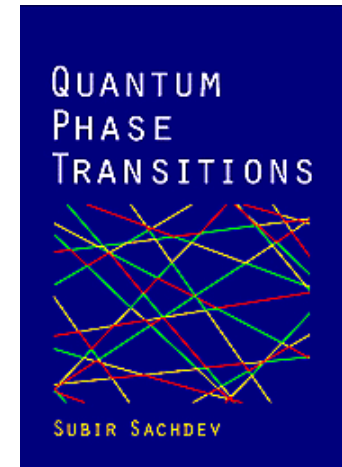


Quantum phase transitions of correlated electrons and atoms

Subir Sachdev
Harvard University

See also: *Quantum phase transitions of correlated electrons in two dimensions*,
cond-mat/0109419.



Quantum Phase Transitions
Cambridge University Press



Outline

A. Magnetic quantum phase transitions in “dimerized” Mott insulators

Landau-Ginzburg-Wilson (LGW) theory

B. Mott insulators with spin $S=1/2$ per unit cell

- 1. Valence-bond-solid (VBS) order in the paramagnet;*
- 2. Mapping to hard-core bosons at half-filling*

C. The superfluid-insulator transition of bosons in lattices

Multiple order parameters in quantum systems

D. Boson-vortex duality

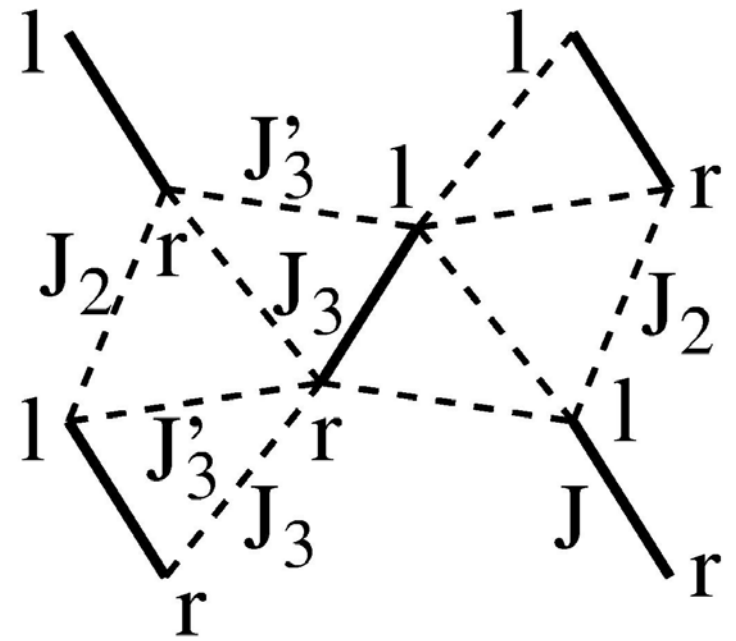
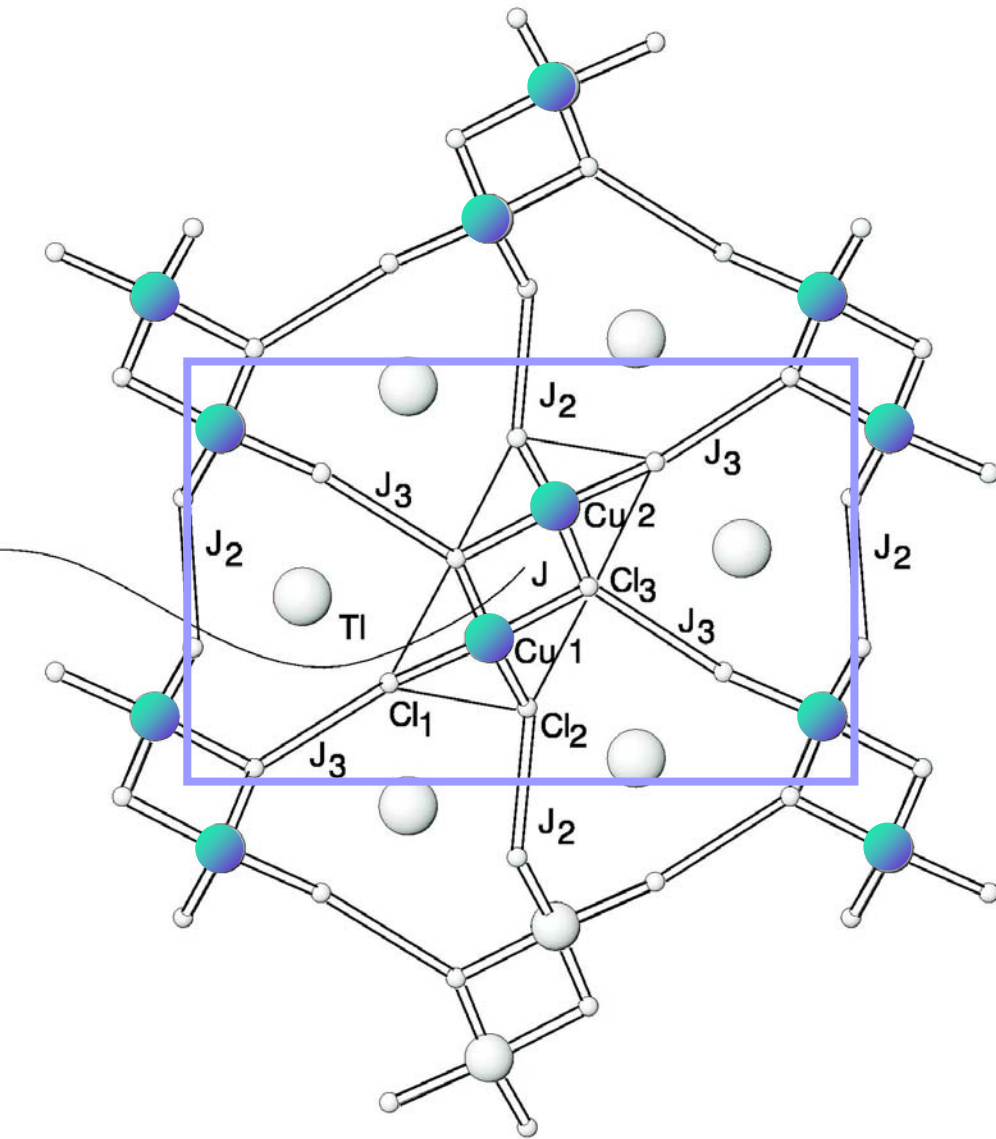
Breakdown of the LGW paradigm

A. Magnetic quantum phase transitions in
“dimerized” Mott insulators:

Landau-Ginzburg-Wilson (LGW) theory:

*Second-order phase transitions described by
fluctuations of an order parameter
associated with a broken symmetry*

TiCuCl₃



Coupled Dimer Antiferromagnet

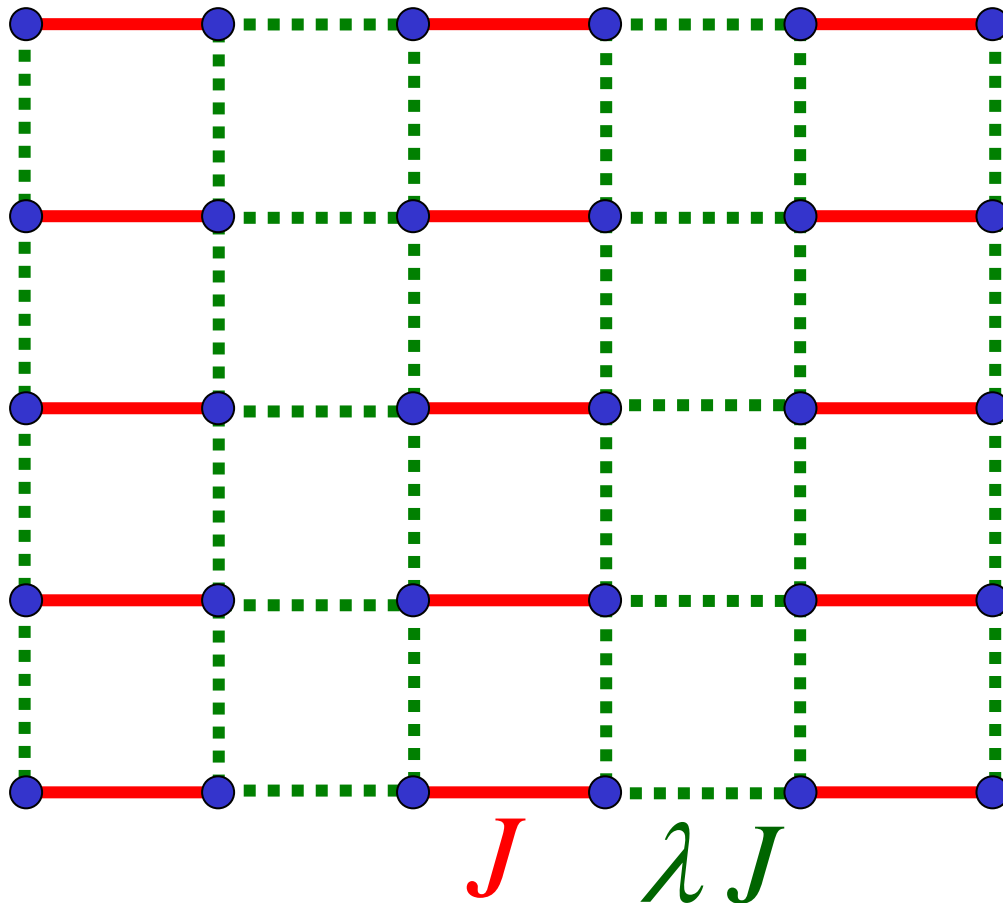
M. P. Gelfand, R. R. P. Singh, and D. A. Huse, *Phys. Rev. B* **40**, 10801-10809 (1989).

N. Katoh and M. Imada, *J. Phys. Soc. Jpn.* **63**, 4529 (1994).

J. Tworzydło, O. Y. Osman, C. N. A. van Duin, J. Zaanen, *Phys. Rev. B* **59**, 115 (1999).

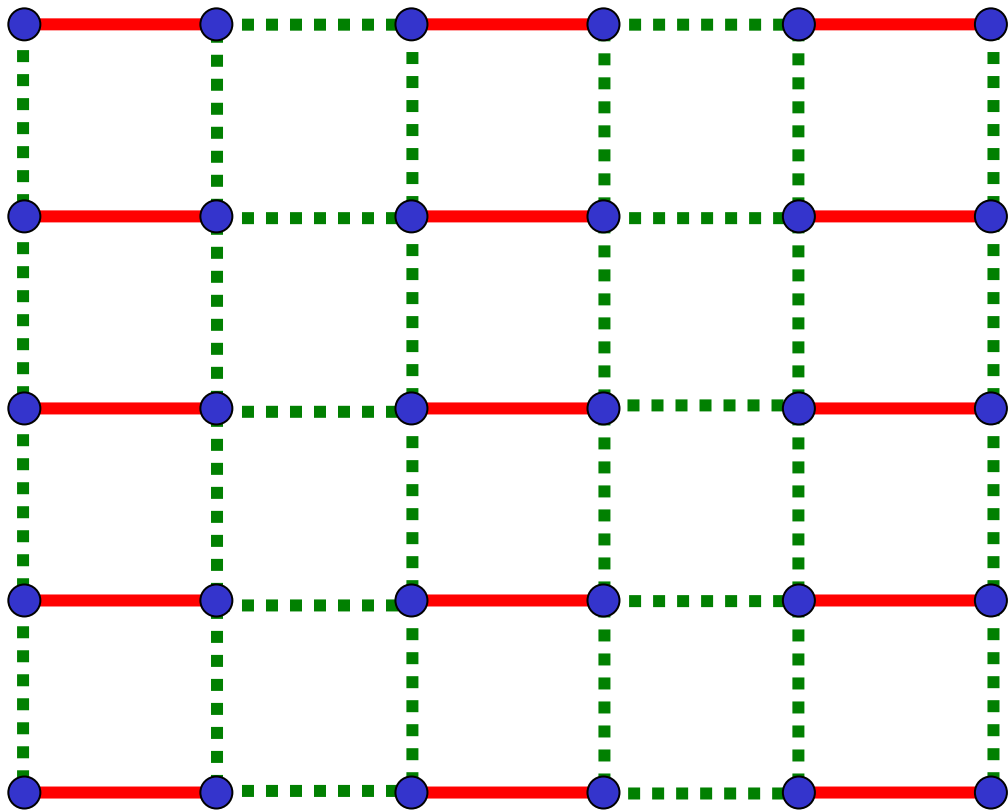
M. Matsumoto, C. Yasuda, S. Todo, and H. Takayama, *Phys. Rev. B* **65**, 014407 (2002).

$S=1/2$ spins on coupled dimers



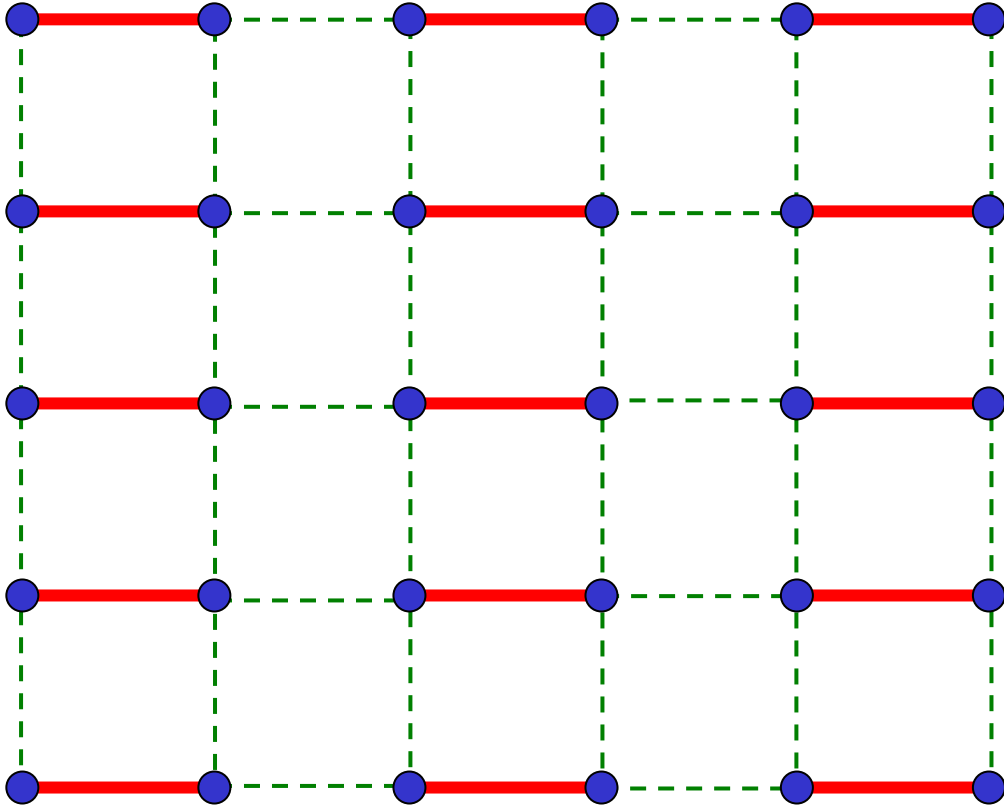
$$H = \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j$$

$$0 \leq \lambda \leq 1$$



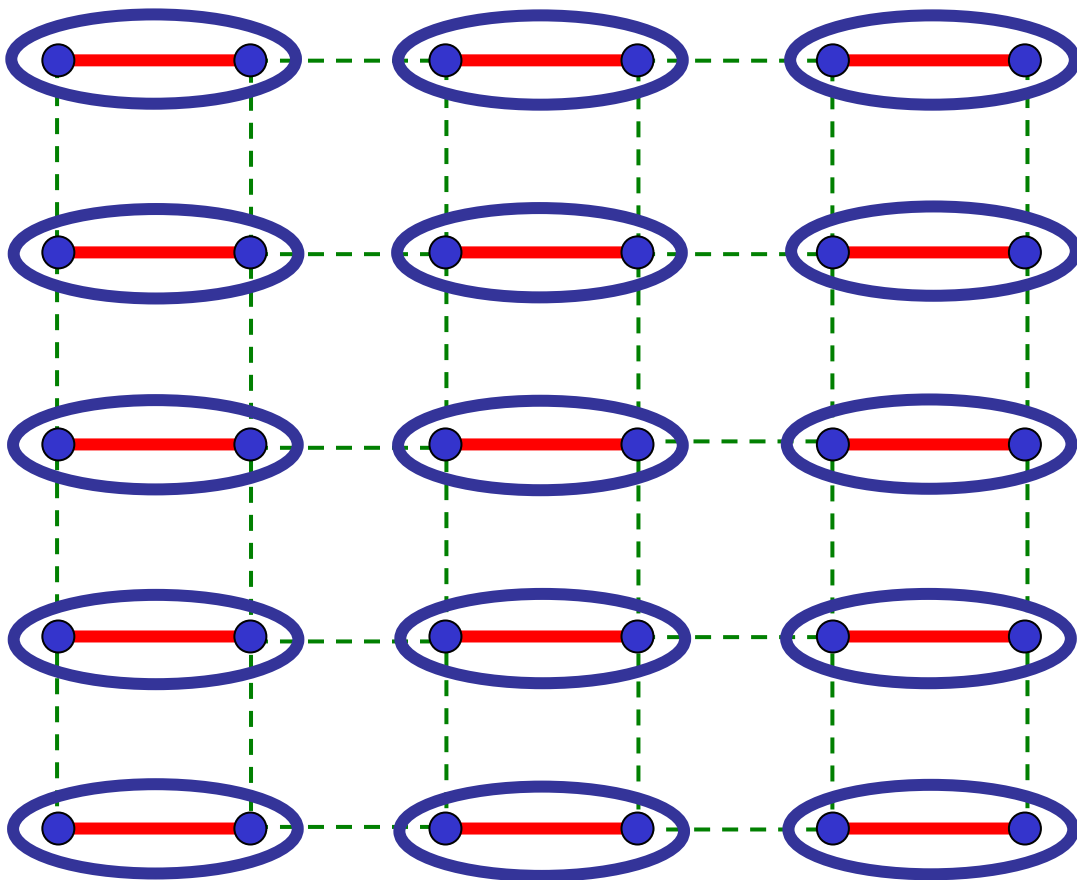
λ close to 0

Weakly coupled dimers



λ close to 0

Weakly coupled dimers



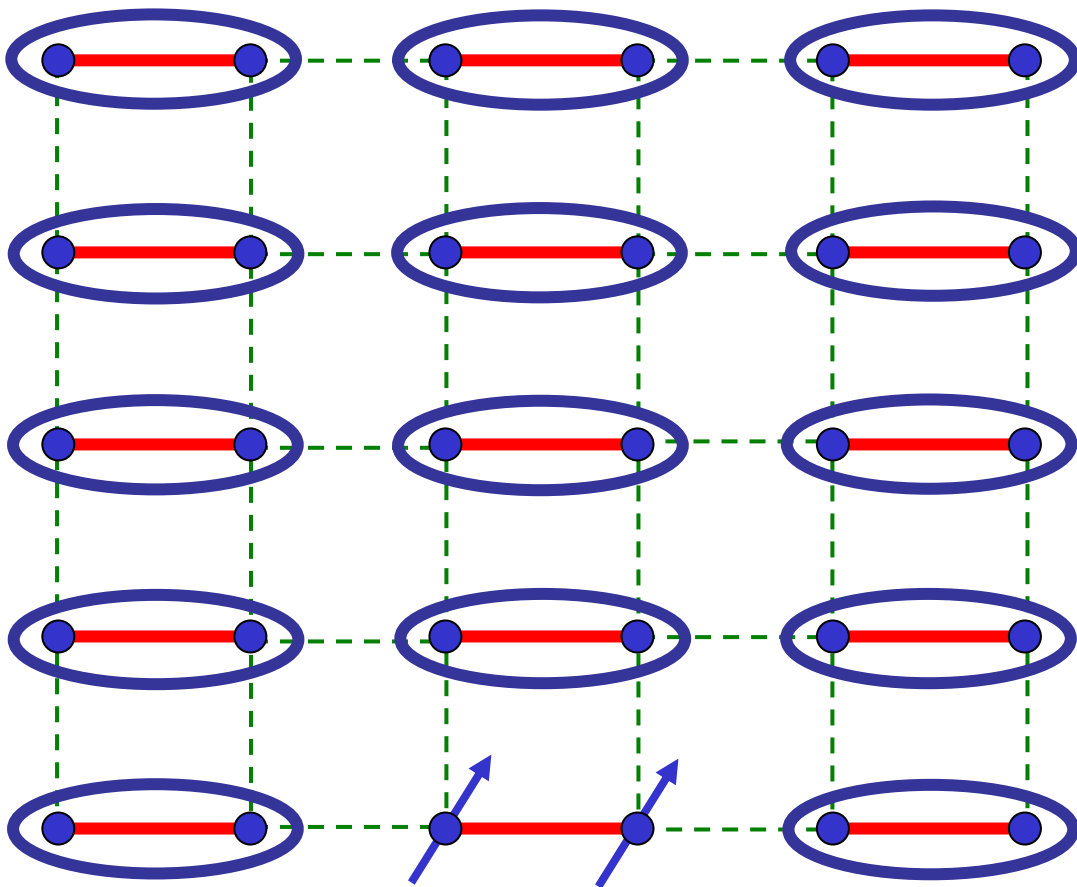
$$\text{dimer} = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

Paramagnetic ground state

$$\langle \vec{S}_i \rangle = 0, \quad \langle \vec{\phi} \rangle = 0$$

λ close to 0

Weakly coupled dimers

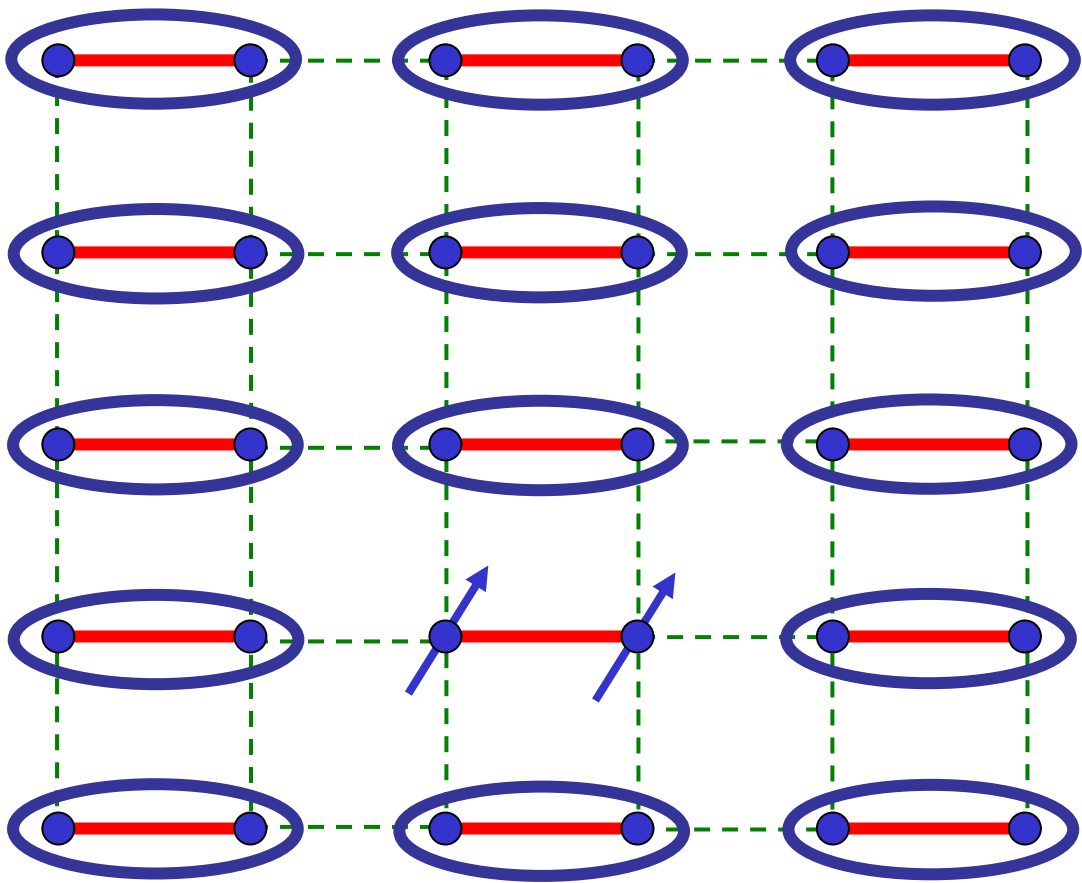


$$\text{dimer} = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

Excitation: $S=1$ *triplon*

λ close to 0

Weakly coupled dimers

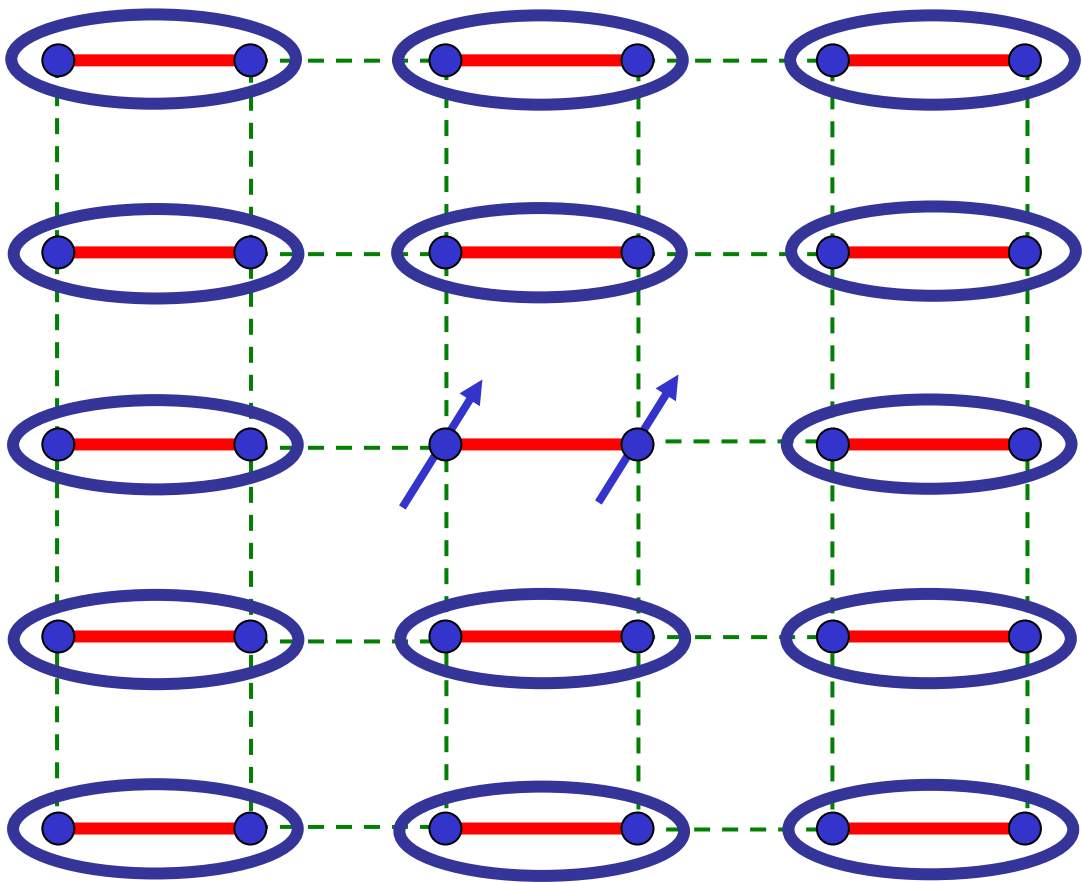


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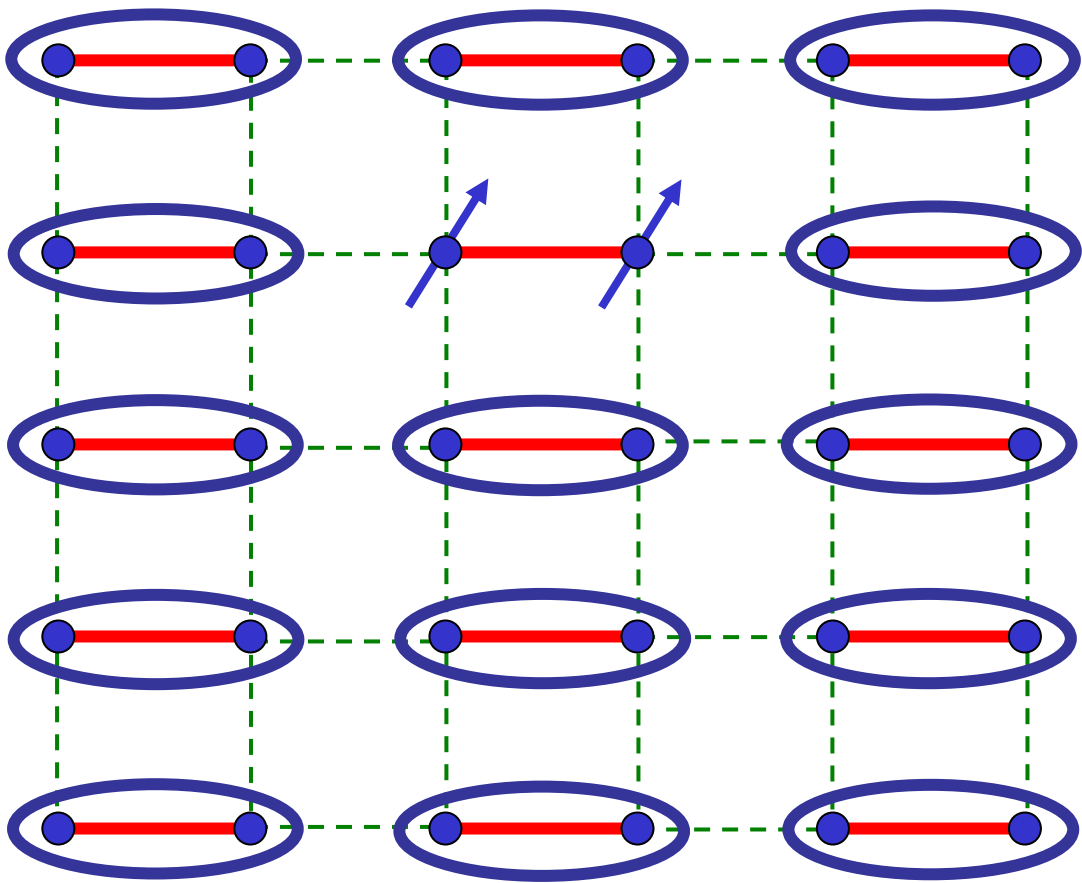


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Weakly coupled dimers

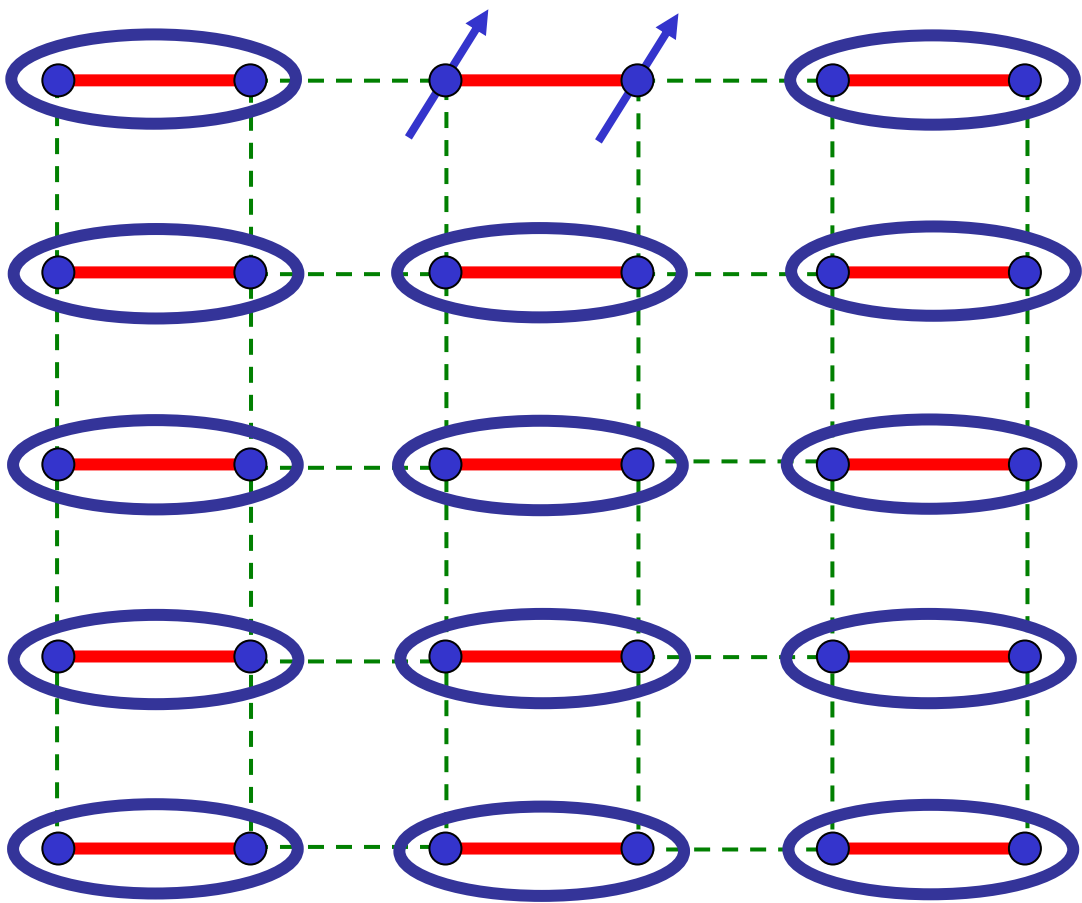


$$\text{dimer} = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

Excitation: $S=1$ *triplon*

λ close to 0

Weakly coupled dimers

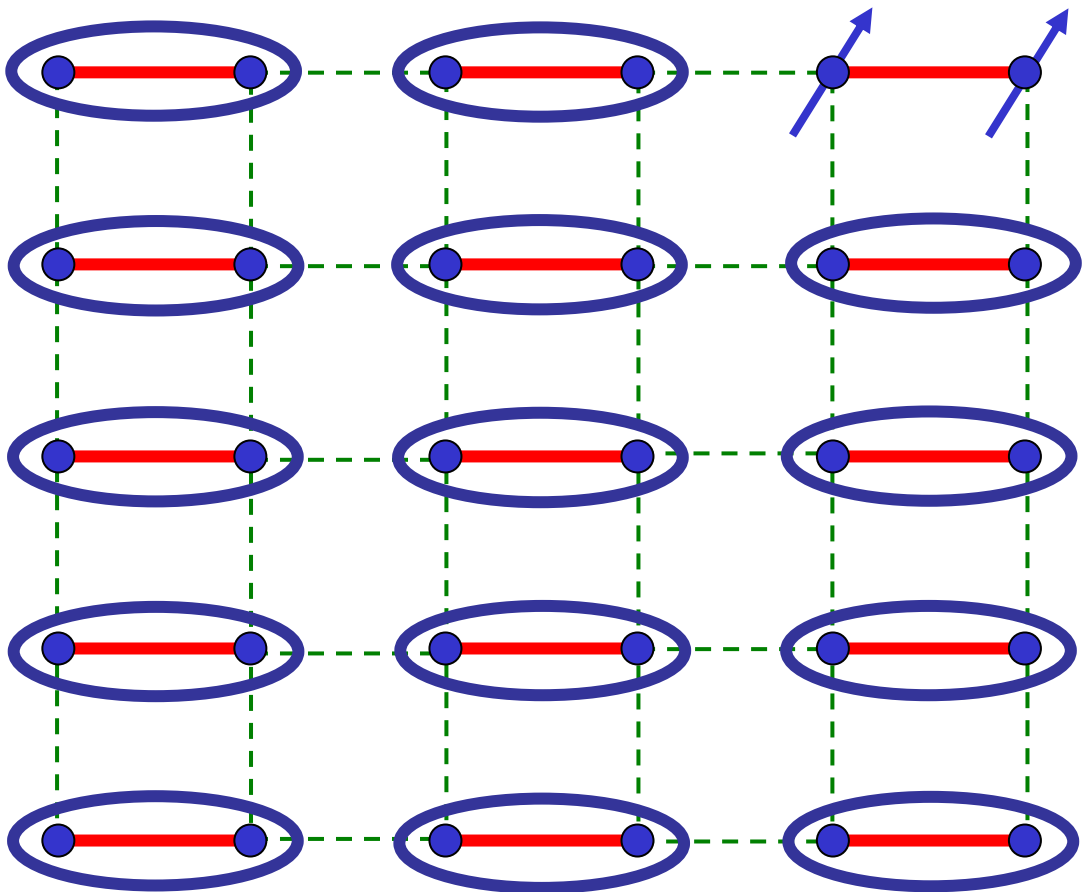


$$\text{Dimer} = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

Excitation: $S=1$ *triplon*

λ close to 0

Weakly coupled dimers



$$\text{dimer} = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

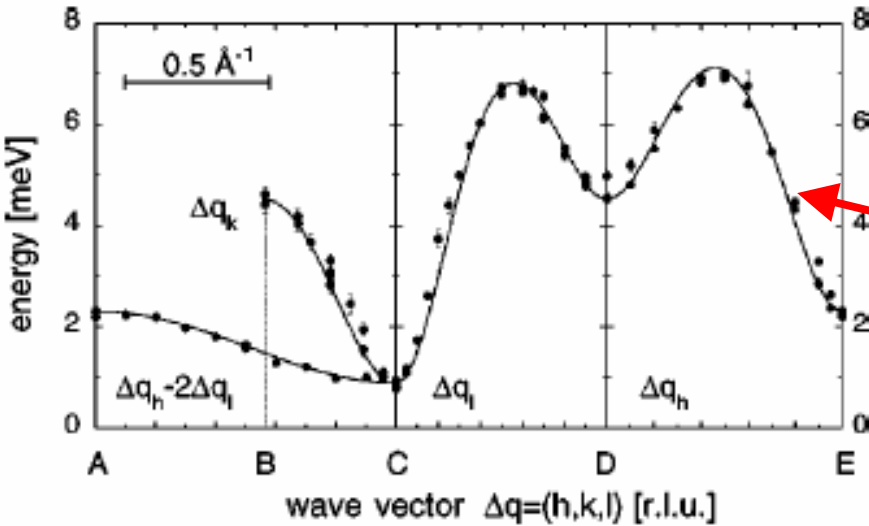
Excitation: $S=1$ *triplon*
(*exciton*, spin collective mode)

Energy dispersion away from
antiferromagnetic wavevector

$$\varepsilon_p = \Delta + \frac{c_x^2 p_x^2 + c_y^2 p_y^2}{2\Delta}$$

$\Delta \rightarrow$ spin gap

TlCuCl₃



N. Cavadini, G. Heigold, W. Henggeler, A. Furrer, H.-U. Güdel, K. Krämer and H. Mutka, *Phys. Rev. B* **63** 172414 (2001).

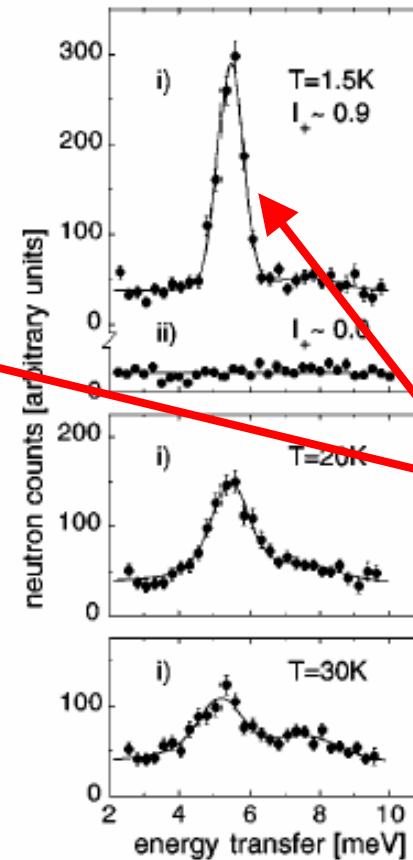


FIG. 1. Measured neutron profiles in the a^*c^* plane of TlCuCl₃ for $i=(1.35,0,0)$, $ii=(0,0,3.15)$ [r.l.u.]. The spectrum at $T=1.5$ K

“triplon”

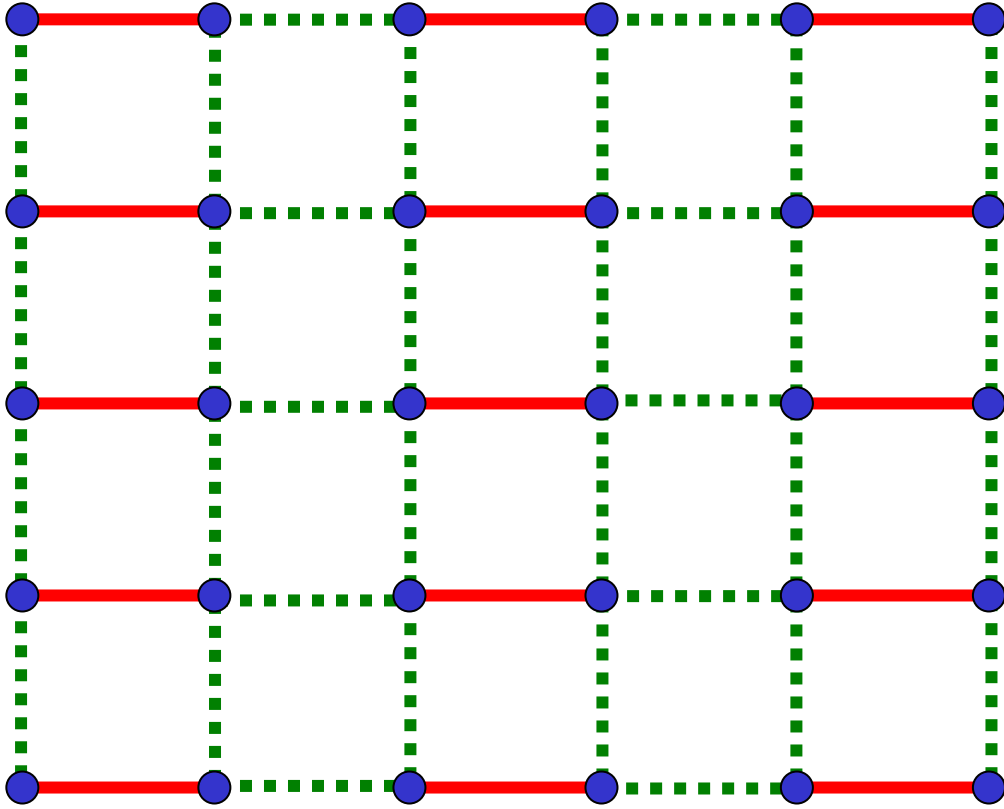
For quasi-one-dimensional systems, the triplon linewidth takes

the exact universal value $= 1.20k_B T e^{-\Delta/k_B T}$ at low T

K. Damle and S. Sachdev, *Phys. Rev. B* **57**, 8307 (1998)

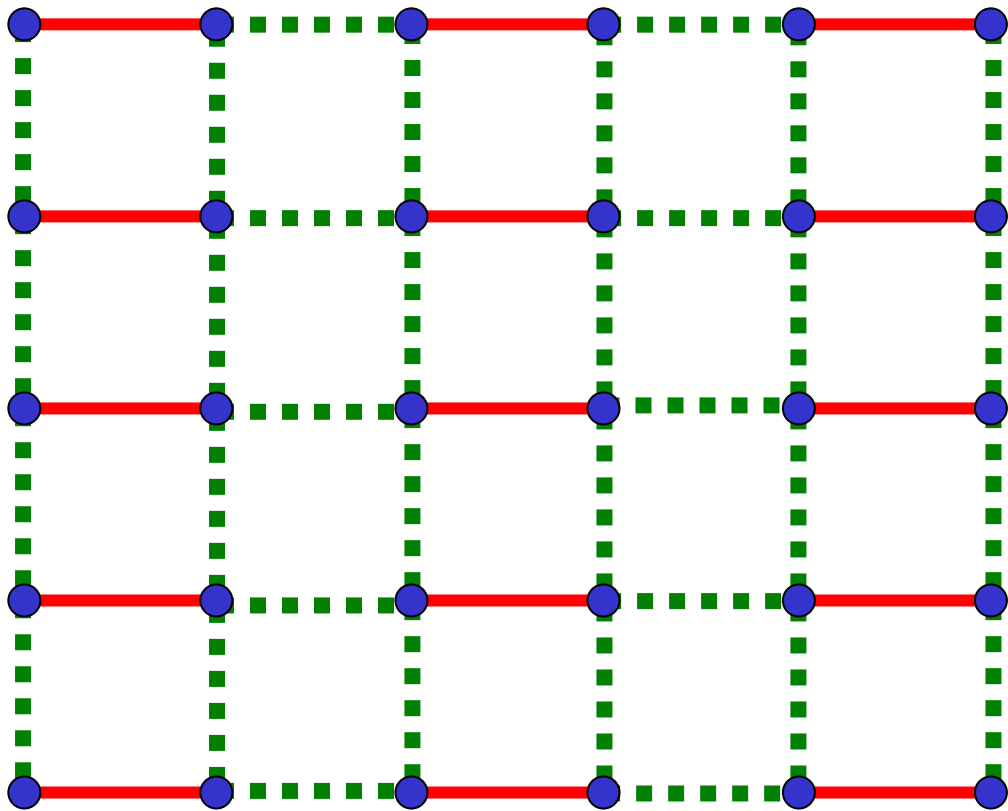
This result is in good agreement with observations in CsNiCl₃ (M. Kenzelmann, R. A. Cowley, W. J. L. Buyers, R. Coldea, M. Enderle, and D. F. McMorrow *Phys. Rev. B* **66**, 174412 (2002)) and Y₂NiBaO₅ (G. Xu, C. Broholm, G. Aeppli, J. F. DiTusa, T. Ito, K. Oka, and H. Takagi, preprint).

Coupled Dimer Antiferromagnet



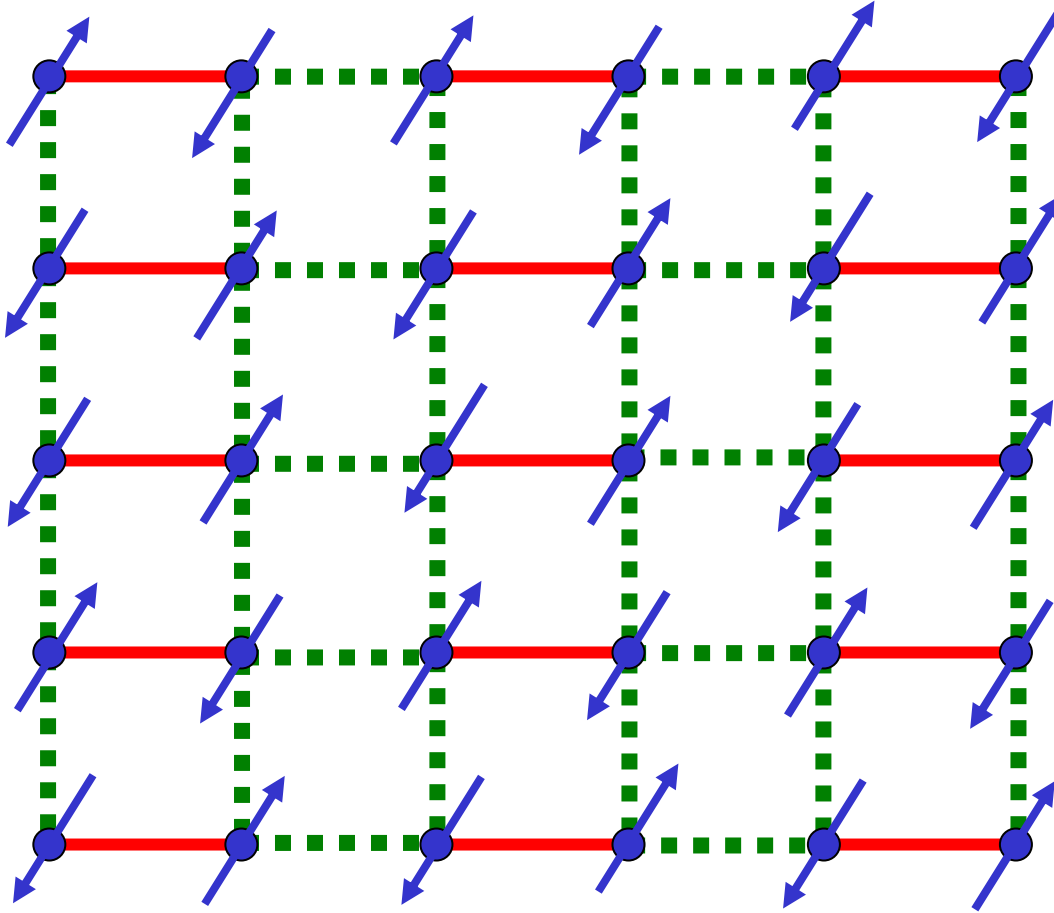
λ close to 1

Weakly dimerized square lattice



λ close to 1

Weakly dimerized square lattice



Excitations:
2 spin waves (*magnons*)

$$\varepsilon_p = \sqrt{c_x^2 p_x^2 + c_y^2 p_y^2}$$

Ground state has long-range spin density wave
(Néel) order at wavevector $\mathbf{K} = (\pi, \pi)$

$$\langle \vec{\phi} \rangle \neq 0$$

spin density wave order parameter: $\vec{\phi} = \eta_i \frac{\vec{S}_i}{S}$; $\eta_i = \pm 1$ on two sublattices



Neutron Diffraction Study of the Pressure-Induced Magnetic Ordering in the Spin Gap System TiCuCl₃

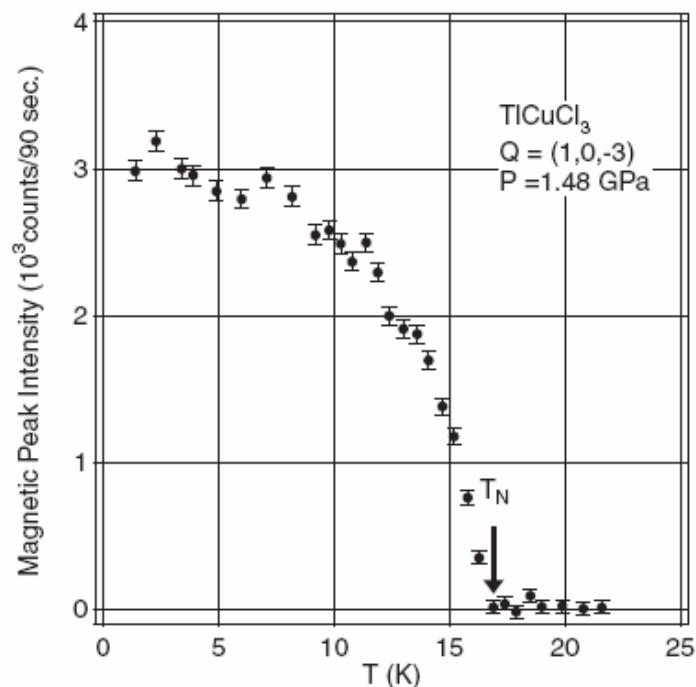
Akira OOSAWA*, Masashi FUJISAWA¹, Toyotaka OSAKABE, Kazuhisa KAKURAI and Hidekazu TANAKA²

Advanced Science Research Center, Japan Atomic Energy Research Institute, Tokai, Ibaraki 319-1195

¹*Department of Physics, Tokyo Institute of Technology, Oh-okayama, Meguro-ku, Tokyo 152-8551*

²*Research Center for Low Temperature Physics, Tokyo Institute of Technology, Oh-okayama, Meguro-ku, Tokyo 152-8551*

(Received February 3, 2003)



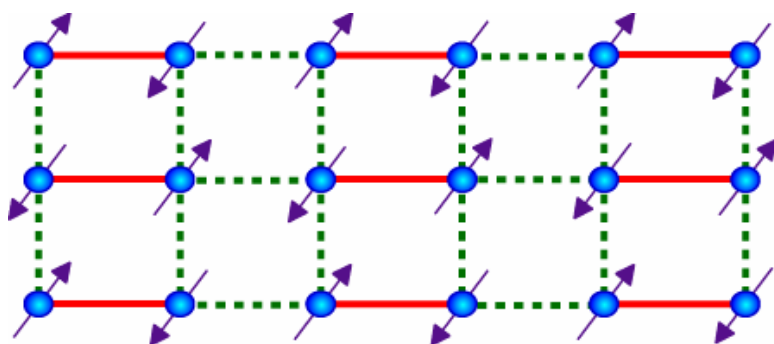
J. Phys. Soc. Jpn **72**, 1026 (2003)

Fig. 3. Temperature dependence of the magnetic Bragg peak intensity for $Q = (1, 0, -3)$ reflection measured at $P = 1.48$ GPa in TiCuCl₃.

$T=0$

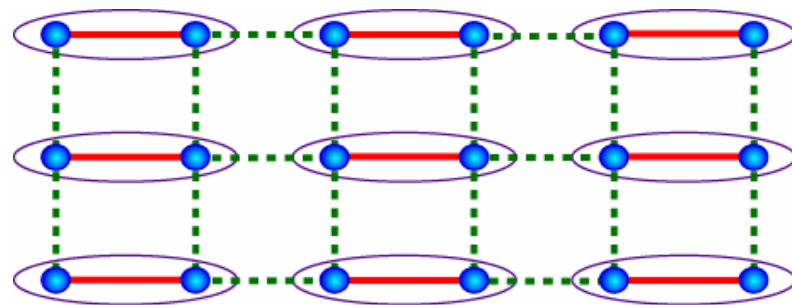
$$\lambda_c = 0.52337(3)$$

M. Matsumoto, C. Yasuda, S. Todo, and H. Takayama,
Phys. Rev. B **65**, 014407 (2002)



Néel state

$$\langle \vec{\phi} \rangle \neq 0$$



Quantum paramagnet

$$\langle \vec{\phi} \rangle = 0$$



The method of bond operators (S. Sachdev and R.N. Bhatt, *Phys. Rev. B* **41**, 9323 (1990)) provides a quantitative description of spin excitations in TlCuCl_3 across the quantum phase transition (M. Matsumoto, B. Normand, T.M. Rice, and M. Sigrist, *Phys. Rev. Lett.* **89**, 077203 (2002))

LGW theory for quantum criticality

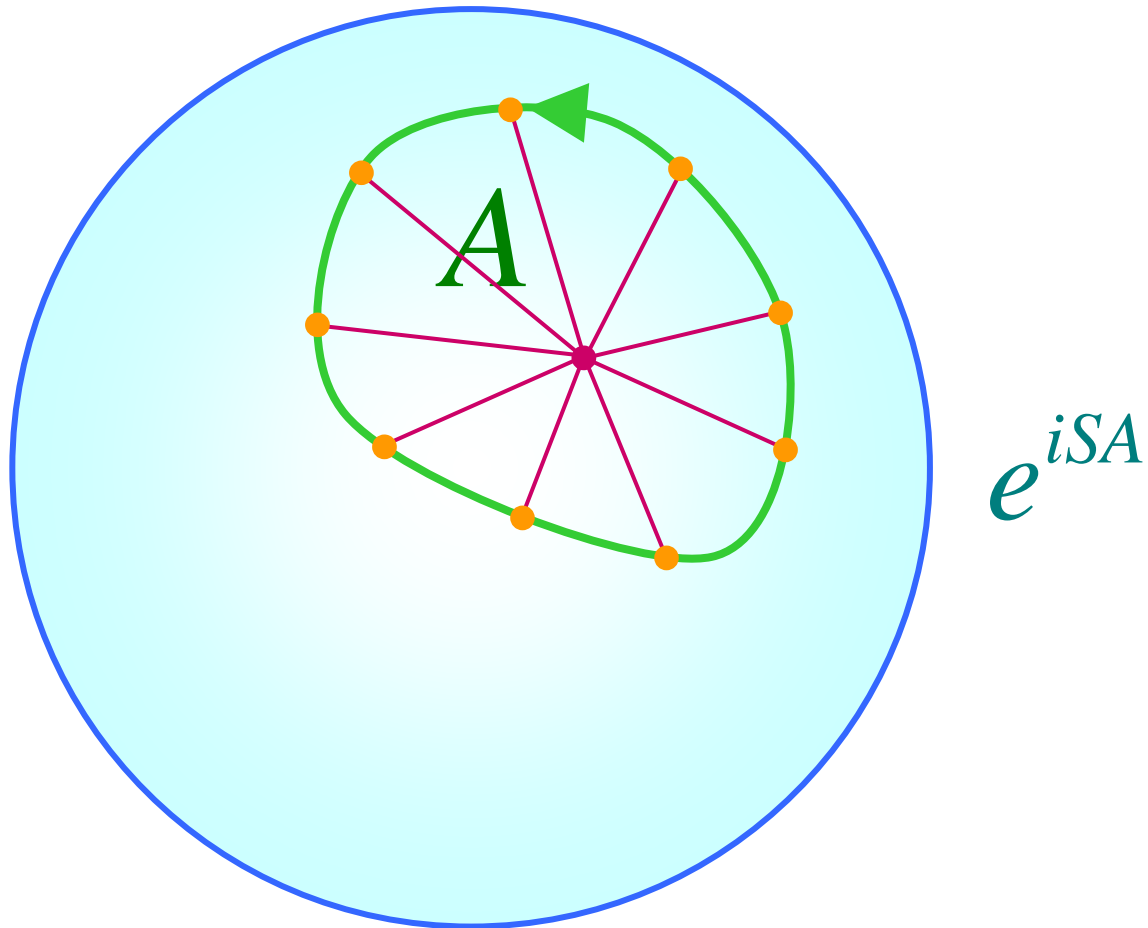
Landau-Ginzburg-Wilson theory: write down an effective action for the antiferromagnetic order parameter $\vec{\phi}$ by expanding in powers of $\vec{\phi}$ and its spatial and temporal derivatives, while preserving all symmetries of the microscopic Hamiltonian

$$S_{\phi} = \int d^2x d\tau \left[\frac{1}{2} \left((\nabla_x \vec{\phi})^2 + \frac{1}{c^2} (\partial_{\tau} \vec{\phi})^2 + (\lambda_c - \lambda) \vec{\phi}^2 \right) + \frac{u}{4!} (\vec{\phi}^2)^2 \right]$$

Coherent state path integral

Path integral for quantum spin fluctuations

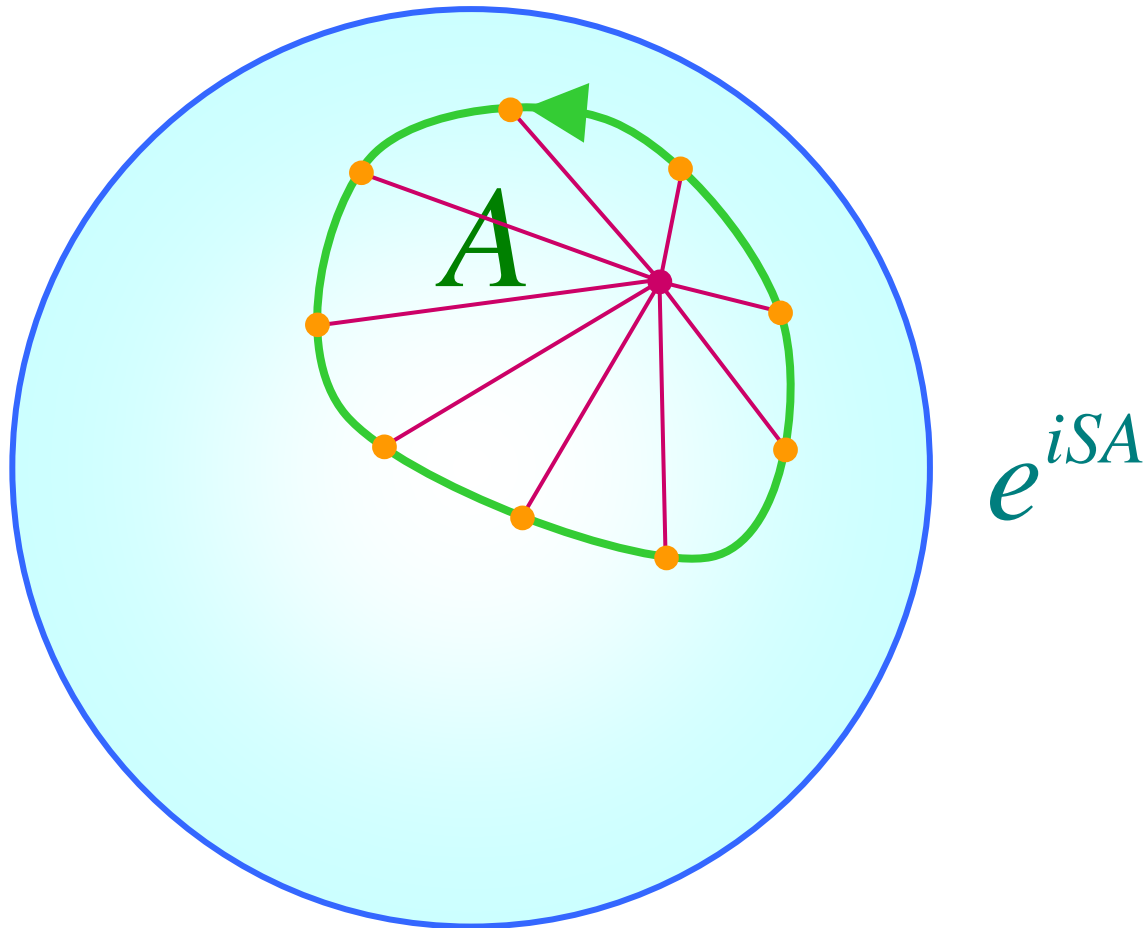
Key ingredient: Spin Berry Phases



Coherent state path integral

Path integral for quantum spin fluctuations

Key ingredient: Spin Berry Phases



Coherent state path integral

See Chapter 13 of *Quantum Phase Transitions*, S. Sachdev, Cambridge University Press (1999).

Path integral for a single spin

$$Z = \text{Tr} \left(e^{-H[S]/T} \right)$$

$$= \int \mathcal{D}N(\tau) \delta(N^2 - 1) \exp \left(-iS \int A_\tau(\tau) d\tau - \int d\tau H[SN(\tau)] \right)$$

$A_\tau(\tau) d\tau$ = Oriented area of triangle on surface of unit sphere bounded by $N(\tau)$, $N(\tau + d\tau)$, and a fixed reference N_0

Action for lattice antiferromagnet

$$N_j(\tau) = \eta_j \mathbf{n}(x_j, \tau) + \mathbf{L}(x_j, \tau)$$

$\eta_j = \pm 1$ identifies sublattices

\mathbf{n} and \mathbf{L} vary slowly in space and time

Integrate out \mathbf{L} and take the continuum limit

$$Z = \int \mathcal{D}\mathbf{n}(x, \tau) \delta(\mathbf{n}^2 - 1) \exp \left(-iS \sum_j \int \eta_j A_\tau(x_j, \tau) d\tau - \frac{1}{2g} \int d^2x d\tau \left((\partial_\tau \mathbf{n})^2 + c^2 (\nabla_x \mathbf{n})^2 \right) \right)$$

$\eta_j = \pm 1$ identifies sublattices

$$\begin{aligned} g < g_c &\Leftrightarrow \lambda > \lambda_c \\ g > g_c &\Leftrightarrow \lambda < \lambda_c \end{aligned}$$

Discretize spacetime into a cubic lattice

$$Z = \prod_a \int d\mathbf{n}_a \delta(\mathbf{n}_a^2 - 1) \exp \left(\frac{1}{g} \sum_{a, \mu} \mathbf{n}_a \cdot \mathbf{n}_{a+\mu} - \frac{i}{2} \sum_a \eta_a A_{a\tau} \right)$$

$a \rightarrow$ cubic lattice sites;

$\mu \rightarrow x, y, \tau;$

$A_{a\mu} \rightarrow$ oriented area of spherical triangle

formed by \mathbf{n}_a , $\mathbf{n}_{a+\mu}$, and an arbitrary reference point \mathbf{n}_0

Integrate out \mathbf{L} and take the continuum limit

$$Z = \int \mathcal{D}\mathbf{n}(x, \tau) \delta(\mathbf{n}^2 - 1) \exp \left(-iS \sum_j \int \eta_j A_\tau(x_j, \tau) d\tau - \frac{1}{2g} \int d^2x d\tau \left((\partial_\tau \mathbf{n})^2 + c^2 (\nabla_x \mathbf{n})^2 \right) \right)$$

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$a \rightarrow$ cubic lattice sites;
 $\mu \rightarrow x, y, \tau;$

Berry phases can be neglected for coupled dimer antiferromagnet

(justified later)

Quantum path integral for two-dimensional quantum antiferromagnet

\Leftrightarrow Partition function of a classical three-dimensional ferromagnet at a “temperature” g

Quantum transition at $\lambda = \lambda_c$ is related to classical Curie transition at $g = g_c$

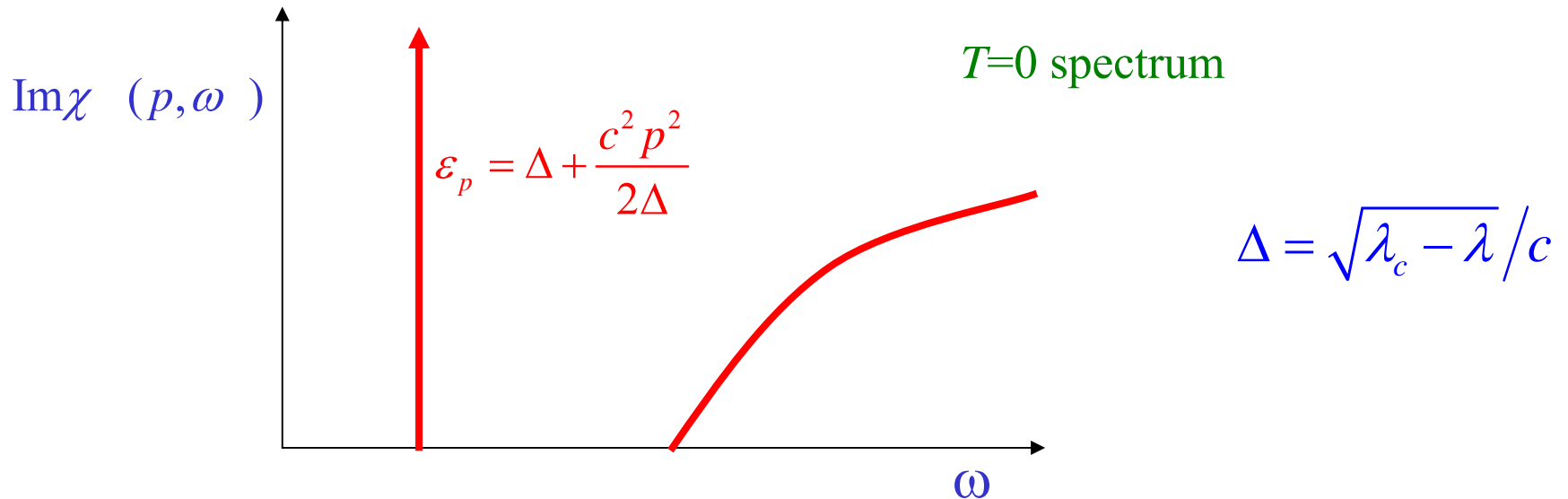
Quantum field theory for critical point

λ close to λ_c : use “soft spin” field

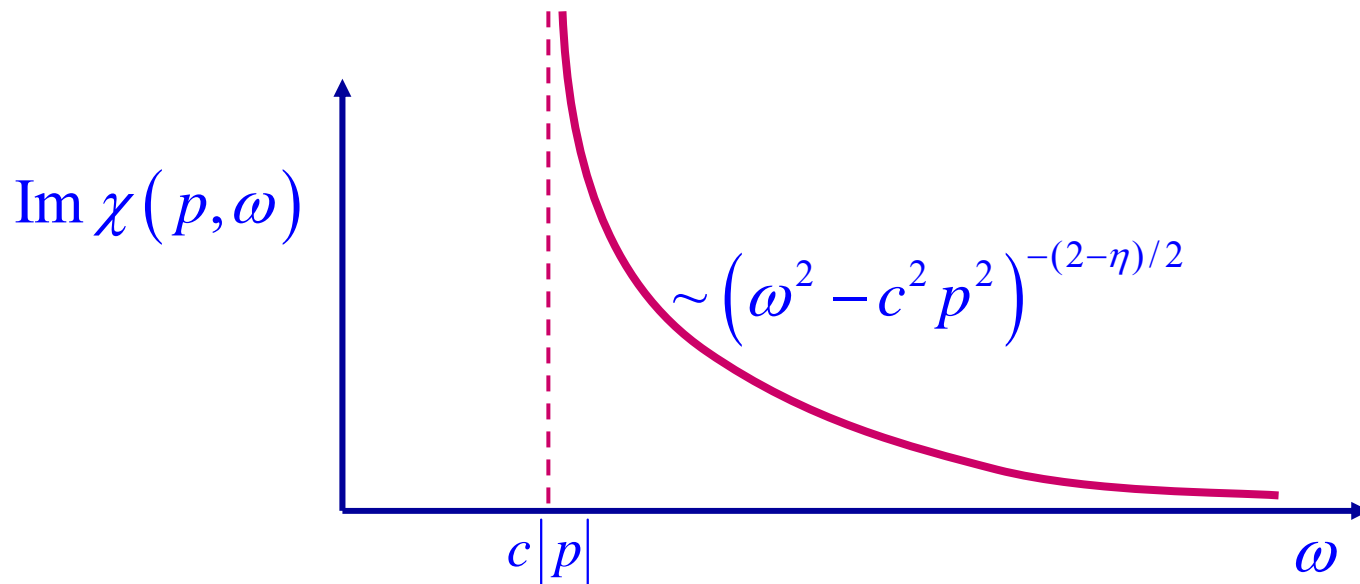
$$\mathcal{S}_b = \int d^2x d\tau \left[\frac{1}{2} \left((\nabla_x \phi_\alpha)^2 + c^2 (\partial_\tau \phi_\alpha)^2 + (\lambda_c - \lambda) \phi_\alpha^2 \right) + \frac{u}{4!} (\phi_\alpha^2)^2 \right]$$

$\phi_\alpha \longrightarrow$ 3-component antiferromagnetic order parameter

Oscillations of ϕ_α about zero (for $\lambda < \lambda_c$)
 \longrightarrow spin-1 collective mode



Dynamic spectrum at the critical point



No quasiparticles --- dissipative critical continuum

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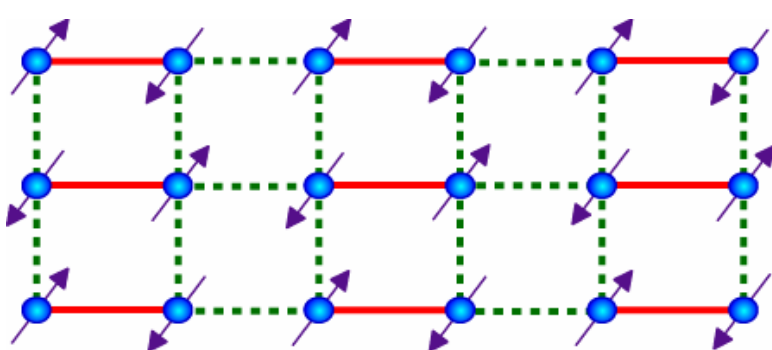
B. Mott insulators with spin $S=1/2$ per unit cell:

*1. Valence-bond-solid order in the
paramagnet.*

Recall: dimerized Mott insulators

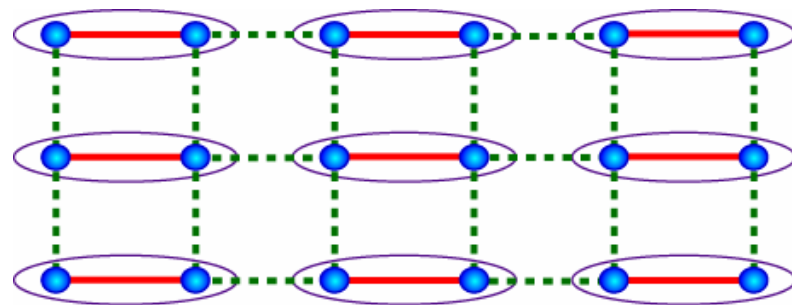
$$\lambda_c = 0.52337(3)$$

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Phys. Rev. B **65**, 014407 (2002)



Néel state

$$\langle \vec{\phi} \rangle \neq 0$$

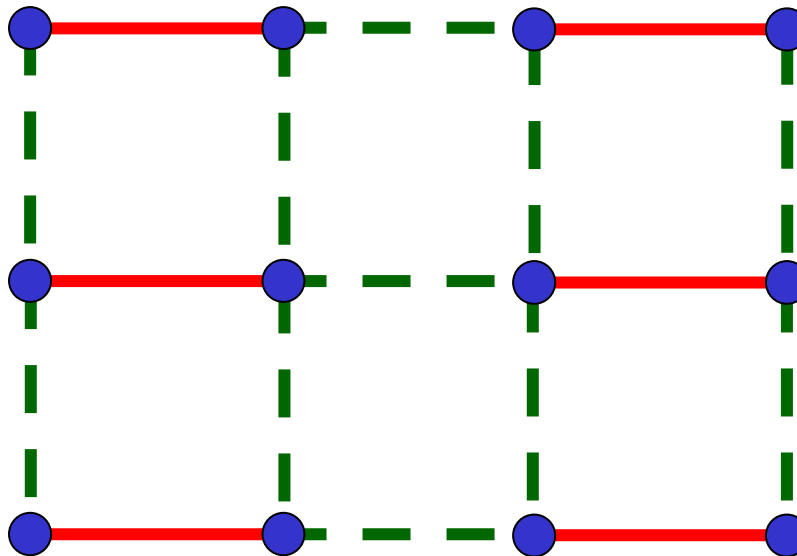


Quantum paramagnet

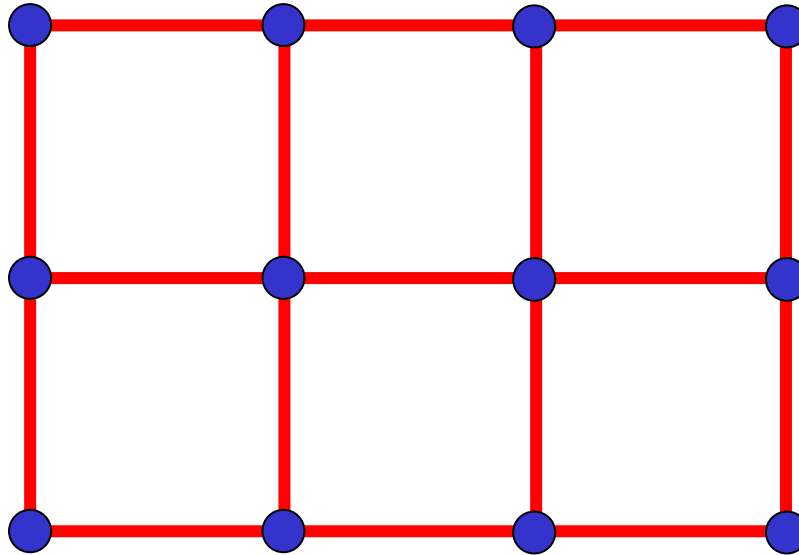
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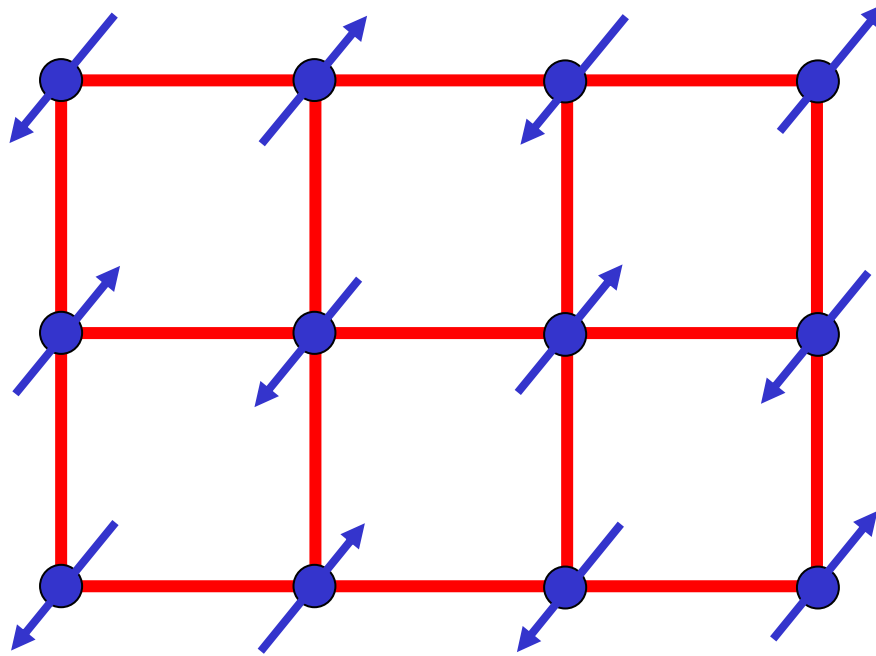
Mott insulator with two $S=1/2$ spins per unit cell



Mott insulator with one $S=1/2$ spin per unit cell

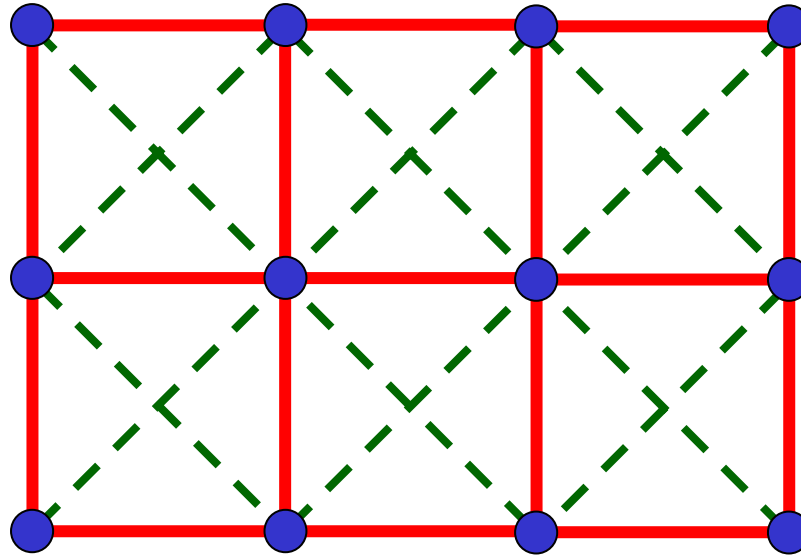


Mott insulator with one $S=1/2$ spin per unit cell



Ground state has Neel order with $\vec{\phi} \neq 0$

Mott insulator with one $S=1/2$ spin per unit cell



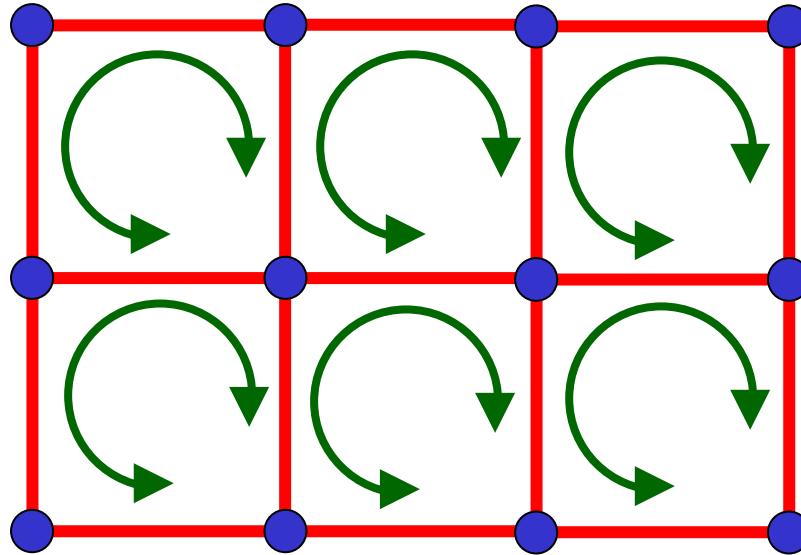
Destroy Neel order by perturbations which preserve full square lattice symmetry *e.g.* second-neighbor or ring exchange.

The strength of this perturbation is measured by a coupling g .

Small $g \Rightarrow$ ground state has Neel order with $\langle \vec{\phi} \rangle \neq 0$

Large $g \Rightarrow$ paramagnetic ground state with $\langle \vec{\phi} \rangle = 0$

Mott insulator with one $S=1/2$ spin per unit cell



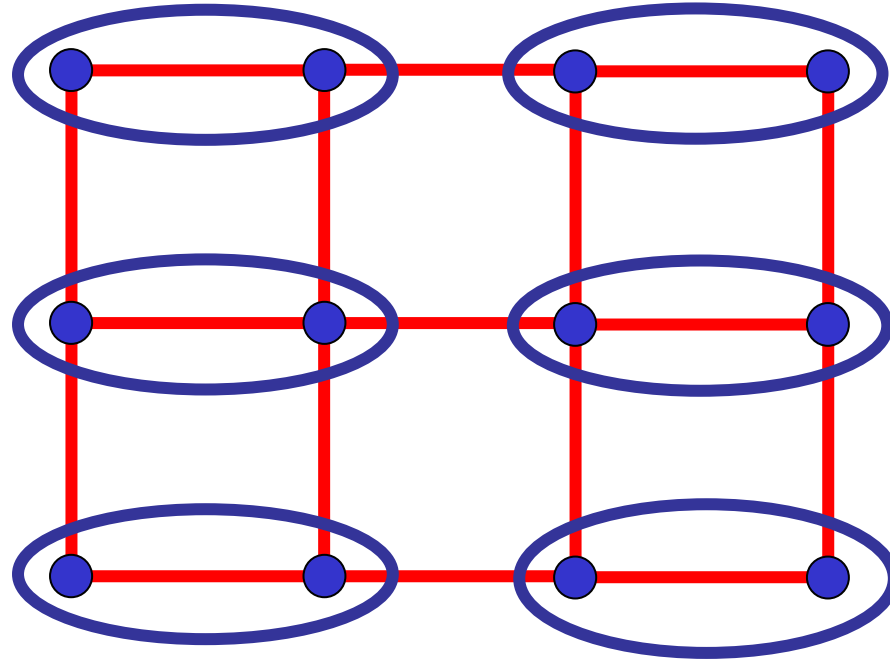
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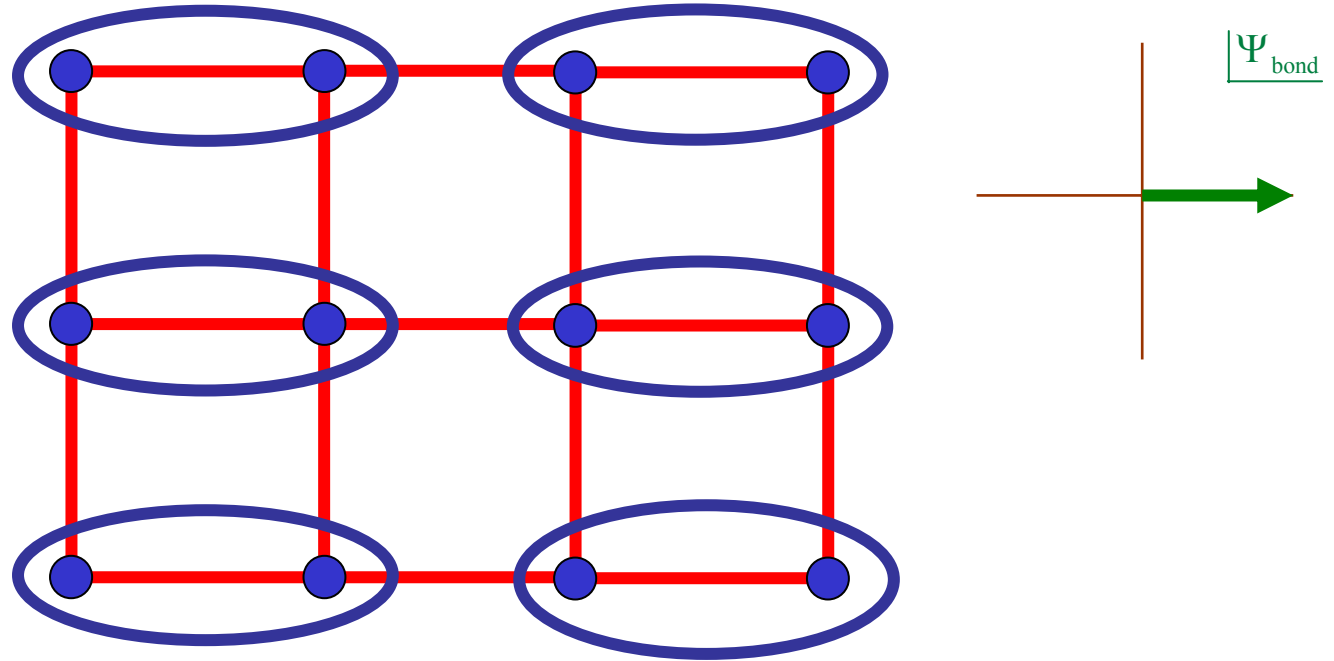
Large $g \Rightarrow$ paramagnetic ground state with $\langle \vec{\phi} \rangle = 0$

Mott insulator with one $S=1/2$ spin per unit cell



Possible large g paramagnetic ground state with $\langle \vec{\phi} \rangle = 0$

Mott insulator with one $S=1/2$ spin per unit cell

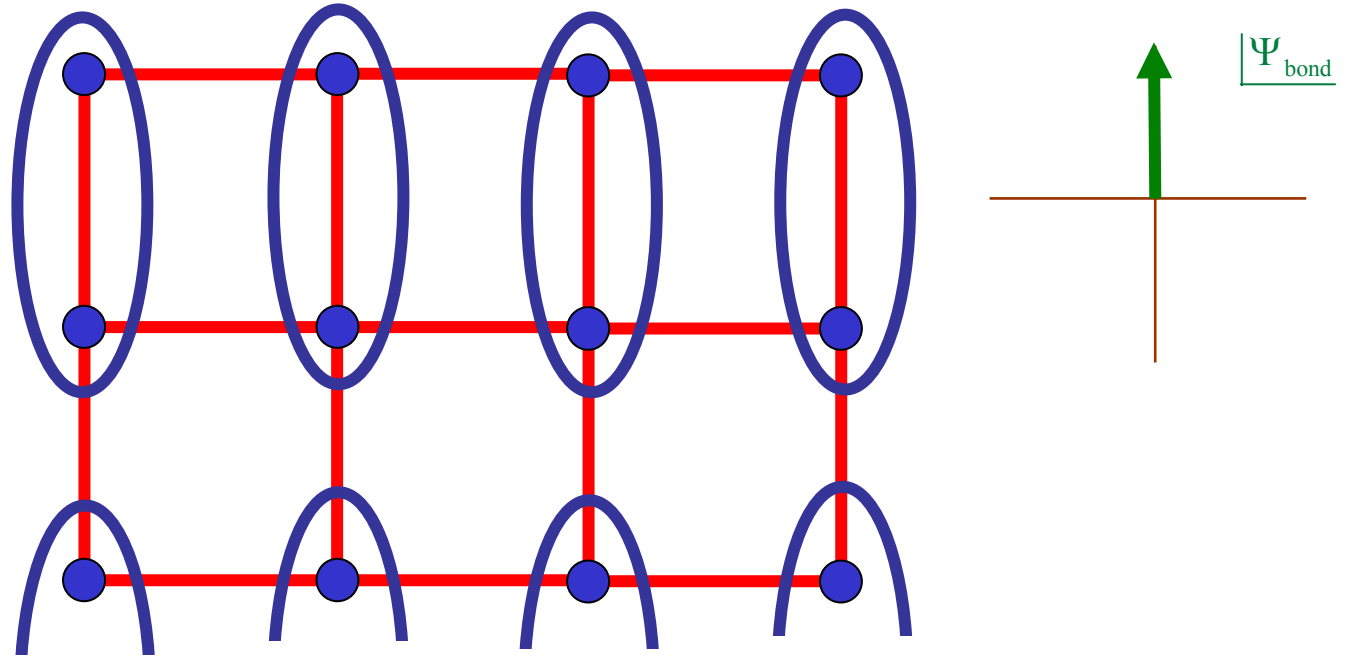


Possible large g paramagnetic ground state with $\langle \vec{\phi} \rangle = 0$

Such a state breaks the symmetry of rotations by $n\pi/2$ about lattice sites,
and has $\langle \Psi_{\text{bond}} \rangle \neq 0$, where Ψ_{bond} is the *bond order parameter*

$$\Psi_{\text{bond}}(i) = \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j e^{i \arctan(\mathbf{r}_j - \mathbf{r}_i)}$$

Mott insulator with one $S=1/2$ spin per unit cell

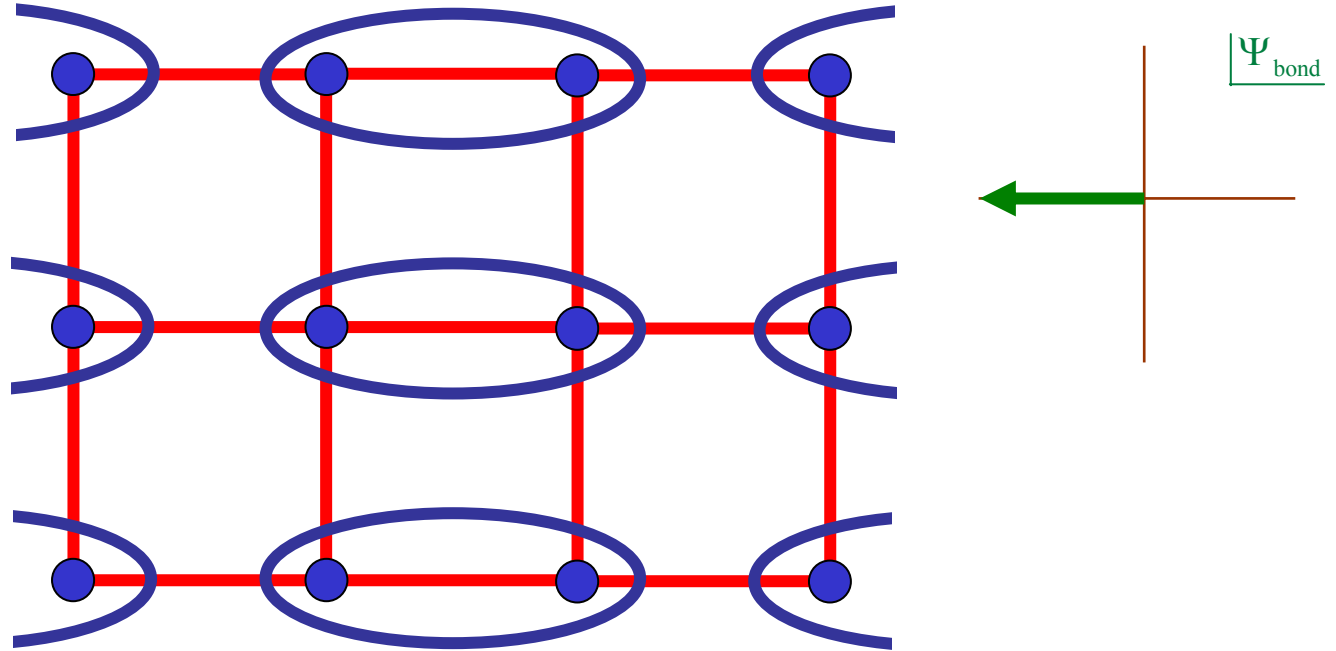


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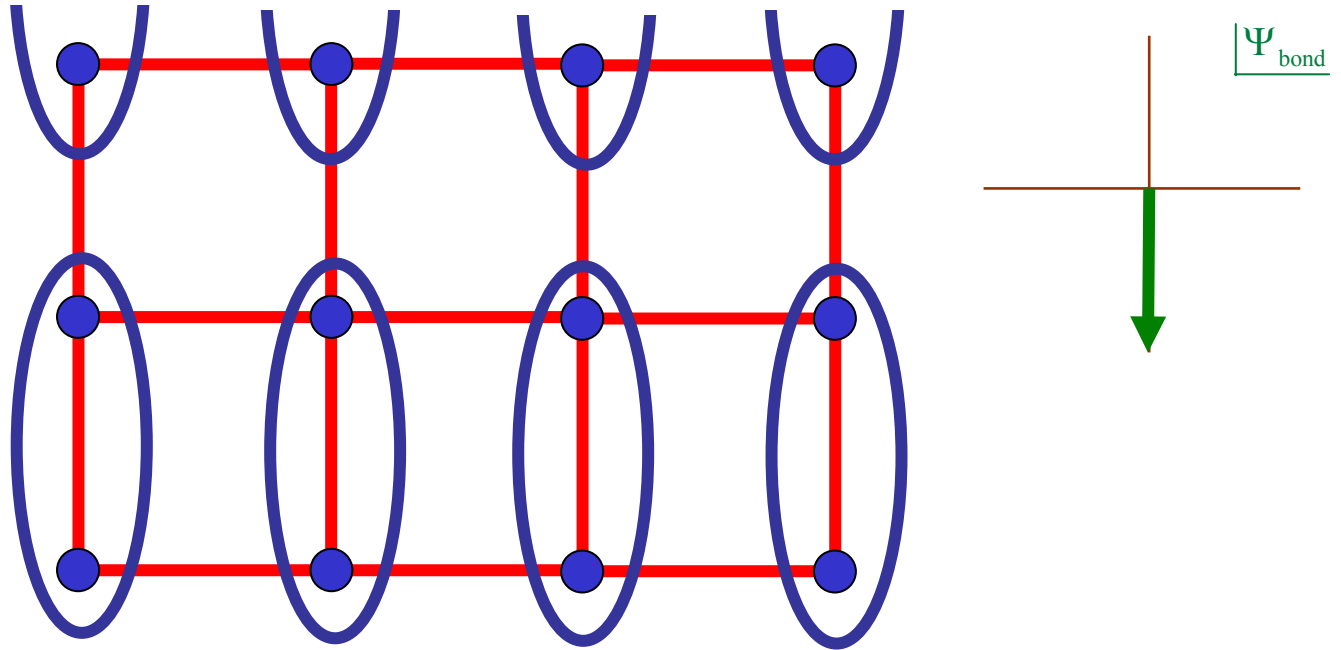


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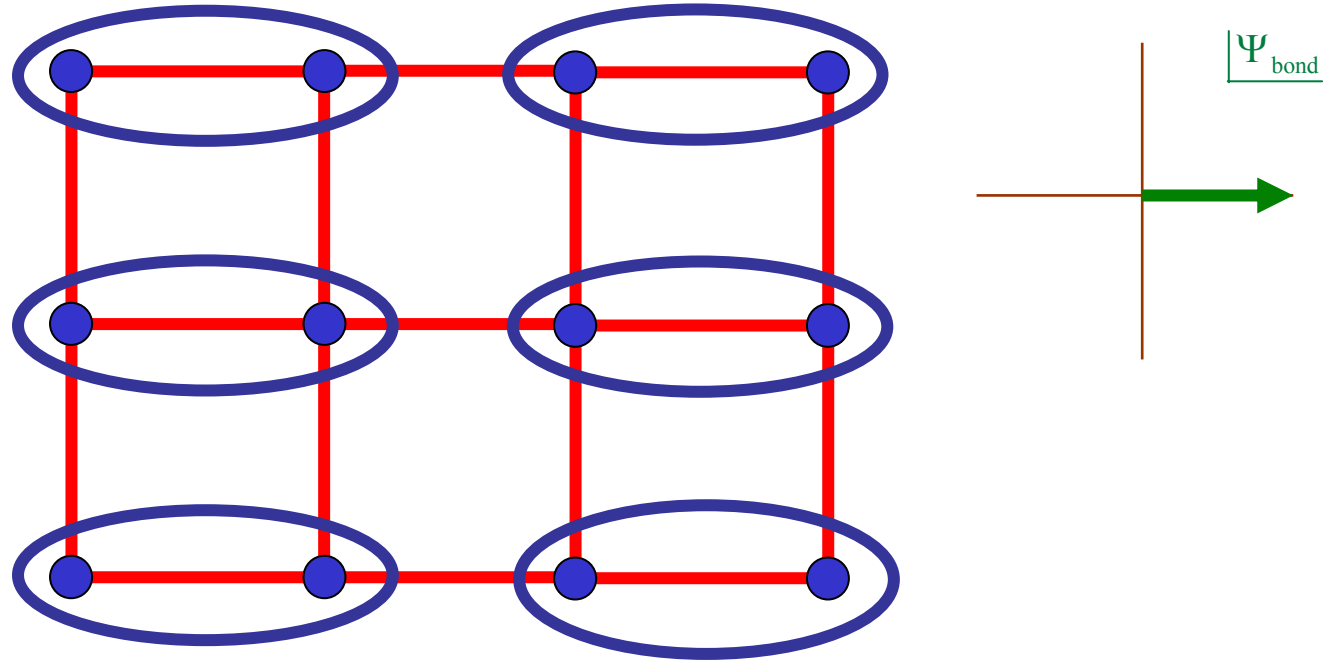


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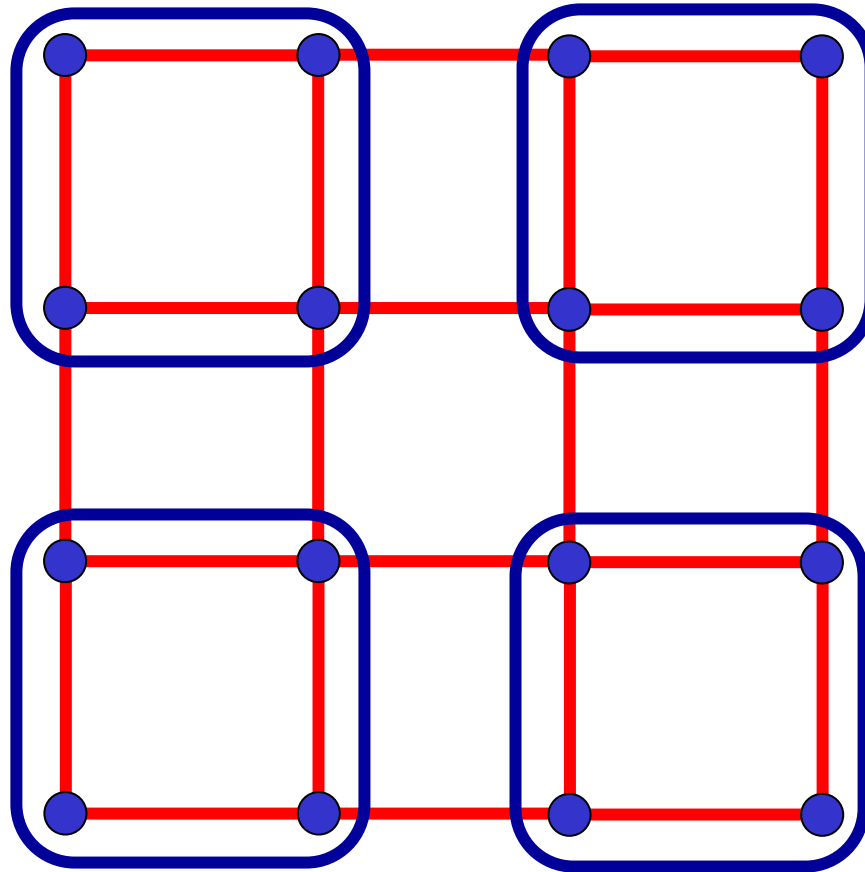


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Mott insulator with one $S=1/2$ spin per unit cell

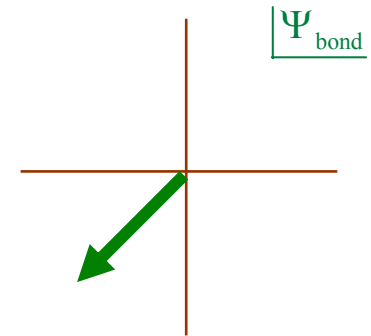
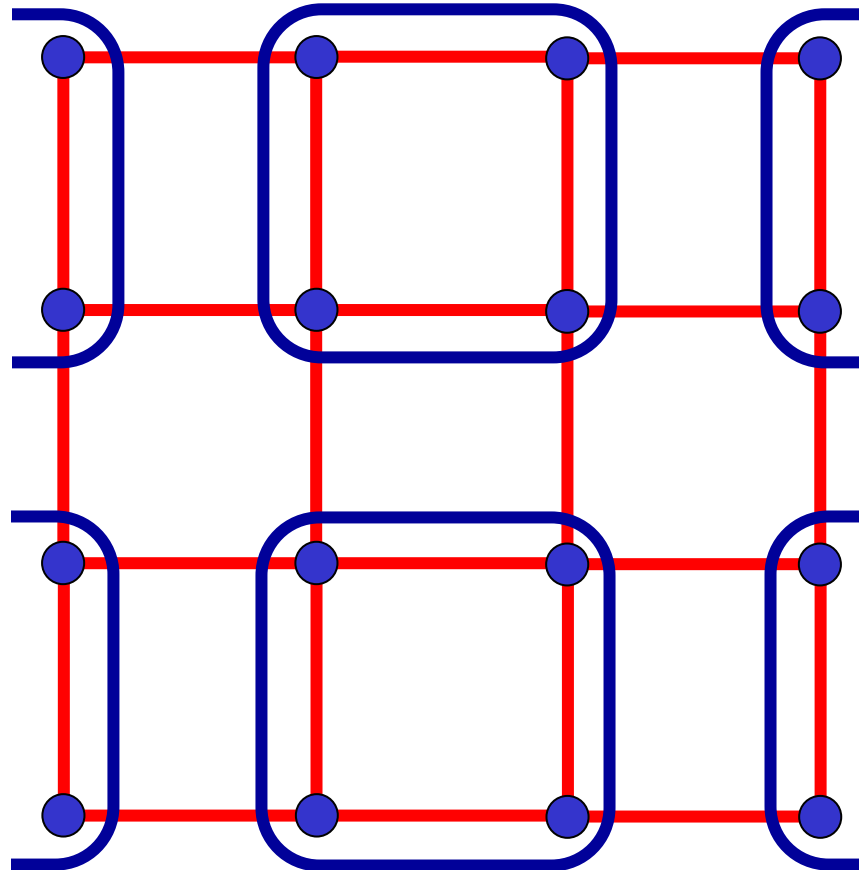


Possible large g paramagnetic ground state with $\langle \vec{\phi} \rangle = 0$

Another state breaking the symmetry of rotations by $n\pi/2$ about lattice sites, which also has $\langle \Psi_{\text{bond}} \rangle \neq 0$, where Ψ_{bond} is the *bond order parameter*

$$\Psi_{\text{bond}}(i) = \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j e^{i \arctan(\mathbf{r}_j - \mathbf{r}_i)}$$

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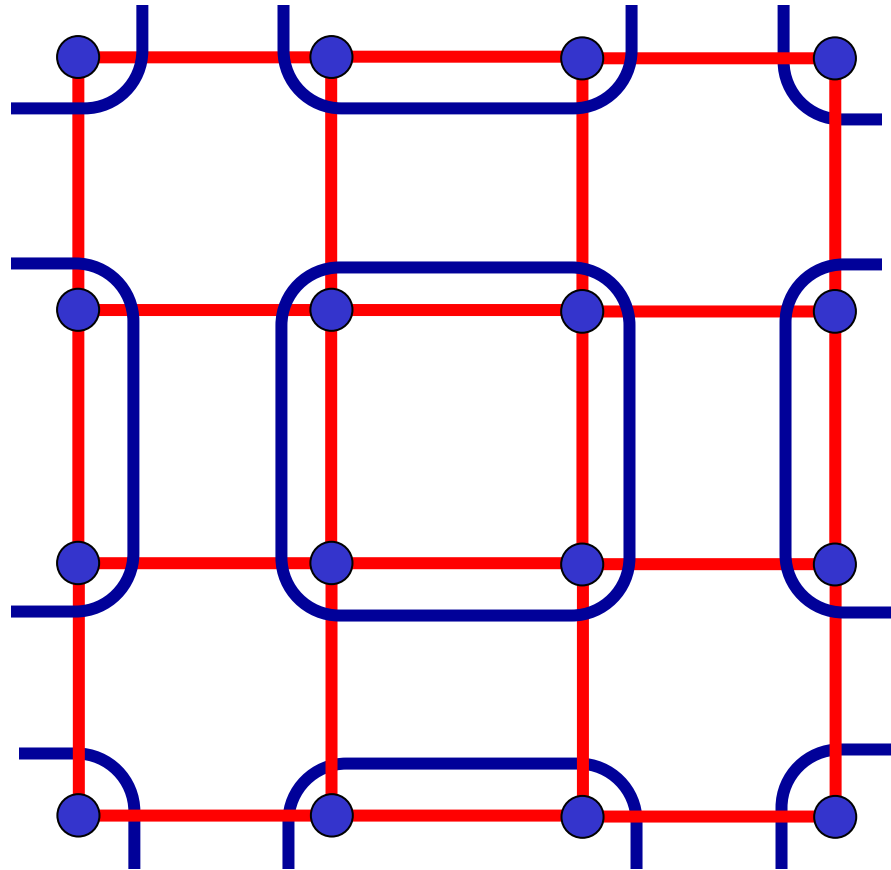


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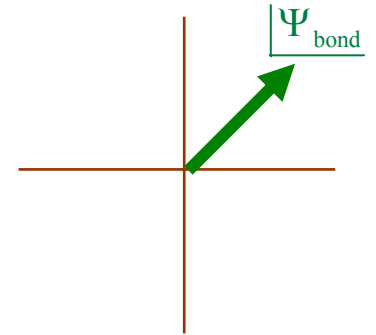
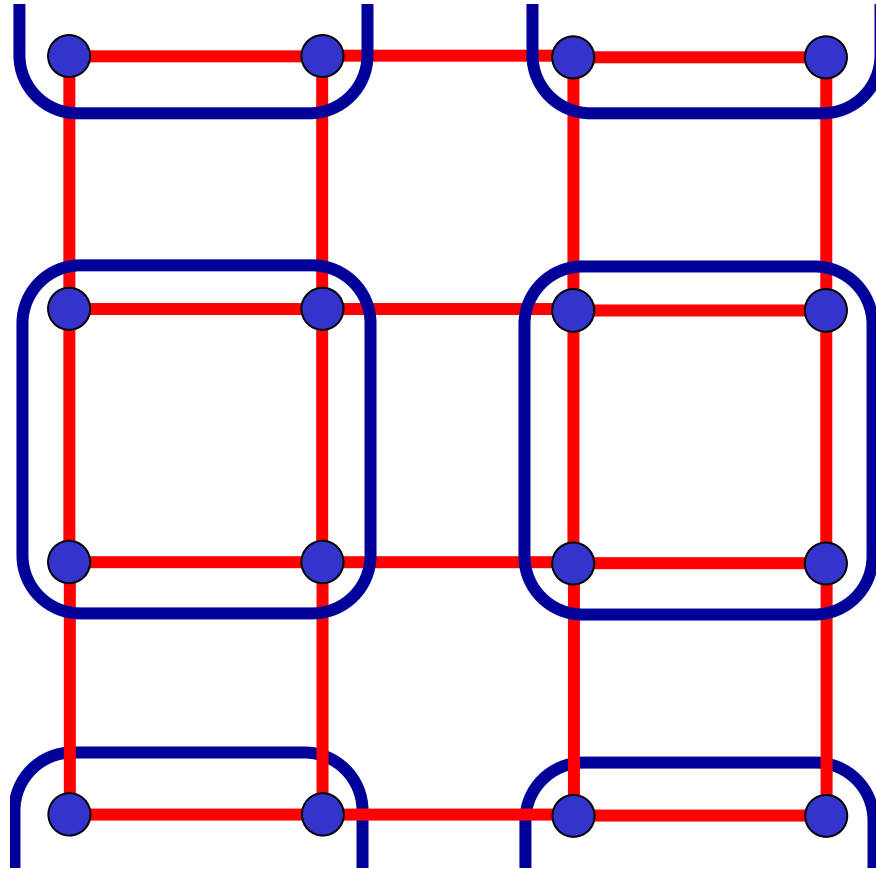


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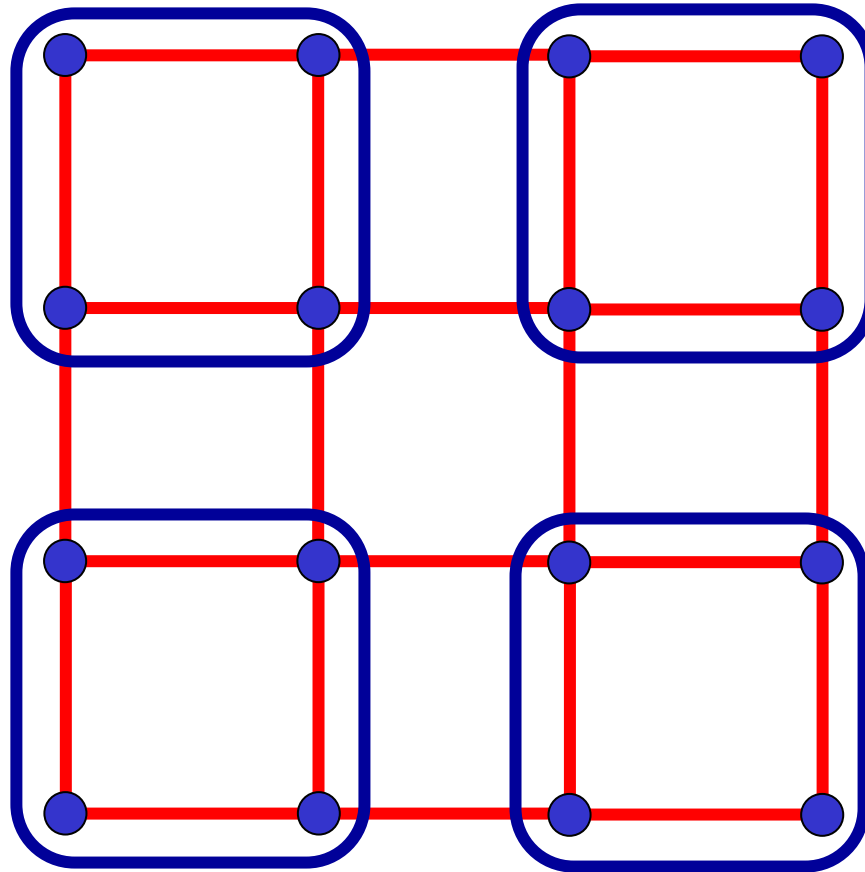


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B. Mott insulators with spin $S=1/2$ per unit cell:

1. Valence-bond-solid order in the paramagnet.

2. Mapping to hard-core bosons at half-filling

At each site, identify the states $|\uparrow\rangle$, $|\downarrow\rangle$, with the occupation number of a hard-core boson:

$$\begin{aligned}|\downarrow\rangle &= |0\rangle \\ |\uparrow\rangle &= b^\dagger|0\rangle\end{aligned}$$

Then the spin operators map as follows

$$\begin{aligned}S_z &= b^\dagger b - 1/2 \\ S_+ &= b^\dagger \\ S_- &= b\end{aligned}$$

Outline

A. Magnetic quantum phase transitions in “dimerized” Mott insulators

Landau-Ginzburg-Wilson (LGW) theory

B. Mott insulators with spin $S=1/2$ per unit cell

1. Valence-bond-solid (VBS) order in the paramagnet;

2. Mapping to hard-core bosons at half-filling

C. The superfluid-insulator transition of bosons in lattices

Multiple order parameters in quantum systems

D. Boson-vortex duality

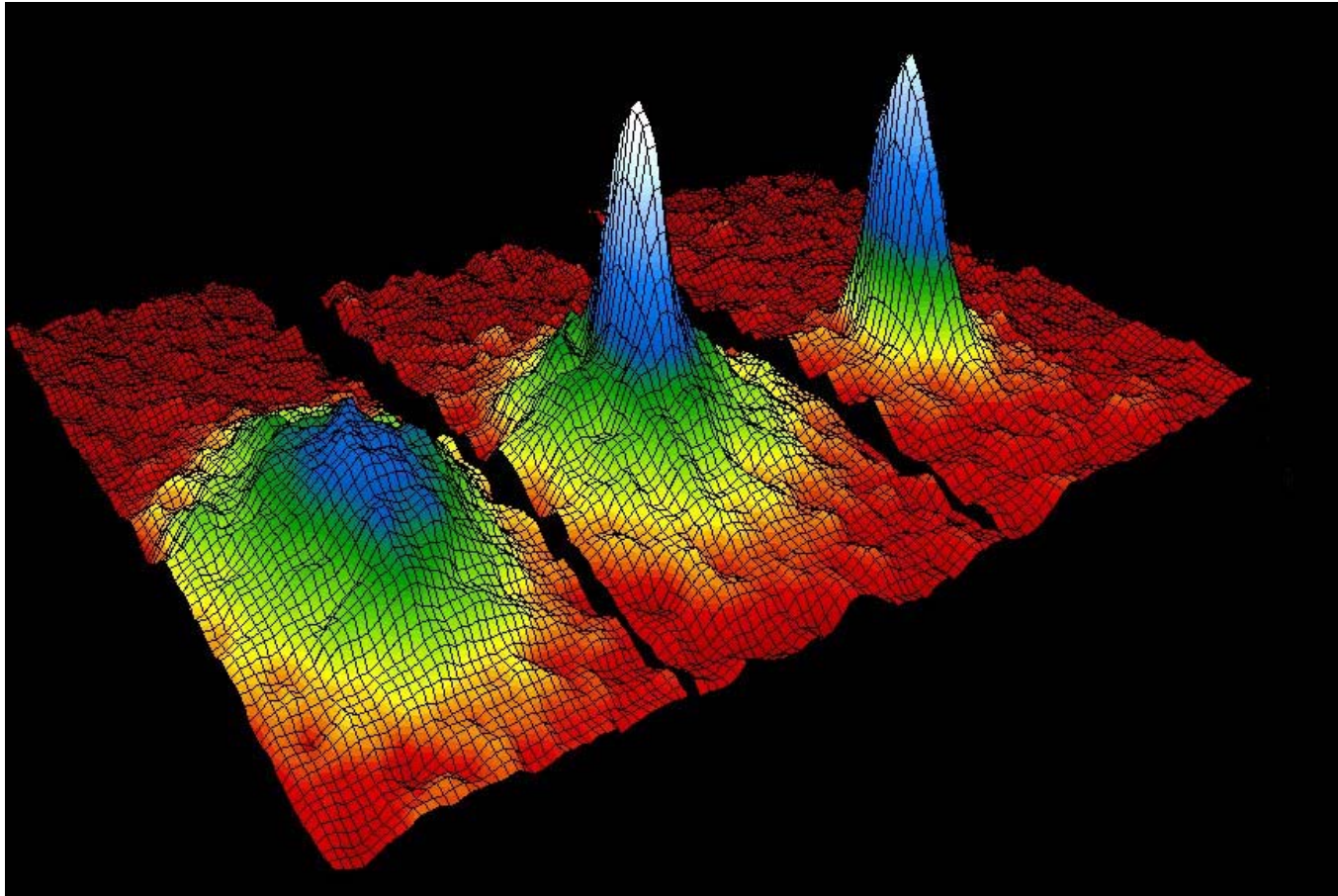
Breakdown of the LGW paradigm

B. Superfluid-insulator transition

1. Bosons in a lattice at integer filling

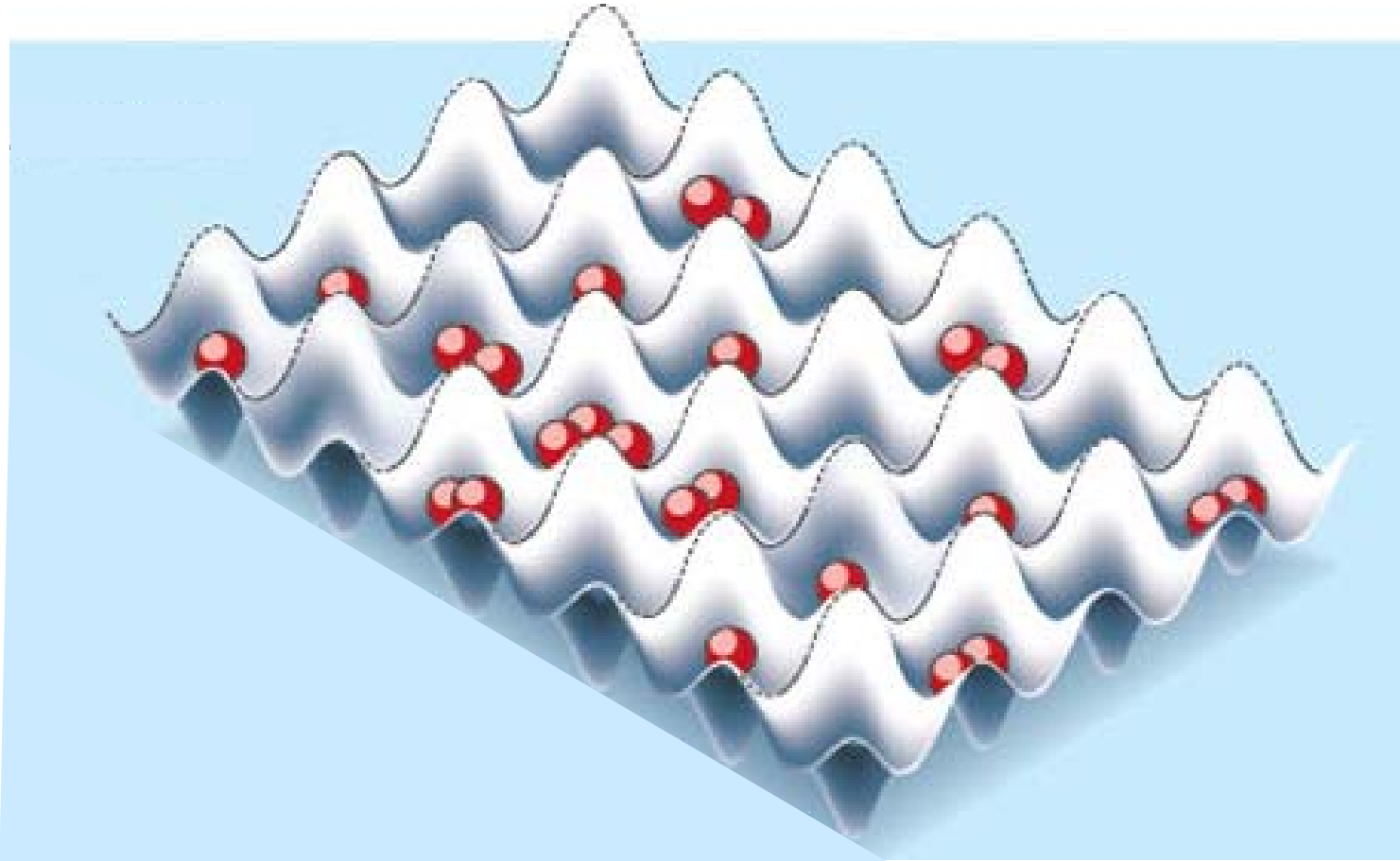
Bose condensation

Velocity distribution function of ultracold ^{87}Rb atoms

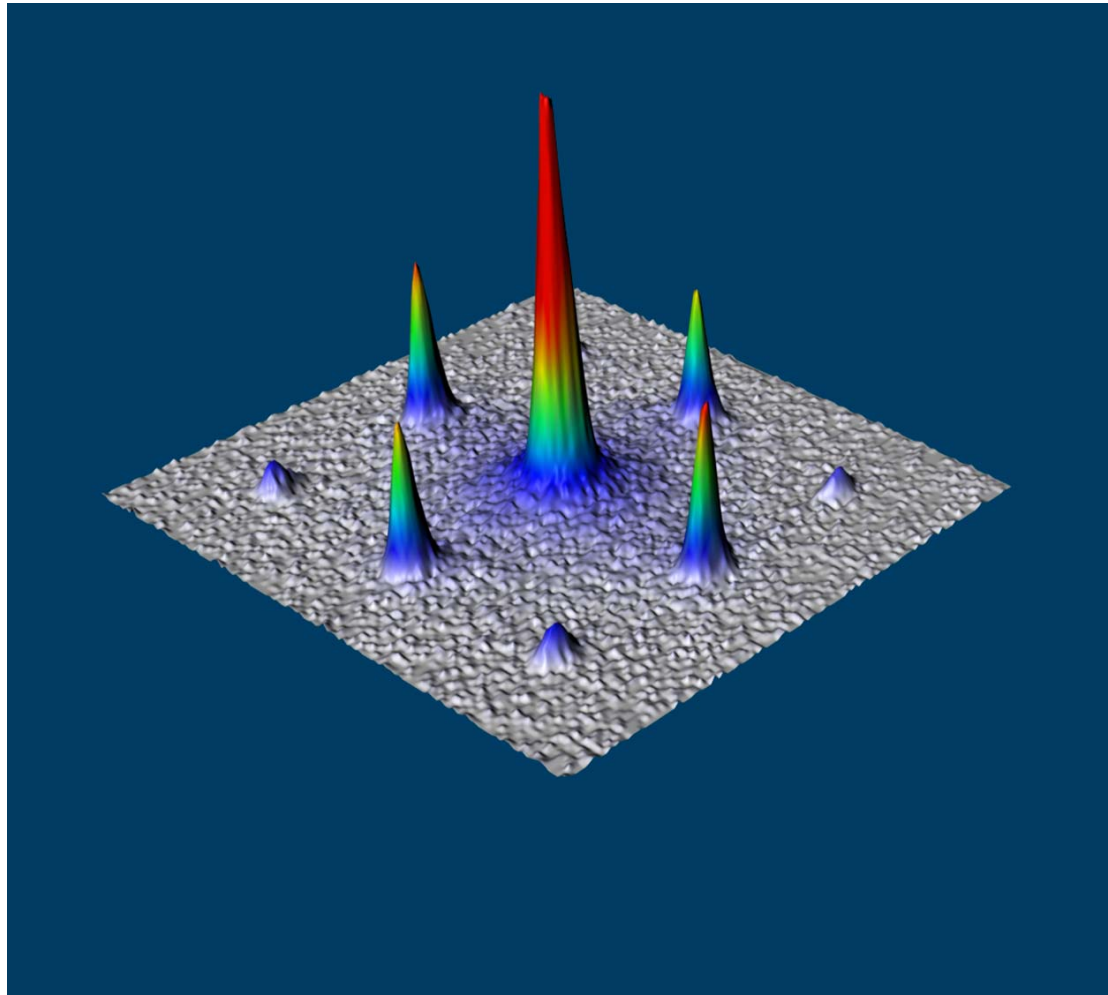


M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman
and E. A. Cornell, *Science* **269**, 198 (1995)

Apply a periodic potential (standing laser beams)
to trapped ultracold bosons (^{87}Rb)

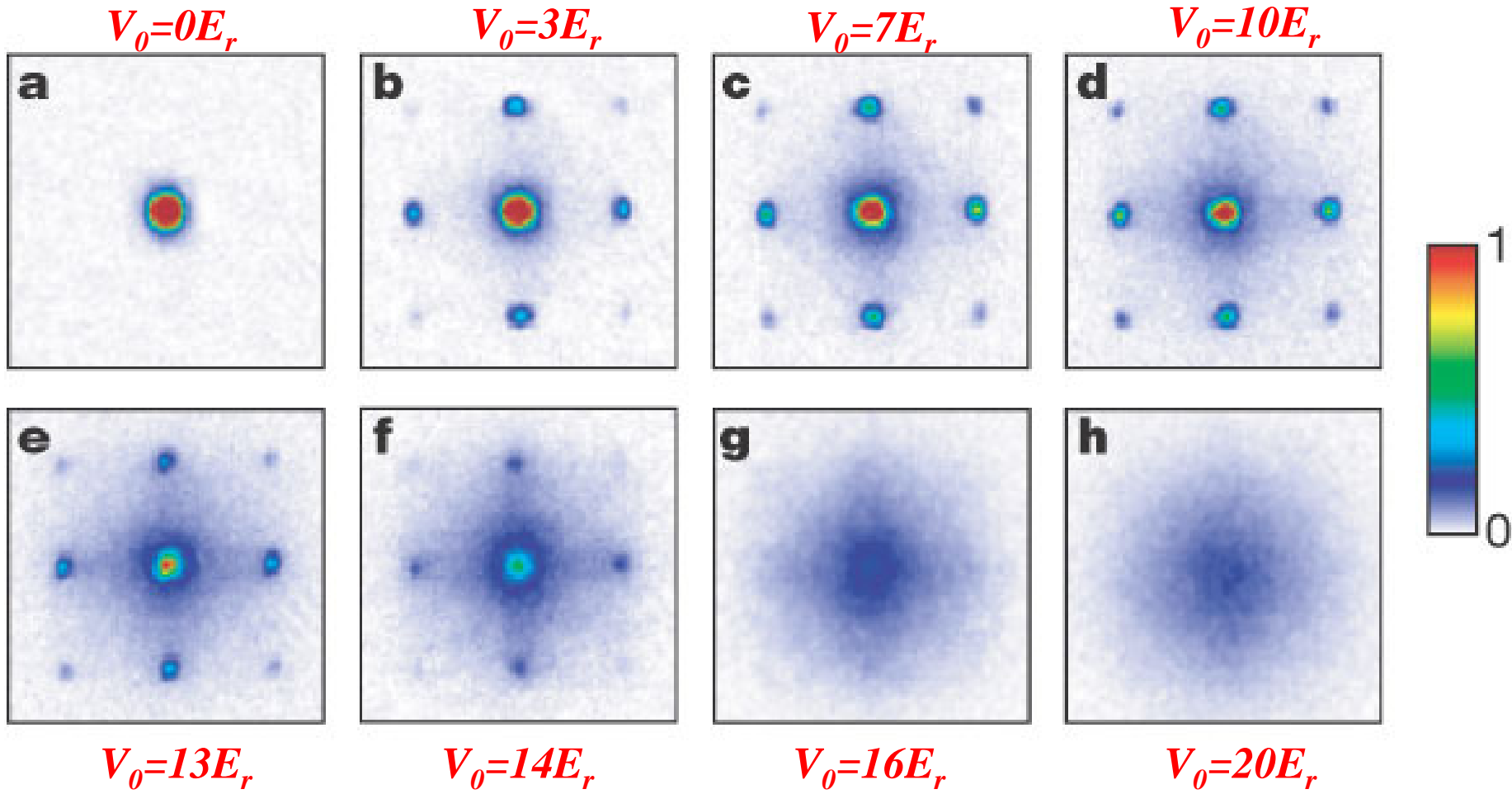
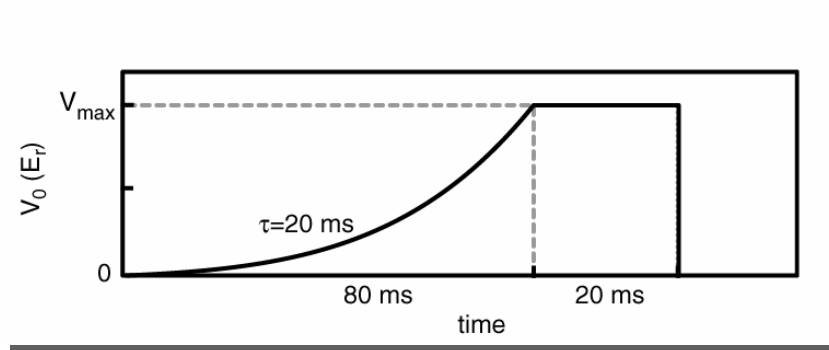


Momentum distribution function of bosons



Bragg reflections of condensate at reciprocal lattice vectors

Superfluid-insulator quantum phase transition at $T=0$



Bosons at filling fraction $f = 1$

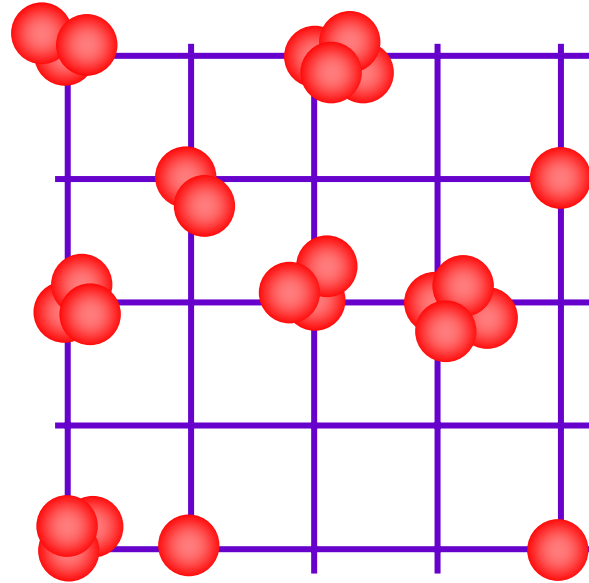
Weak interactions:
superfluidity

a Superfluid state

b Insulating state

Strong interactions:
Mott insulator which
preserves all lattice
symmetries

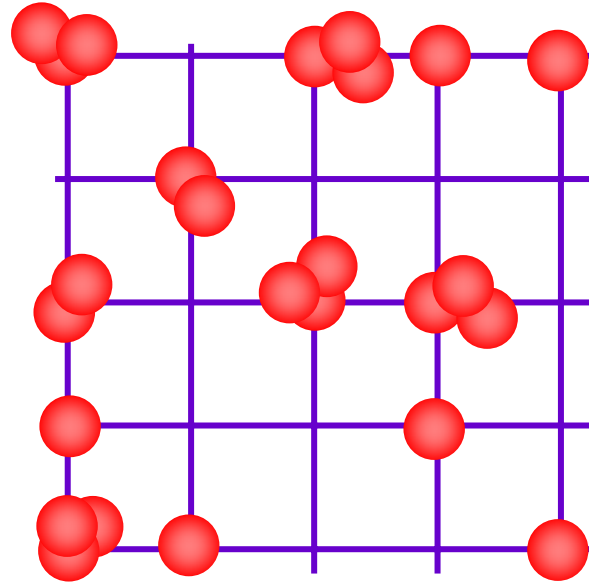
Bosons at filling fraction $f = 1$



$$\langle \Psi \rangle \neq 0$$

Weak interactions: superfluidity

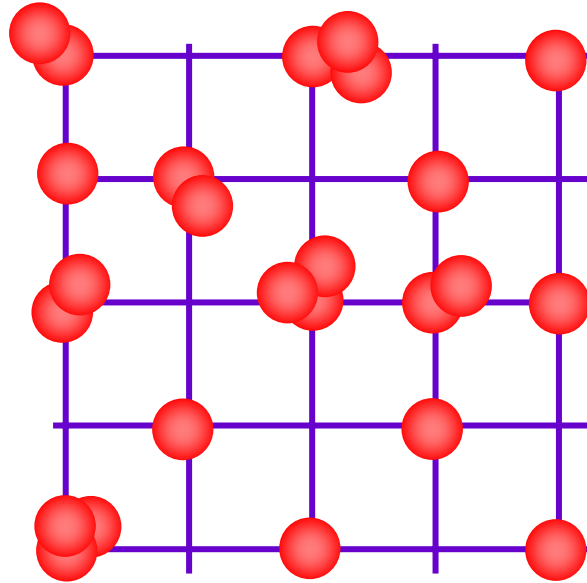
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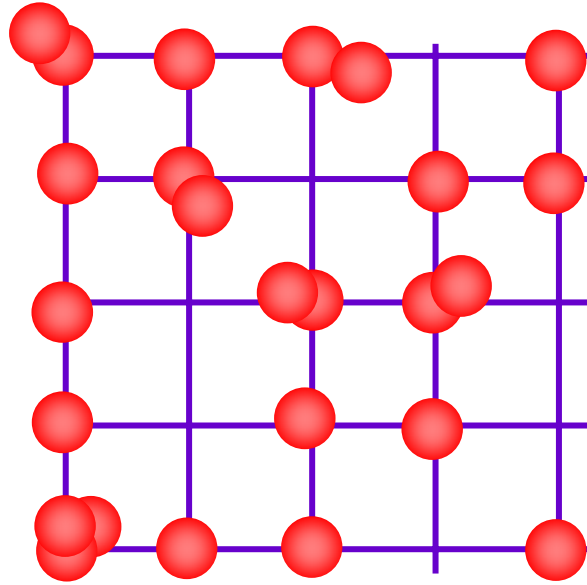
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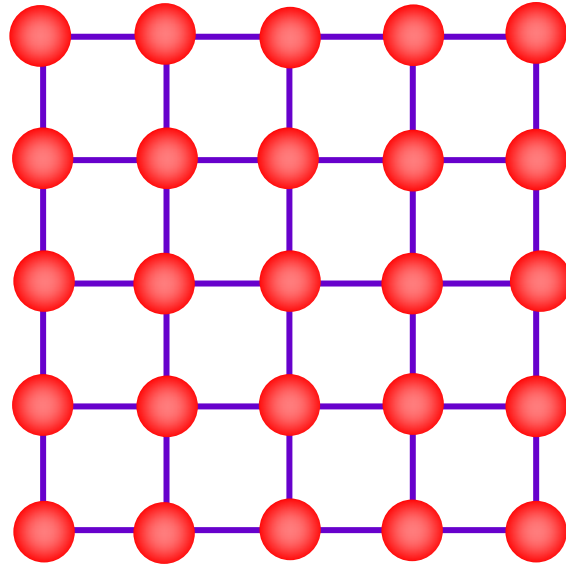
Bosons at filling fraction $f = 1$



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Weak interactions: superfluidity

Bosons at filling fraction $f = 1$



$$\langle \Psi \rangle = 0$$

Strong interactions: insulator

The Superfluid-Insulator transition

Boson Hubbard model

Degrees of freedom: Bosons, b_j^\dagger , hopping between the sites, j , of a lattice, with short-range repulsive interactions.

$$H = -t \sum_{\langle ij \rangle} b_i^\dagger b_j - \mu \sum_j n_j + \frac{U}{2} \sum_j n_j (n_j - 1) + \dots$$

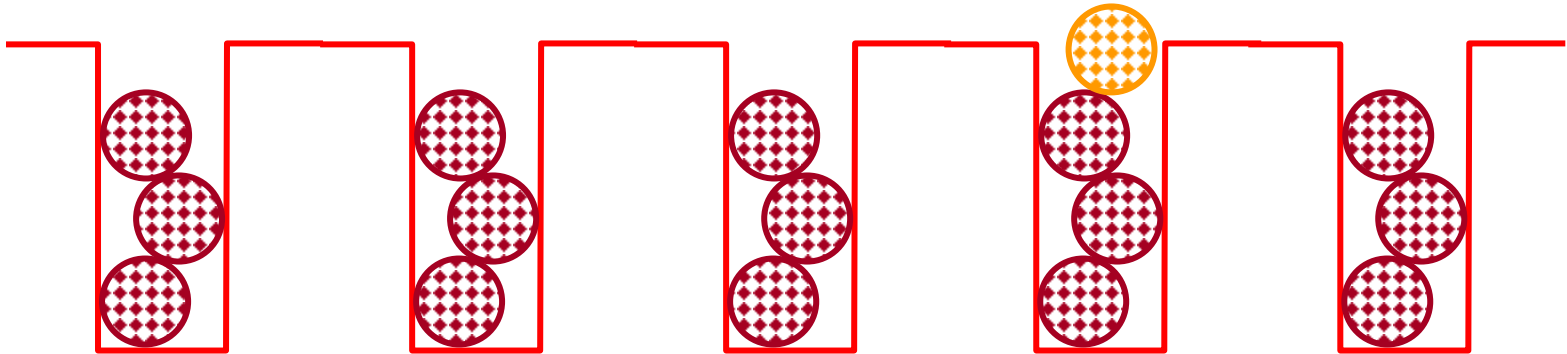
$$n_j \equiv b_j^\dagger b_j$$

M.P.A. Fisher, P.B. Weichmann,
G. Grinstein, and D.S. Fisher
Phys. Rev. B **40**, 546 (1989).

For small U/t , ground state is a superfluid BEC with
superfluid density \approx density of bosons

What is the ground state for large U/t ?

Typically, the ground state remains a superfluid, but with
superfluid density \ll density of bosons

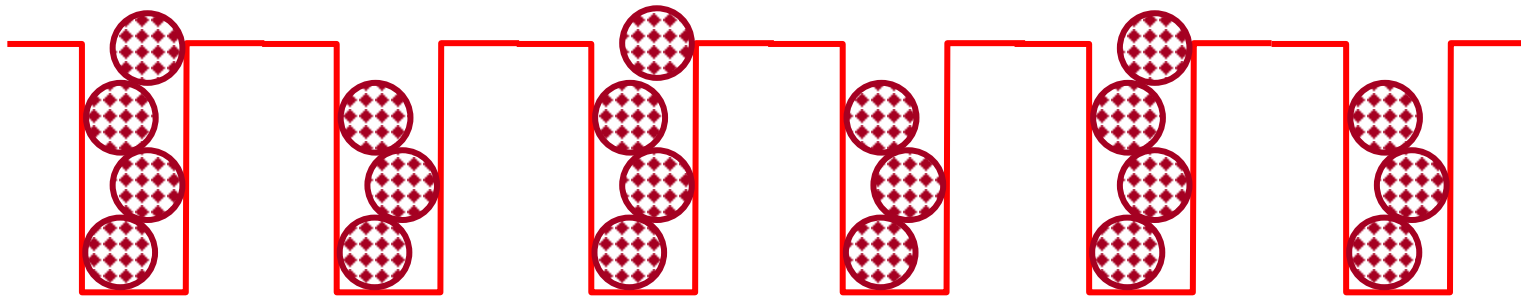
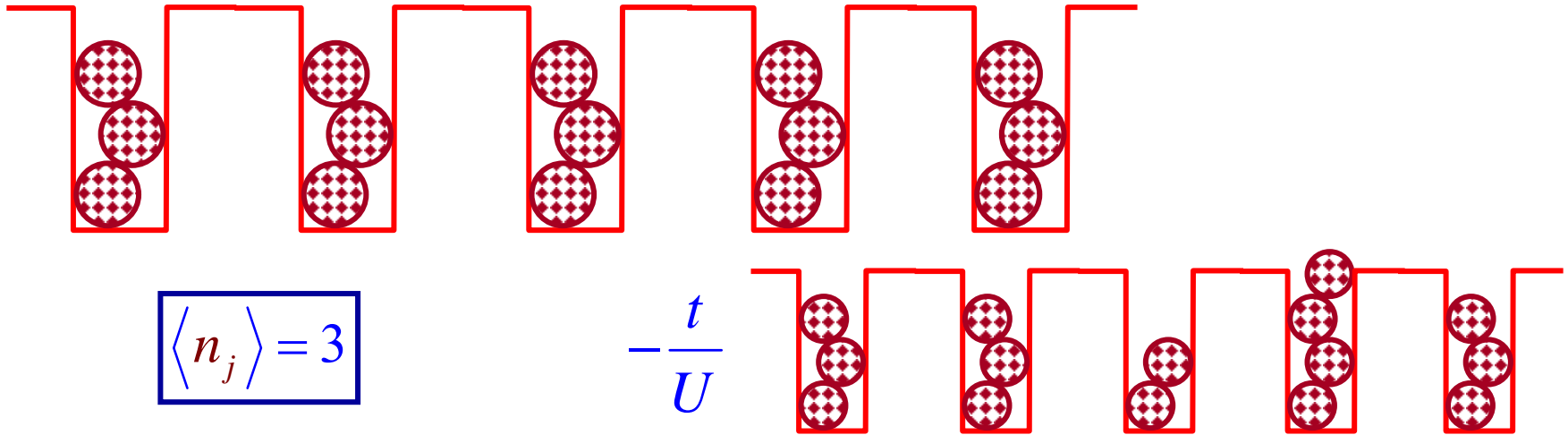


The superfluid density evolves smoothly from large values at small U/t , to small values at large U/t , and there is no quantum phase transition at any intermediate value of U/t .

(In systems with Galilean invariance and at zero temperature, superfluid density=density of bosons always, independent of the strength of the interactions)

What is the ground state for large U/t ?

Incompressible, insulating ground states, with zero superfluid density, appear at special commensurate densities



$$\langle n_j \rangle = 7/2$$

Ground state has “density wave” order, which spontaneously breaks lattice symmetries

LGW theory of the superfluid insulator transition

- Identify order parameter $\Psi(x, \tau) \sim b_j^\dagger$
- Symmetries:

$$\text{Gauge invariance:} \quad \Psi \rightarrow \Psi e^{i\theta}$$

$$\text{Time reversal} \quad \tau \rightarrow -\tau \quad ; \quad \Psi \rightarrow \Psi^*$$

$$\text{Spatial inversion} \quad x \rightarrow -x$$

- Write down most general Lagrangian consistent with symmetries

$$\mathcal{Z} = \int \mathcal{D}\Psi(x, \tau) \exp \left(- \int d^d x \int d\tau \mathcal{L}[\Psi] \right)$$
$$\mathcal{L}[\Psi] = K \Psi^* \frac{\partial \Psi}{\partial \tau} + |\partial_\tau \Psi|^2 + c^2 |\nabla_x \Psi|^2 + r |\Psi|^2 + \frac{u}{2} |\Psi|^4 + \dots$$

- Identify phases at $r \gg 0$ and $r \ll 0$ with the insulator and the superfluid respectively.
- For $K \neq 0$, the particle and hole excitations have different energies.

- Gauge-invariance of the underlying boson Hamiltonian shows that

$$K = -\frac{\partial r}{\partial \mu}$$

- In mean-field theory, the ground state energy, E , across the superfluid-insulator transition has the non-analytic term

$$E = E_0 - \frac{r^2}{2u}\theta(-r)$$

(Beyond mean-field theory, the non-analytic term is $E \sim r^{(d+z)\nu}$).

- Because the density of bosons $= -\partial E/\partial \mu$, this implies a change in the boson density across the transition *unless* $\partial r/\partial \mu = 0$
- A superfluid-insulator transition at fixed boson density must have.

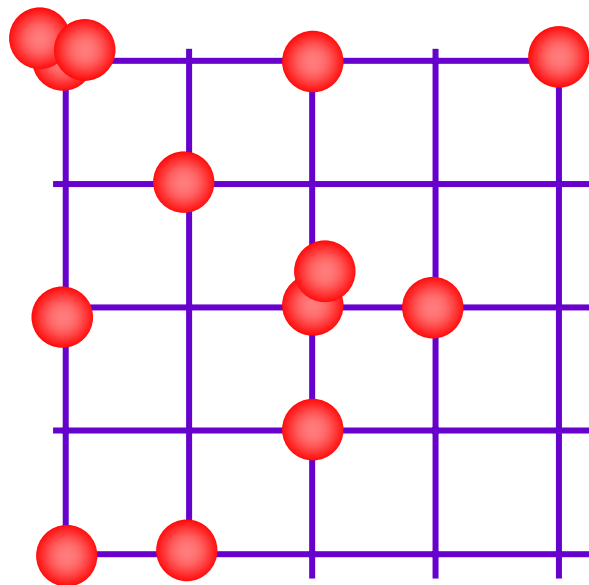
$$K = 0$$

B. Superfluid-insulator transition

2. Bosons in a lattice at fractional filling

L. Balents, L. Bartosch, A. Burkov, S. Sachdev, K. Sengupta,
Physical Review B **71**, 144508 and 144509 (2005),
cond-mat/0502002, and cond-mat/0504692.

Bosons at filling fraction $f = 1/2$



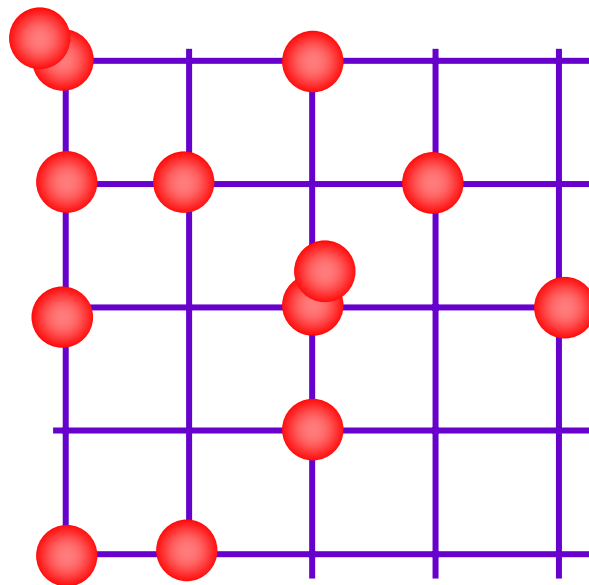
$$\langle \Psi \rangle \neq 0$$

Weak interactions: superfluidity

C. Lannert, M.P.A. Fisher, and T. Senthil, *Phys. Rev. B* **63**, 134510 (2001)

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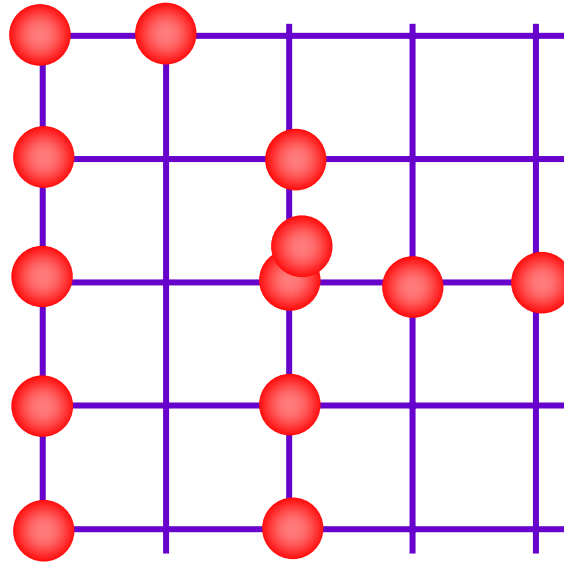
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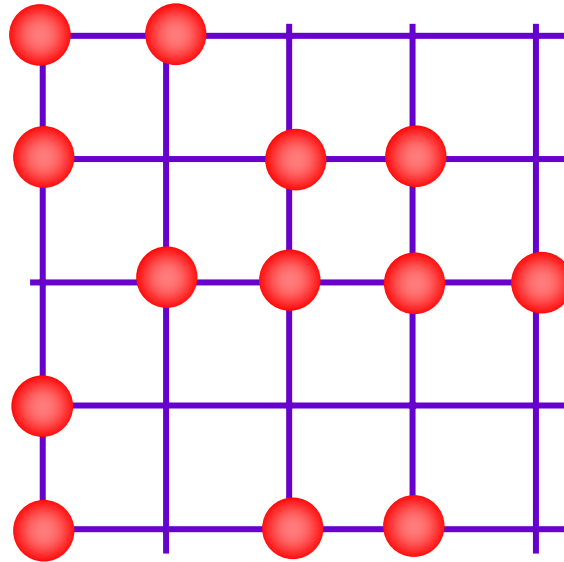
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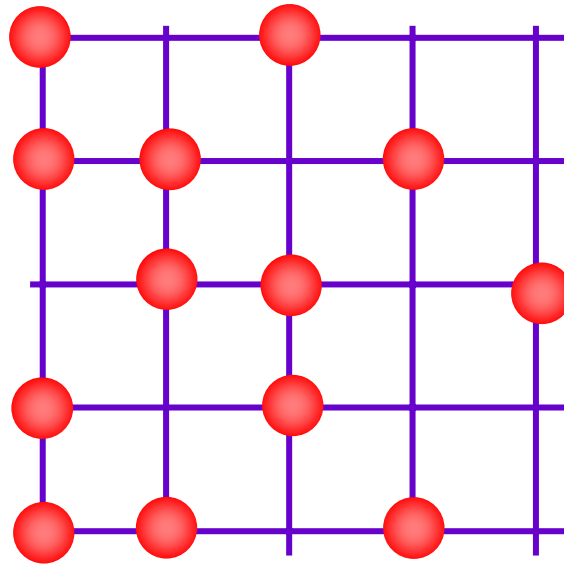
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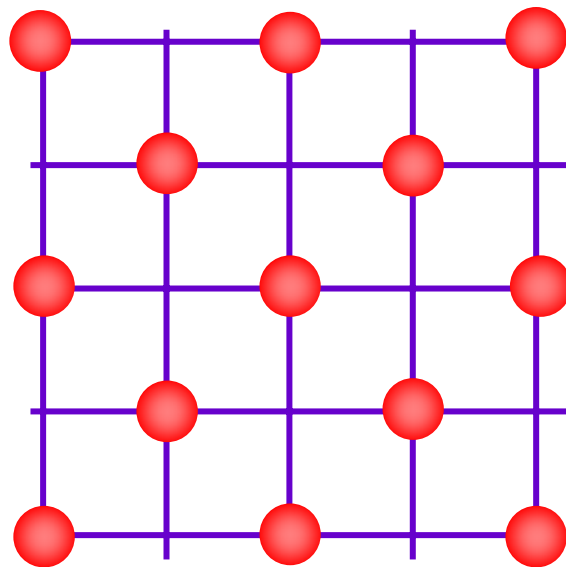
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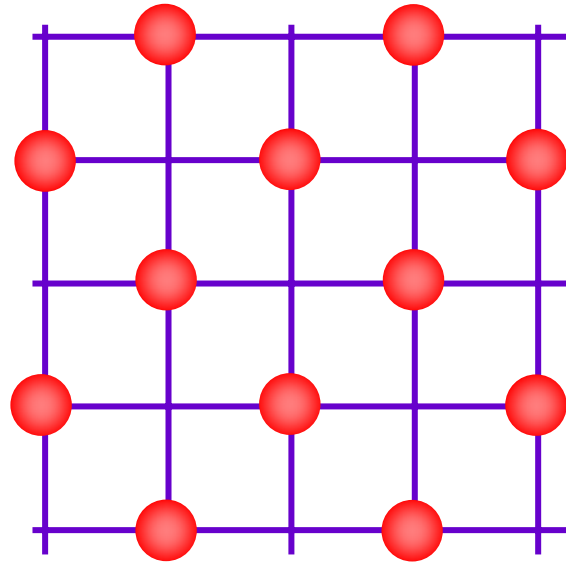
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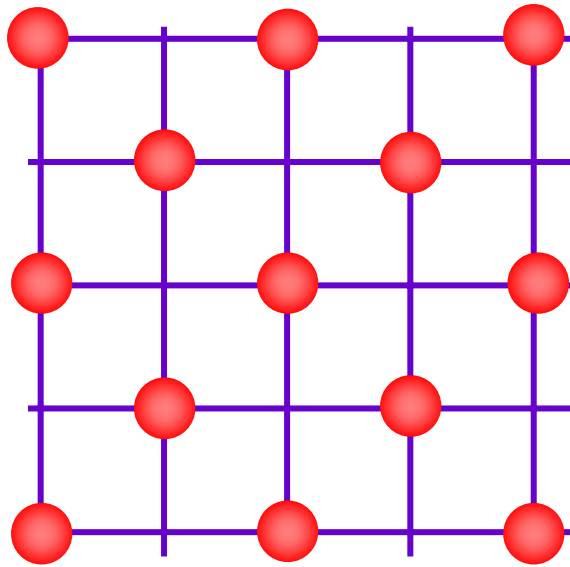
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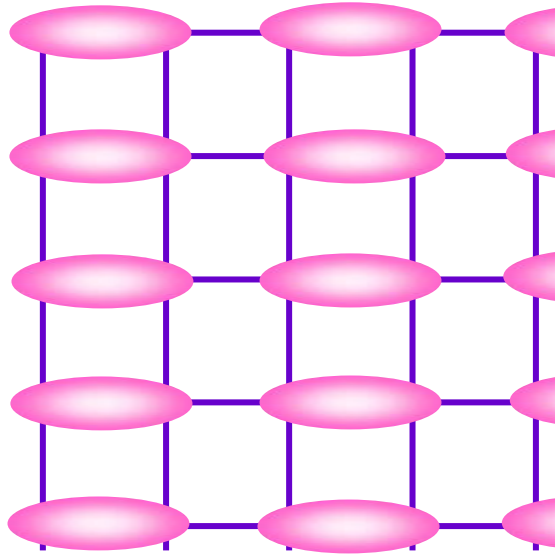
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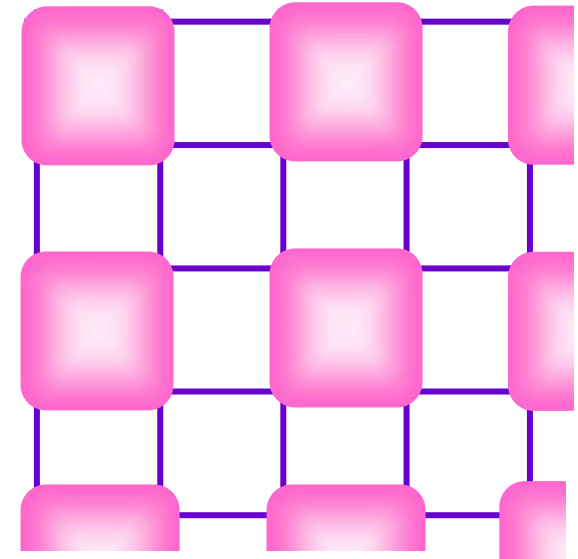
Insulating phases of bosons at filling fraction $f = 1/2$



Charge density wave (CDW) order



Valence bond solid (VBS) order



Valence bond solid (VBS) order

$$\text{pink oval} = \frac{1}{\sqrt{2}} \left(\text{red sphere} - \text{red sphere} \right)$$

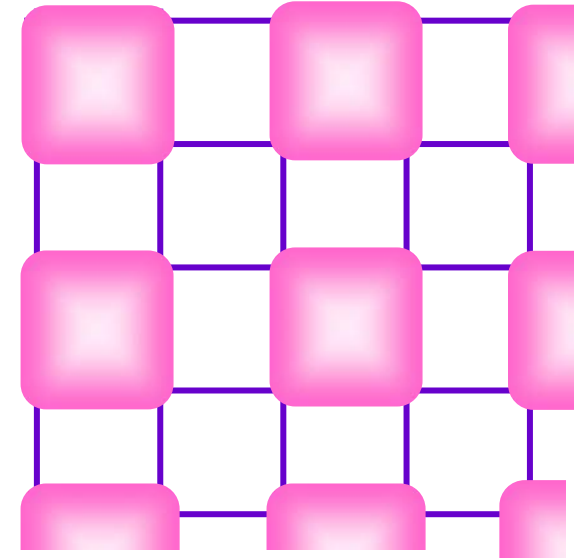
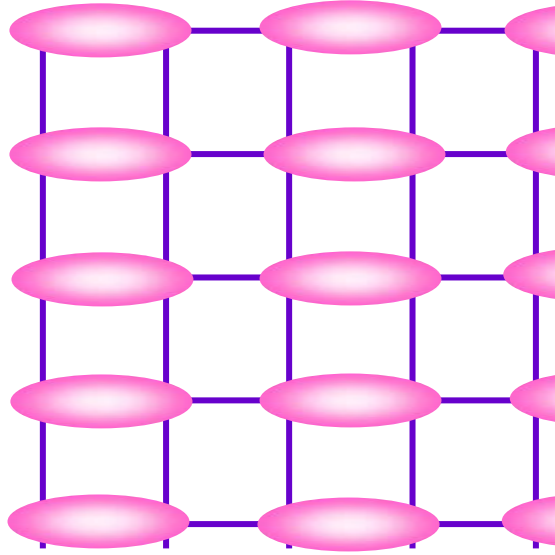
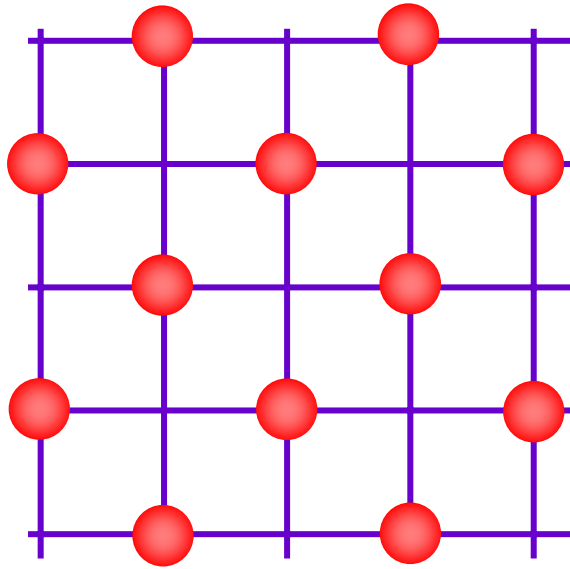
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All insulators have $\langle \Psi \rangle = 0$ and $\langle \rho_{\mathbf{Q}} \rangle \neq 0$ for certain \mathbf{Q}

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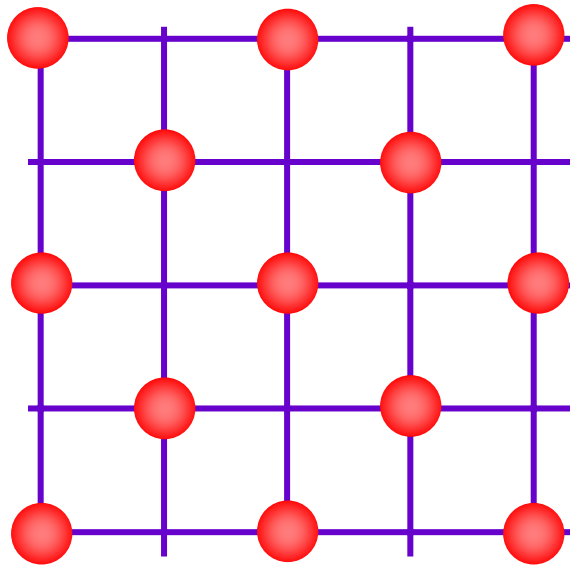
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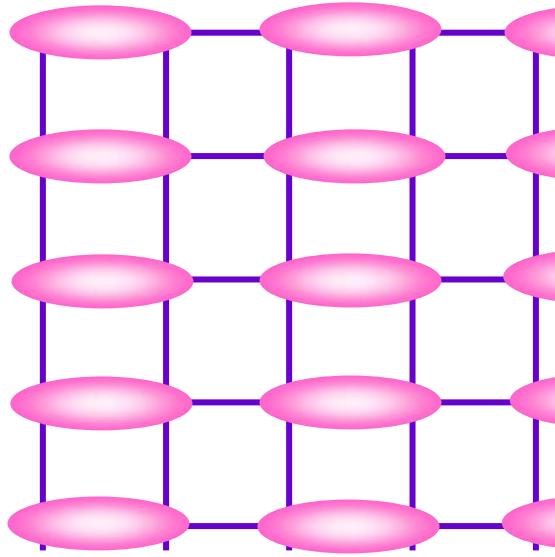
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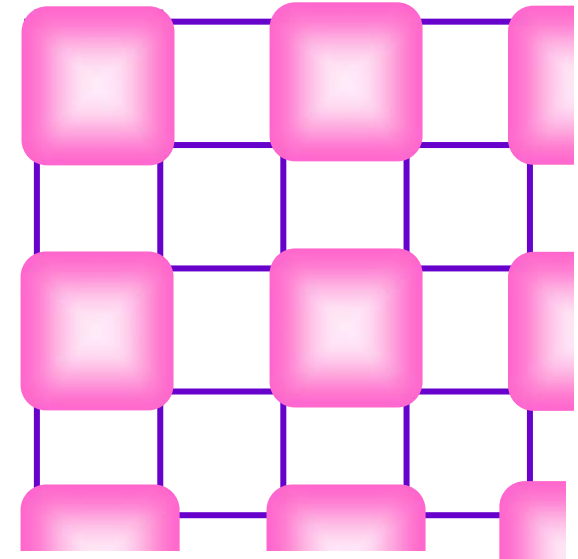
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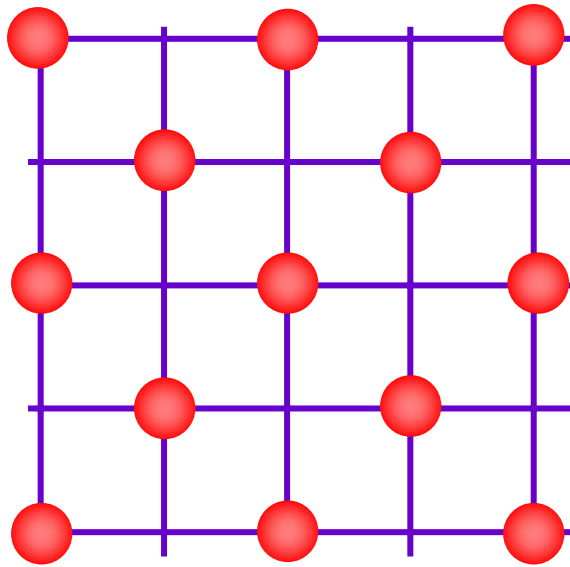
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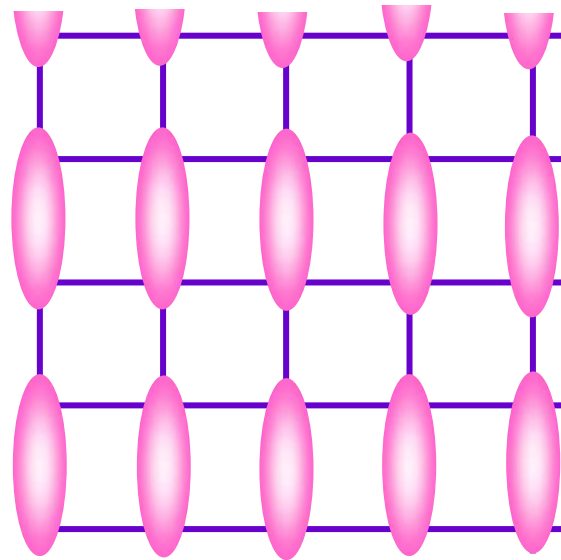
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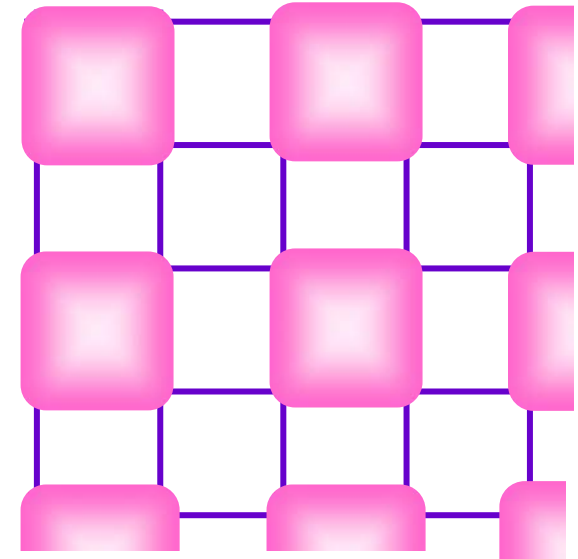
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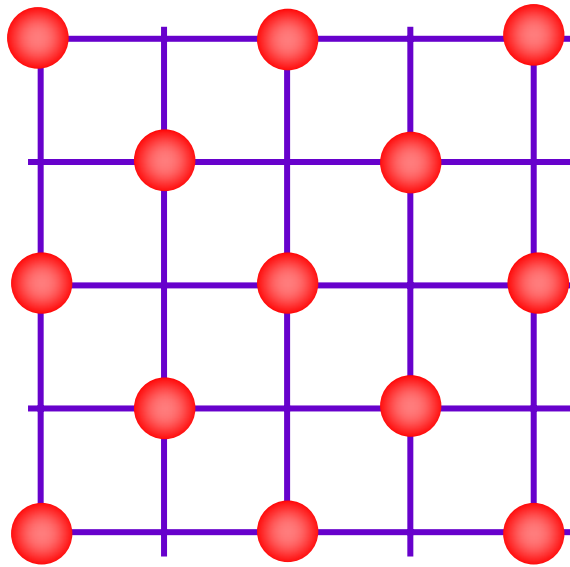
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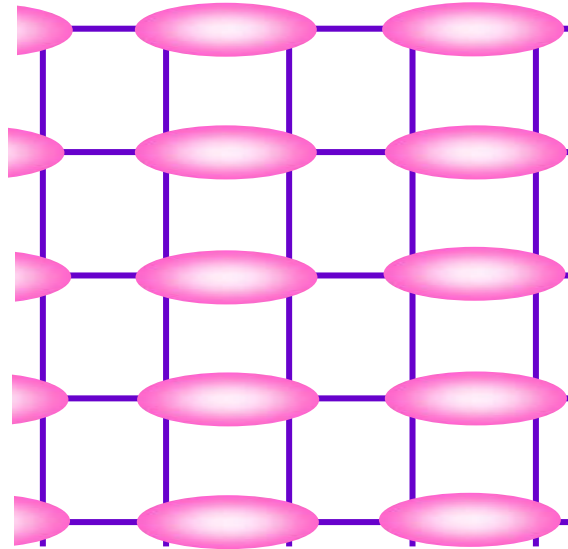
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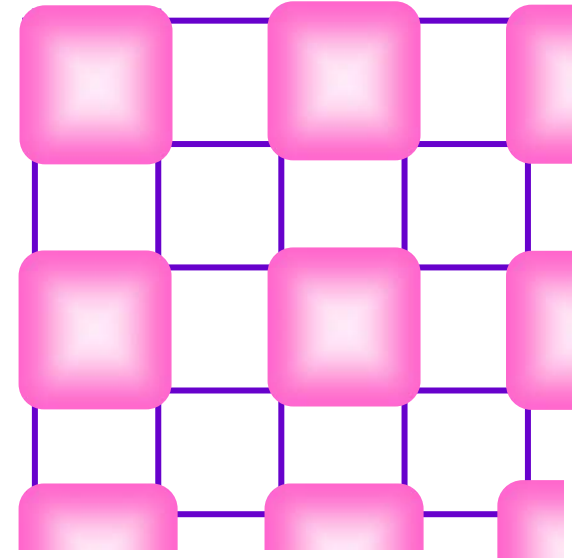
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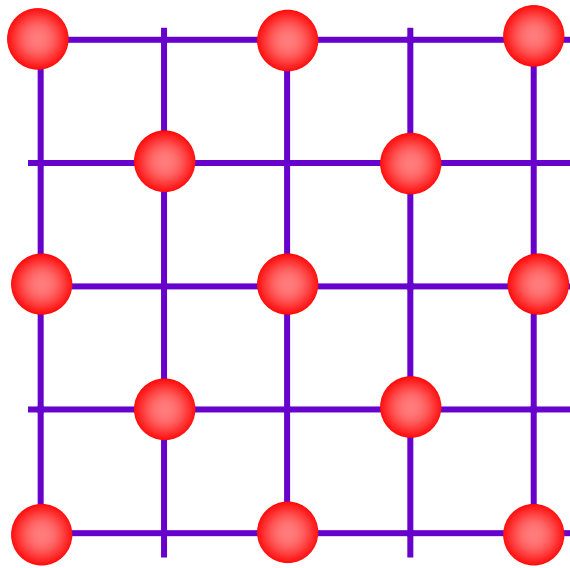
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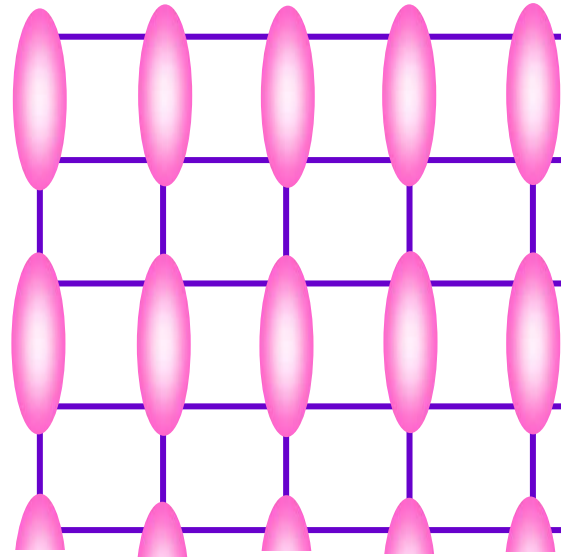
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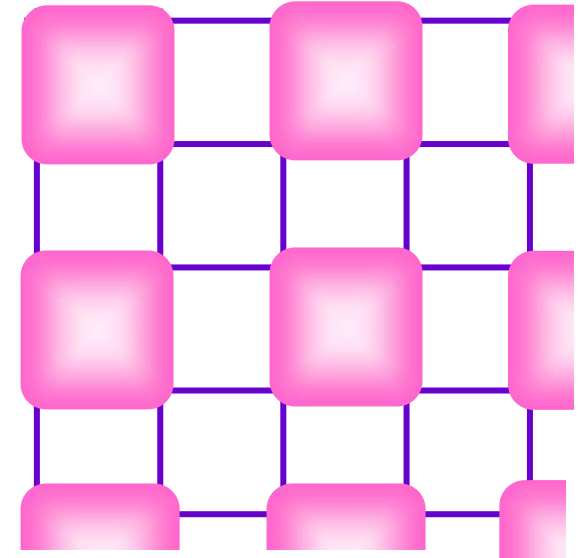
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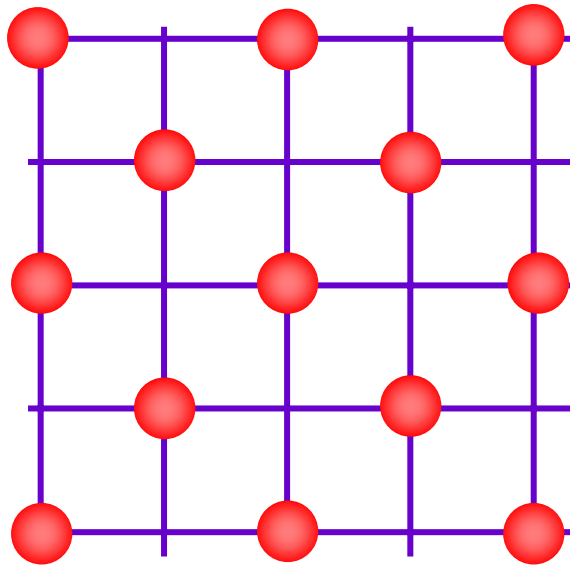
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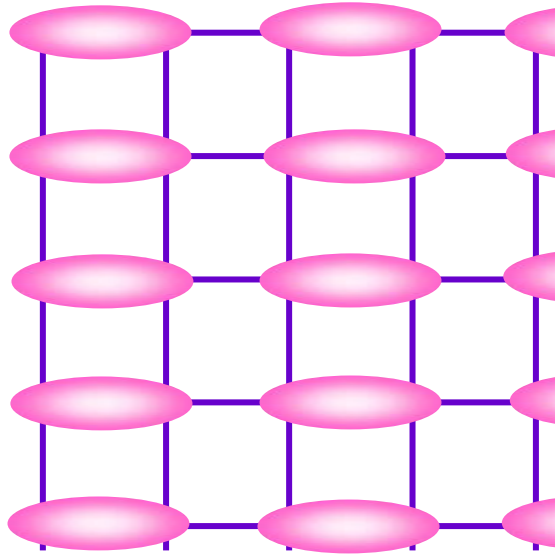
C. Lannert, M.P.A. Fisher, and T. Senthil, *Phys. Rev. B* **63**, 134510 (2001)

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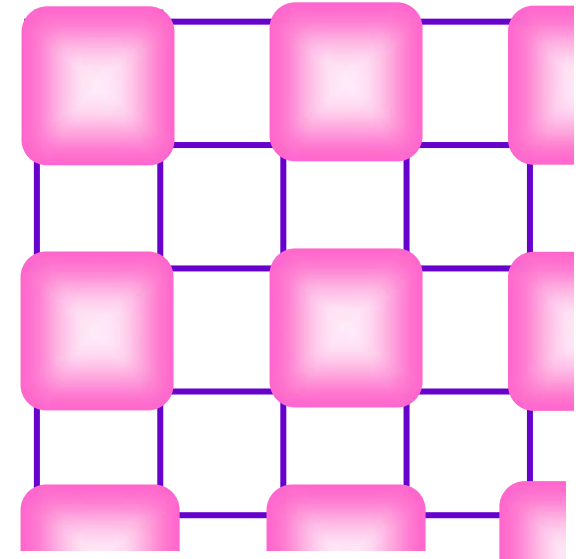
Insulating phases of bosons at filling fraction $f = 1/2$



Charge density wave (CDW) order



Valence bond solid (VBS) order



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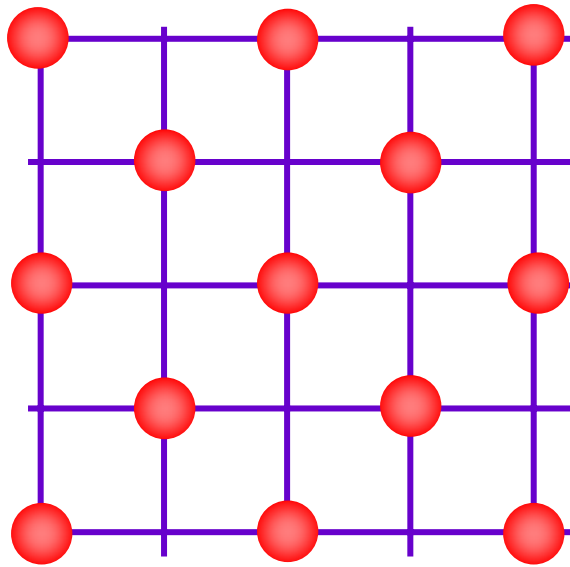
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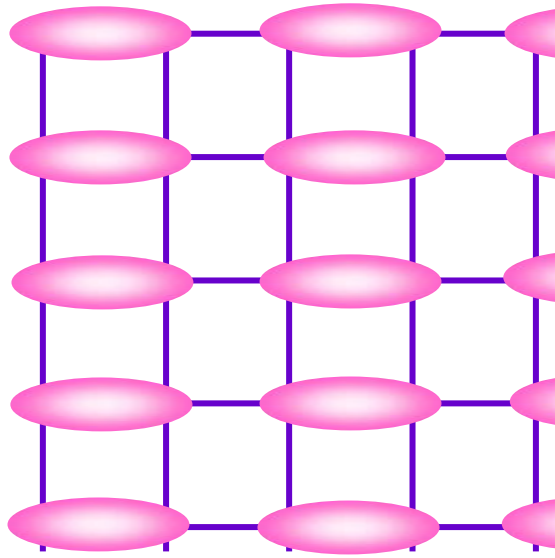
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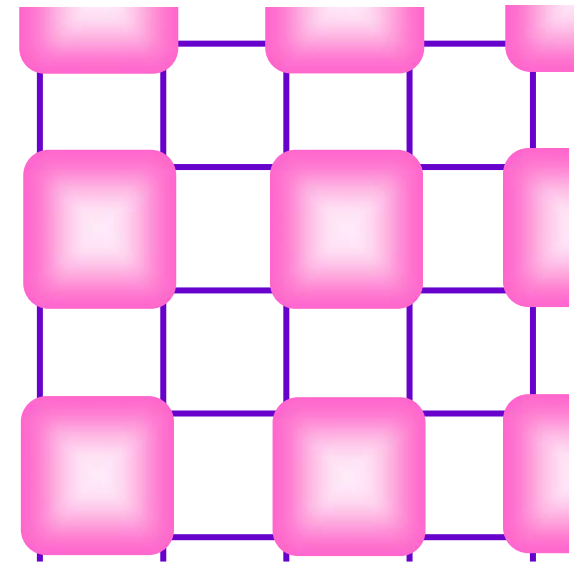
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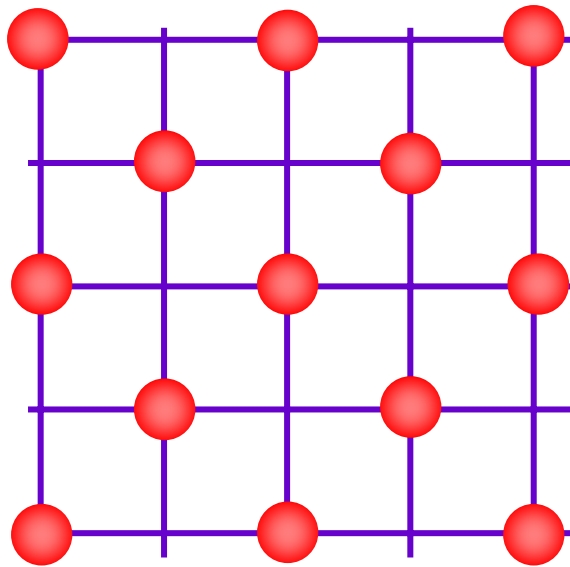
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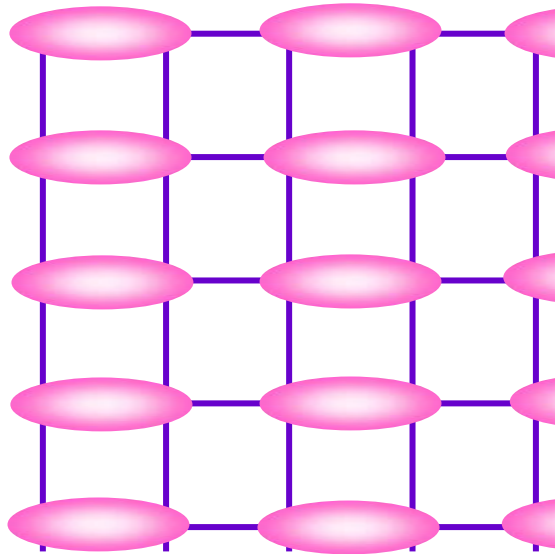
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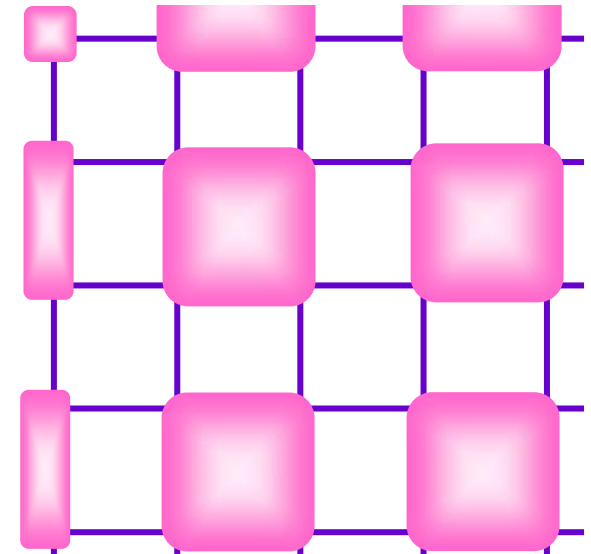
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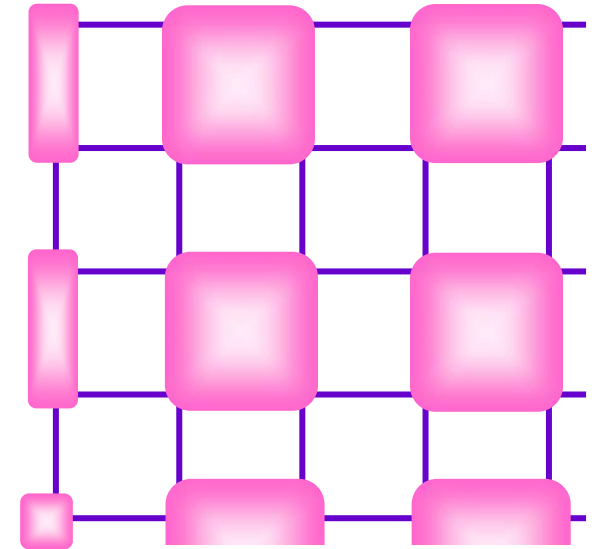
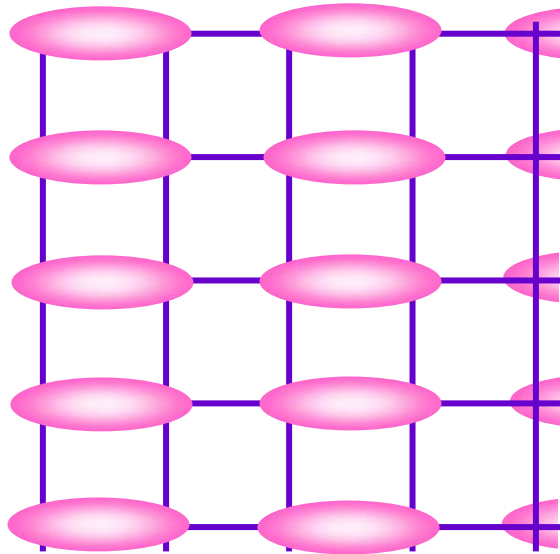
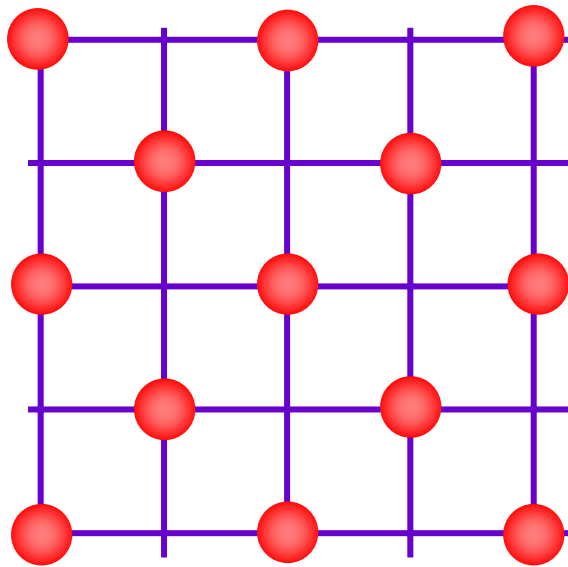
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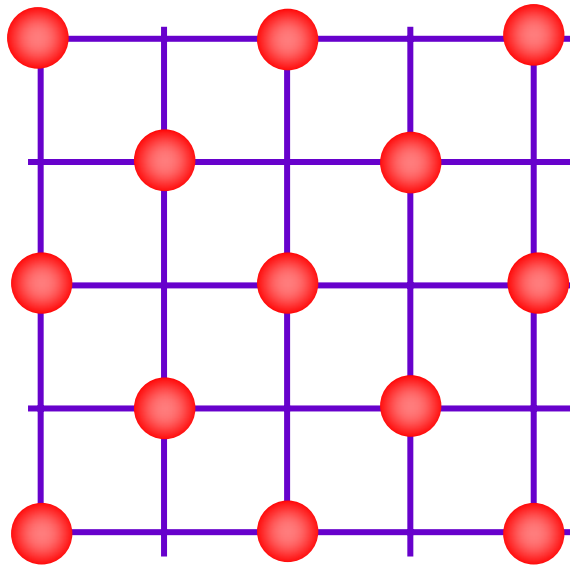
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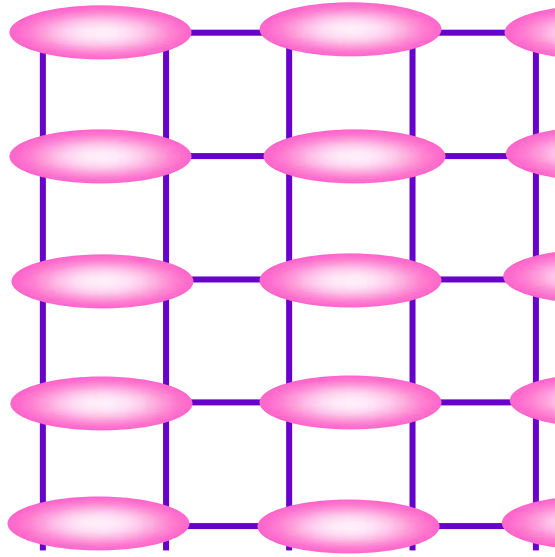
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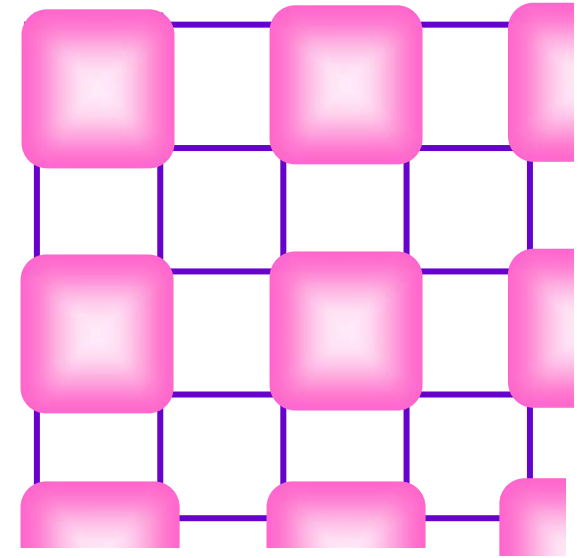
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Ginzburg-Landau-Wilson approach to multiple order parameters:

$$F = F_{sc} [\Psi_{sc}] + F_{\text{charge}} [\rho_Q] + F_{\text{int}}$$

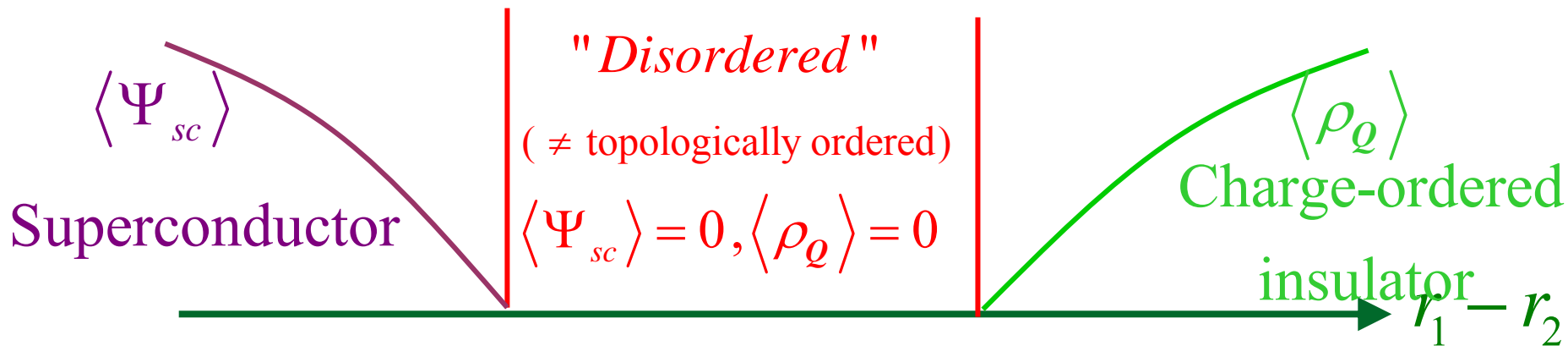
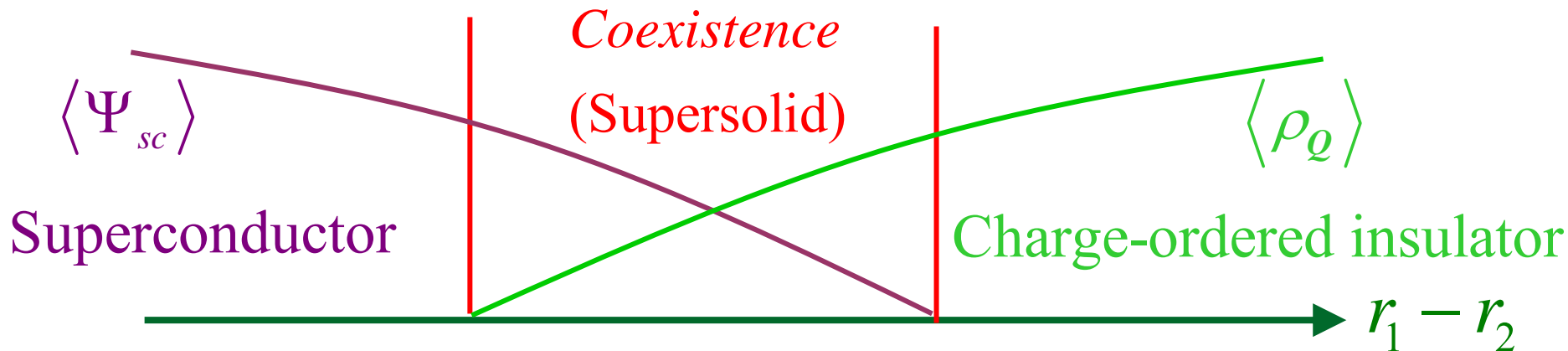
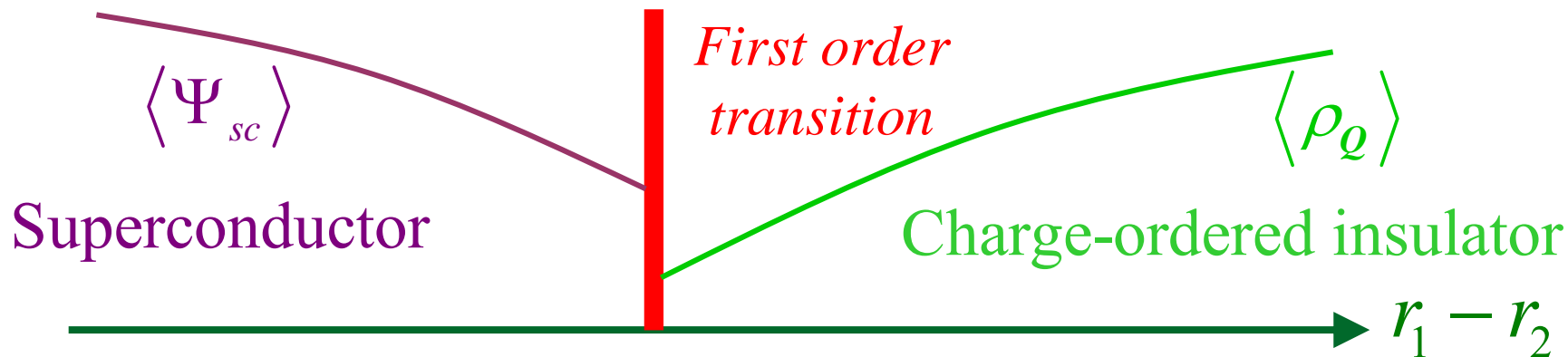
$$F_{sc} [\Psi_{sc}] = r_1 |\Psi_{sc}|^2 + u_1 |\Psi_{sc}|^4 + \dots$$

$$F_{\text{charge}} [\rho_Q] = r_2 |\rho_Q|^2 + u_2 |\rho_Q|^4 + \dots$$

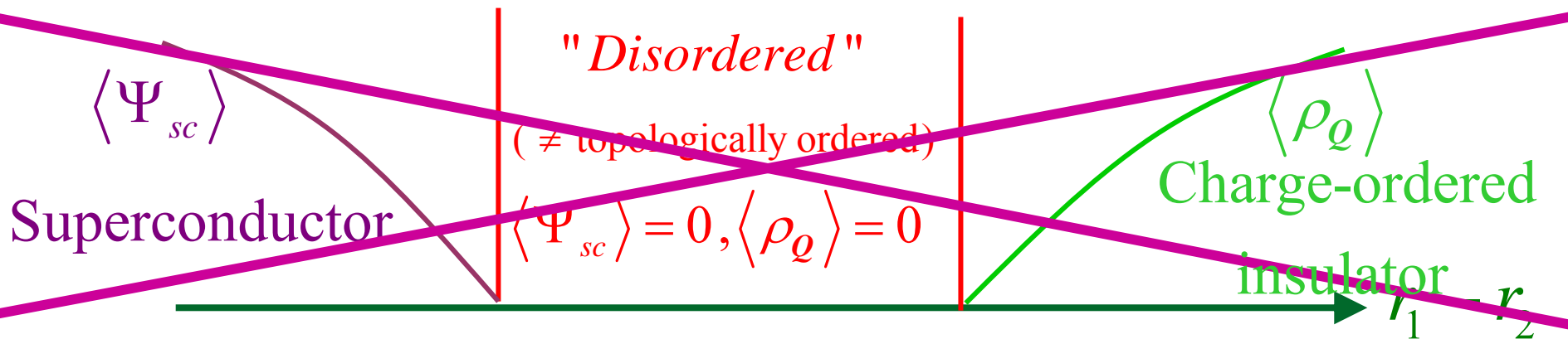
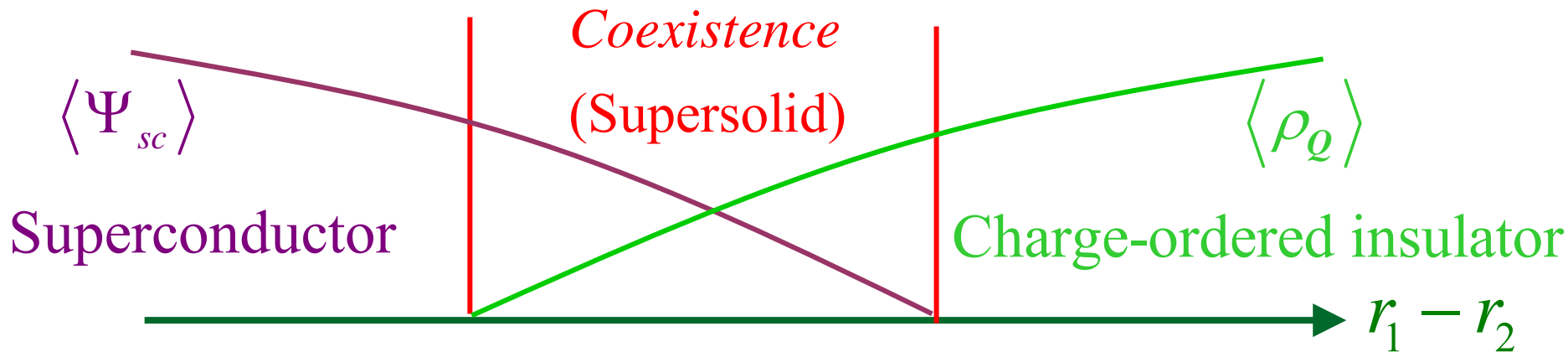
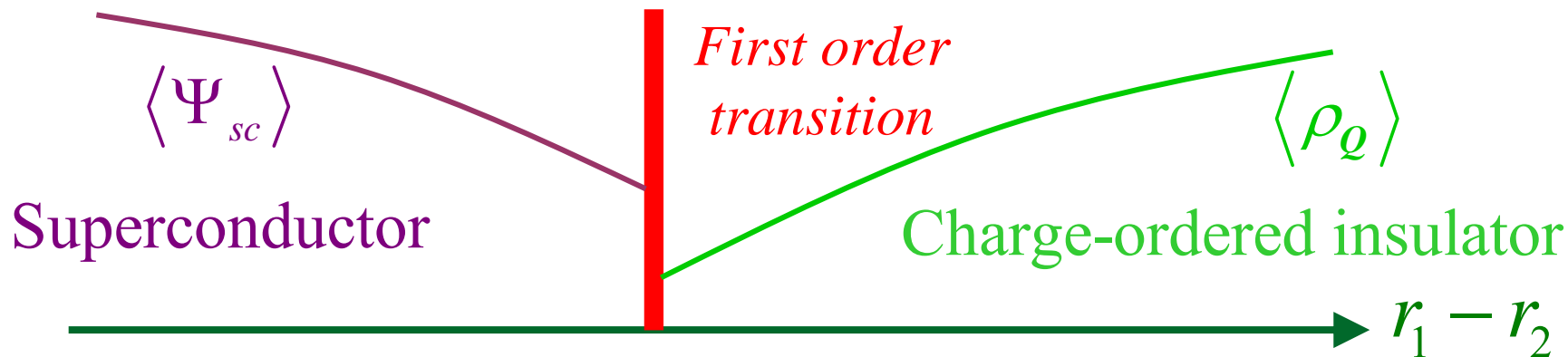
$$F_{\text{int}} = v |\Psi_{sc}|^2 |\rho_Q|^2 + \dots$$

Distinct symmetries of order parameters permit couplings only between their energy densities

Predictions of LGW theory



Predictions of LGW theory



Outline

A. Magnetic quantum phase transitions in “dimerized” Mott insulators

Landau-Ginzburg-Wilson (LGW) theory

B. Mott insulators with spin $S=1/2$ per unit cell

1. Valence-bond-solid (VBS) order in the paramagnet;

2. Mapping to hard-core bosons at half-filling

C. The superfluid-insulator transition of bosons in lattices

Multiple order parameters in quantum systems

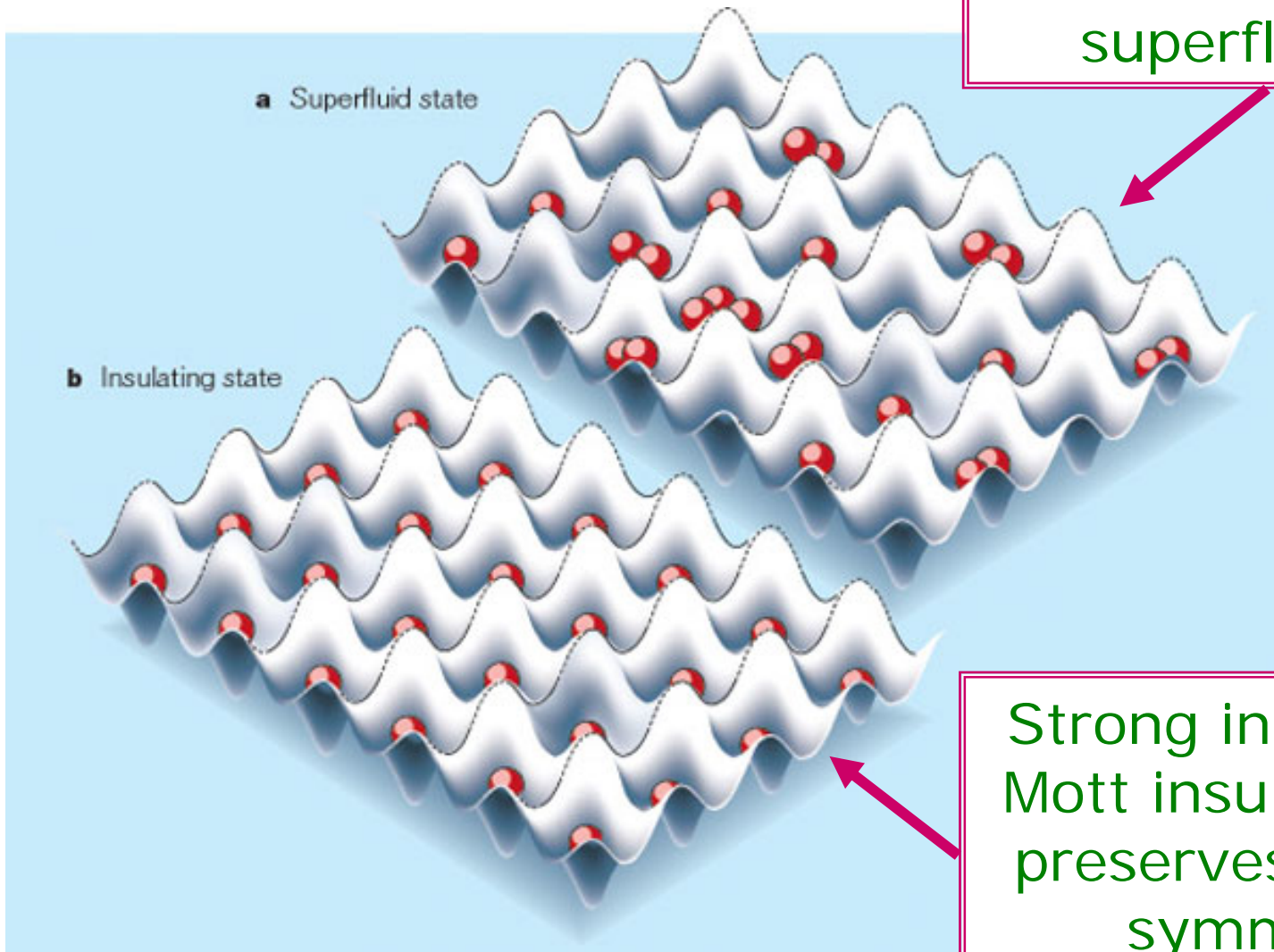
D. Boson-vortex duality

Breakdown of the LGW paradigm

D. Boson-vortex duality

1. Bosons in a lattice at integer filling

Bosons at density $f = 1$

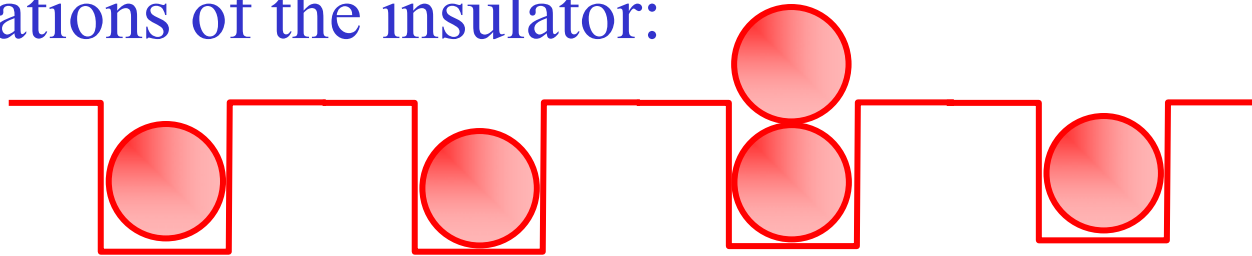


Weak interactions:
superfluidity

Strong interactions:
Mott insulator which
preserves all lattice
symmetries

Approaching the transition from the insulator ($f=1$)

Excitations of the insulator:



Particles $\sim \psi^\dagger$



Holes $\sim \psi$

Density of particles = density of holes \Rightarrow

“relativistic” field theory for ψ :

$$\mathcal{S} = \int d^2r d\tau \left[|\partial_\tau \psi|^2 + |\nabla_r \psi|^2 + s|\psi|^2 + \frac{u}{2}|\psi|^4 \right]$$

Insulator $\Leftrightarrow \langle \psi \rangle = 0$

Superfluid $\Leftrightarrow \langle \psi \rangle \neq 0$

Approaching the transition from the superfluid ($f=1$)

Excitations of the superfluid: (A) **Spin waves**

With $\psi \sim e^{i\theta}$, action for spin waves is

$$\mathcal{S}_{sw} = \frac{\rho_s}{2} \int d^3x (\partial_\mu \theta)^2$$

Dual form: After a Hubbard-Stratonovich transformation, write

$$\mathcal{S}_{sw} = \int d^3x \left[\frac{1}{2\rho_s} J_\mu^2 + iJ_\mu \partial_\mu \theta \right]$$

Integrating over θ yields $\partial_\mu J_\mu = 0$. Solve, by writing

$$J_\mu = \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda$$

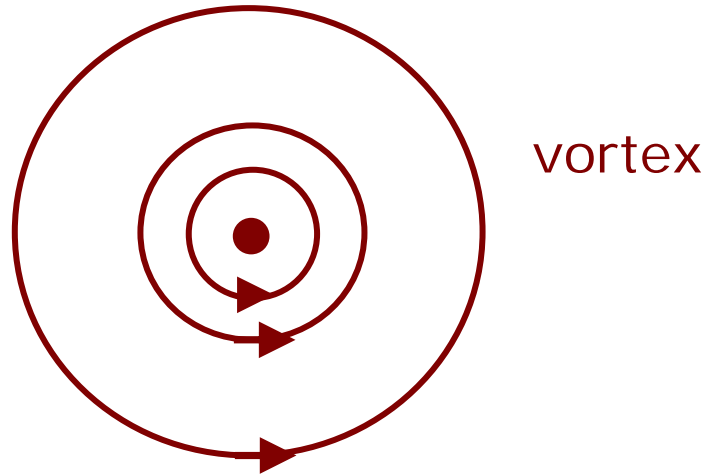
leading to

$$\mathcal{S}_{sw} = \int d^3x \left[\frac{1}{2\rho_s} (\epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda)^2 \right]$$

Spin waves are dual to a U(1) gauge theory in 2+1 dimensions

Approaching the transition from the superfluid ($f=1$)

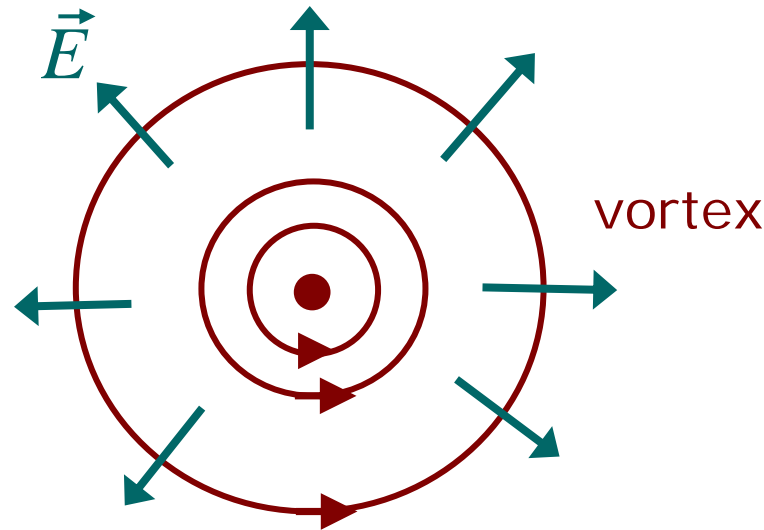
Excitations of the superfluid: (B) **Vortices**



A vortex is a point-like object. We can therefore define a local field operator, φ , which annihilates a vortex.

Approaching the transition from the superfluid ($f=1$)

Excitations of the superfluid: (B) **Vortices**



A vortex is a point-like object. We can therefore define a local field operator, φ , which annihilates a vortex.

Each vortex is the source of an 'electric field' \vec{E} associated with the U(1) gauge field A_μ .

Consequently, φ carries +1 U(1) gauge charge.

Approaching the transition from the superfluid ($f=1$)

Excitations of the superfluid: **Spin wave and vortices**

φ : vortex annihilation operator.

$\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda$: boson current $\sim i\psi^*\partial_\mu\psi - i\partial_\mu\psi^*\psi$.

Density of vortices = density of antivortices \Rightarrow
“relativistic” field theory for φ :

$$\mathcal{S}_{\text{dual}} = \int d^3x \left[|(\partial_\mu - iA_\mu)\varphi|^2 + \tilde{s}|\varphi|^2 + \frac{\tilde{u}}{2}|\varphi|^4 + \frac{1}{2\rho_s}(\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda)^2 \right]$$

Superfluid $\Leftrightarrow \langle \varphi \rangle = 0$

Insulator $\Leftrightarrow \langle \varphi \rangle \neq 0$

Dual theories of the superfluid-insulator transition ($f=1$)

Excitations of the superfluid: **Spin wave and vortices**

Using the boson quasiparticle excitations of the insulator $\sim \psi$

$$\mathcal{S} = \int d^2r d\tau \left[|\partial_\tau \psi|^2 + |\nabla_r \psi|^2 + s|\psi|^2 + \frac{u}{2}|\psi|^4 \right]$$

$$\text{Insulator} \Leftrightarrow \langle \psi \rangle = 0$$

$$\text{Superfluid} \Leftrightarrow \langle \psi \rangle \neq 0$$

is dual to

Using the vortex quasiparticle excitations of the superfluid $\sim \varphi$

$$\mathcal{S}_{\text{dual}} = \int d^3x \left[|(\partial_\mu - iA_\mu)\varphi|^2 + \tilde{s}|\varphi|^2 + \frac{\tilde{u}}{2}|\varphi|^4 + \frac{1}{2\rho_s}(\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda)^2 \right]$$

$$\text{Superfluid} \Leftrightarrow \langle \varphi \rangle = 0$$

$$\text{Insulator} \Leftrightarrow \langle \varphi \rangle \neq 0$$

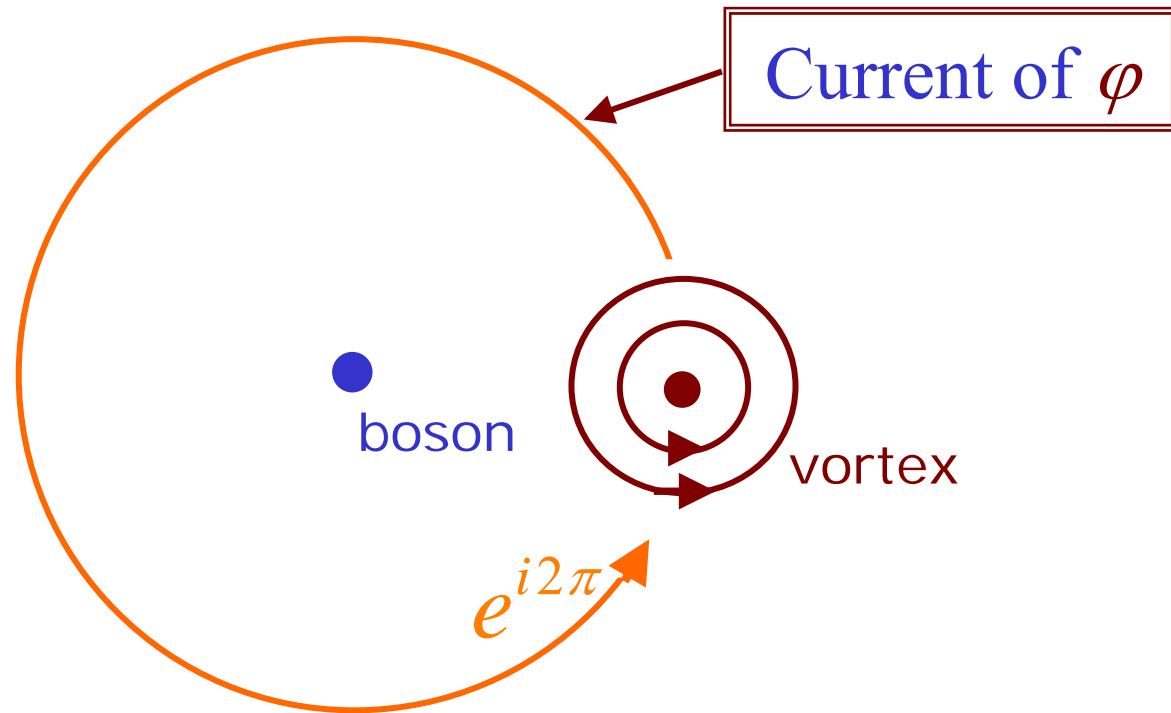
A vortex in the vortex field is the original boson

A vortex in φ carries 2π flux in the ‘magnetic field’

$B = \epsilon_{\tau\mu\nu}\partial_\mu A_\nu$. But this is just the original boson number operator. Consequently, in the path integral viewpoint, the world line of the vortex in φ is just the world line of the original boson.

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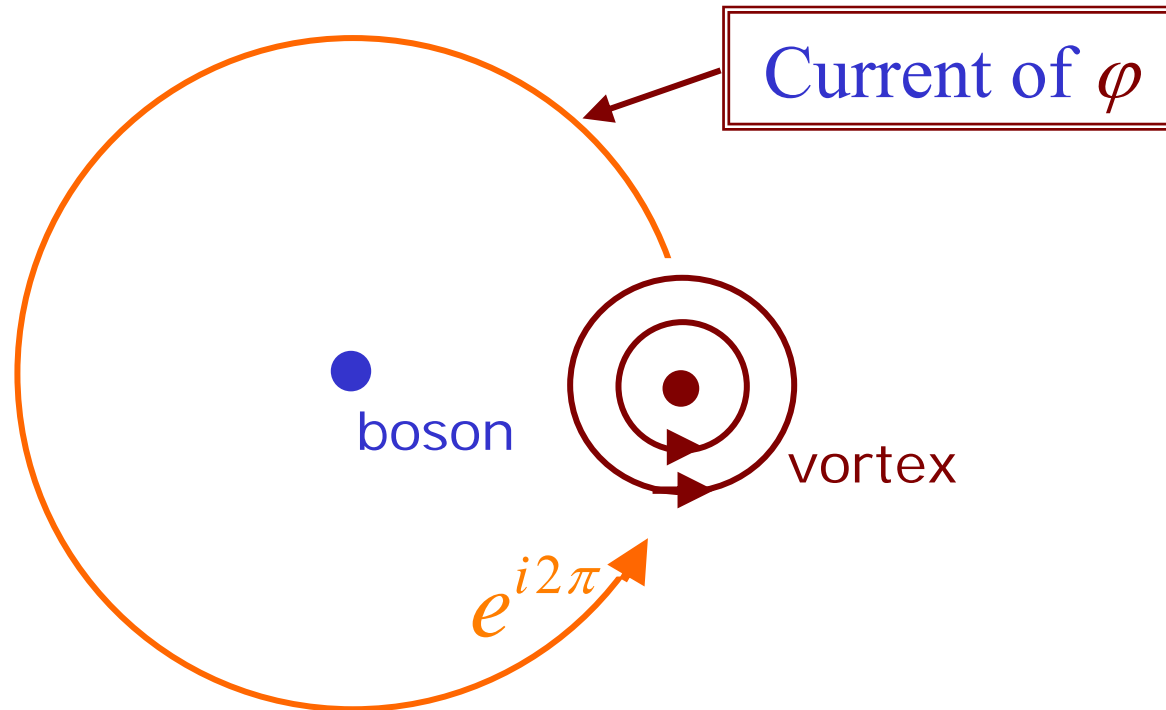
The wavefunction of a vortex acquires a phase of 2π each time the vortex encircles a boson

D. Boson-vortex duality

2. Bosons in a lattice at fractional filling f

L. Balents, L. Bartosch, A. Burkov, S. Sachdev, K. Sengupta,
Physical Review B **71**, 144508 and 144509 (2005),
cond-mat/0502002, and cond-mat/0504692.

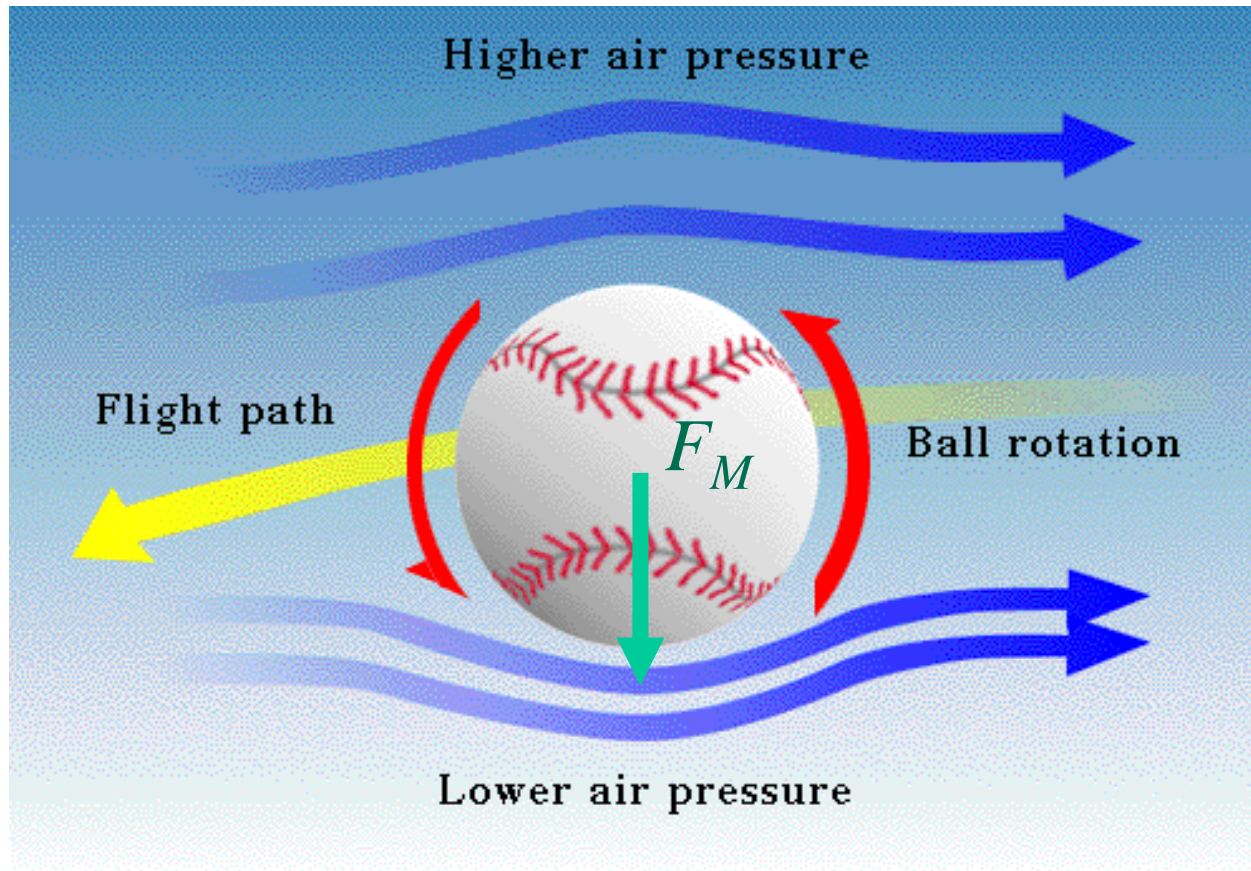
Boson-vortex duality



The wavefunction of a vortex acquires a phase of 2π each time the vortex encircles a boson

Strength of “magnetic” field on vortex field φ
= density of bosons = f flux quanta per plaquette

In ordinary fluids, vortices experience the Magnus Force



$$F_M = (\text{mass density of air}) \cdot (\text{velocity of ball}) \cdot (\text{circulation})$$

For a vortex in a superfluid, this is

$$\begin{aligned}\mathbf{F}_M &= (m\rho) \left(\left(\mathbf{v}_s - \frac{d\mathbf{r}_v}{dt} \right) \times \hat{\mathbf{z}} \right) \left(\oint \mathbf{v}_s \cdot d\mathbf{r} \right) \\ &= nh\rho \left(\mathbf{v}_s - \frac{d\mathbf{r}_v}{dt} \right) \times \hat{\mathbf{z}}\end{aligned}$$

where ρ = number density of bosons

\mathbf{v}_s = local velocity of superfluid

\mathbf{r}_v = position of vortex

For a vortex in a superfluid, this is

$$\begin{aligned}\mathbf{F}_M &= (m\rho) \left(\left(\mathbf{v}_s - \frac{d\mathbf{r}_v}{dt} \right) \times \hat{\mathbf{z}} \right) \left(\oint \mathbf{v}_s \cdot d\mathbf{r} \right) \\ &= nh\rho \left(\mathbf{v}_s - \frac{d\mathbf{r}_v}{dt} \right) \times \hat{\mathbf{z}} \\ &= n \left(\mathbf{E} + \frac{d\mathbf{r}_v}{dt} \times \mathbf{B} \right)\end{aligned}$$

where $\mathbf{E} = \rho\mathbf{v}_s \times \hat{\mathbf{z}}$ and $\mathbf{B} = -h\rho\hat{\mathbf{z}}$

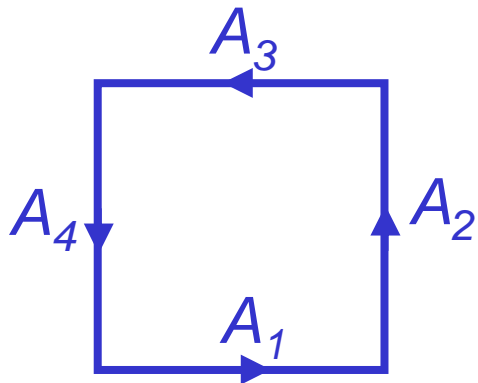
Dual picture:

The vortex is a quantum particle with dual “electric” charge n , moving in a dual “magnetic” field of strength = $h \times$ (number density of Bose particles)

- The vortices are quantum particles moving in a periodic potential with the symmetry of the square lattice, and in the presence of a dual “magnetic” field of strength $= h\rho$, where ρ is the number density of bosons per unit cell.
- The vortex motion can be described by the effective Hofstadter Hamiltonian:

$$\mathcal{H}_v = -t \sum_{\langle ij \rangle} (e^{iA_{ij}} \varphi_i^* \varphi_j + \text{c.c.})$$

where φ_i is an operator which annihilates a vortex particle at site i of a square lattice.



$$A_1 + A_2 + A_3 + A_4 = 2\pi f$$

where f is the boson filling fraction.

Bosons at filling fraction $f = 1$

- At $f=1$, the “magnetic” flux per unit cell is 2π , and the vortex does not pick up any phase from the boson density.
- The effective dual “magnetic” field acting on the vortex is zero, and the corresponding component of the Magnus force vanishes.

Bosons at rational filling fraction $f=p/q$

Quantum mechanics of the vortex “particle” in a periodic potential with f flux quanta per unit cell

Space group symmetries of Hofstadter Hamiltonian:

T_x, T_y : Translations by a lattice spacing in the x, y directions

R : Rotation by 90 degrees.

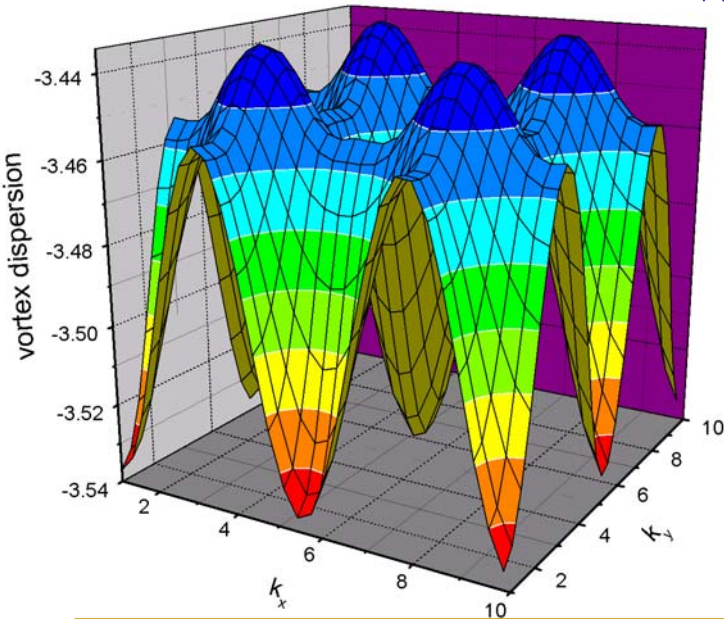
Magnetic space group:

$$T_x T_y = e^{2\pi i f} T_y T_x \ ;$$

$$R^{-1} T_y R = T_x \ ; \ R^{-1} T_x R = T_y^{-1} \ ; \ R^4 = 1$$

The low energy vortex states must form a representation of this algebra

Vortices in a superfluid near a Mott insulator at filling $f=p/q$ Hofstadter spectrum of the quantum vortex “particle” with field operator φ



At filling $f=p/q$, there are q species of vortices, φ_ℓ (with $\ell=1\dots q$), associated with q degenerate minima in the vortex spectrum. These vortices realize the smallest, q -dimensional, representation of the magnetic algebra.

The q vortices form a *projective* representation of the space group

$$T_x : \varphi_\ell \rightarrow \varphi_{\ell+1} \quad ; \quad T_y : \varphi_\ell \rightarrow e^{2\pi i \ell f} \varphi_\ell$$

$$R : \varphi_\ell \rightarrow \frac{1}{\sqrt{q}} \sum_{m=1}^q \varphi_m e^{2\pi i \ell m f}$$

Boson-vortex duality

The $q \varphi_\ell$ vortices characterize *both* superconducting and density wave orders

Superconductor/insulator : $\langle \varphi_\ell \rangle = 0 / \langle \varphi_\ell \rangle \neq 0$

Boson-vortex duality

The q φ_ℓ vortices characterize *both* superconducting and density wave orders

Density wave order:

Status of space group symmetry determined by

density operators $\rho_{\mathbf{Q}}$ at wavevectors $\mathbf{Q}_{mn} = \frac{2\pi p}{q}(m, n)$

$$\rho_{mn} = e^{i\pi mnf} \sum_{\ell=1}^q \varphi_\ell^* \varphi_{\ell+n} e^{2\pi i \ell m f}$$

$$T_x : \rho_{\mathbf{Q}} \rightarrow \rho_{\mathbf{Q}} e^{i\mathbf{Q} \cdot \hat{x}} \quad ; \quad T_y : \rho_{\mathbf{Q}} \rightarrow \rho_{\mathbf{Q}} e^{i\mathbf{Q} \cdot \hat{y}}$$

$$R : \rho(\mathbf{Q}) \rightarrow \rho(R\mathbf{Q})$$

Field theory with projective symmetry

Degrees of freedom:

q complex φ_ℓ vortex fields

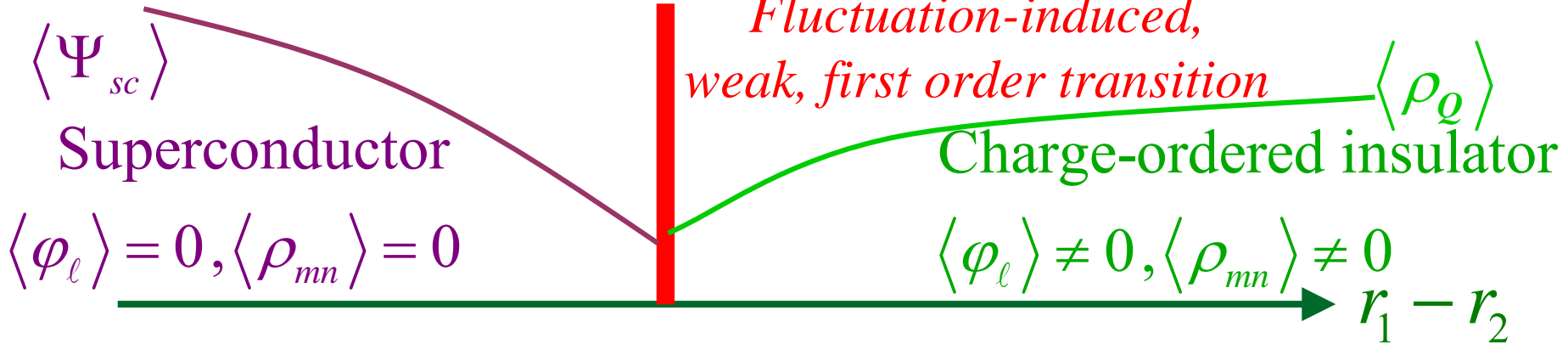
1 non-compact U(1) gauge field A_μ

$$\mathcal{S} = \int d^2x d\tau \left[\sum_\ell \{ |(\partial_\mu - iA_\mu)\varphi_\ell|^2 + s|\varphi_\ell|^2 \} + \frac{1}{2e^2} (\epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda)^2 + \sum_{lmn} \gamma_{lmn} \varphi_\ell^* \varphi_{\ell+m}^* \varphi_{\ell+n} \varphi_{\ell+m-n} \right]$$

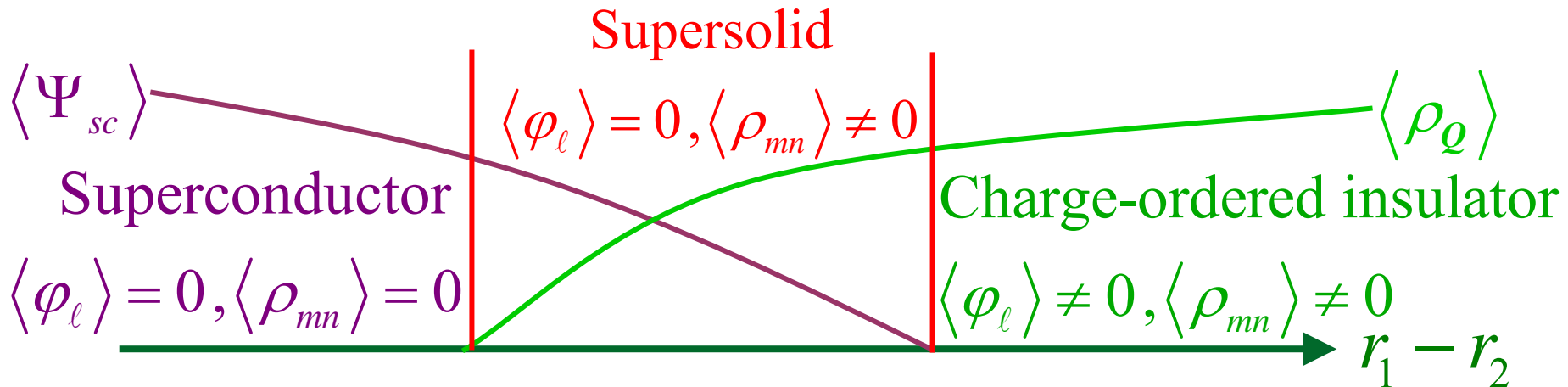
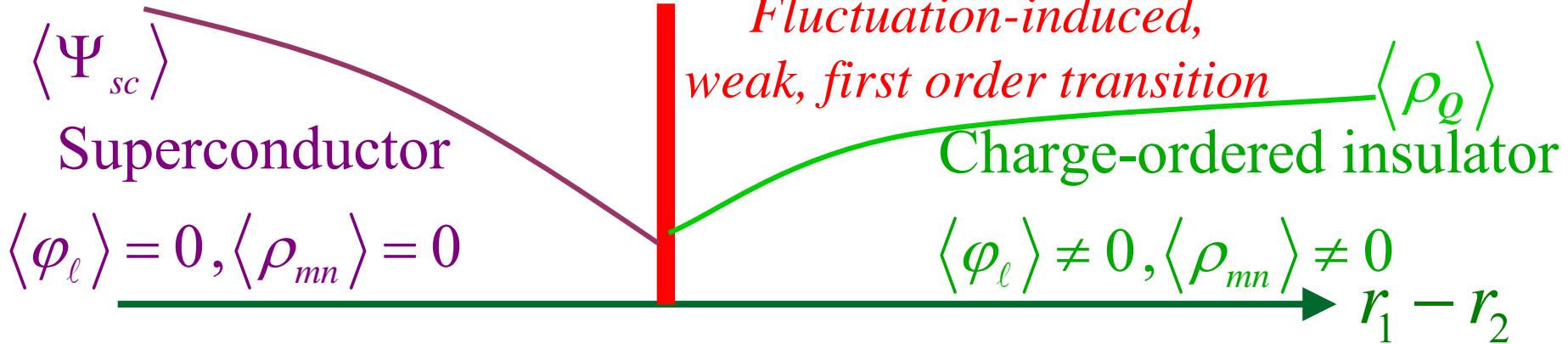
The projective symmetries constrain the couplings γ_{mn} to obey

$$\begin{aligned} \gamma_{mn} &= \gamma_{-m,-n} \quad ; \quad \gamma_{mn} = \gamma_{m,m-n} \quad ; \quad \gamma_{mn} = \gamma_{m-2n,-n} \\ \gamma_{\bar{m}\bar{n}} &= \frac{1}{q} \sum_{mn} \gamma_{mn} e^{-2\pi i f [n(\bar{m}-\bar{n}) + \bar{n}(m-n)]} \end{aligned}$$

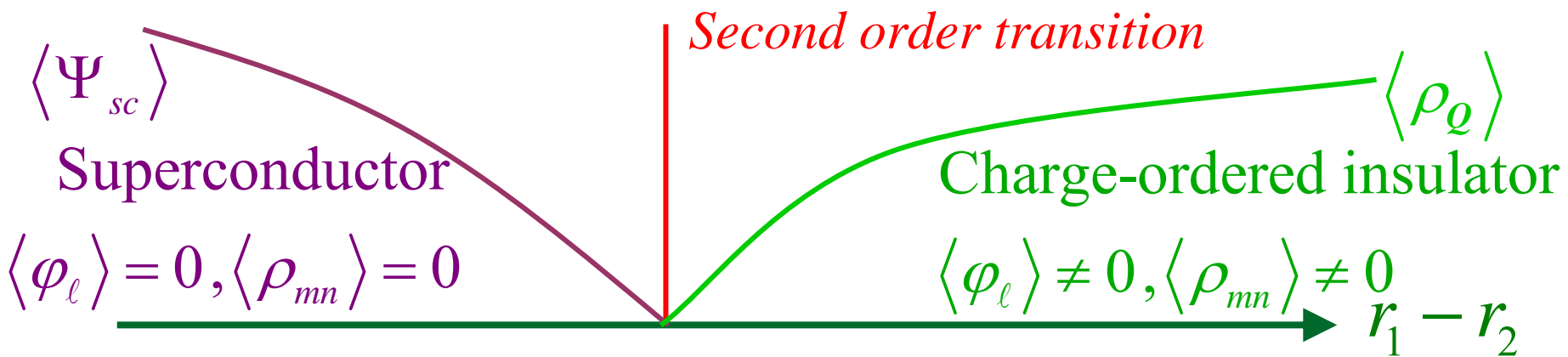
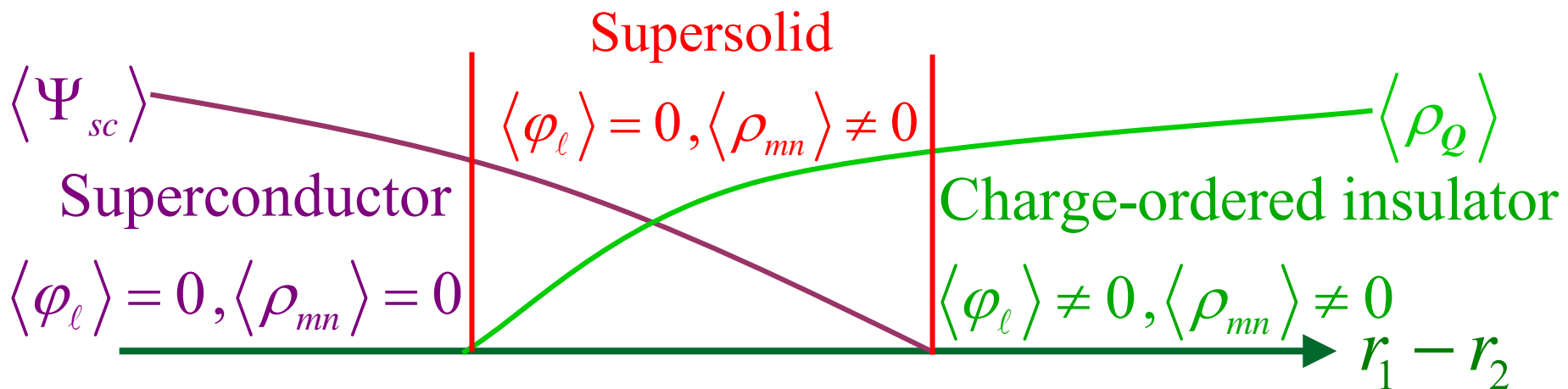
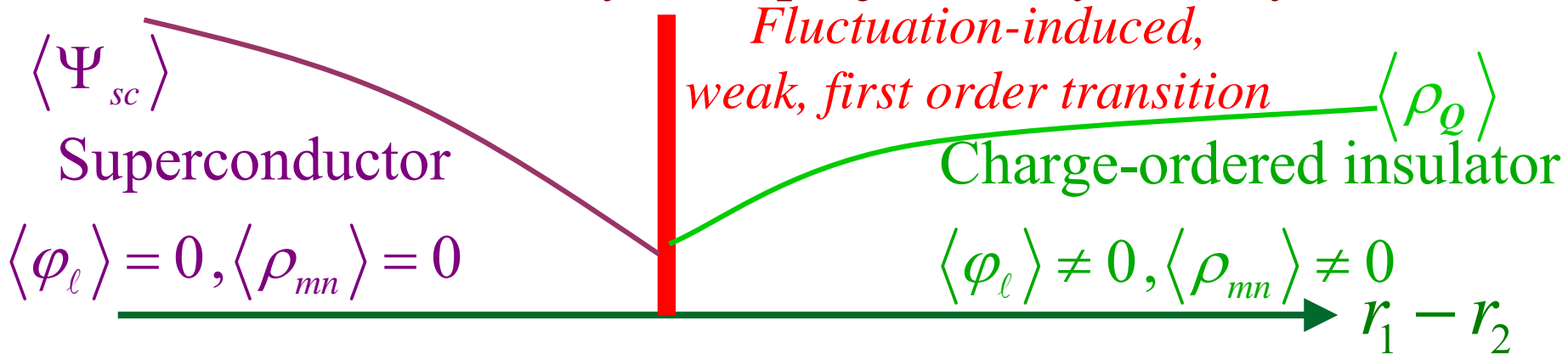
Field theory with projective symmetry



Field theory with projective symmetry

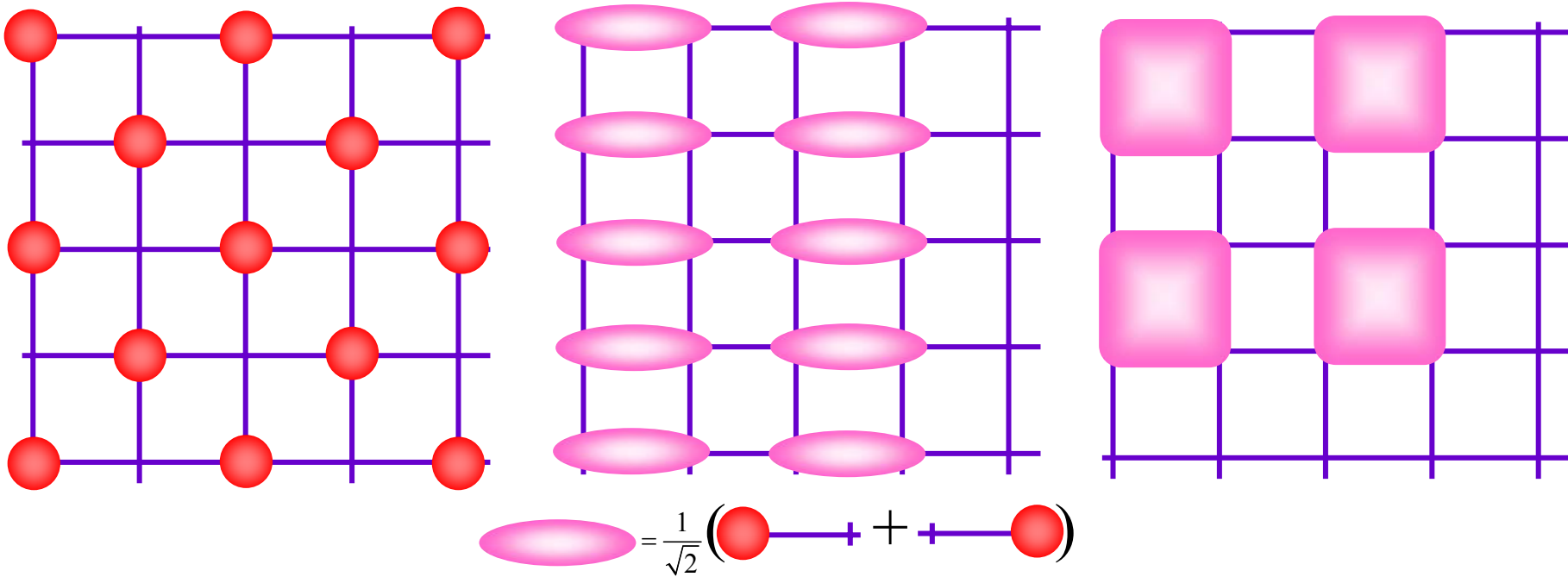


Field theory with projective symmetry



Field theory with projective symmetry

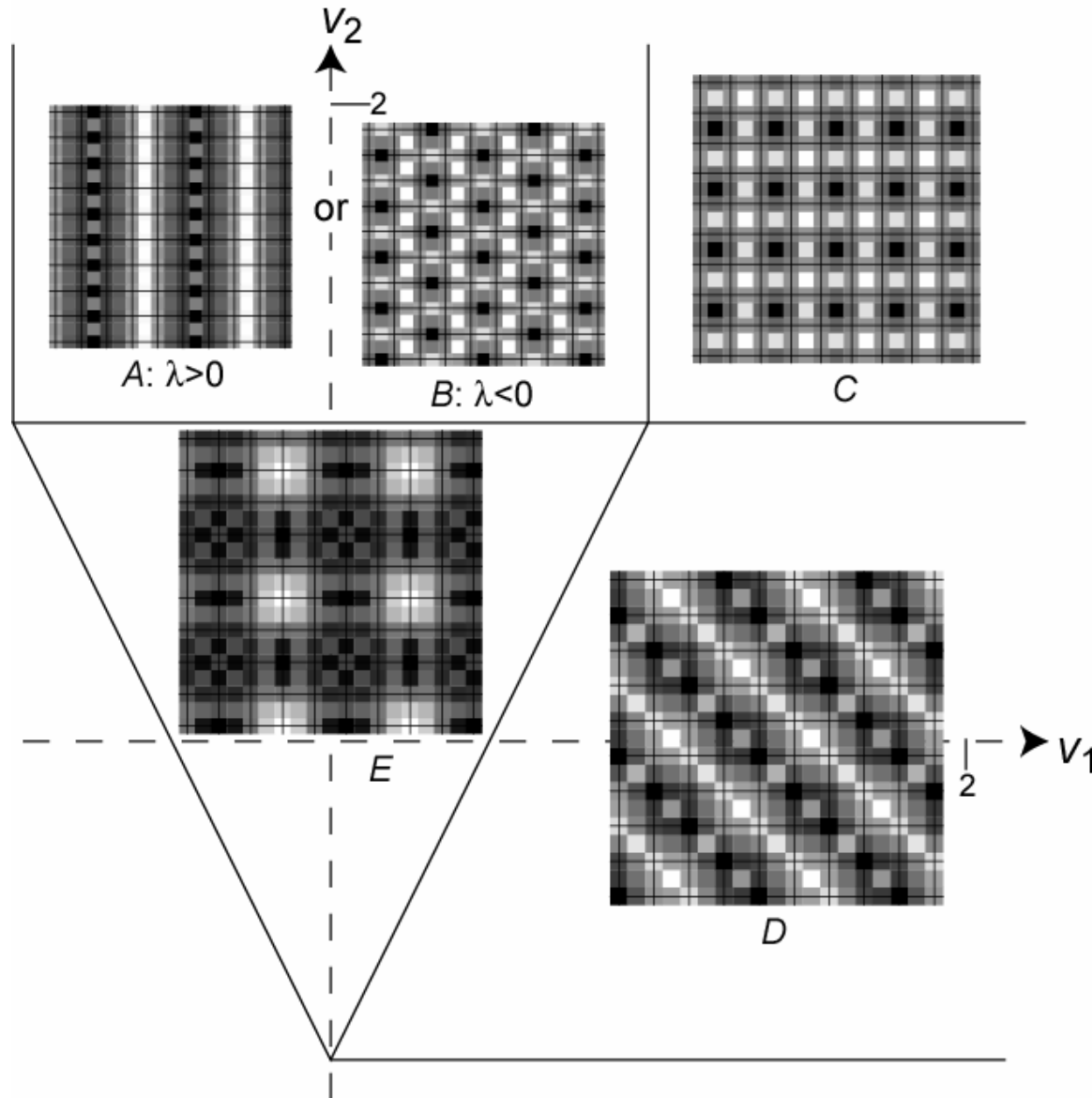
Spatial structure of insulators for $q=2$ ($f=1/2$)



All insulating phases have density-wave order $\rho(\mathbf{r}) = \sum_{\mathbf{Q}} \rho_{\mathbf{Q}} e^{i\mathbf{Q}\cdot\mathbf{r}}$ with $\langle \rho_{\mathbf{Q}} \rangle \neq 0$

Field theory with projective symmetry

Spatial structure of insulators for $q=4$ ($f=1/4$ or $3/4$)



$a \times b$ unit cells;
 $\frac{q}{a}, \frac{q}{b}, \frac{ab}{q}$,
all integers

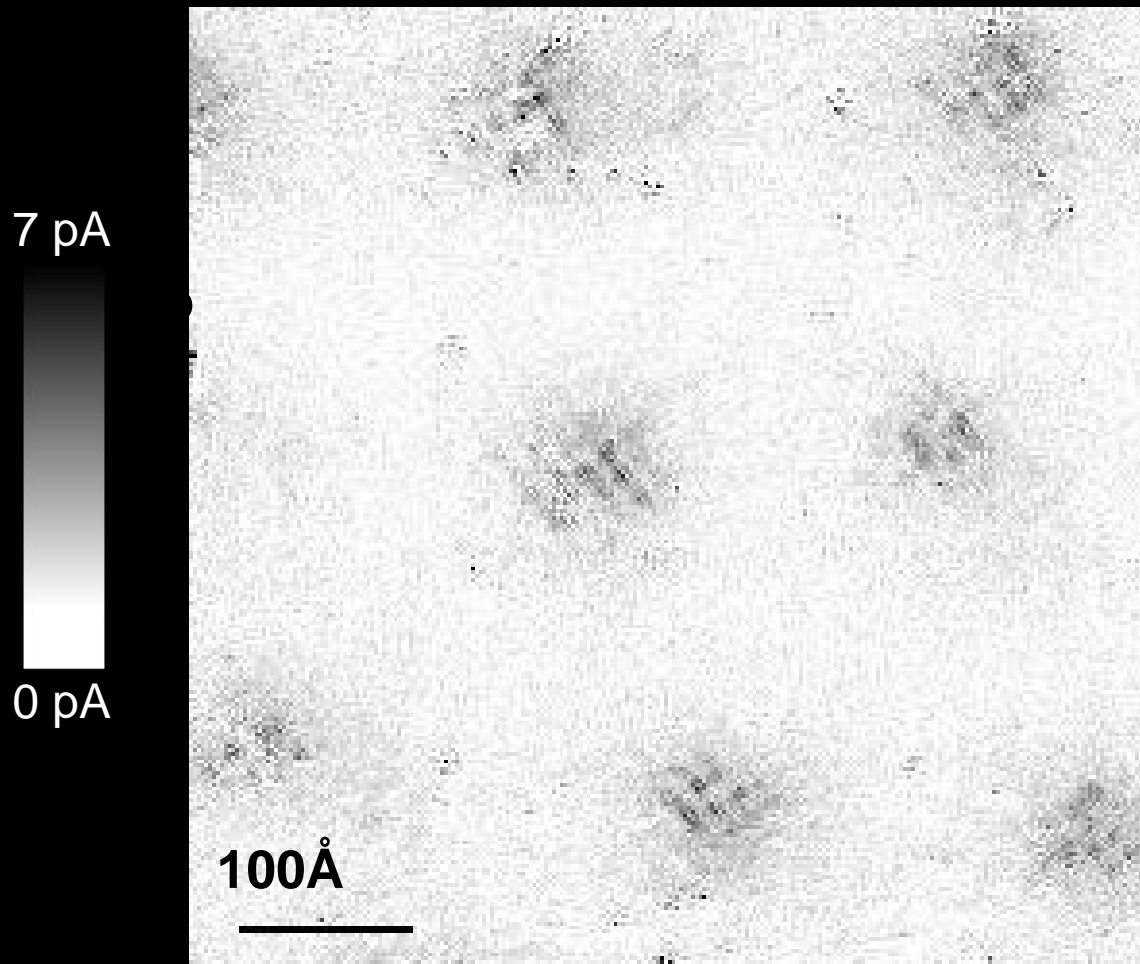
Field theory with projective symmetry

Density operators ρ_Q at wavevectors $Q_{mn} = \frac{2\pi p}{q}(m, n)$

$$\rho_{mn} = e^{i\pi mnf} \sum_{\ell=1}^q \varphi_{\ell}^* \varphi_{\ell+n} e^{2\pi i \ell mf}$$

Each pinned vortex in the superfluid has a halo of density wave order over a length scale \approx the zero-point quantum motion of the vortex. This scale diverges upon approaching the insulator

Vortex-induced LDOS of $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}$ integrated from 1meV to 12meV at 4K



Vortices have halos with LDOS modulations at a period ≈ 4 lattice spacings

J. Hoffman E. W. Hudson, K. M. Lang, V. Madhavan, S. H. Pan, H. Eisaki, S. Uchida, and J. C. Davis, *Science* 295, 466 (2002).

Prediction of VBS order near vortices: K. Park and S. Sachdev, *Phys. Rev. B* 64, 184510 (2001).

Superfluids near Mott insulators

The Mott insulator has average Cooper pair density, $f = p/q$ per site, while the density of the superfluid is close (but need not be identical) to this value

- Vortices with flux $h/(2e)$ come in multiple (usually q) “flavors”
- The lattice space group acts in a projective representation on the vortex flavor space.
- These flavor quantum numbers provide a distinction between superfluids: they constitute a “quantum order”
- Any pinned vortex must choose an orientation in flavor space. This necessarily leads to modulations in the local density of states over the spatial region where the vortex executes its quantum zero point motion.