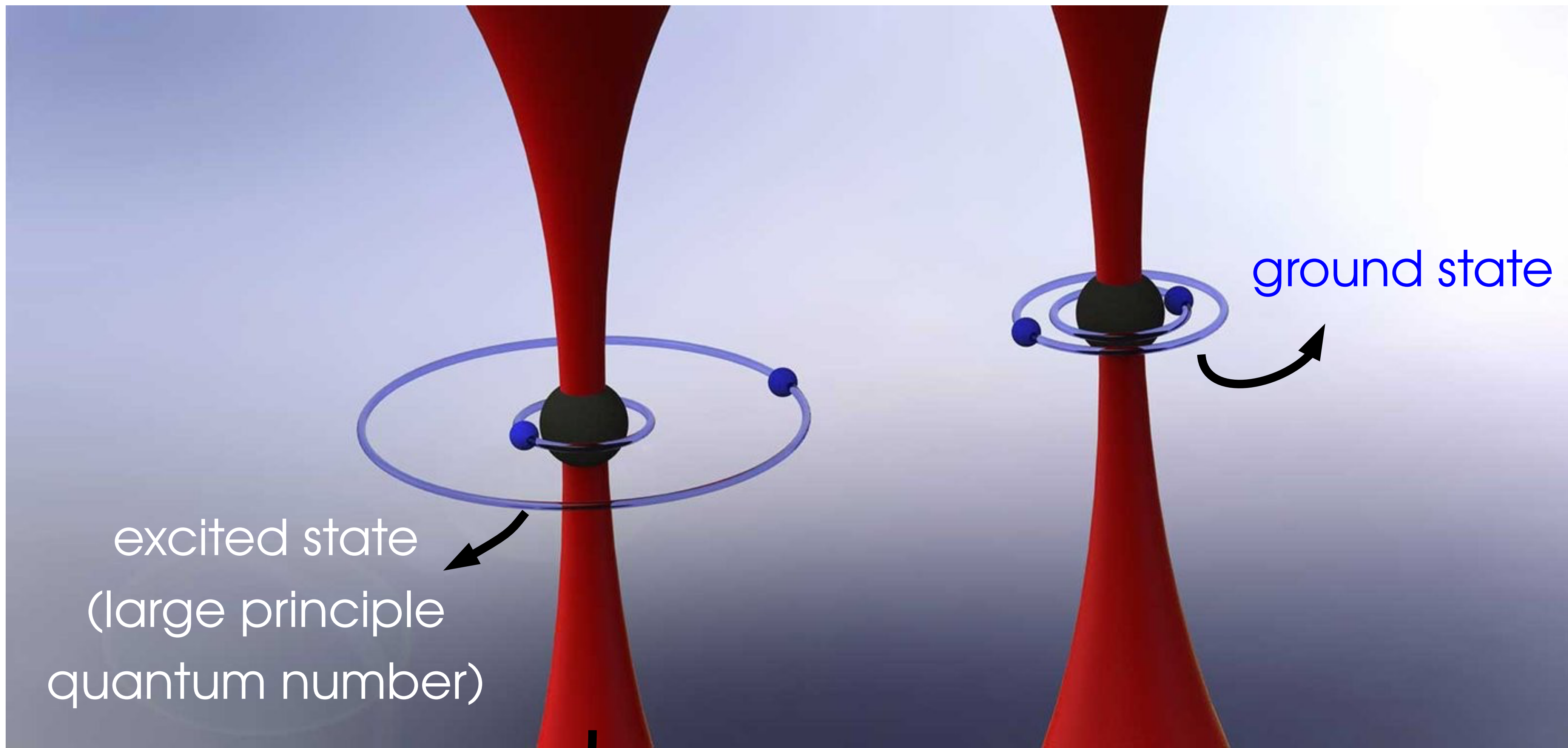


# Quantum critical dynamics of the Ising model in 2 spatial dimensions

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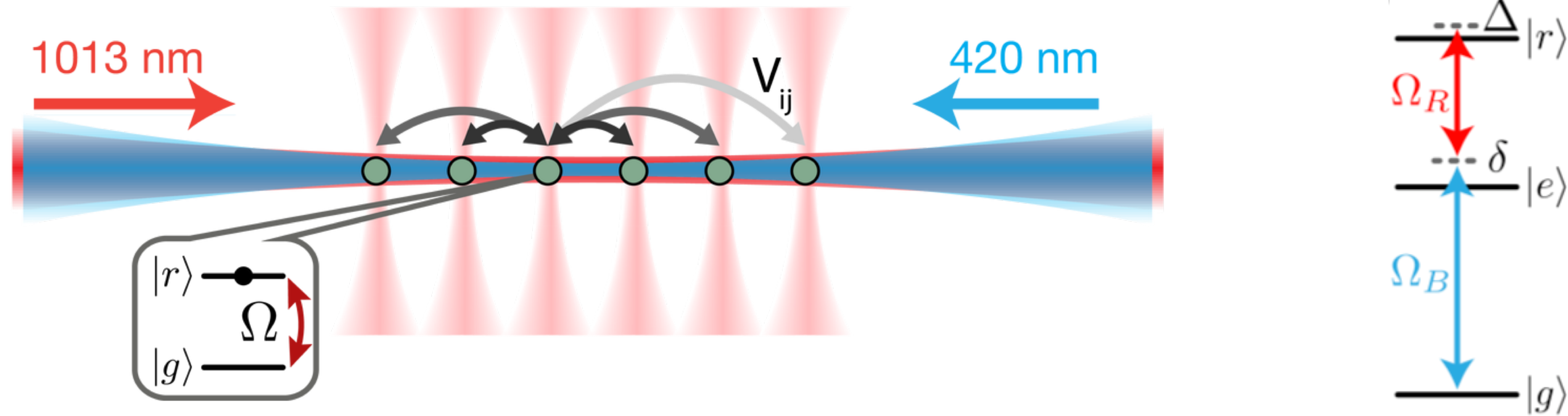
$$V_{|i-j|} \sim |i-j|^{-6}$$

optical tweezer (traps atom)

Fig: <https://www.caltech.edu/about/news/quantum-innovations-achieved-using-alkaline-earth-atoms>

$$H_{\text{Ryd}} = \sum_i \left[ \frac{\Omega}{2} (|g\rangle\langle r| + |r\rangle\langle g|)_i - \Delta |r\rangle\langle r| \right] + \sum_{(i,j)} V_{|i-j|} (|r\rangle\langle r|_i \otimes |r\rangle\langle r|_j)$$

# QPTs in a Rydberg quantum simulator



$$\mathcal{H} = \sum_j \left[ \frac{\Omega}{2} (b_j + b_j^\dagger) - \Delta n_j \right] + \sum_{i < j} V_{|i-j|} n_i n_j$$

$$n_j \equiv b_j^\dagger b_j$$

$n_j = 0, 1$  'hard core' bosons

$$V_{|i-j|} \sim \frac{1}{|r_i - r_j|^6}$$

**The PXP model:** ( $V_1 = \infty, V_{j \geq 2} = 0$ ,  $\mathcal{P}$  projects out nearest-neighbor boson states)

$$\mathcal{H}_{\text{PXP}} = \sum_j \left[ \frac{\Omega}{2} \mathcal{P} (b_j + b_j^\dagger) \mathcal{P} - \Delta b_j^\dagger b_j \right]$$

These models were motivated by ‘tilted lattices’ of bosonic atoms, and predicted a  $\mathbb{Z}_2$  quantum transition which was observed in J. Simon, W. S. Bakr, Ruichao Ma, M. Eric Tai, P. M. Preiss, M. Greiner, Nature **472**, 307 (2011).

$$\mathcal{H} = \sum_j \left[ \frac{\Omega}{2} (b_j + b_j^\dagger) - \Delta n_j \right] + \sum_{i < j} V_{|i-j|} n_i n_j \quad V_{|i-j|} \sim \frac{1}{|r_i - r_j|^6}$$

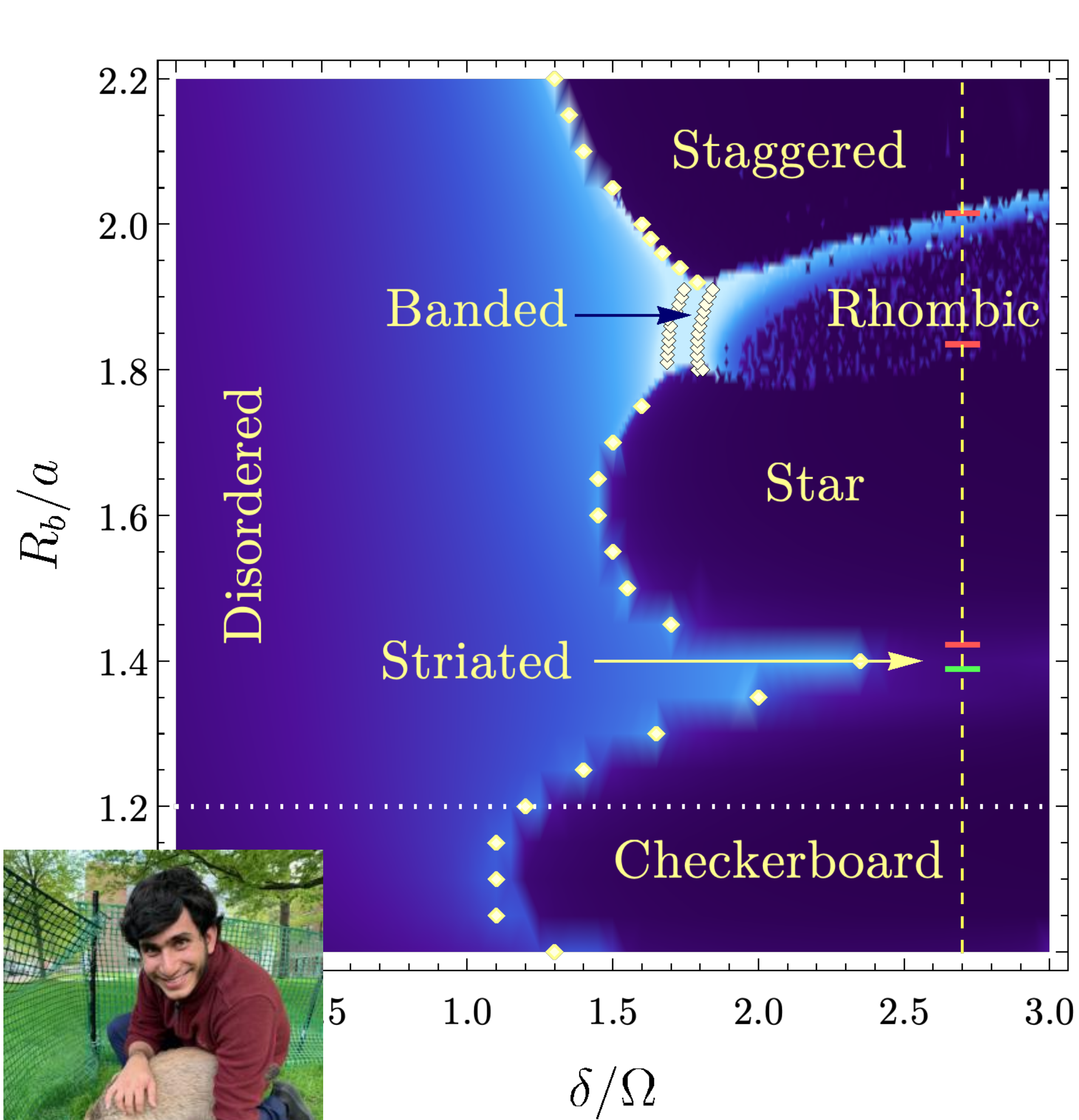
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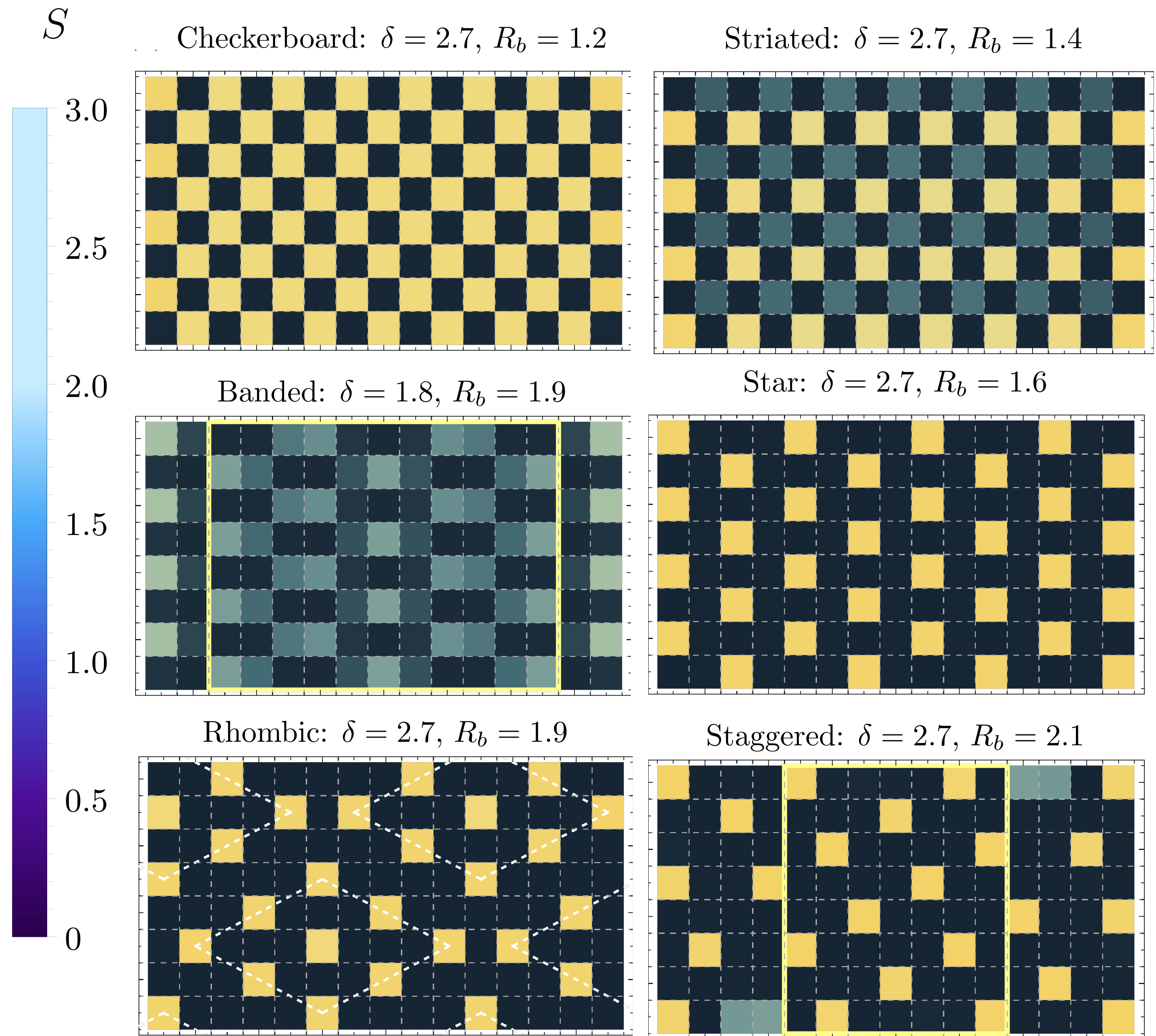
S. Sachdev, K. Sengupta, and S.M. Girvin, PRB **66**, 075128 (2002)

P. Fendley, K. Sengupta, S. Sachdev, PRB **69**, 075106 (2004)

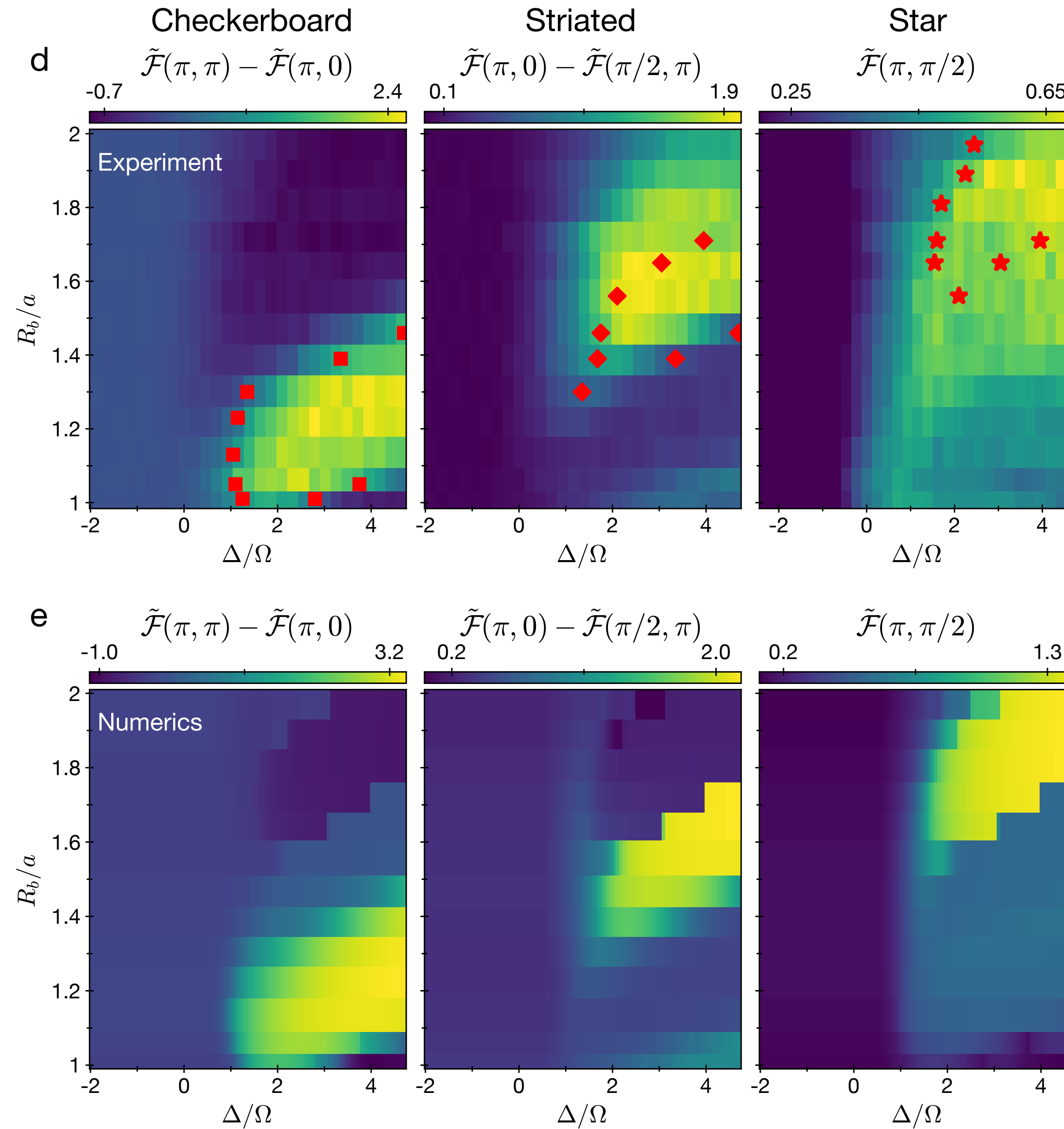
# Rydberg atoms on the square lattice: theory



R. Samajdar, Wen Wei Ho, H. Pichler, M. D. Lukin, S. Sachdev, PRL **124**, 103601 (2020)

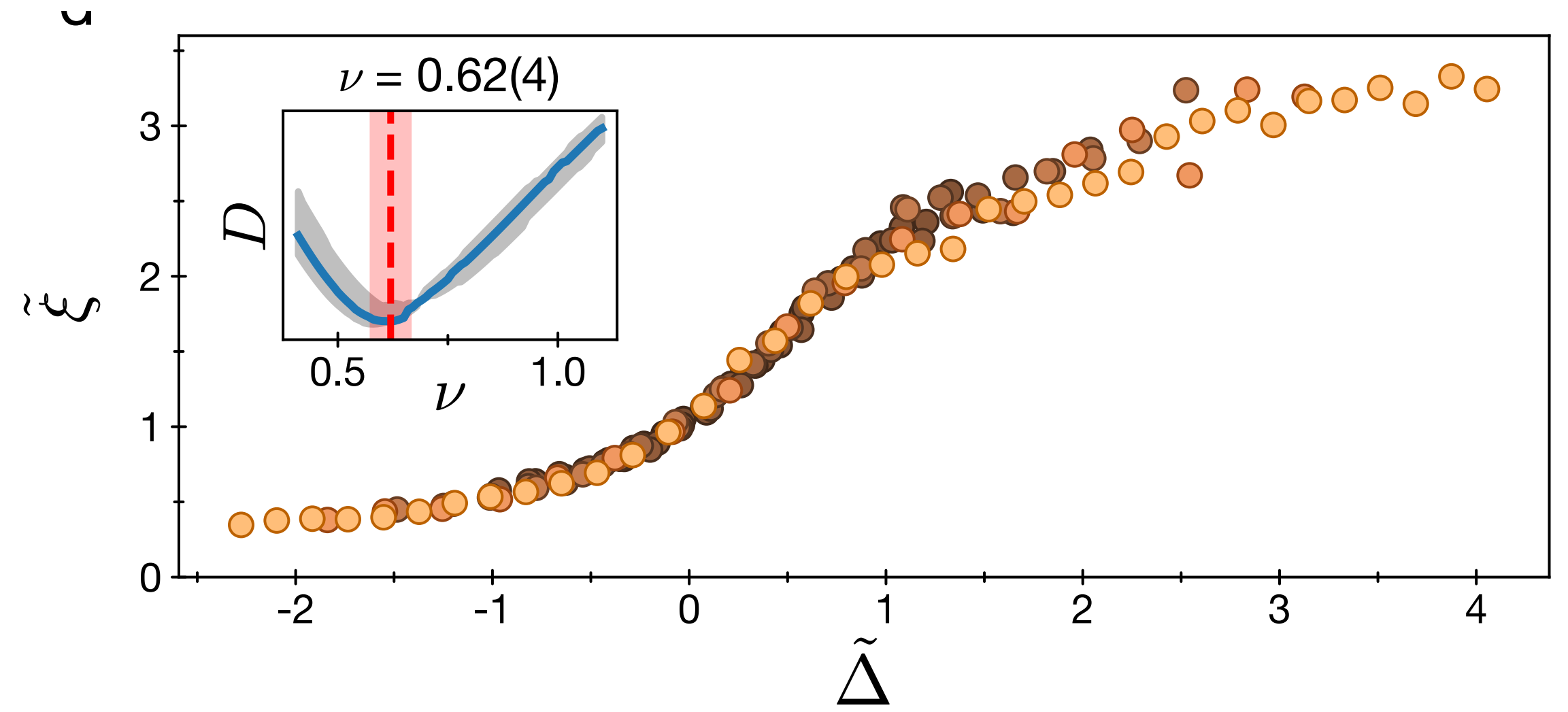
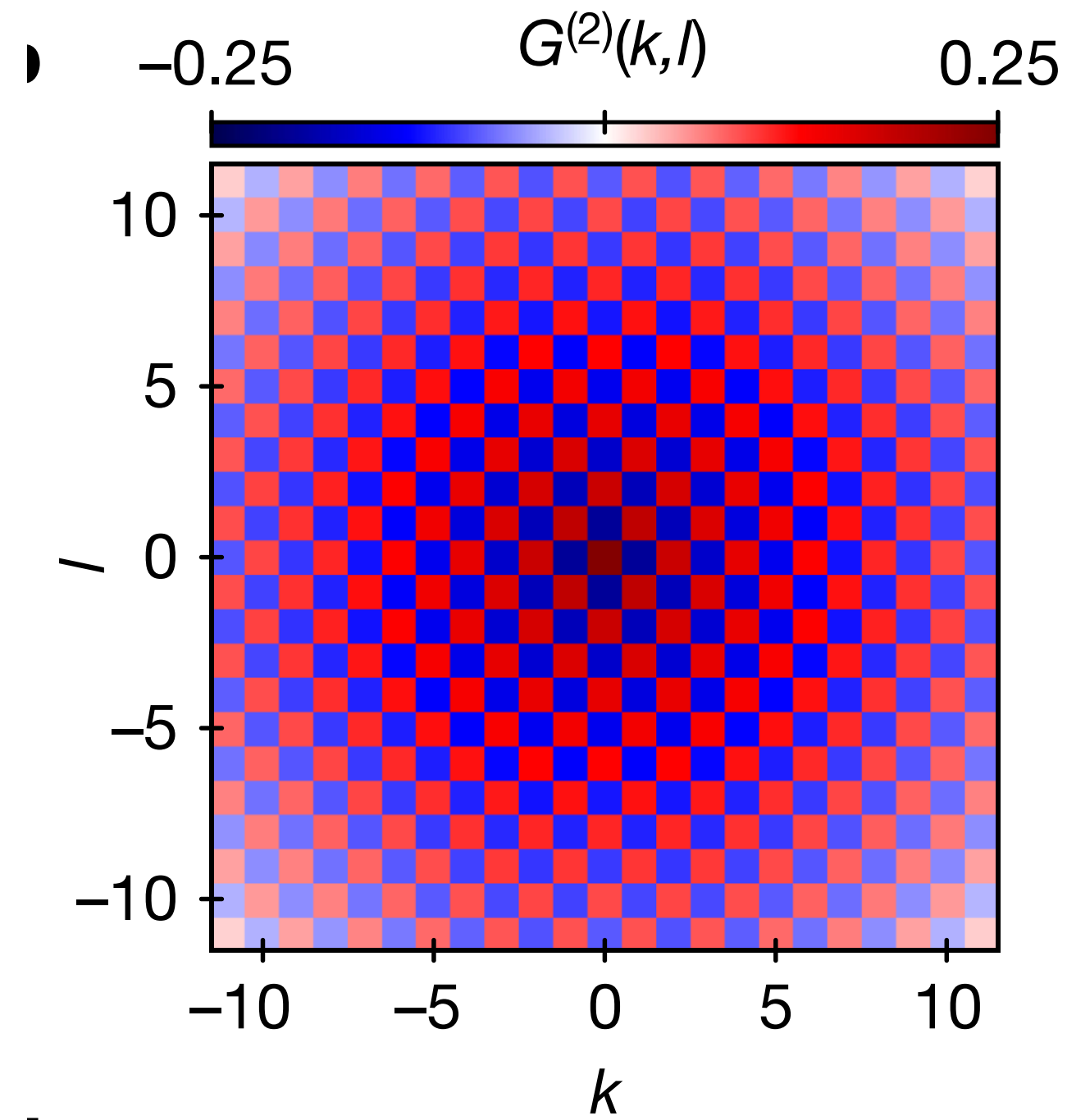
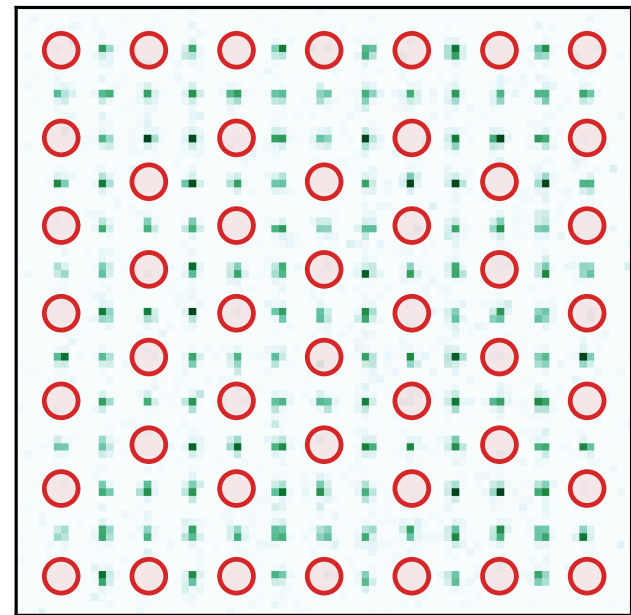
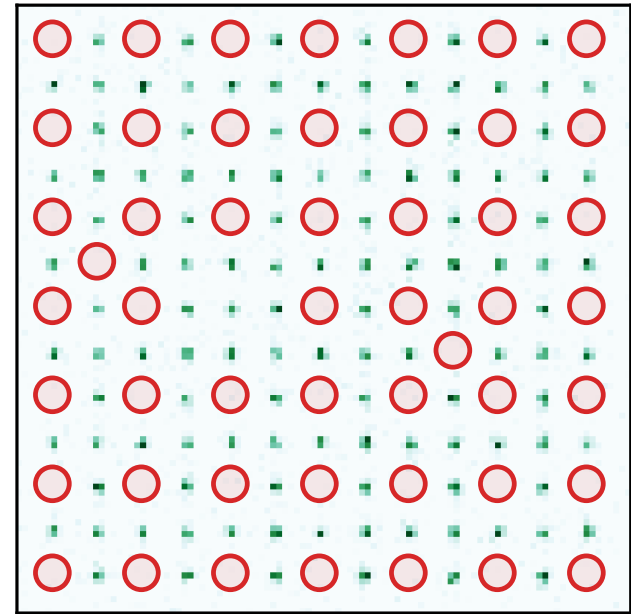
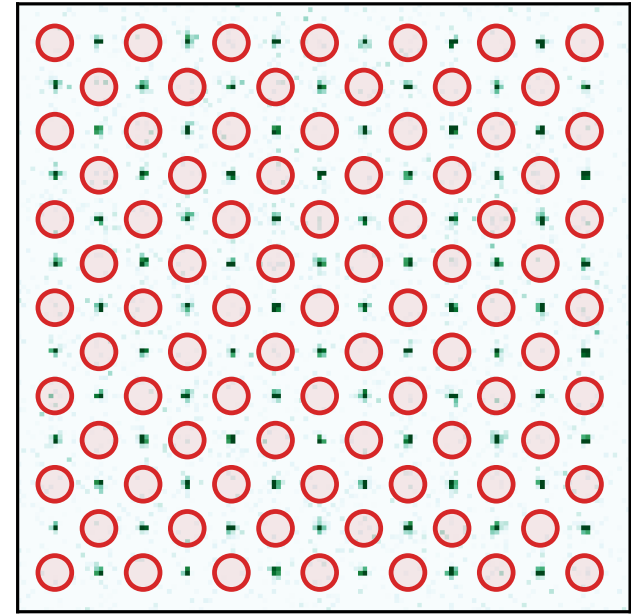


# Rydberg atoms on the square lattice: experiment



Quantum Phases of Matter on a 256-Atom Programmable Quantum Simulator, Sepehr Ebadi, Tout T. Wang, Harry Levine, Alexander Keesling, Giulia Semeghini, Ahmed Omran, Dolev Bluvstein, Rhine Samajdar, Hannes Pichler, Wen Wei Ho, Soonwon Choi, Subir Sachdev, Markus Greiner, Vladan Vuletic, and Mikhail D. Lukin, Nature **595**, 227 (2021); Pascal Scholl et al. Nature **595**, 233 (2021)

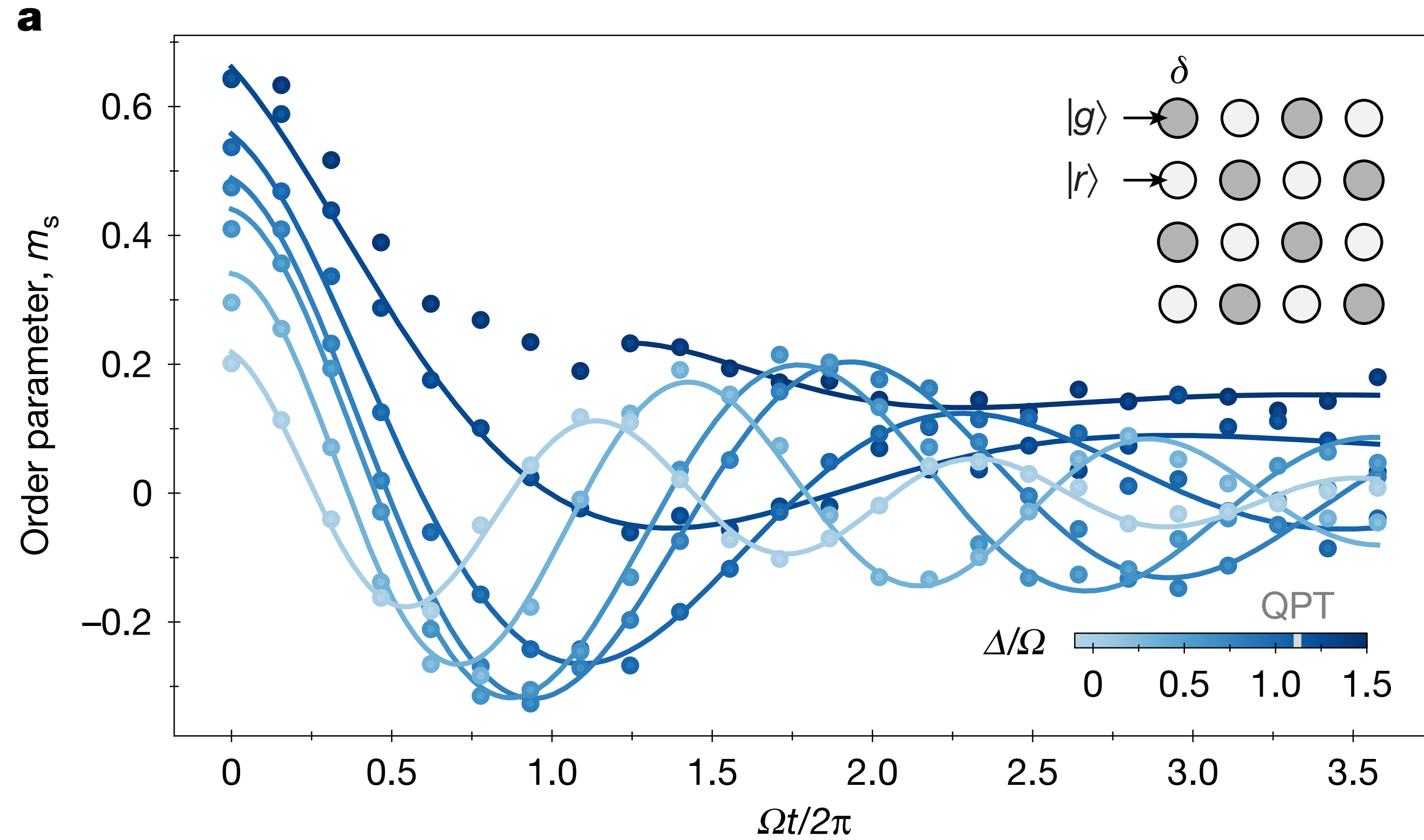
# Rydberg atoms on the square lattice: experiment



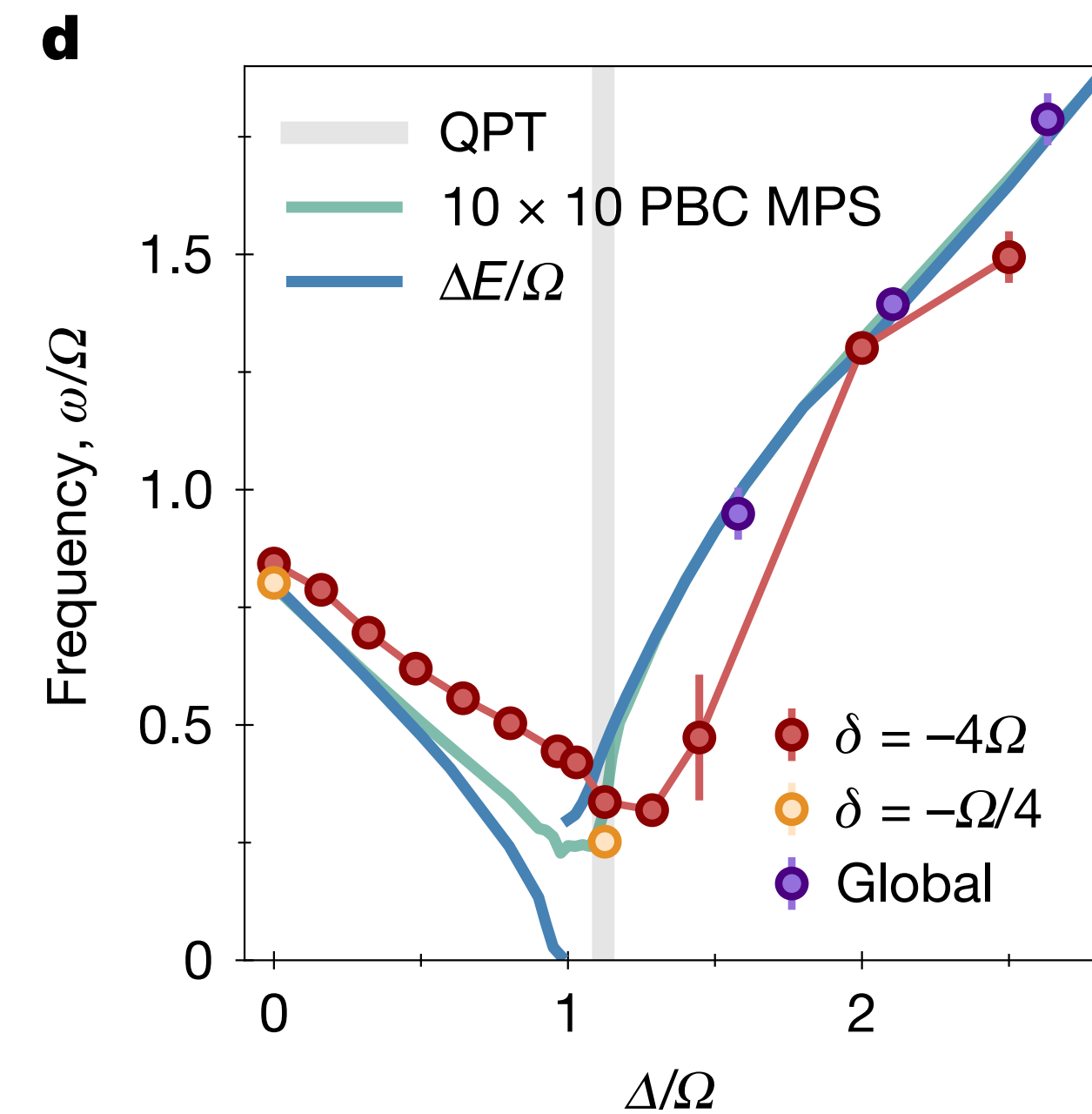
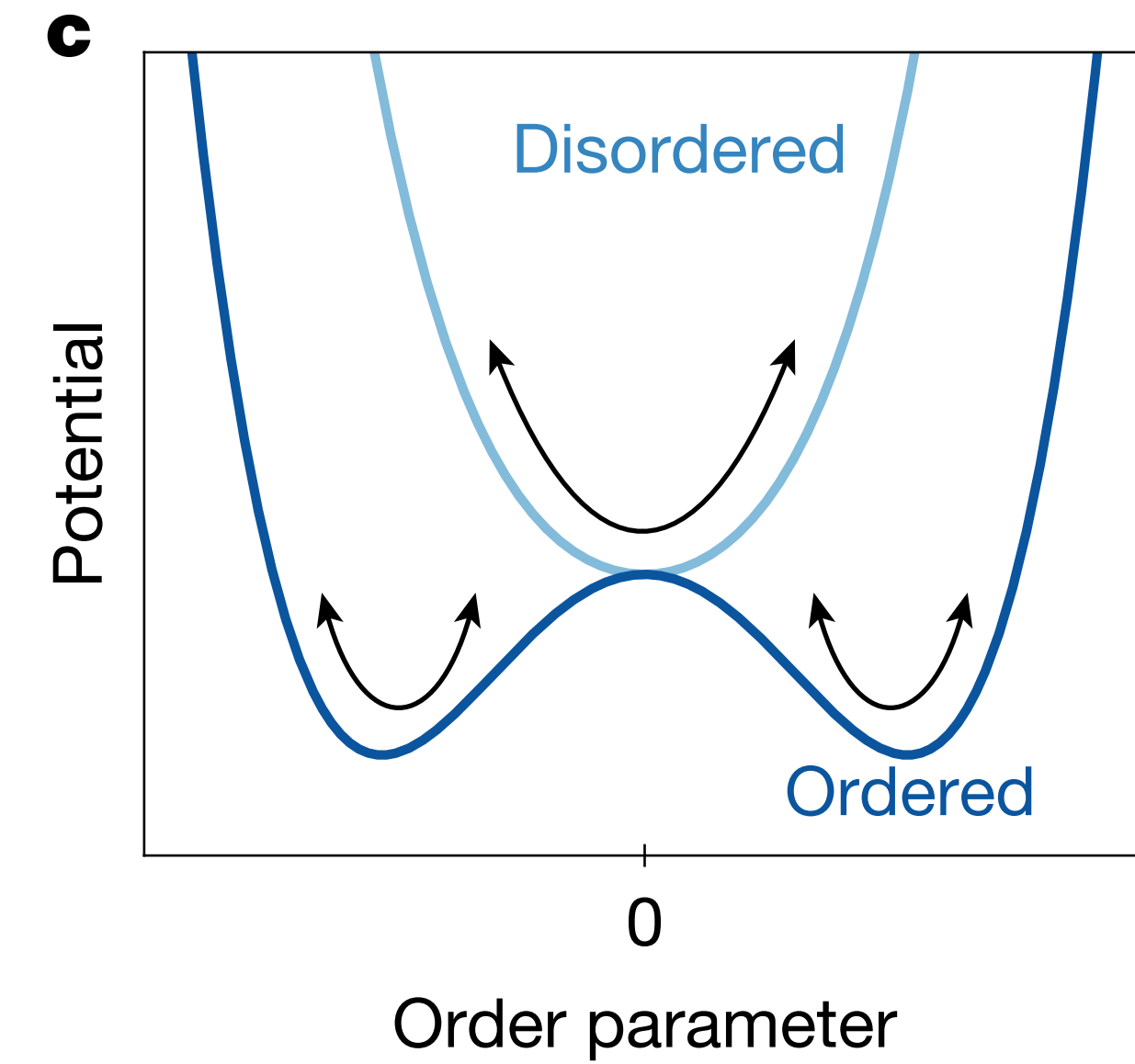
First observation of Ising quantum phase transition in 2+1 dimensions

# Quantum coarsening and collective dynamics on a programmable simulator

Tom Manovitz<sup>1,8</sup>, Sophie H. Li<sup>1,8</sup>, Sepehr Ebadi<sup>1,7,8</sup>, Rhine Samajdar<sup>2,3</sup>, Alexandra A. Geim<sup>1</sup>, Simon J. Evered<sup>1</sup>, Dolev Bluvstein<sup>1</sup>, Hengyun Zhou<sup>1,4</sup>, Nazli Ugur Koyluoglu<sup>1,5</sup>, Johannes Feldmeier<sup>1</sup>, Pavel E. Dolgirev<sup>1</sup>, Nishad Maskara<sup>1</sup>, Marcin Kalinowski<sup>1</sup>, Subir Sachdev<sup>1</sup>, David A. Huse<sup>2</sup>, Markus Greiner<sup>1</sup>, Vladan Vuletić<sup>6</sup> & Mikhail D. Lukin<sup>1</sup>✉



$$f(x) \equiv \frac{\omega(\Delta - \Delta_c = x)}{\omega(\Delta_c - \Delta = x)} \approx 1.9$$



$$H = -J \sum_{\langle ij \rangle} Z_i Z_j - Jg \sum_i X_i$$

$i \rightarrow$   $d$ -dimensional hypercubic lattice

$X_i, Z_i \rightarrow$  Pauli operators :

$$J > 0, g > 0.$$

Determine phase diagram as a function of  $g$ .

Will find a quantum phase transition at

$$g = g_c \text{ for all } d.$$

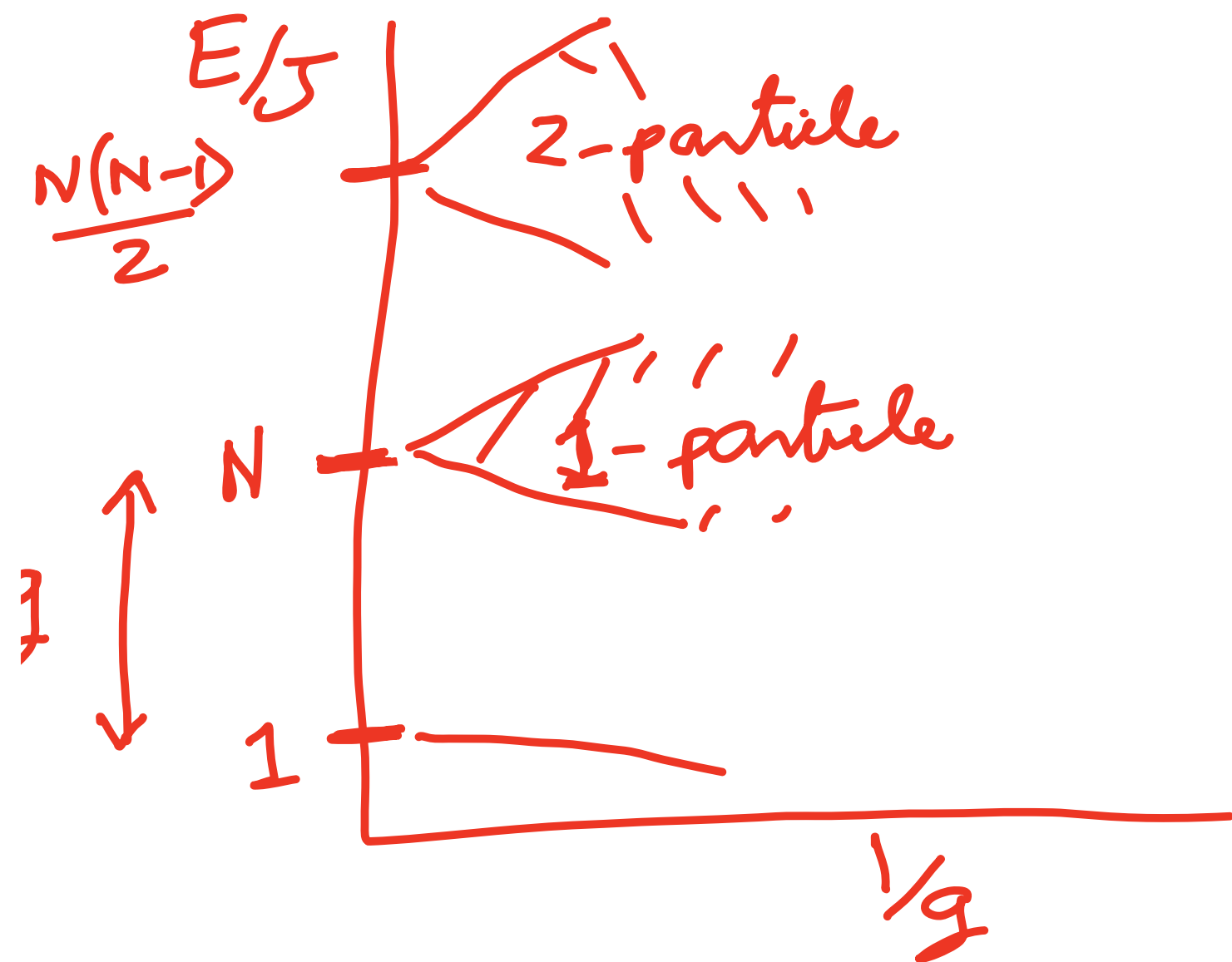
(i)  $g \gg 1$ .

Unique ground state  $|\rightarrow \rightarrow \rightarrow \rightarrow \dots\rangle$

"Quantum paramagnet" ;  $|\rightarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$   
 $|\leftarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$

$N$ -fold degenerate first excited states ( $N$  sites)  
 at  $g = \infty$

$|i\rangle = |\rightarrow \rightarrow \rightarrow \leftarrow_i \rightarrow \rightarrow \rightarrow \dots\rangle$



Use degenerate perturbation theory in  $1/g$ .

1- particle states

$$|\vec{k}\rangle = \frac{1}{\sqrt{N}} \sum_i e^{i\vec{k}\cdot\vec{r}_i} |i\rangle$$

$$\frac{\mathcal{E}_k}{J} = -Ng + 2g - 2 \sum_{\alpha=1}^d \cos(k_\alpha a)$$

2- particle states

$$|\vec{k}_1, \vec{k}_2\rangle$$

Scattering and bound states...

(ii)  $g \ll 1$

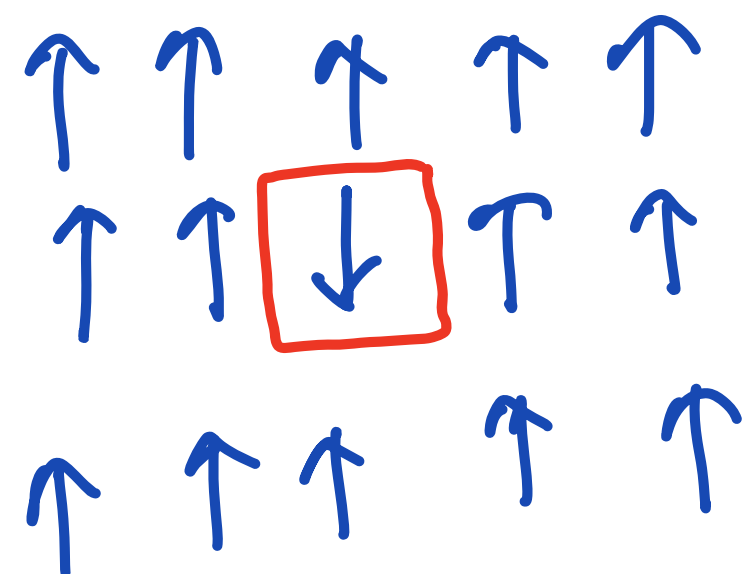
2-fold degenerate ferromagnetic ground states.

$$|\uparrow\uparrow\rangle = |\uparrow\uparrow\uparrow\uparrow\dots\rangle$$

$$|\downarrow\downarrow\rangle = |\downarrow\downarrow\downarrow\downarrow\dots\rangle$$

Cannot be smoothly connected to  $g \gg 1$  at  $N = \infty$   
 $\Rightarrow$  must be a quantum phase transition(s) at intermediate  $g$

First excited states



"Flipped spin"  
particle

$$E = 2dJ$$

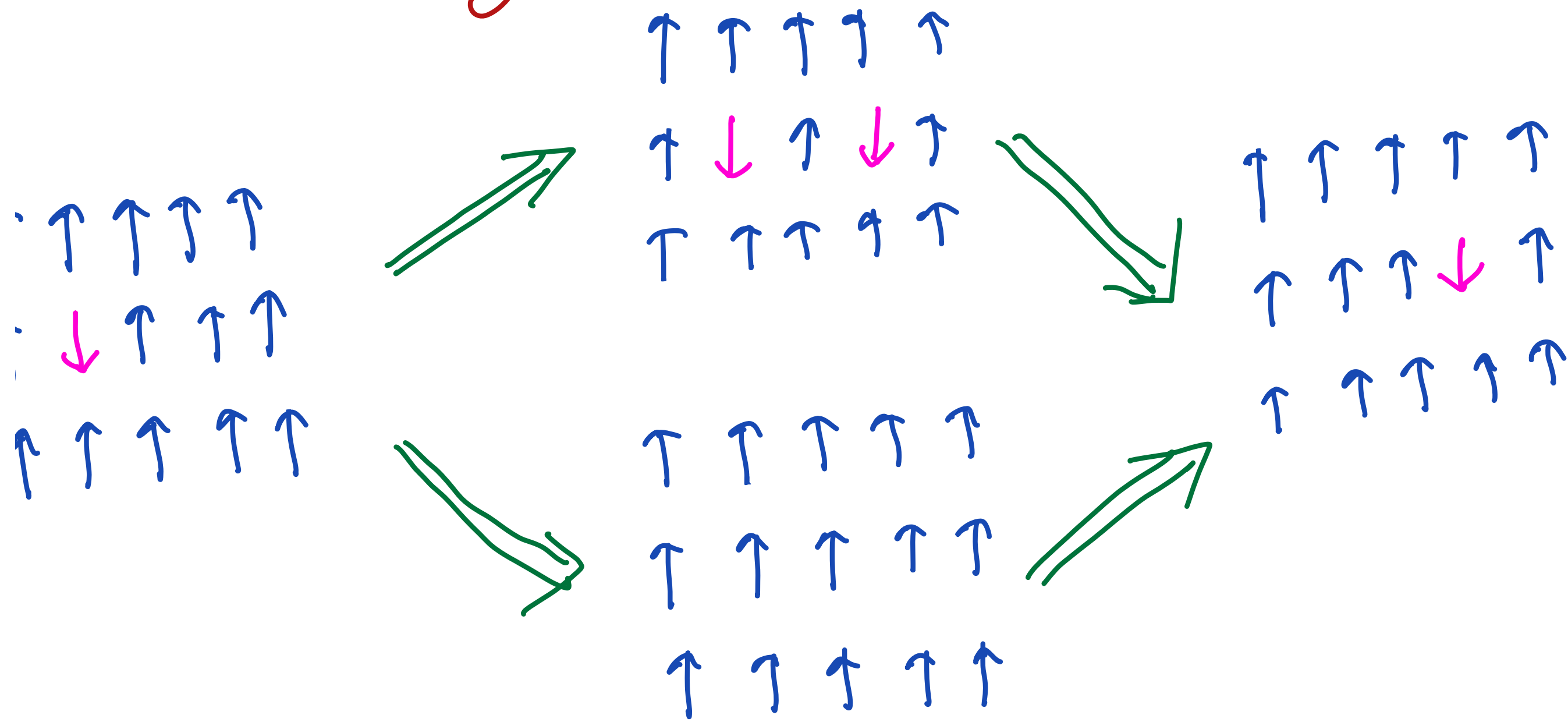
d=1 is special

$\uparrow\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\uparrow\uparrow \implies \uparrow\uparrow\downarrow\downarrow\downarrow\downarrow\downarrow\uparrow\uparrow\uparrow$

Flipped spin "fractionalizes" into  
2 domain-wall particles.

## Dispersion of flipped spin particle.

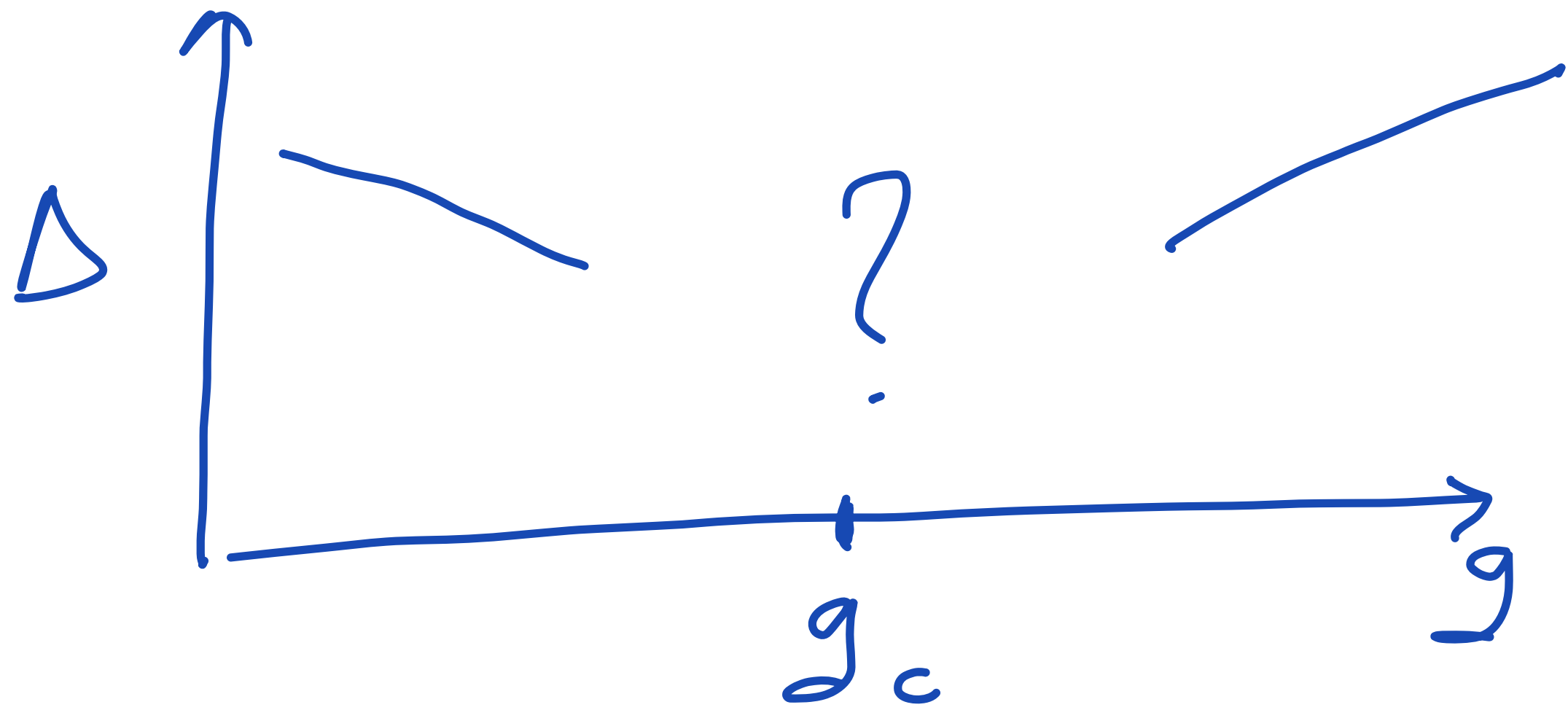
Have to use effective Hamiltonian method to perform perturbation theory to order  $g^2$ .



Total matrix element is zero unless initial and final flipped spins are nearest neighbors.

$$\epsilon_k = 2dJ + cJg^2 \sum_{\alpha=1}^d \cos(k_\alpha a)$$

$\Delta =$  gap to first excited state



A single quantum phase transition at  $g = g_c$  where  $\Delta$  vanishes as

$$\Delta \sim |g - g_c|^z$$

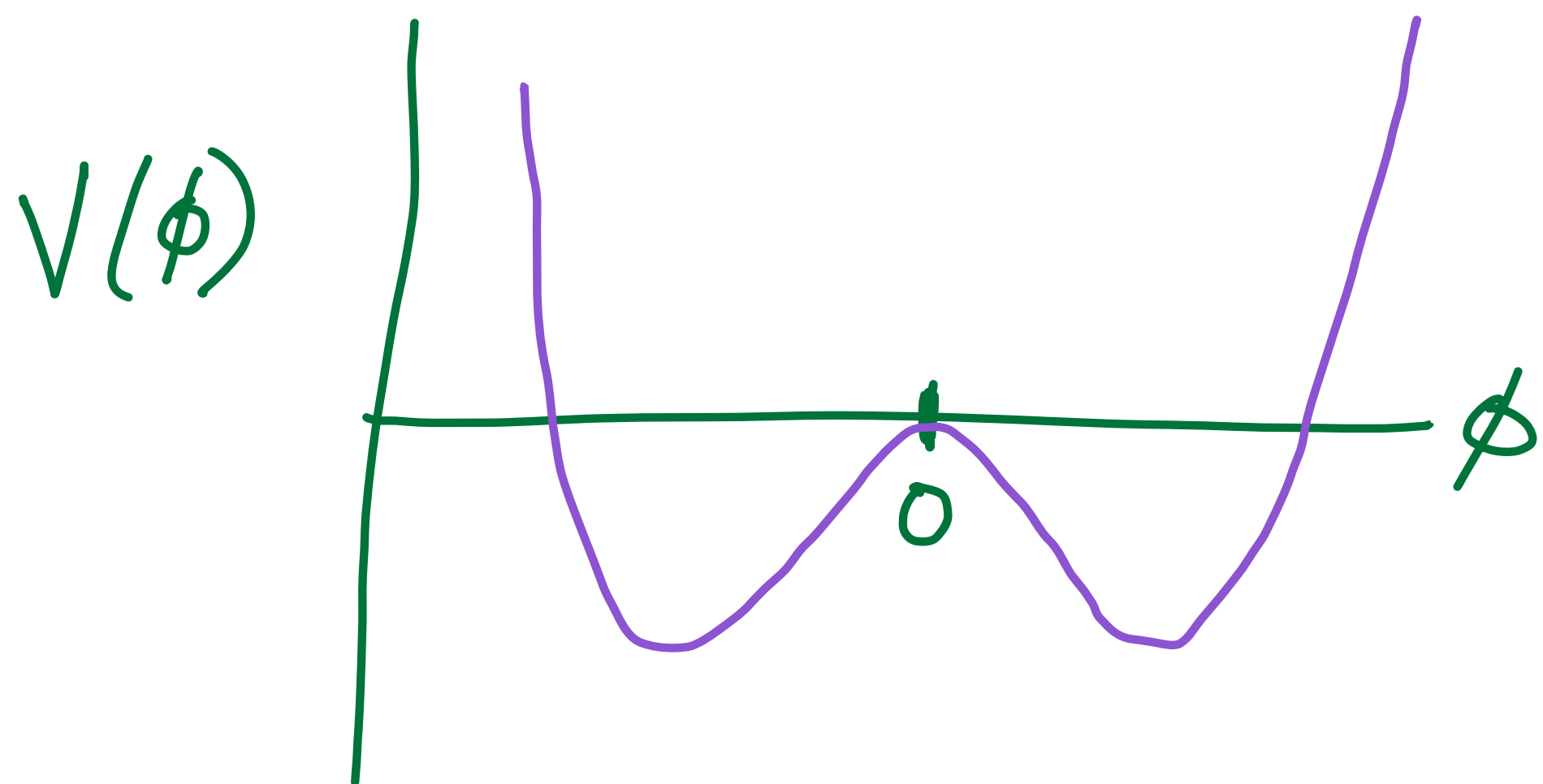
with  $z = 1$ .

# Mapping to $\phi^4$ quantum field theory

Consider the following non-linear oscillator

$$H = \frac{1}{2m} \pi_\phi^2 + V(\phi)$$

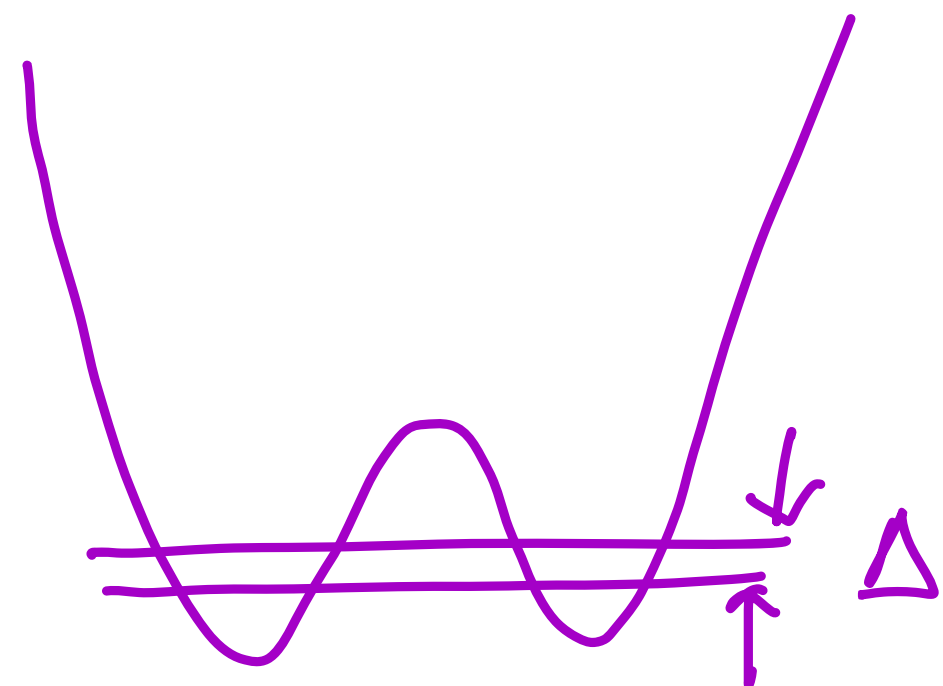
$$[\phi, \pi_\phi] = i$$

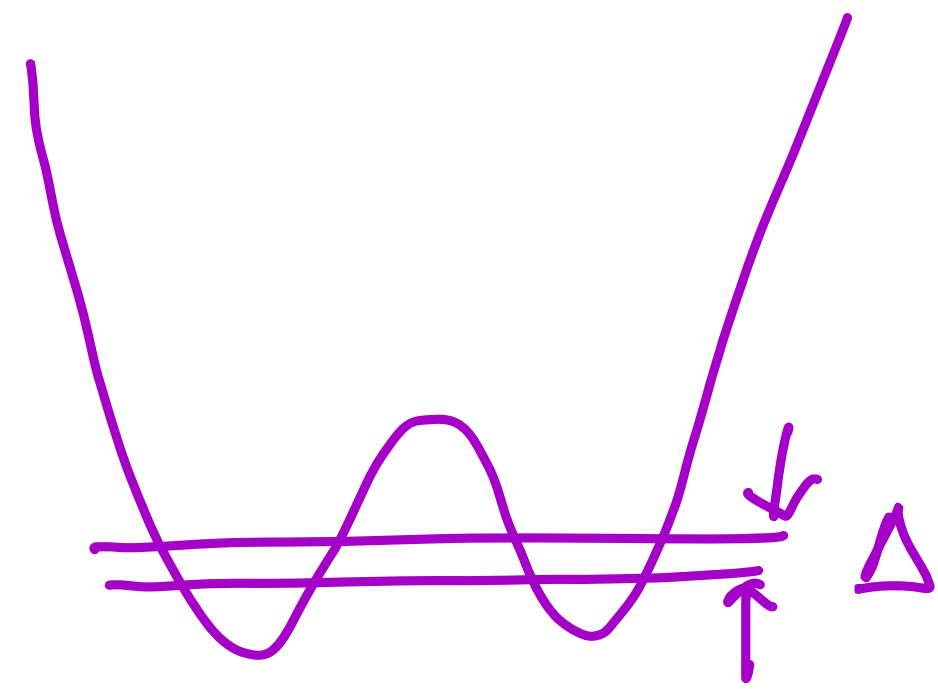


Focus on 2 lowest energy states



Tunneling under the barrier will lift the degeneracy and produce a gap  $\Delta$





ground state

$$|\rightarrow\rangle = \frac{1}{\sqrt{2}} (\psi_u + \psi_D) \equiv \frac{1}{\sqrt{2}} (|\uparrow\rangle + |\downarrow\rangle)$$

$$|\leftarrow\rangle = \frac{1}{\sqrt{2}} (\psi_u - \psi_D) \equiv \frac{1}{\sqrt{2}} (|\uparrow\rangle - |\downarrow\rangle)$$

low energy Hamiltonian

$$H = -\frac{\Delta}{2} X$$

What is the operator  $\phi$  in this low energy subspace?

$$|\phi\rangle \Rightarrow \frac{\phi}{\sqrt{2}} (\psi_u(\phi) + \psi_d(\phi))$$

$\pi$  odd under  $\phi \rightarrow -\phi$

$$\langle \rightarrow | \phi | \rightarrow \rangle = 0$$

$$\langle \leftarrow | \phi | \leftarrow \rangle = 0$$

$$\langle \leftarrow | \phi | \rightarrow \rangle = \int_{-\infty}^{\infty} d\phi \frac{\phi}{2} (\psi_u^*(\phi) - \psi_d^*(\phi)) (\psi_u(\phi) + \psi_d(\phi))$$

$$= K \neq 0$$

$\Rightarrow \phi$  maps to  $Z$ !

Now consider a lattice of such  
non-linear oscillators and couple them  
together

$$H = \sum_i \frac{1}{2m} \pi_i^2 + \sum_i V(\phi_i) - \sum_{\langle ij \rangle} \tilde{J} \phi_i \phi_j$$

Provided the  $\tilde{J}$  is not too-large,  
we can use the single-site mapping  
to obtain

$$H_{\text{eff}} = -\frac{\Delta}{2} \sum_i X_i - \tilde{J} K^2 \sum_{\langle ij \rangle} Z_i Z_j$$

From now on we will consider

$$H_{\phi} = \sum_i \frac{1}{2m} \pi_{\phi_i}^2 + \sum_i \left( \frac{g}{2} \phi_i^2 + \frac{u}{24} \phi_i^4 \right) - J \sum_{\langle ij \rangle} \phi_i \phi_j$$

In momentum space we can write

$$\phi_k = \frac{1}{\sqrt{N}} \sum_i \phi_i e^{i k r_i}$$

$$-J \sum_{\langle ij \rangle} \phi_i \phi_j = -J \sum_k |\phi_k|^2 \left( \sum_{\alpha} 2 \omega(k_{\perp} a) \right)$$

$$\approx -2J \sum_k |\phi_k|^2 (2d - k^2 \dots) \quad \text{as } k \rightarrow 0.$$

Now take the continuous limit while  
defining

$$\phi(x_i) = \frac{1}{\sqrt{a^d}} \phi_i$$

$$\Pi_\phi(x_i) = \frac{1}{\sqrt{a^d}} \Pi_{\phi_i}$$

so that

$$[\phi(x), \Pi_\phi(y)] = i \delta(x-y)$$

$\text{as } a \rightarrow 0.$

Then after various re-scalings

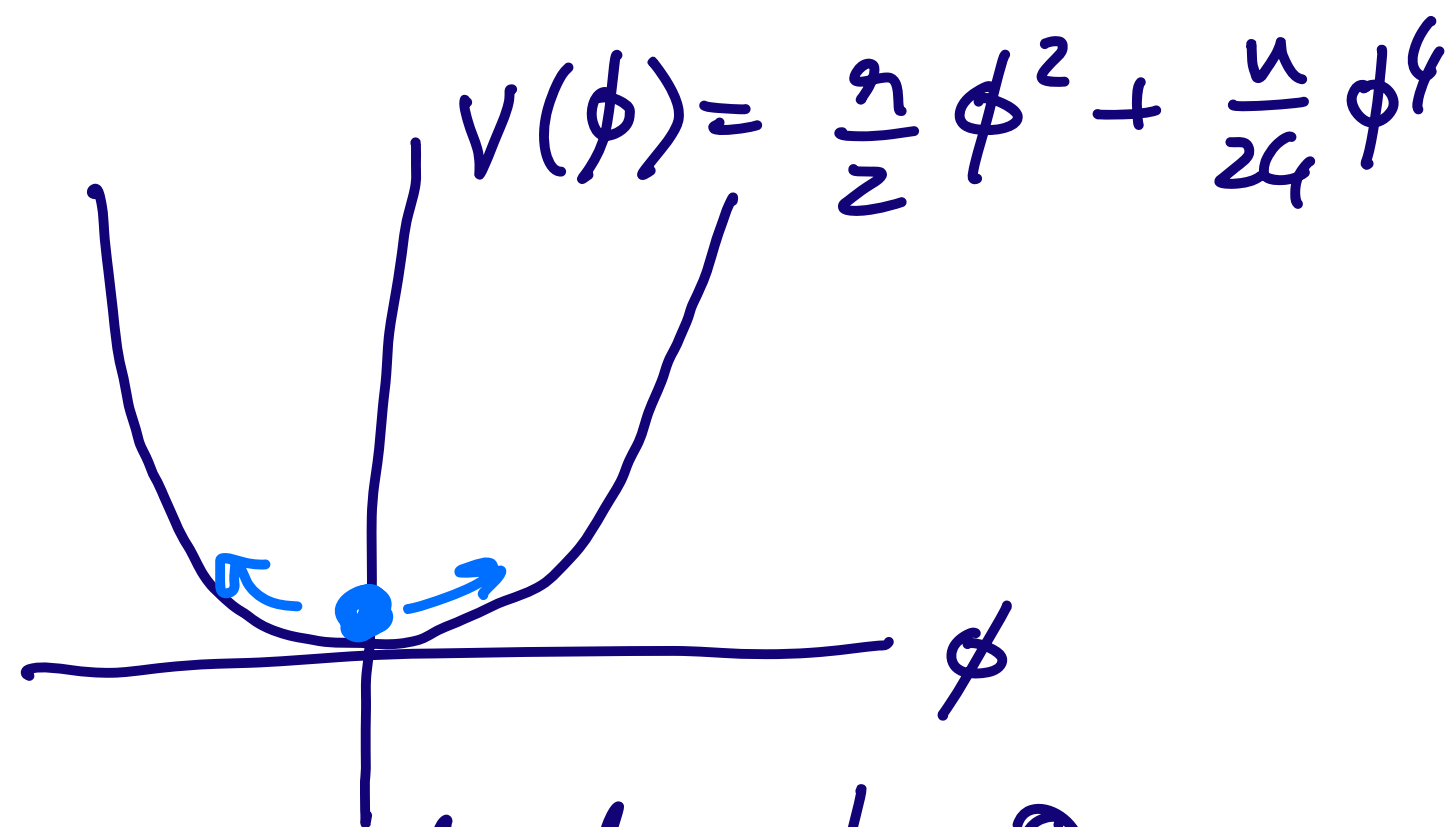
$$H_\phi = \int d^d x \left[ \frac{1}{2} \pi_\phi^2 + \frac{1}{2} (\vec{\nabla} \phi)^2 + \frac{\eta}{2} \phi^2 + \frac{u}{24} \phi^4 \right]$$

$\phi^4$  quantum field theory!

Examine perturbation theory in  $u$  for

$\eta < 0$  and  $\eta > 0$ .

(i)  $r > 0$



$\phi$  will fluctuate about  $\phi = 0$ .

$$H = \sum_k \left( \frac{\pi_k^2}{2} + \frac{\phi_k^2}{2} (r + k^2) \right)$$

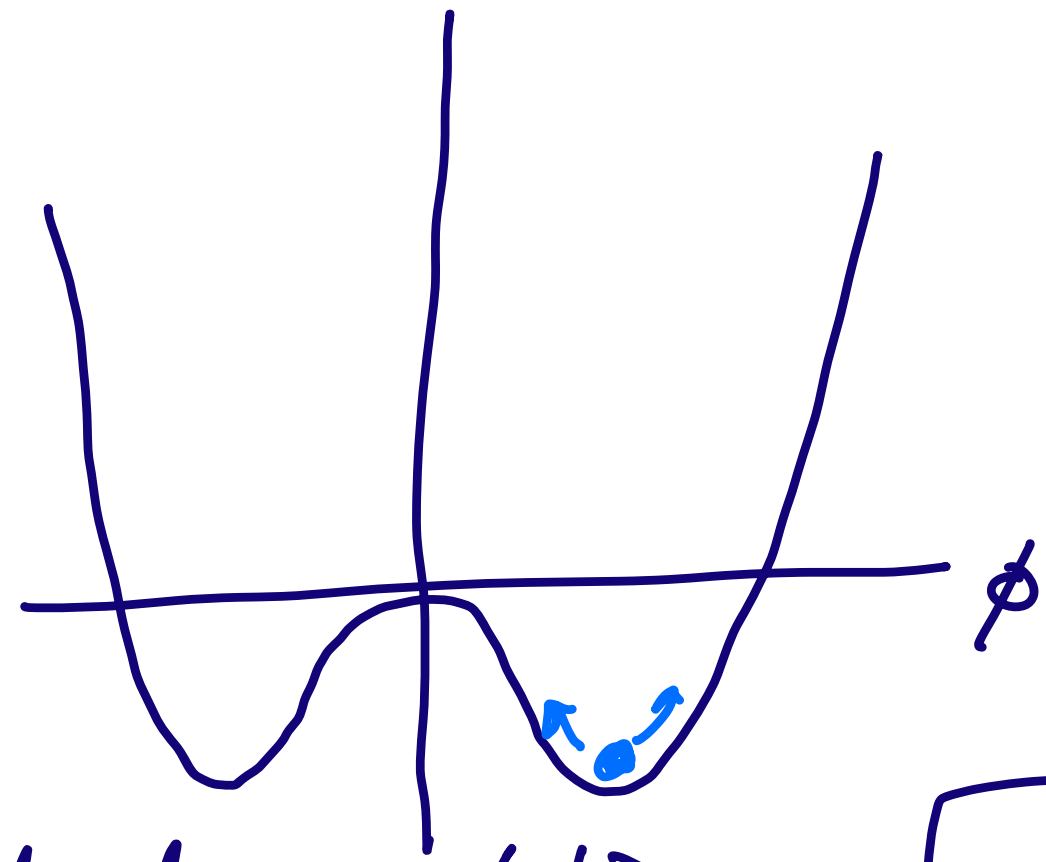
One oscillator for each  $k$

$$\epsilon_k = \sqrt{r + k^2}$$

Very similar to  $g \gg 1$  Ising model  
gap vanishes as  $r \rightarrow 0$

$$\Delta = \sqrt{r} \quad ; \quad Z \nu = \frac{1}{2}$$

(ii)  $\eta < 0$



$\phi$  fluctuates about  $\langle \phi \rangle = \sqrt{-\frac{6\eta}{u}}$

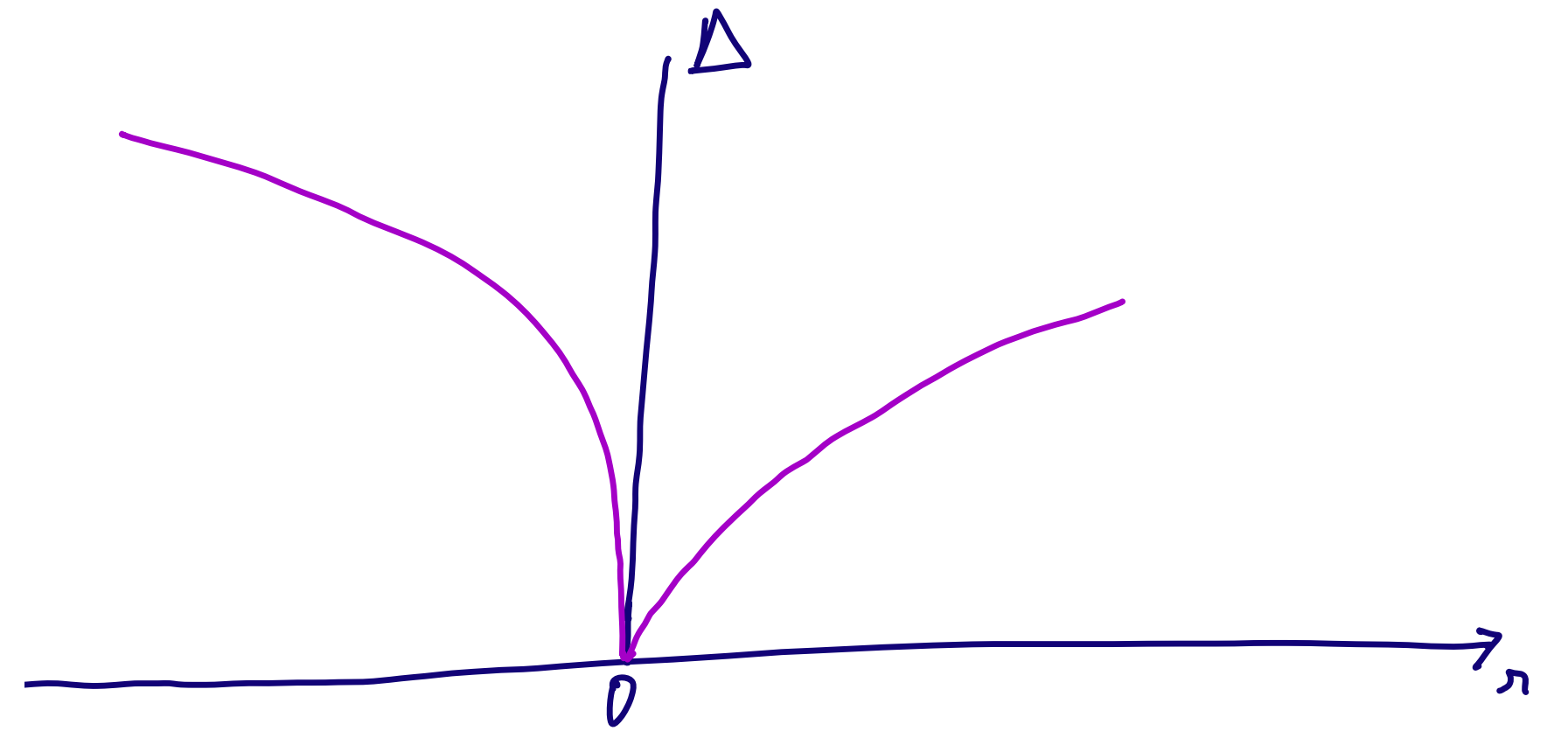
$$H_\phi = \sum_k \frac{|\phi_k|^2}{2} (2|\eta| + k^2) + \sum_k \frac{\pi_k^2}{2}$$

Similar to flipped spin quasiparticles at  $g \ll 1$ .

$$\epsilon_k = \sqrt{k^2 + 2|\eta|}$$

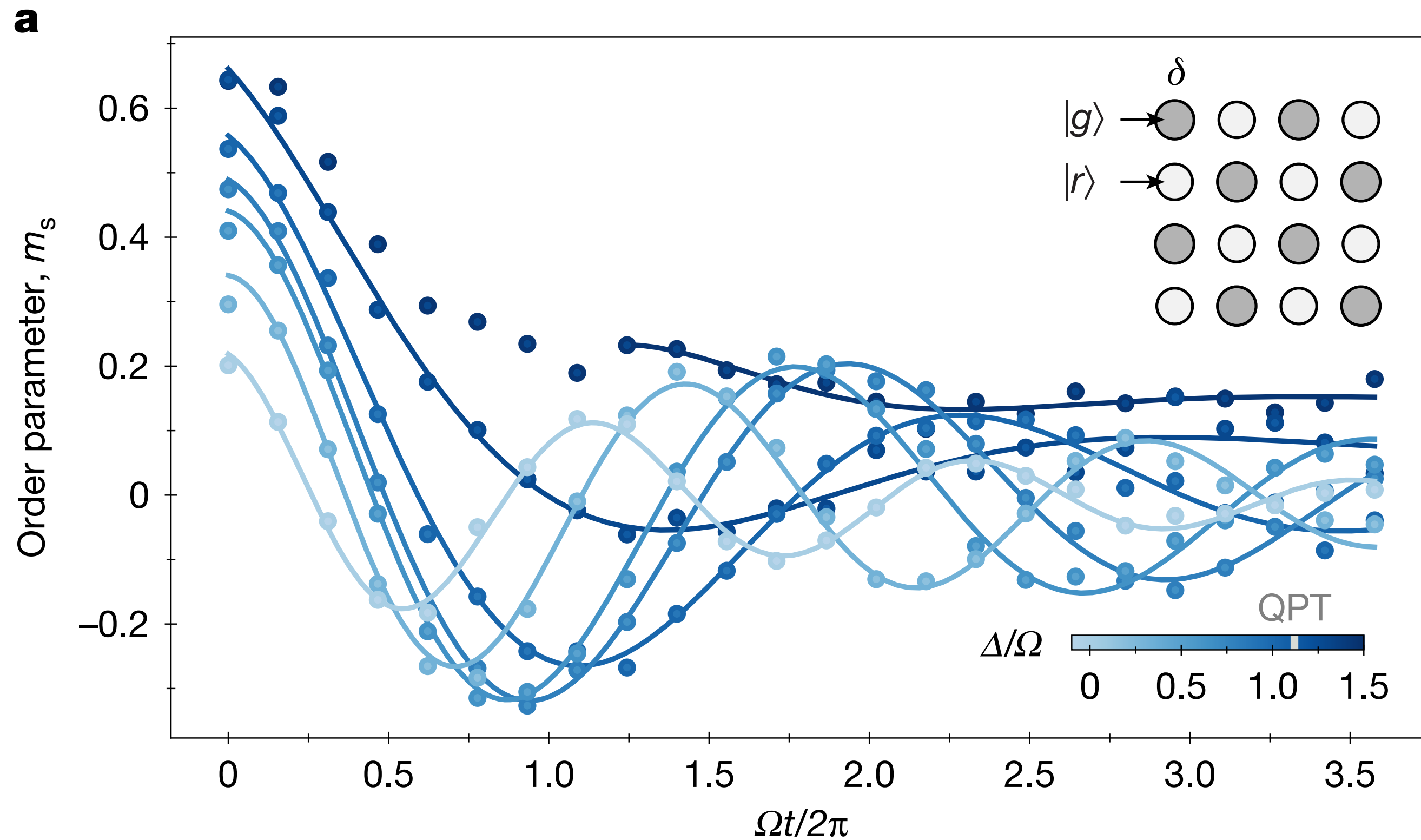
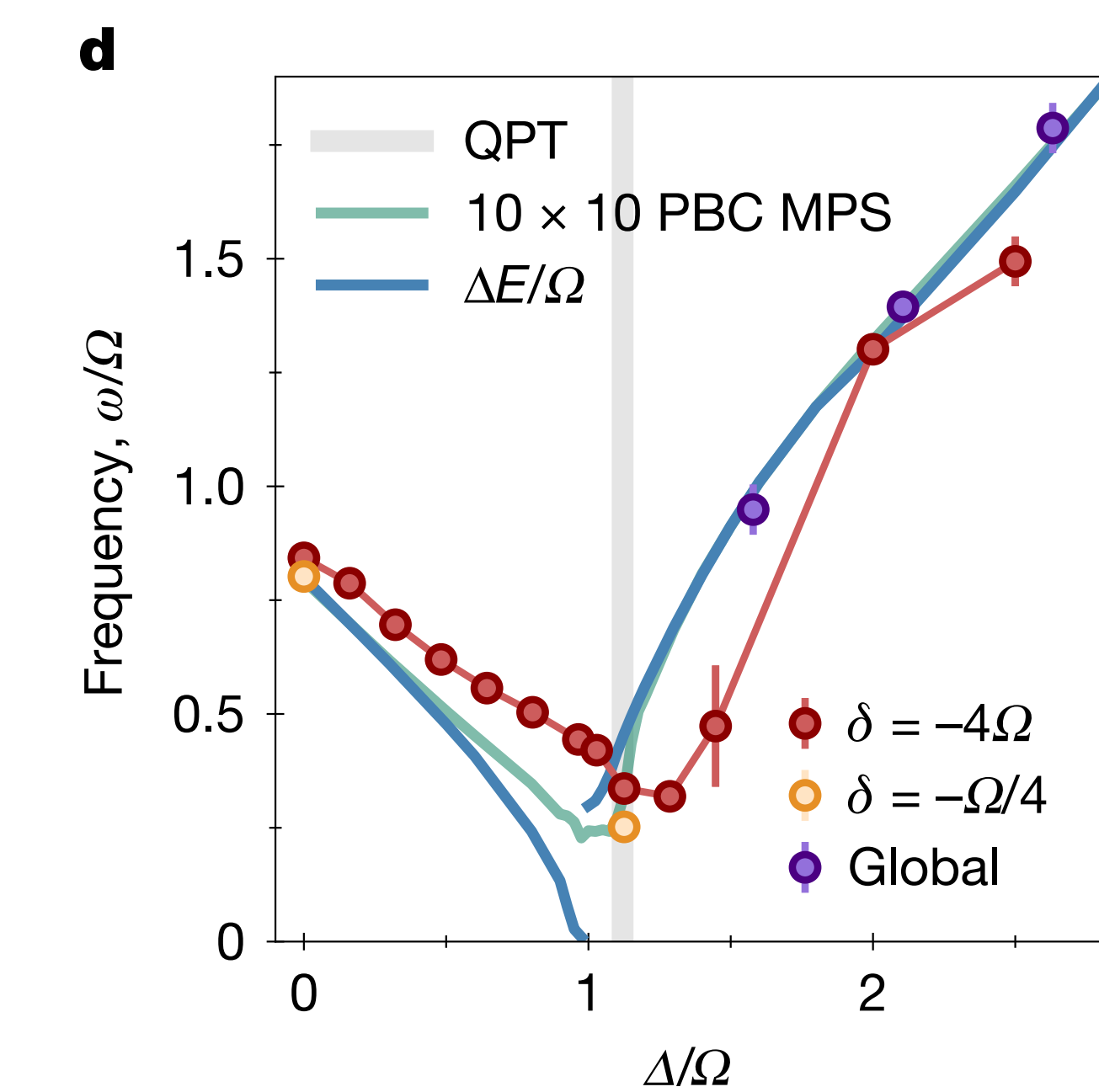
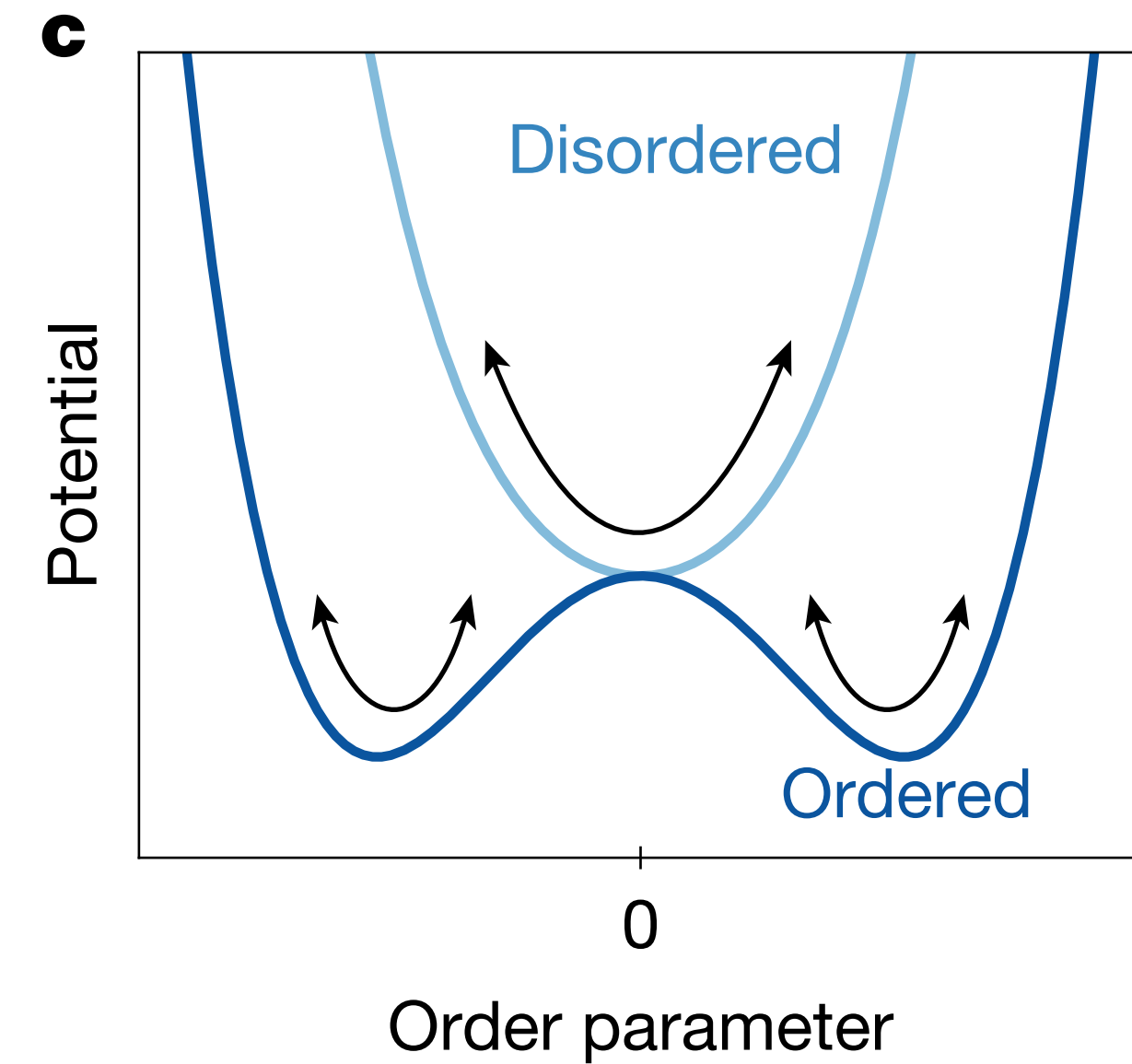
Energy gap vanishes as

$$\Delta = \sqrt{2|\eta|}$$



# Quantum coarsening and collective dynamics on a programmable simulator

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$$f(x) \equiv \frac{\omega(\Delta - \Delta_c = x)}{\omega(\Delta_c - \Delta = x)} \approx 1.9$$

$$f(x) = \sqrt{2} \text{ in mean-field theory.}$$

Imaginary time path integral

Single particle path integral

$$H = \frac{m}{2} \Pi_\phi^2 + V(\phi)$$

$$Z = \text{Tr} e^{-\beta H} = \int_{\phi(0)=\phi(\beta)} \mathcal{D}\phi(\tau) e^{-S[\phi]}$$

$$S[\phi] = \int_0^\beta d\tau \left[ \frac{m}{2} \left( \frac{d\phi}{d\tau} \right)^2 + V(\phi) \right]$$

$$H_\phi = \int d^d x \left[ \frac{1}{2} \Pi_\phi^2 + \frac{1}{2} (\nabla\phi)^2 + \frac{g}{2} \phi^2 + \frac{u}{24} \phi^4 \right]$$

$$Z_\phi = \text{Tr} e^{-\beta H_\phi} = \int_{\phi(x,\beta)=\phi(x,0)} \mathcal{D}\phi(x,\tau) \exp\left(-\int d^d x d\tau \mathcal{L}[\phi]\right)$$

Quantum critical point at  $g=0$

$$\epsilon_k = k$$

Massless relativistic boson

$$z=1$$

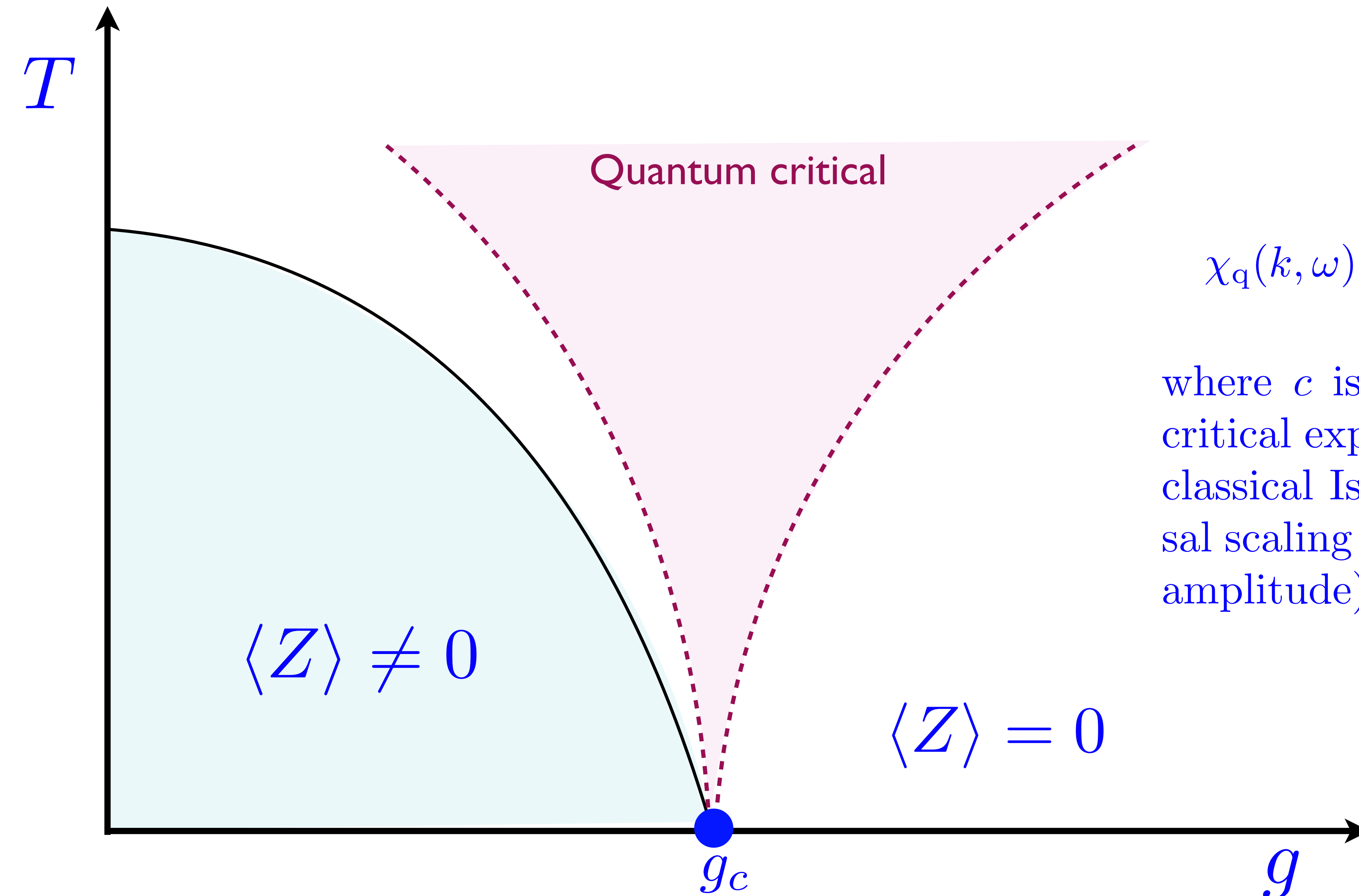
$$Z_\phi = \text{Tr} e^{-\beta H_\phi} = \int_{\phi(x, \beta) = \phi(x, 0)} \mathcal{D}\phi(x, \tau) \exp\left(-\int_0^\beta \int_{\mathbb{R}^d} \mathcal{L}[\phi]\right)$$

$$\mathcal{L}[\phi] = \frac{1}{2} (\partial_\tau \phi)^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{g}{2} \phi^2 + \frac{u}{24} \phi^4$$

$$= \frac{1}{2} (\partial_\mu \phi)^2 + \frac{g}{2} \phi^2 + \frac{u}{24} \phi^4$$

Relativistic quantum field theory in  
 $d+1$  spacetime dimensions.

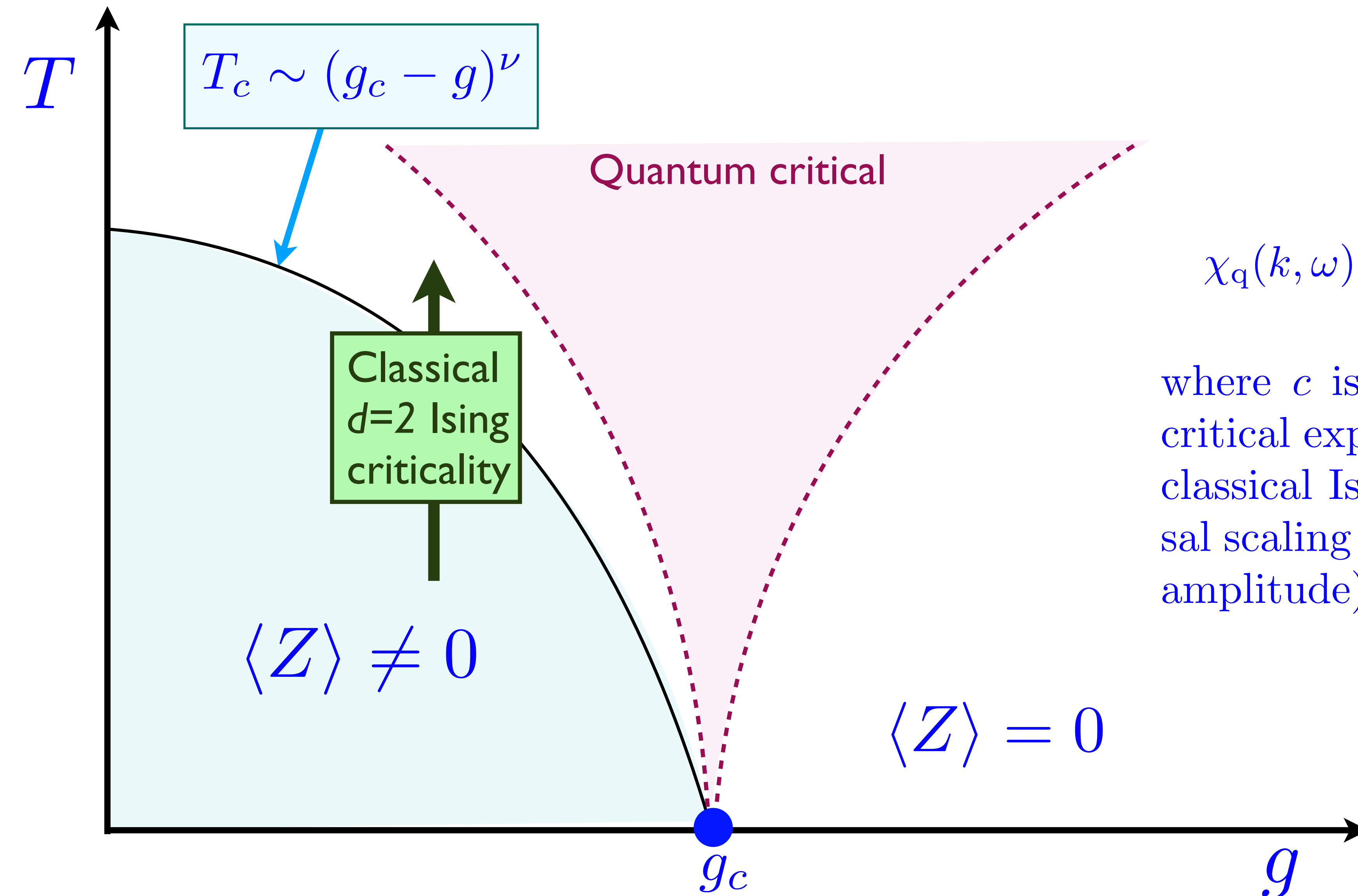
# Quantum criticality of Ising model in 2+1 dimensions



$$\chi_q(k, \omega) = \frac{1}{T^{2-\eta}} \Phi_I \left( \frac{ck}{T}, \frac{\omega}{T}, \frac{g - g_c}{T^{1/\nu}} \right)$$

where  $c$  is the velocity of 'light',  $\nu$  is a critical exponent of the three-dimensional classical Ising model, and  $\Phi_I$  is a universal scaling function (apart from an overall amplitude).

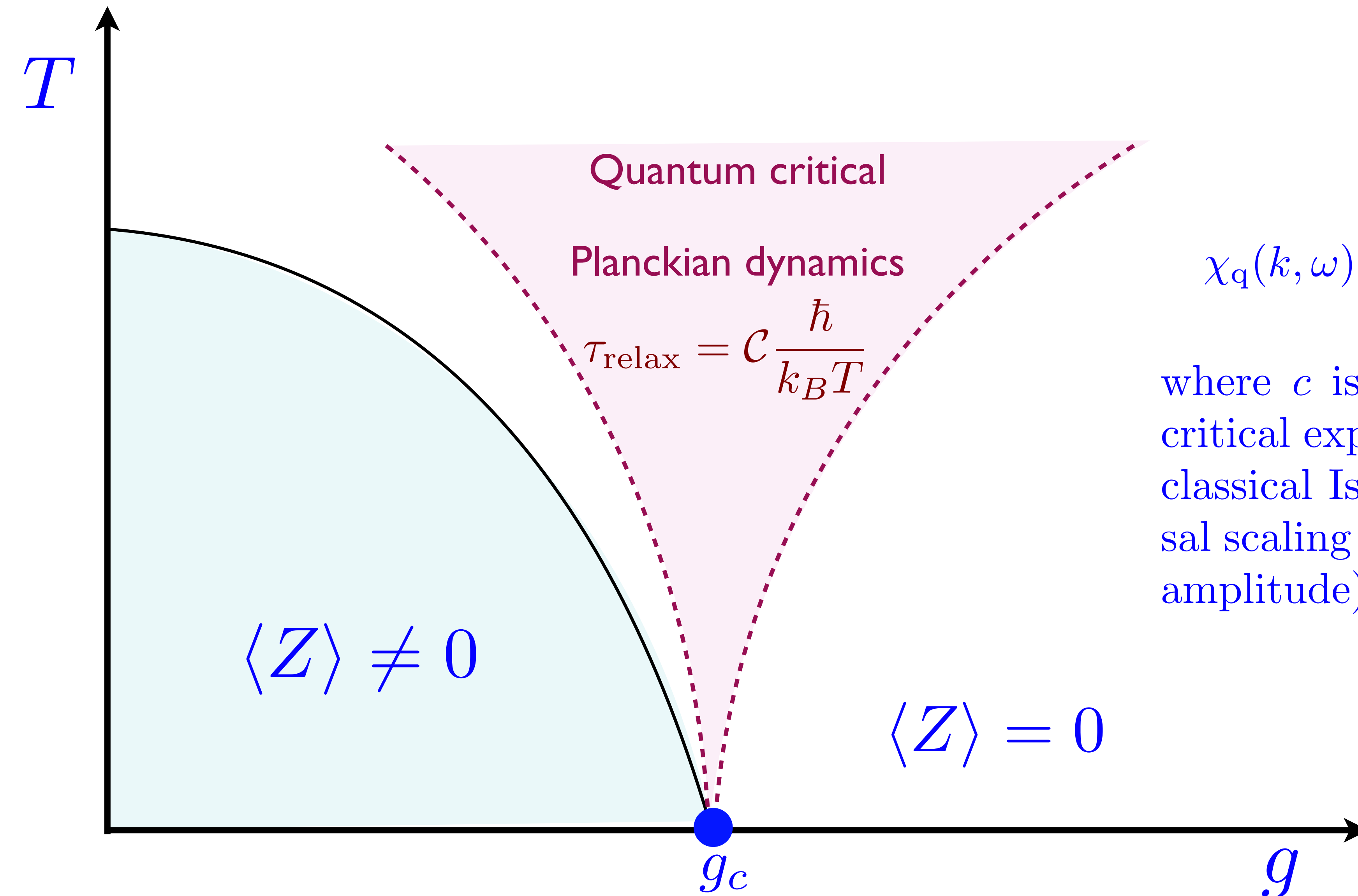
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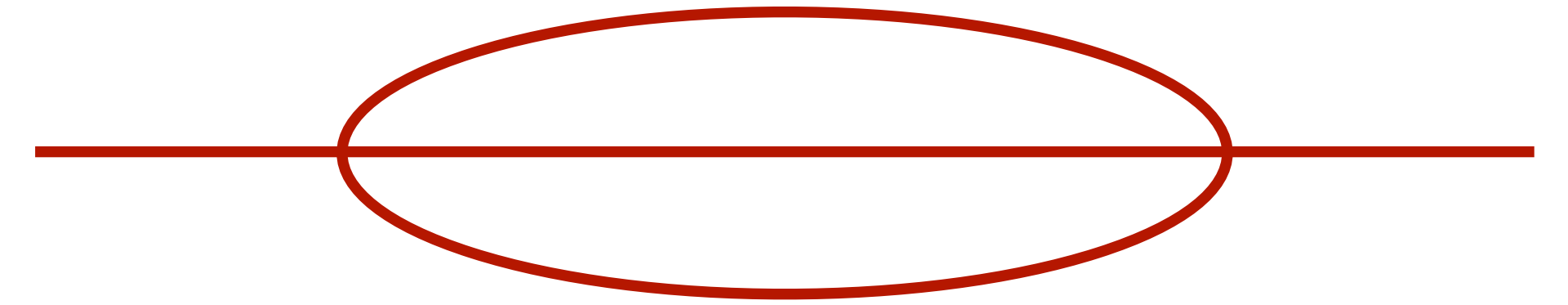
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# Quantum criticality of Ising model in 2+1 dimensions

Quantum critical dynamic susceptibility in an expansion in  $\epsilon = 3 - d$

$$\chi(k, \omega) = \frac{1}{c^2 k^2 - \omega^2 + \epsilon \frac{2\pi^2 T^2}{9} - i \operatorname{sgn}(\omega) \epsilon^2 \frac{4\pi^3 T^2}{27}}$$



# Quantum criticality of Ising model in 2+1 dimensions

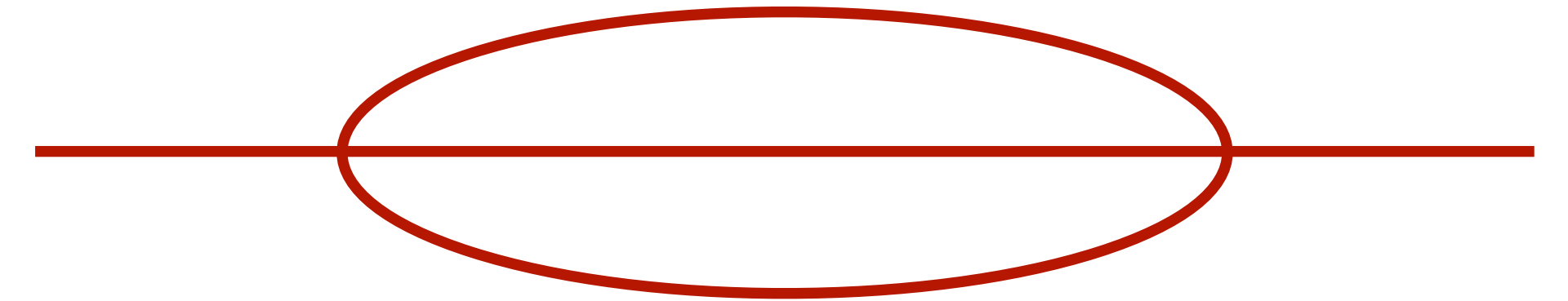
Quantum critical dynamic susceptibility in an expansion in  $\epsilon = 3 - d$

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This indicates a breakdown of the  $\epsilon$  expansion for  $\hbar\omega \sim \sqrt{\epsilon} k_B T$ . Expected low frequency form is

$$\chi(k, \omega) = \frac{1}{c^2 k^2 - \omega^2 + \epsilon \frac{2\pi^2 T^2}{9} - i\Gamma\omega T}$$

with  $\Gamma$  a universal constant.



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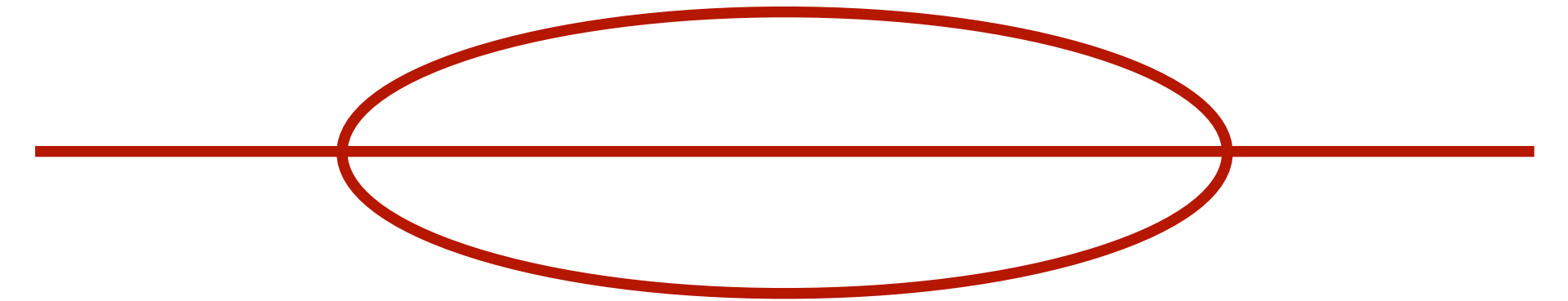
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More generally

$$\chi(k=0, \omega) = T^{-2+\eta} \Phi(\omega/T)$$

and

$$\tilde{\Gamma} = iT\chi \left. \frac{\partial \chi^{-1}}{\partial \omega} \right|_{\omega=0}$$

is a universal number

# Quantum criticality of Ising model in 2+1 dimensions

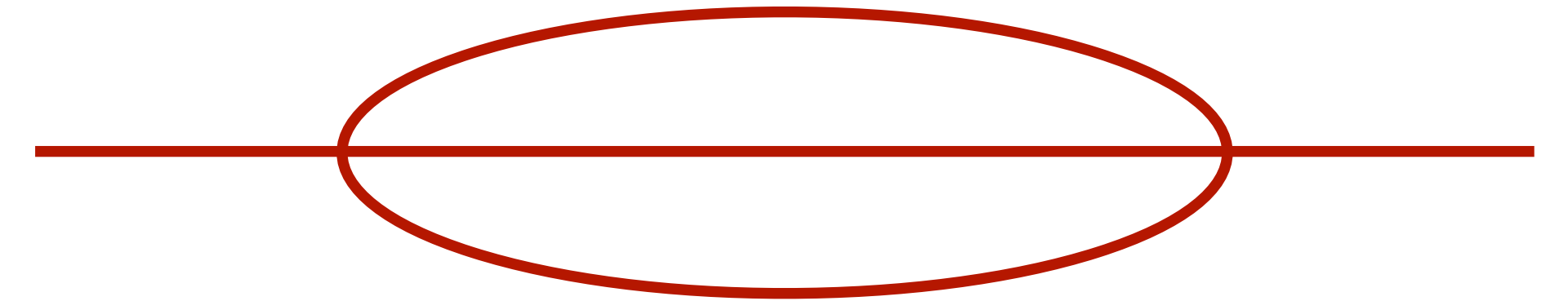
Quantum critical dynamic susceptibility in an expansion in  $\epsilon = 3 - d$

$$\chi(k, \omega) = \frac{1}{c^2 k^2 - \omega^2 + \epsilon \frac{2\pi^2 T^2}{9} - i \operatorname{sgn}(\omega) \epsilon^2 \frac{4\pi^3 T^2}{27}}$$

This indicates a breakdown of the  $\epsilon$  expansion for  $\hbar\omega \sim \sqrt{\epsilon} k_B T$ . Expected low frequency form is

$$\chi(k, \omega) = \frac{1}{c^2 k^2 - \omega^2 + \epsilon \frac{2\pi^2 T^2}{9} - i\Gamma\omega T}$$

with  $\Gamma$  a universal constant.



However, the  $\epsilon$  expansion works well at  $T = 0$ , and yields systematic corrections to  $f(x) \equiv \frac{\omega(\Delta - \Delta_c = x)}{\omega(\Delta_c - \Delta = x)}$  which agree with observations.

# Quantum criticality of Ising model in 2+1 dimensions

For  $\hbar\omega \sim \sqrt{\epsilon}k_B T$  and  $\epsilon$  small, we can use a **semi-classical approach**.

## **Equal time correlators**

In a classical theory, equal time correlators are independent of the dynamics, and depend only upon the potential energy. The classical  $\Phi^4$  field theory has potential energy

$$\mathcal{V}[\Phi] = \int d^2x \left[ \frac{1}{2} (\nabla\Phi)^2 + \frac{\tilde{R}}{2} \Phi^2 + \frac{U}{24} \Phi^4 \right]$$

59, 14054 (1999)

The partition function at a temperature  $T$  is

$$\mathcal{Z} = \int \mathcal{D}\Phi \exp\left(-\frac{\mathcal{V}[\Phi]}{T}\right)$$

and a typical correlator of interest is the equal time structure factor

$$S_{\text{cl}}(k) = \frac{1}{\mathcal{Z}} \int \mathcal{D}\Phi \exp\left(-\frac{\mathcal{V}[\Phi]}{T}\right) \int d^2x e^{-ikx} \Phi(x) \Phi(0)$$

# Quantum criticality of Ising model in 2+1 dimensions

## Matching to determine couplings

Let us assume a square lattice of spacing  $a$ . A key assertion is that results are independent of the lattice as  $a \rightarrow 0$  a single mass renormalization. This renormalization is equivalent to replacing  $\tilde{R}$  by  $R$ , where

$$\tilde{R} = R - \frac{TU}{2} \int_{-\pi/a}^{\pi/a} \frac{dk_x dk_y}{4\pi^2} \frac{1}{(2/a^2)[2 - \cos(k_x a) - \cos(k_y a)] + R}$$

After all lattice results are expressed in terms of  $R$ , then the claim is that no further renormalizations are needed, and the limit  $a \rightarrow 0$  exists, and is independent of the lattice choice. Notice that we assume that  $R > 0$ , and that as  $R$  ranges from 0 to  $+\infty$ ,  $\tilde{R}$  ranges from  $-\infty$  to  $+\infty$ . The values of  $R$  and  $U$  are determined by matching to static properties of the quantum critical point.

# Quantum criticality of Ising model in 2+1 dimensions

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## Classical dynamics

We introduce a conjugate momentum field  $\Pi(x)$  with the Poisson bracket

$$\{\Phi(x), \Pi(y)\}_{P.B.} = \delta^2(x - y)$$

and the Hamiltonian

$$\mathcal{H}[\Phi, \Pi] = \int d^2x \frac{c^2 \Pi^2}{2} + \mathcal{V}[\Phi]$$

# Quantum criticality of Ising model in 2+1 dimensions

where  $c$  is a velocity. The partition function is now extended to

$$\mathcal{Z} = \int \mathcal{D}\Phi \mathcal{D}\Pi \exp \left( -\frac{\mathcal{H}[\Phi, \Pi]}{T} \right).$$

The equations of motion are

$$\begin{aligned} \frac{\partial \Phi}{\partial t} &= c^2 \Pi \\ \frac{\partial \Pi}{\partial t} &= \nabla^2 \Phi - \tilde{R} \Phi - \frac{U}{6} \Phi^3 \end{aligned}$$

By integrating these equations of motion, we can obtain the dynamic structure factor

$$S_{\text{cl}}(k, \omega) = \int d^2x \int dt \langle \Phi(x, t) \Phi(0, 0) \rangle e^{-i(kx - \omega t)}.$$

Here the average  $\langle \cdot \rangle$  is over thermal initial conditions defined by the partition function  $\mathcal{Z}$ .

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Precise numerical results to appear by  
Julia Wei, Matthew Dodelson, Anders Sandvik...

S. Sachdev, PRB **59**, 14054 (1999)

# Planckian Dynamics

$$\tau_r \geq C \frac{\hbar}{k_B T} \quad (\text{SS, } \textit{Quantum Phase Transitions}, 1999)$$

1. Quantum critical dynamics of the Ising model in 2 spatial dimensions
2. Quantum critical transport at the superfluid-insulator transition in 2 spatial dimensions

Thermal Seminars  
DESY  
September 2, 2025

Subir Sachdev



# Quantum critical transport at the superfluid-insulator transition in 2 spatial dimensions

Thermal Seminars  
DESY  
September 2, 2025

Subir Sachdev

PHYSICS



HARVARD





Emanuel Katz  
Boston University



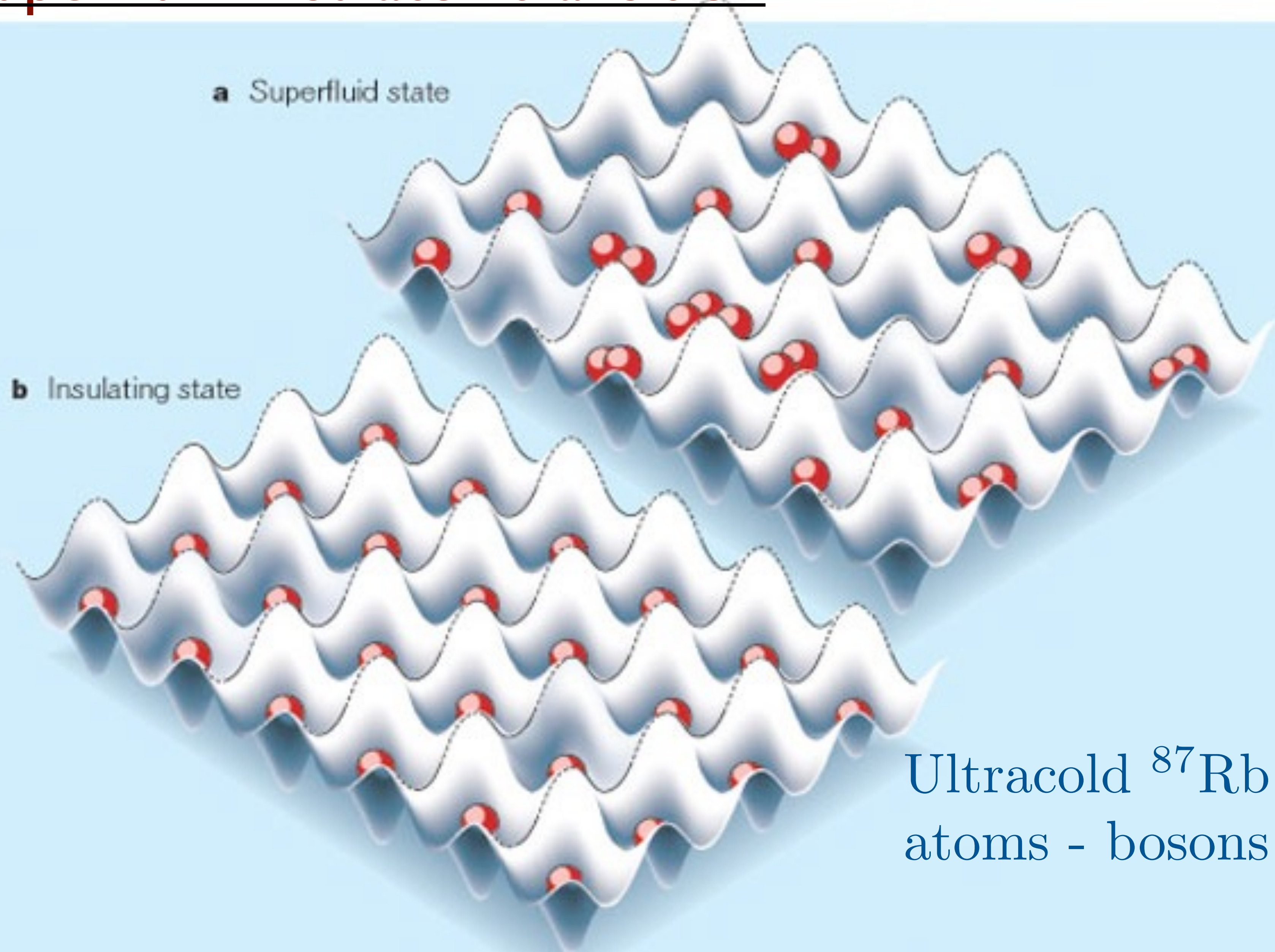
William Witczak-Krempa  
Perimeter



Erik Sorensen  
McMaster

E. Katz, S. Sachdev, E. Sorensen, and W. Witczak-Krempa, Physical Review B **90**, 245109 (2014)

# Superfluid-insulator transition



M. Greiner, O. Mandel, T. Esslinger, T. W. Hänsch, and I. Bloch, *Nature* **415**, 39 (2002).

# The Superfluid-Insulator transition

## Boson Hubbard model

Bosons,  $b_j$  hopping on the sites  $j$  of a square lattice with Hamiltonian

$$H = -t \sum_{\langle ij \rangle} b_i^\dagger b_j + \frac{U}{2} \sum_j n_j (n_j - 1)$$
$$n_j \equiv b_j^\dagger b_j$$

The boson operators obey the commutation relation

$$[b_j, b_k^\dagger] = \delta_{jk}$$

We restrict attention to the sector of the Fock space with

$$\sum_j n_j = \text{integer multiple of the number of sites}$$

$$\underline{U \gg t}$$

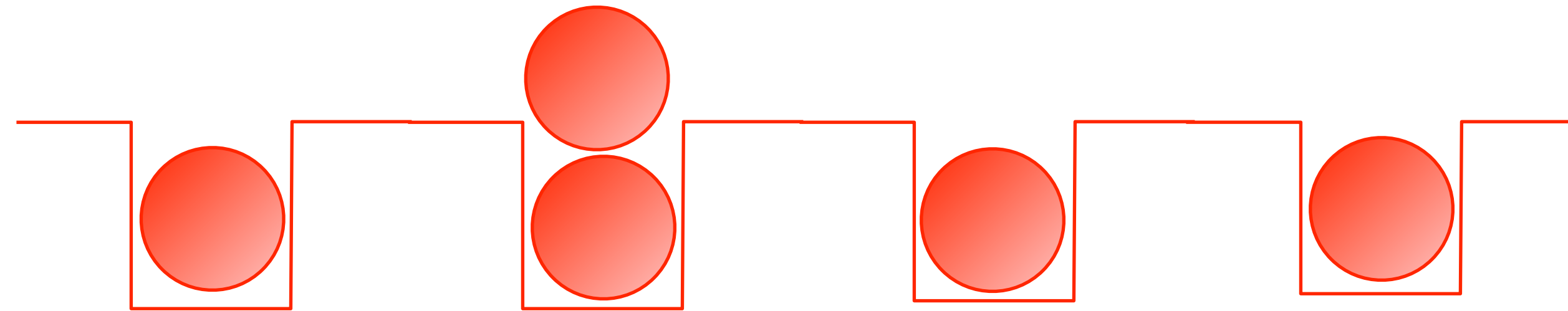


Insulator (the vacuum)  
at large repulsion between bosons

$$|\text{Ground state}\rangle = \prod_i b_i^\dagger |0\rangle$$

$$\underline{U \gg t}$$

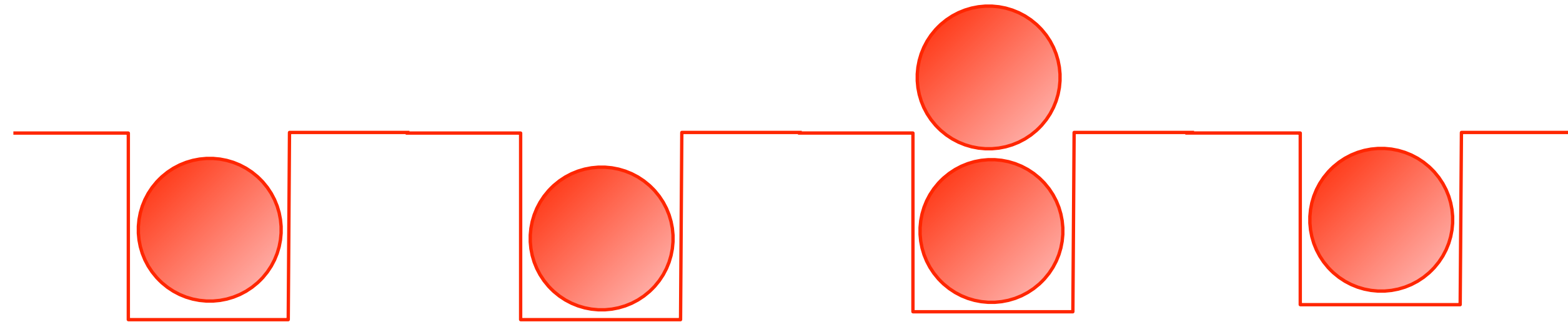
Excitations of the insulator:



Particles  $\sim \psi^\dagger$

$$\underline{U \gg t}$$

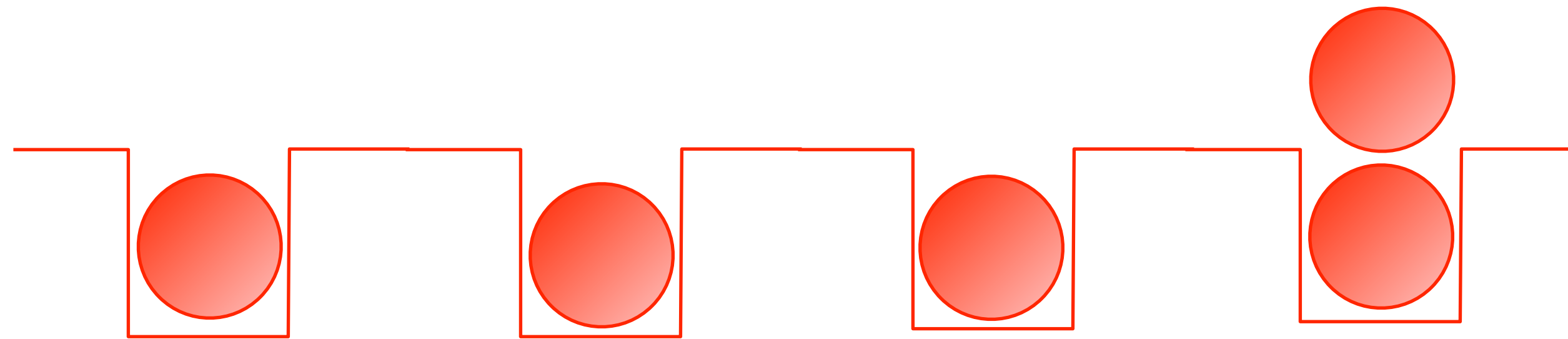
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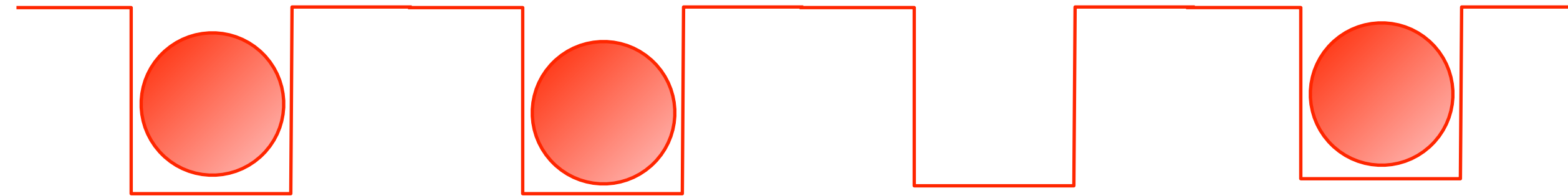
Excitations of the insulator:



Particles  $\sim \psi^\dagger$

$$\underline{U \gg t}$$

Excitations of the insulator:



Holes  $\sim \psi$

$$\underline{U \gg t}$$

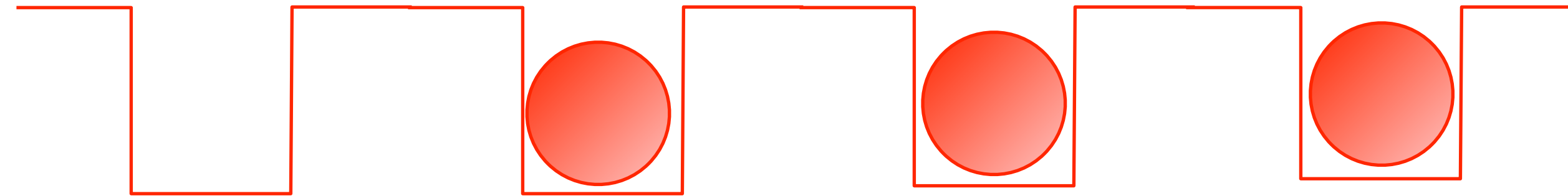
Excitations of the insulator:



Holes  $\sim \psi$

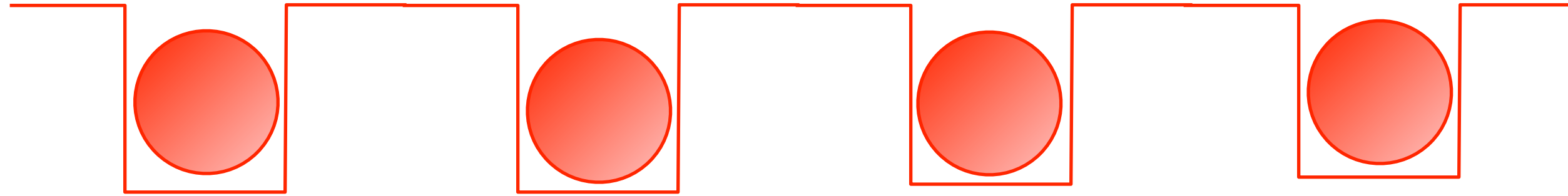
$$\underline{U \gg t}$$

Excitations of the insulator:



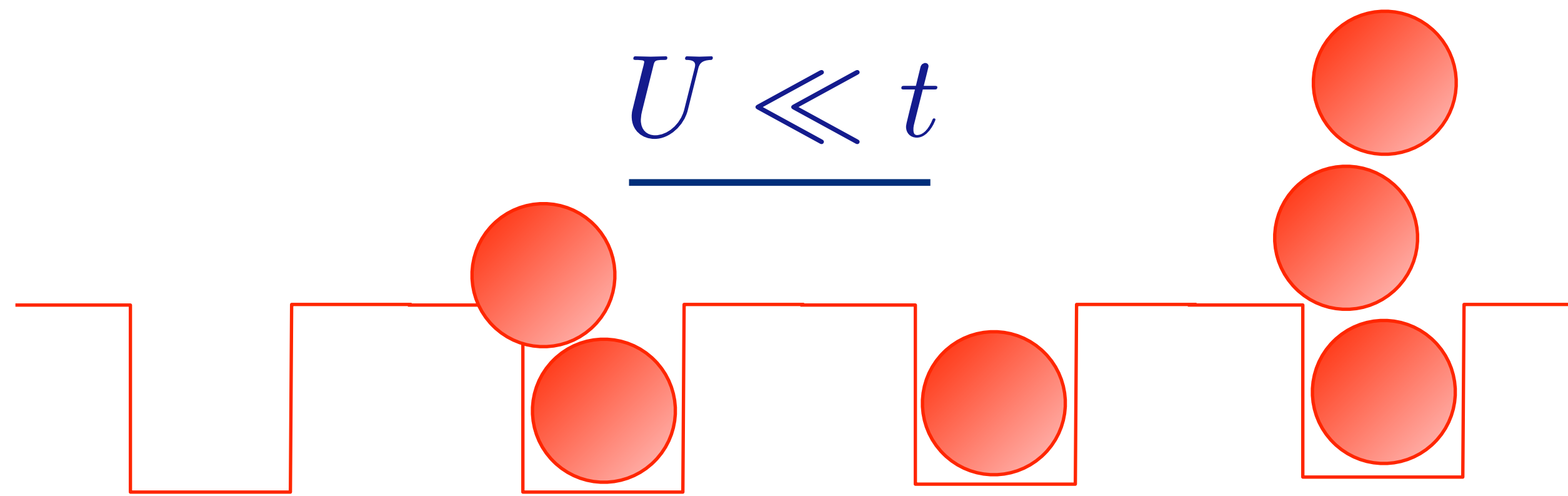
Holes  $\sim \psi$

$$\underline{U \gg t}$$



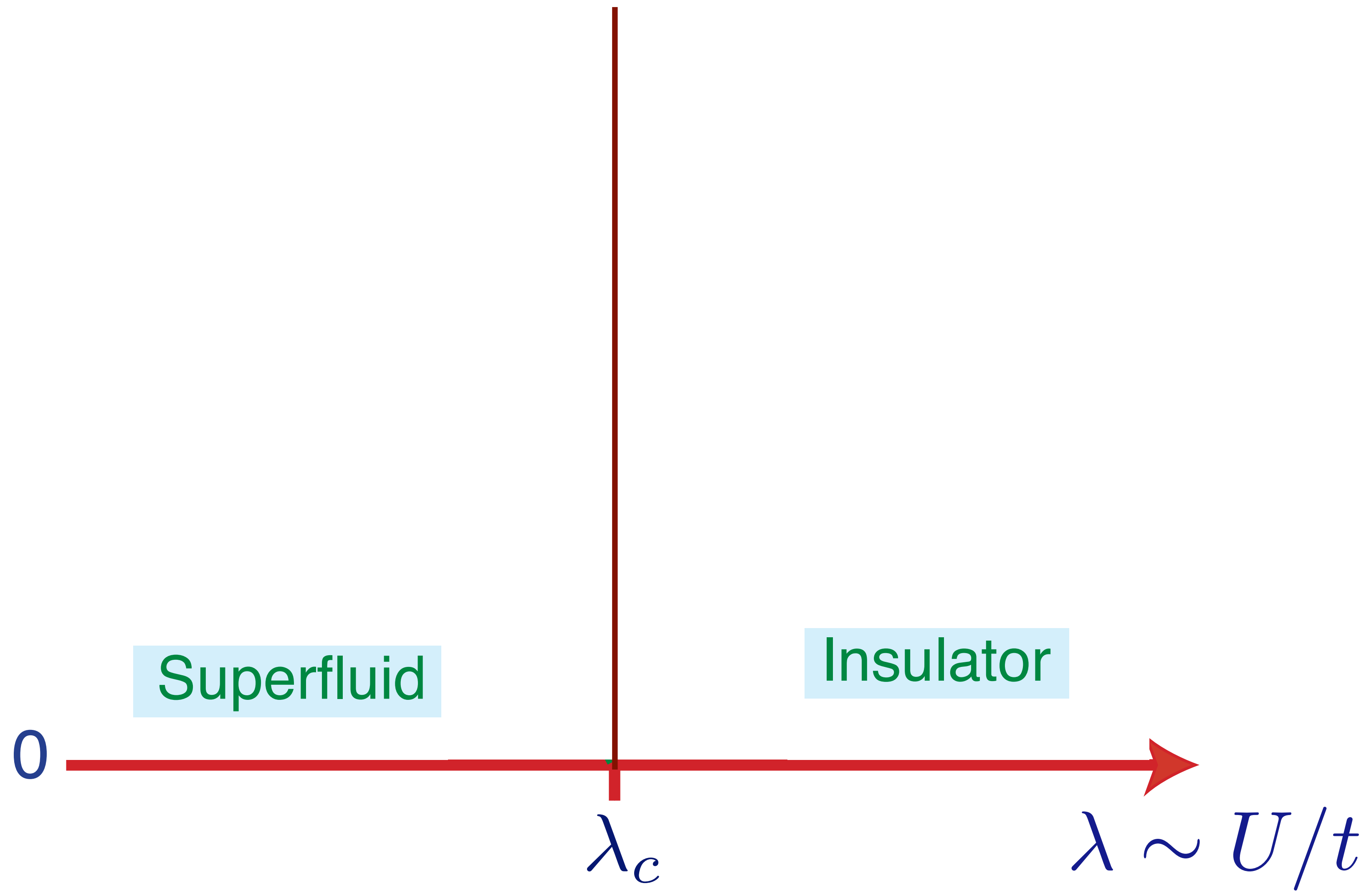
Insulator (the vacuum)  
at large repulsion between bosons

$$|\text{Ground state}\rangle = \prod_i b_i^\dagger |0\rangle$$

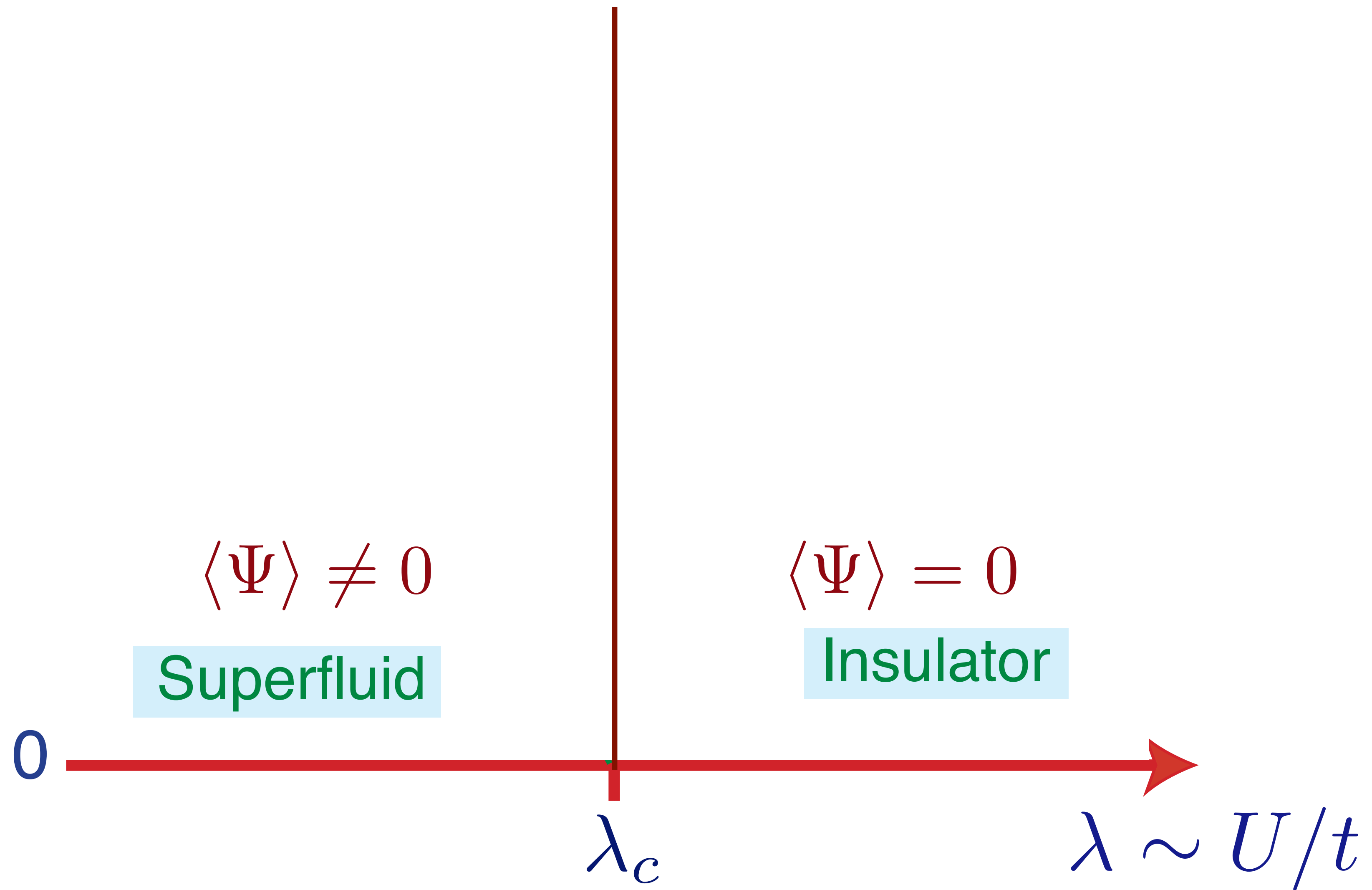


Superfluid  
at small repulsion between bosons

$$|\text{Ground state}\rangle = \left[ \sum_i b_i^\dagger \right]^N |0\rangle$$

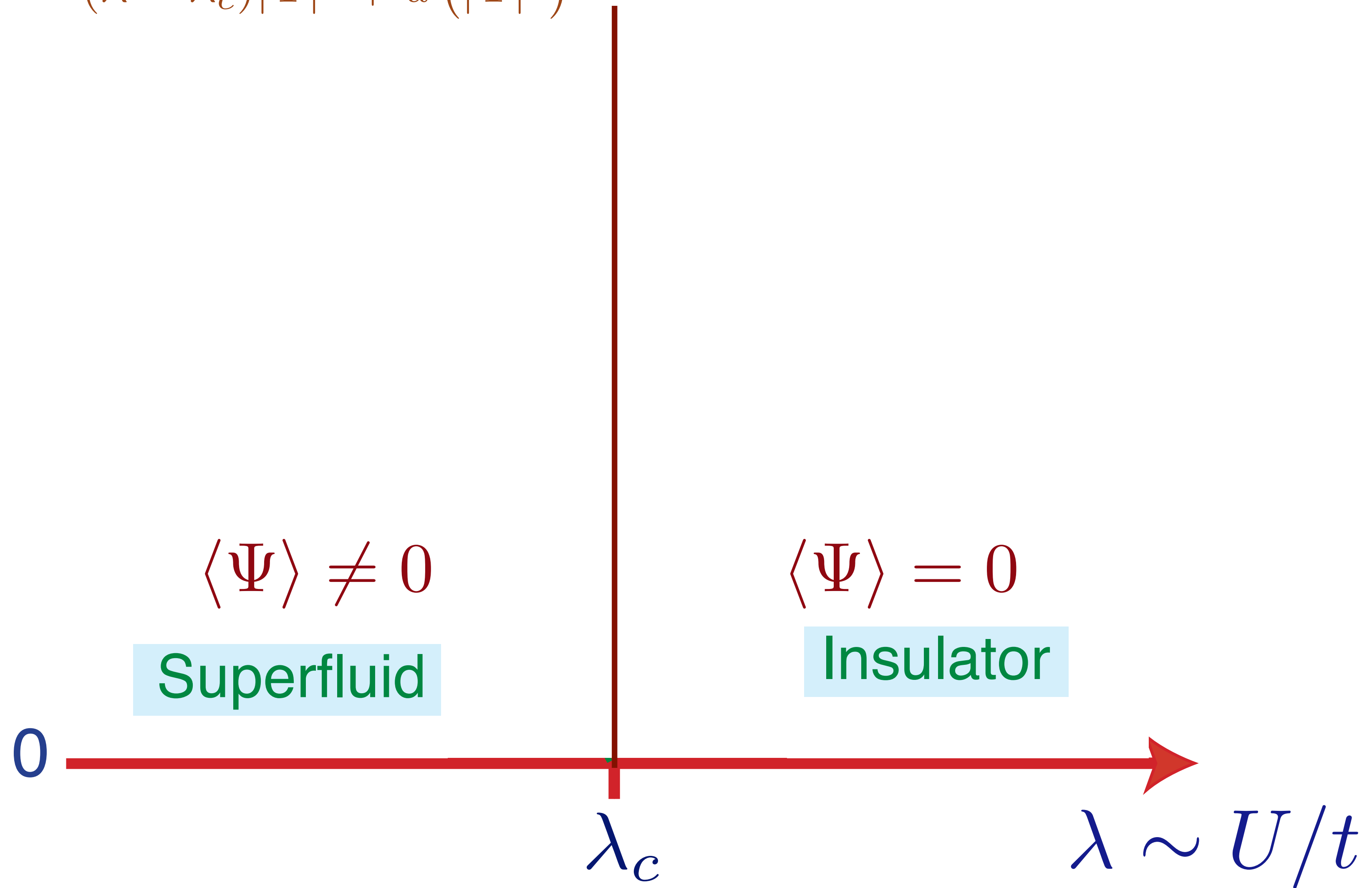


$\Psi \sim b_{k=0} \rightarrow$  a complex field representing the Bose-Einstein condensate of the superfluid



$$\mathcal{Z} = \int \mathcal{D}\Psi(r, \tau) \exp \left( - \int d^2r d\tau [|\partial_\tau \Psi|^2 + c^2 |\nabla_r \Psi|^2 + V(\Psi)] \right)$$

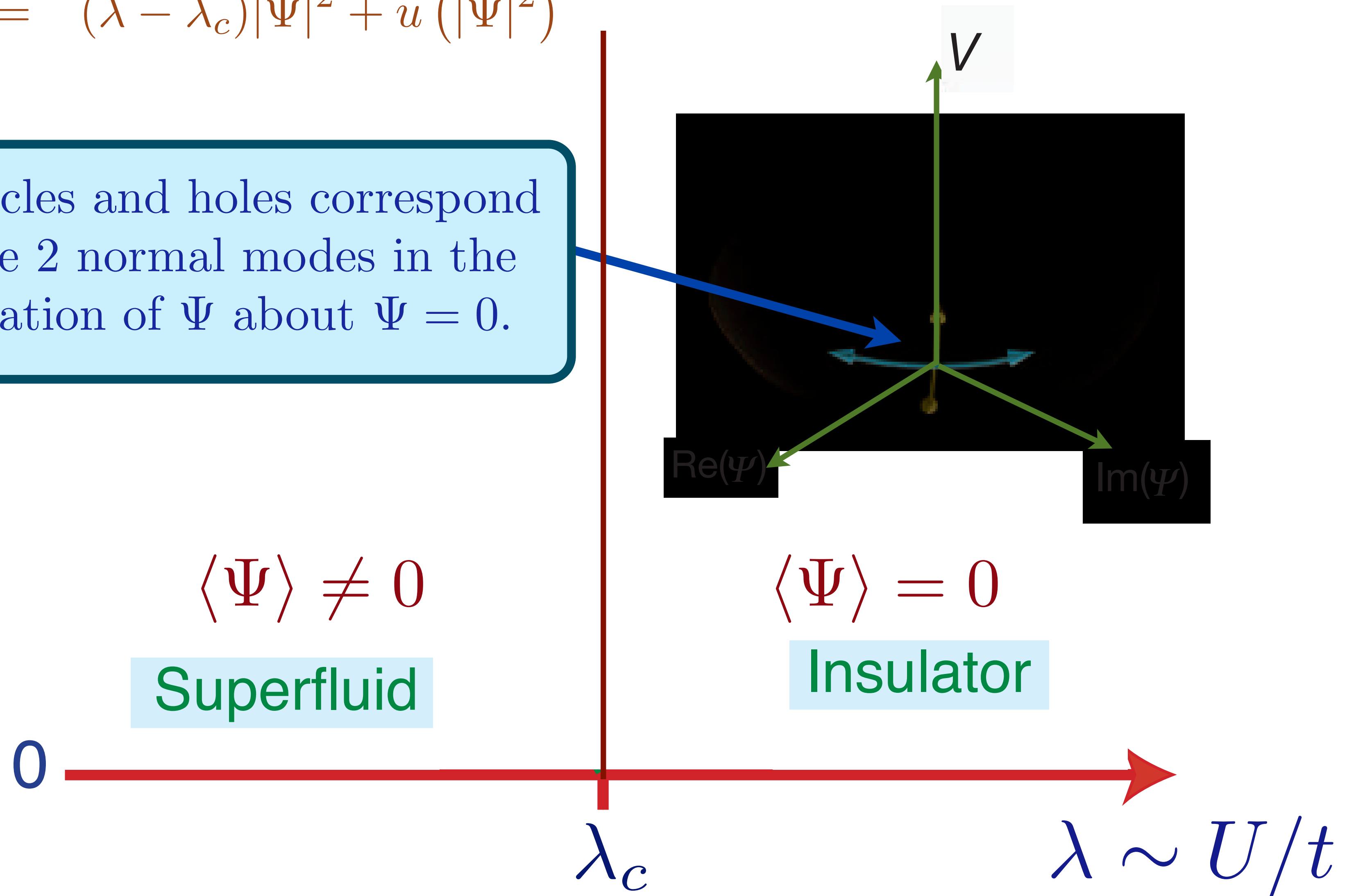
$$V(\Psi) = (\lambda - \lambda_c) |\Psi|^2 + u (|\Psi|^2)^2$$



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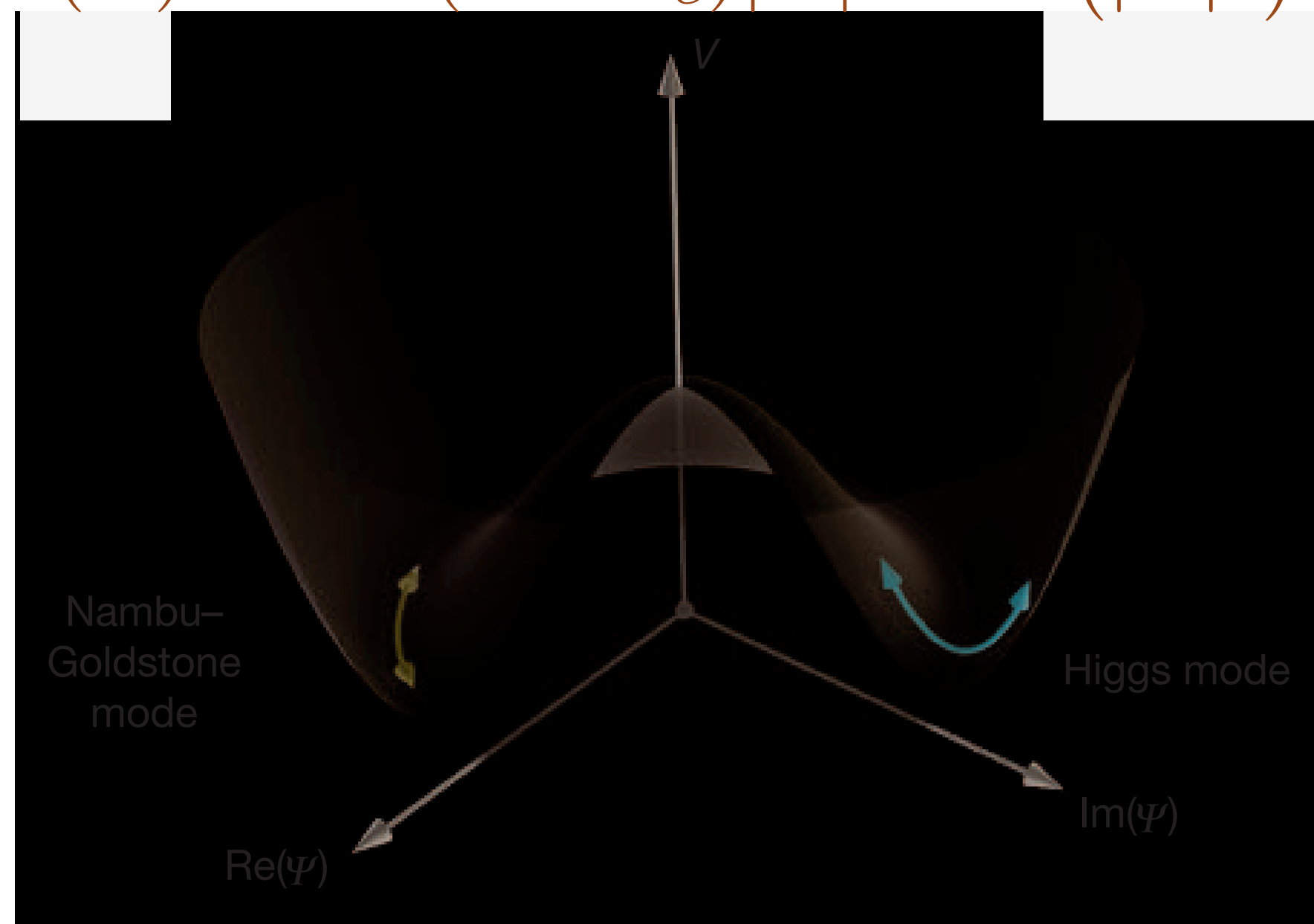
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Particles and holes correspond to the 2 normal modes in the oscillation of  $\Psi$  about  $\Psi = 0$ .



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$$V(\Psi) = (\lambda - \lambda_c) |\Psi|^2 + u (|\Psi|^2)^2$$



$$\langle \Psi \rangle \neq 0$$

Superfluid

$$\langle \Psi \rangle = 0$$

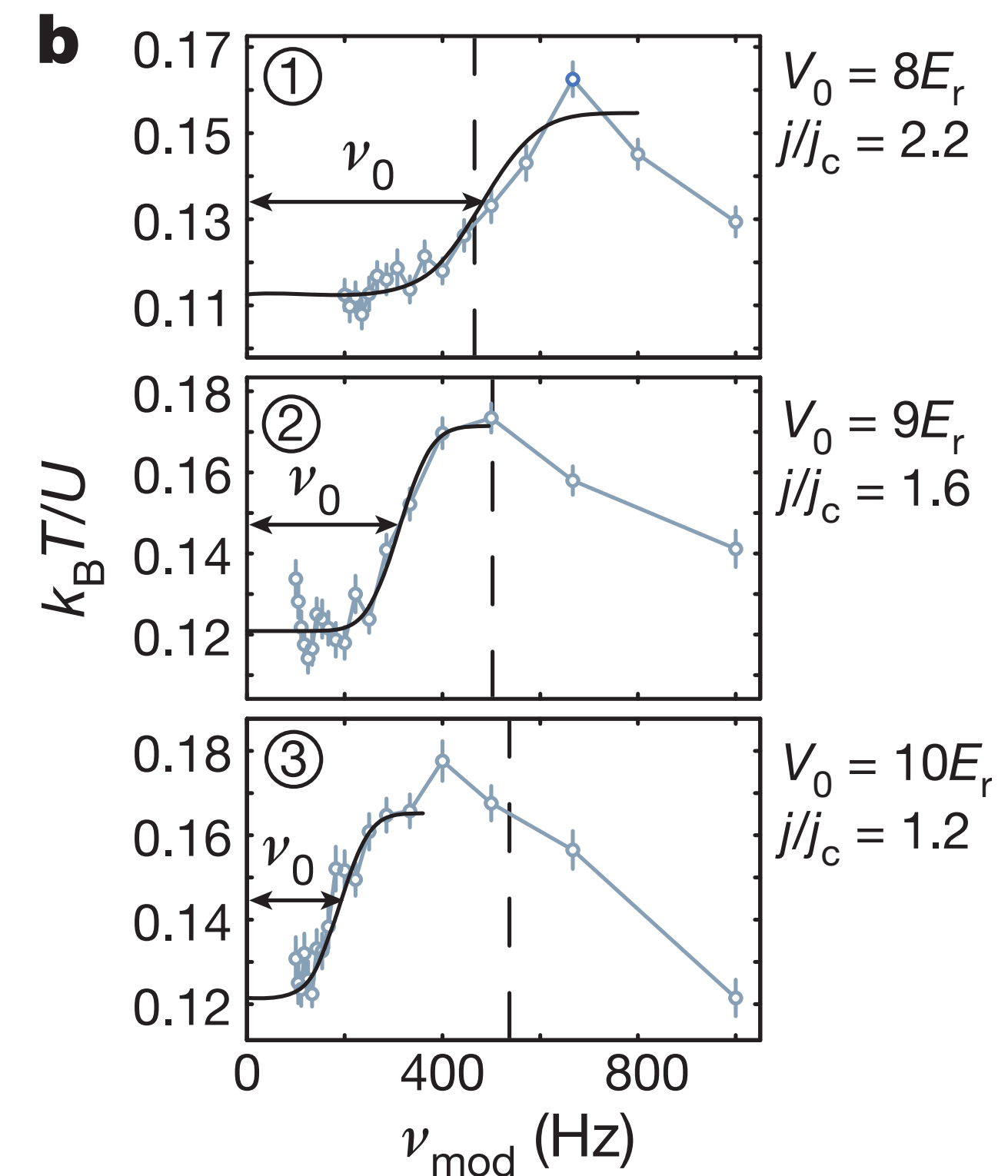
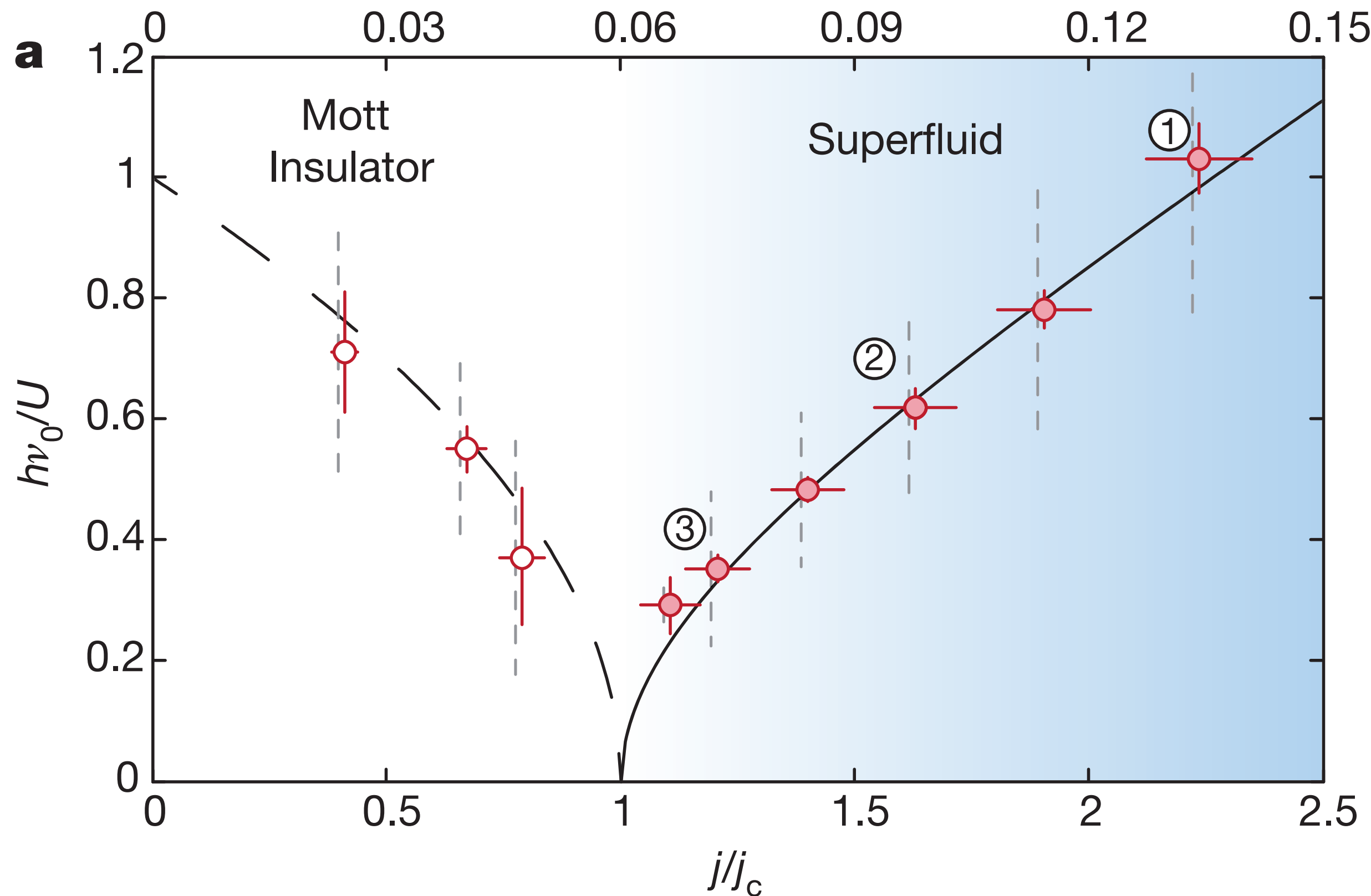
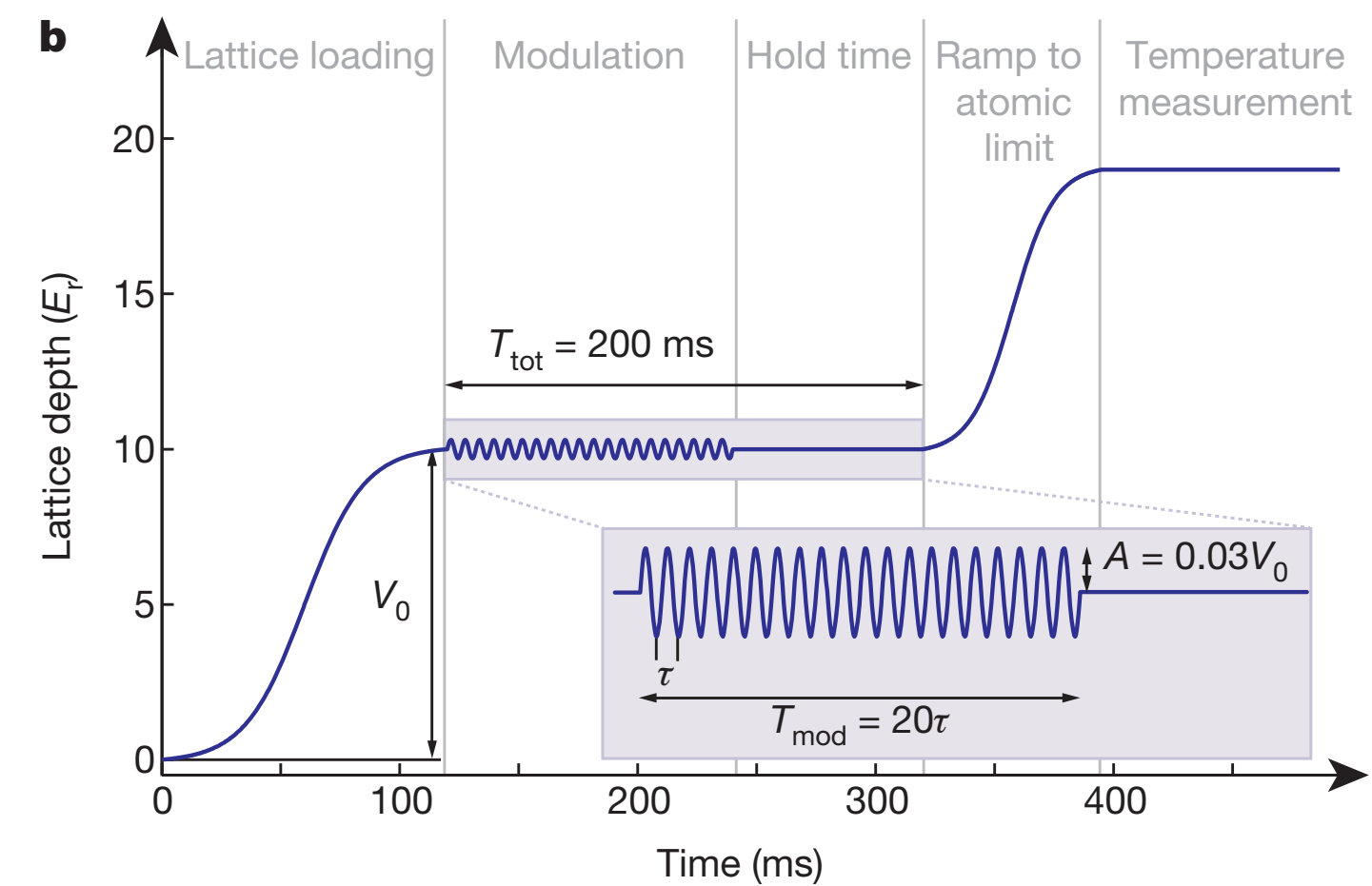
Insulator

0

$\lambda_c$

$\lambda \sim U/t$

Observation of Higgs quasi-normal mode across the superfluid-insulator transition of ultracold atoms in a 2-dimensional optical lattice:  
 Response to modulation of lattice depth scales as expected from the LHP pole

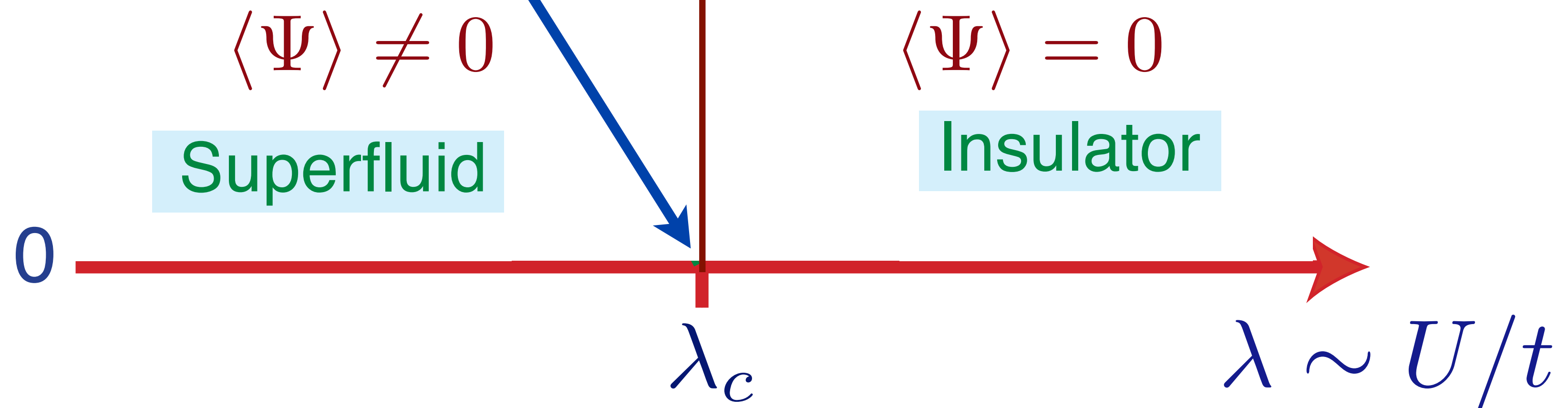


Manuel Endres, Takeshi Fukuhara, David Pekker, Marc Cheneau, Peter Schaub, Christian Gross, Eugene Demler, Stefan Kuhr, and Immanuel Bloch, *Nature* **487**, 454 (2012).

$$\mathcal{Z} = \int \mathcal{D}\Psi(r, \tau) \exp \left( - \int d^2r d\tau [|\partial_\tau \Psi|^2 + c^2 |\nabla_r \Psi|^2 + V(\Psi)] \right)$$

$$V(\Psi) = (\lambda - \lambda_c) |\Psi|^2 + u (|\Psi|^2)^2$$

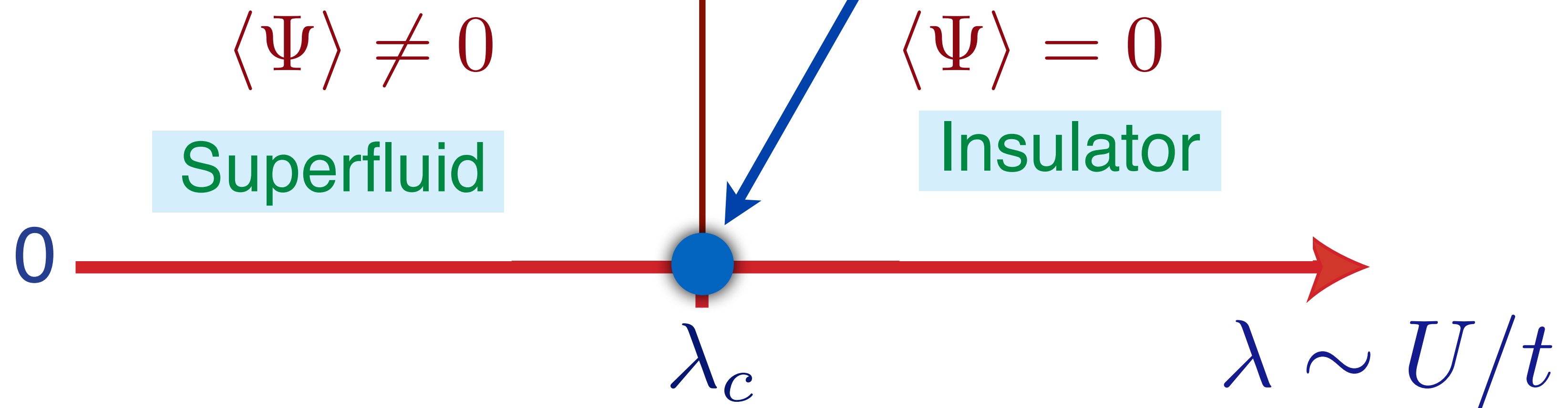
Quantum state with  
complex, many-body,  
“long-range” quantum entanglement



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A conformal field theory  
in 2+1 spacetime dimensions (CFT3):  
the O(2) Wilson-Fisher CFT3

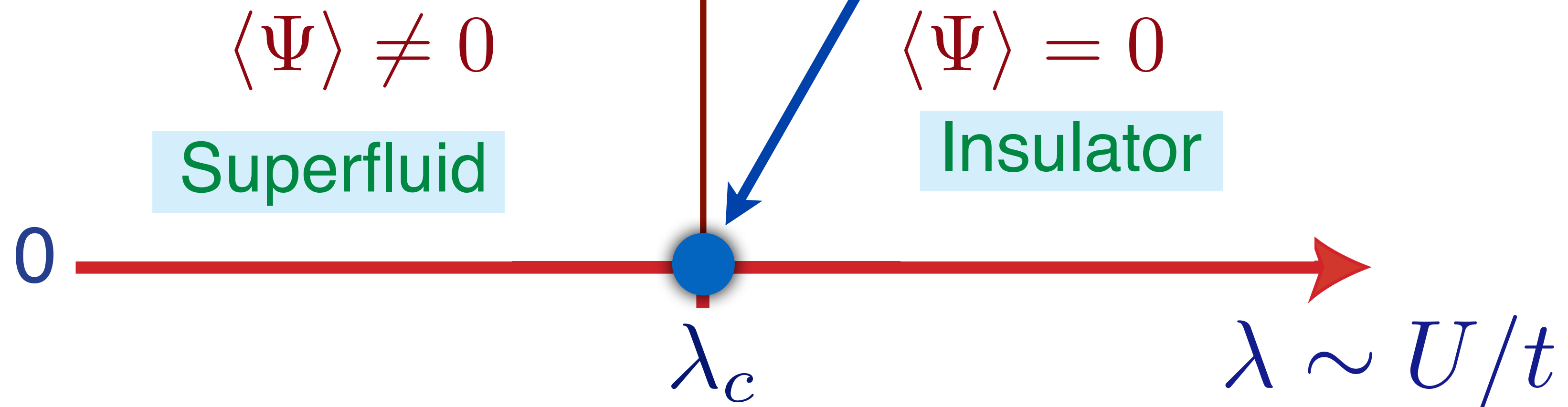


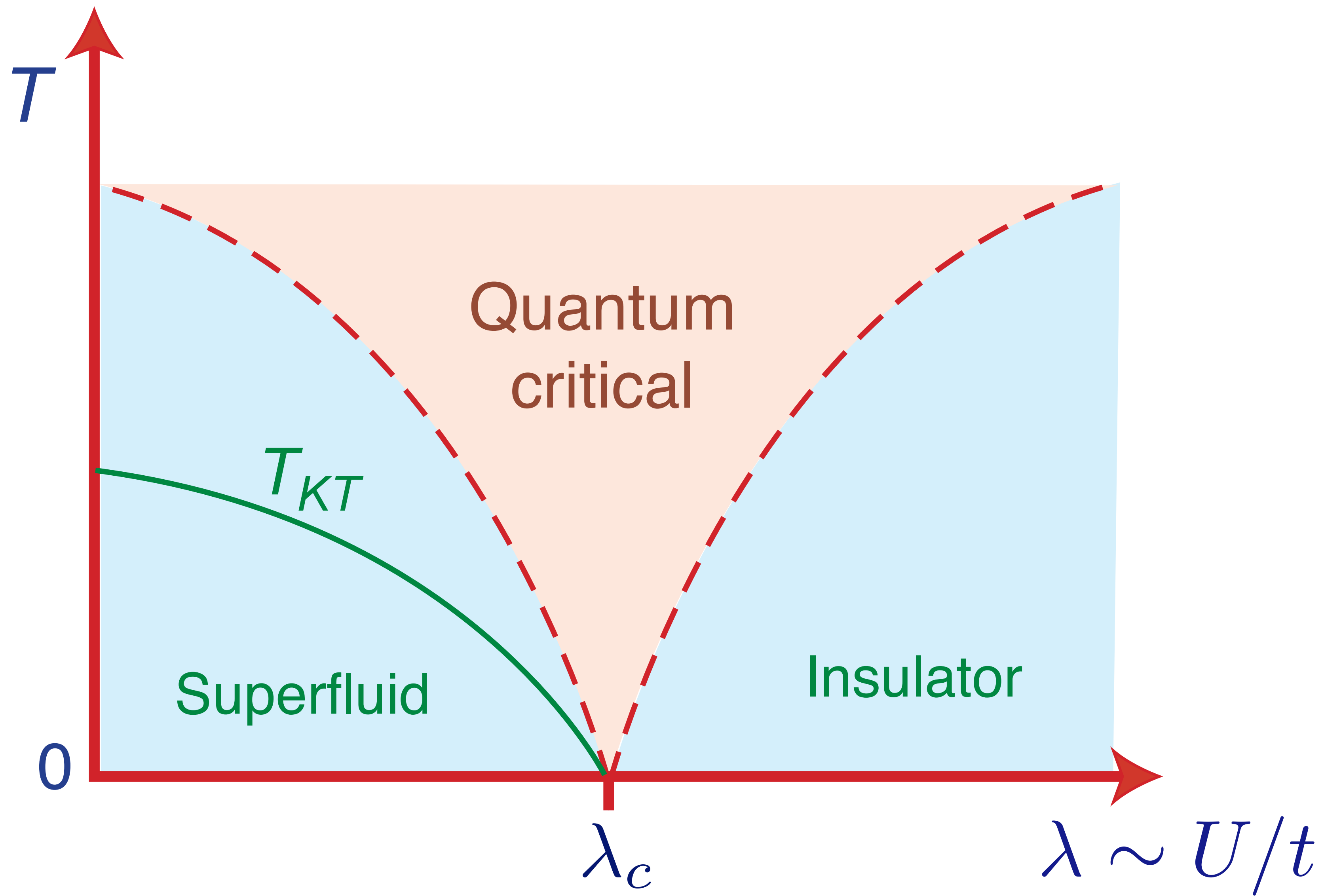
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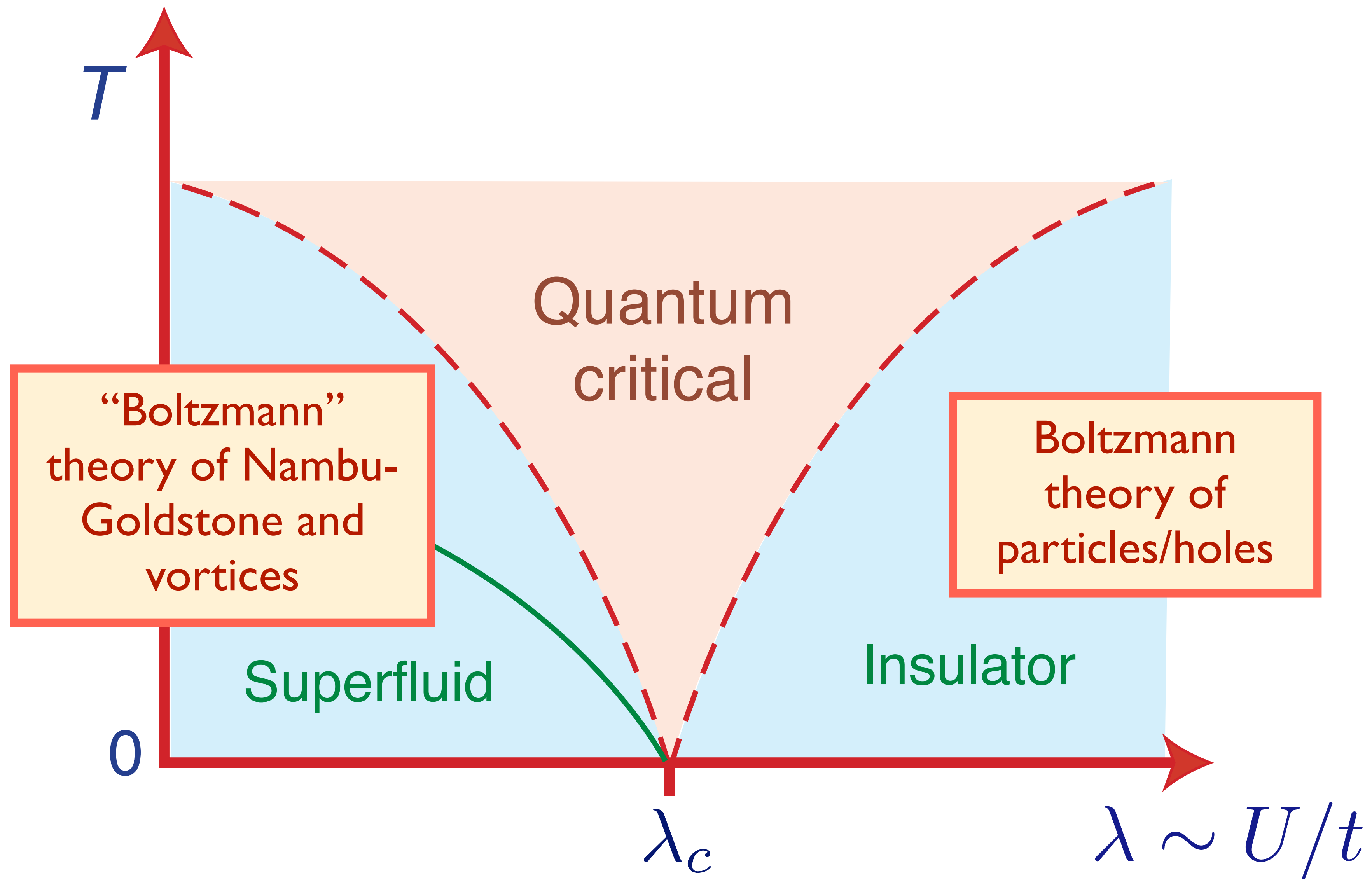
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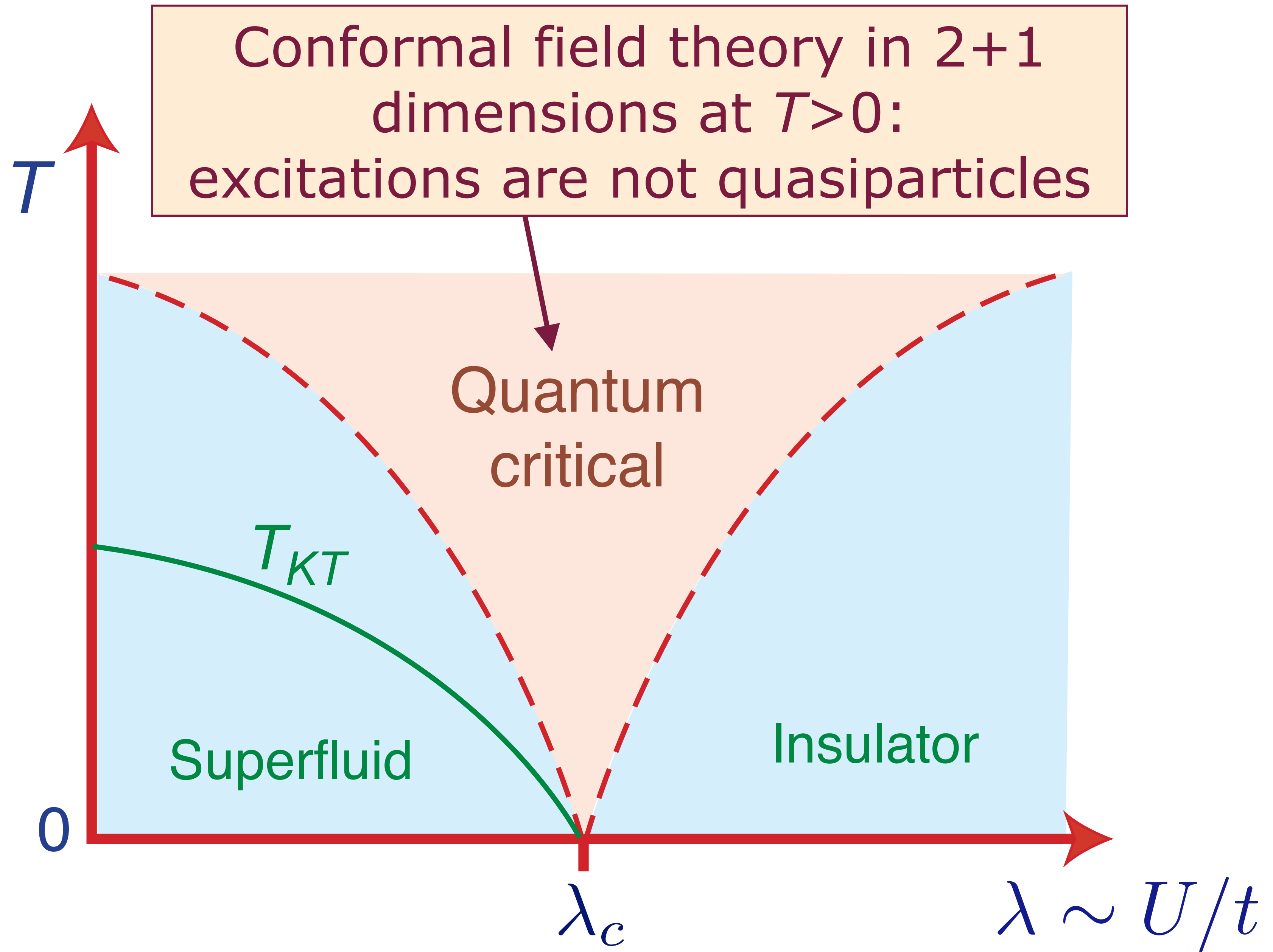
The coupling  $u \rightarrow u^*$ ,  
the renormalization group  
fixed point, for the CFT3.

A conformal field theory  
in 2+1 spacetime dimensions (CFT3):  
the O(2) Wilson-Fisher CFT3

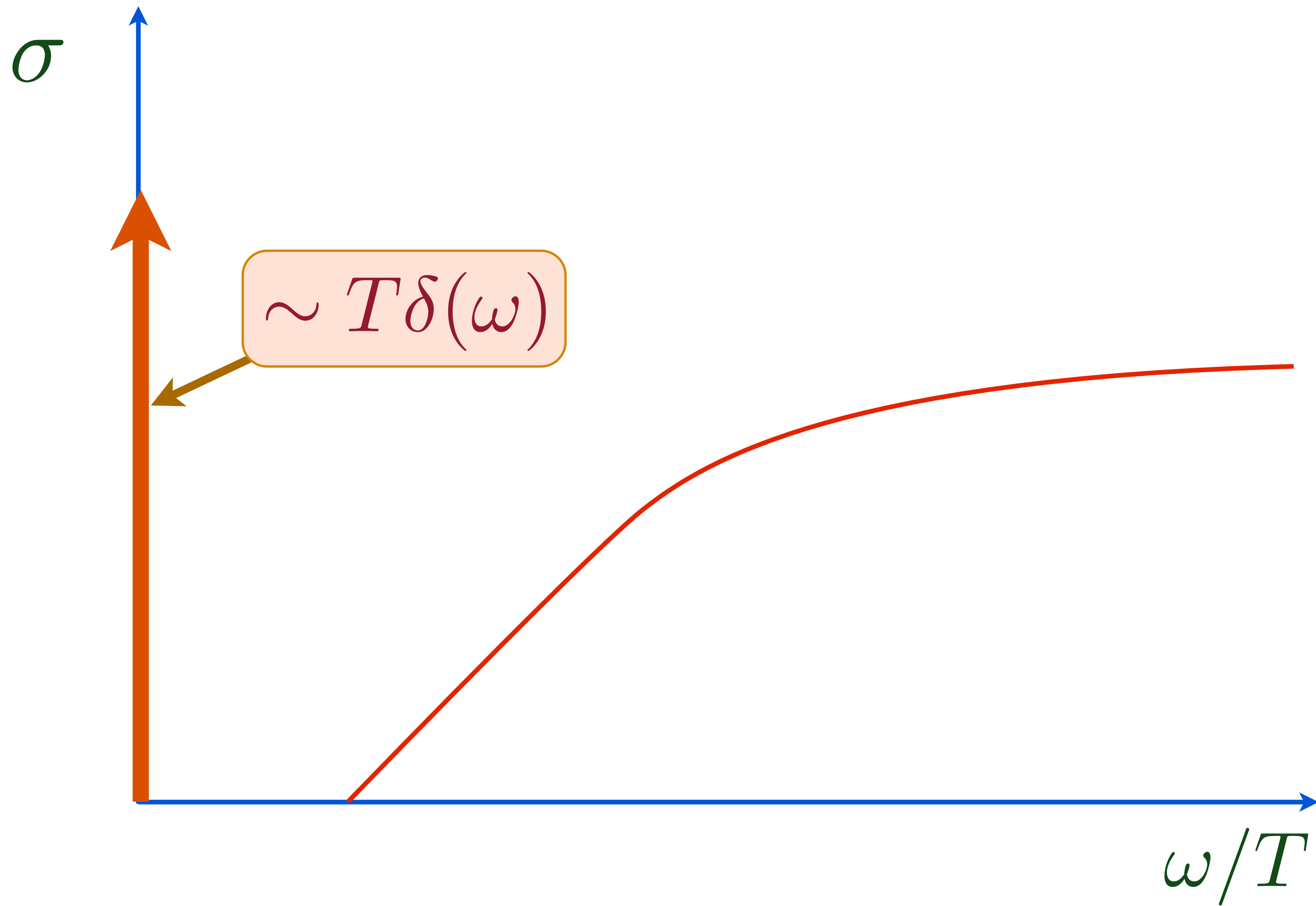




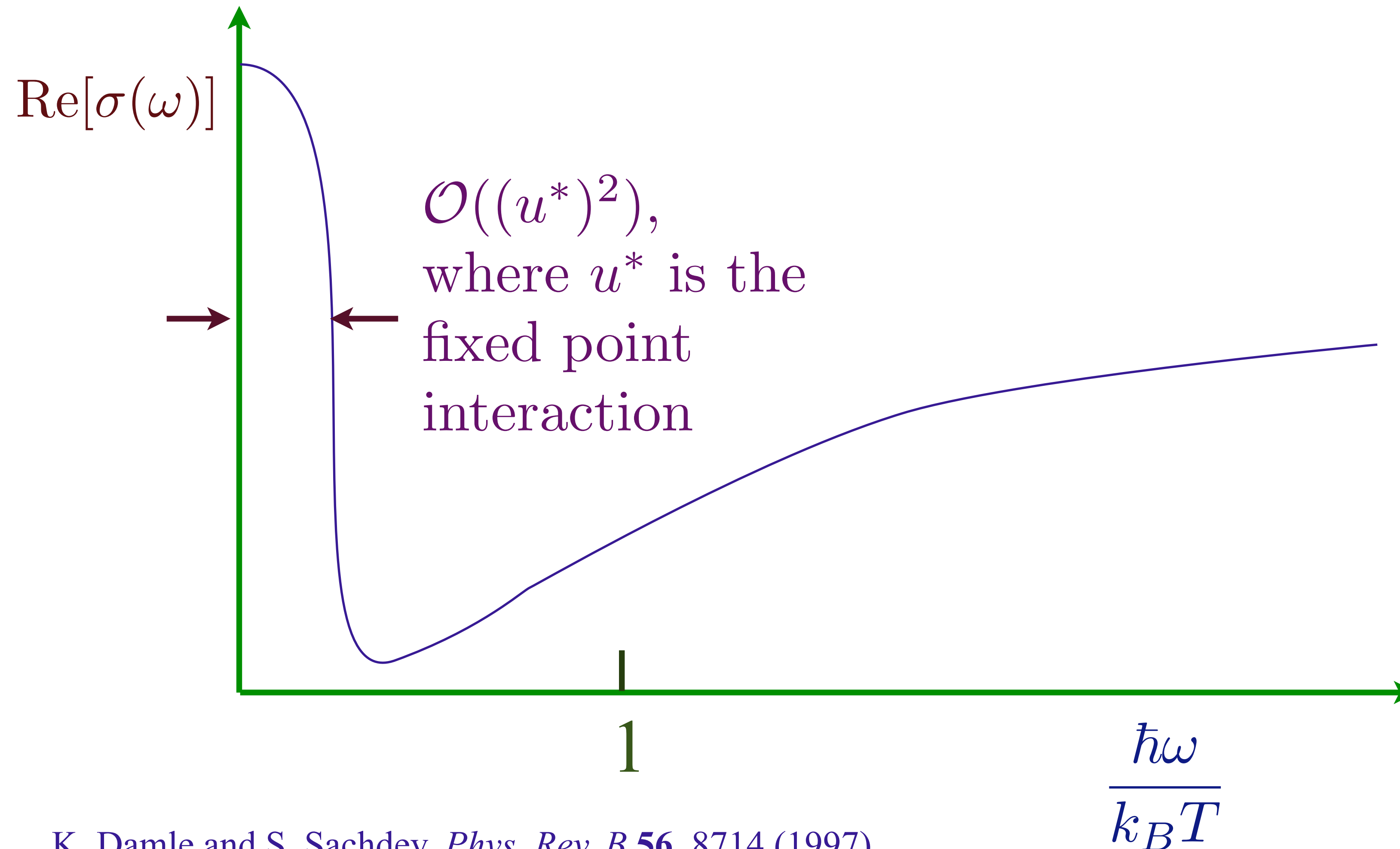




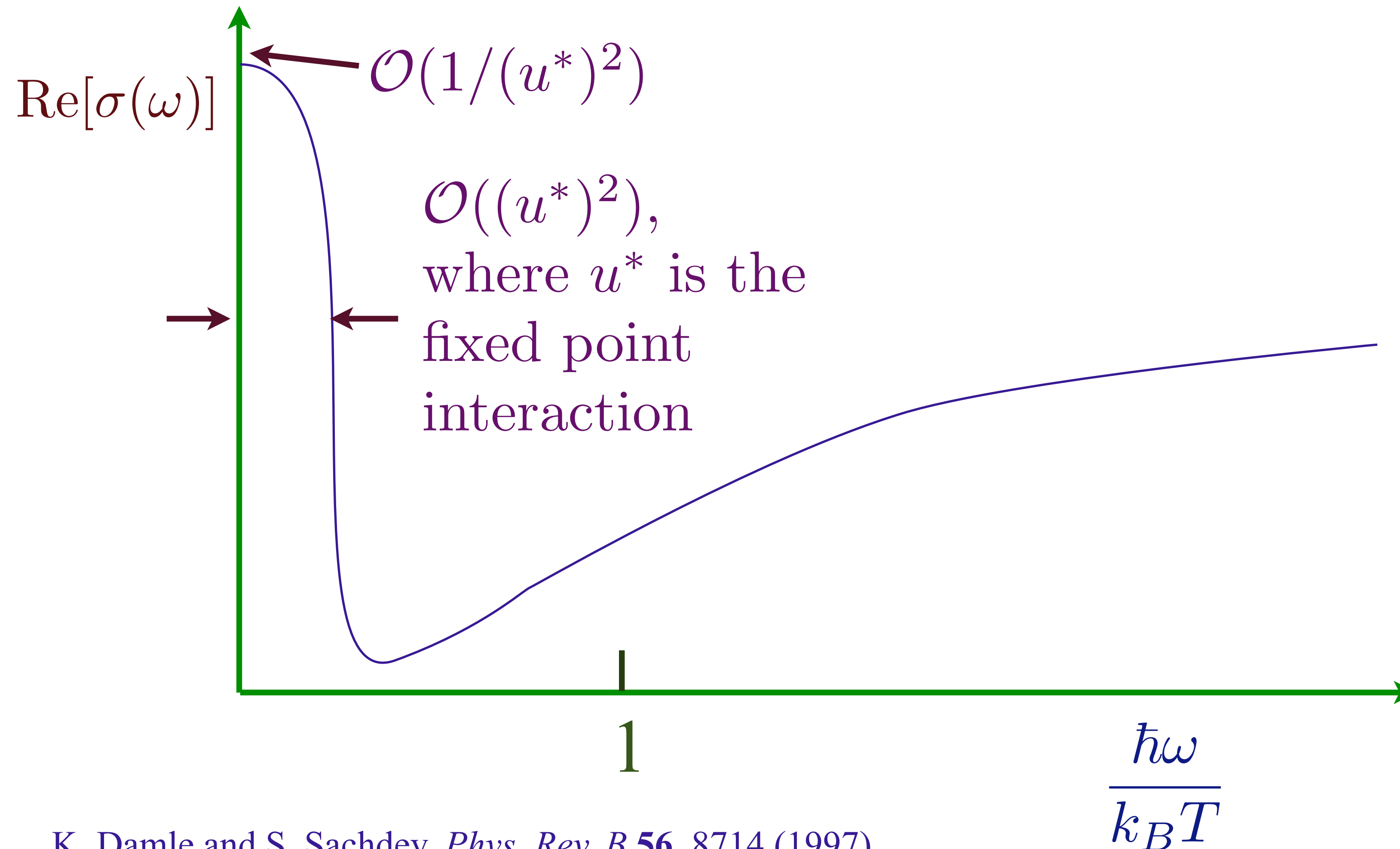
Electrical transport in a free quasiparticle  
CFT3 for  $T > 0$



Quasiparticle view of quantum criticality (Boltzmann equation):  
Electrical transport for a (weakly) interacting CFT3



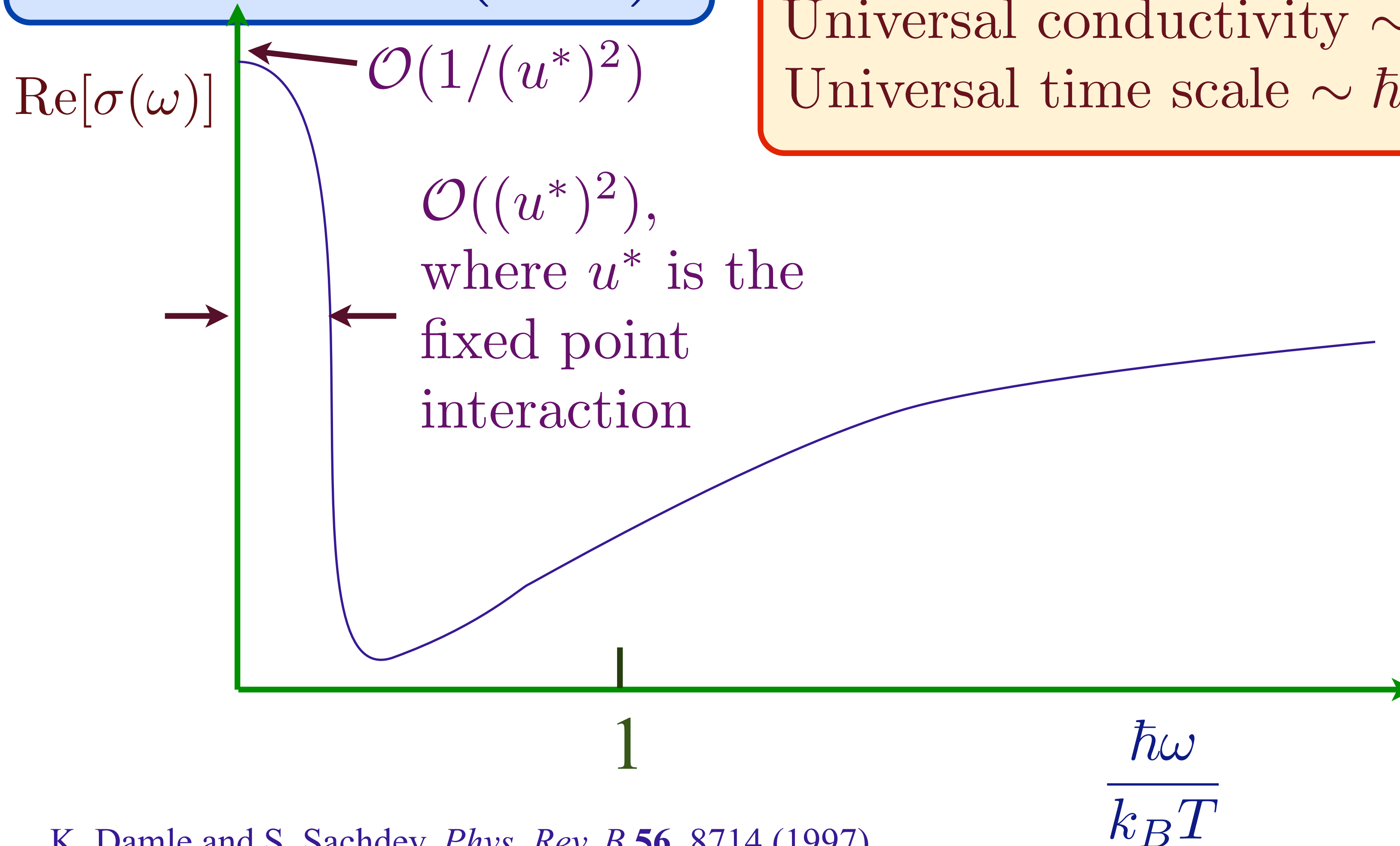
Quasiparticle view of quantum criticality (Boltzmann equation):  
Electrical transport for a (weakly) interacting CFT3



Quasiparticle view of quantum criticality (Boltzmann equation):  
Electrical transport for a (weakly) interacting CFT3

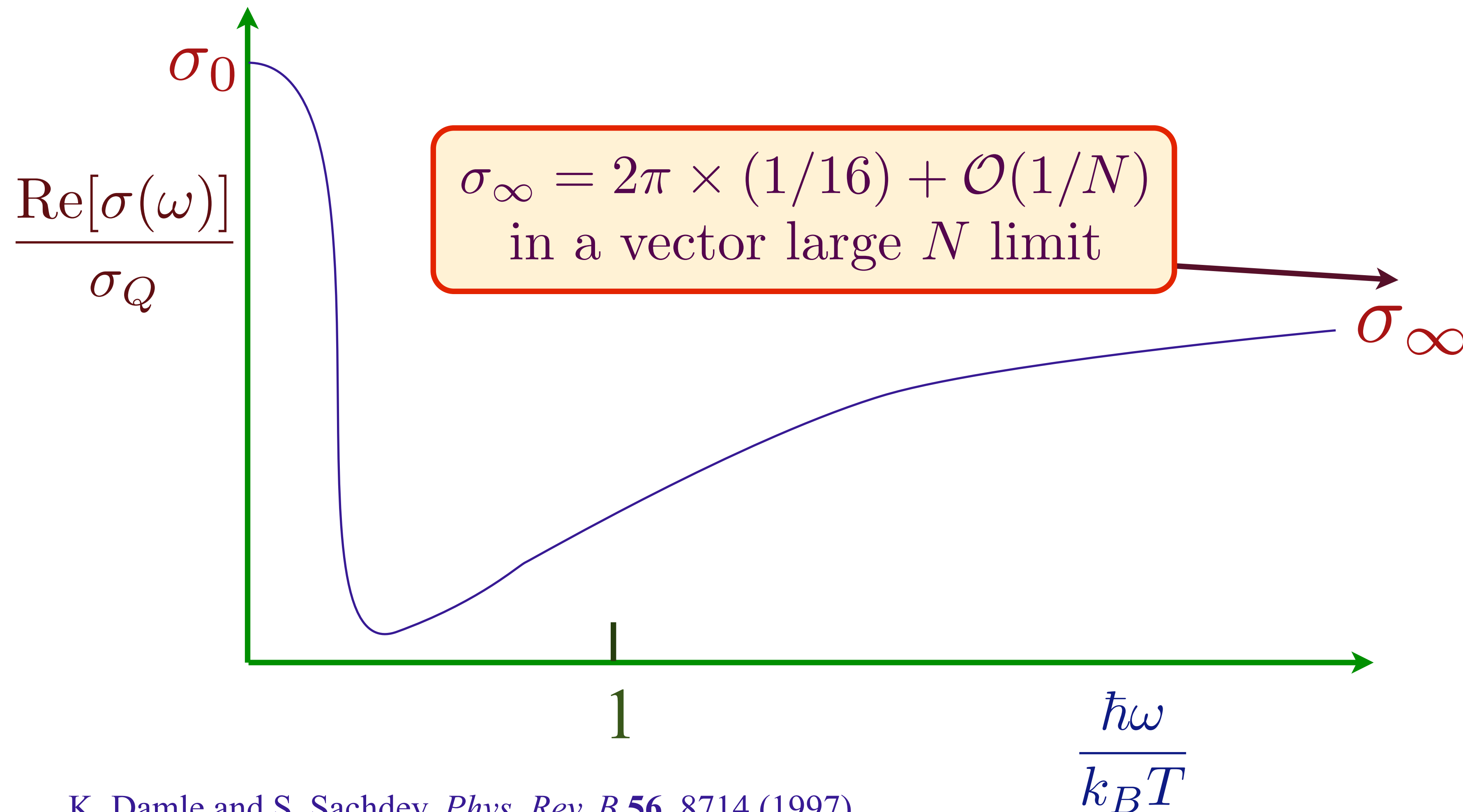
$$\sigma(\omega, T) = \frac{e^2}{h} \Sigma \left( \frac{\hbar\omega}{k_B T} \right)$$

$\Sigma \rightarrow$  a universal function  
 Universal conductivity  $\sim e^2/h$   
 Universal time scale  $\sim \hbar/k_B T$



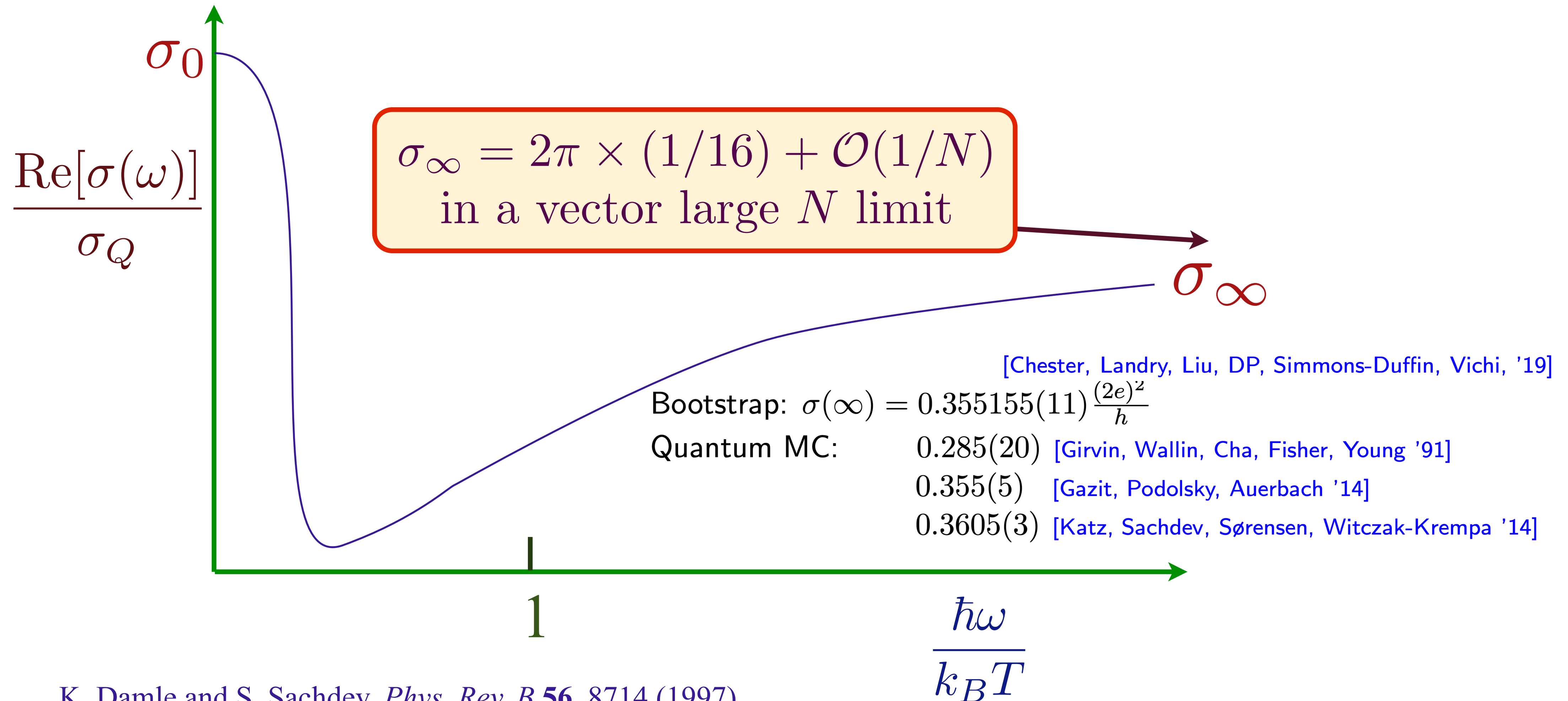
Quasiparticle view of quantum criticality (Boltzmann equation):  
Transport of  $O(N)$  current for a (weakly) interacting CFT3

$\sigma_Q = e^2/h$ , the quantum unit of conductance



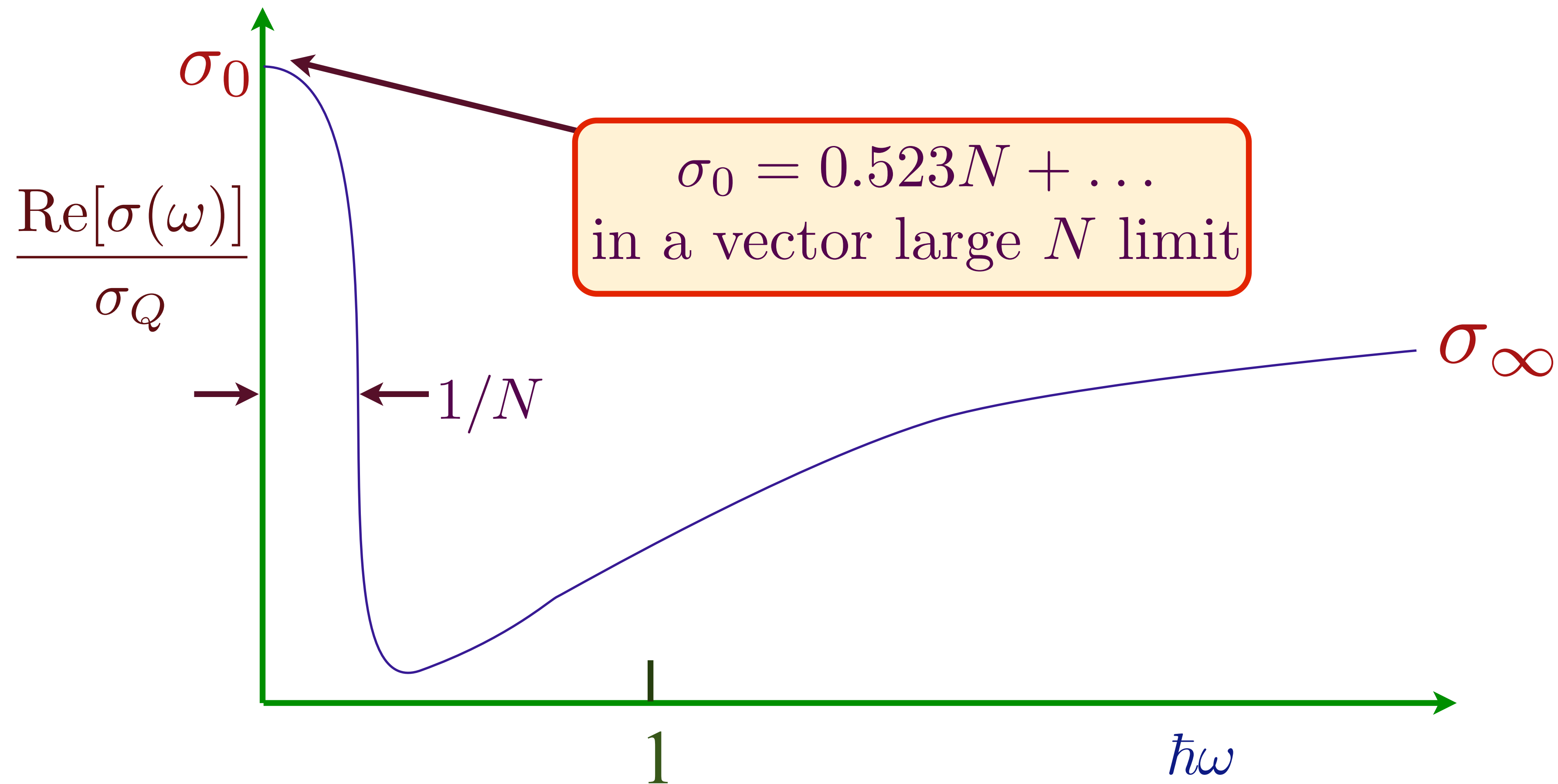
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$\sigma_Q = e^2/h$ , the quantum unit of conductance



# Quasiparticle view of quantum criticality (Boltzmann equation): Transport of $O(N)$ current for a (weakly) interacting CFT3

$\sigma_Q = e^2/h$ , the quantum unit of conductance



S. Sachdev, *Phys. Rev. B* **57**, 7157 (1998)  
Witczak-Krempa, P. Ghaemi, T. Senthil, and Yong Baek Kim,  
*Rev. B* **86**, 24102 (2012)

W.  
*Phys.*  $\frac{\hbar\omega}{k_B T}$

## Traditional CMT

- ◇ Identify quasiparticles and their dispersions
- ◇ Compute scattering matrix elements of quasiparticles
- ◇ Input parameters into a quantum Boltzmann equation
- ◇ Compute dissipative properties at  $\omega \ll$  quasiparticle-collision-rate

## Dynamics without quasiparticles

- ★ Start with strongly interacting CFT without quasiparticles
- ★ Using scaling dimensions and operator product expansions (OPE) of the CFT, compute conductivity at  $\hbar\omega \gg k_B T$
- ★ Relate OPE coefficients to couplings of an effective gravitational theory on AdS
- ★ Dynamics of a “horizon” in gravitational theory yields info at  $\hbar\omega \ll k_B T$ .

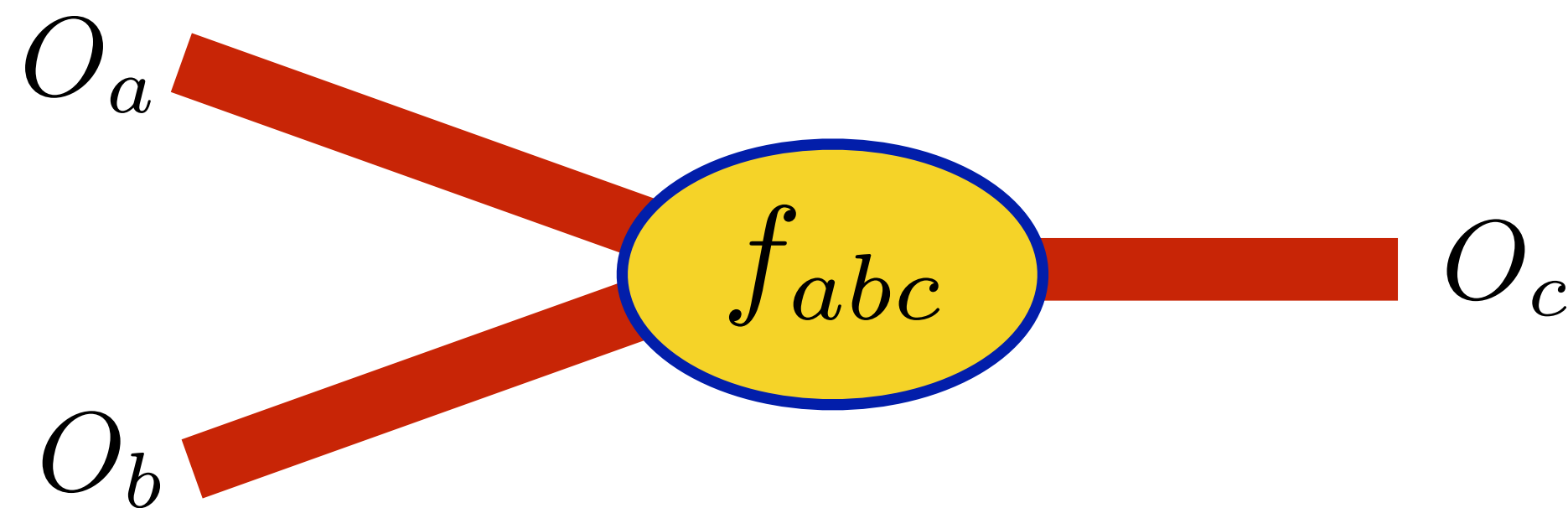
## Basic characteristics of CFTs

Primary operators of CFT,  $O_a(x)$ , obey ( at  $T = 0$ ):

$$\langle O_a(x)O_b(0) \rangle = \frac{\delta_{ab}}{|x|^{2\Delta_a}}$$

where  $\Delta_a$  is their scaling dimension. Their “interactions” are determined by the OPE (considering scalar operators only)

$$\lim_{x' \rightarrow x} \langle O_a(x')O_b(x)O_c(0) \rangle = \frac{f_{abc}}{|x|^{\Delta_a + \Delta_b + \Delta_c}}$$



The values of  $\{\Delta_a, f_{abc}\}$  determine (in principle) all observable properties of the CFT, as constrained by conformal Ward identities. For the Wilson-Fisher CFT<sub>3</sub>, systematic methods exist to compute (in principle) all the  $\{\Delta_a, f_{abc}\}$ , and we will assume this data is *known*. This knowledge will be taken as an *input* to the computation of the finite  $T$  dynamics

# Basic characteristics of CFT3s

The OPE of the current operator is

$$\lim_{|\Omega| \gg p} J_x(\omega) J_x(-\omega + \mathbf{p}) = -|\Omega| \sigma_\infty \delta^{(3)}(\mathbf{p}) - \frac{\mathcal{C}}{|\Omega|^{\Delta-1}} \mathcal{O}(\mathbf{p})$$

The most important  $\mathcal{O}$  is the thermal operator  $\mathcal{O} \sim |\Psi|^2$  with scaling dimensions  $\Delta = 3 - 1/\nu$ . We also need

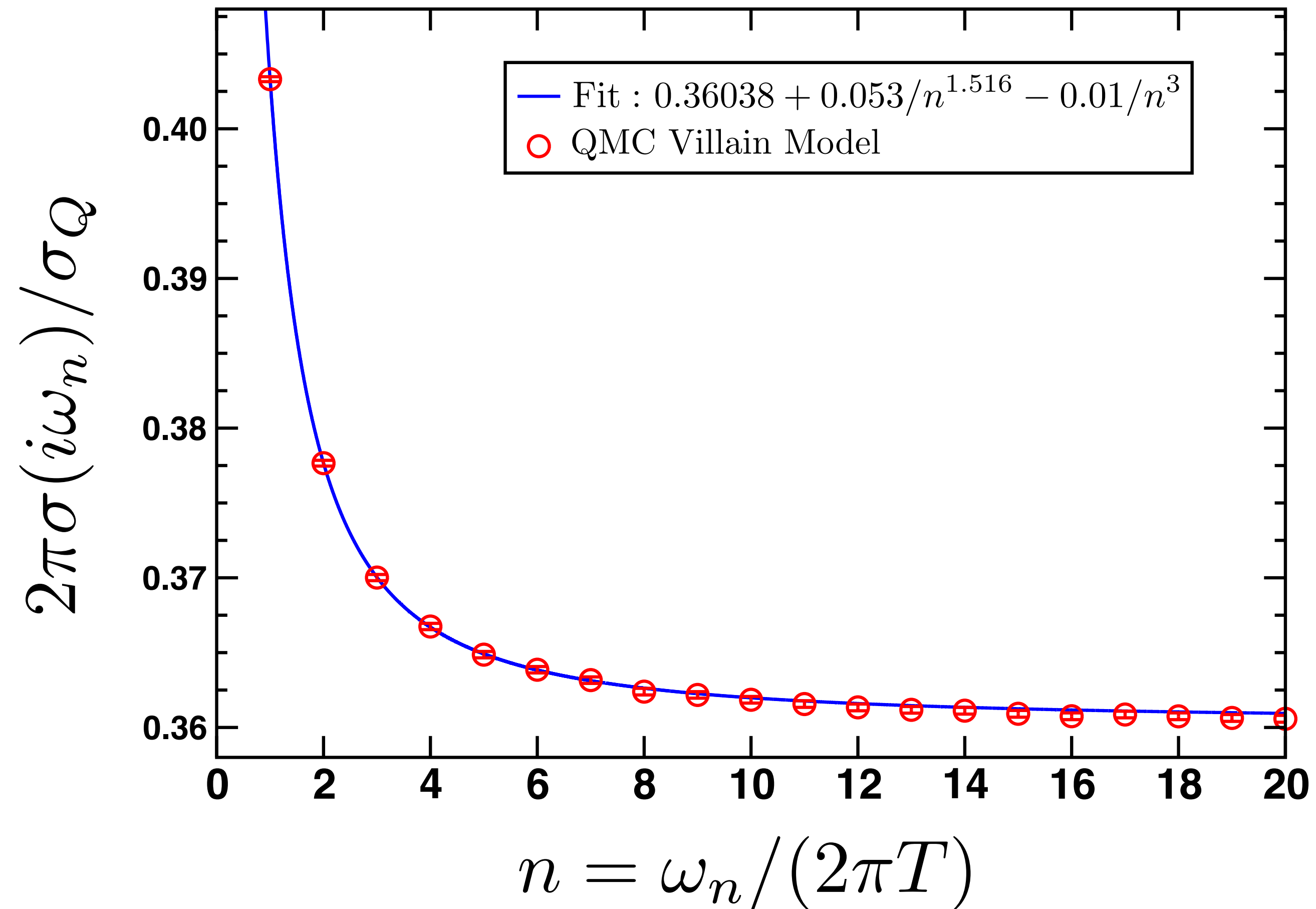
$$\langle \mathcal{O} \rangle_T - \langle \mathcal{O} \rangle_{T=0} \equiv BT^{3-1/\nu}$$

Then the thermal average of the OPE of two O(2) current operators yields for  $\omega \gg T$

$$\frac{\sigma(\omega)}{\sigma_Q} = \sigma_\infty + b_1 \left( \frac{T}{\omega} \right)^{3-1/\nu} + b_2 \left( \frac{T}{\omega} \right)^3 + \dots$$

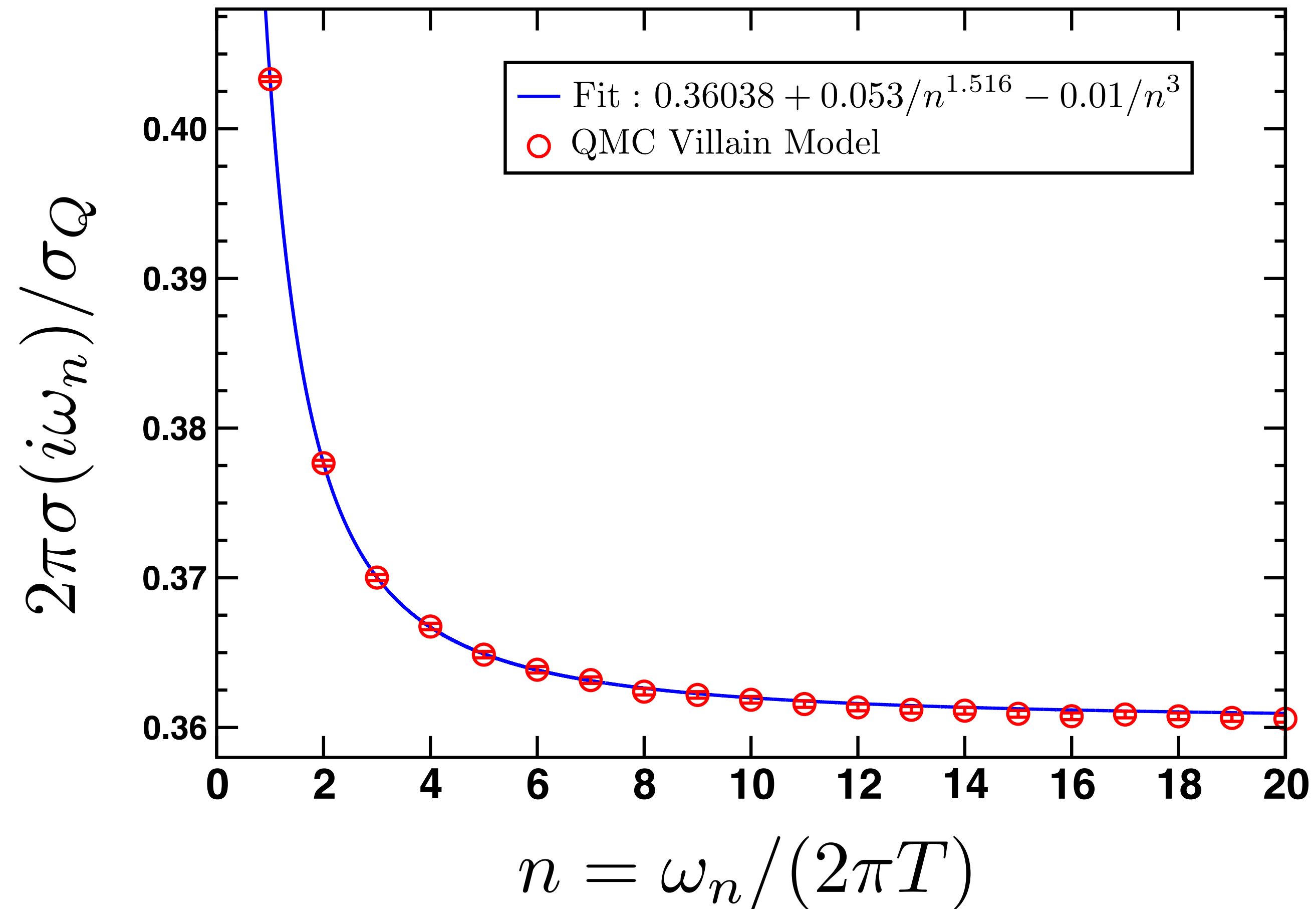
where  $b_1 = BC$  and  $b_2$  depends on OPE with the energy-momentum tensor.

# Quantum Monte Carlo for lattice model of integer currents (Villain model) in Euclidean time



**Excellent agreement with OPE**

# Quantum Monte Carlo for lattice model of integer currents (Villain model) in Euclidean time



QMC fails for Minkowski frequencies  $\hbar\omega \ll k_B T$

# Extending OPE using holography

We match the OPE results to a holographic theory of a U(1) gauge field  $F$ , which is dual to the conserved current  $J$ , in a AdS<sub>4</sub> background.

The  $T > 0$  AdS<sub>4</sub>-Schwarzschild metric is

$$ds_{\text{Sch}}^2 = \frac{r_0^2}{L^2 u^2} [-f(u)dt^2 + dx^2 + dy^2] + \frac{L^2 du^2}{u^2 f(u)},$$

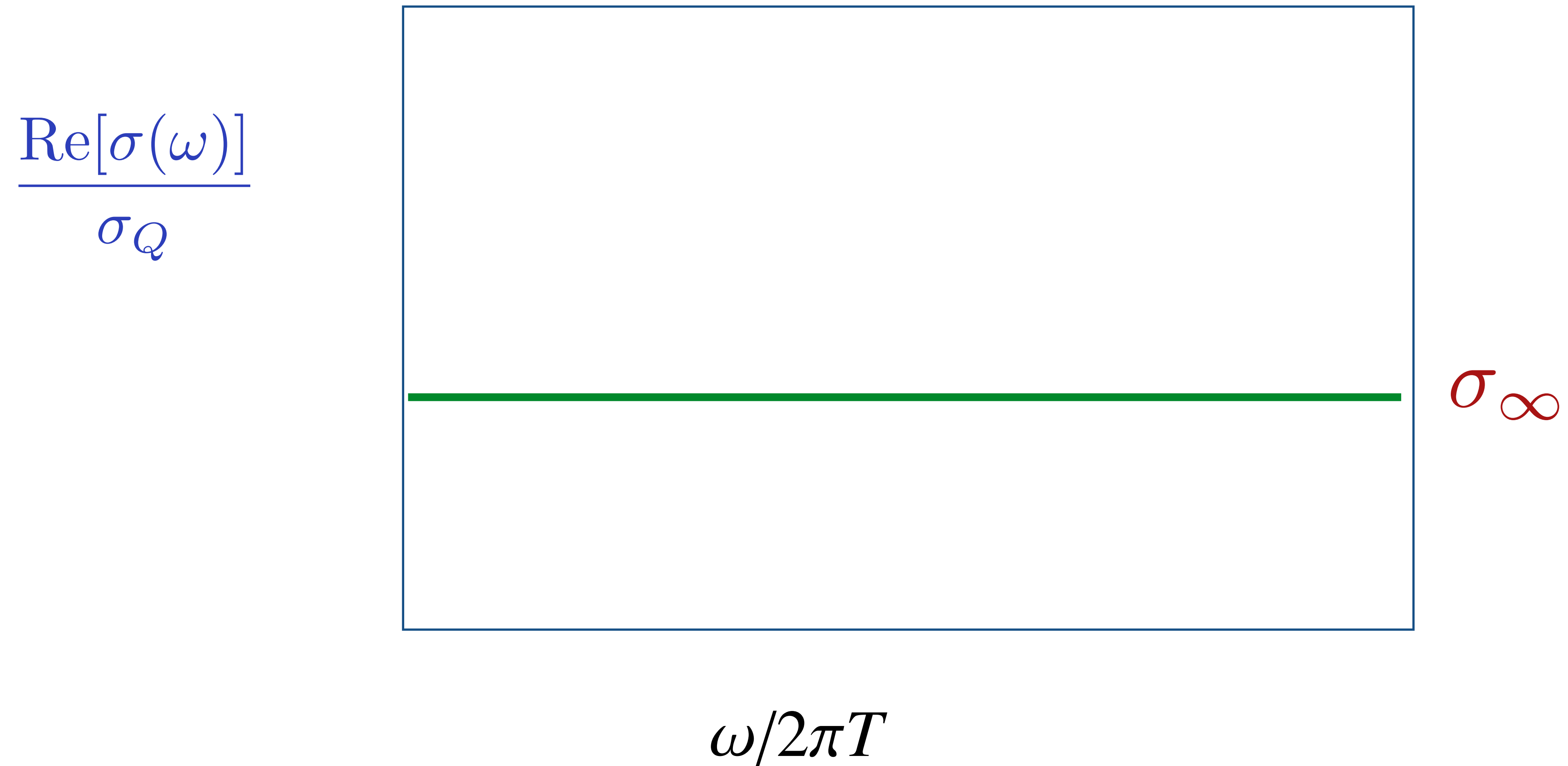
where  $u$  is the holographic dimension with  $f(u) = 1 - u^3$ . We also introduce a holographic scalar field  $\varphi$  dual to  $\mathcal{O}$  which has the profile near the boundary

$$\varphi(\mathbf{x}, \tilde{u} \rightarrow 0) = \frac{\tilde{u}^\Delta}{(2\Delta - D)} (\langle \mathcal{O} \rangle_T - \langle \mathcal{O} \rangle_{T=0})$$

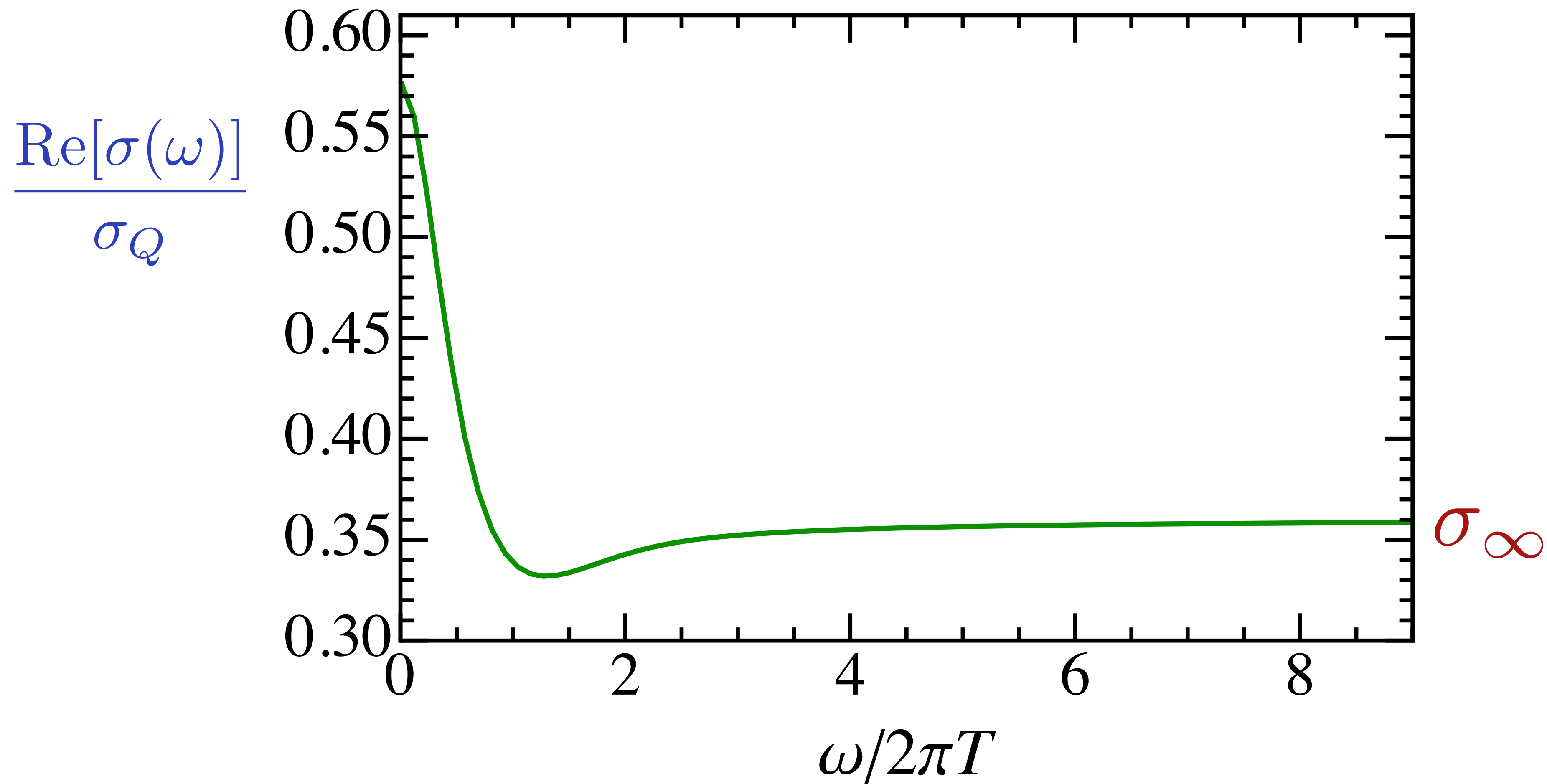
where  $\tilde{u} = uL^2/r_0$  with  $r_0 = 4\pi TL^2/3$ . Then the current correlations are determined by the holographic theory for  $F$ :

$$\mathcal{S} = \int d^4x \sqrt{-g_{\text{Sch}}} \left\{ -\frac{1}{4g_4^2} [1 + \alpha\varphi(u)] F_{ab} F^{ab} \right\}$$

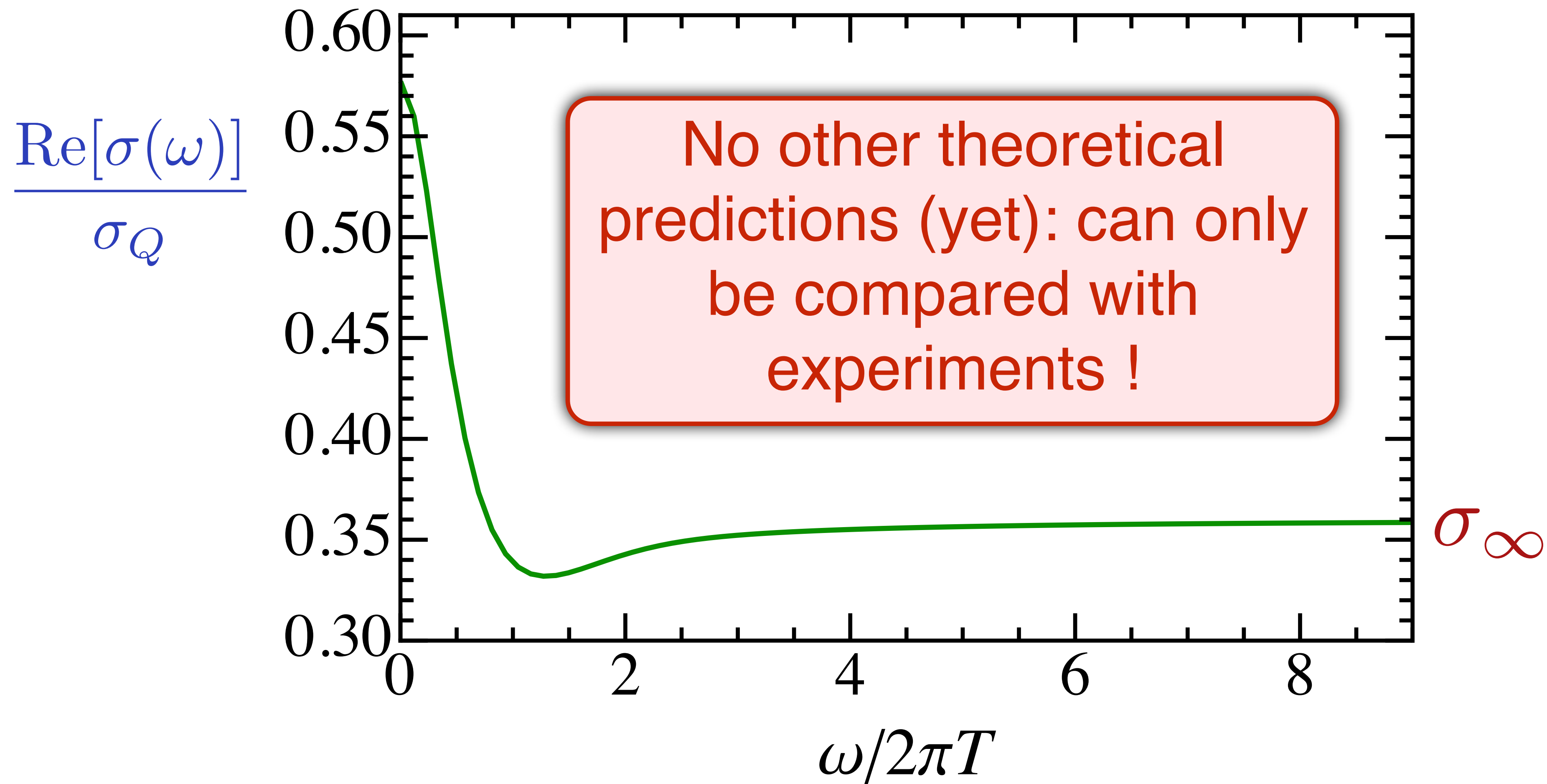
# Conductivity of Einstein-Maxwell theory



# Numerical solution of Einstein-Maxwell-Weyl-scalar theory + OPE info from QMC



# Numerical solution of Einstein-Maxwell-Weyl-scalar theory + OPE info from QMC



## Traditional CMT

- ◇ Identify quasiparticles and their dispersions
- ◇ Compute scattering matrix elements of quasiparticles
- ◇ Input parameters into a quantum Boltzmann equation
- ◇ Compute dissipative properties at  $\omega \ll$  quasiparticle-collision-rate

## Dynamics without quasiparticles

- ★ Start with strongly interacting CFT without quasiparticles
- ★ Using scaling dimensions and operator product expansions (OPE) of the CFT, compute conductivity at  $\hbar\omega \gg k_B T$
- ★ Relate OPE coefficients to couplings of an effective gravitational theory on AdS
- ★ Dynamics of a “horizon” in gravitational theory yields info at  $\hbar\omega \ll k_B T$ .