The "normal" states of the cuprates

Cornell University
August 4, 2015

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PHYSICS
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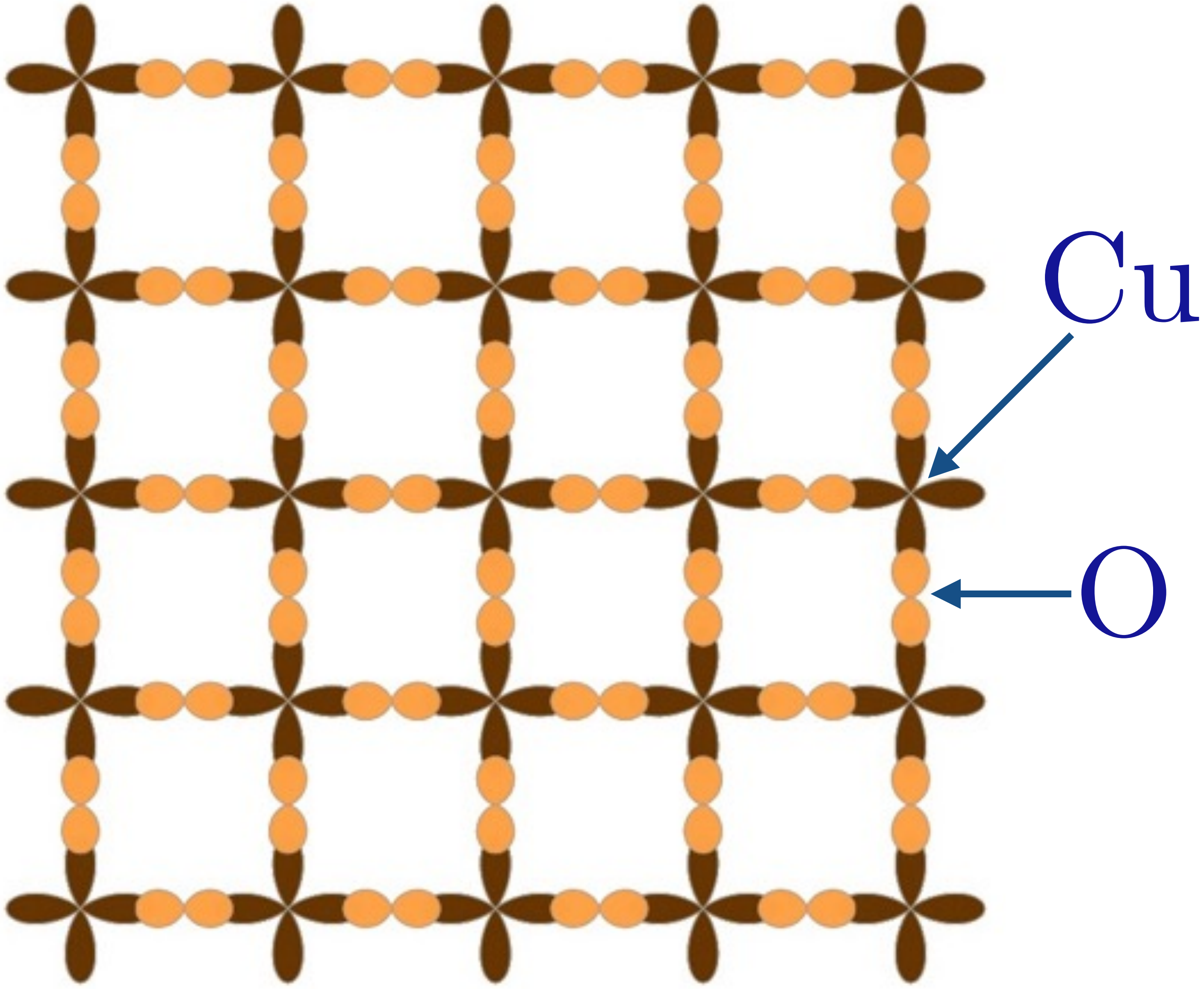
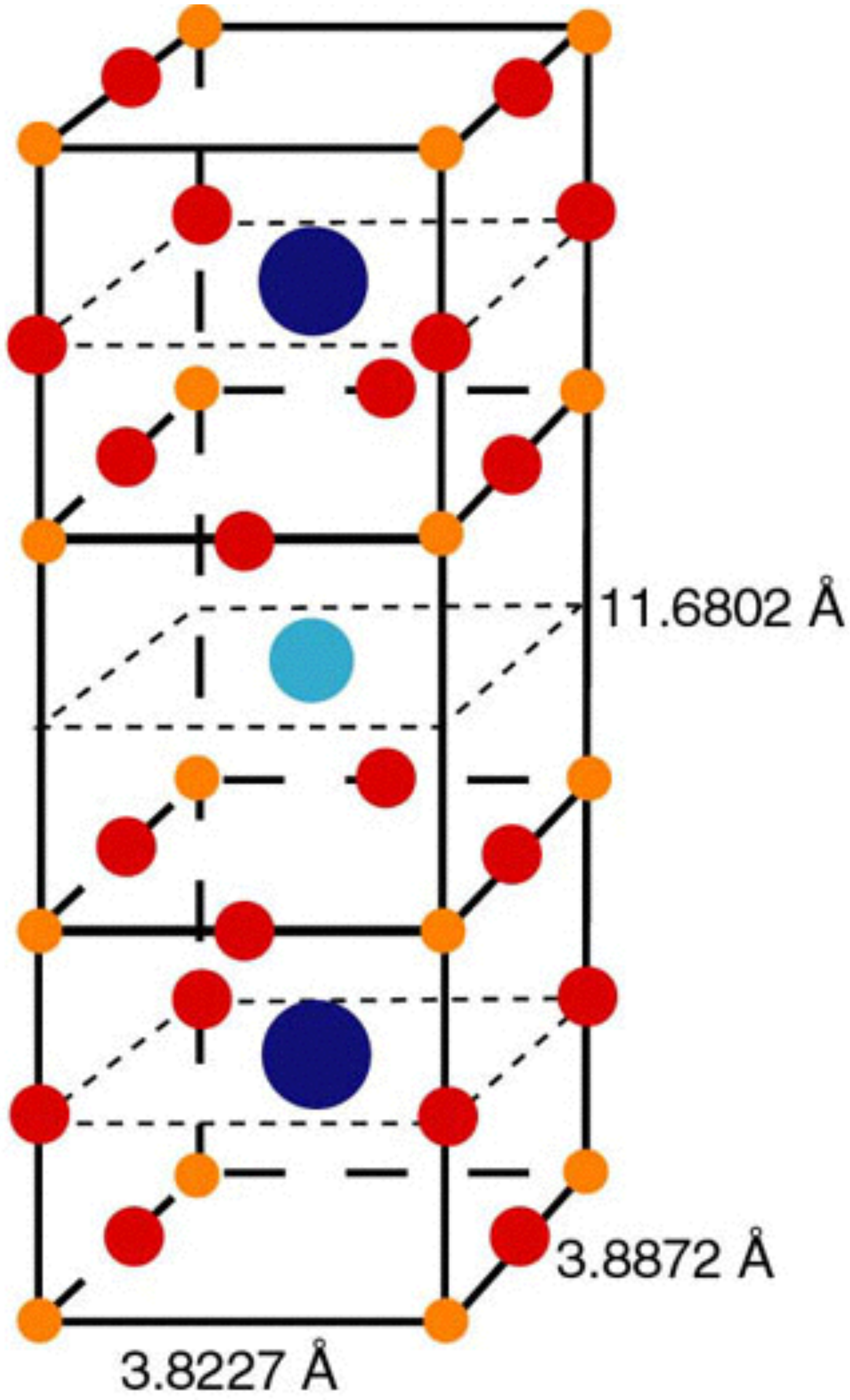
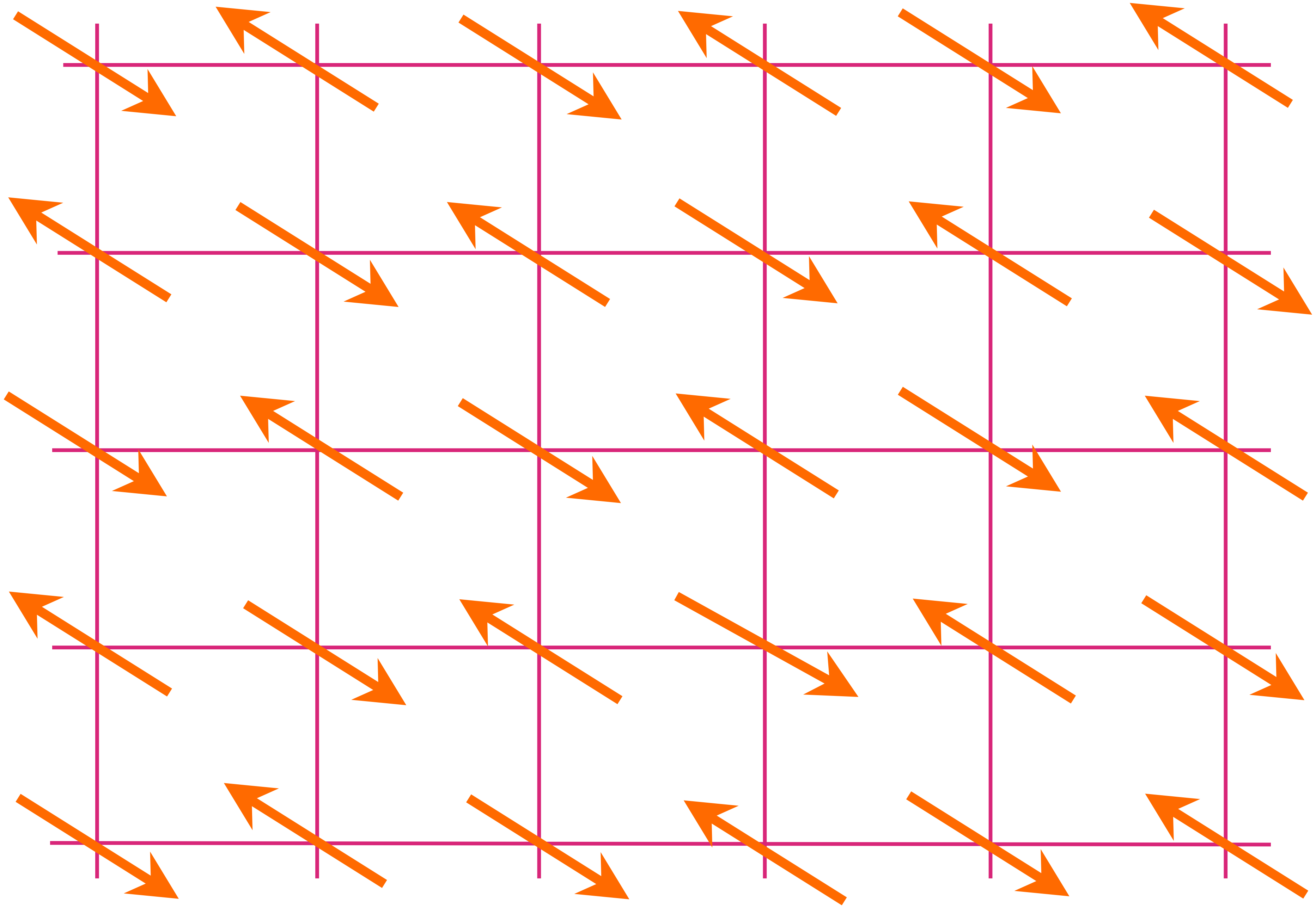
High temperature superconductors



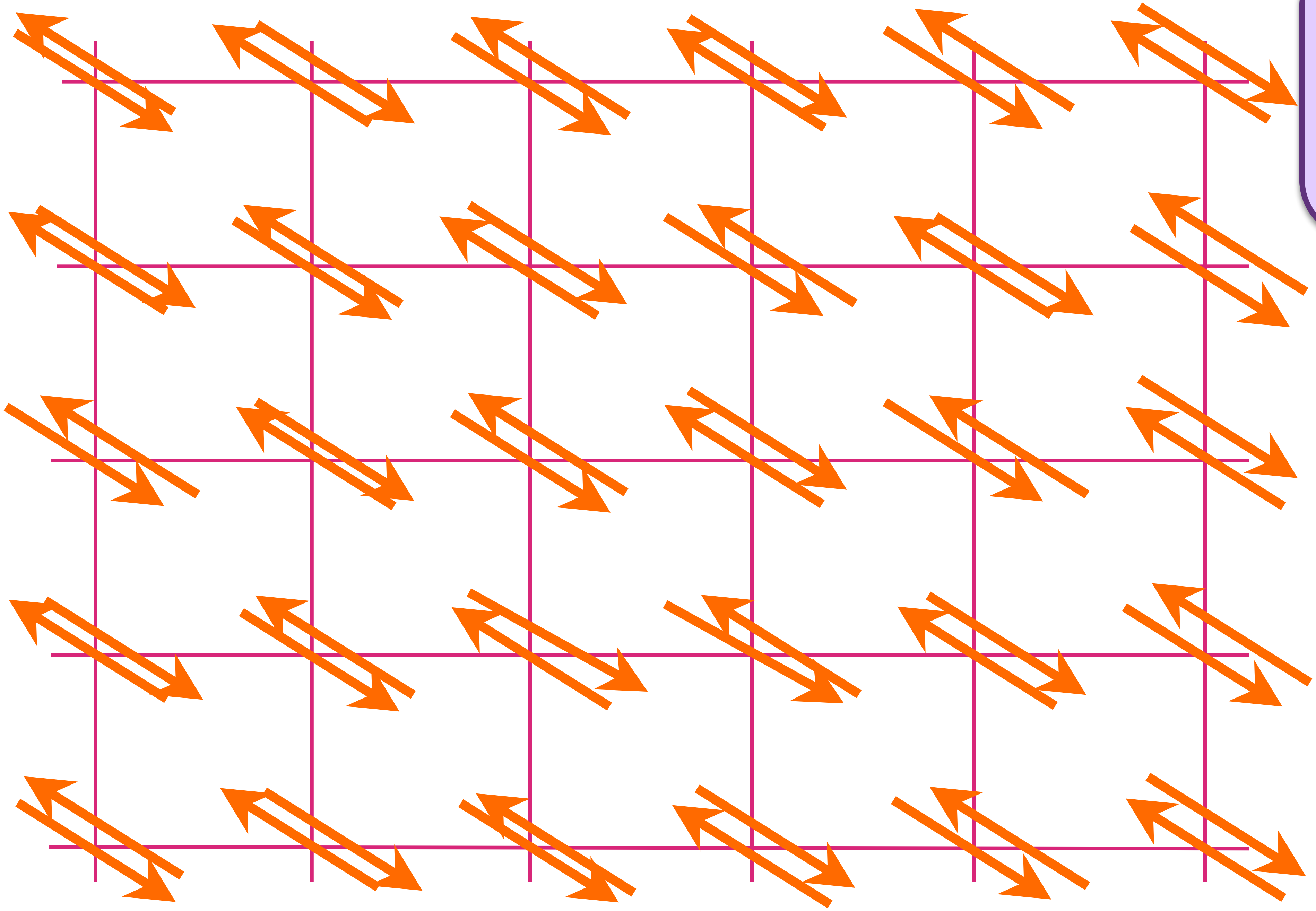
CuO_2 plane



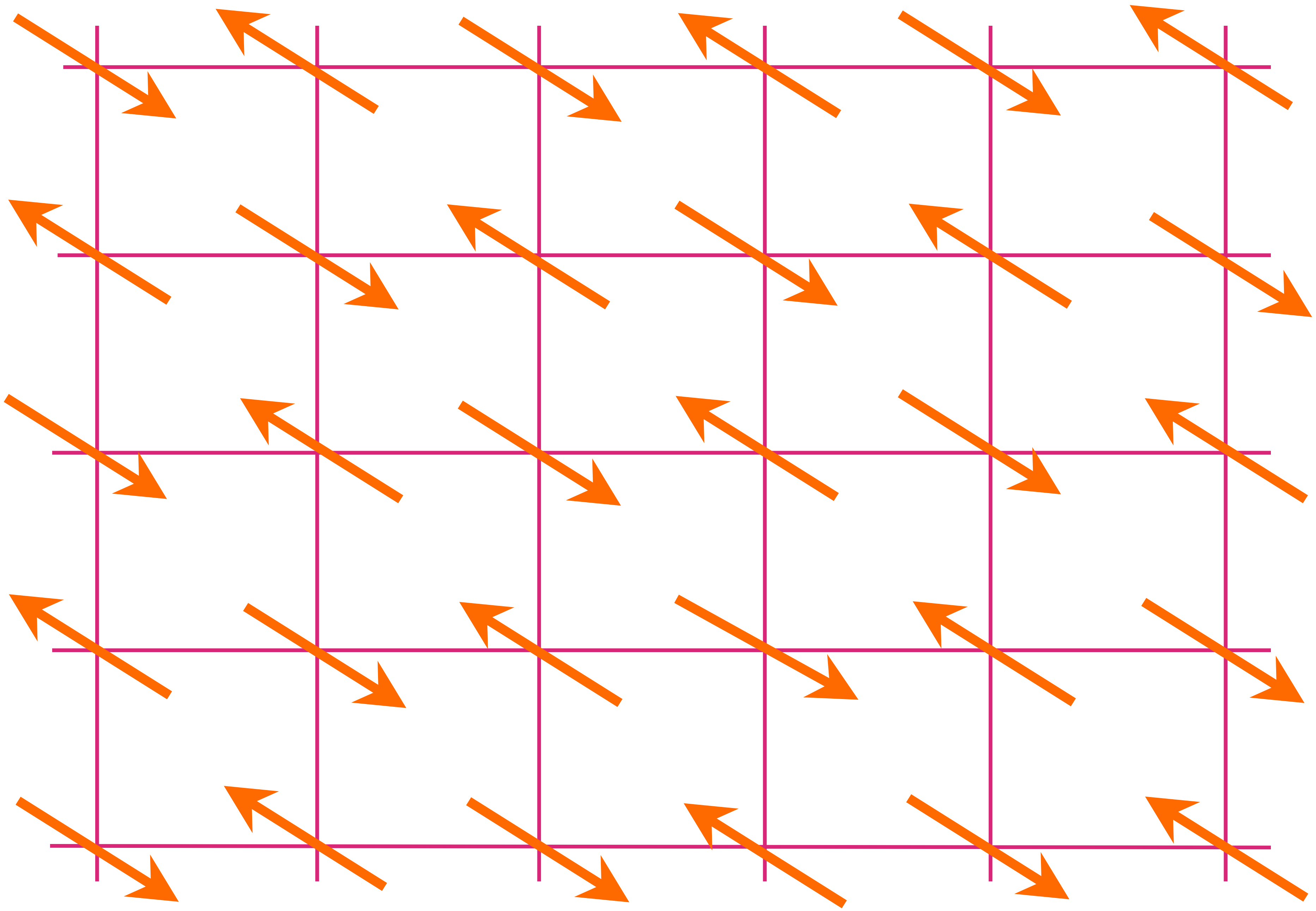
$YBa_2Cu_3O_{6+x}$



“Undoped” Anti-ferromagnet



Filled Band



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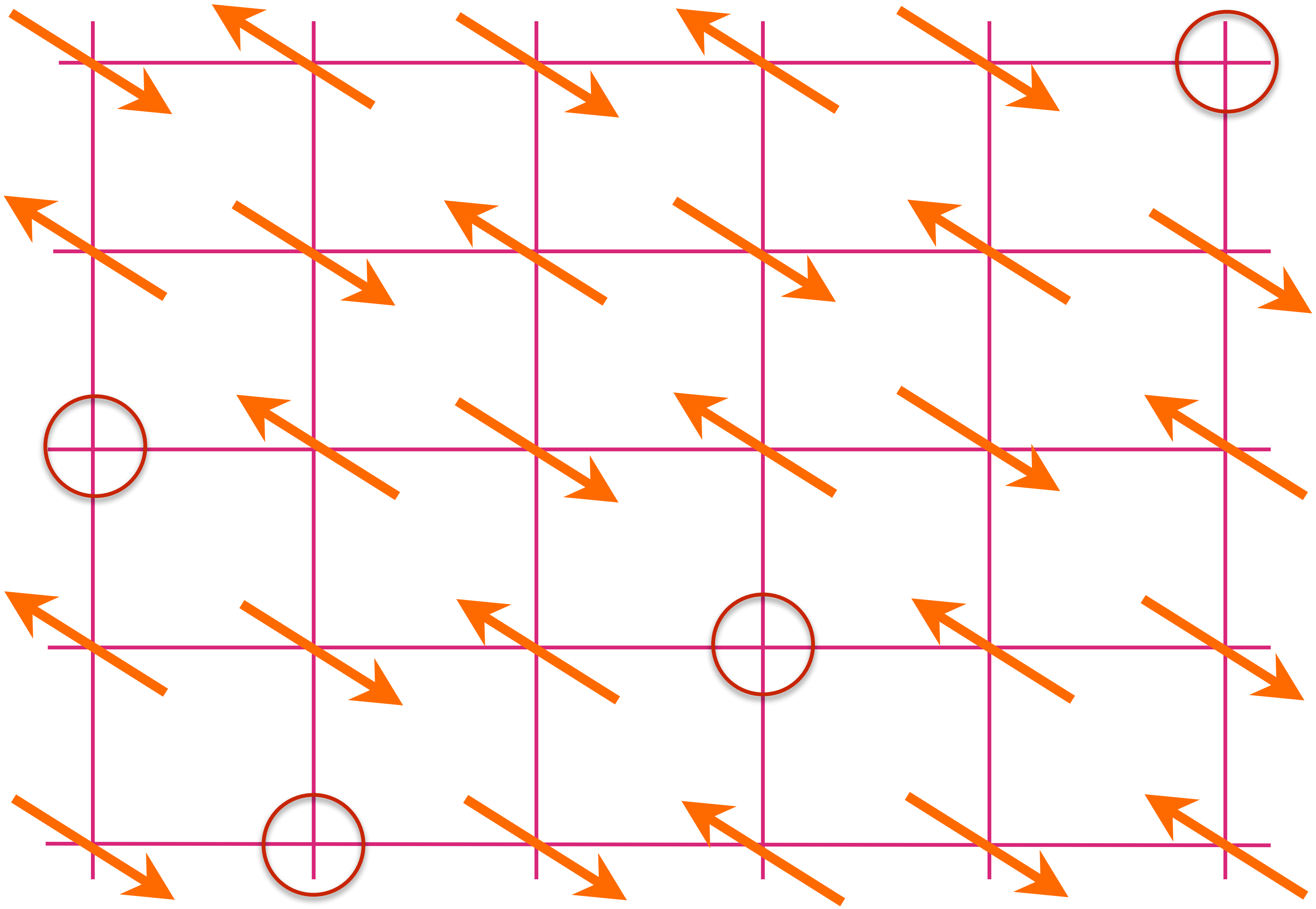
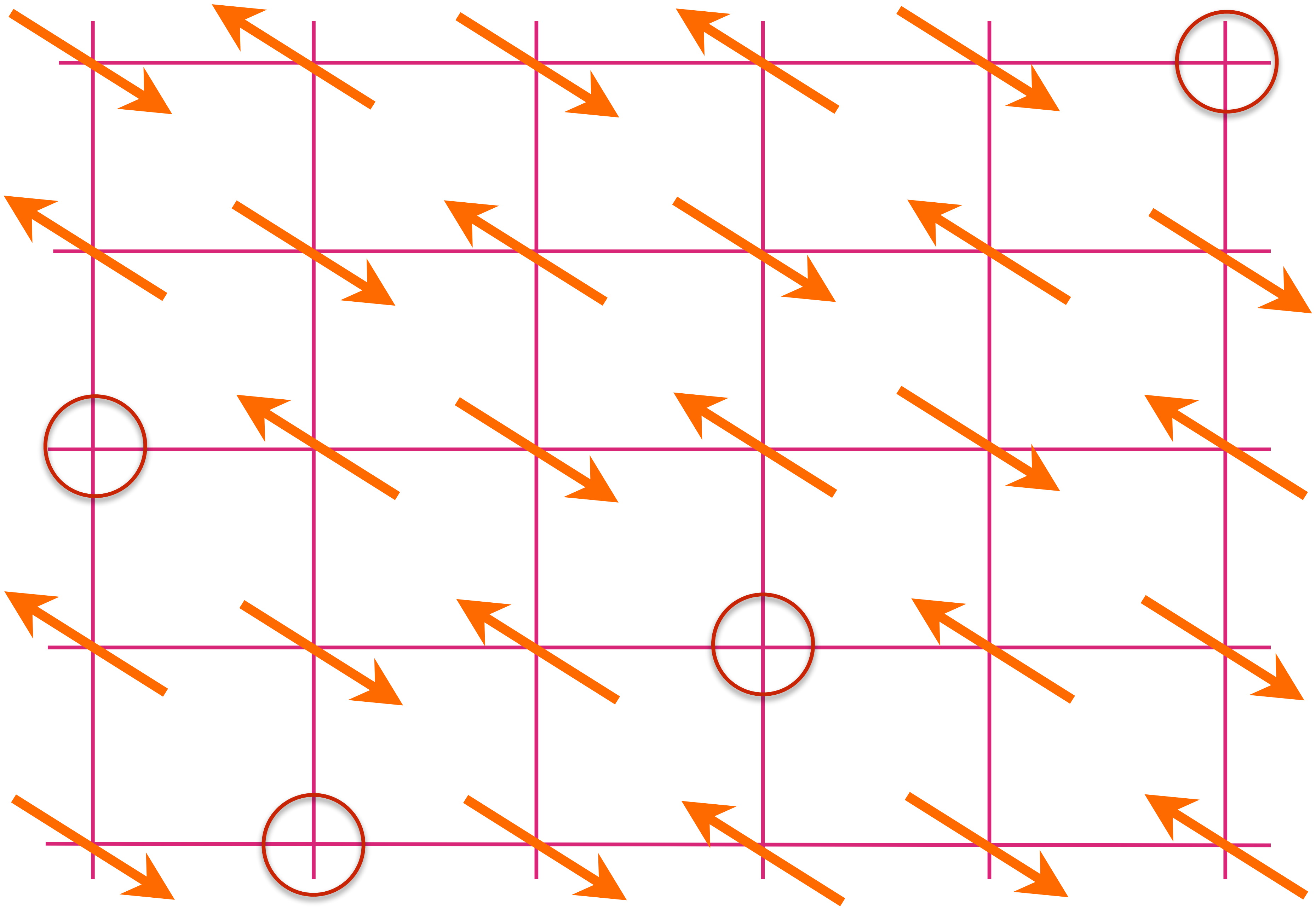


Anti-
ferromagnet
with *p* holes
per square



Anti-ferromagnet with p holes per square

But relative to the band insulator, there are $1+p$ holes per square, and so a Fermi liquid has a Fermi surface of size $1+p$

A. Review of Fermi liquid theory

Metals with with quasiparticle excitations

B. Fractionalized Fermi liquid

A Fermi liquid co-existing with topological order

C. Phase diagram of cuprates

The pseudogap metal and the strange metal

D. Quantum matter without quasiparticles

A mean-field model of a non-Fermi liquid

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Ordinary quantum matter: the Fermi liquid (FL)

- Fermi surface separates empty and occupied states in momentum space.

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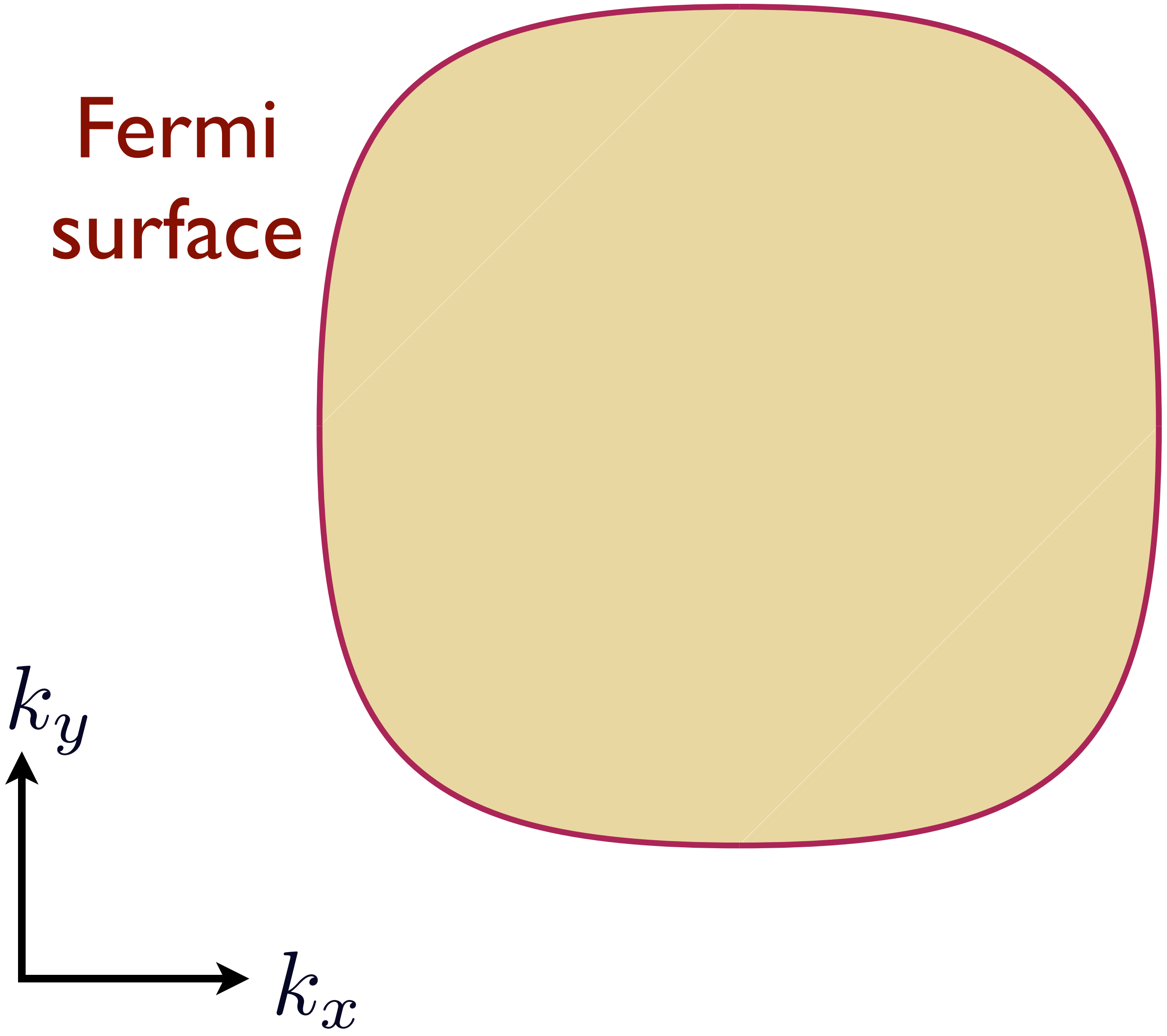

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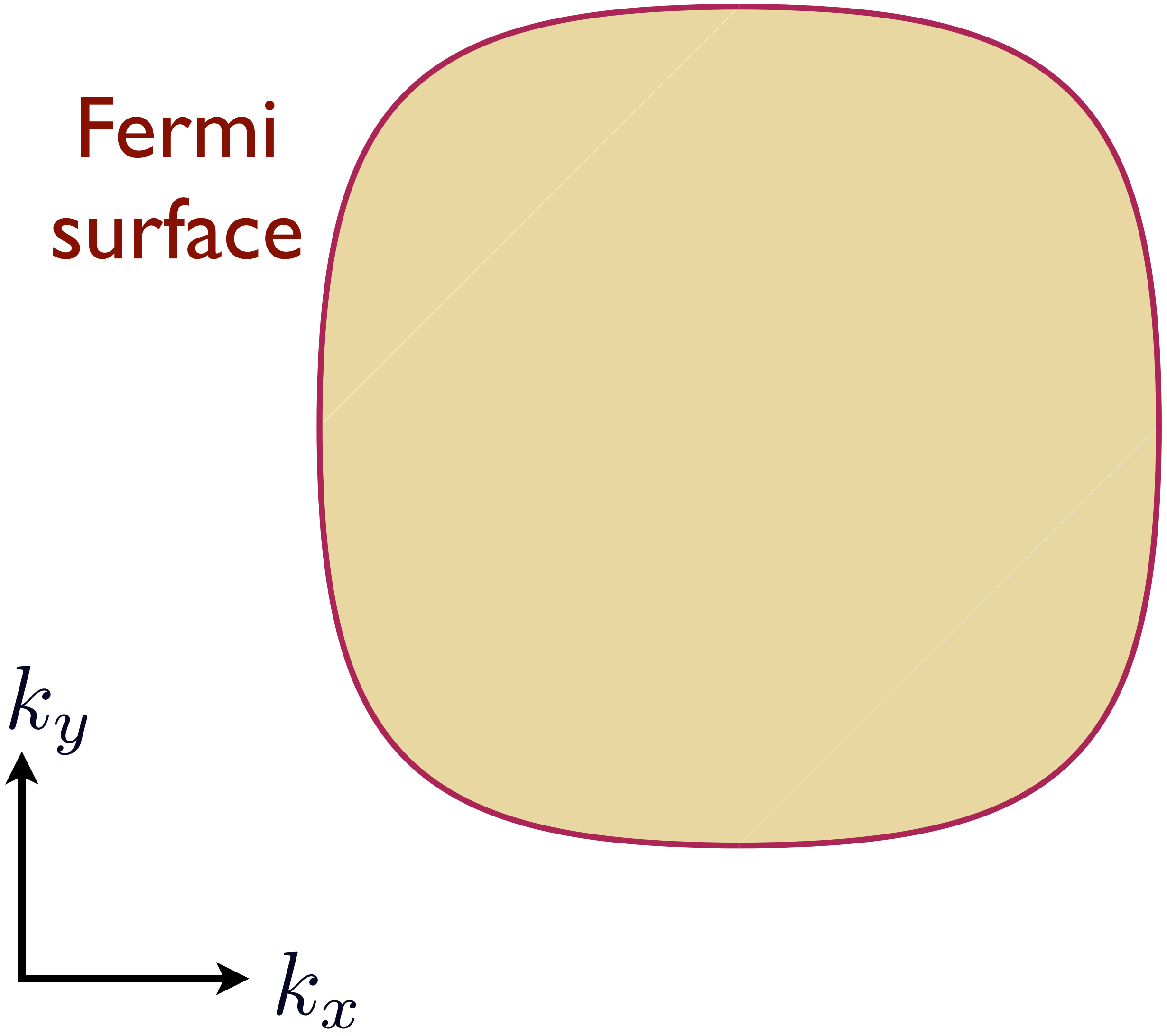

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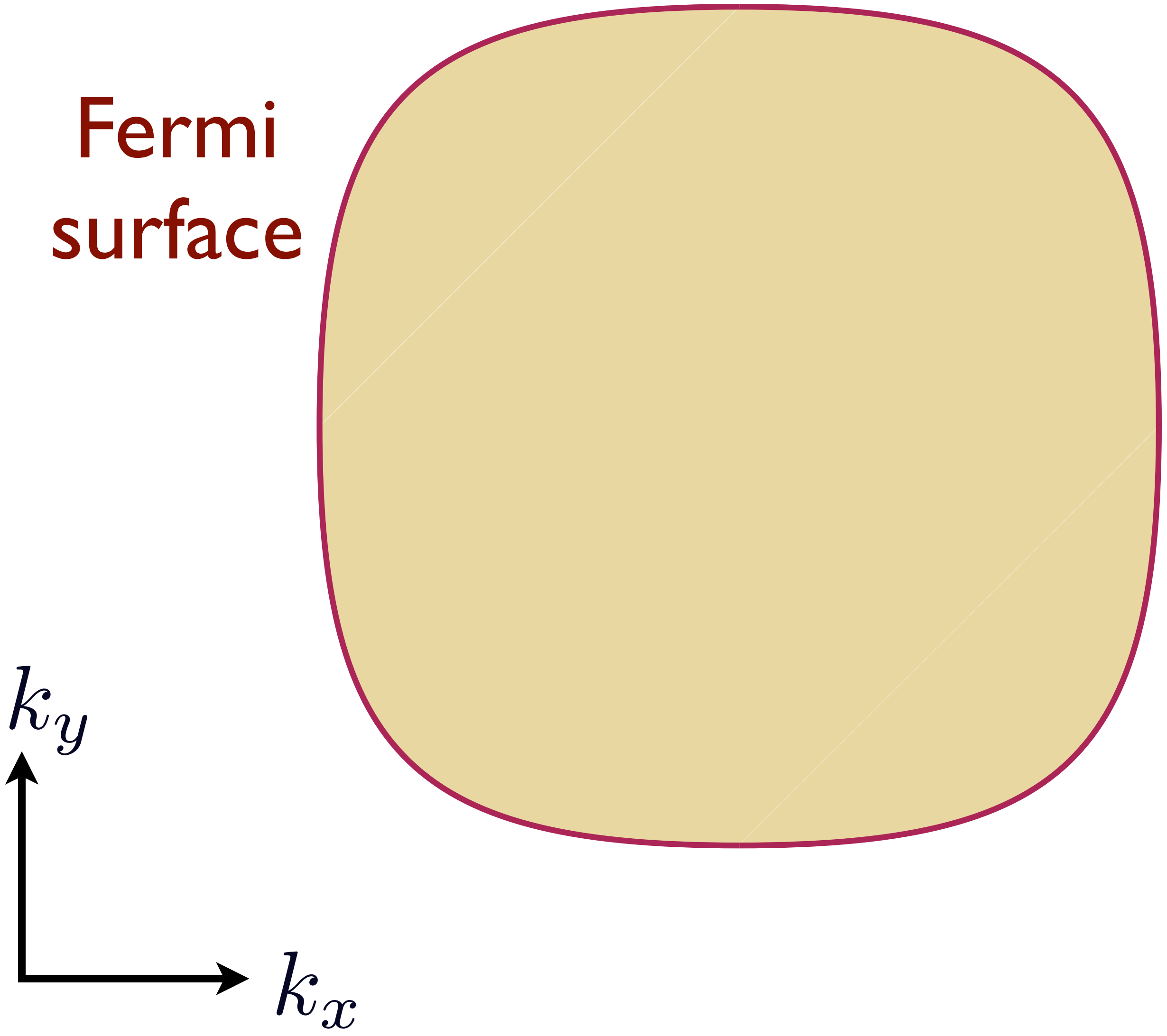


- Fermi surface separates empty and occupied states in momentum space.
- Area enclosed by Fermi surface = total density of electrons (mod 2)
- Density of electrons can be continuously varied at zero temperature.
- Long-lived electron-like quasiparticle excitations near the Fermi surface: lifetime of quasiparticles $\sim 1/T^2$.

Fermi Liquid Theory

(Chapter 15, Bruus + Flensberg).

In Hartree-Fock theory

$$G(k, i\omega_n) = \frac{Z}{i\omega_n - v_F(k - k_F)}$$

where $v_F = \frac{k_F}{m^*}$

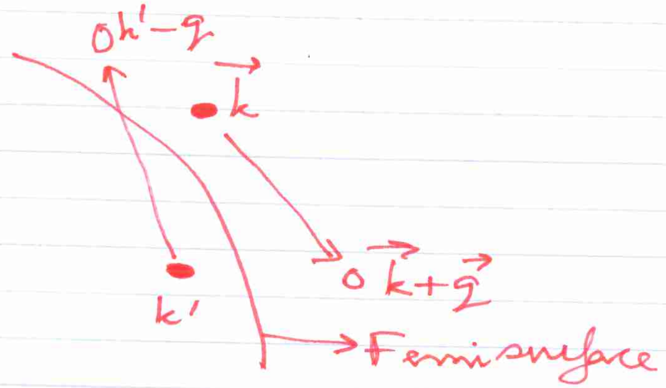
and $2\int^{k_F} \frac{d^3k}{8\pi^3} = N$ (Luttinger relation).

Beyond Hartree-Fock

Z and m^* will have corrections at order U^2.

However more significant is the imaginary of the self energy, representing decay of particle excitations.

By Fermi's Golden rule decay rate for a particle with momentum $\vec{k}$.



$$\frac{1}{\tau_k} = |U|^2 \int_{k',q} \left(1-f(\varepsilon_{k+q})\right) f(\varepsilon_{k'}) \left(1-f(\varepsilon_{k'-q})\right) \delta\left(\varepsilon_k + \varepsilon_{k'} - \varepsilon_{k+q} - \varepsilon_{k'-q}\right)$$

$$\sim |U|^2 [g(E_F)]^3 \int_{-\infty}^{0} d\varepsilon' \int_{0}^{\infty} d\varepsilon'' \, \theta\left(\varepsilon_k + \varepsilon' - \varepsilon''\right).$$

$$\sim |U|^2 [g(E_F)]^3 \, \varepsilon_k^2$$

So lifetime → ∞ as $1/\varepsilon^2$ as $\varepsilon \to 0$.

Particle-decay effects can be neglected near the Fermi surface!

Scattering rate of fermions vanishes on the Fermi surface.

$$\Rightarrow \quad \mathrm{Im}\,\Sigma(k_F, \omega \to 0) = 0.$$

$$\Rightarrow \quad G^{-1}(k = k_F, \omega = 0) = 0$$

defines the Fermi surface for an interacting system.

In a Fermi liquid

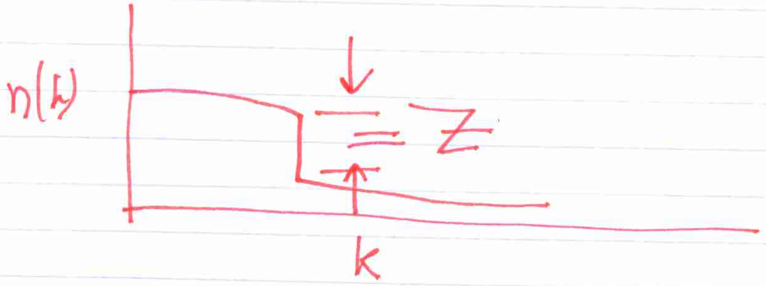
$$G(k, i\omega) = \frac{Z}{i\omega - v_F(k - k_F) + i\omega^2 \mathrm{sgn}(\omega)}$$

As $k \to k_F$ and $\omega \to 0$.

Use this to compute

$$n(k) = \int \frac{d\omega}{2\pi} G(k, i\omega) e^{i\omega 0^+}$$

Then



Luttinger Relation.

The relationship

$$N = \oint 2\int^{k_F} \frac{d^3k}{8\pi^3} \quad \text{is exact!}$$

Write

$$N = +2\int \frac{d^3k}{8\pi^3}\,\frac{d\omega}{2\pi}\, G(k, i\omega)\, e^{i\omega 0^+}$$

$$= +2i\int \frac{d^3k}{8\pi^3}\,\frac{d\omega}{2\pi}\left[-i G_k(i\omega)\frac{\partial}{\partial\omega}\Sigma_k(i\omega) - \frac{\partial}{\partial\omega}\ln G(i\omega)\right]$$

where $$G(i\omega) = \frac{1}{i\omega - \epsilon_k - \Sigma_k(i\omega)}$$

We will show below that

$$\int \frac{d^3k}{8\pi^3}\,\frac{d\omega}{2\pi}\, G_k(i\omega)\frac{\partial}{\partial\omega}\Sigma_k(i\omega) = 0 \quad (\star).$$

So

$$N = -2i\int \frac{d^3k\,d\omega}{(2\pi)^4}\, e^{i\omega 0^+}\frac{\partial}{\partial \omega}\ln G_k(i\omega)$$

$$= -2i\int \frac{d^3k}{8\pi^3}\int_{-\infty}^{0}\frac{dz}{2\pi}\frac{\partial}{\partial z}\ln\frac{G_k(z+i0^+)}{G_k(z+i0^-)}.$$

$$= -2i\int \frac{d^3k}{(2\pi)^4}\ln\frac{G_k(i0^+)}{G_k(i0^-)}.$$

$$= 2\int \frac{d^3k}{(2\pi)^3}\,\theta\left(-\varepsilon_k-\Sigma_k(i0^+)\right)$$

$$= 2\int^{k_F}\frac{d^3k}{8\pi^3}$$

where k_F is defined by

$$G^{-1}(k=k_F,\ \omega=0) = 0.$$

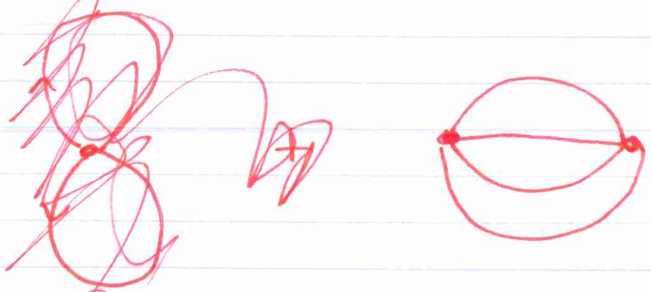
↑ location of Fermi surface.

Finally we need to establish (☆).

(☆) can be shown if there exists a functional $\Phi[G_k(i\omega)]$ (Luttinger-Ward functional) with 2 properties

(i) $$\Sigma_k(i\omega) = \frac{\delta \Phi}{G_k(i\omega)}$$

(ii) $$\Phi\left[G_k(i\omega + i\varepsilon)\right] = \Phi\left[G_k(i\omega)\right]$$

$\Phi =$  $+ \cdots$

$=$ ~~$U \int G G$~~ $+ \; U^2 \int G G G G + \ldots$

Sum of 2-particle irreducible graphs.

For more details, see cond-mat/0406671.

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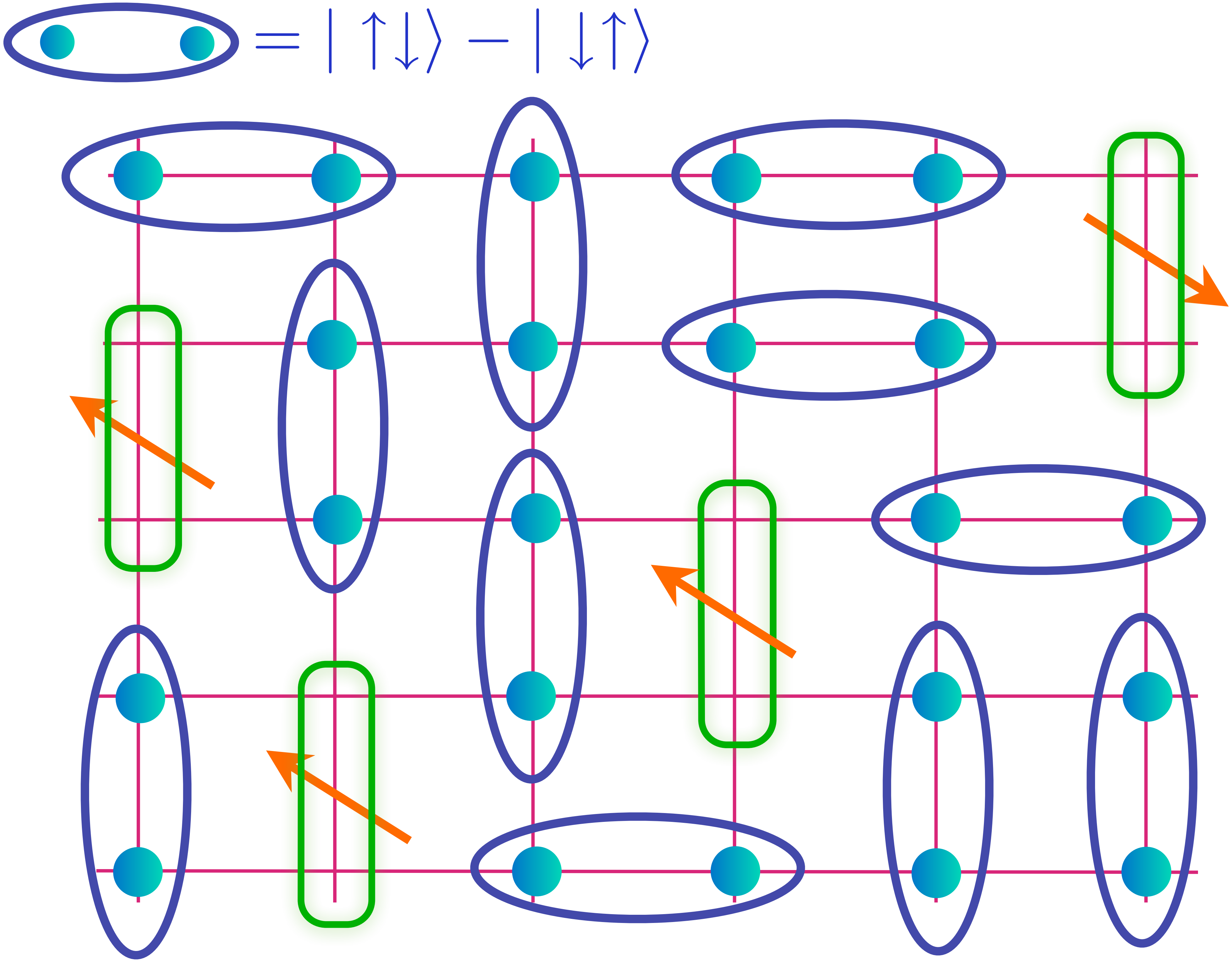

$= |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$

Realizes a metal with a Fermi surface of area p co-existing with "topological order"

M. Punk, A. Allais, and S. S., arXiv:1501.00978, PNAS to appear

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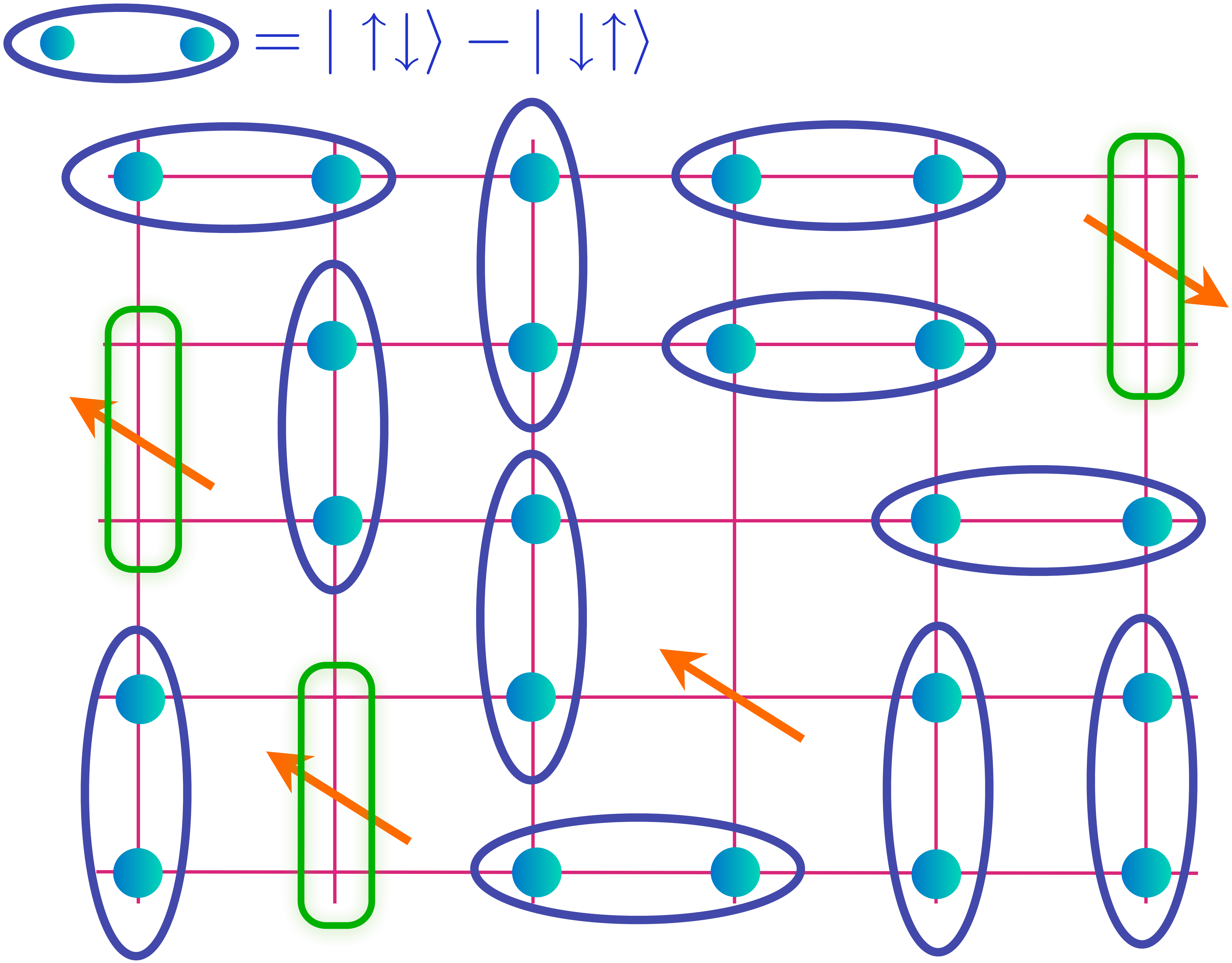

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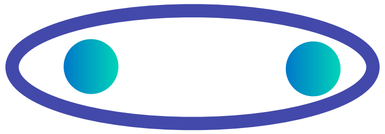

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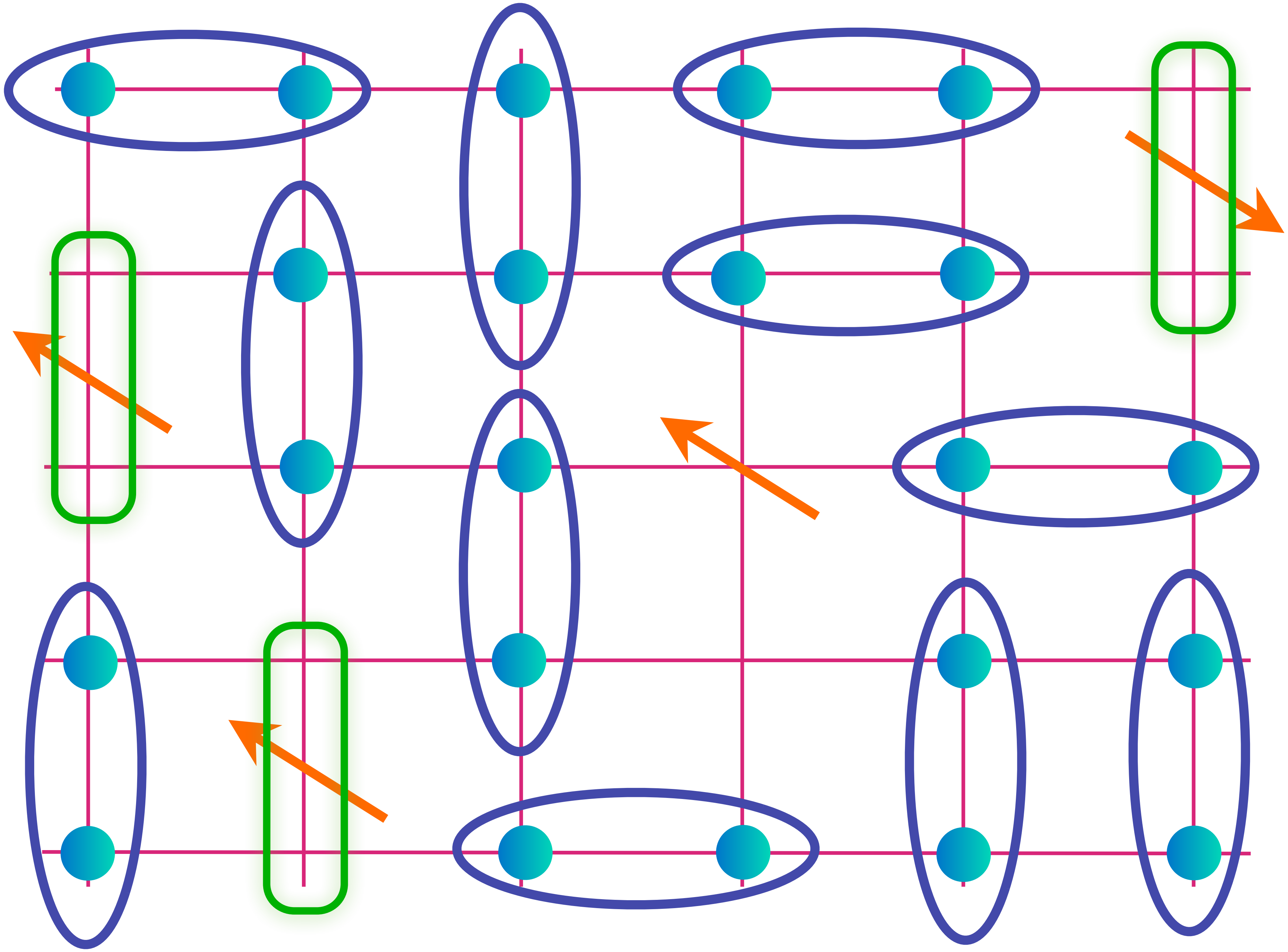

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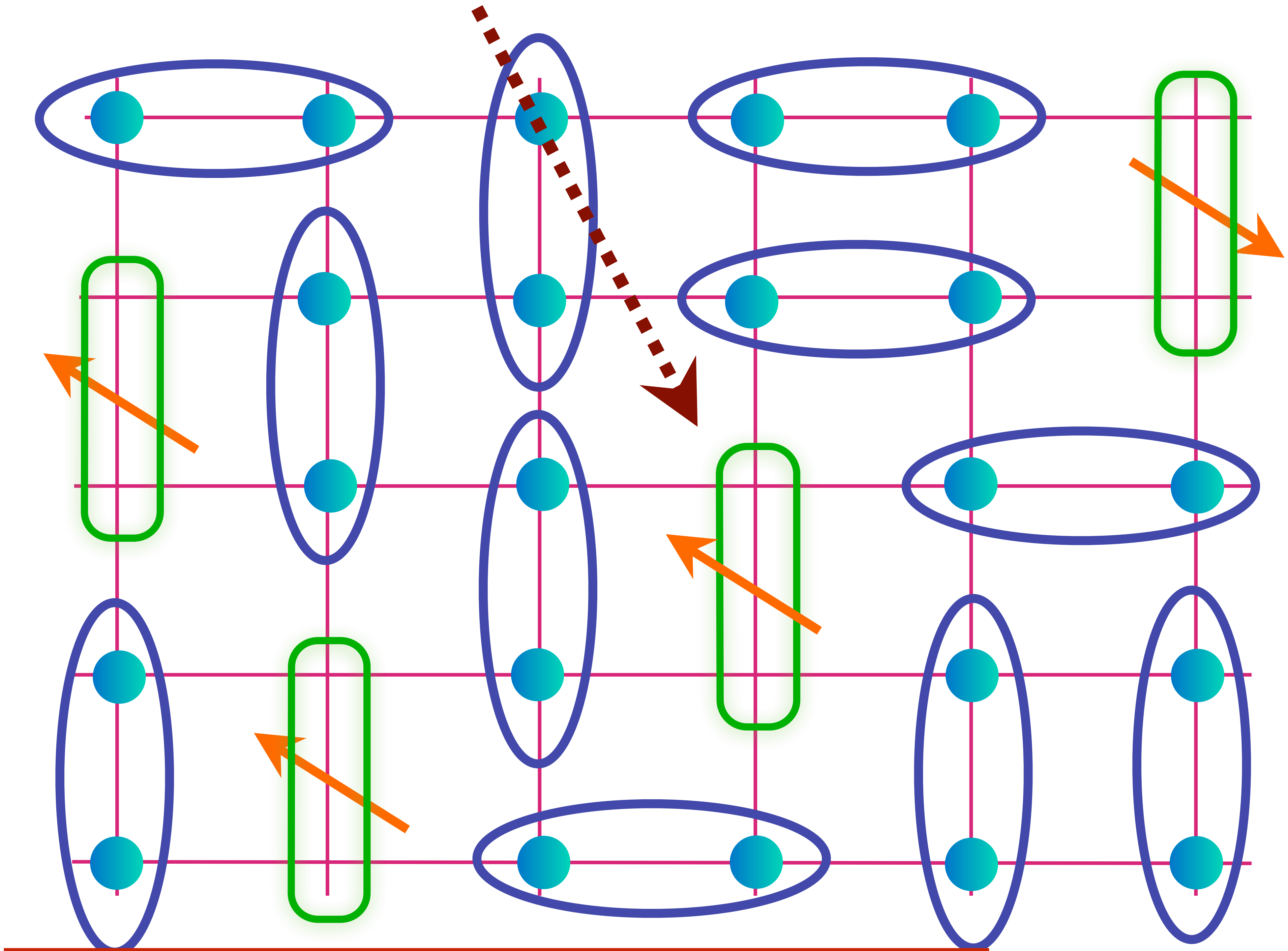
A fermionic “dimer” describing a “bonding” orbital between two sites



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Density of fermionic dimers = p;
density of holes relative to filled band = $1+p$

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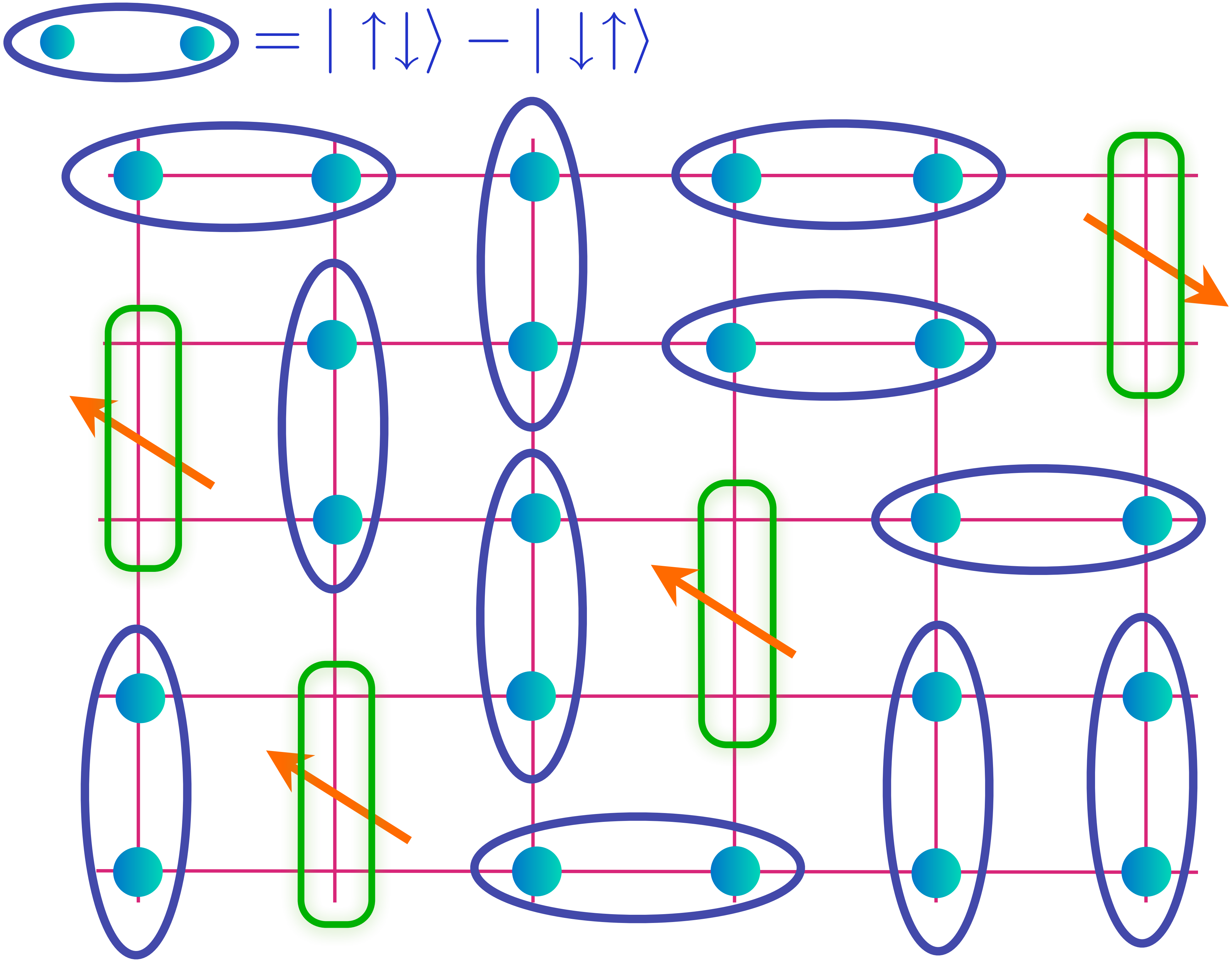

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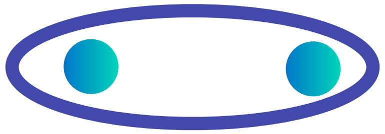

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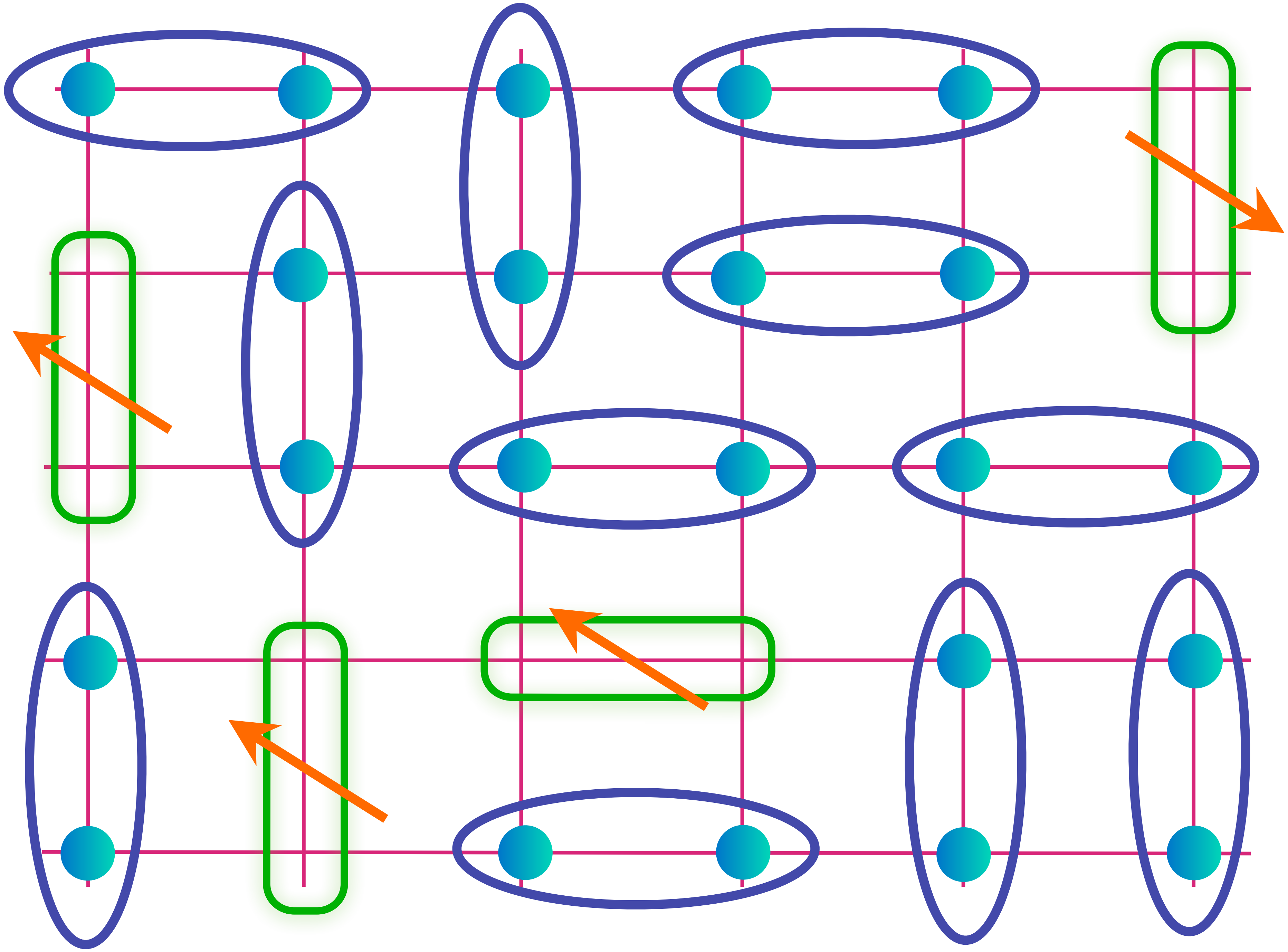

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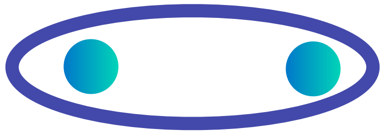
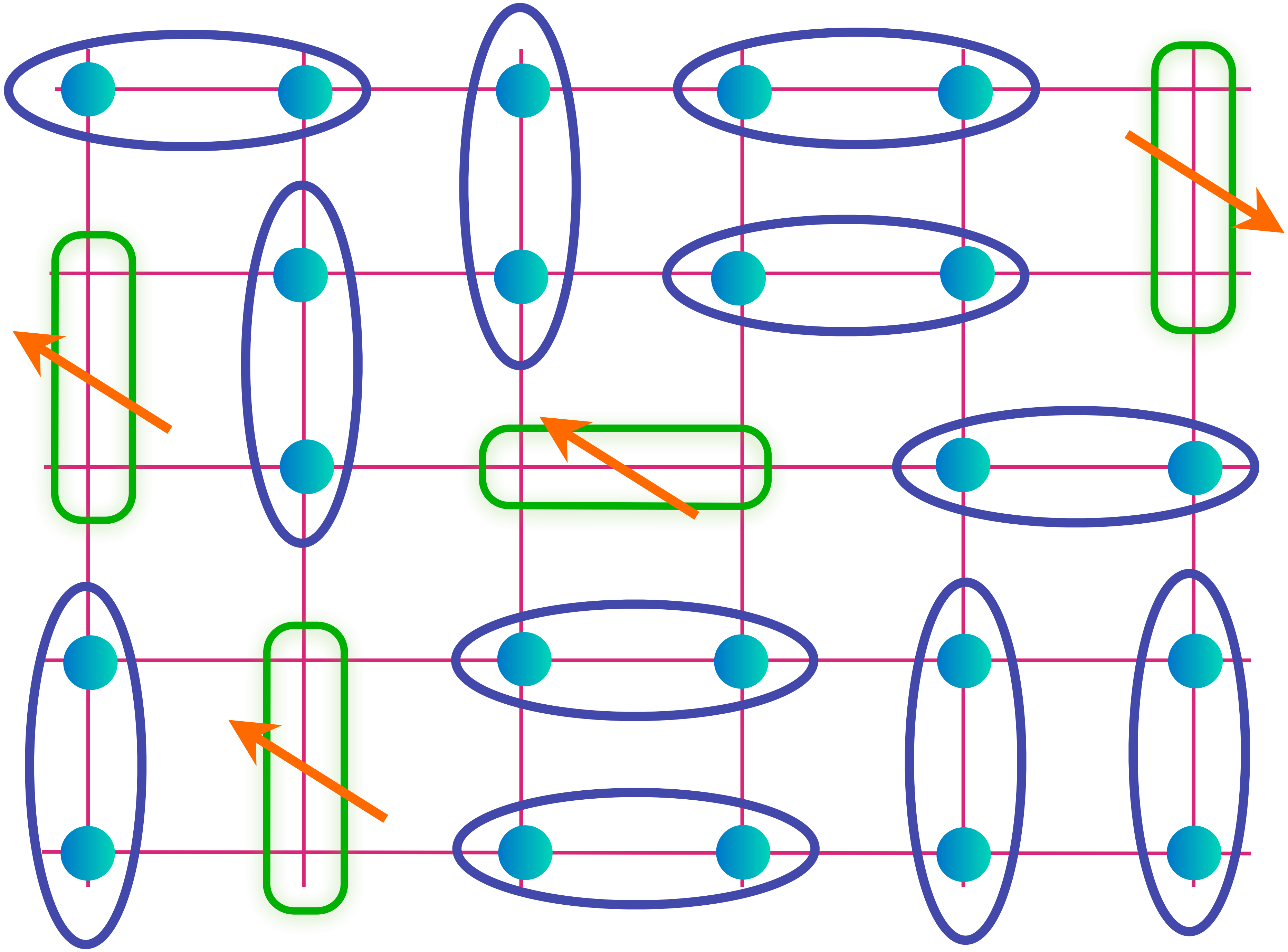
 $= |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$



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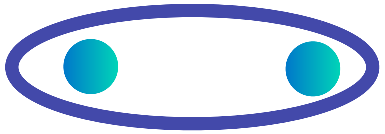

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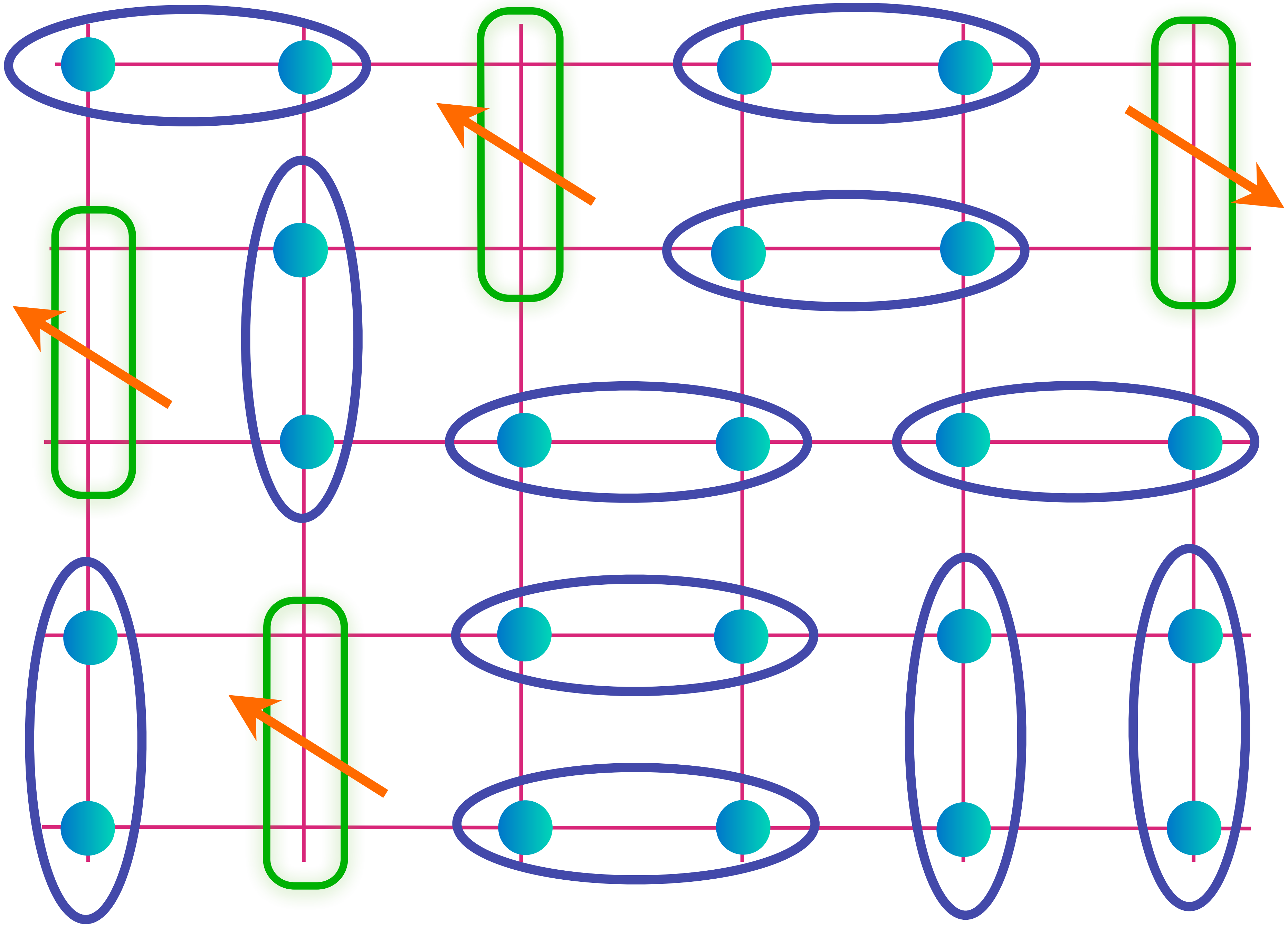

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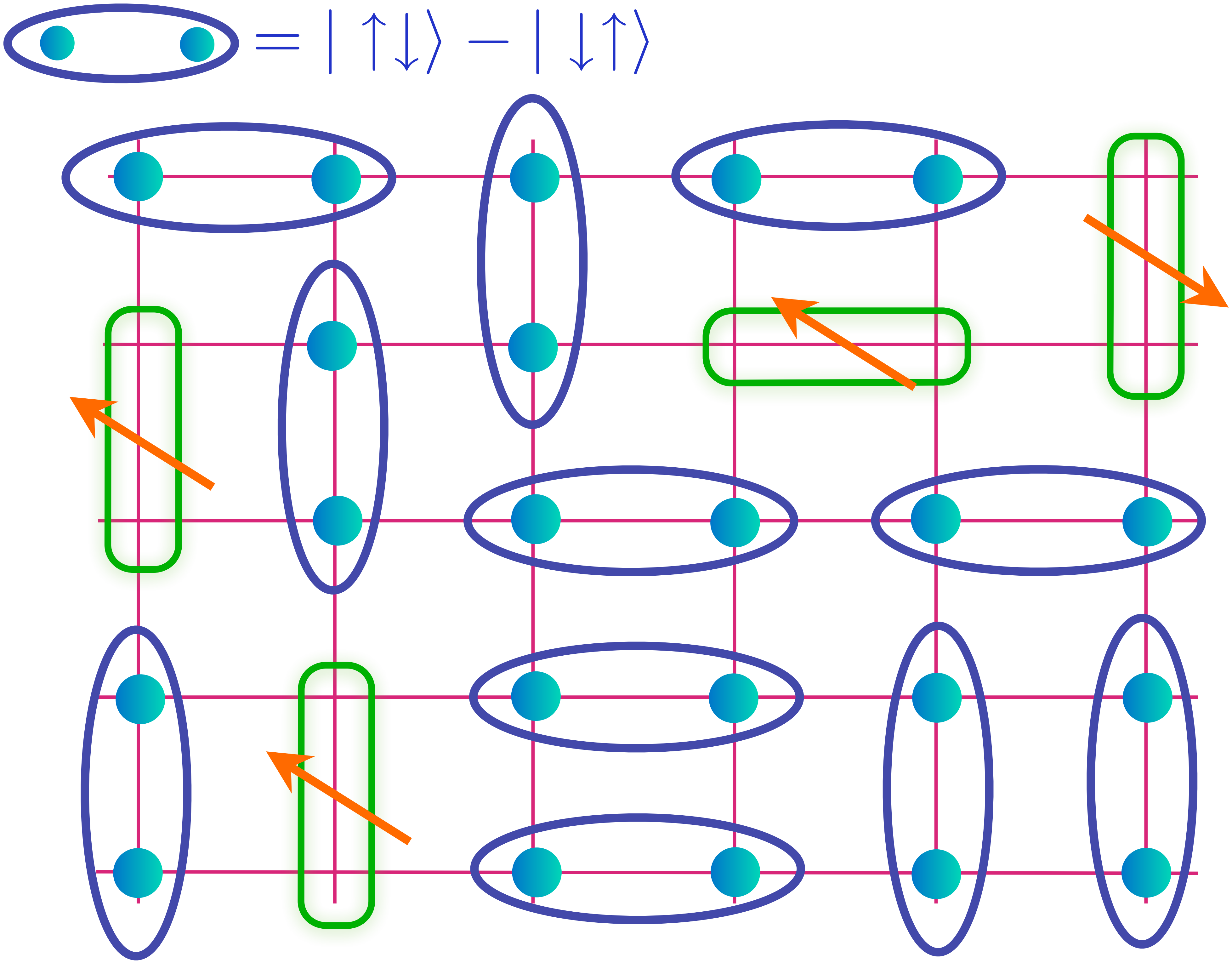

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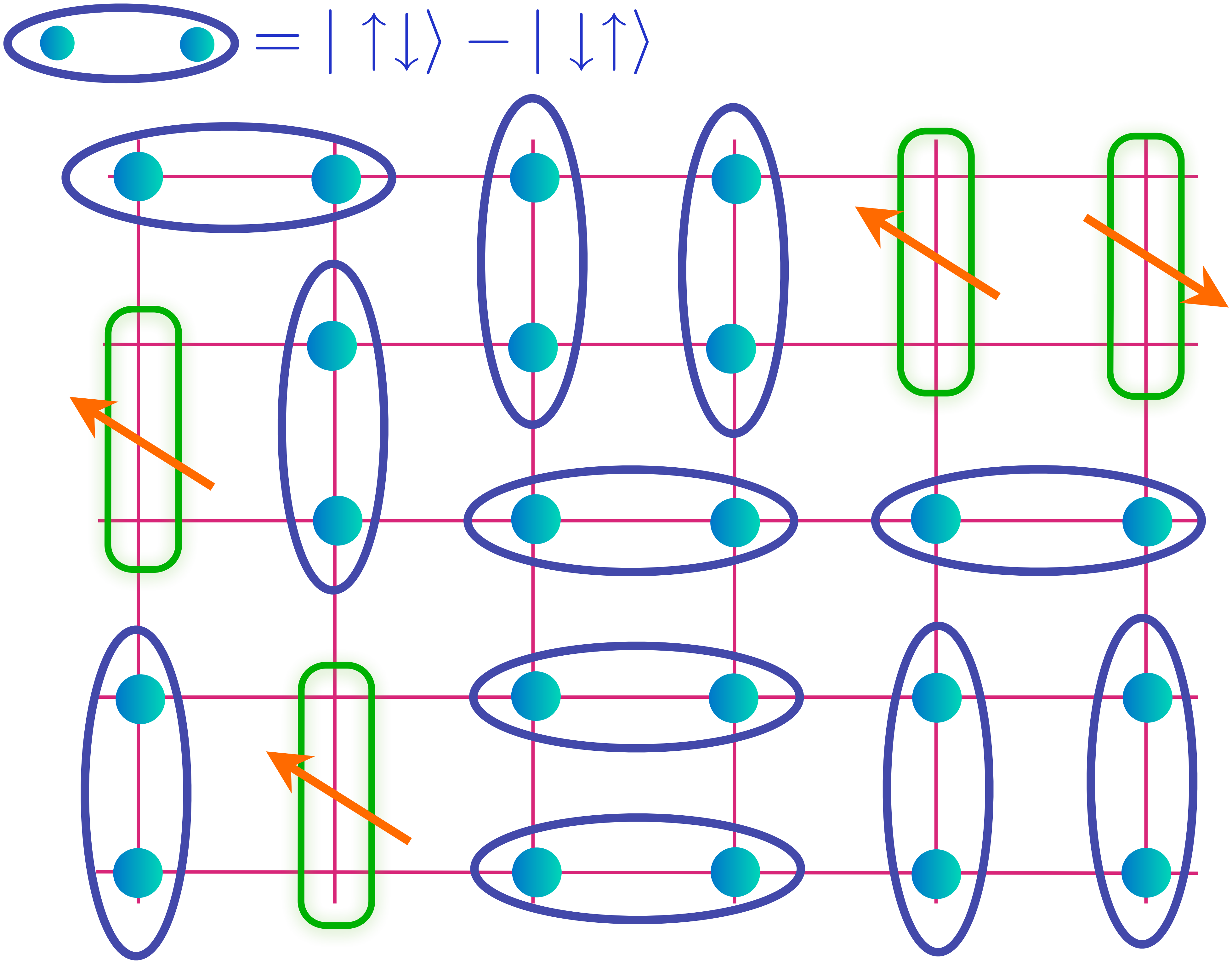

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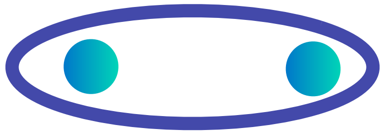

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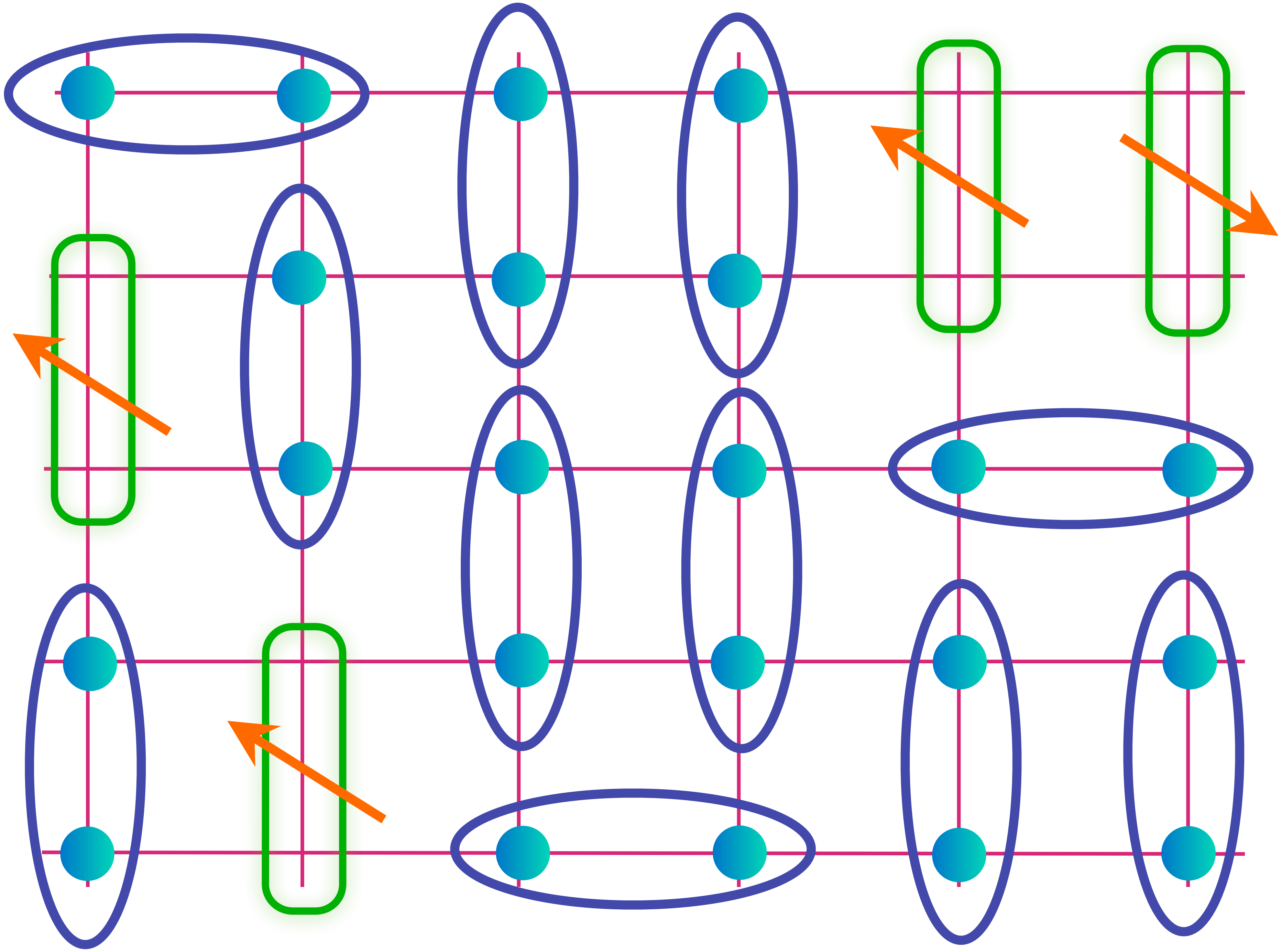

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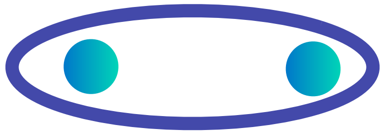
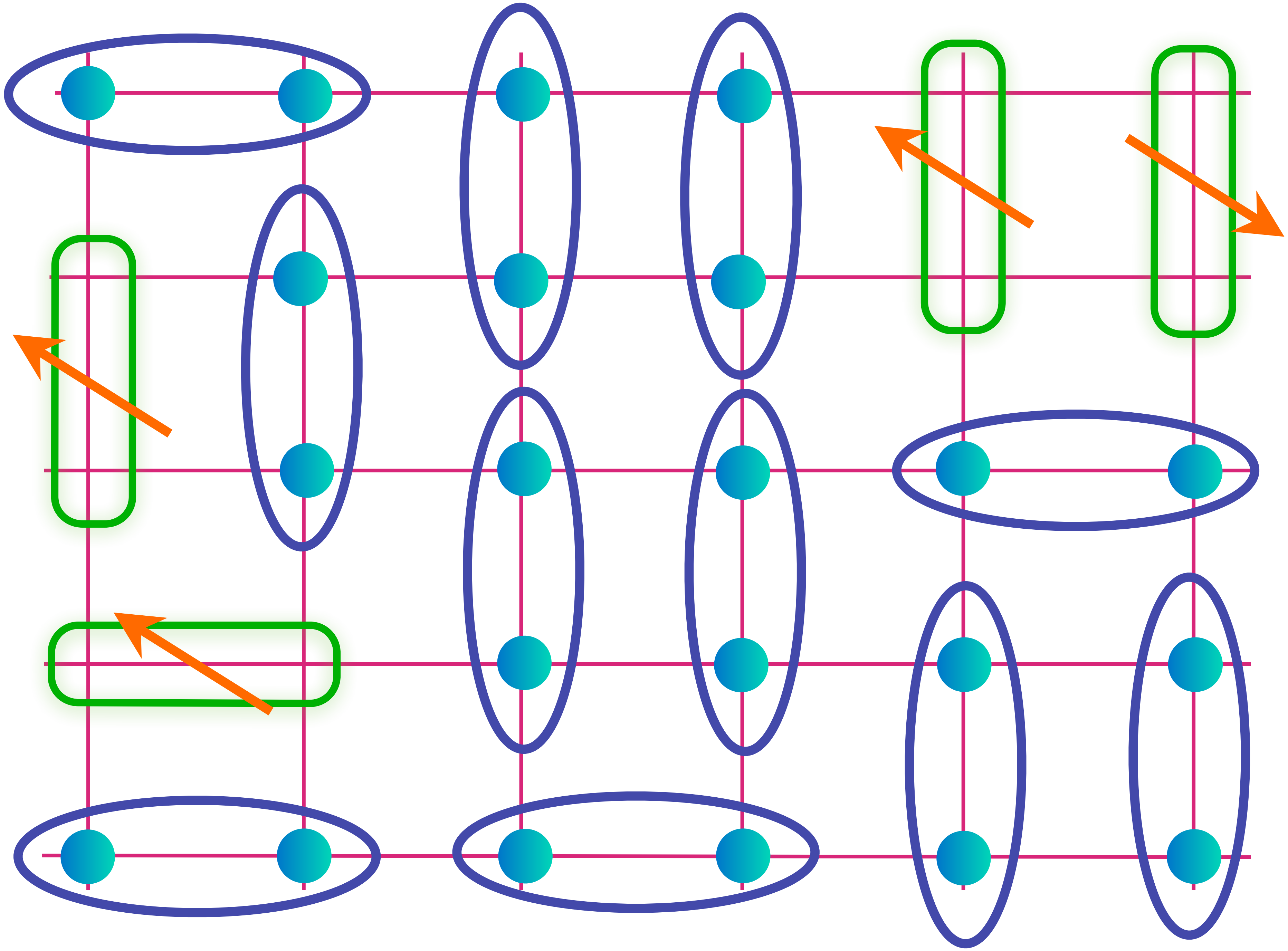
 $= |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$



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Topological order

Place pseudogap metal on a torus;

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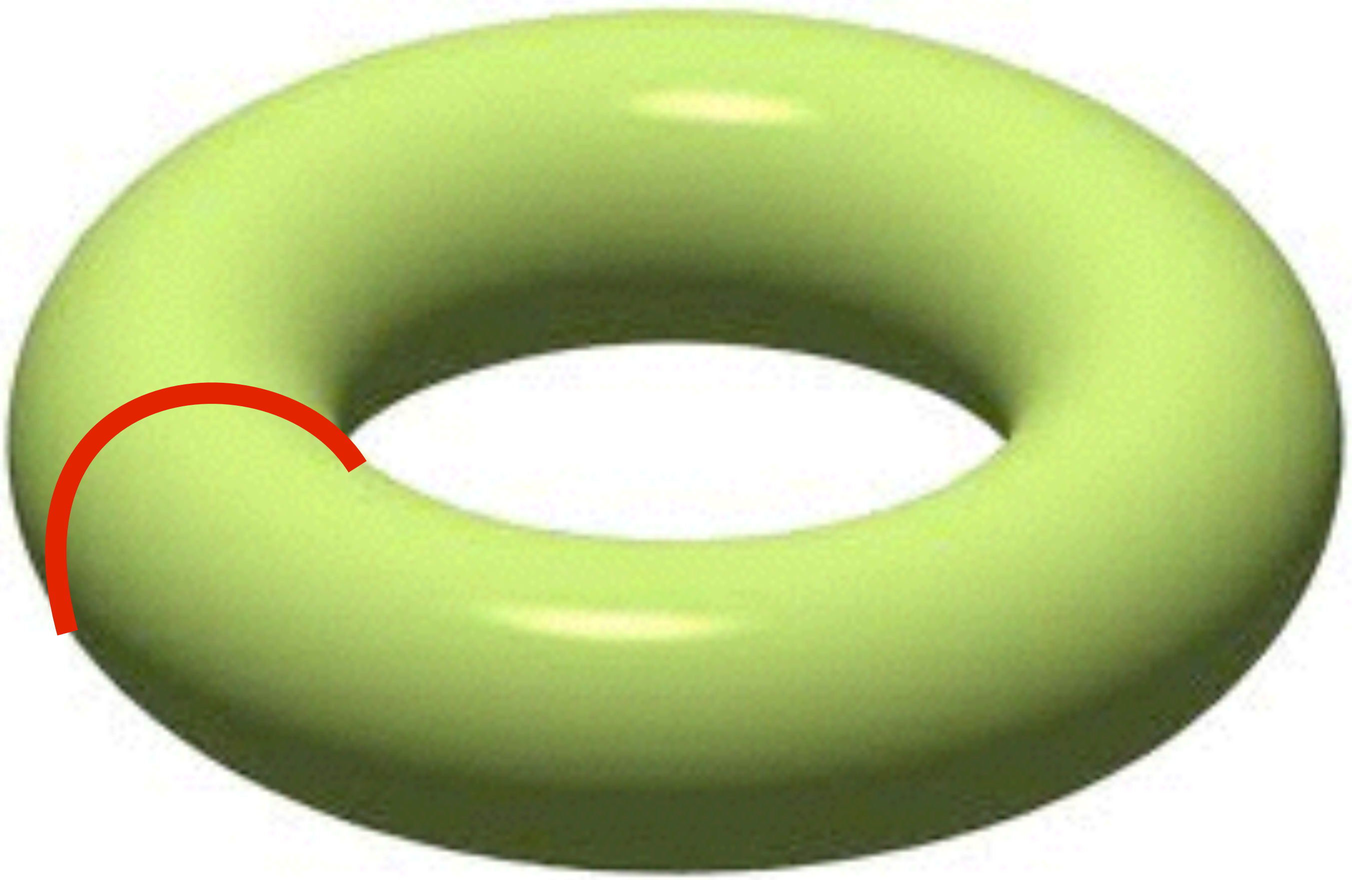

Place pseudogap metal on a torus;

obtain "topological" states nearly degenerate with the ground state: number of dimers crossing red line is (almost) conserved modulo 2

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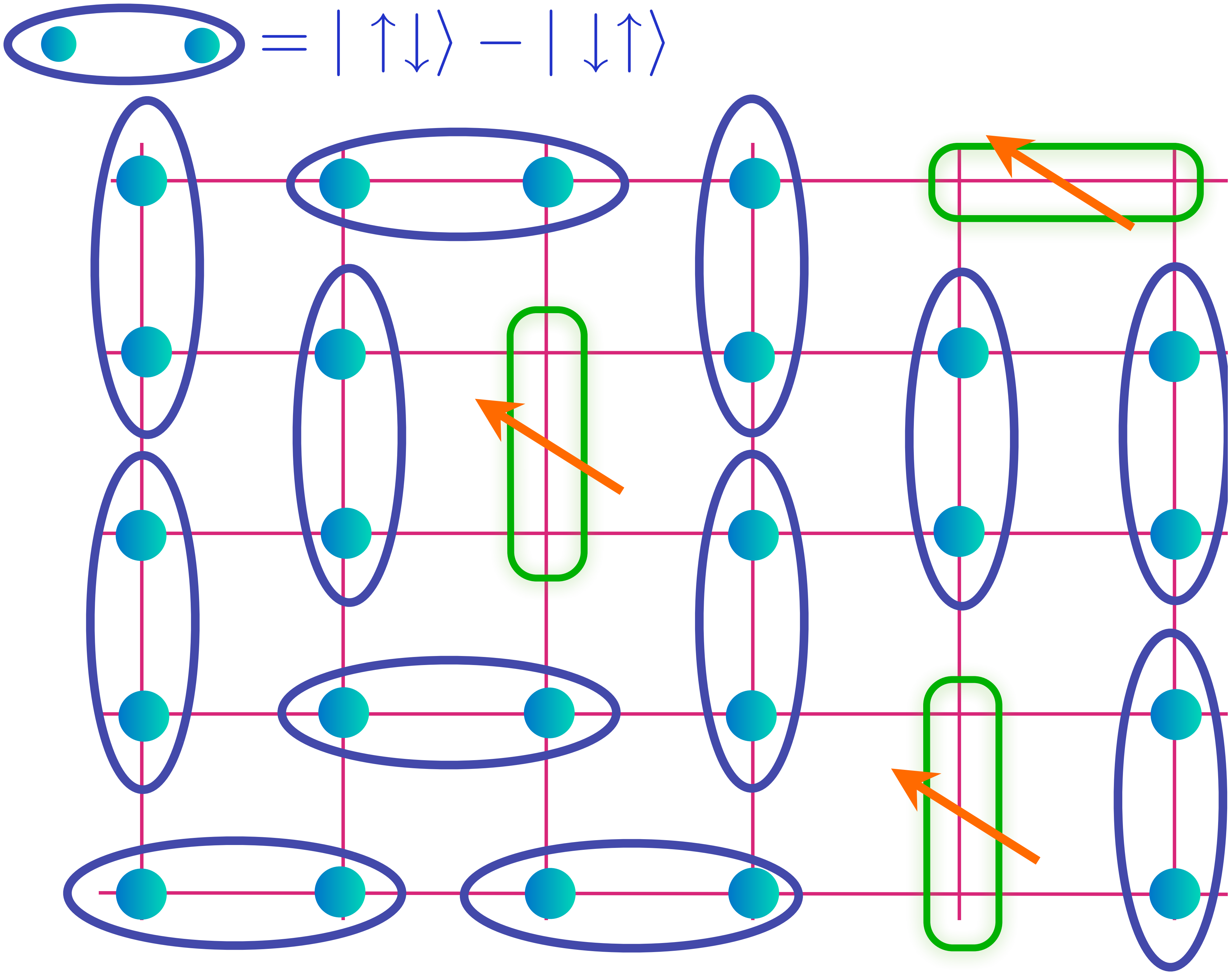
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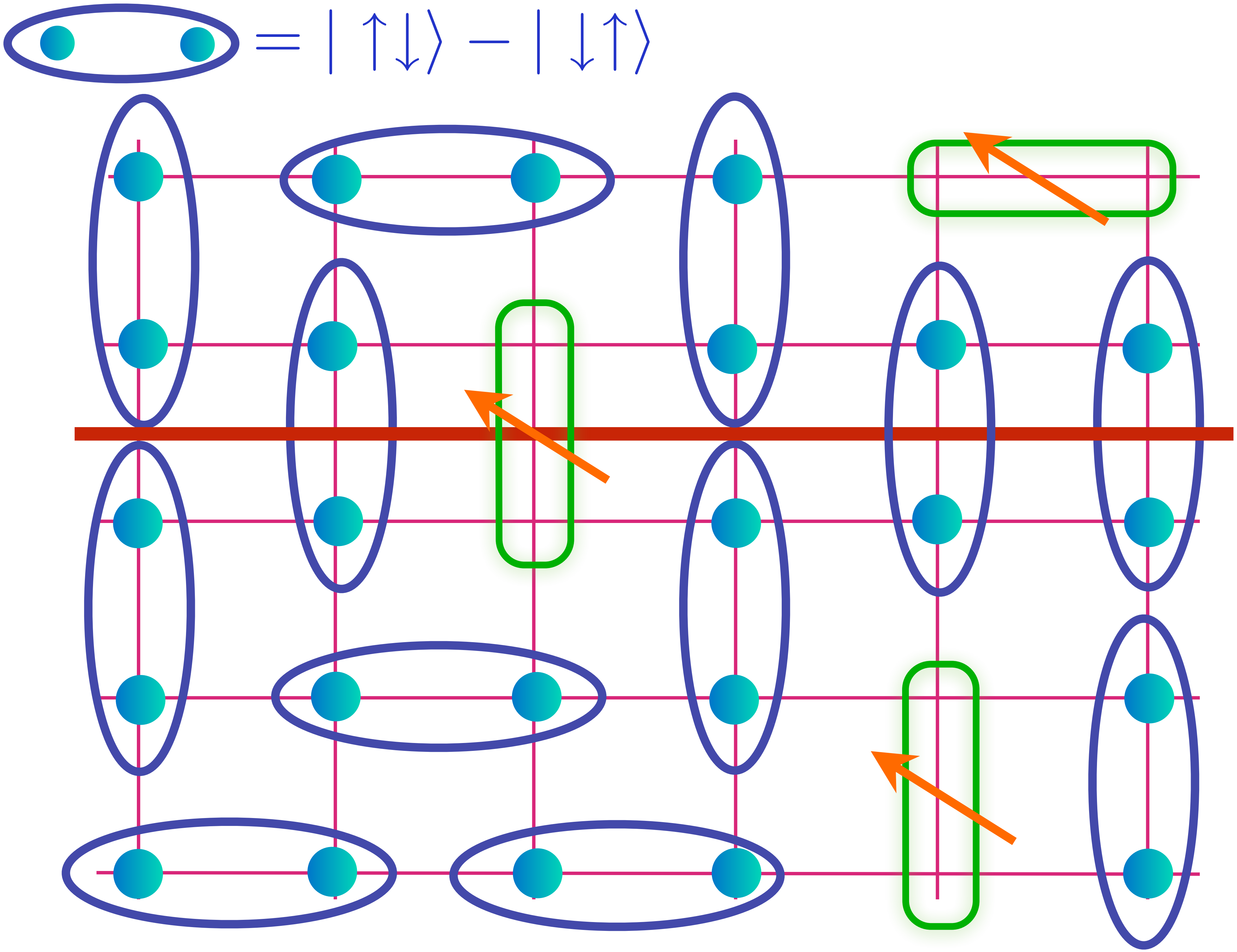

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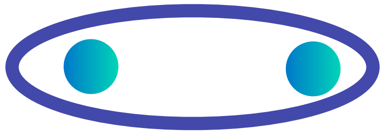

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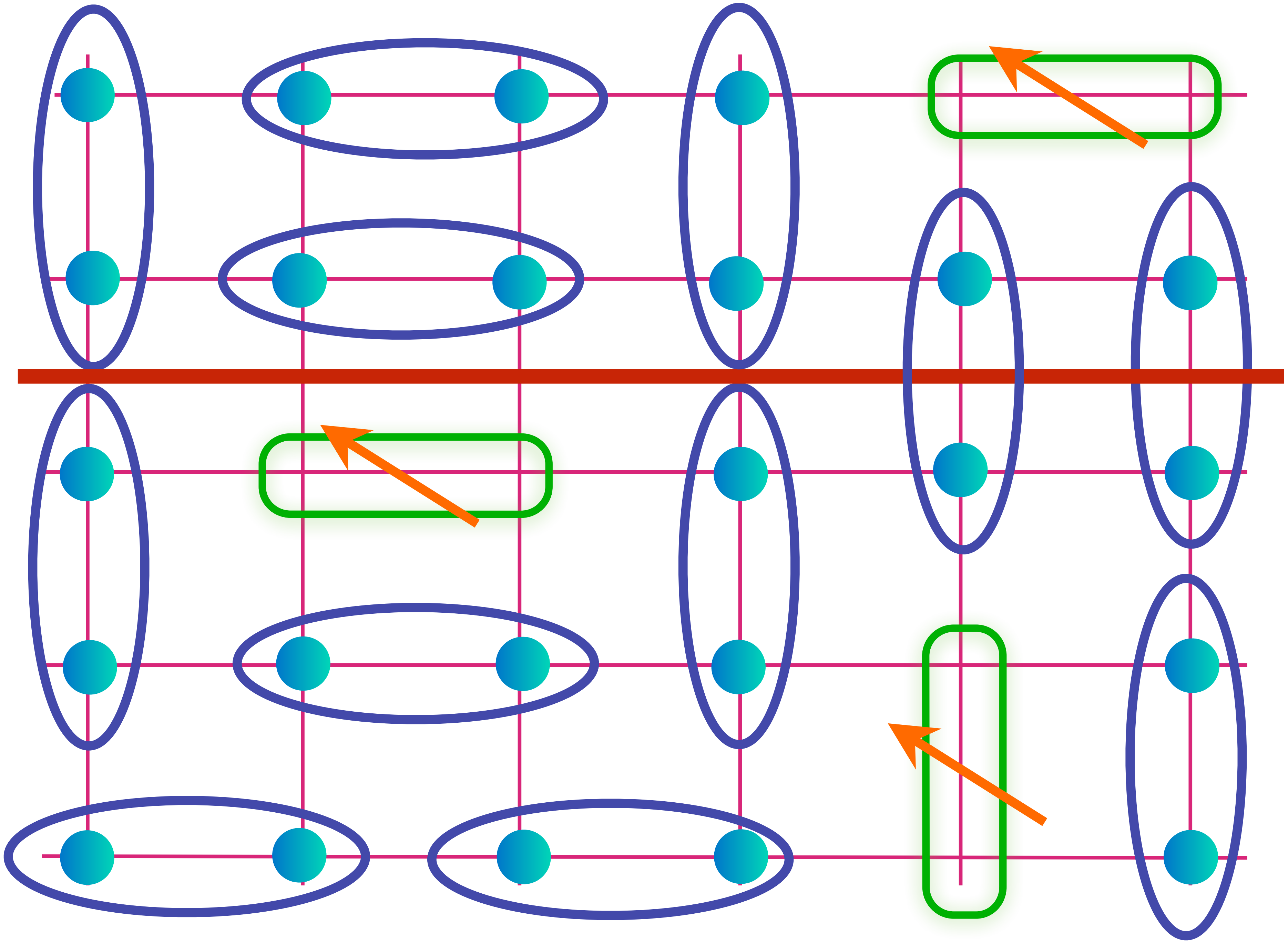

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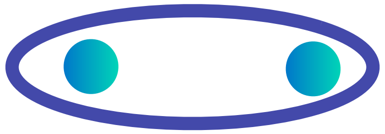

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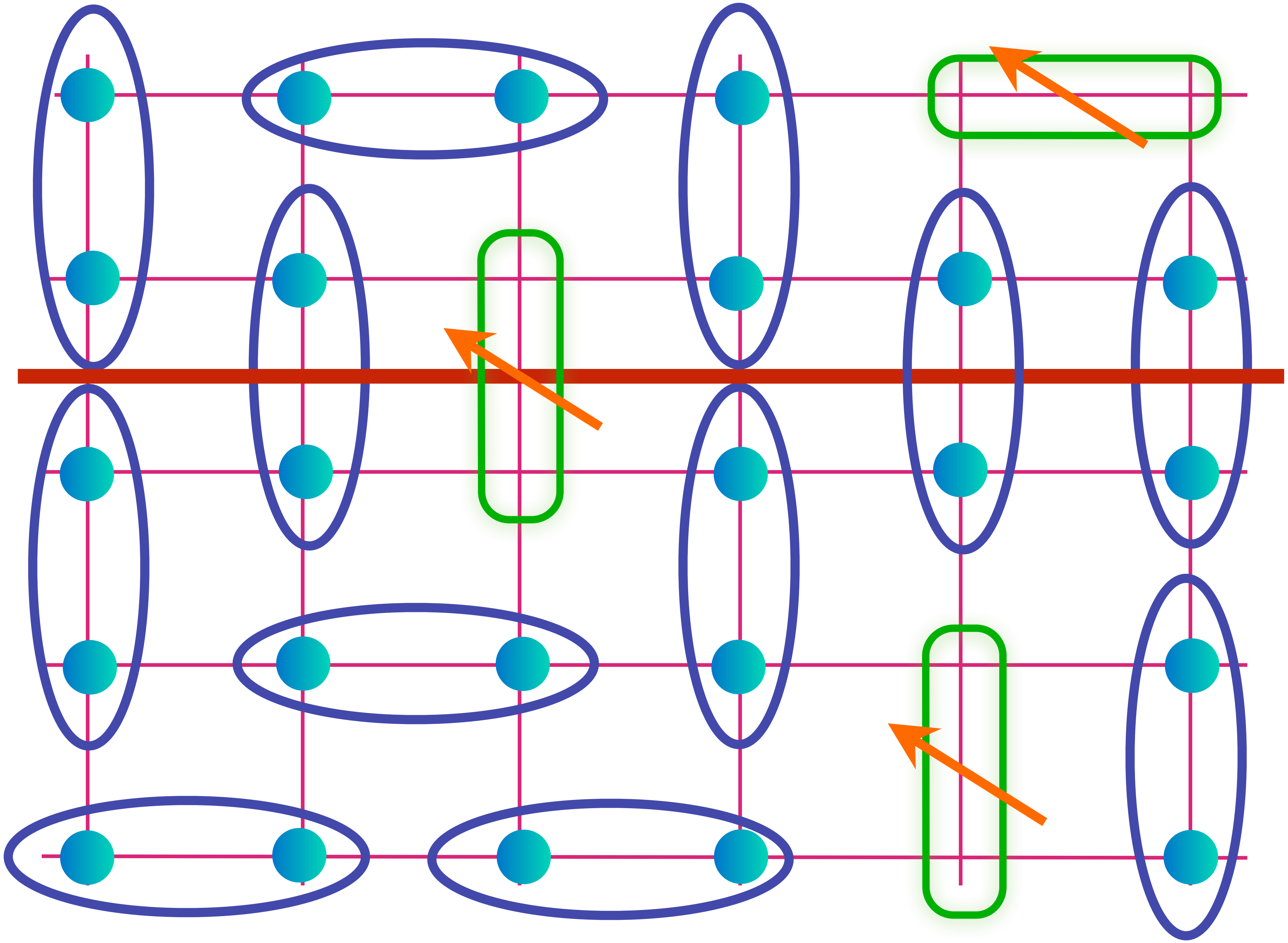

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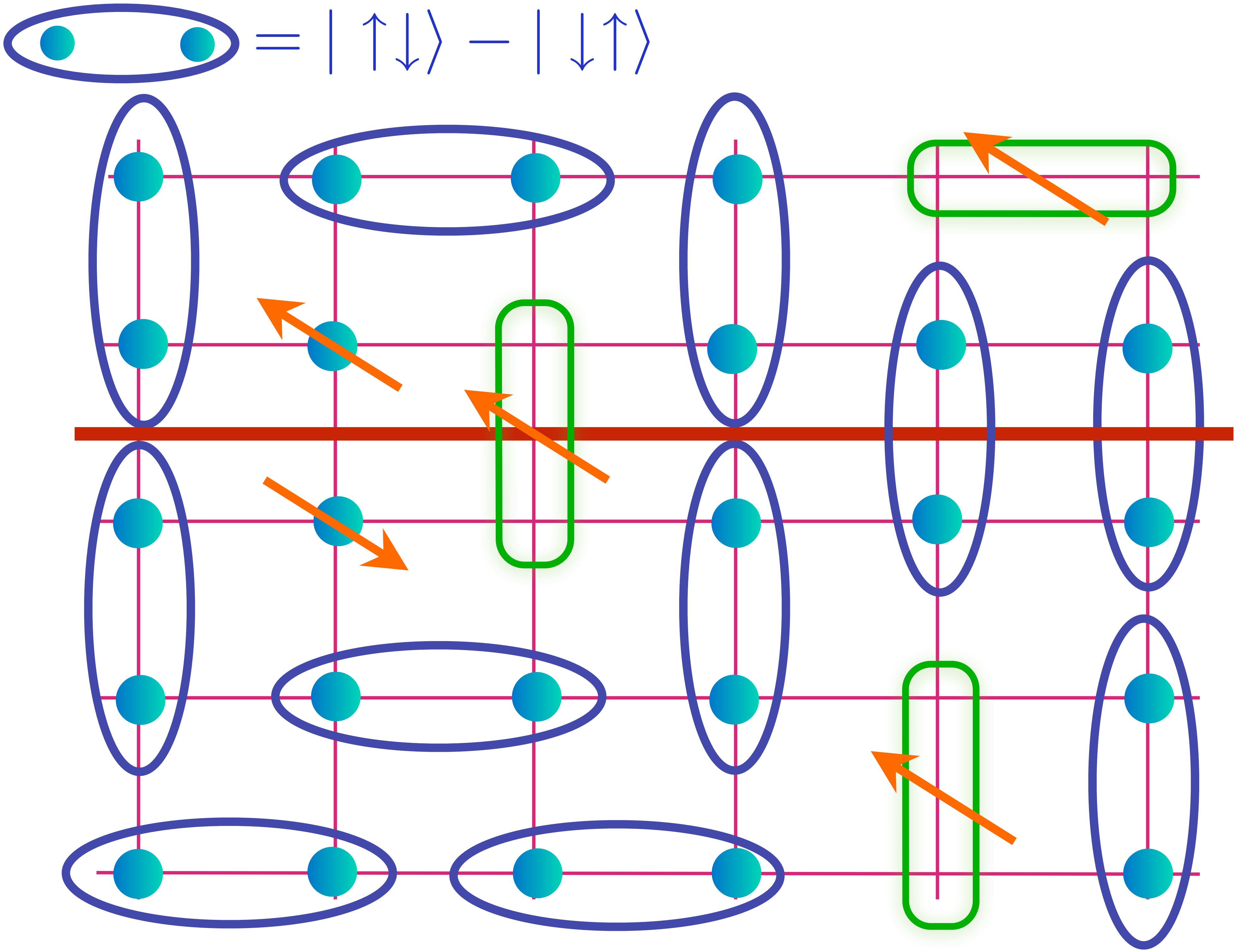

Place pseudogap metal on a torus;

to change dimer number parity across red line, it is necessary to create a pair of unpaired spins and move them around the sample.

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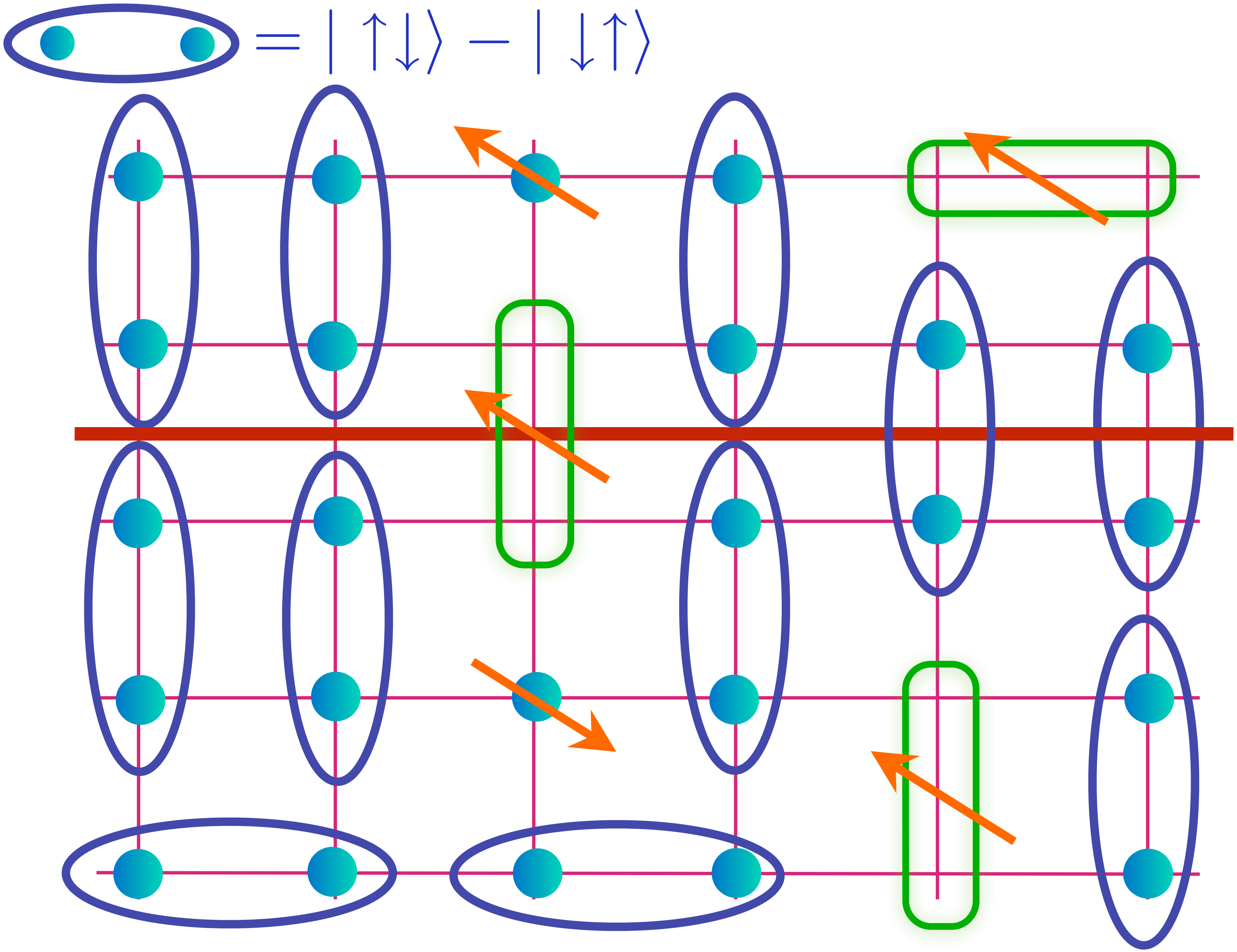

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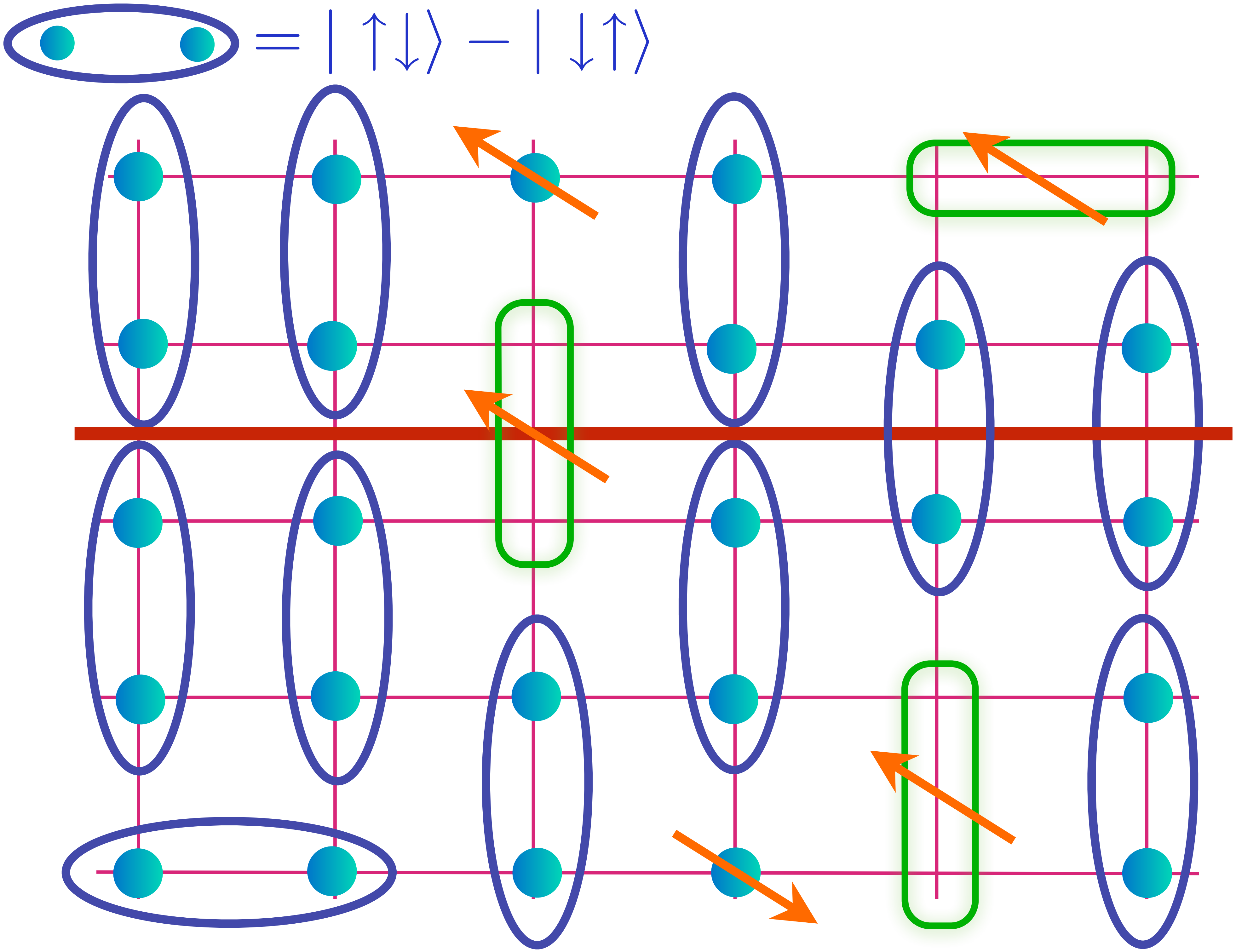
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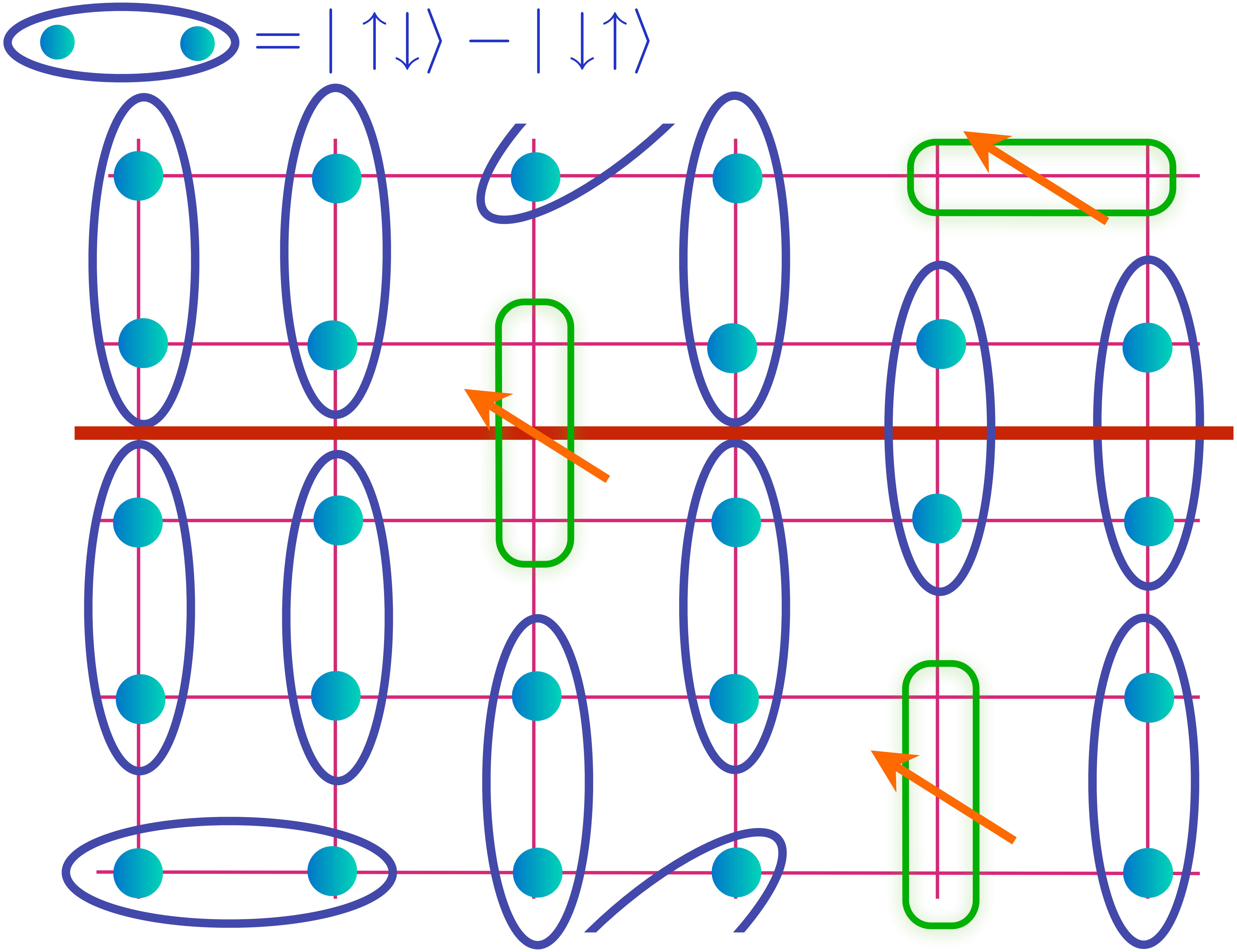

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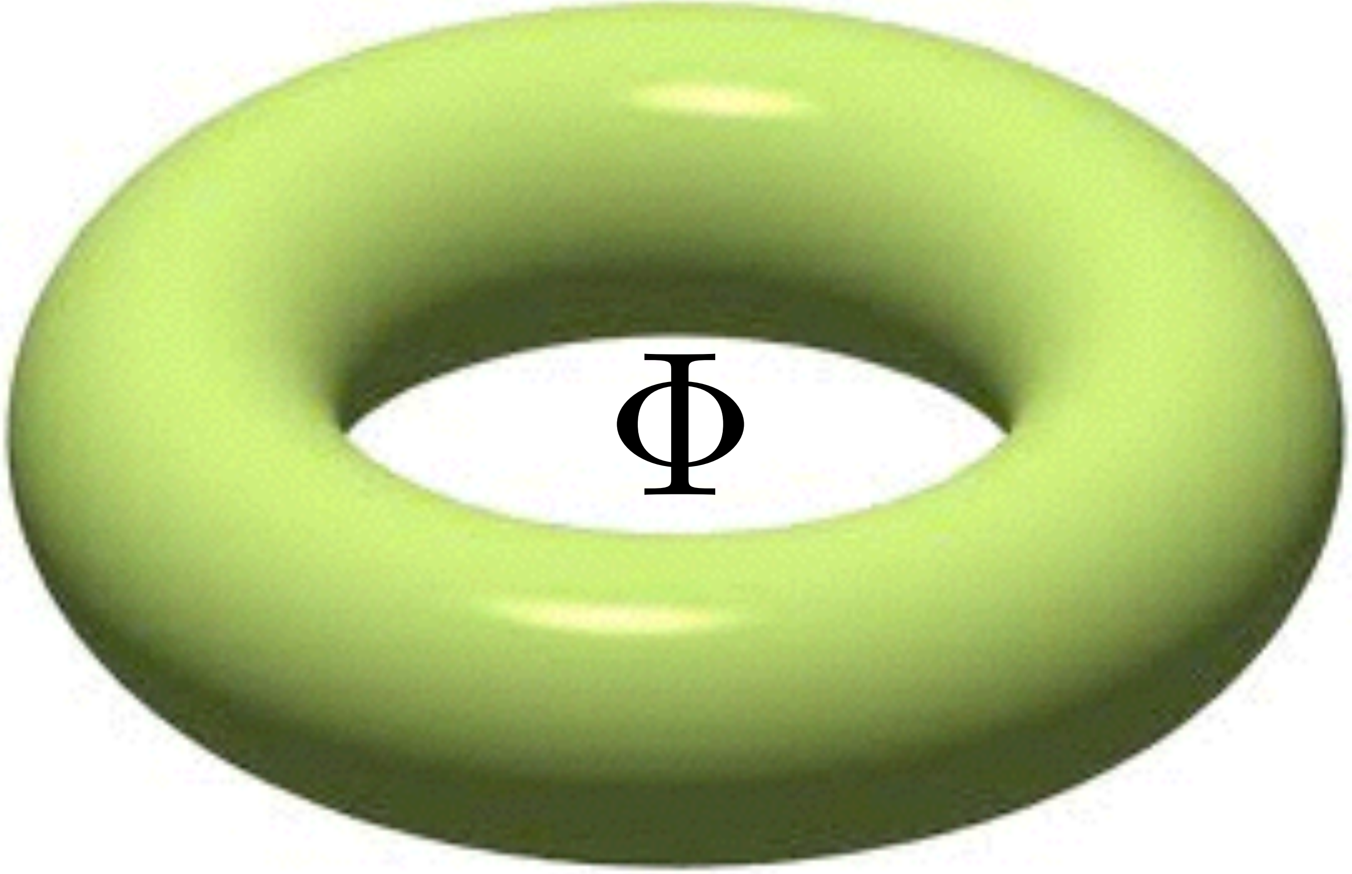
Fermi liquid (FL)

Topological argument for the area of Fermi surface

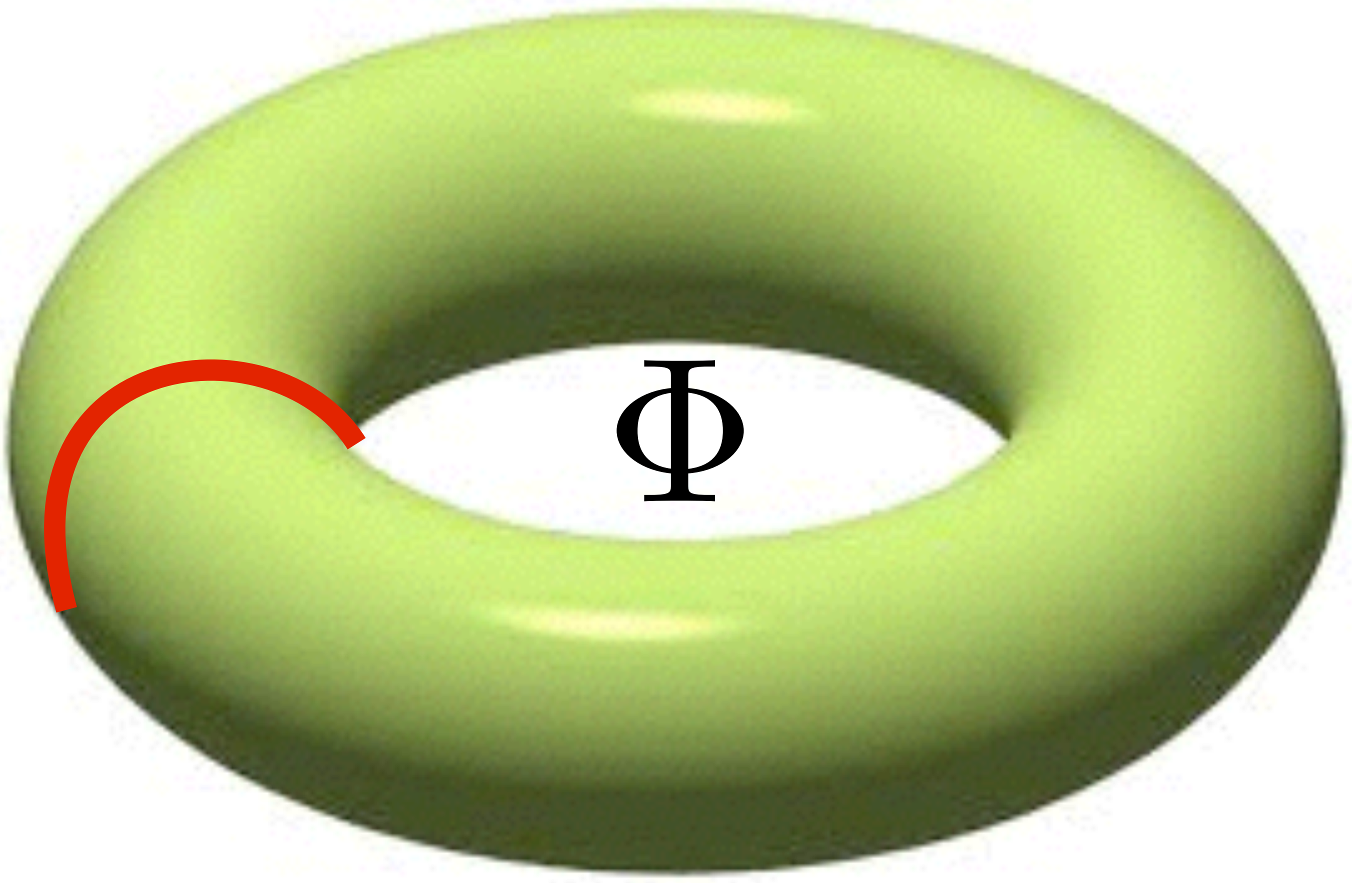


Put metal on a torus, adiabatically insert flux $\Phi = h/e$ through hole, and measure change in momentum. In a FL, we can assume the only low energy excitations are quasiparticles near the Fermi surface, and this leads to a non-perturbative proof of the Luttinger relation on the area enclosed by the Fermi surface.

M. Oshikawa, *Phys. Rev. Lett.* **84**, 3370 (2000)

Fractionalized Fermi liquid (FL*)

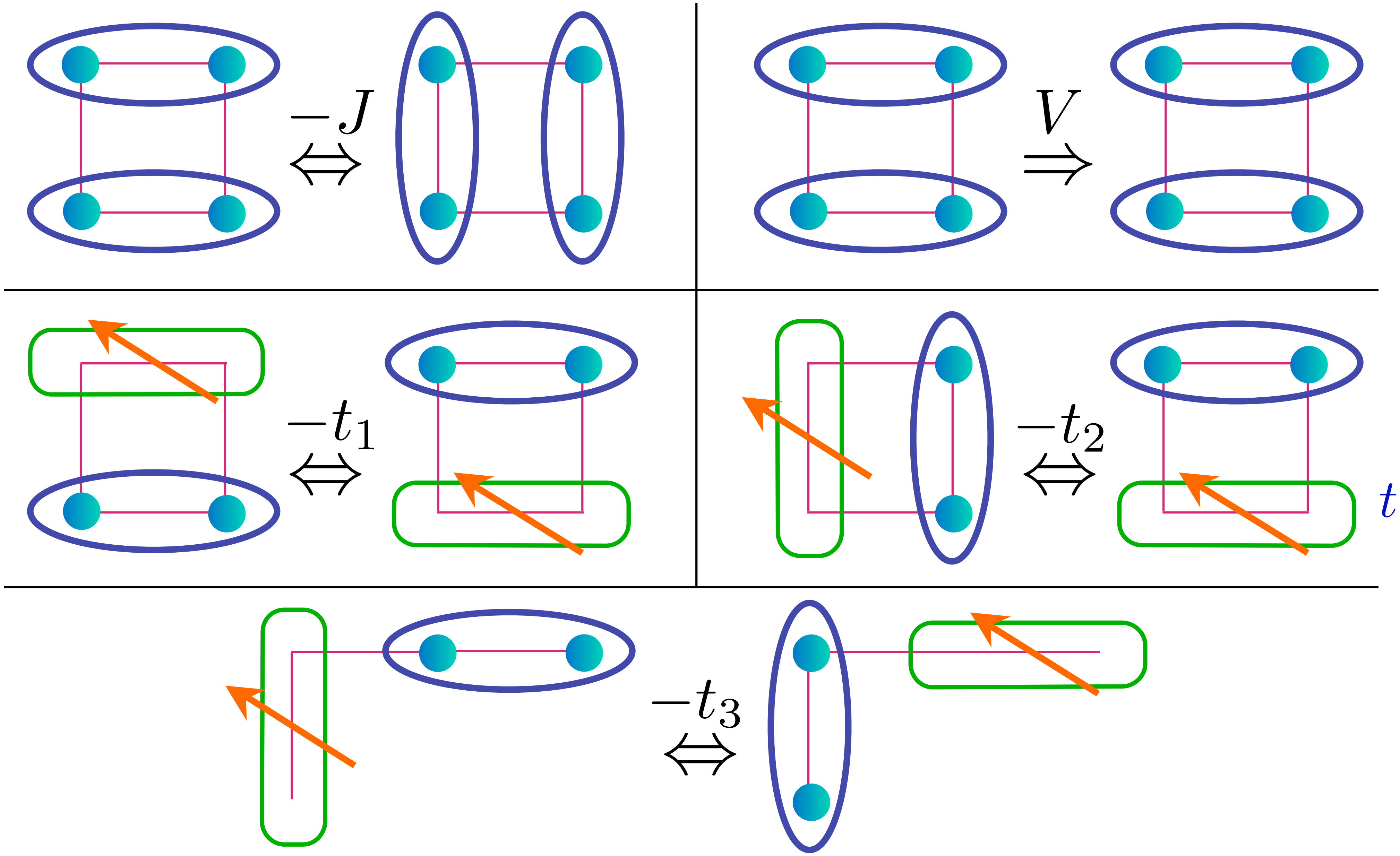
Topological argument for the area of Fermi surface



Violations of the Luttinger relation are possible in a fractionalized Fermi liquid (FL*) because there are "topological" low energy excitations associated with a flux of the emergent gauge field in the hole of the torus.

T. Senthil, M. Vojta, and S. Sachdev, *Phys. Rev. B* **69**, 035111 (2004)

Quantum dimer model with bosonic and fermionic dimers

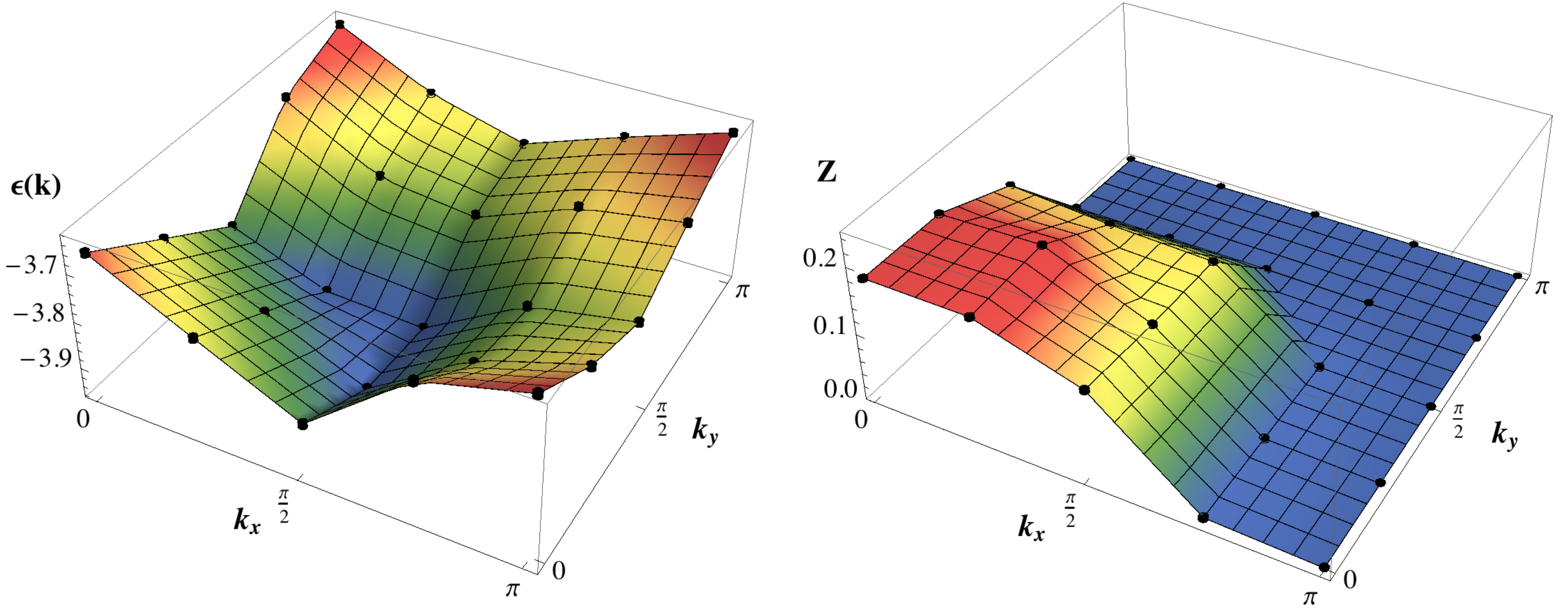


Connection to the
t-t'-t''-J model:

$$t_1 = -(t + t')/2$$
$$t_2 = (t - t')/2$$
$$t_3 = -(t + t' + t'')/4$$

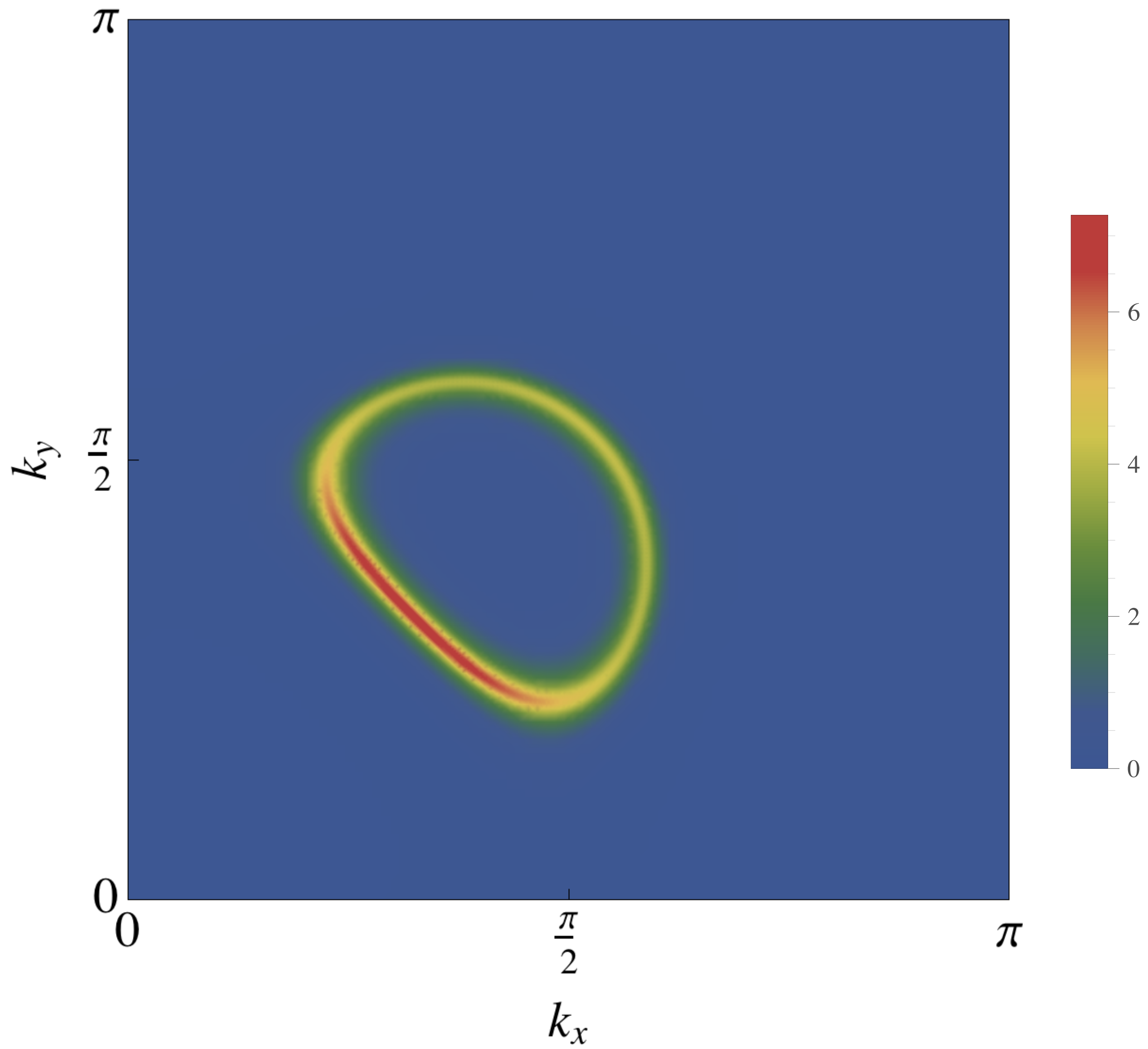
M. Punk, A. Allais, and S. S., arXiv:1501.00978, PNAS to appear

Quantum dimer model with bosonic and fermionic dimers

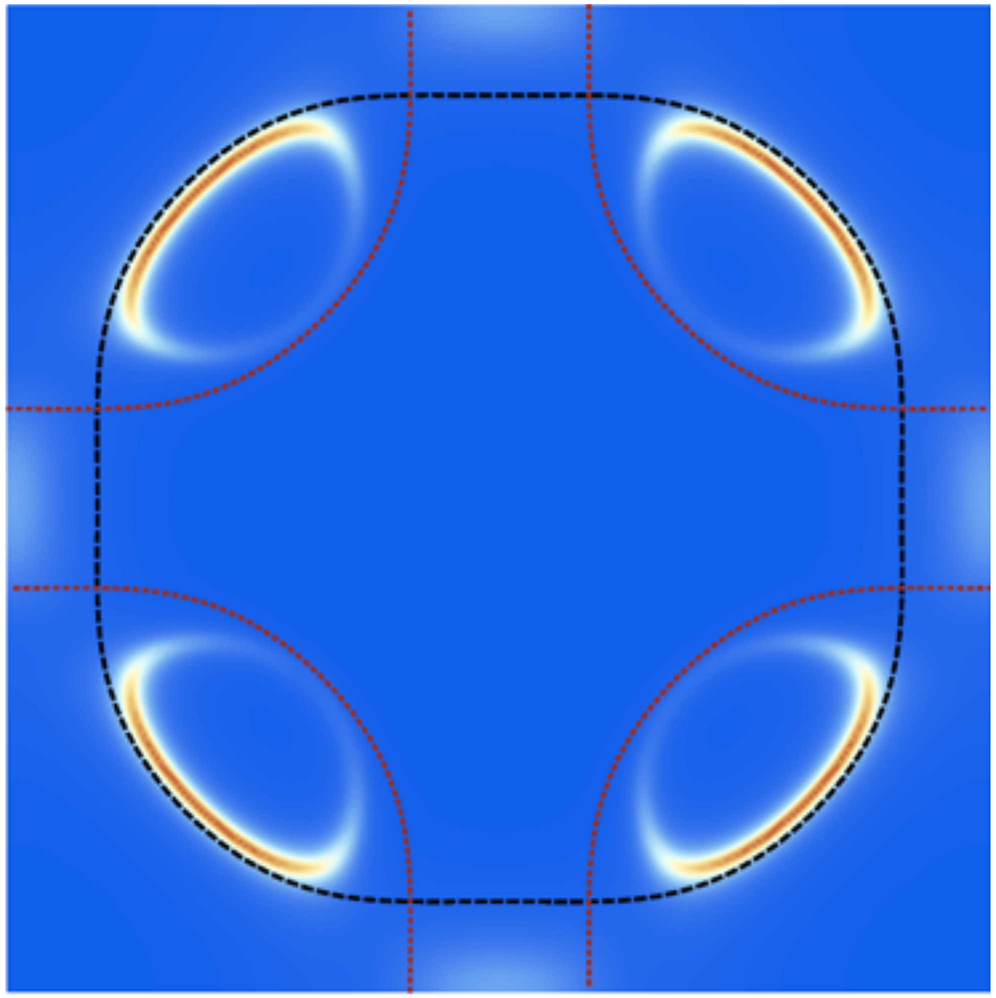


Dispersion and quasiparticle residue of a single fermionic dimer for $J = V = 1$, and hopping parameters obtained from the t-J model for the cuprates, $t_1 = -1.05$, $t_2 = 1.95$ and $t_3 = -0.6$, on a 8×8 lattice.

M. Punk, A. Allais, and S. Sachdev, arXiv:1501.00978



M. Punk, A. Allais, and S. S.,
arXiv:1501.00978, PNAS to appear



Y. Qi and S. Sachdev,
Phys. Rev. B **81**, 115129 (2010)

"Back side" of Fermi surface is suppressed for observables which change electron number in the square lattice

A. Review of Fermi liquid theory

Metals with with quasiparticle excitations

B. Fractionalized Fermi liquid

A Fermi liquid co-existing with topological order

C. Phase diagram of cuprates

The pseudogap metal and the strange metal

D. Quantum matter without quasiparticles

A mean-field model of a non-Fermi liquid

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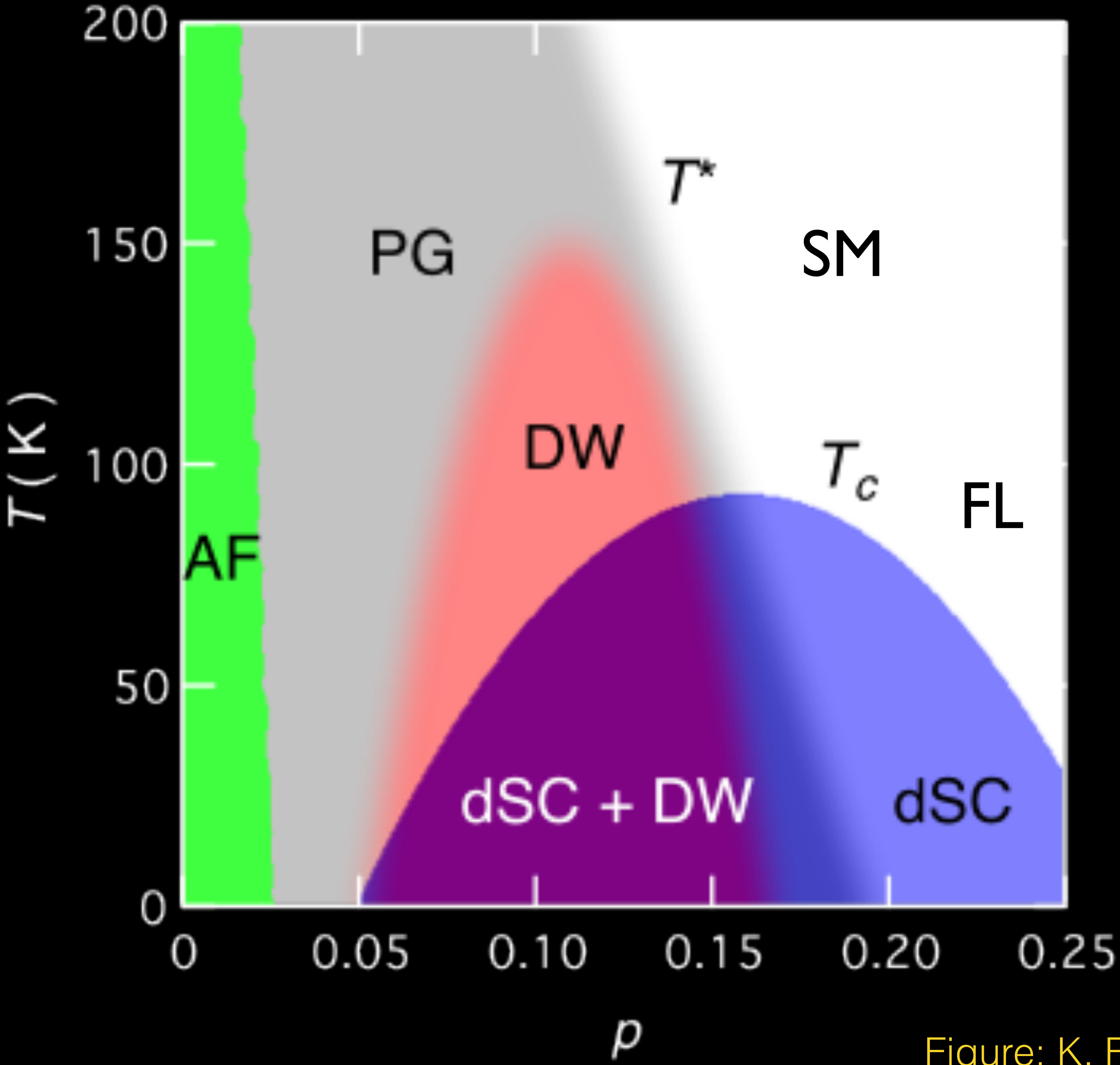


Figure: K. Fujita and J. C. Seamus Davis

Antiferromagnet

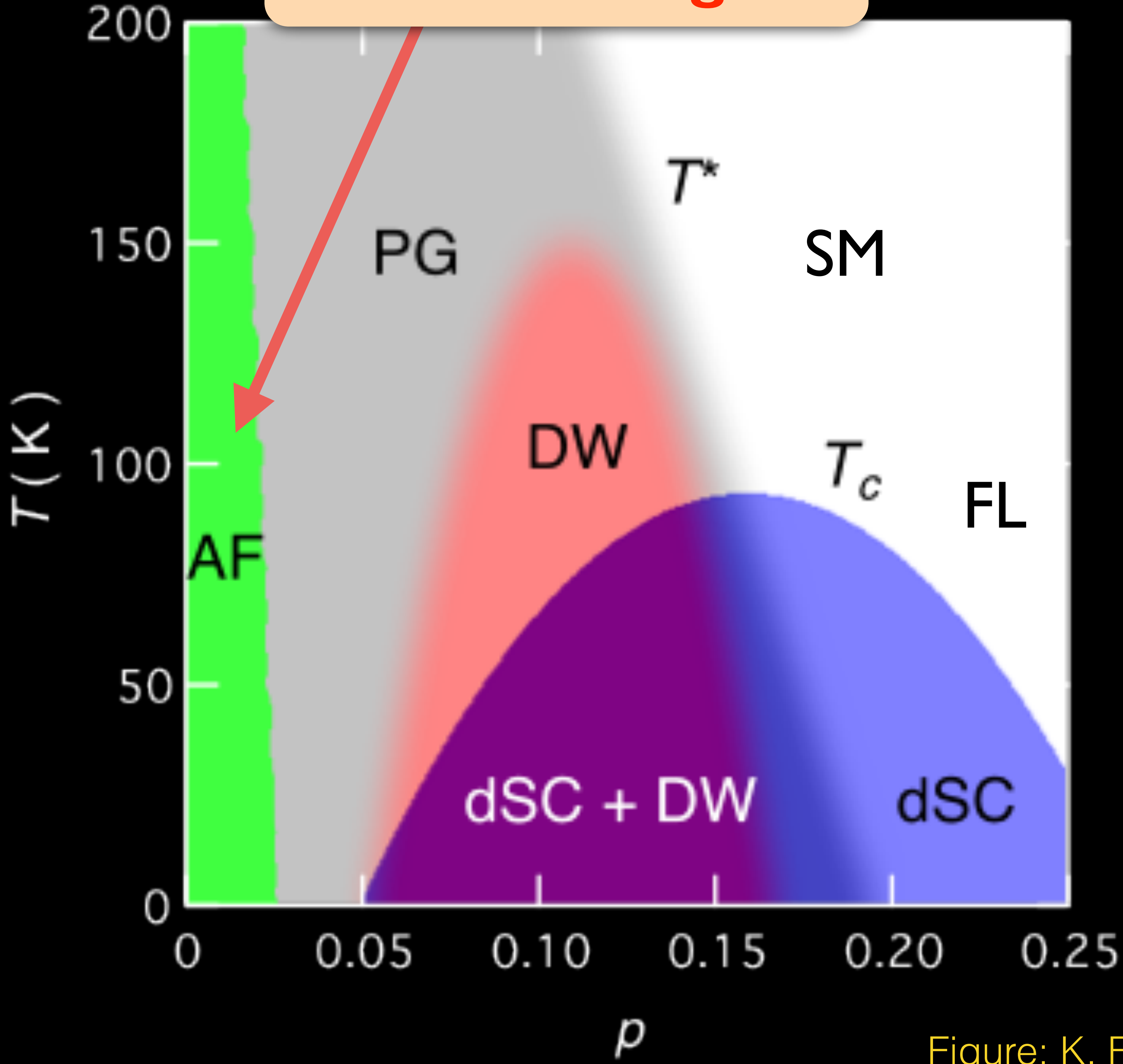
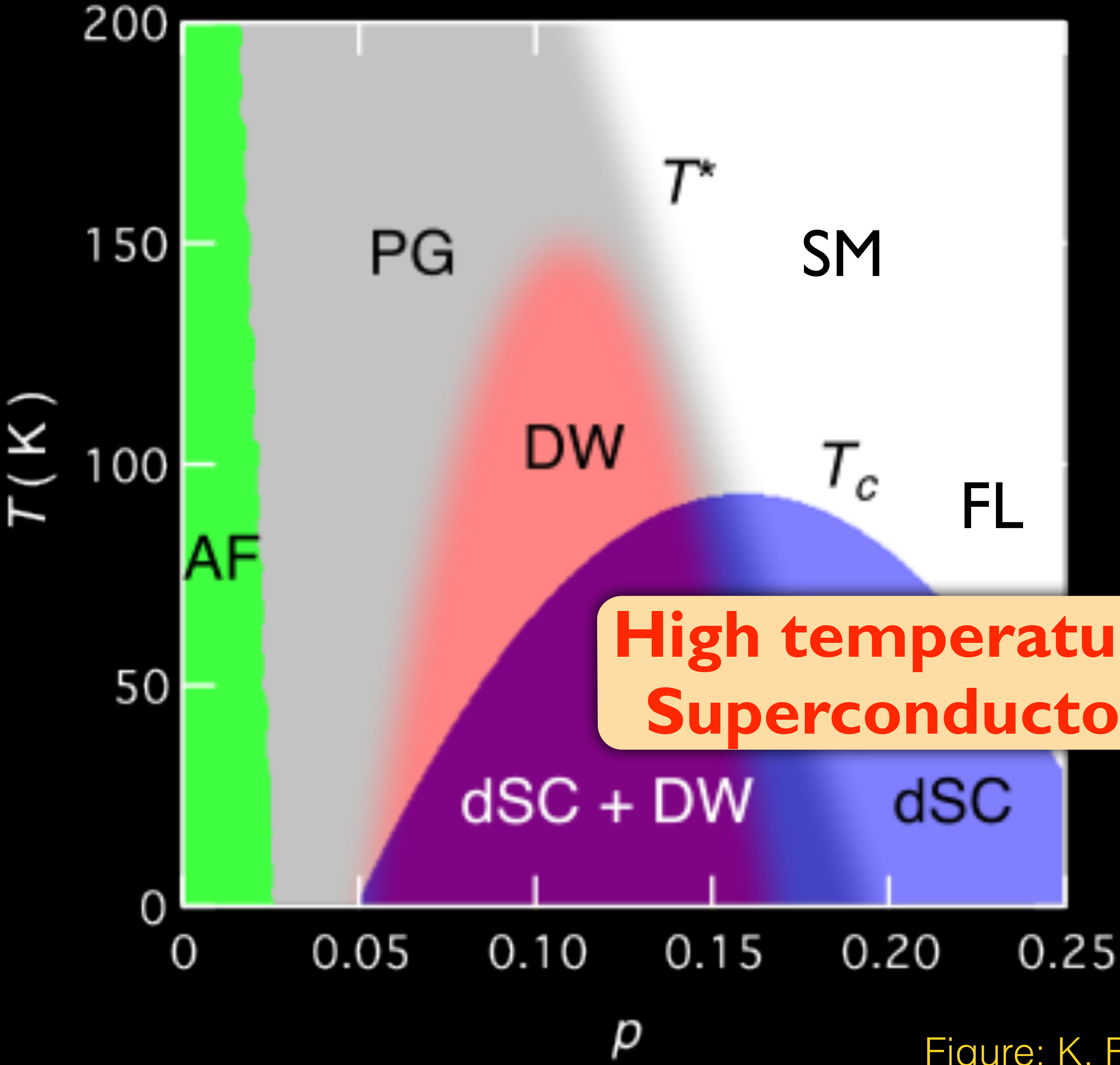


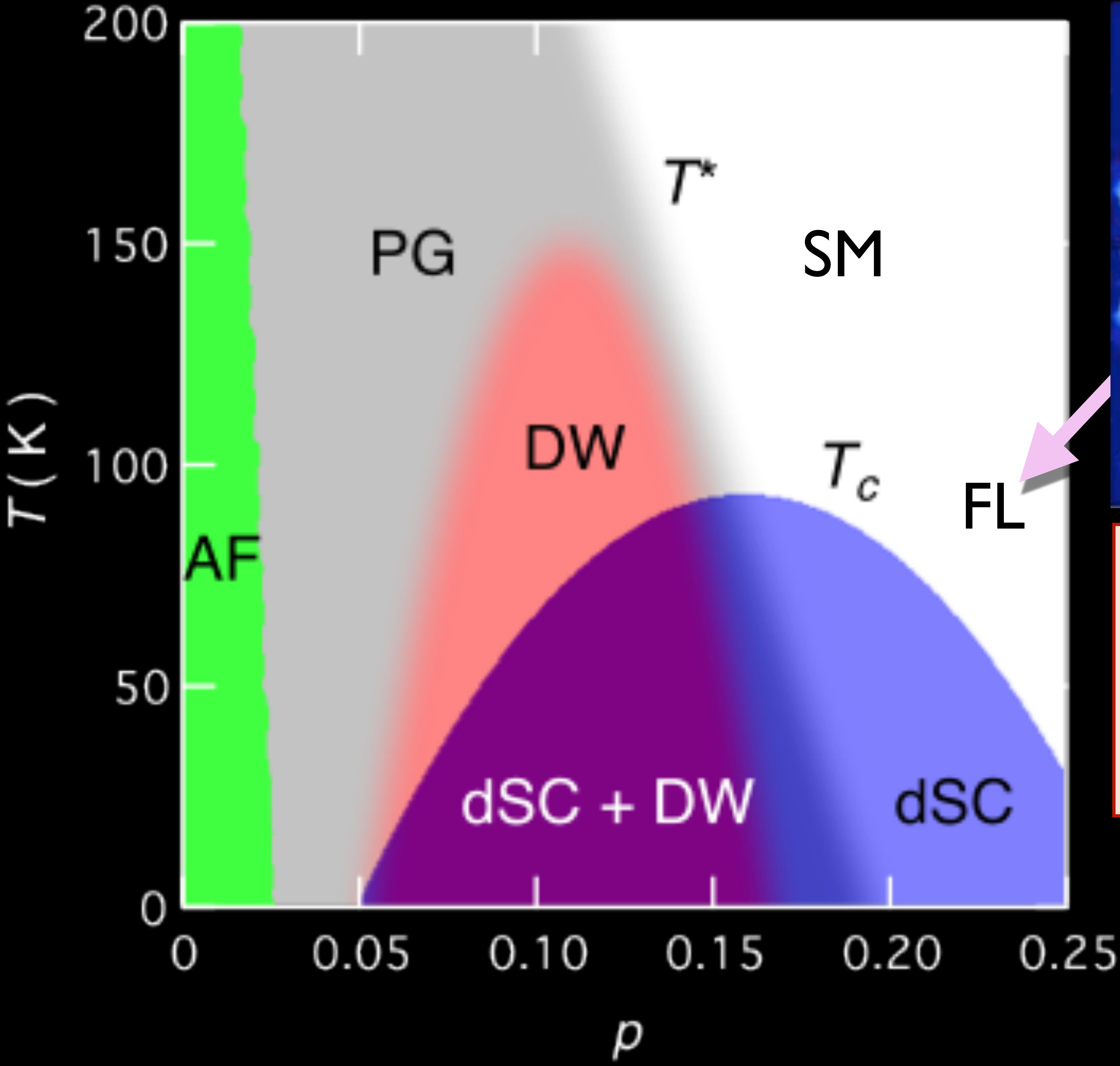
Figure: K. Fujita and J. C. Seamus Davis

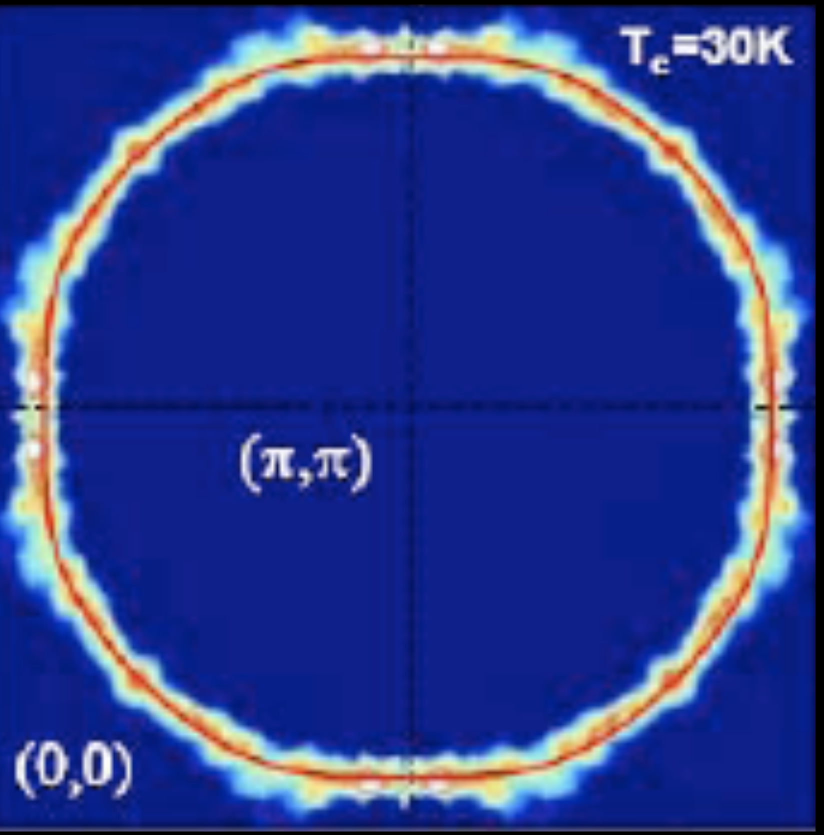


High temperature Superconductor

Figure: K. Fujita and J. C. Seamus Davis

M. Platé, J. D. F. Mottershead, I. S. Elfimov, D. C. Peets, Ruixing Liang, D. A. Bonn, W. N. Hardy, S. Chiuzbaian, M. Falub, M. Shi, L. Patthey, and A. Damascelli, Phys. Rev. Lett. **95**, 077001 (2005)

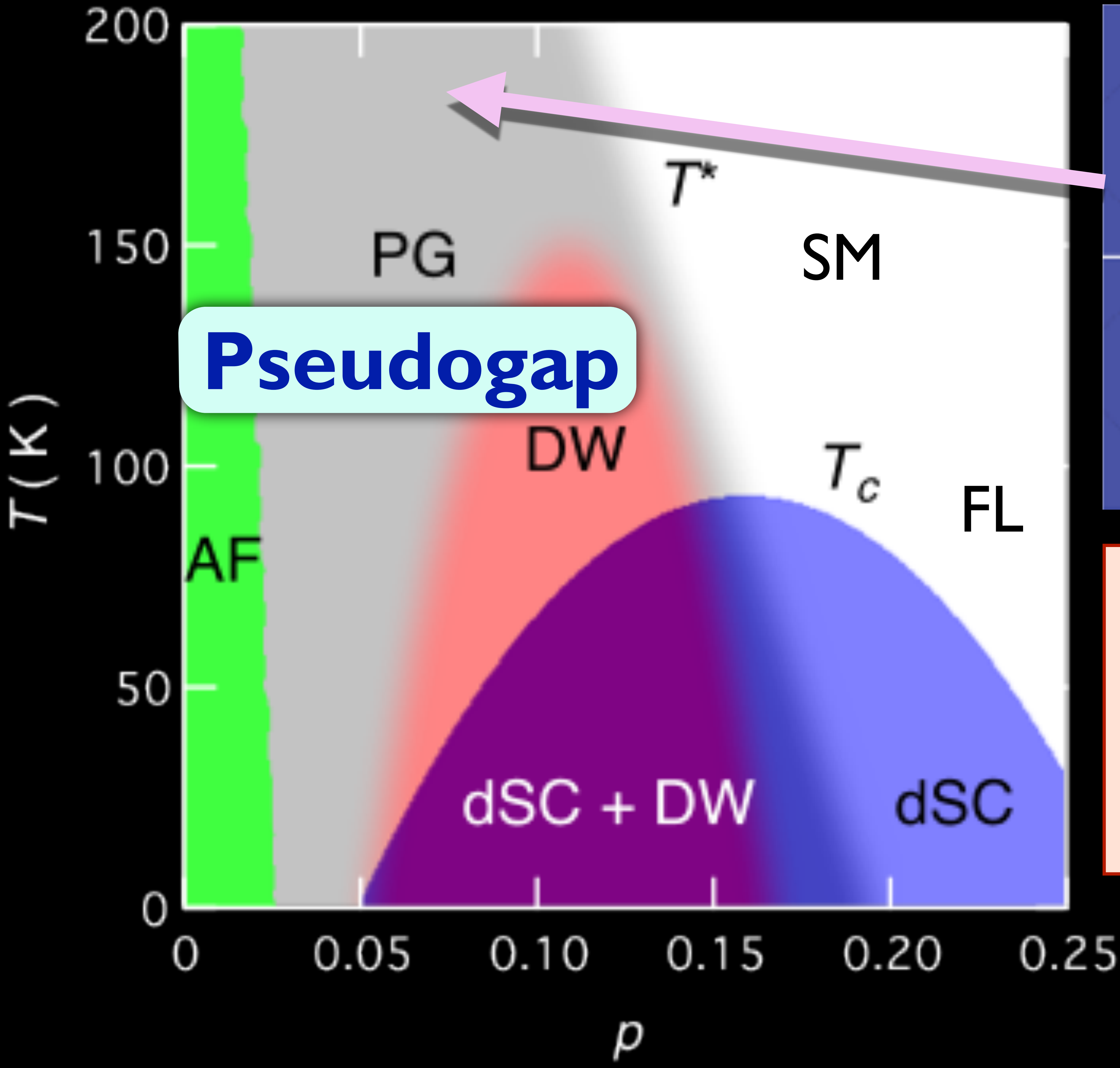




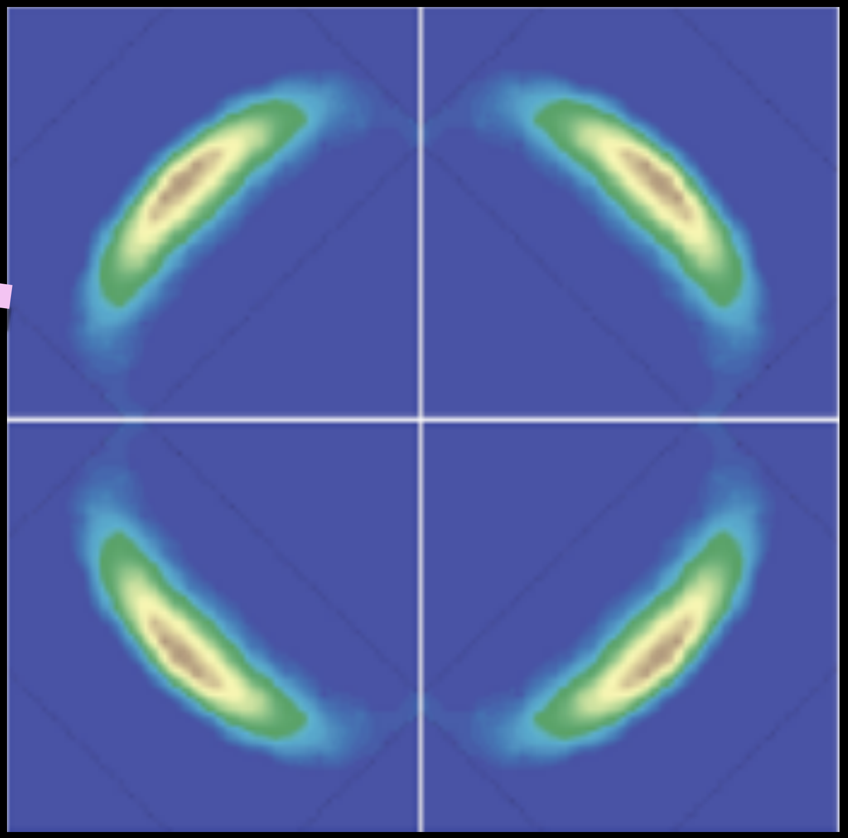
Conventional metal

Area enclosed by Fermi surface =1+p

Kyle M. Shen, F. Ronning, D. H. Lu, F. Baumberger, N. J. C. Ingle, W. S. Lee, W. Meevasana, Y. Kohsaka, M. Azuma, M. Takano, H. Takagi, Z.-X. Shen, Science **307**, 901 (2005)



Pseudogap



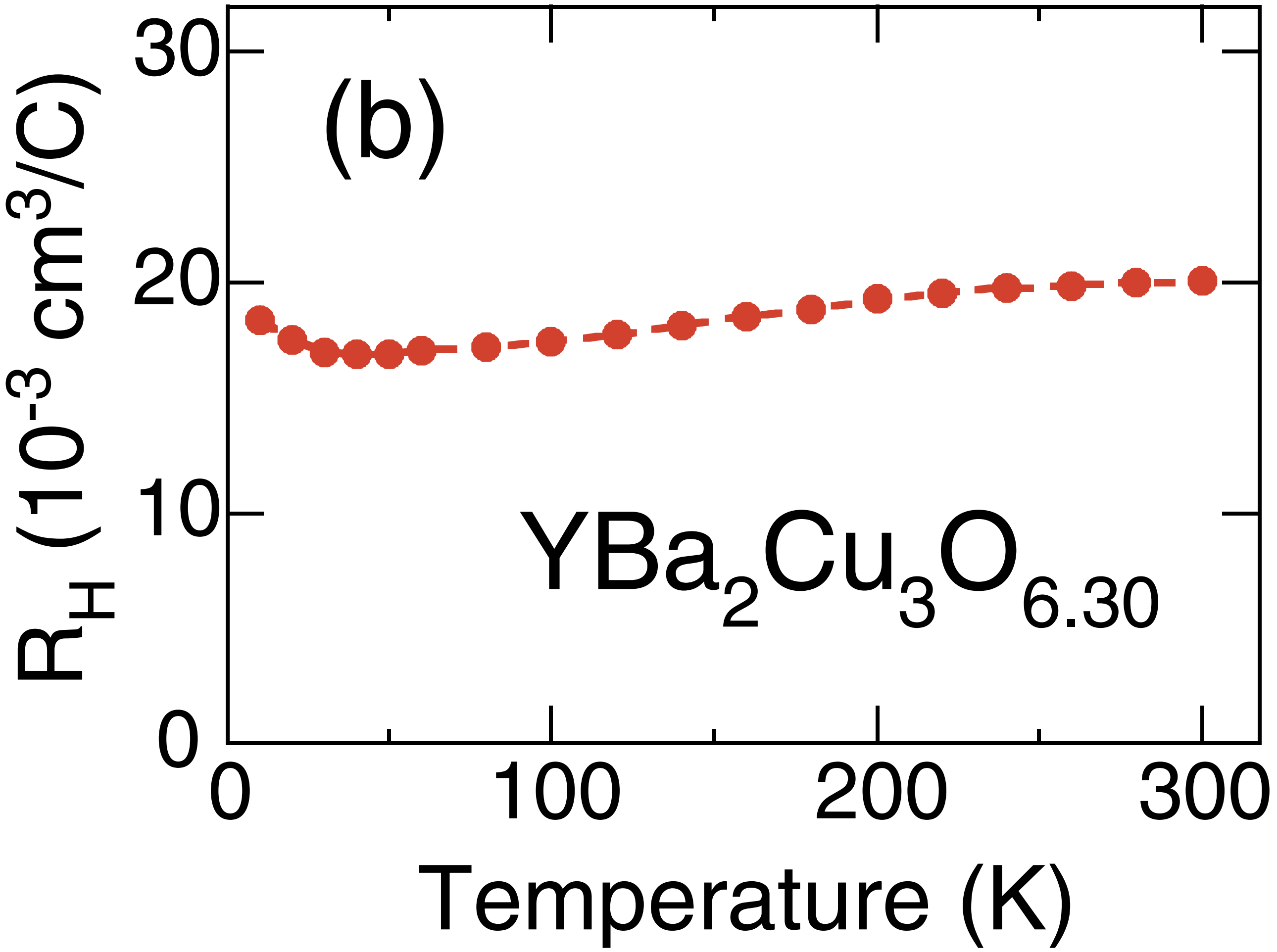
“Fermi arcs”
at
low p

Electrical and optical evidence for Fermi surface of long-lived quasiparticles of density *p*

Evolution of the Hall Coefficient and the Peculiar Electronic Structure of the Cuprate Superconductors

Yoichi Ando,* Y. Kurita,† Seiki Komiya, S. Ono, and Kouji Segawa

PRL **92**, 197001 (2004)



T-independent Hall effect in a magnetic field of fermions of charge +e and density *p*

Electrical and optical evidence for Fermi surface of long-lived quasiparticles of density p

Spectroscopic evidence for Fermi liquid-like energy and temperature dependence of the relaxation rate in the pseudogap phase of the cuprates

Seyed Iman Mirzaei[a], Damien Stricker[a], Jason N. Hancock[a,b], Christophe Berthod[a], Antoine Georges[a,c,d], Erik van Heumen[a,e], Mun K. Chan[f], Xudong Zhao[f,g], Yuan Li[h], Martin Greven[f], Neven Barišić[f,i,j], and Dirk van der Marel[a,1]

PNAS **110**, 5774 (2013)

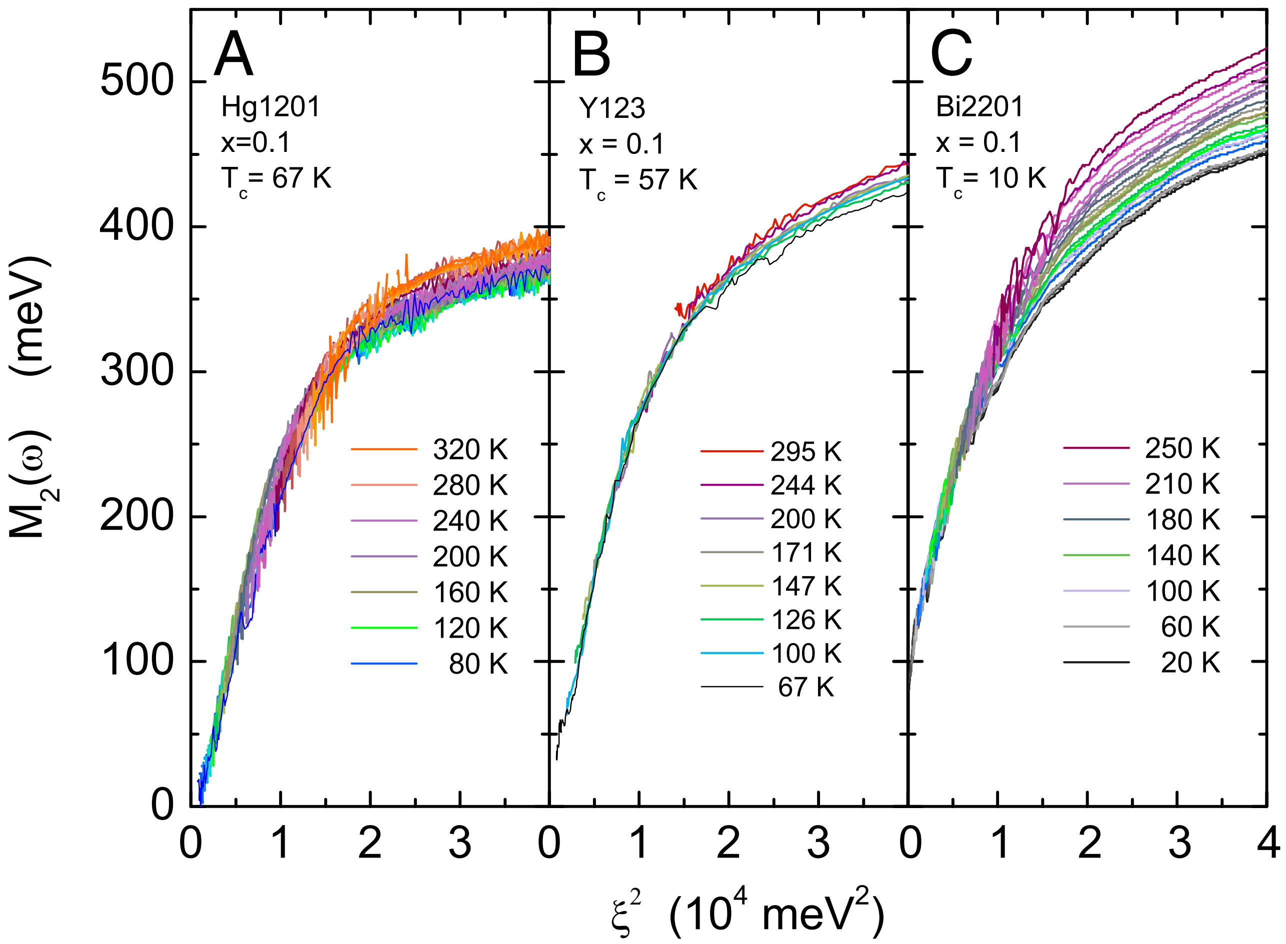


Fig. 6. Collapse of the frequency and temperature dependence of the relaxation rate of underdoped cuprate materials. Normal state $M_2(\omega, T)$ as a function of $\xi^2 \equiv (\hbar\omega)^2 + (p\pi k_B T)^2$

$$\sigma_{xx} \sim \frac{1}{(-i\omega + 1/\tau)}$$

$$\text{with } \frac{1}{\tau} \sim \omega^2 + T^2$$

Electrical and optical evidence for Fermi surface of long-lived quasiparticles of density *p*

In-Plane Magnetoresistance Obeys Kohler's Rule in the Pseudogap Phase of Cuprate Superconductors

M. K. Chan,[1,*] M. J. Veit,[1] C. J. Dorow,[1,†] Y. Ge,[1] Y. Li,[1] W. Tabis,[1,2] Y. Tang,[1] X. Zhao,[1,3]
N. Barišić,[1,4,5,‡] and M. Greven[1,§]

PRL **113**, 177005 (2014)

We report in-plane resistivity (ρ) and transverse magnetoresistance (MR) measurements for underdoped $\mathrm{HgBa_2CuO_{4+\delta}}$ (Hg1201). Contrary to the long-standing view that Kohler's rule is strongly violated in underdoped cuprates, we find that it is in fact satisfied in the pseudogap phase of Hg1201. The transverse MR shows a quadratic field dependence, $\delta\rho/\rho_0 = aH^2$, with $a(T) \propto T^{-4}$. In combination with the observed $\rho \propto T^2$ dependence, this is consistent with a single Fermi-liquid quasiparticle scattering rate. We show that this behavior is typically masked in cuprates with lower structural symmetry or strong disorder effects.

$$\rho_{xx} \sim \frac{1}{\tau}(1 + aH^2\tau^2 + \ldots)$$

$$\text{with } \frac{1}{\tau} \sim T^2$$

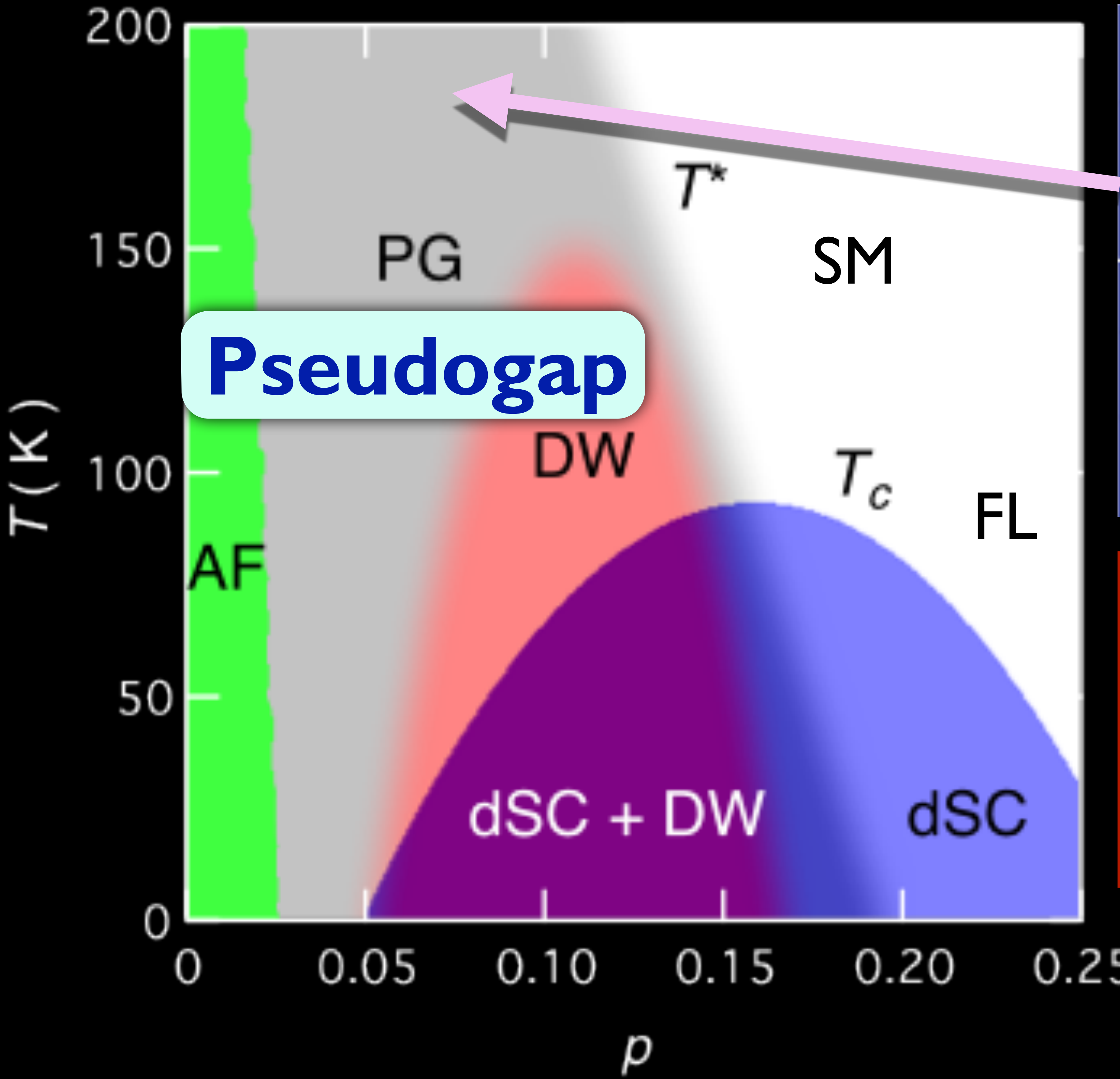
Can we have a metal with no broken translational symmetry, and with long-lived electron-like quasiparticles on a Fermi surface of size p ?

Can we have a metal with no broken translational symmetry, and with long-lived electron-like quasiparticles on a Fermi surface of size p ?

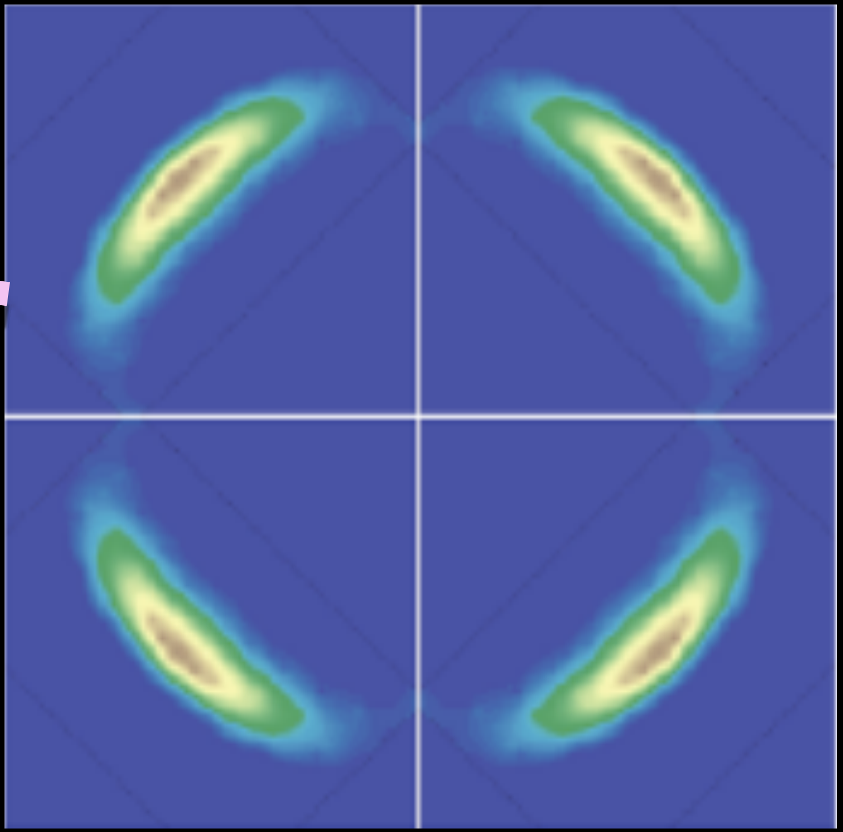
Answer: **Yes.**
There can be a Fermi surface of size p,
but it must be accompanied by
topological order, in a
"fractionalized Fermi liquid".

At $T=0$, such a metal must be separated from
a Fermi liquid (with a Fermi surface of size
$1+p$) by a quantum phase transition

Kyle M. Shen, F. Ronning, D. H. Lu, F. Baumberger, N. J. C. Ingle, W. S. Lee, W. Meevasana, Y. Kohsaka, M. Azuma, M. Takano, H. Takagi, Z.-X. Shen, Science **307**, 901 (2005)

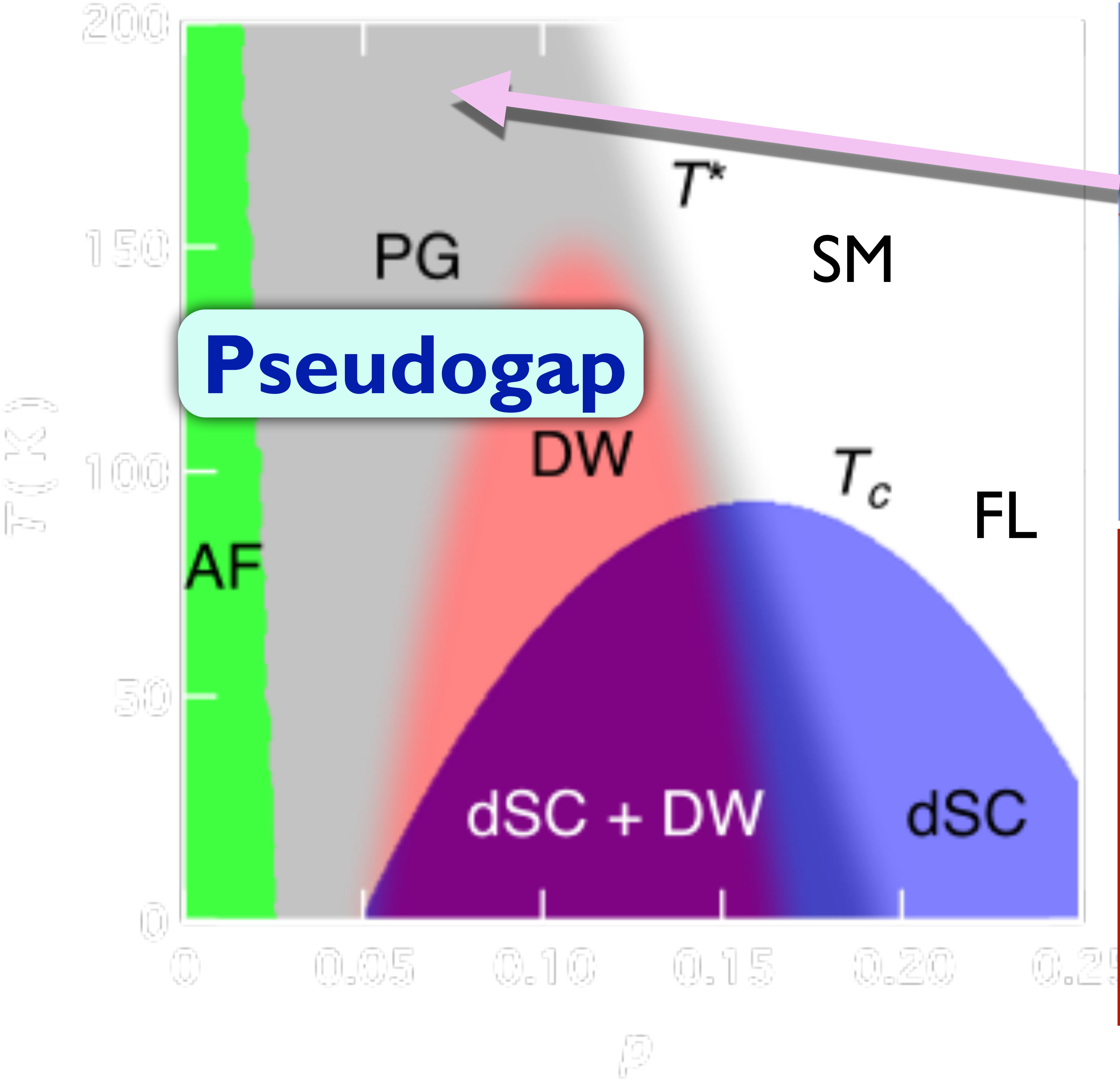


Pseudogap

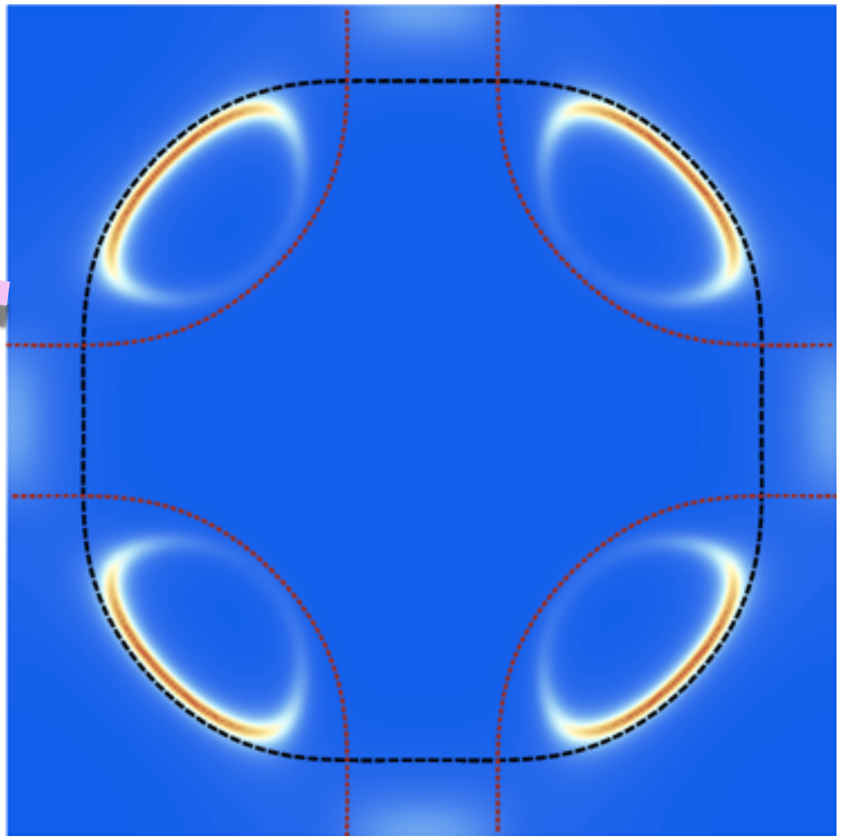


“Fermi arcs”
at
low p

Y. Qi and S. Sachdev, Phys. Rev. B **81**, 115129 (2010)
M. Punk, A. Allais, and S. Sachdev, arXiv:1501.00978, PNAS to appear

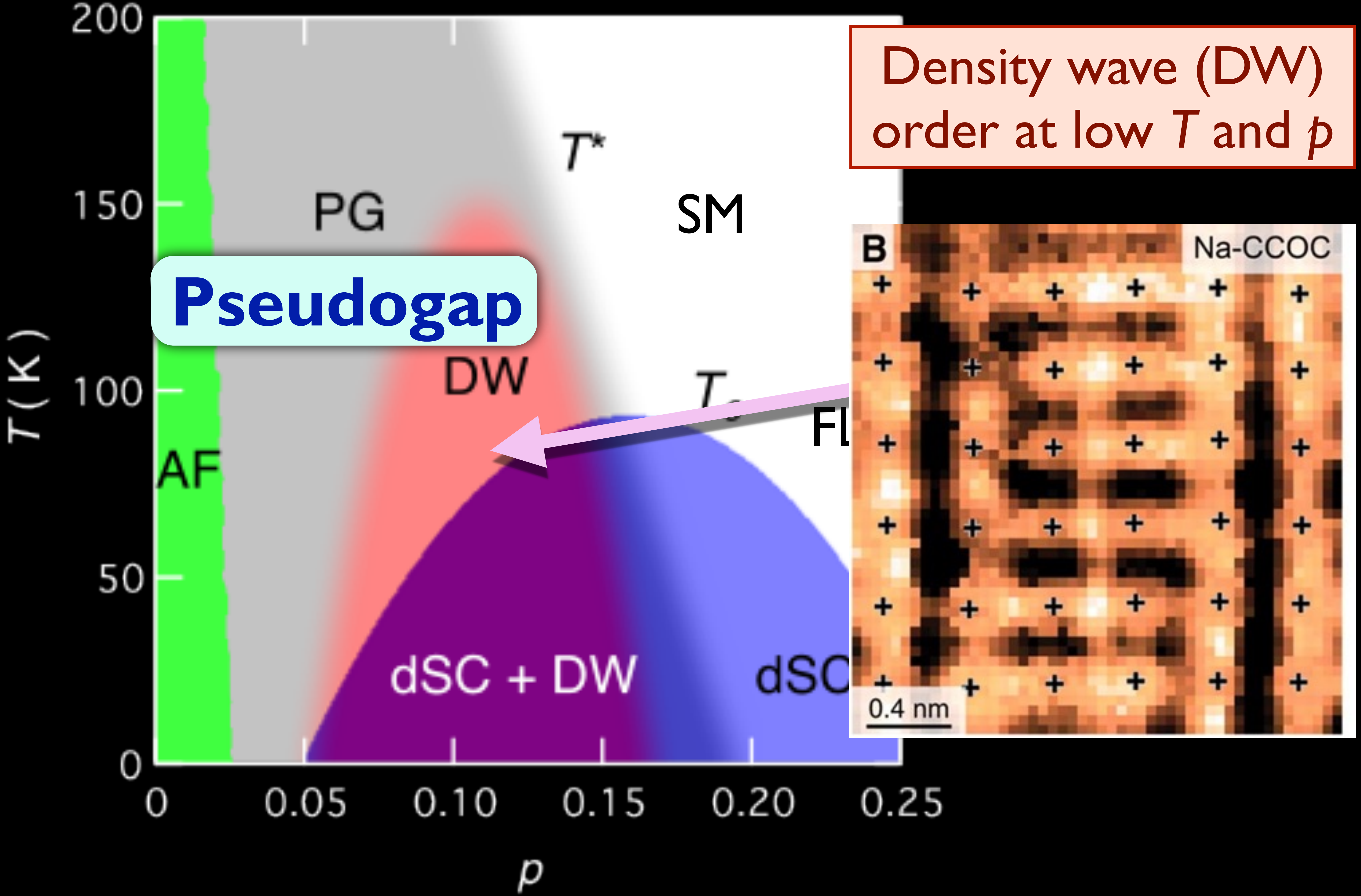


Pseudogap



A new metal — a fractionalized Fermi liquid (FL*) — with electron-like quasiparticles on a Fermi surface of size *p* ?

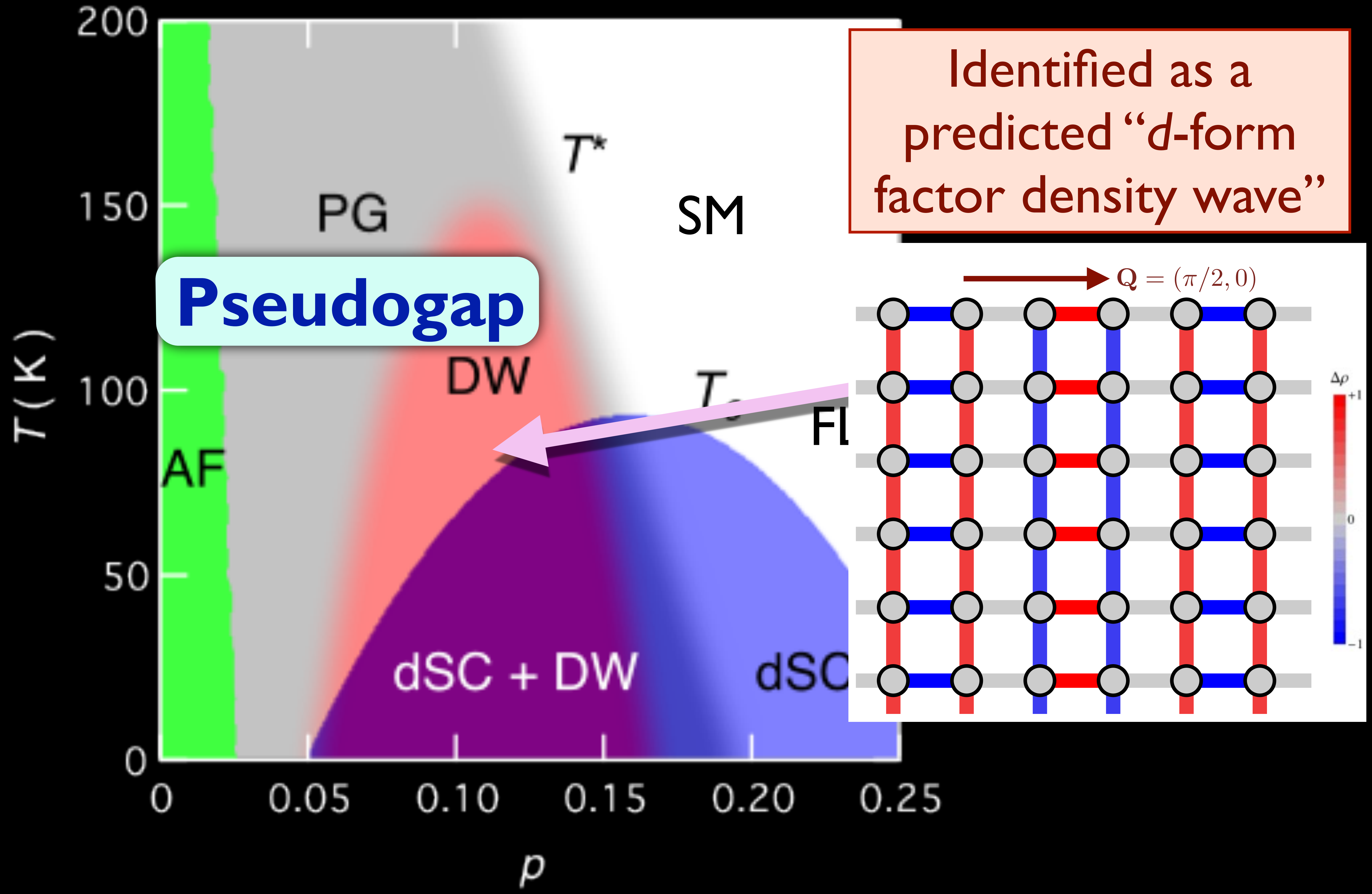
Y. Kohsaka *et al.*, Science **315**, 1380 (2007)



Density wave (DW) order at low T and p

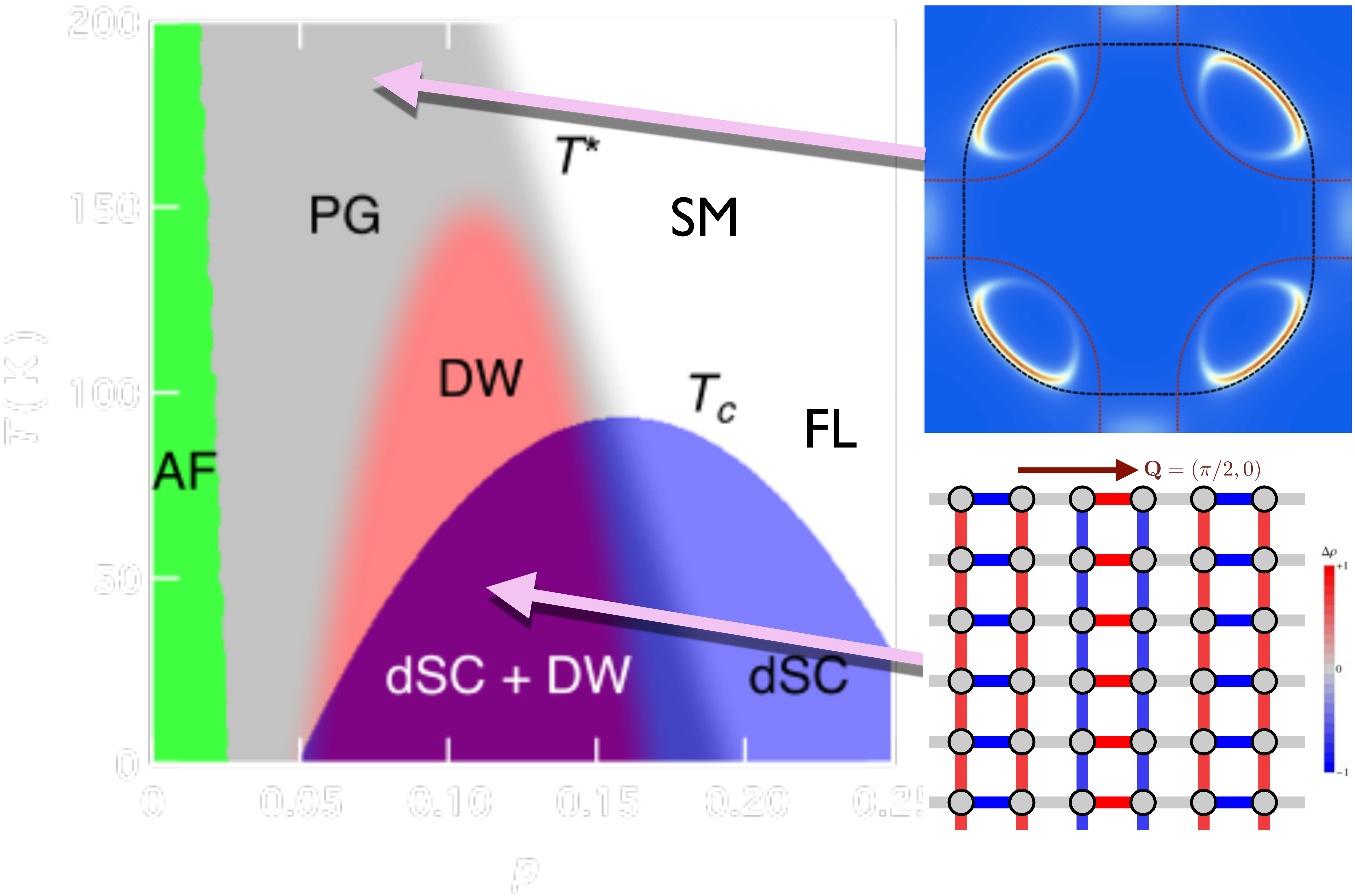
Pseudogap

M. A. Metlitski and S. Sachdev, PRB **82**, 075128 (2010). S. Sachdev R. La Placa, PRL **111**, 027202 (2013).
K. Fujita, M. H Hamidian, S. D. Edkins, Chung Koo Kim, Y. Kohsaka, M. Azuma, M. Takano, H. Takagi, H. Eisaki, S. Uchida, A. Allais, M. J. Lawler, E.-A. Kim, S. Sachdev, and J. C. Davis, PNAS **111**, E3026 (2014)



Pseudogap

Identified as a predicted “*d*-form factor density wave”


200
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T (K)
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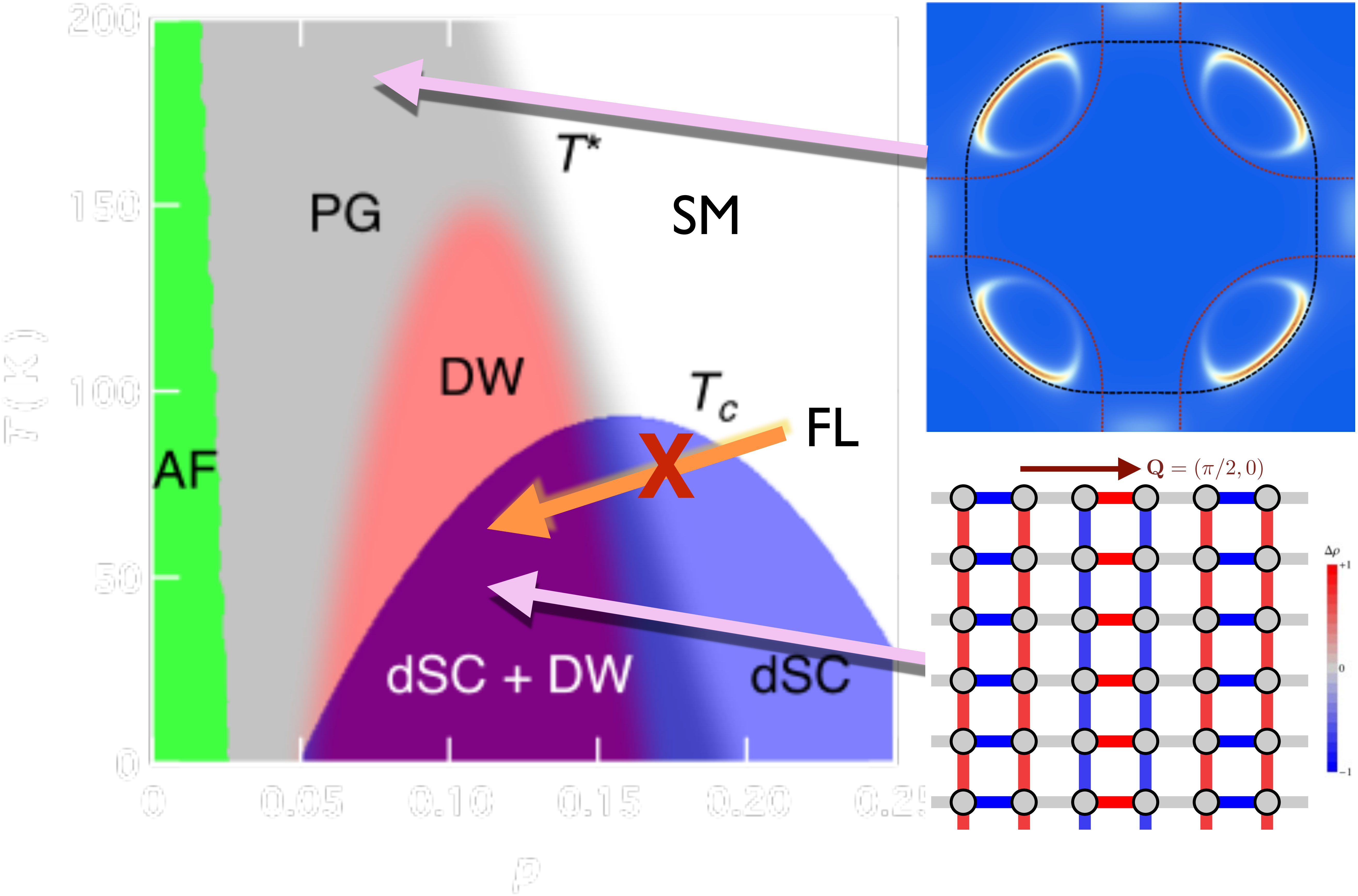
0.05
0.10
0.15
0.20
0.25
p
AF
PG
T^*
SM
DW
T_c
FL
dSC + DW
dSC
$\mathbf{Q} = (\pi/2, 0)$
$\Delta\rho$
+1
0
−1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T (K)
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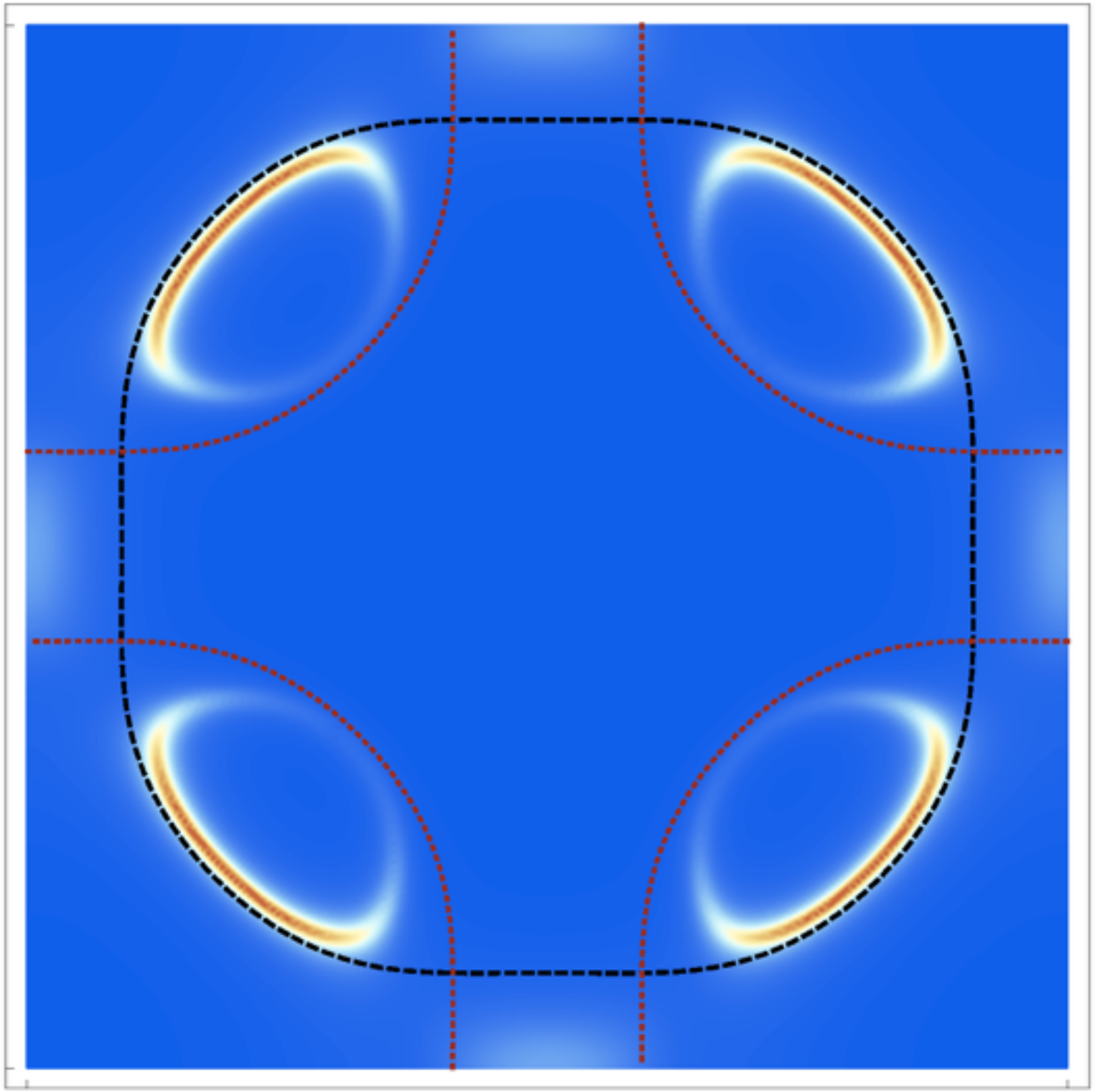
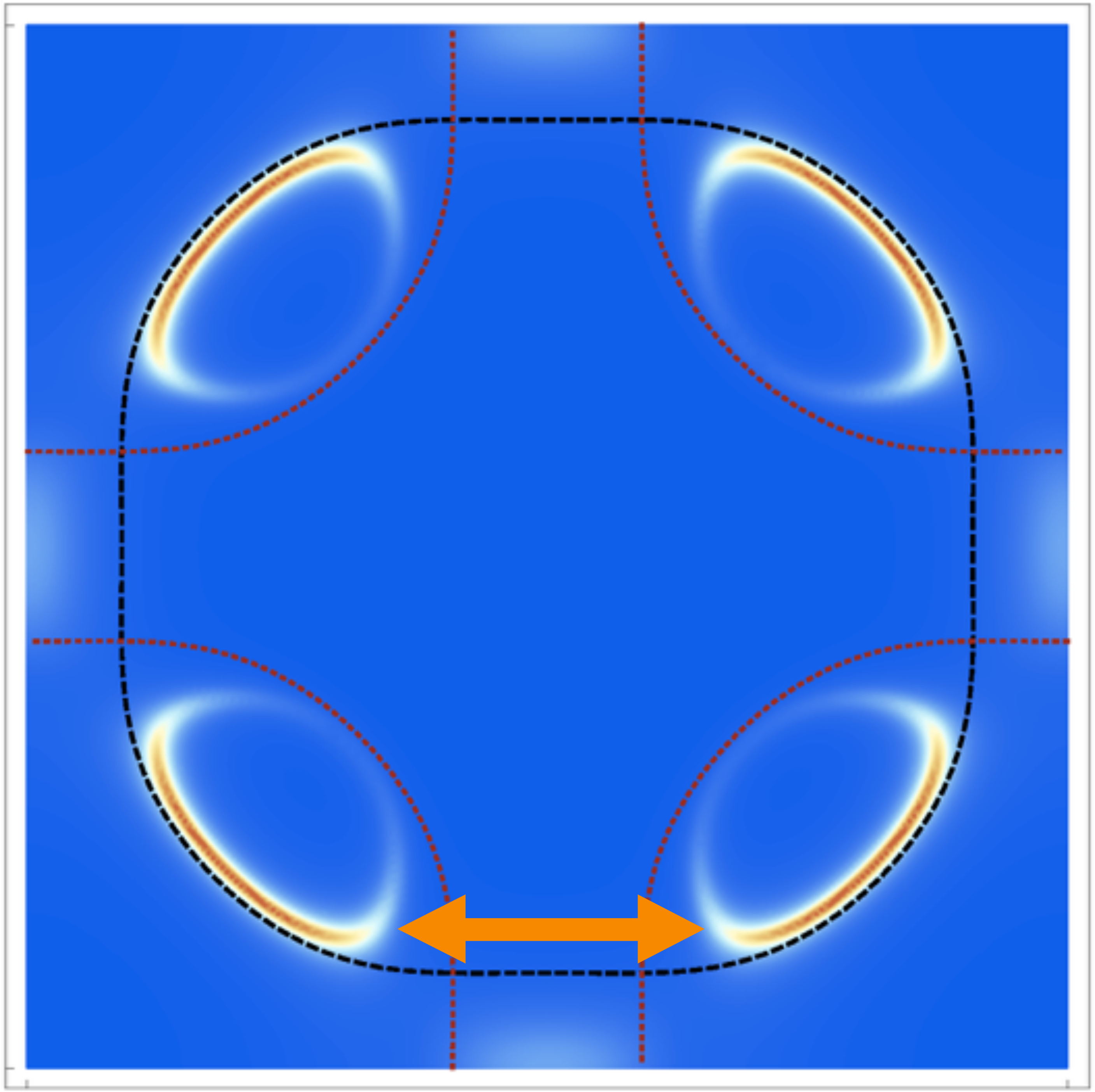
dSC + DW
dSC
$\mathbf{Q} = (\pi/2, 0)$
$\Delta\rho$
+1
0
−1



Fermi surface of a fractionalized Fermi liquid (FL*)

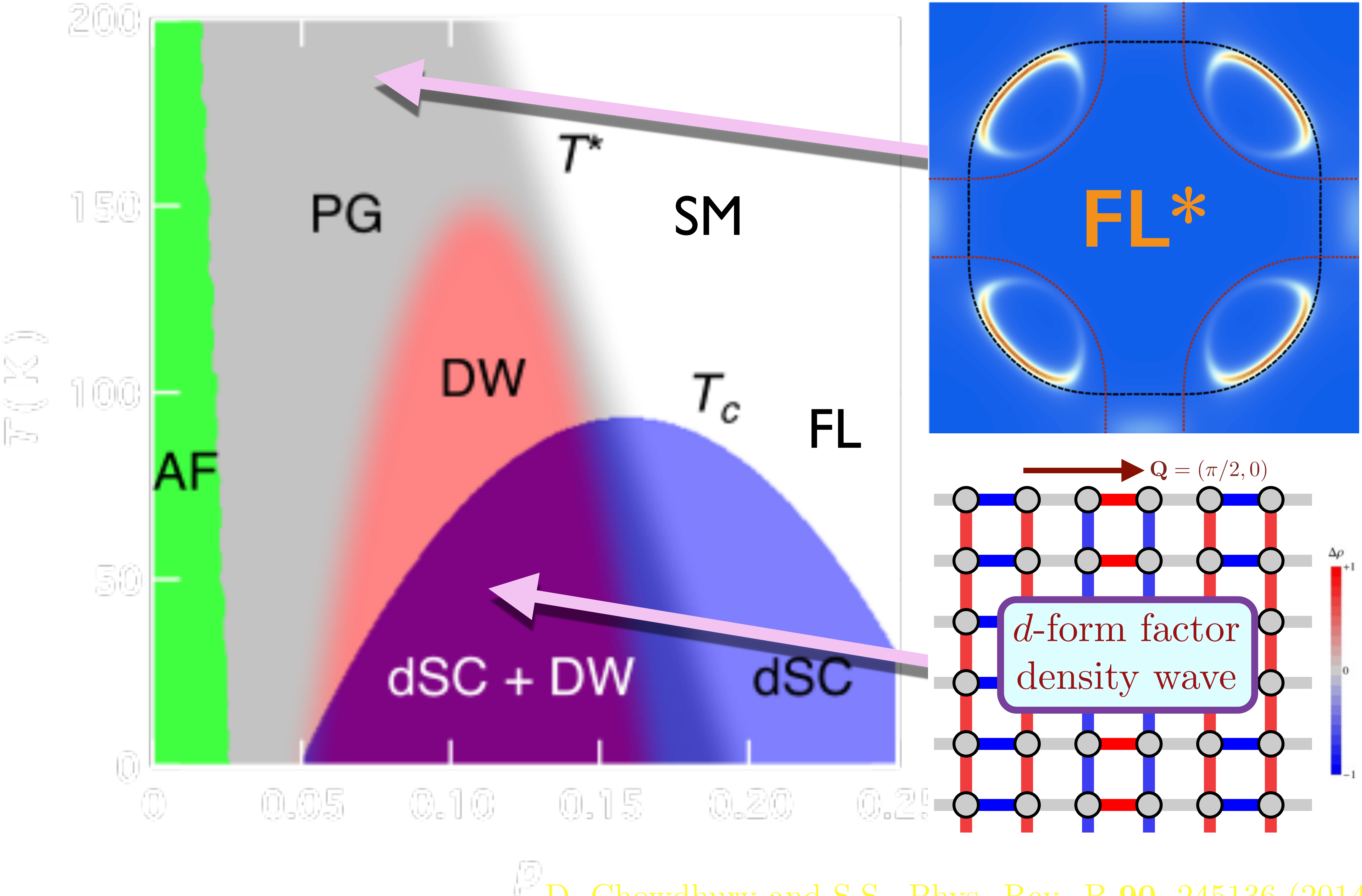
Y. Qi and S. Sachdev, Phys. Rev. B **81**, 115129 (2010)



Density wave instability of FL* leads to the observed wavevector and form-factor

D. Chowdhury and S. Sachdev, Physical Review B **90**, 245136 (2014).

The high *T* FL* can help explain the “d-form factor density wave” observed at low *T*



D. Chowdhury and S.S., Phys. Rev. B **90**, 245136 (2014)

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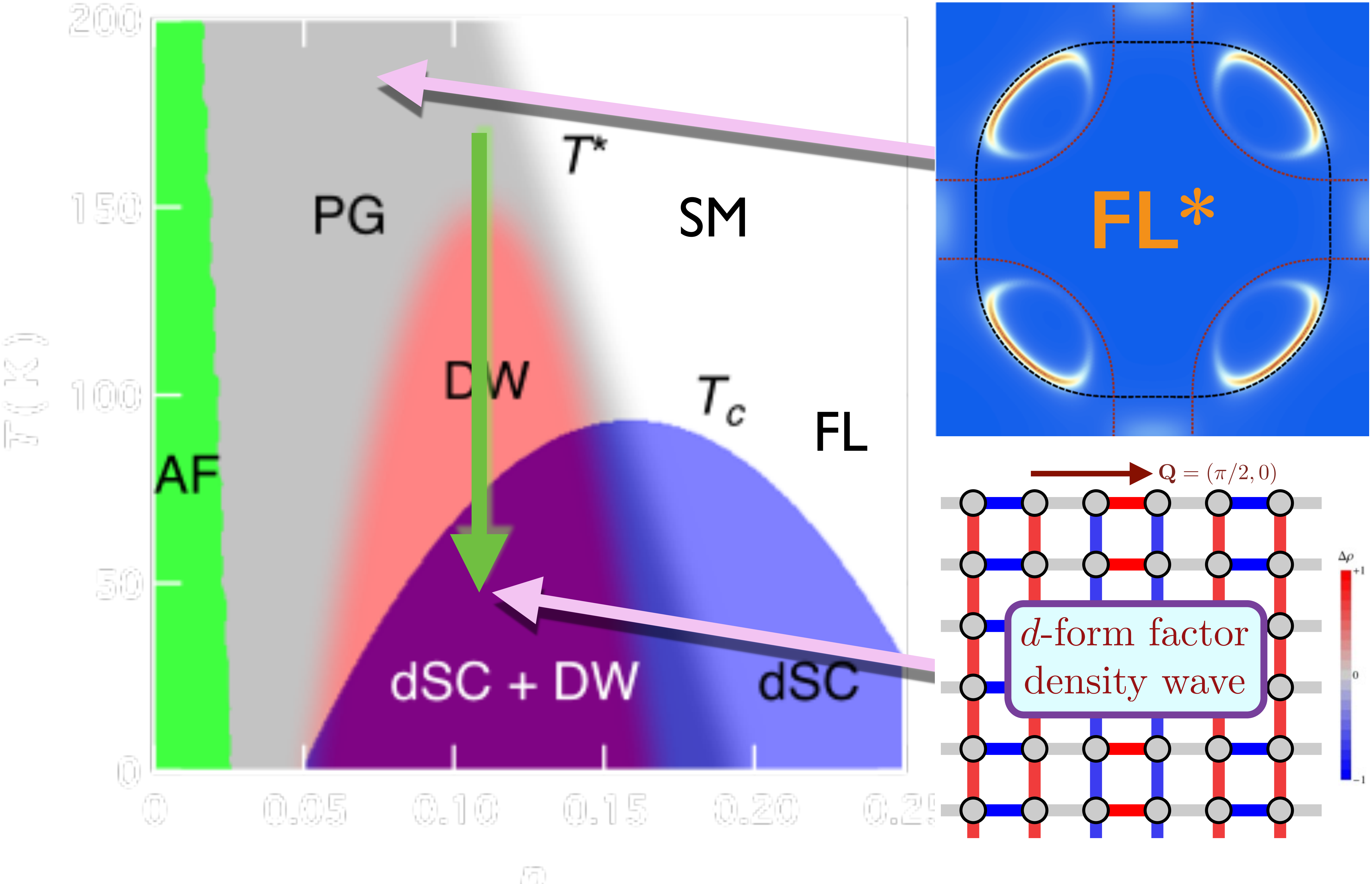

D. Chowdhury and S.S., Phys. Rev. B **90**, 245136 (2014)

A. Review of Fermi liquid theory

Metals with with quasiparticle excitations

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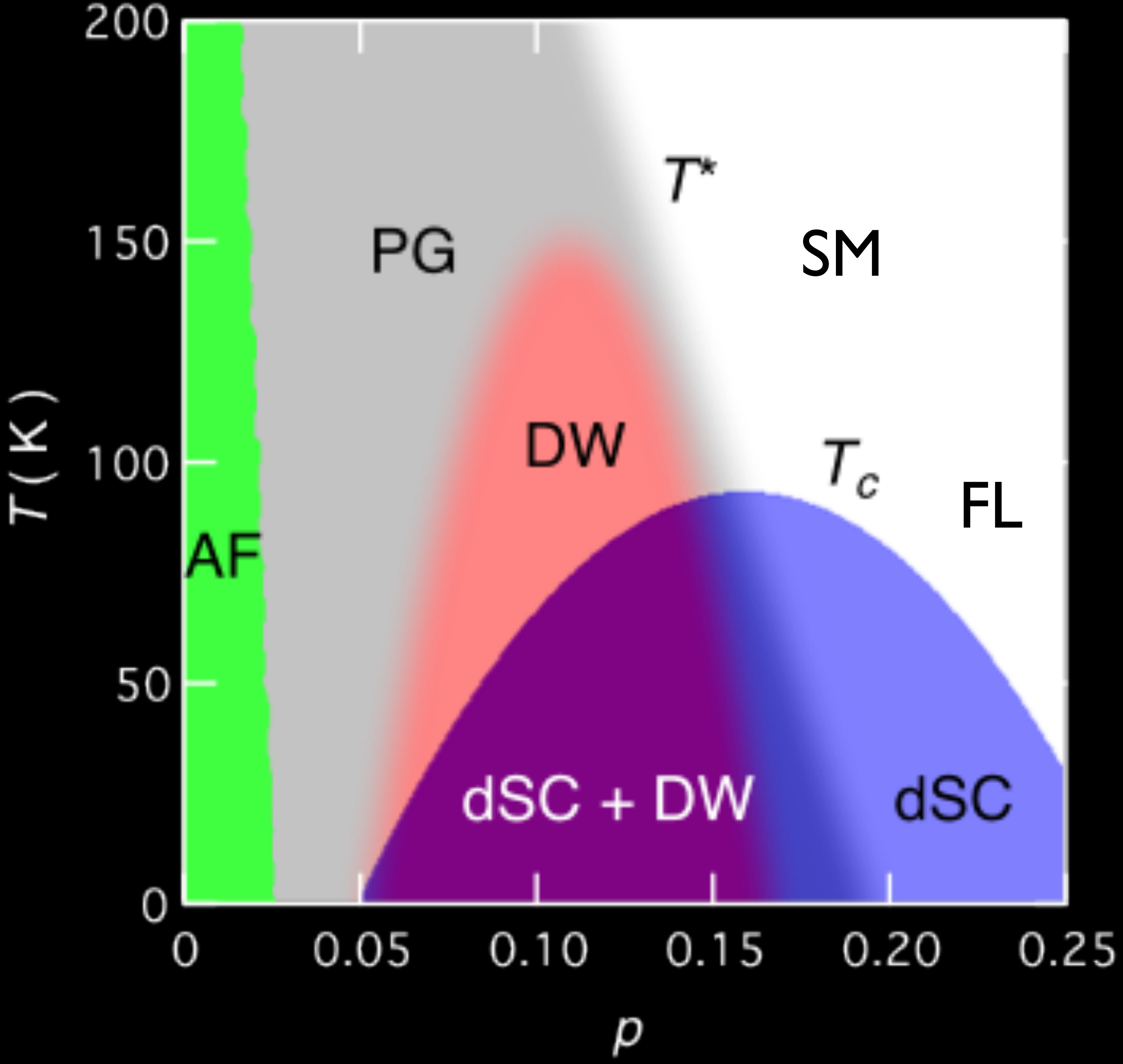
200
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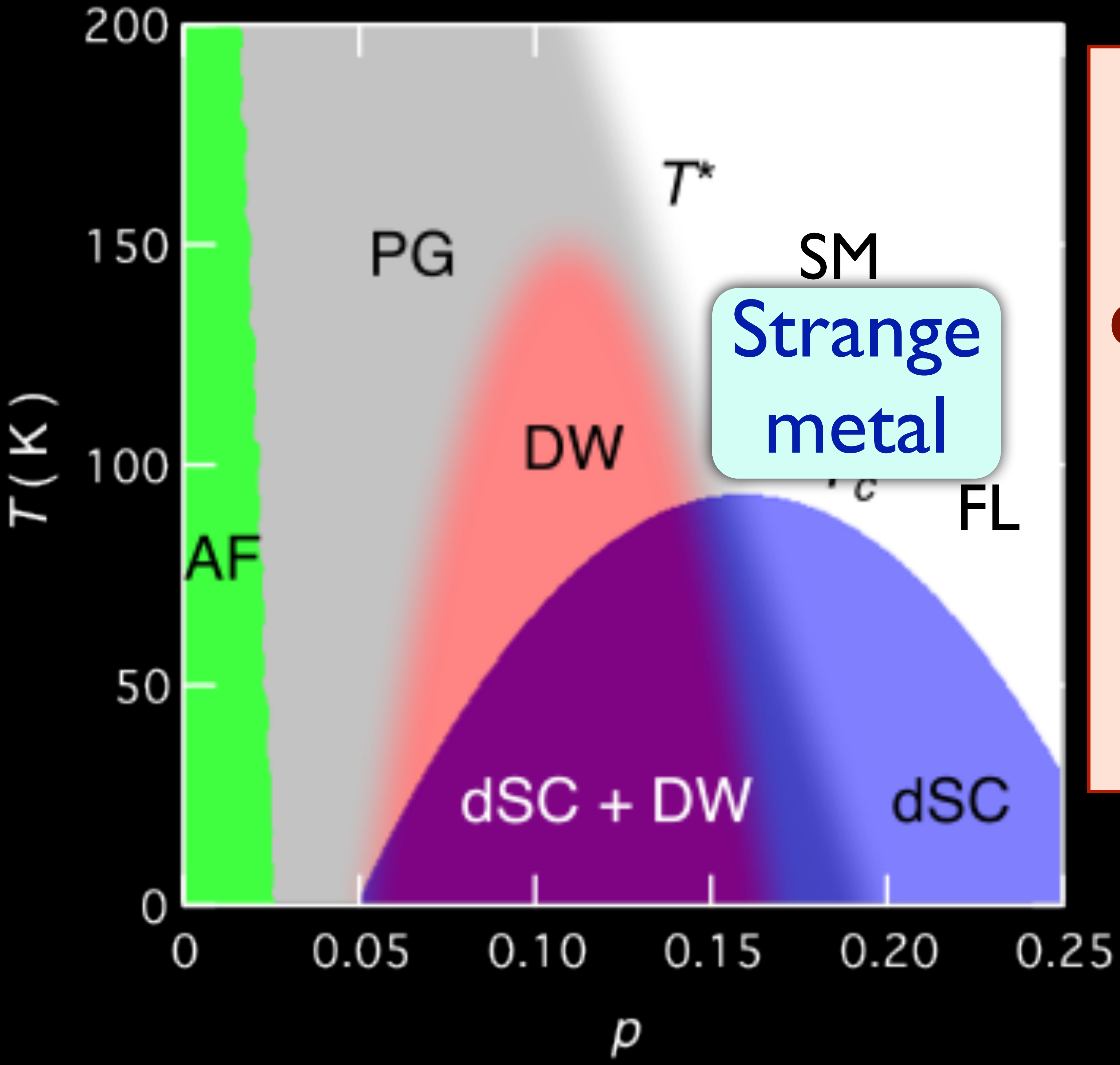


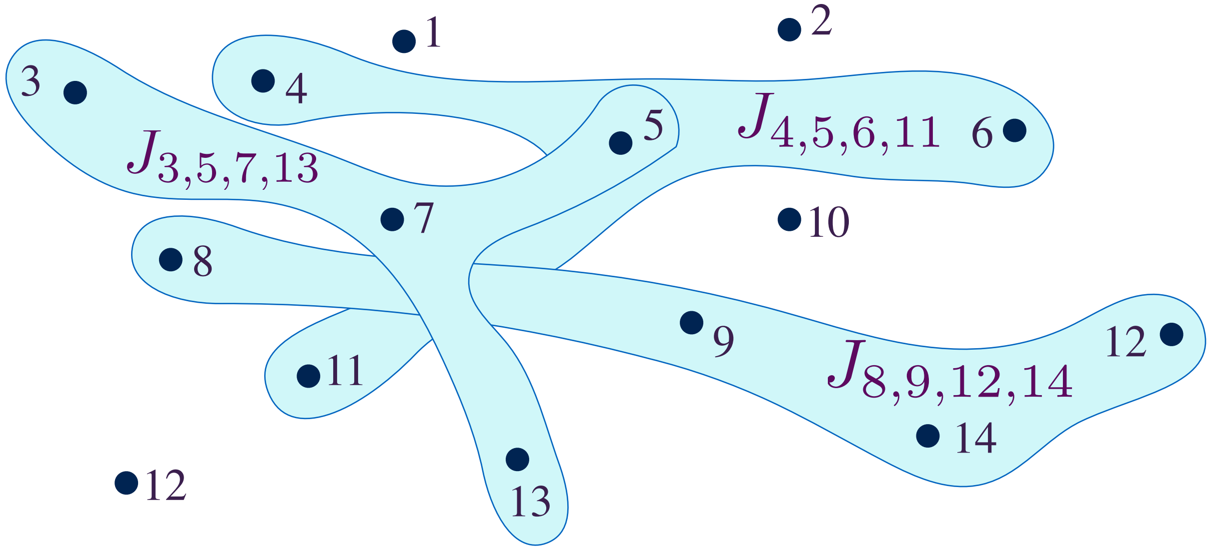
Metal
(gapless,
compressible
state)
without
quasi-
particles

$$H = \frac{1}{(2N)^{3/2}} \sum_{i,j,k,\ell=1}^{N} J_{ij;k\ell}\, c_i^\dagger c_j^\dagger c_k c_\ell$$

$$c_i c_j + c_j c_i = 0$$

$$c_i c_j^\dagger + c_j^\dagger c_i = \delta_{ij}$$

$J_{ij;k\ell}$ independent random numbers

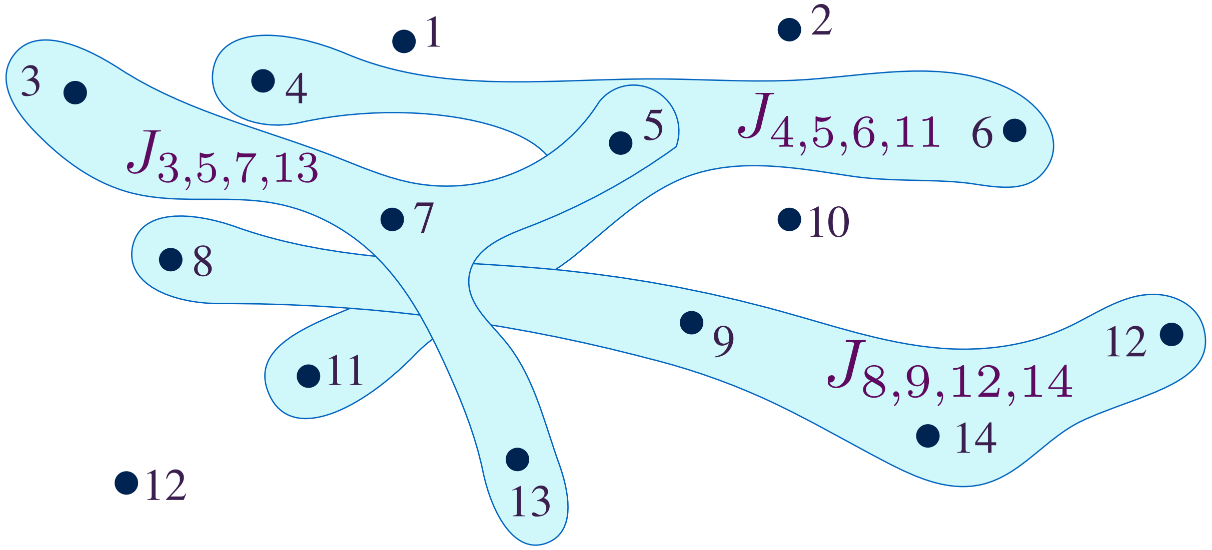


$$\mathcal{Q} = \frac{1}{N} \sum_i \left\langle c_i^\dagger c_i \right\rangle .$$

An infinite-range model of a strange metal

S. Sachdev and J. Ye, Phys. Rev. Lett. **70**, 3339 (1993)
A. Kitaev, unpublished
S. Sachdev, arXiv:1506.05111

$$H = \frac{1}{(2N)^{3/2}} \sum_{i,j,k,\ell=1}^{N} J_{ij;k\ell}\, c_i^\dagger c_j^\dagger c_k c_\ell$$



$$\mathcal{Q} = \frac{1}{N} \sum_i \left\langle c_i^\dagger c_i \right\rangle .$$

Local fermion density of states

$$\rho(\omega) \sim \left\{ \begin{array}{ll} \omega^{-1/2} & , \ \omega > 0 \\ e^{-2\pi\mathcal{E}}\, |\omega|^{-1/2} & , \ \omega < 0. \end{array} \right.$$

S. Sachdev and J. Ye, Phys. Rev. Lett. **70**, 3339 (1993)

$$c_i c_j + c_j c_i = 0$$

$$c_i c_j^\dagger + c_j^\dagger c_i = \delta_{ij}$$

$J_{ij;k\ell}$ independent random numbers

$$H = \frac{1}{(2N)^{3/2}} \sum_{i,j,k,\ell=1}^{N} J_{ij;k\ell}\, c_i^\dagger c_j^\dagger c_k c_\ell$$

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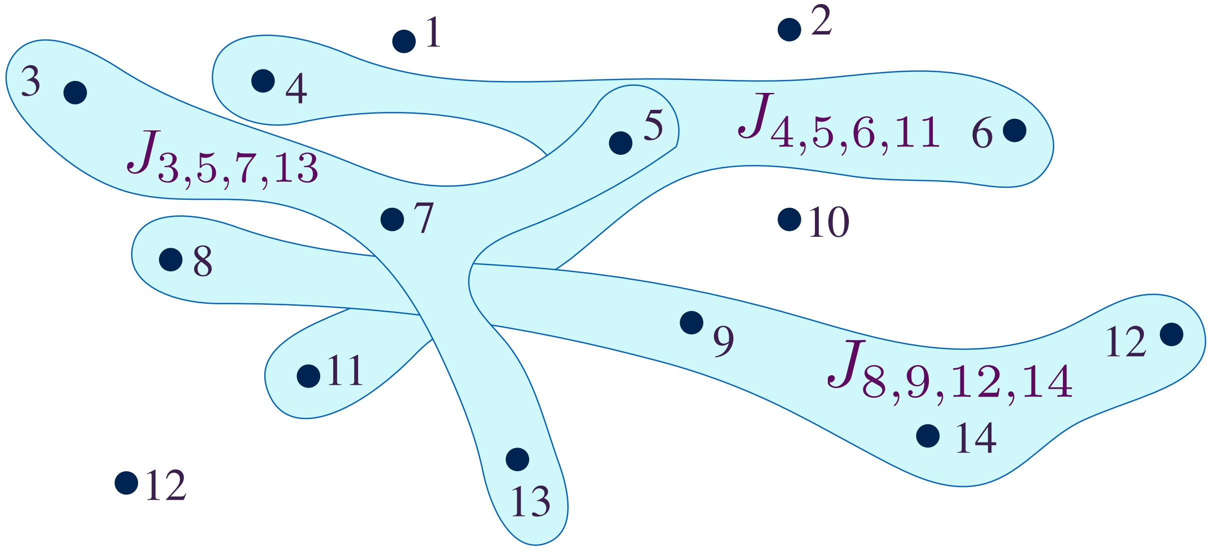

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Known ‘equation of state’ determines $\mathcal{E}$ as a function of $\mathcal{Q}$

$$\mathcal{Q} = \frac{1}{4}(3 - \tanh(2\pi\mathcal{E})) - \frac{1}{\pi} \tan^{-1}\left(e^{2\pi\mathcal{E}}\right)$$

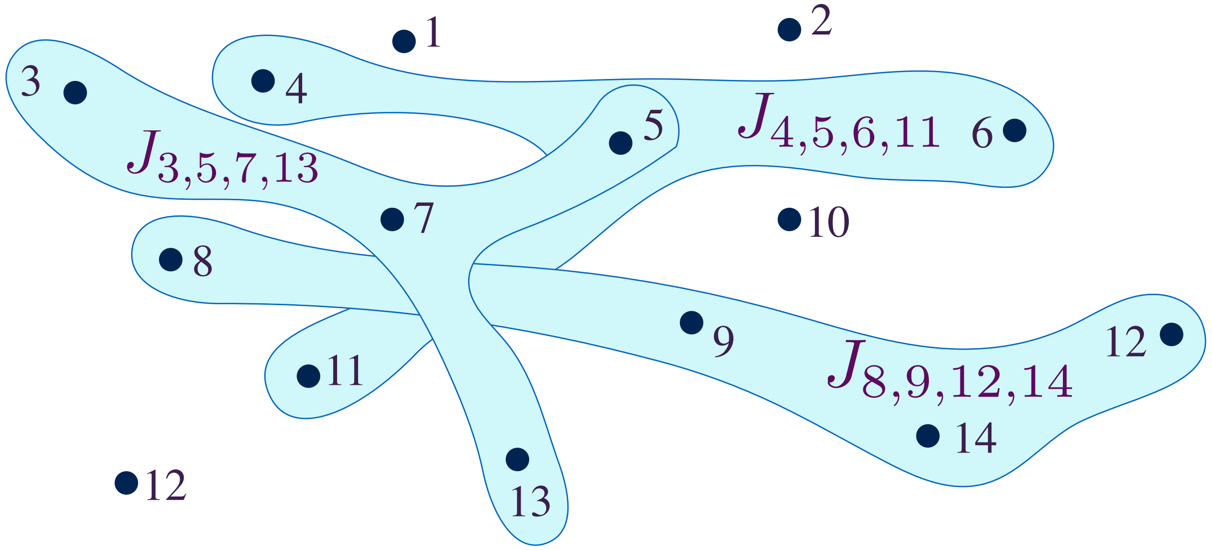
A. Georges, O. Parcollet, and S. Sachdev
Phys. Rev. B **63**, 134406 (2001)

$$H = \frac{1}{(2N)^{3/2}} \sum_{i,j,k,\ell=1}^{N} J_{ij;k\ell}\, c_i^\dagger c_j^\dagger c_k c_\ell$$

$$c_i c_j + c_j c_i = 0$$

$$c_i c_j^\dagger + c_j^\dagger c_i = \delta_{ij}$$

$J_{ij;k\ell}$ independent random numbers



$$\mathcal{Q} = \frac{1}{N} \sum_i \left\langle c_i^\dagger c_i \right\rangle .$$

Local fermion density of states

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Known 'equation of state' determines $\mathcal{E}$ as a function of $\mathcal{Q}$

Microscopic zero temperature entropy density, $\mathcal{S}$, obeys

$$\frac{\partial \mathcal{S}}{\partial \mathcal{Q}} = 2\pi \mathcal{E}$$

O. Parcollet, A. Georges, G. Kotliar, and A. Sengupta
Phys. Rev. B **58**, 3794 (1998)

A. Georges, O. Parcollet, and S. Sachdev
Phys. Rev. B **63**, 134406 (2001)

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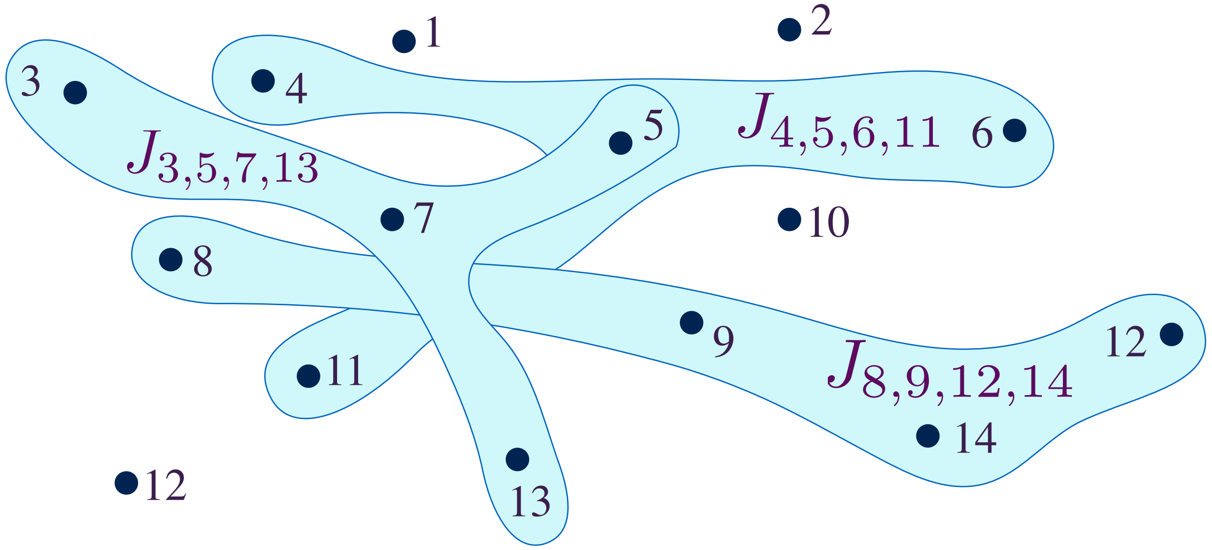

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Known 'equation of state'
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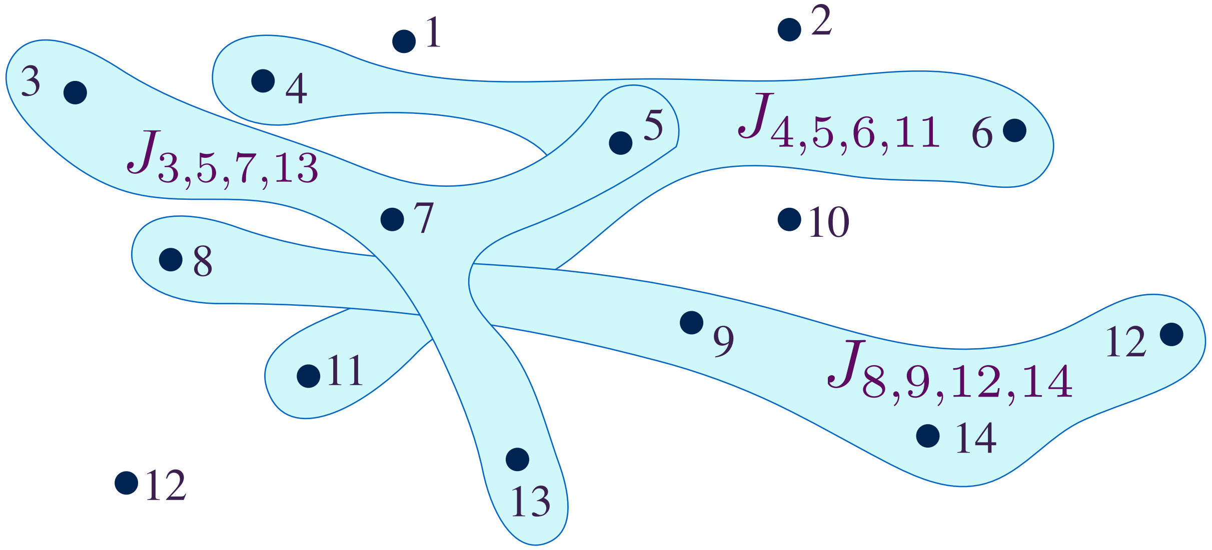
Microscopic zero temperature
entropy density, $\mathcal{S}$, obeys

$$\frac{\partial \mathcal{S}}{\partial \mathcal{Q}} = 2\pi\mathcal{E}$$

Einstein-Maxwell theory
+ cosmological constant

$$H = \frac{1}{(2N)^{3/2}} \sum_{i,j,k,\ell=1}^{N} J_{ij;k\ell}\, c_i^\dagger c_j^\dagger c_k c_\ell$$



$$\mathcal{Q} = \frac{1}{N} \sum_i \left\langle c_i^\dagger c_i \right\rangle .$$

Local fermion density of states

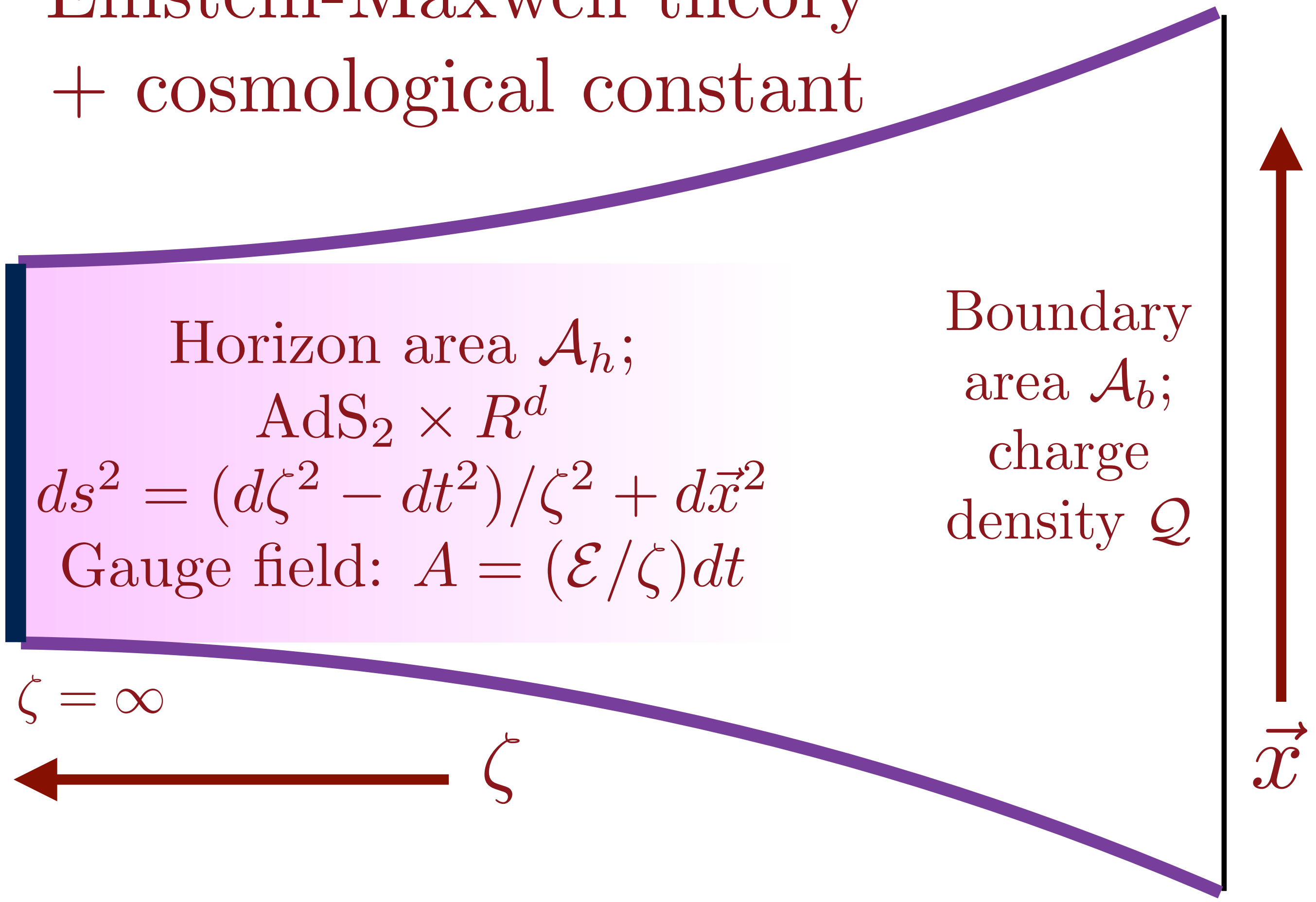
$$\rho(\omega) \sim \begin{cases} \omega^{-1/2} & , \ \omega > 0 \\ e^{-2\pi\mathcal{E}}\, |\omega|^{-1/2} & , \ \omega < 0. \end{cases}$$

Known ‘equation of state’
determines $\mathcal{E}$ as a function of $\mathcal{Q}$

Microscopic zero temperature
entropy density, $\mathcal{S}$, obeys

$$\frac{\partial \mathcal{S}}{\partial \mathcal{Q}} = 2\pi\mathcal{E}$$

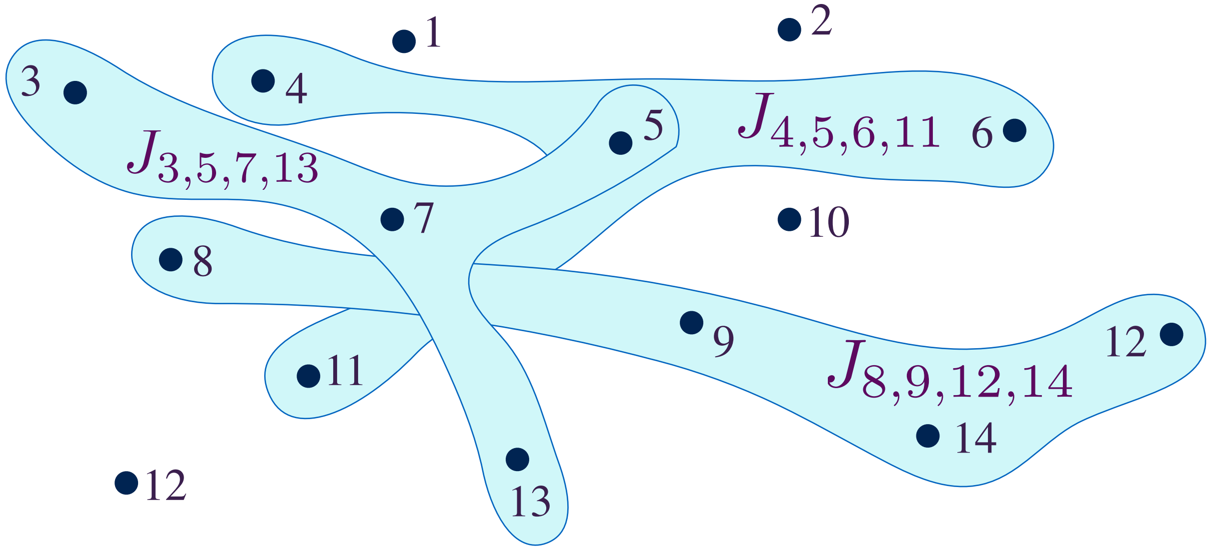
Einstein-Maxwell theory
+ cosmological constant


Horizon area $\mathcal{A}_h$;
AdS$_2 \times R^d$
$ds^2 = (d\zeta^2 - dt^2)/\zeta^2 + d\vec{x}^2$
Gauge field: $A = (\mathcal{E}/\zeta)dt$

Boundary
area $\mathcal{A}_b$;
charge
density $\mathcal{Q}$

A. Chamblin, R. Emparan, C.V. Johnson, and R.C. Myers
Phys. Rev. D **60**, 064018 (1999)

$$H = \frac{1}{(2N)^{3/2}} \sum_{i,j,k,\ell=1}^{N} J_{ij;k\ell} \, c_i^\dagger c_j^\dagger c_k c_\ell$$



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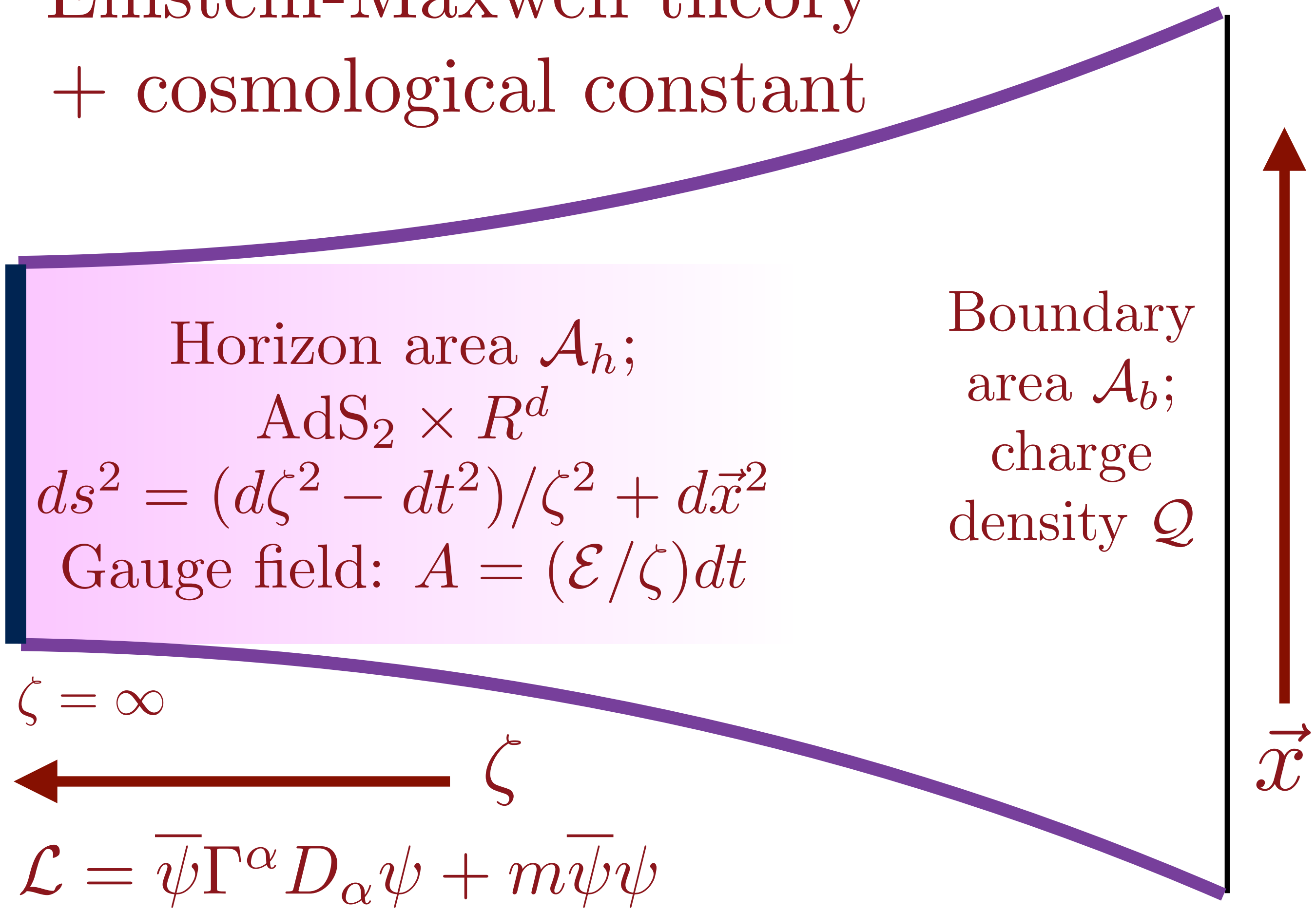

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T. Faulkner, Hong Liu, J. McGreevy, and D. Vegh
Phys. Rev. D **83**, 125002 (2011)

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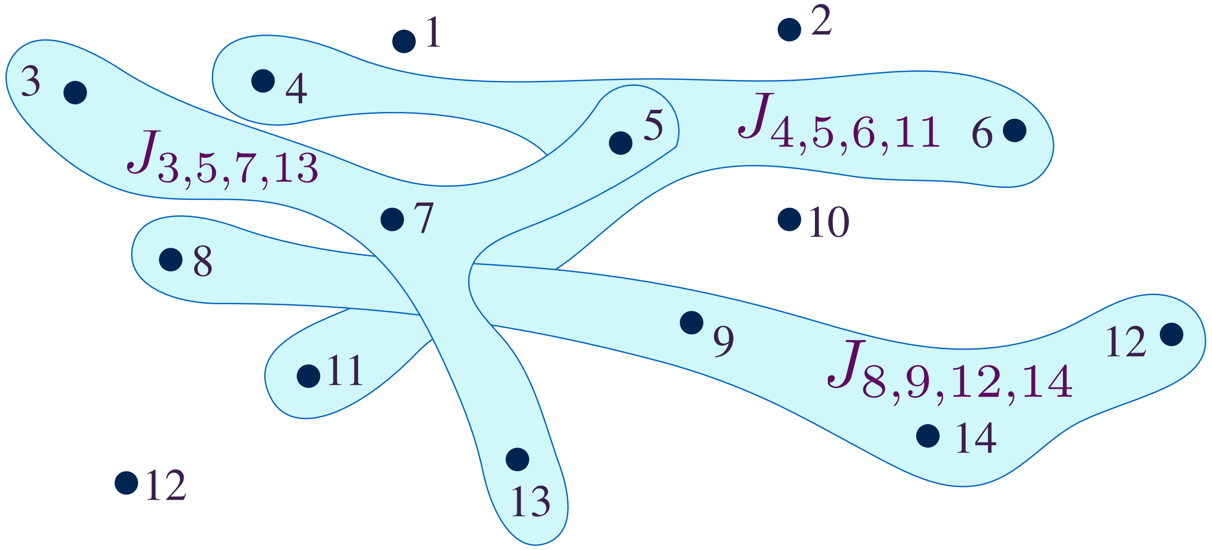

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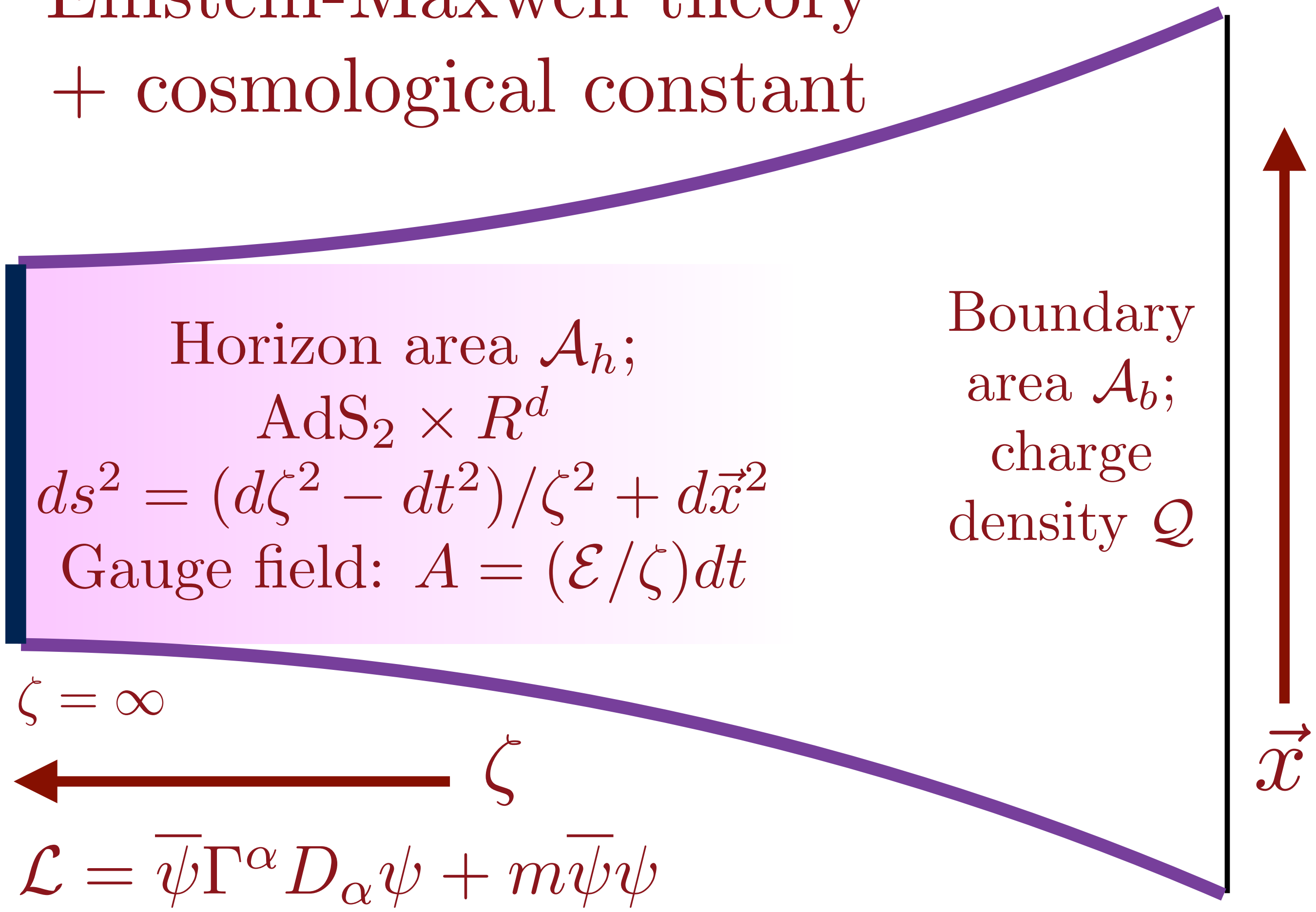

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‘Equation of state’ relating $\mathcal{E}$ and $\mathcal{Q}$ depends upon the geometry of spacetime far from the AdS_2

Eliminate r_0 between

$$\mathcal{Q} = \frac{r_0^{d-1}\sqrt{2d\left[(d-1)R^2 + (d+1)r_0^2\right]}}{\kappa^2 g_F}$$

$$\mathcal{E} = \frac{g_F r_0 \sqrt{2d\left[(d-1)R^2 + (d+1)r_0^2\right]}}{2\left[(d-1)^2 R^2 + d(d+1)r_0^2\right]}$$

S. Sachdev, arXiv:1506.05111

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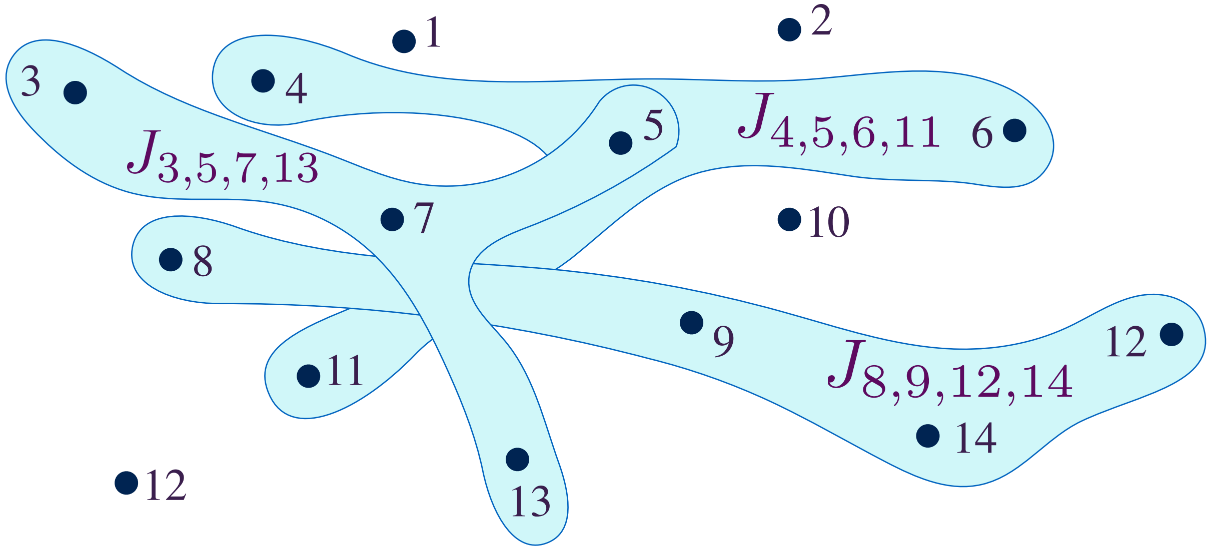

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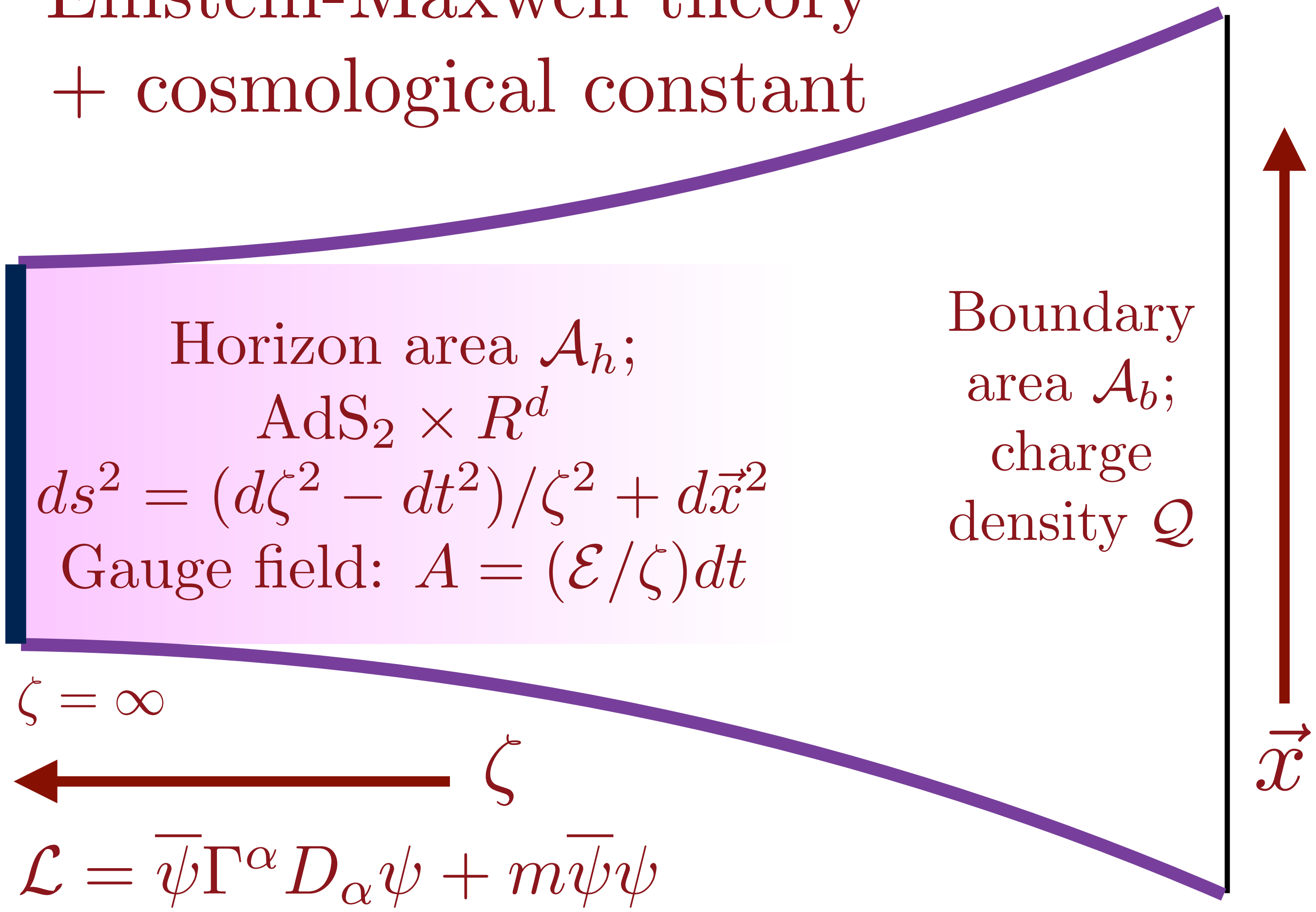

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'Equation of state' relating $\mathcal{E}$
and $\mathcal{Q}$ depends upon the geometry
of spacetime far from the AdS$_2$

Black hole thermodynamics
(classical GR) yields

$$\frac{1}{\mathcal{A}_b} \frac{\partial \mathcal{A}_h}{\partial \mathcal{Q}} = 8\pi G_N \mathcal{E}$$

A. Sen, arXiv:hep-th/0506177; S. Sachdev, arXiv:1506.0511

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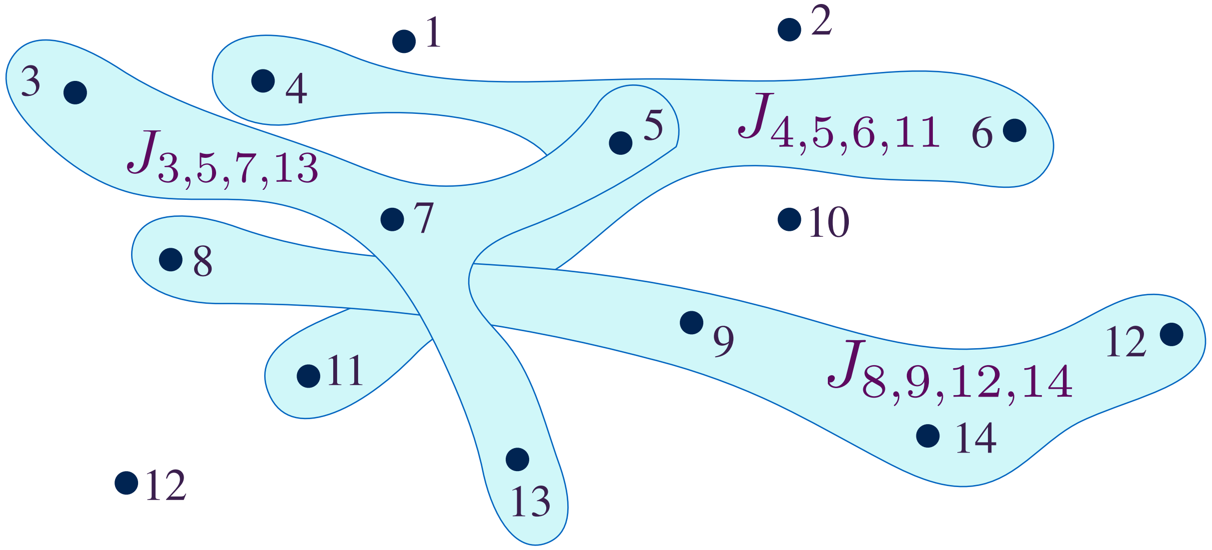

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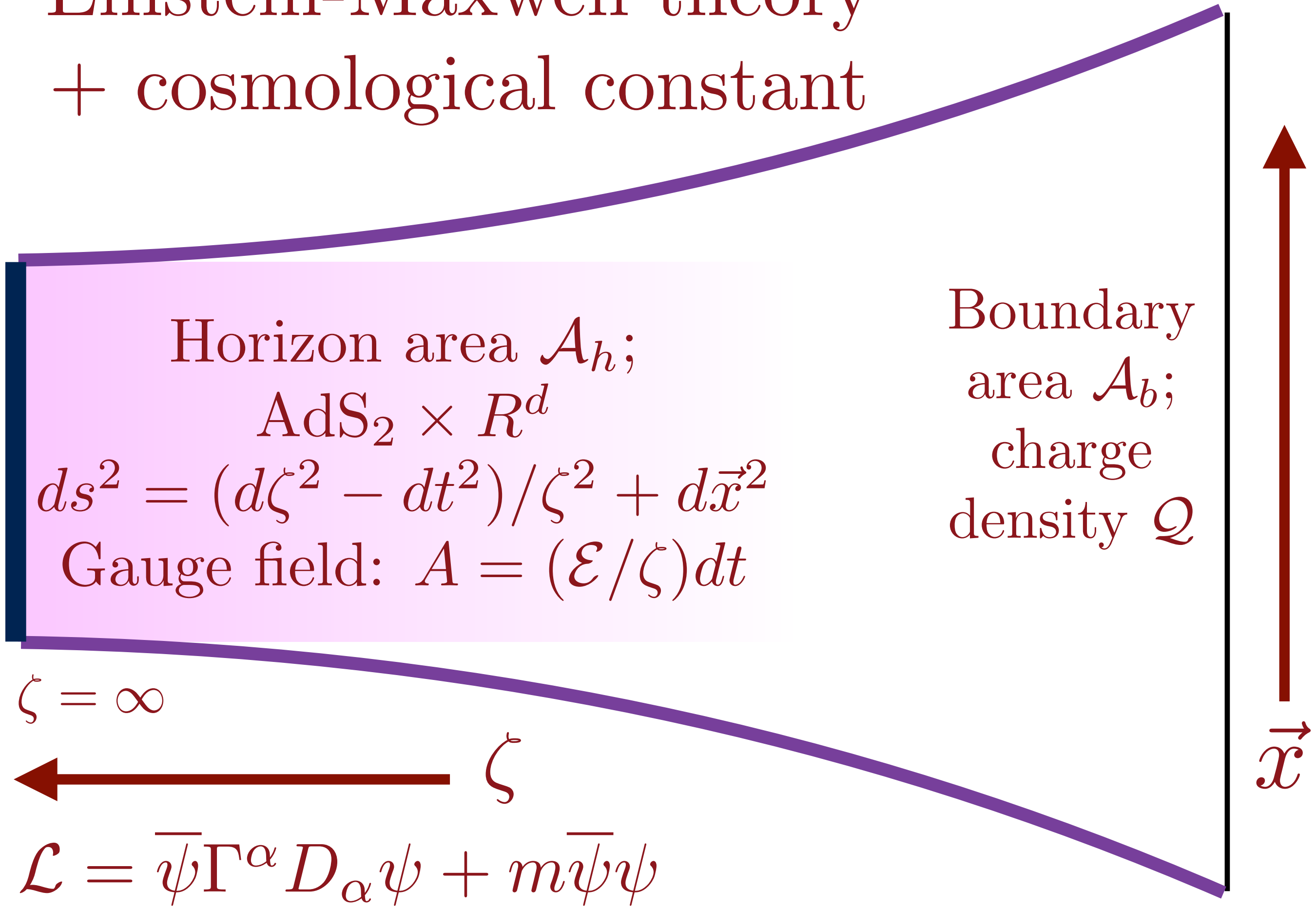
Einstein-Maxwell theory + cosmological constant



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Boundary area $\mathcal{A}_b$; charge density $\mathcal{Q}$

$\zeta = \infty$

ζ

$\vec{x}$

$$\mathcal{L} = \overline{\psi}\Gamma^\alpha D_\alpha \psi + m\overline{\psi}\psi$$

Local fermion density of states

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Microscopic zero temperatu

entropy density, $\mathcal{S}$, obeys

$$\frac{\partial \mathcal{S}}{\partial \mathcal{Q}} = 2\pi\mathcal{E}$$

Combination:

$$\mathcal{S} = \frac{\mathcal{A}_h}{4 G_N \mathcal{A}_b}$$

lack hole thermodynamics

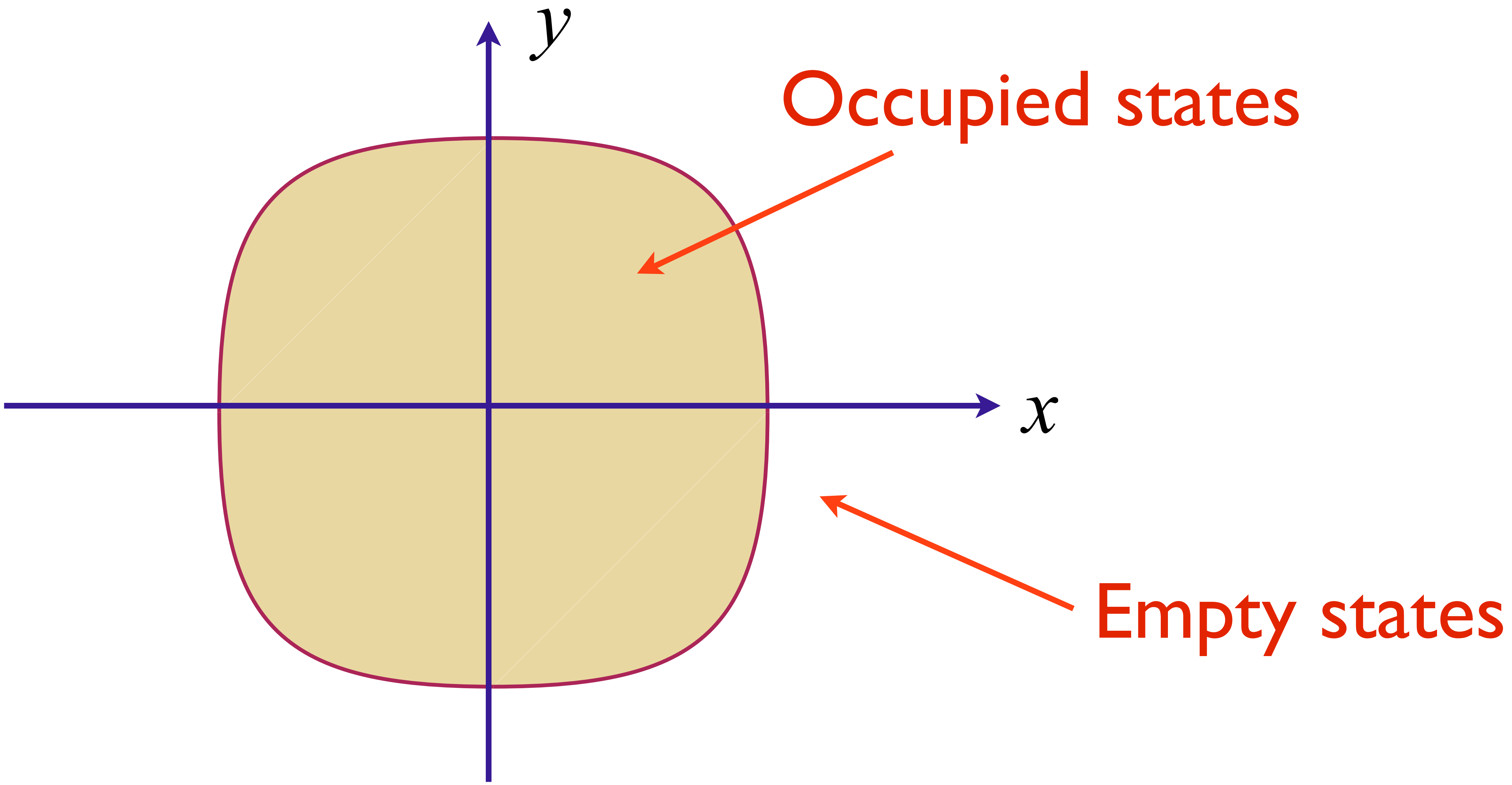
(classical GR) yields

$$\frac{1}{\mathcal{A}_b} \frac{\partial \mathcal{A}_h}{\partial \mathcal{Q}} = 8\pi G_N \mathcal{E}$$

S. Sachdev, arXiv:1506.05111

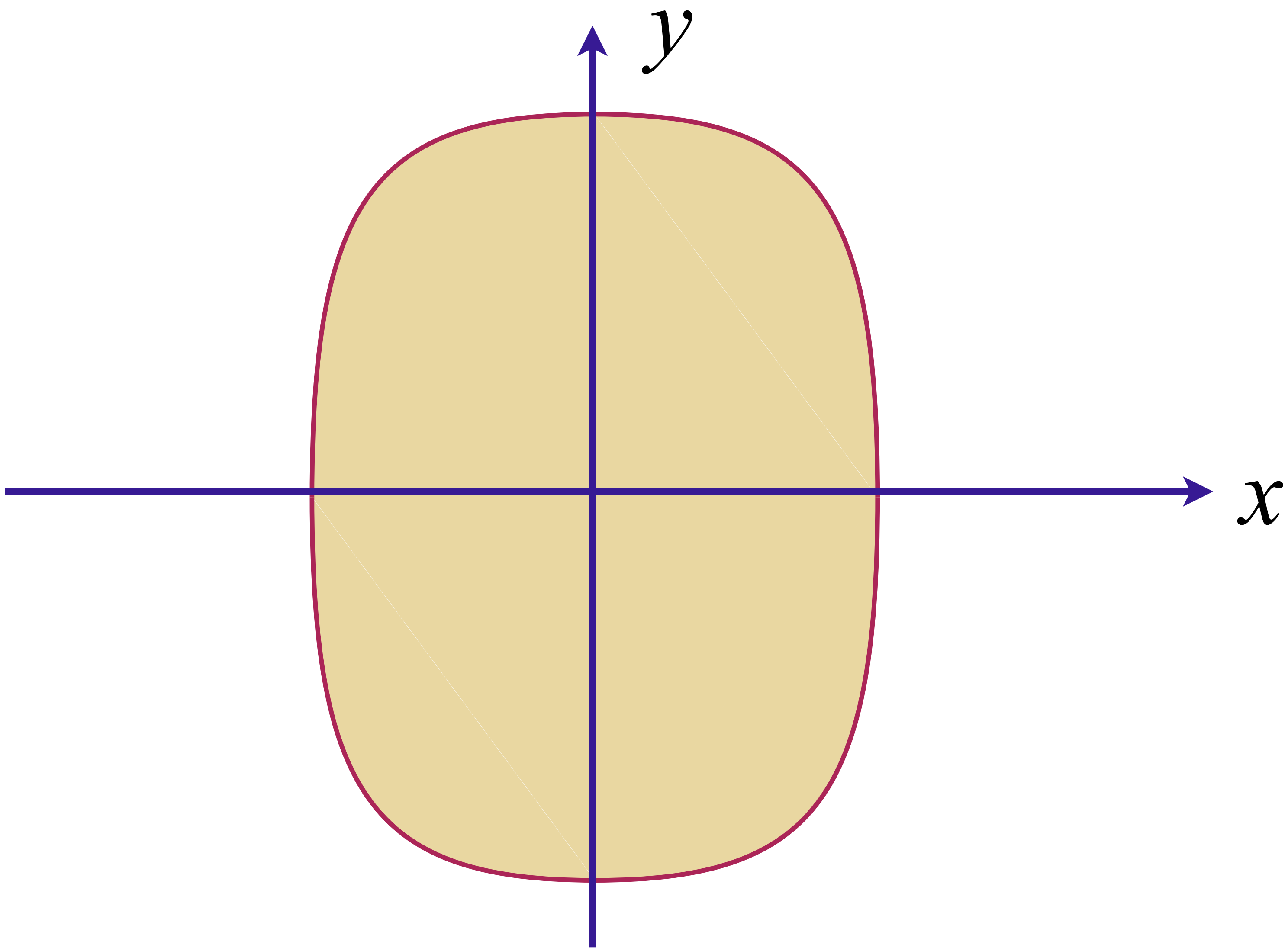
Quantum criticality of Ising-nematic ordering in a metal

Quantum criticality of Ising-nematic ordering in a metal



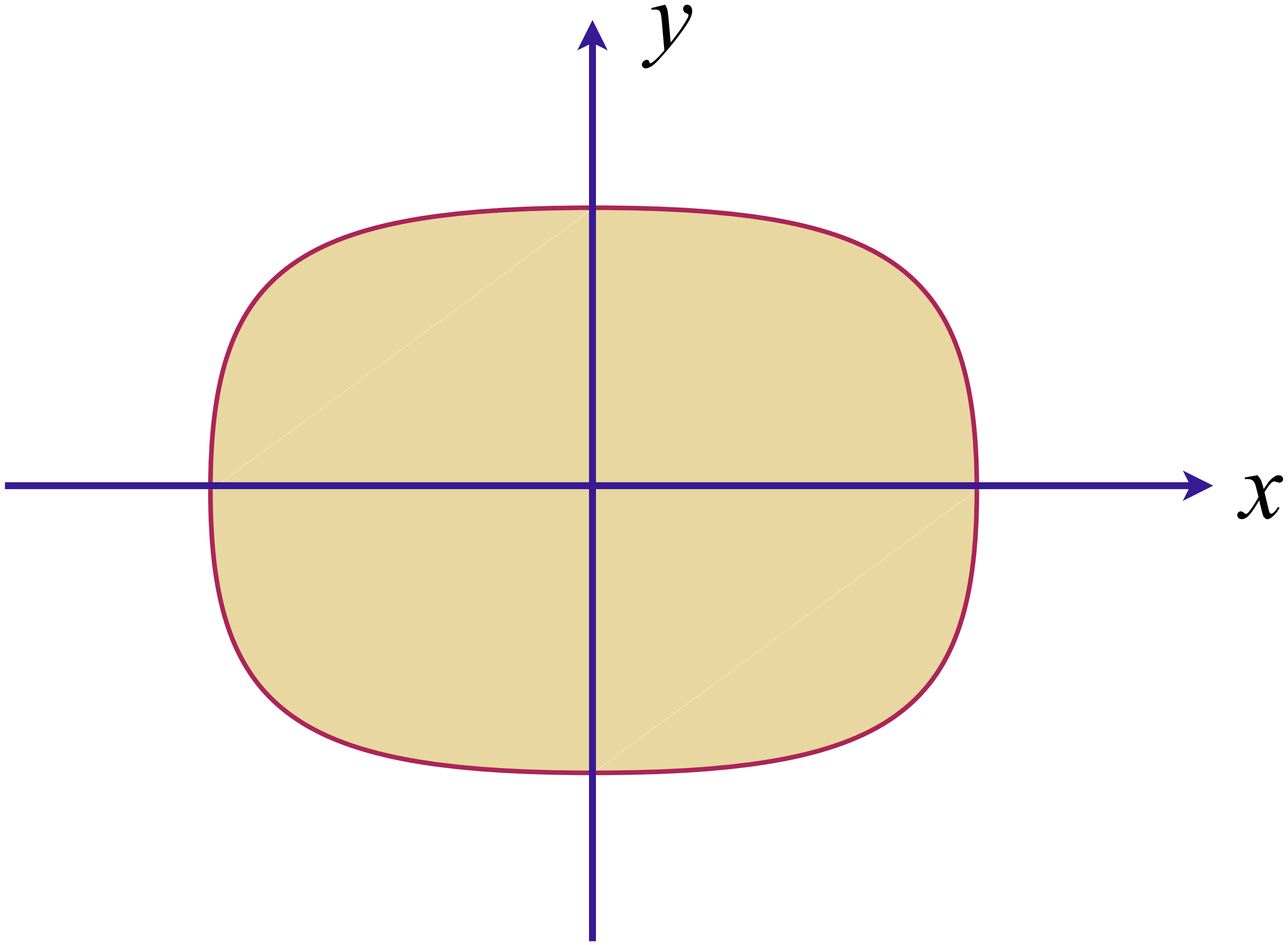
A metal with a Fermi surface
with full square lattice symmetry

Quantum criticality of Ising-nematic ordering in a metal



Spontaneous elongation along y direction:

Quantum criticality of Ising-nematic ordering in a metal



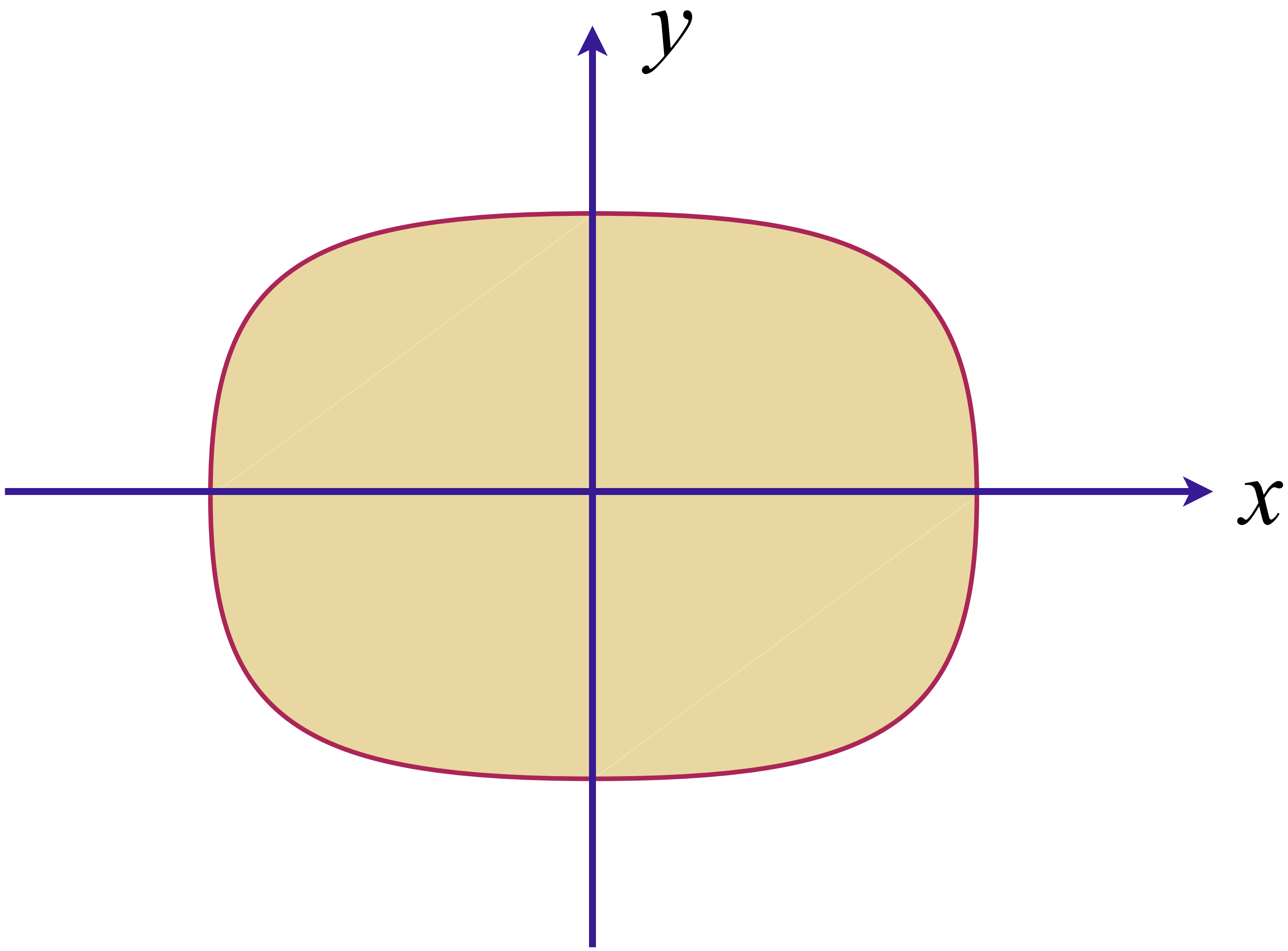
Spontaneous elongation along x direction:

Ising-nematic order parameter

$$\phi \sim \int d^2k \, (\cos k_x - \cos k_y) \, c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma}$$

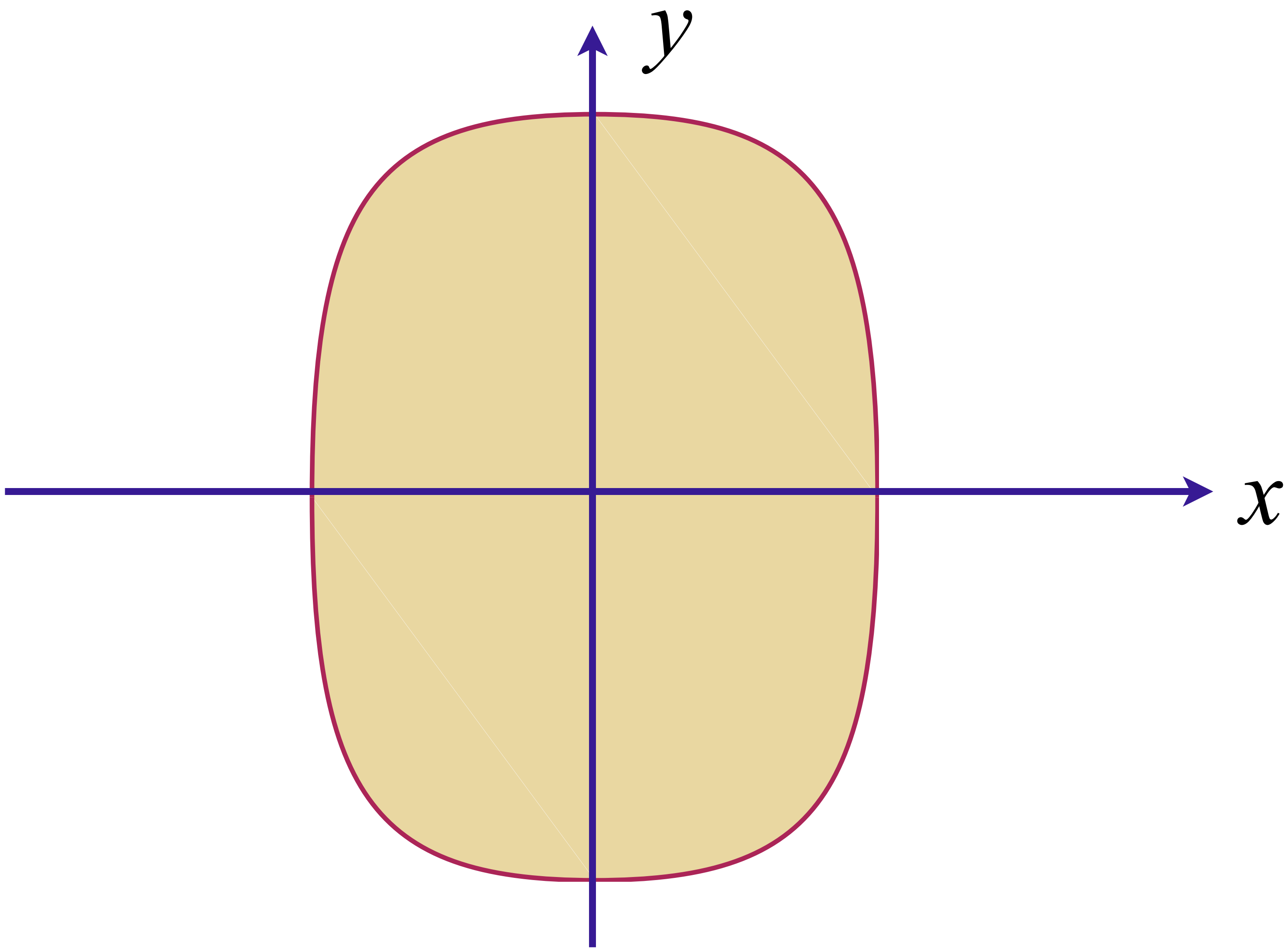
Measures spontaneous breaking of square lattice point-group symmetry of underlying Hamiltonian

Quantum criticality of Ising-nematic ordering in a metal



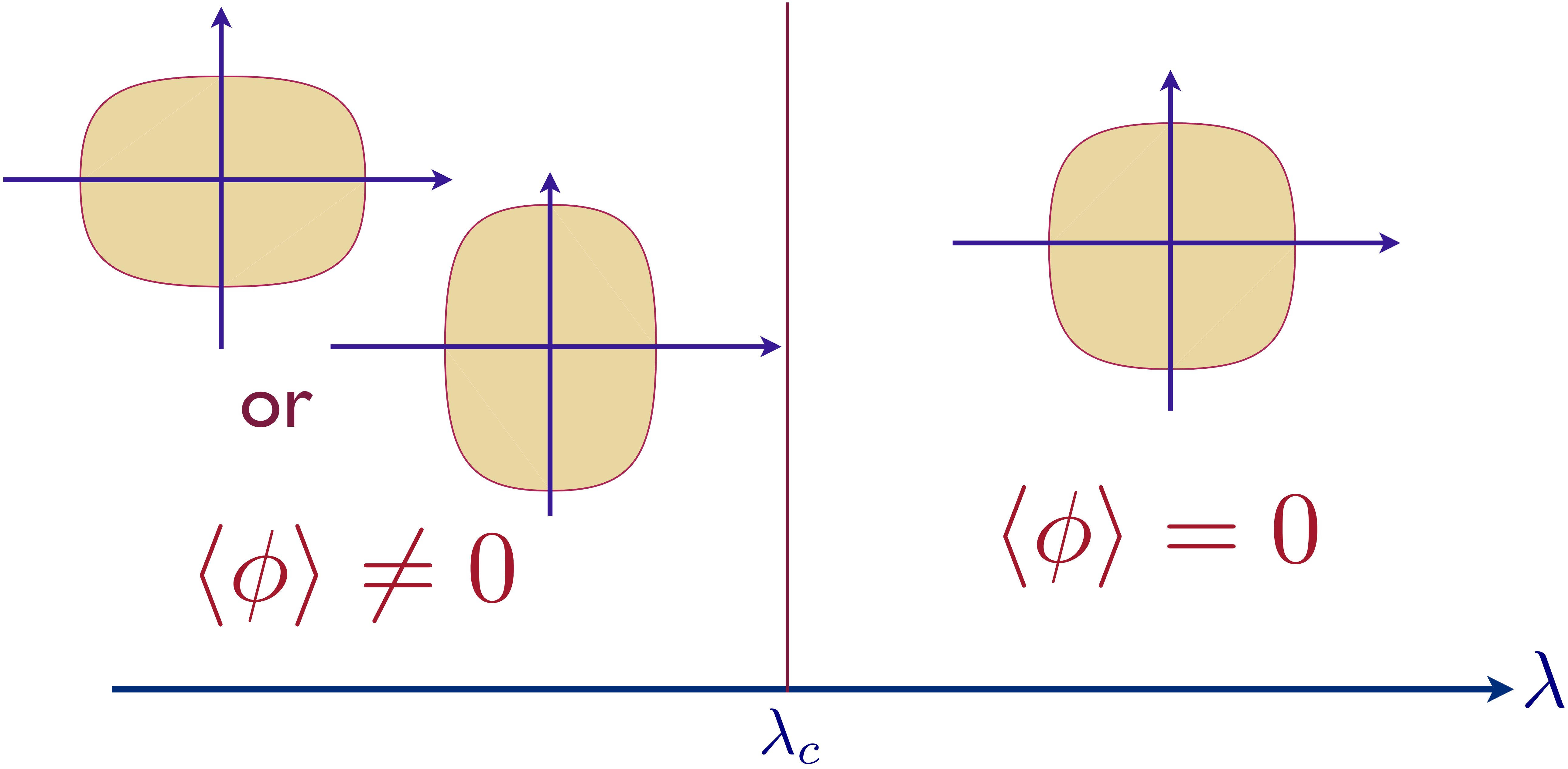
Spontaneous elongation along x direction:
Ising order parameter $\phi > 0$.

Quantum criticality of Ising-nematic ordering in a metal



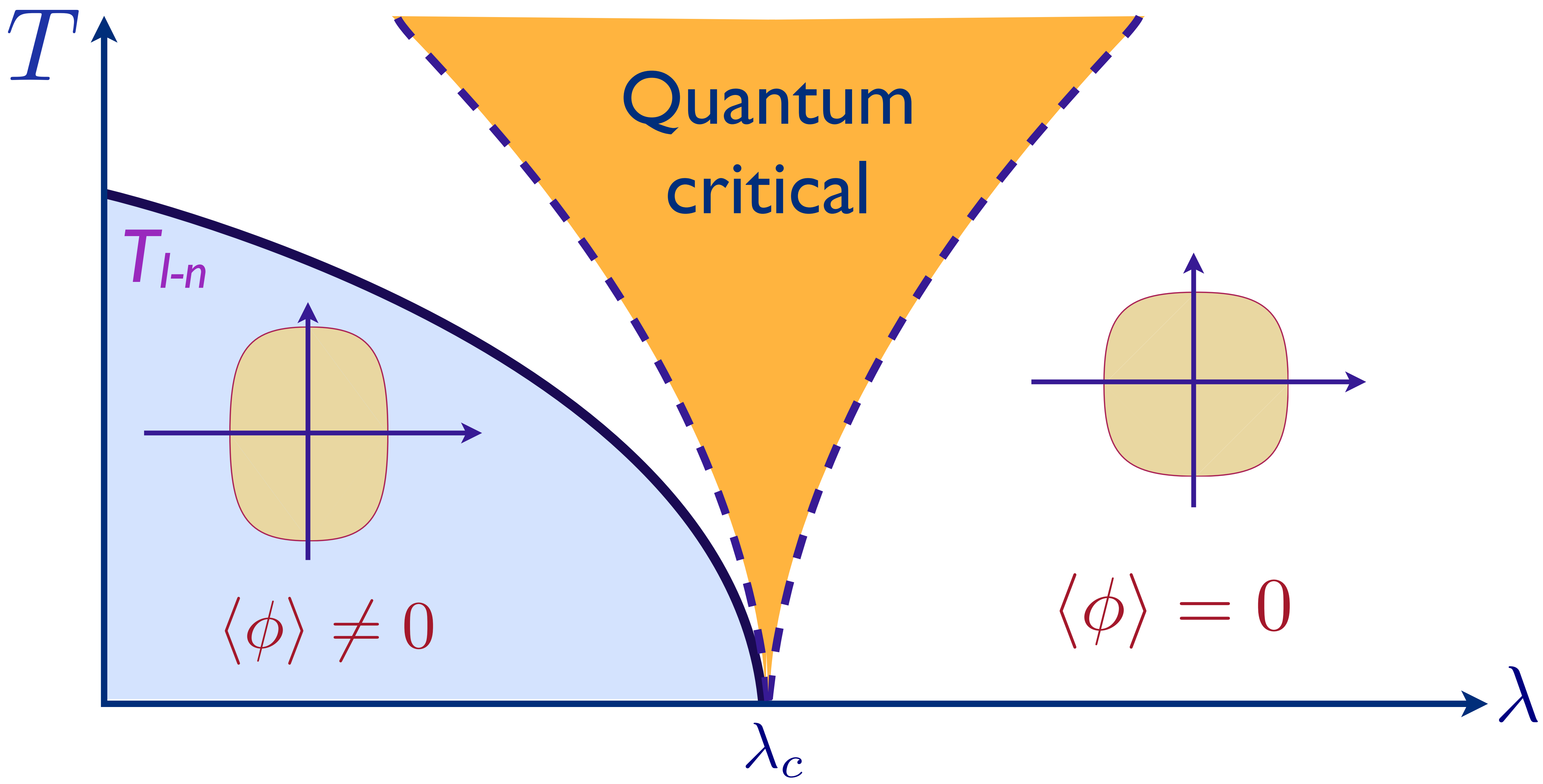
Spontaneous elongation along y direction:
Ising order parameter $\phi < 0$.

Quantum criticality of Ising-nematic ordering in a metal



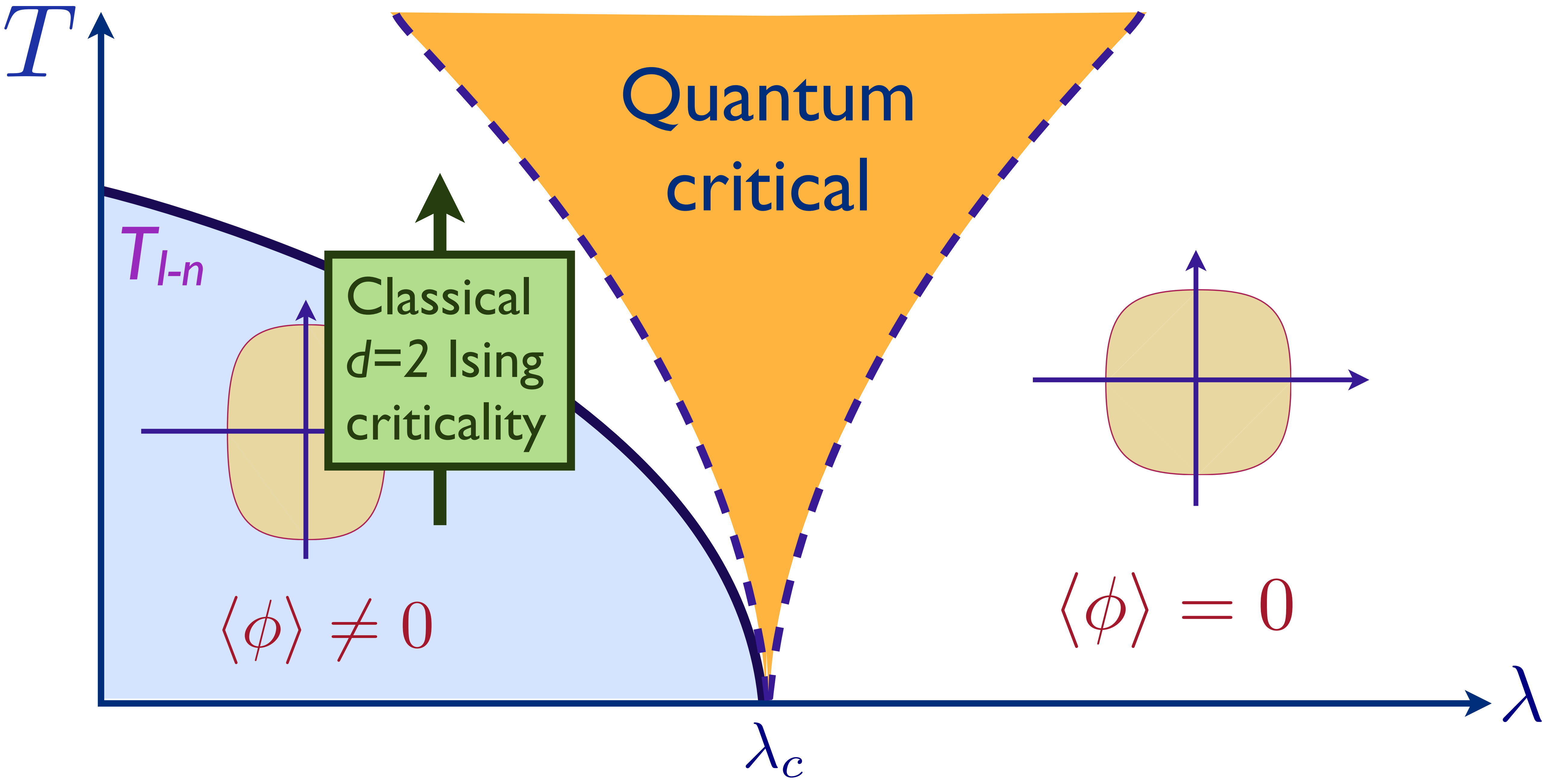
Pomeranchuk instability as a function of coupling λ

Quantum criticality of Ising-nematic ordering in a metal



Phase diagram as a function of T and λ

Quantum criticality of Ising-nematic ordering in a metal



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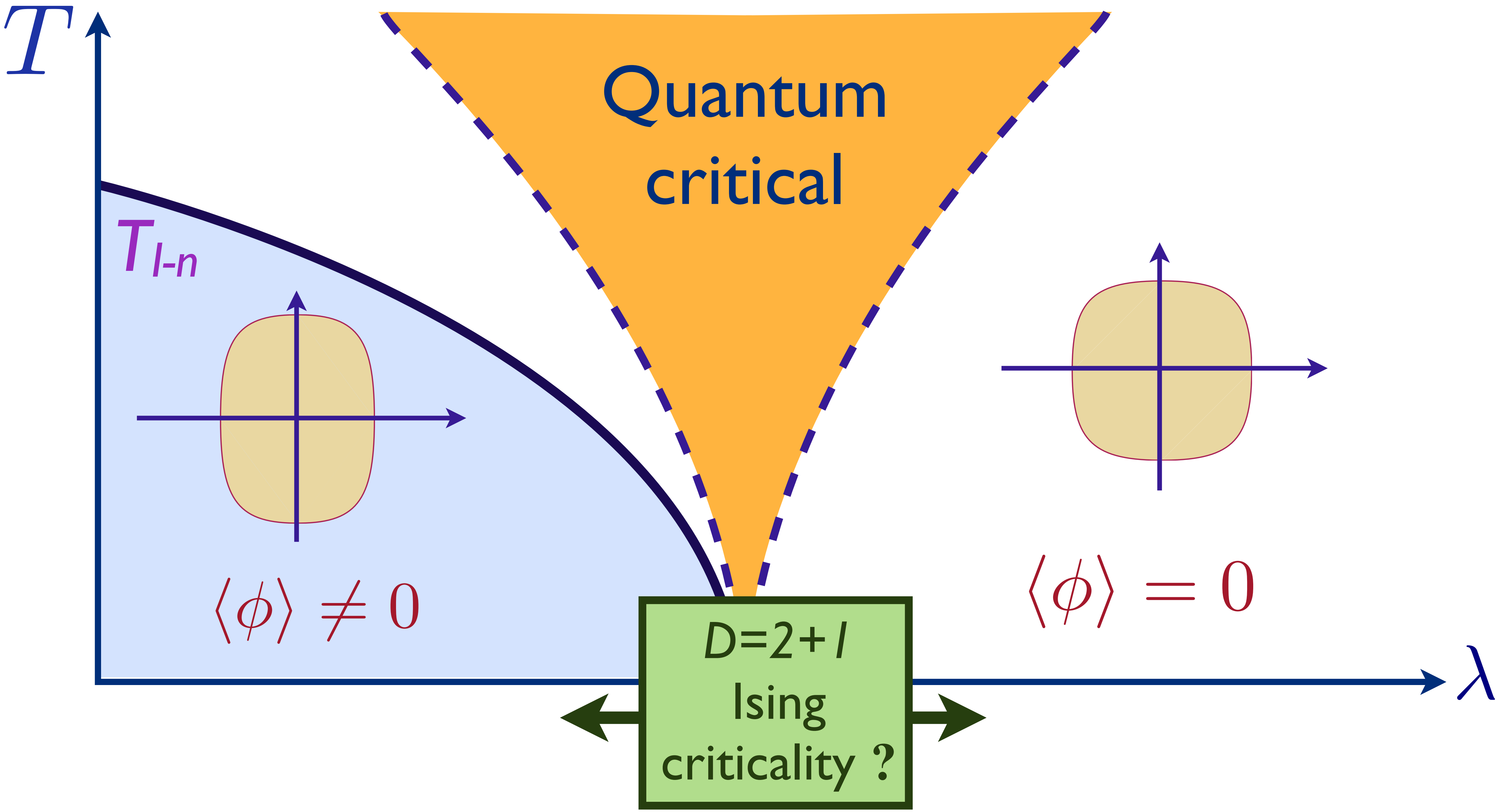

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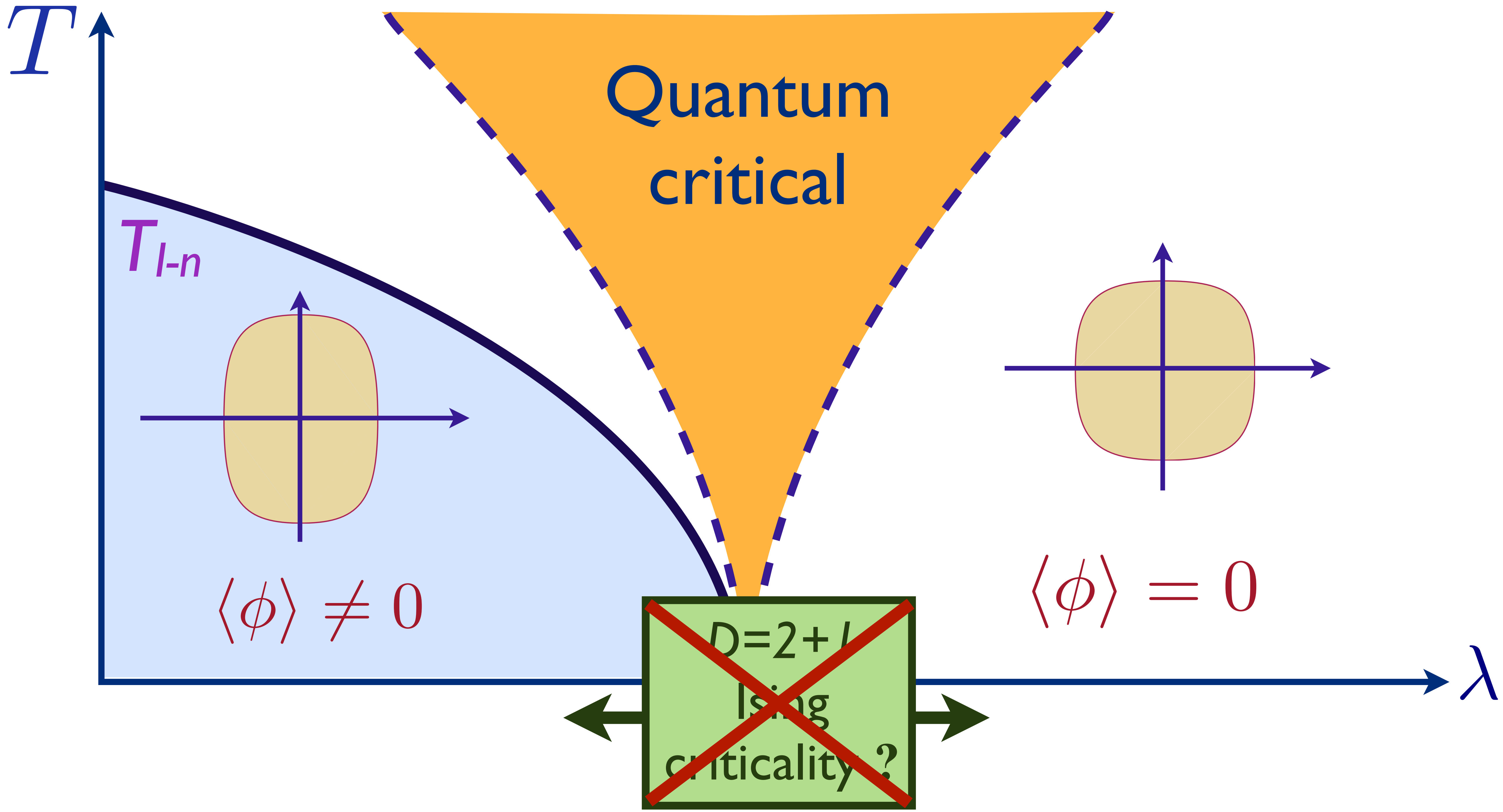

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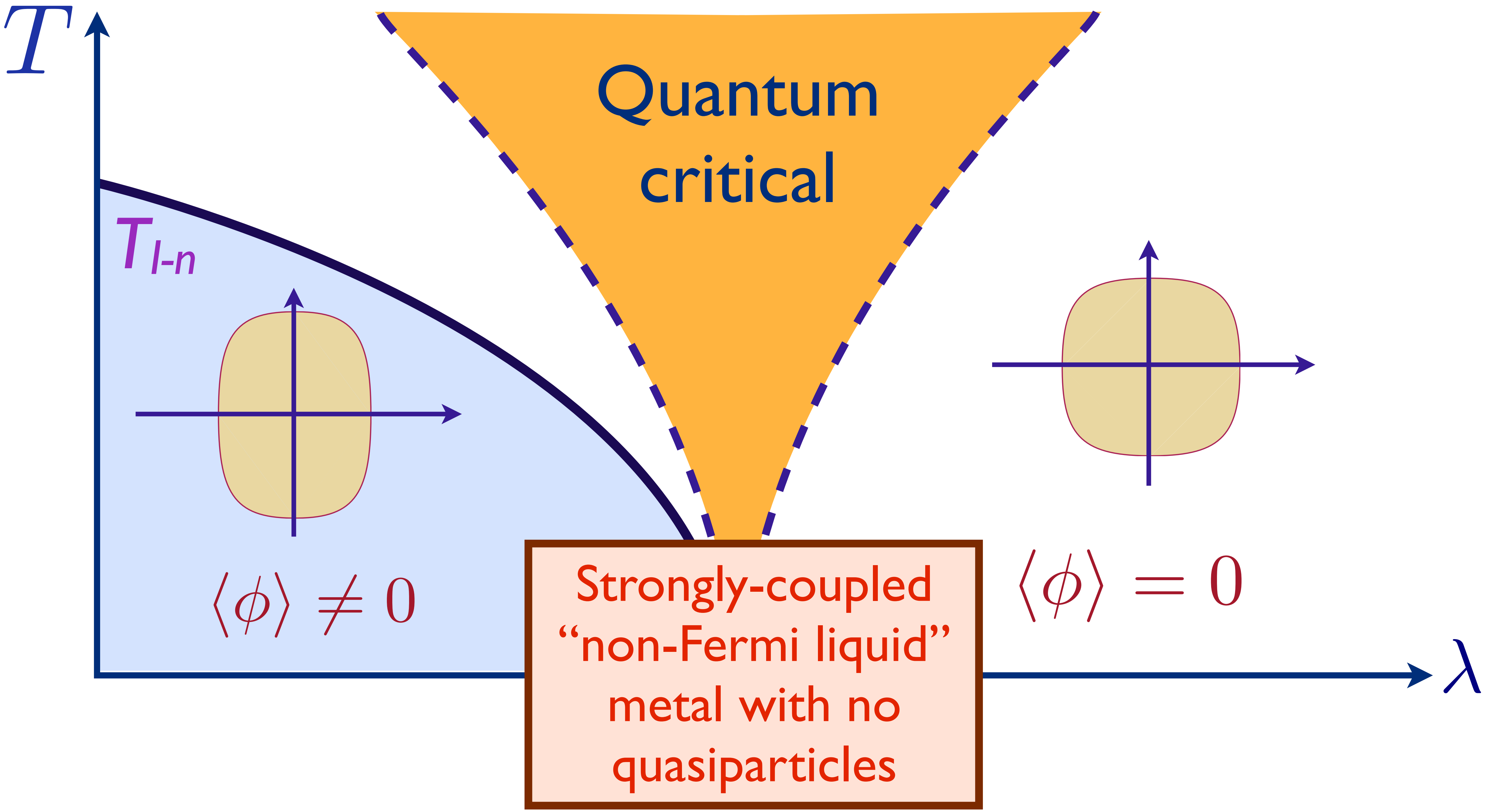
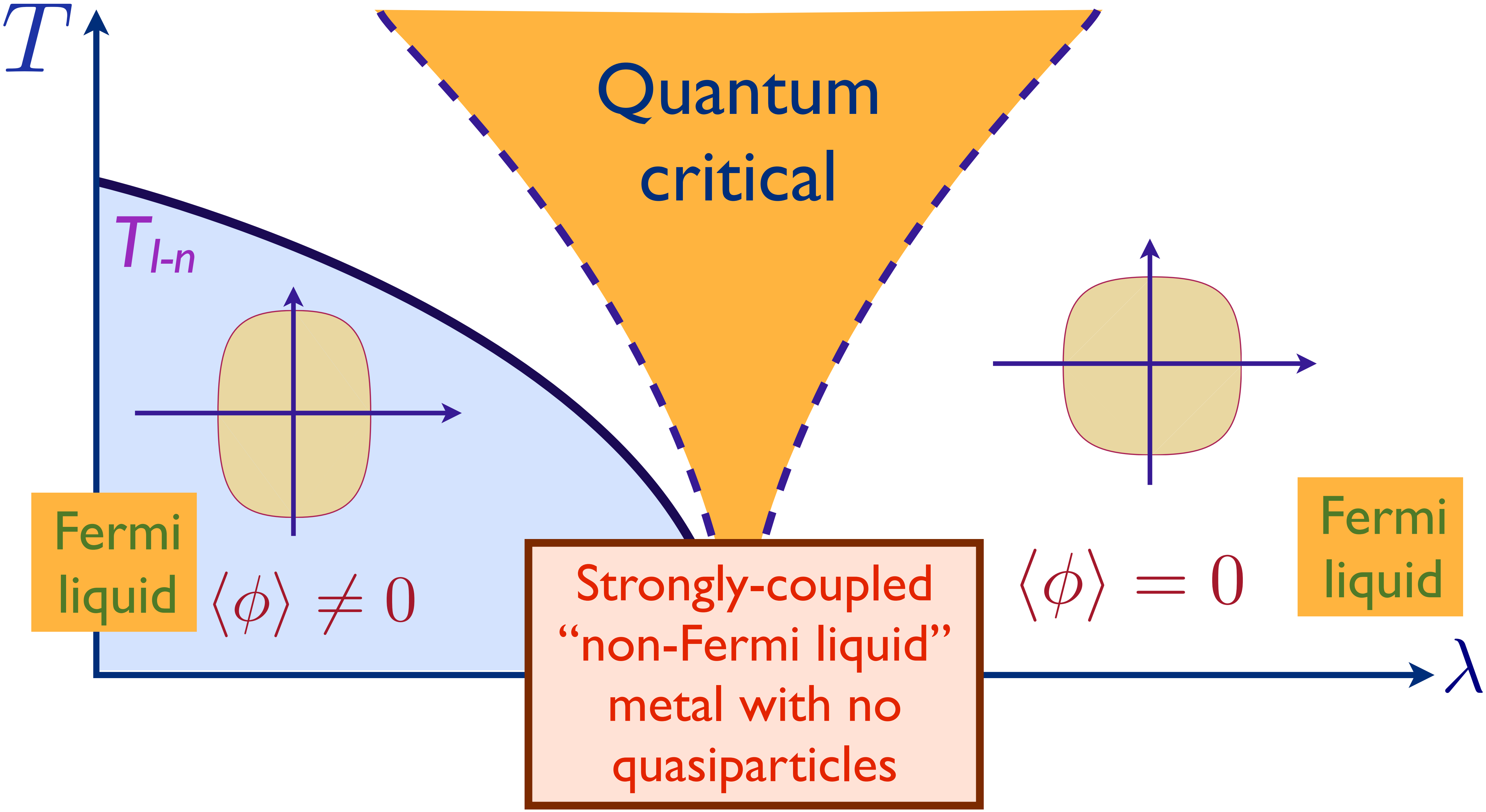


Strongly-coupled "non-Fermi liquid" metal with no quasiparticles

Phase diagram as a function of T and λ

Quantum criticality of Ising-nematic ordering in a metal



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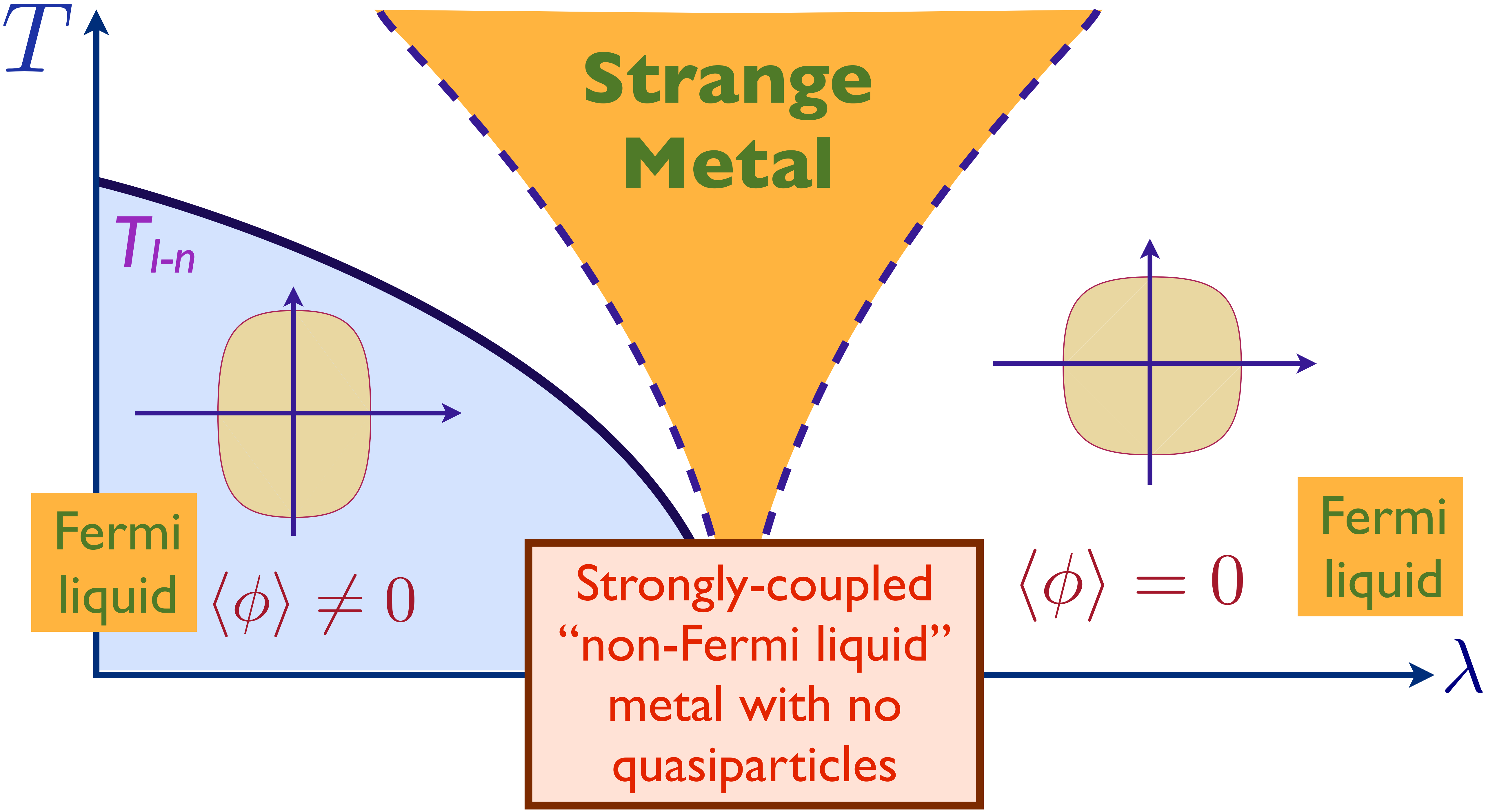


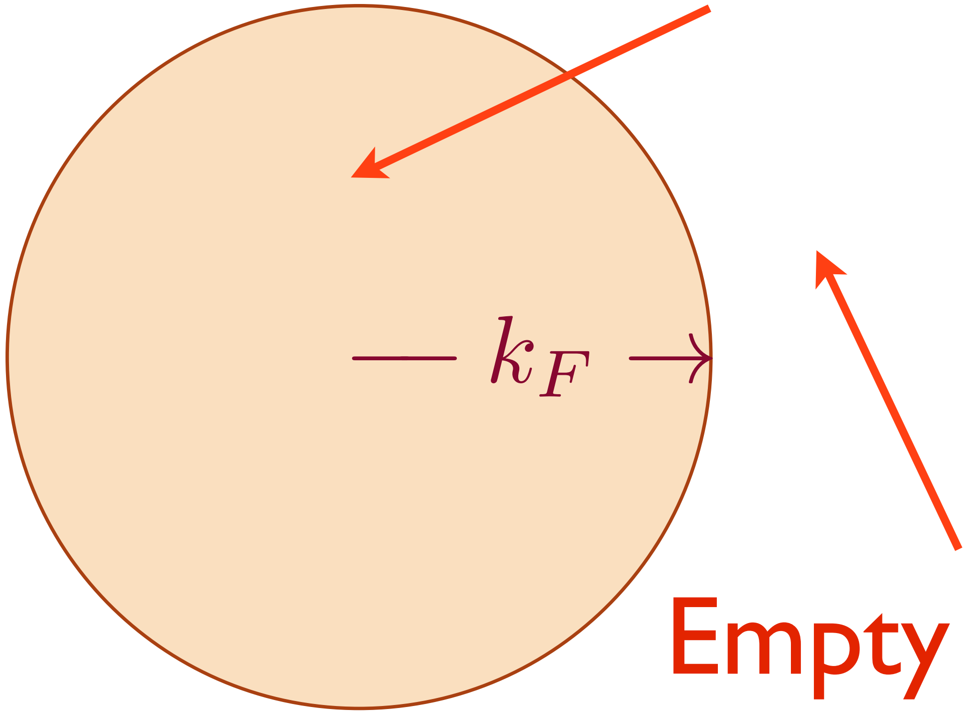
Strongly-coupled “non-Fermi liquid” metal with no quasiparticles

Phase diagram as a function of T and λ

The Fermi liquid

$$\mathcal{L} = f_\alpha^\dagger \left(\partial_\tau - \frac{\nabla^2}{2m} - \mu \right) f_\alpha$$

$$+ u \, f_\alpha^\dagger f_\beta^\dagger f_\beta f_\alpha$$



Occupied states

Empty states

The Fermi liquid: RG

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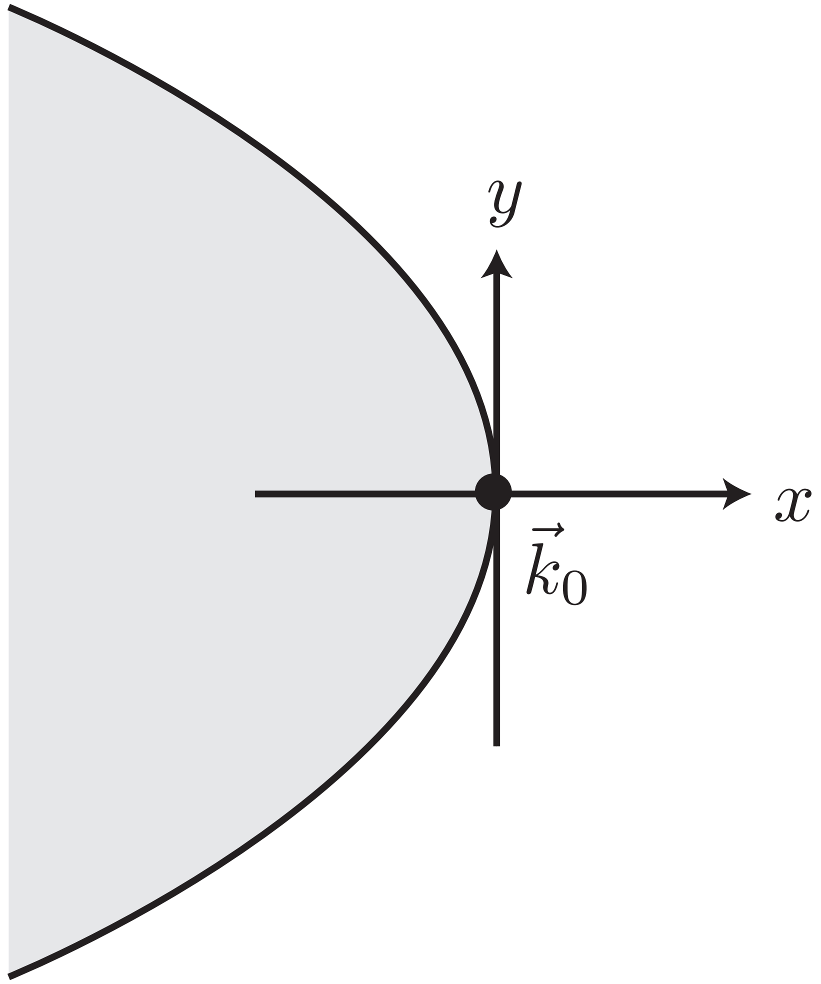

- Expand fermion kinetic energy at wavevectors about $\vec{k}_0$, by writing $f_\alpha(\vec{k}_0 + \vec{q}) = \psi_\alpha(\vec{q})$

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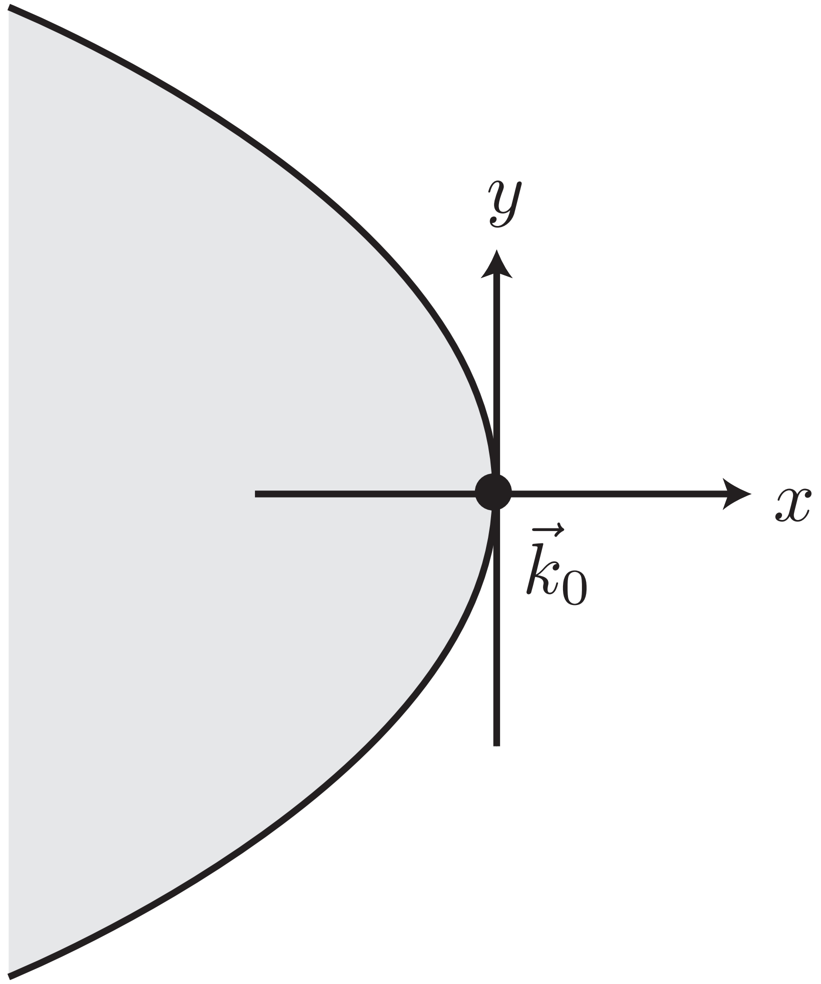

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$$\mathcal{L}[\psi_\alpha] = \psi_\alpha^\dagger \left(\partial_\tau - i\partial_x - \partial_y^2 \right) \psi_\alpha + u \, \psi_\alpha^\dagger \psi_\beta^\dagger \psi_\beta \psi_\alpha$$

The Fermi liquid: RG

$$\mathcal{S}[\psi_\alpha] = \int d^{d-1}y\, dx\, d\tau \left[\psi_\alpha^\dagger \left(\partial_\tau - i\partial_x - \partial_y^2\right)\psi_\alpha + u\, \psi_\alpha^\dagger \psi_\beta^\dagger \psi_\beta \psi_\alpha\right]$$

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The kinetic energy is invariant under the rescaling $x \to x/s$, $y \to y/s^{1/2}$, and $\tau \to \tau/s^z$, provided $z = 1$ and

$$\psi \quad \to \quad \psi\, s^{(d+1)/4}.$$

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$$\psi \quad \to \quad \psi\, s^{(d+1)/4}.$$

Then we find $u \to u s^{(1-d)/2}$, and so we have the RG flow

$$\frac{du}{d\ell} \quad = \quad \frac{(1-d)}{2} u$$

Interactions are *irrelevant* in $d = 2$!

The Fermi liquid: RG

$$\mathcal{S}[\psi_\alpha] = \int d^{d-1}y\, dx\, d\tau \left[\psi_\alpha^\dagger \left(\partial_\tau - i\partial_x - \partial_y^2\right)\psi_\alpha + u\,\psi_\alpha^\dagger \psi_\beta^\dagger \psi_\beta \psi_\alpha\right]$$

The fermion Green's function to order u^2 has the form (upto logs)

$$G(\vec{q},\omega) = \frac{\mathcal{A}}{\omega - q_x - q_y^2 + ic\,\omega^2}$$

So the quasiparticle pole is sharp.

The Fermi liquid: RG

$$\mathcal{S}[\psi_\alpha] = \int d^{d-1}y \, dx \, d\tau \left[\psi_\alpha^\dagger \left(\partial_\tau - i\partial_x - \partial_y^2 \right) \psi_\alpha + u \, \psi_\alpha^\dagger \psi_\beta^\dagger \psi_\beta \psi_\alpha \right]$$

The fermion Green's function to order u^2 has the form (upto logs)

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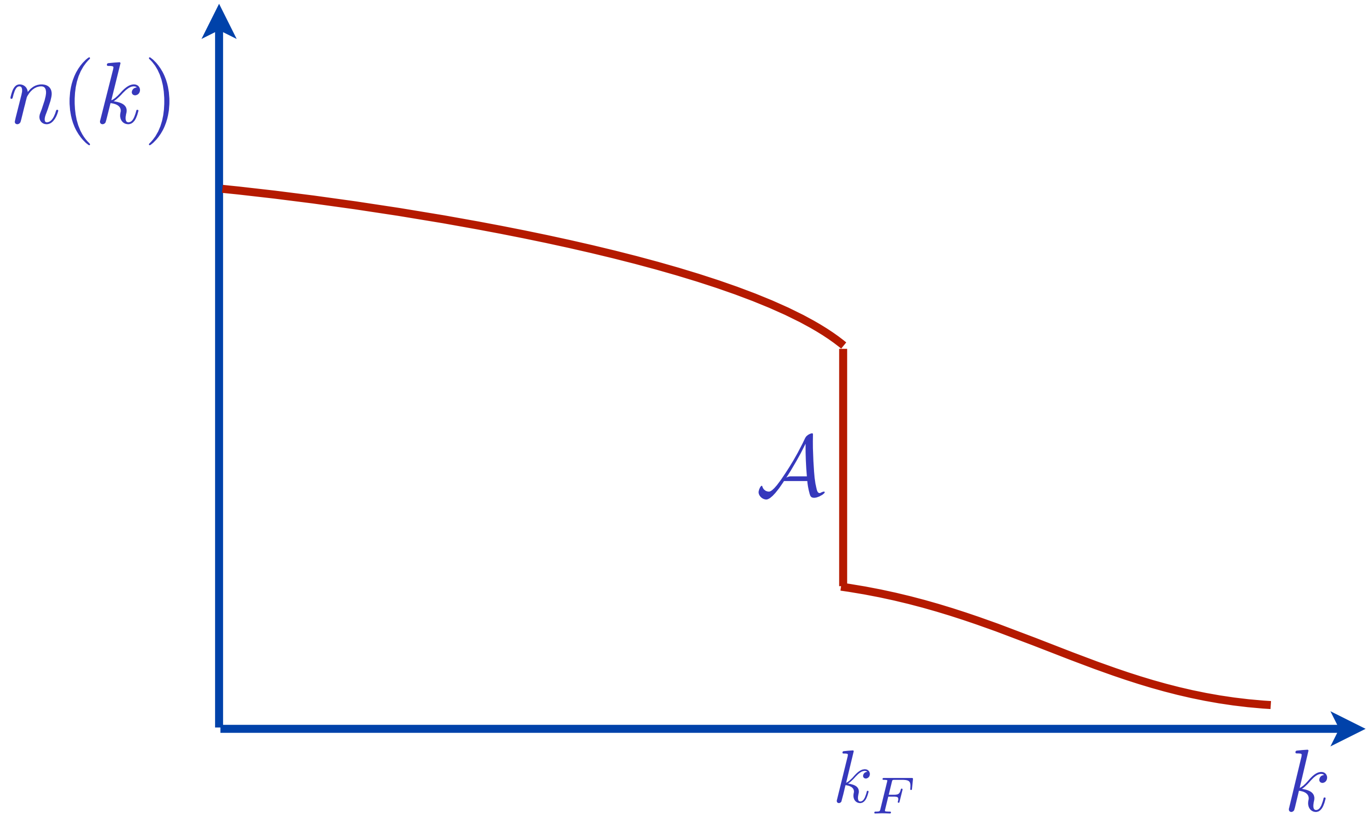
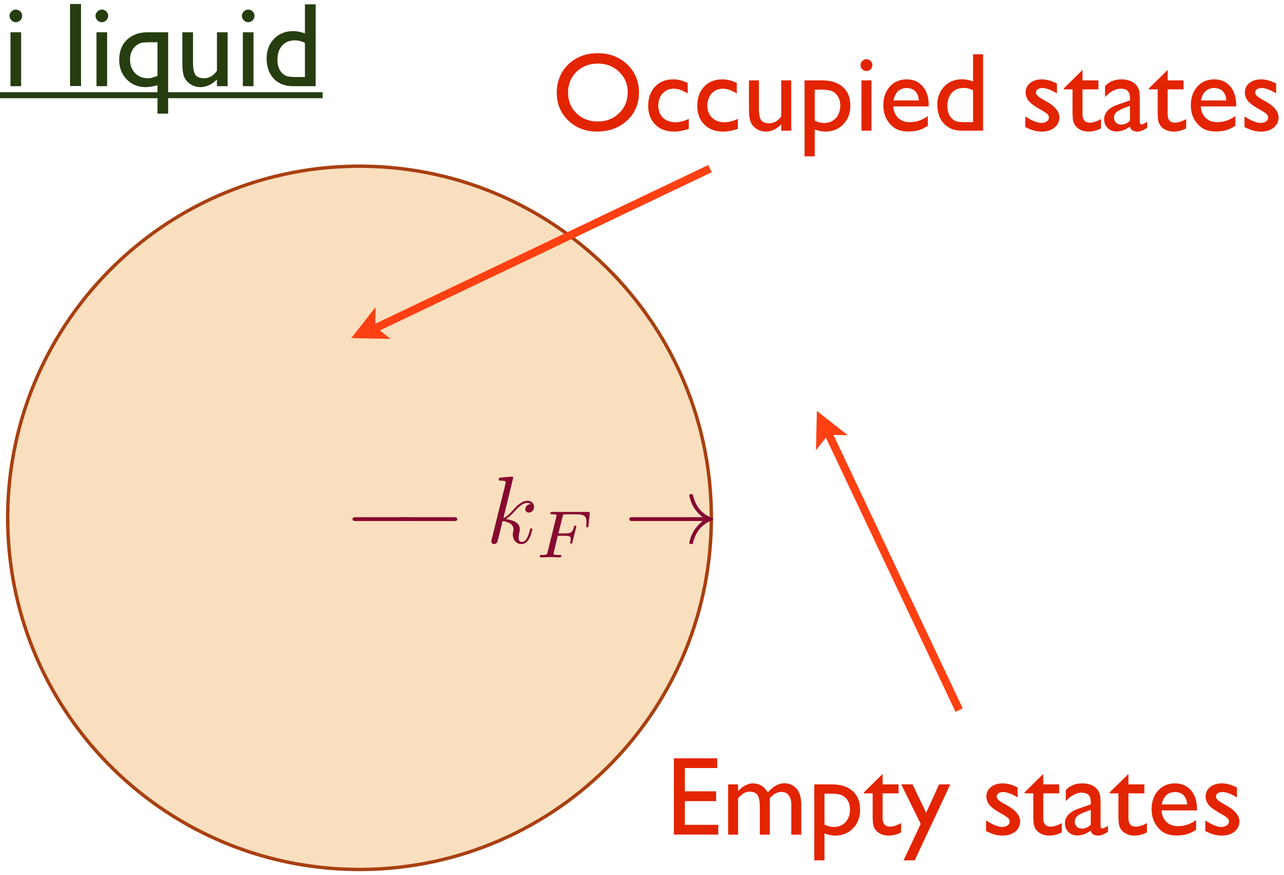
So the quasiparticle pole is sharp. And fermion momentum distribution function $n(\vec{k}) = \left\langle f_\alpha^\dagger(\vec{k}) f_\alpha(\vec{k}) \right\rangle$ had the following form:



The Fermi liquid

$$\mathcal{L} = f^\dagger \left(\partial_\tau - \frac{\nabla^2}{2m} - \mu \right) f$$

$$+ 4 \text{ Fermi terms}$$



- Fermi wavevector obeys the Luttinger relation $k_F^d \sim \mathcal{Q}$, the fermion density

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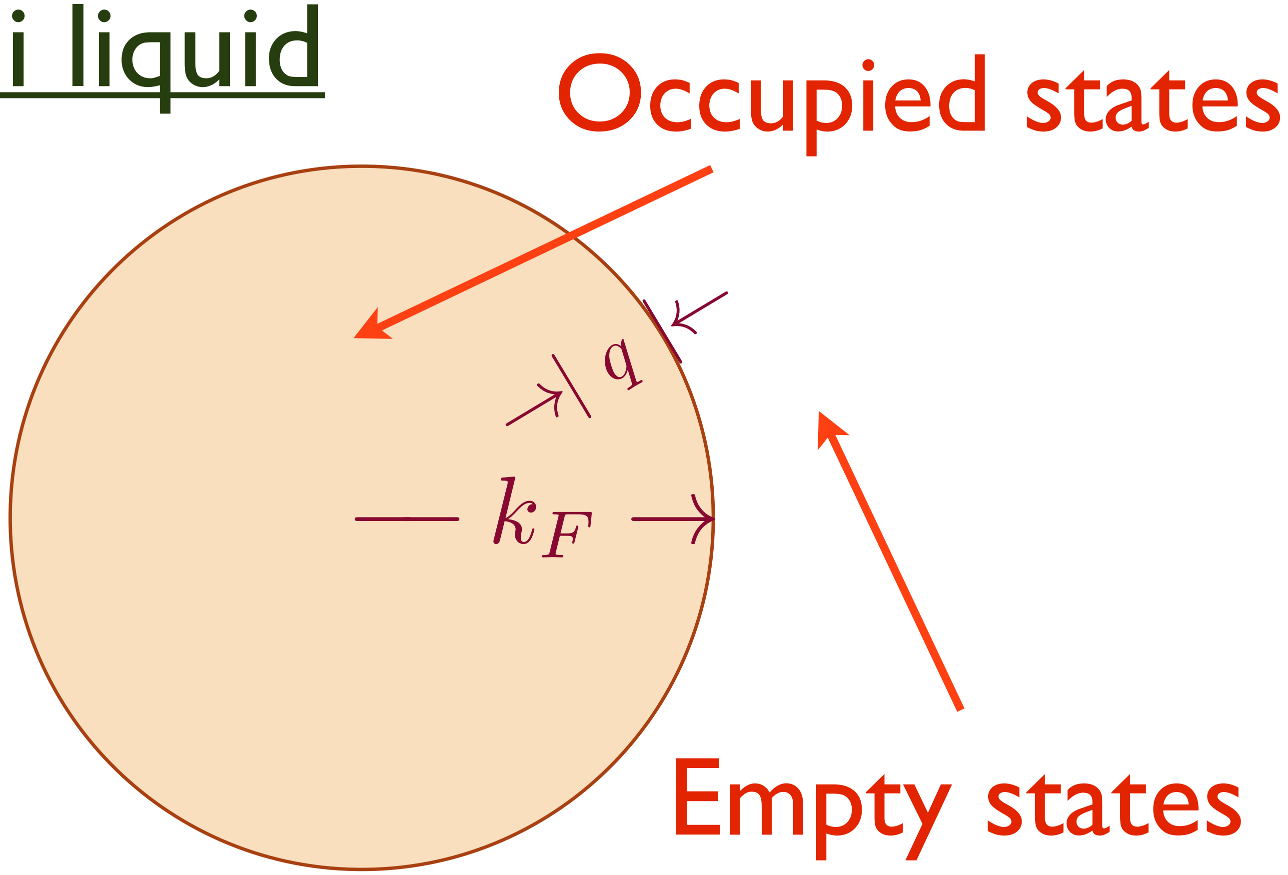

Occupied states

Empty states

- Fermi wavevector obeys the Luttinger relation $k_F^d \sim \mathcal{Q}$, the fermion density
- Sharp particle and hole of excitations near the Fermi surface with energy $\omega \sim |q|^z$, with dynamic exponent $z = 1$.

The Fermi liquid

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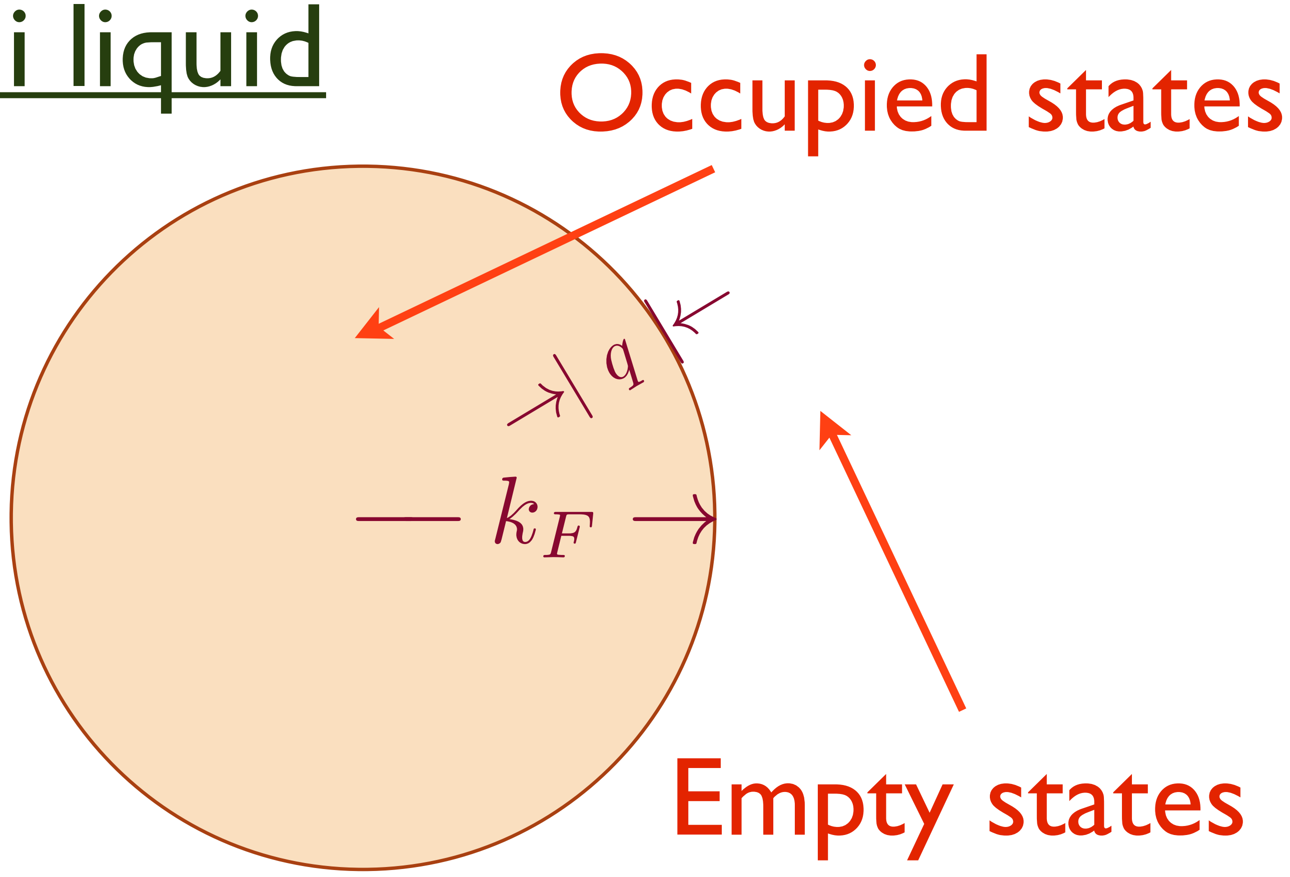

Occupied states

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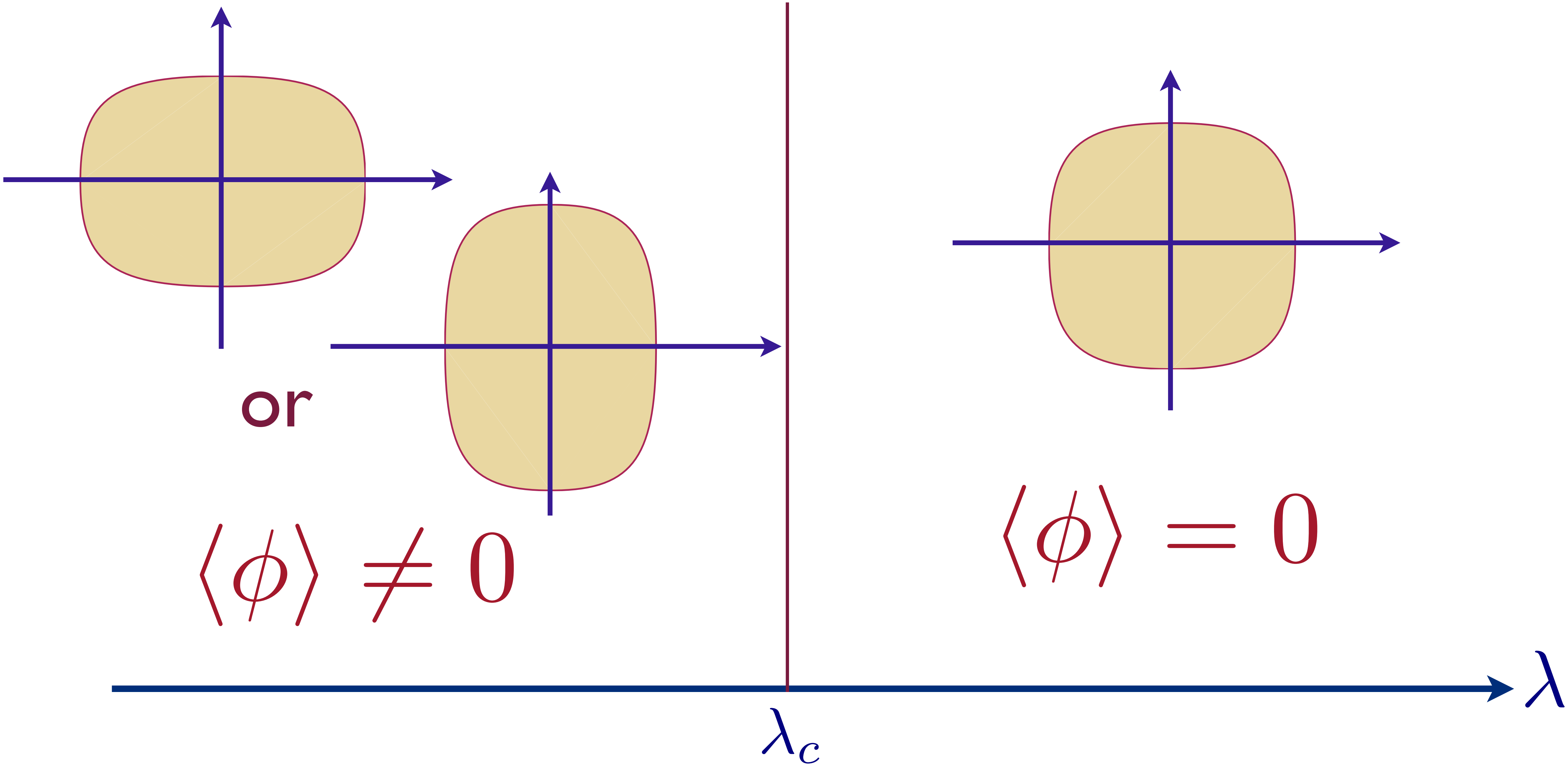
- Fermi wavevector obeys the Luttinger relation $k_F^d \sim \mathcal{Q}$, the fermion density
- Sharp particle and hole of excitations near the Fermi surface with energy $\omega \sim |q|^z$, with dynamic exponent $z = 1$.
- The phase space density of fermions is effectively one-dimensional, so the entropy density $S \sim T$. It is useful to write this is as $S \sim T^{(d-\theta)/z}$, with violation of hyperscaling exponent $\theta = d - 1$.

Quantum criticality of Ising-nematic ordering in a metal



Pomeranchuk instability as a function of coupling λ

Quantum criticality of Ising-nematic ordering in a metal

Effective action for Ising order parameter

$$\mathcal{S}_\phi = \int d^2 r d\tau \left[(\partial_\tau \phi)^2 + c^2 (\nabla \phi)^2 + (\lambda - \lambda_c) \phi^2 + u \phi^4 \right]$$

Quantum criticality of Ising-nematic ordering in a metal

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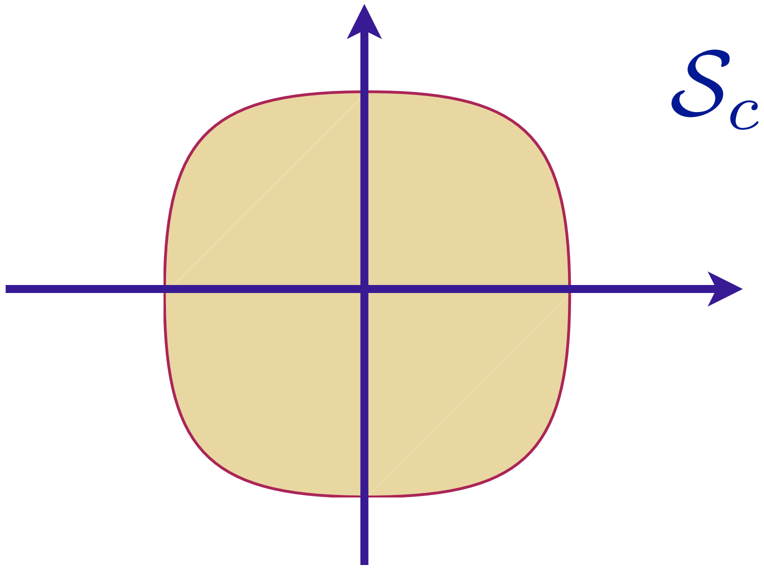
$$\mathcal{S}_\phi = \int d^2r d\tau \left[(\partial_\tau \phi)^2 + c^2 (\nabla \phi)^2 + (\lambda - \lambda_c)\phi^2 + u\phi^4 \right]$$

Effective action for electrons:



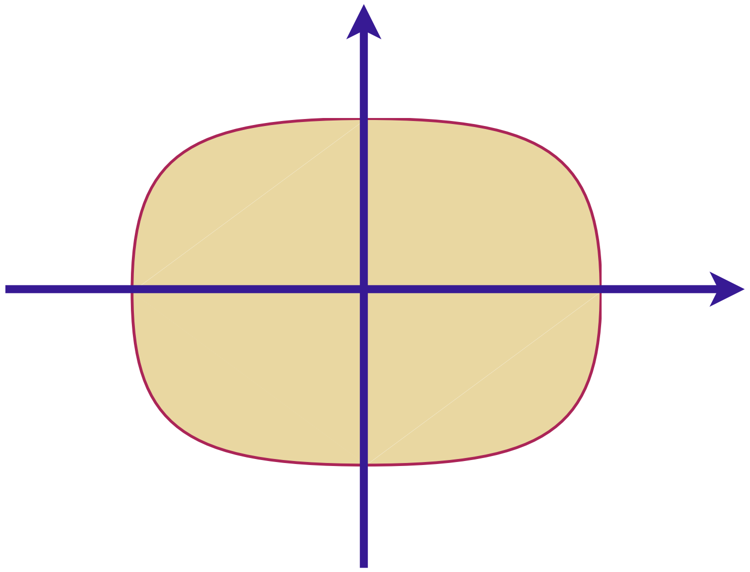
$$\begin{aligned} \mathcal{S}_c &= \int d\tau \sum_{\alpha=1}^{N_f} \left[\sum_i c_{i\alpha}^\dagger \partial_\tau c_{i\alpha} - \sum_{i<j} t_{ij} c_{i\alpha}^\dagger c_{i\alpha} \right] \\ &\equiv \sum_{\alpha=1}^{N_f} \sum_{\mathbf{k}} \int d\tau c_{\mathbf{k}\alpha}^\dagger \left(\partial_\tau + \varepsilon_{\mathbf{k}} \right) c_{\mathbf{k}\alpha} \end{aligned}$$

Quantum criticality of Ising-nematic ordering in a metal

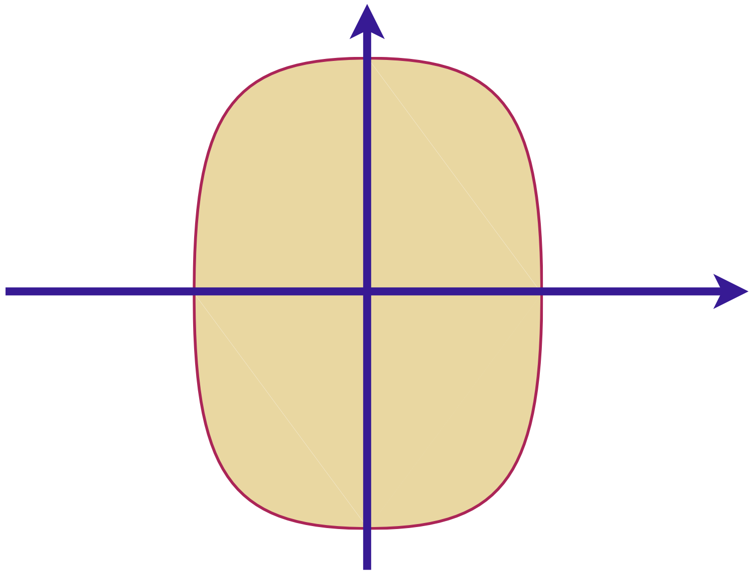
Coupling between Ising order and electrons

$$\mathcal{S}_{\phi c} = -\,g \int d\tau \sum_{\alpha=1}^{N_f} \sum_{\mathbf{k},\mathbf{q}} \phi_{\mathbf{q}} \left(\cos k_x - \cos k_y\right) c^{\dagger}_{\mathbf{k}+\mathbf{q}/2,\alpha} c_{\mathbf{k}-\mathbf{q}/2,\alpha}$$

for spatially dependent ϕ



$$\langle \phi \rangle > 0$$



$$\langle \phi \rangle < 0$$

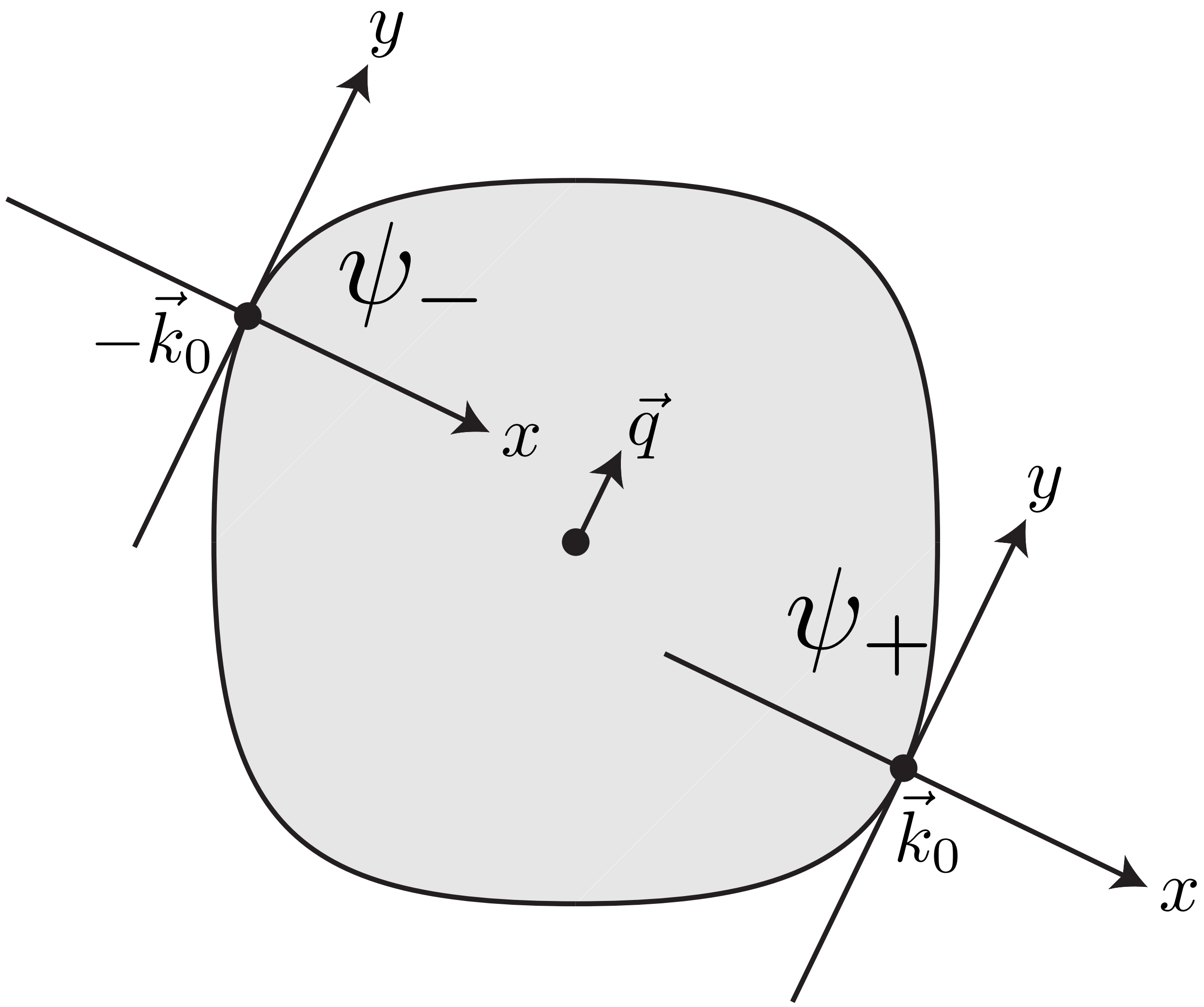
Quantum criticality of Ising-nematic ordering in a metal

$$\mathcal{S}_\phi = \int d^2r d\tau \left[(\partial_\tau \phi)^2 + c^2 (\nabla \phi)^2 + (\lambda - \lambda_c)\phi^2 + u\phi^4 \right]$$

$$\mathcal{S}_c = \sum_{\alpha=1}^{N_f} \sum_{\mathbf{k}} \int d\tau c_{\mathbf{k}\alpha}^\dagger \left(\partial_\tau + \varepsilon_{\mathbf{k}} \right) c_{\mathbf{k}\alpha}$$

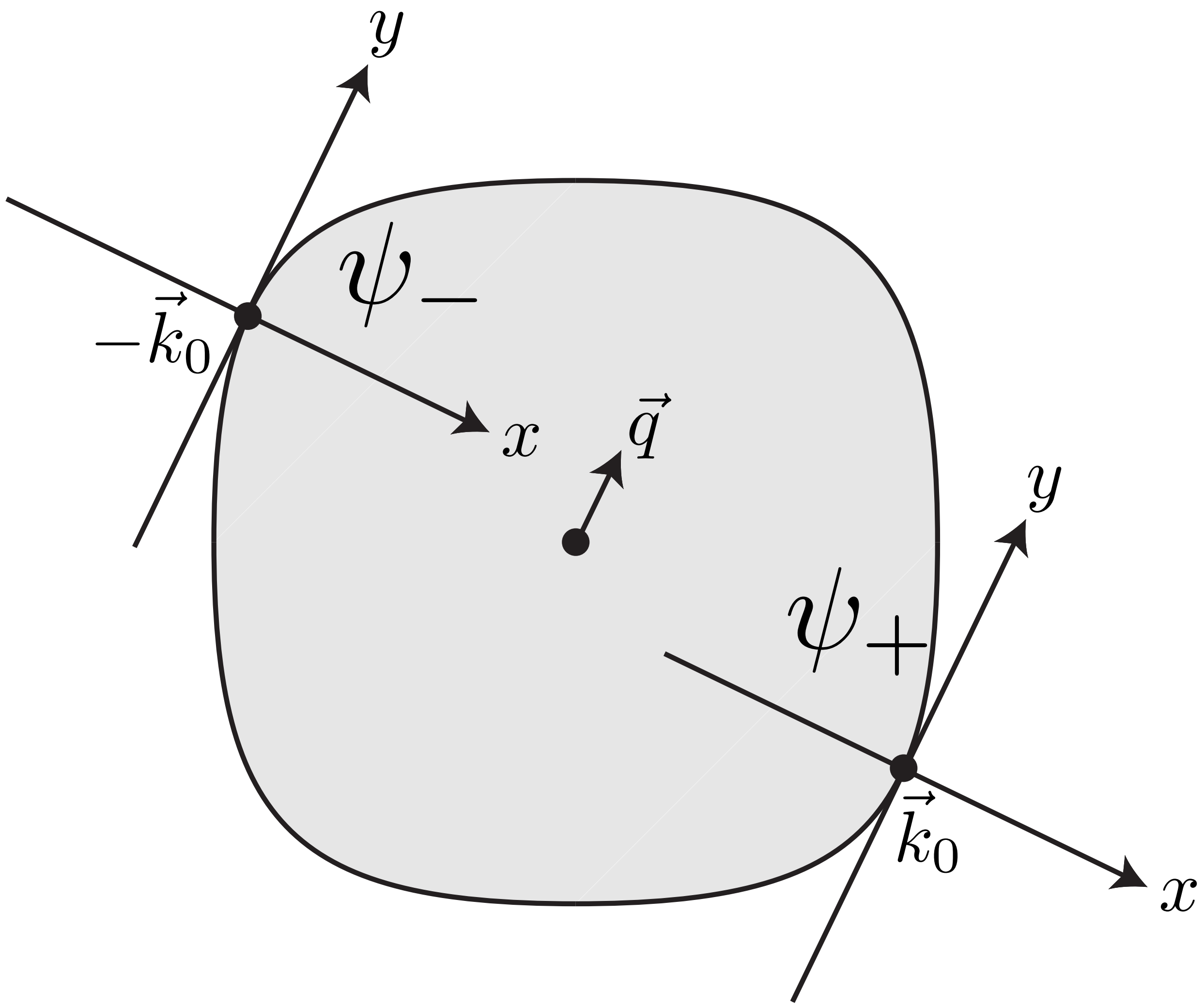
$$\mathcal{S}_{\phi c} = - g \int d\tau \sum_{\alpha=1}^{N_f} \sum_{\mathbf{k},\mathbf{q}} \phi_{\mathbf{q}} \left(\cos k_x - \cos k_y \right) c_{\mathbf{k}+\mathbf{q}/2,\alpha}^\dagger c_{\mathbf{k}-\mathbf{q}/2,\alpha}$$

Quantum criticality of Ising-nematic ordering in a metal



- ϕ fluctuation at wavevector $\vec{q}$ couples most efficiently to fermions near $\pm\vec{k}_0$.

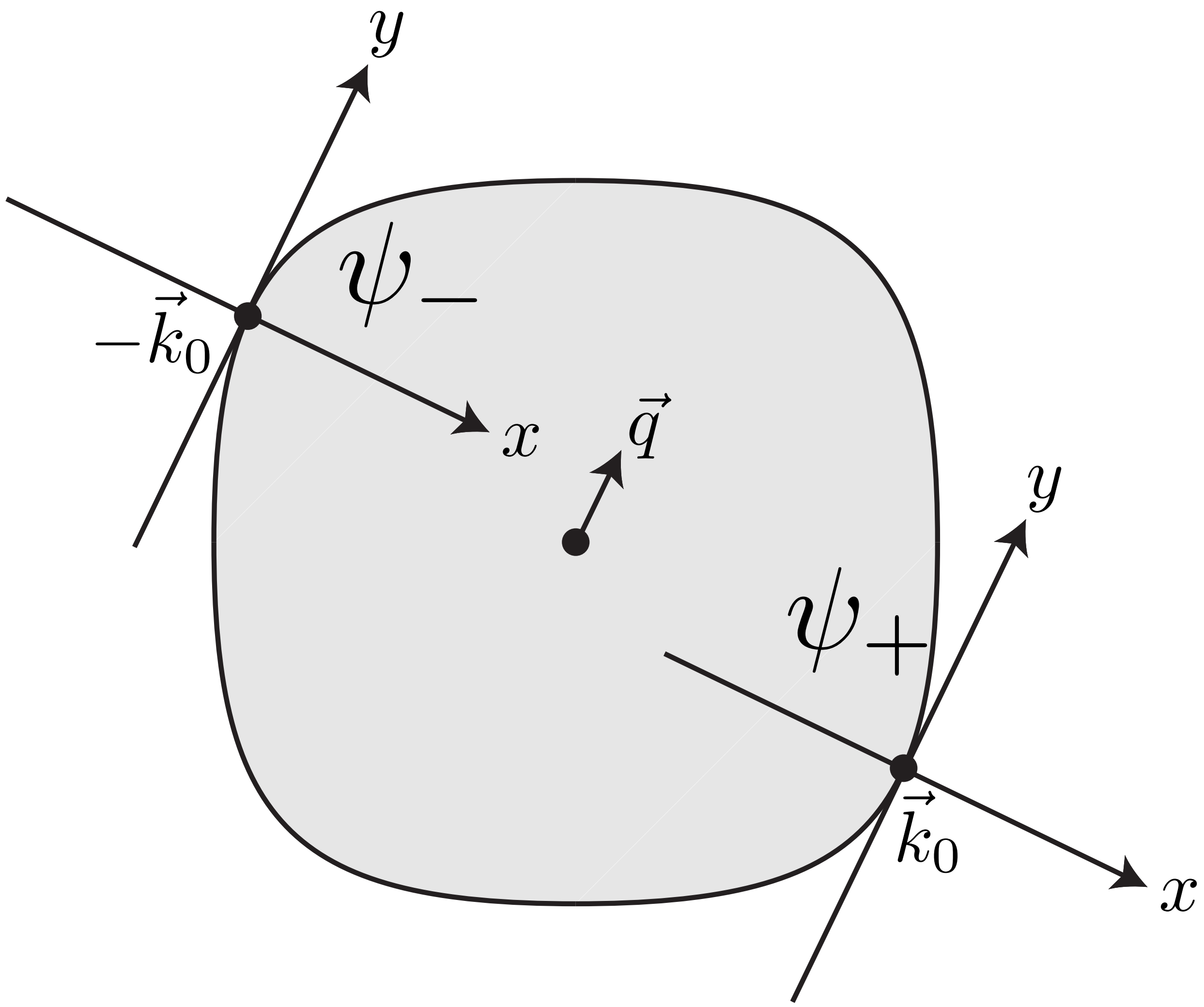
Quantum criticality of Ising-nematic ordering in a metal



- ϕ fluctuation at wavevector $\vec{q}$ couples most efficiently to fermions near $\pm\vec{k}_0$.

- Expand fermion kinetic energy at wavevectors about $\pm\vec{k}_0$ and boson (ϕ) kinetic energy about $\vec{q} = 0$.

Quantum criticality of Ising-nematic ordering in a metal



$$\mathcal{L}[\psi_\pm, \phi] =$$

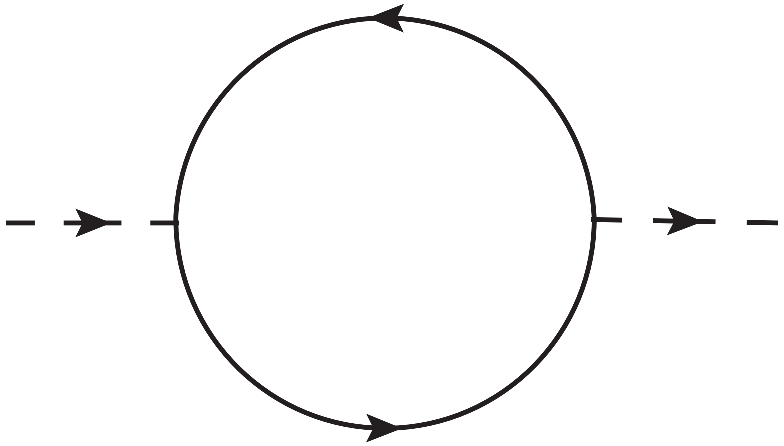
$$\psi_+^\dagger \left(\partial_\tau - i\partial_x - \partial_y^2\right) \psi_+ + \psi_-^\dagger \left(\partial_\tau + i\partial_x - \partial_y^2\right) \psi_-$$

$$-\phi \left(\psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_-\right) + \frac{1}{2g^2} \left(\partial_y \phi\right)^2$$

M. A. Metlitski and S. Sachdev, Phys. Rev. B **82**, 075127 (2010)

Quantum criticality of Ising-nematic ordering in a metal

$$\boxed{\begin{aligned}\mathcal{L} &= \psi_+^\dagger\left(\partial_\tau - i\partial_x - \partial_y^2\right)\psi_+ + \psi_-^\dagger\left(\partial_\tau + i\partial_x - \partial_y^2\right)\psi_- \\ &\quad - \phi\left(\psi_+^\dagger\psi_+ + \psi_-^\dagger\psi_-\right) + \frac{1}{2g^2}\left(\partial_y\phi\right)^2\end{aligned}}$$



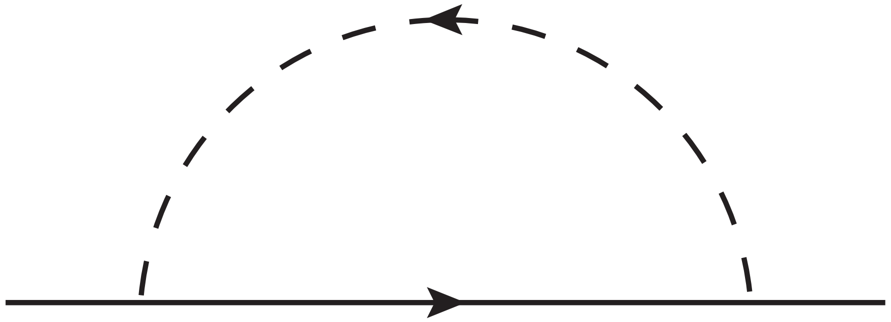
One loop ϕ self-energy with N_f fermion flavors:

$$\begin{aligned}\Sigma_\phi(\vec{q},\omega) &= N_f\int\frac{d^2k}{4\pi^2}\frac{d\Omega}{2\pi}\frac{1}{\left[-i(\Omega+\omega)+k_x+q_x+(k_y+q_y)^2\right]\left[-i\Omega-k_x+k_y^2\right]} \\ &= \frac{N_f}{4\pi}\frac{|\omega|}{|q_y|}\end{aligned}$$

Landau-damping

Quantum criticality of Ising-nematic ordering in a metal

$$\begin{aligned}\mathcal{L} \;&=\; \psi_+^\dagger \left(\partial_\tau - i\partial_x - \partial_y^2\right)\psi_+ + \psi_-^\dagger \left(\partial_\tau + i\partial_x - \partial_y^2\right)\psi_- \\ &\quad - \phi\left(\psi_+^\dagger\psi_+ + \psi_-^\dagger\psi_-\right) + \frac{1}{2g^2}\left(\partial_y\phi\right)^2\end{aligned}$$

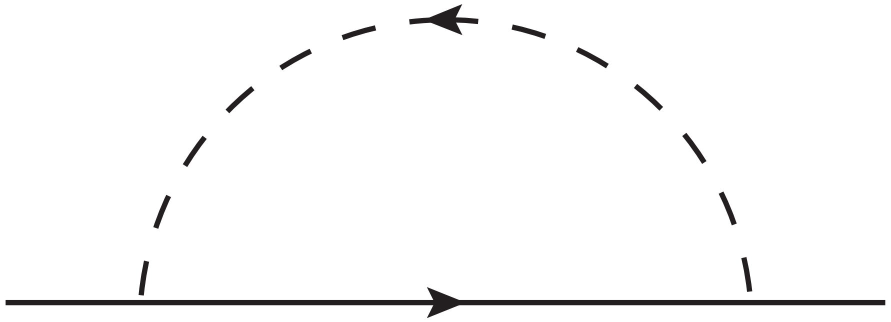


Electron self-energy at order $1/N_f$:

$$\begin{aligned}\Sigma(\vec{k},\Omega) \;&=\; -\frac{1}{N_f}\int \frac{d^2q}{4\pi^2}\frac{d\omega}{2\pi}\frac{1}{\left[-i(\omega+\Omega)+k_x+q_x+(k_y+q_y)^2\right]\left[\dfrac{q_y^2}{g^2}+\dfrac{|\omega|}{|q_y|}\right]} \\ &=\; -i\frac{2}{\sqrt{3}N_f}\left(\frac{g^2}{4\pi}\right)^{2/3}\mathrm{sgn}(\Omega)|\Omega|^{2/3}\end{aligned}$$

Quantum criticality of Ising-nematic ordering in a metal

$$\begin{aligned}\mathcal{L} &= \psi_+^\dagger \left(\partial_\tau - i\partial_x - \partial_y^2\right)\psi_+ + \psi_-^\dagger \left(\partial_\tau + i\partial_x - \partial_y^2\right)\psi_- \\ &\quad - \phi\left(\psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_-\right) + \frac{1}{2g^2}\left(\partial_y \phi\right)^2\end{aligned}$$



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$\sim |\Omega|^{d/3}$ in dimension d.

Quantum criticality of Ising-nematic ordering in a metal

$$\begin{aligned}\mathcal{L} \;&=\; \psi_+^\dagger\left(\partial_\tau - i\partial_x - \partial_y^2\right)\psi_+ + \psi_-^\dagger\left(\partial_\tau + i\partial_x - \partial_y^2\right)\psi_- \\ &\quad - \phi\left(\psi_+^\dagger\psi_+ + \psi_-^\dagger\psi_-\right) + \frac{1}{2g^2}\left(\partial_y\phi\right)^2\end{aligned}$$

Schematic form of ϕ and fermion Green's functions in d dimensions

$$D(\vec{q},\omega) = \frac{1/N_f}{q_\perp^2 + \dfrac{|\omega|}{|q_\perp|}} \quad , \quad G_f(\vec{q},\omega) = \frac{1}{q_x + q_\perp^2 - i\,\mathrm{sgn}(\omega)|\omega|^{d/3}/N_f}$$

In the boson case, $q_\perp^2 \sim \omega^{1/z_b}$ with $z_b = 3/2$.
In the fermion case, $q_x \sim q_\perp^2 \sim \omega^{1/z_f}$ with $z_f = 3/d$.

Note $z_f < z_b$ for $d > 2$ $\Rightarrow$ Fermions have *higher* energy than bosons, and perturbation theory in g is OK.
Strongly-coupled theory in $d = 2$.

Quantum criticality of Ising-nematic ordering in a metal

$$\begin{aligned}\mathcal{L} &= \psi_+^\dagger \left(\partial_\tau - i\partial_x - \partial_y^2\right)\psi_+ + \psi_-^\dagger\left(\partial_\tau + i\partial_x - \partial_y^2\right)\psi_- \\ &\quad - \phi\left(\psi_+^\dagger\psi_+ + \psi_-^\dagger\psi_-\right) + \frac{1}{2g^2}\left(\partial_y\phi\right)^2\end{aligned}$$

Schematic form of ϕ and fermion Green's functions in $d = 2$

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In *both* cases $q_x \sim q_y^2 \sim \omega^{1/z}$, with $z = 3/2$. Note that the bare term $\sim \omega$ in G_f^{-1} is irrelevant.

Strongly-coupled theory without quasiparticles.

Quantum criticality of Ising-nematic ordering in a metal

$$\begin{aligned}\mathcal{L} \;=\;& \psi_+^\dagger \left(\partial_\tau - i\partial_x - \partial_y^2\right) \psi_+ + \psi_-^\dagger \left(\partial_\tau + i\partial_x - \partial_y^2\right) \psi_- \\ & - \phi \left(\psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_-\right) + \frac{1}{2g^2} \left(\partial_y \phi\right)^2\end{aligned}$$

Simple scaling argument for $z = 3/2$.

Quantum criticality of Ising-nematic ordering in a metal

$$
\begin{aligned}
\mathcal{L} \;=\; & \psi_+^\dagger \left(\xcancel{\partial_\tau} - i\partial_x - \partial_y^2 \right) \psi_+ + \psi_-^\dagger \left(\xcancel{\partial_\tau} + i\partial_x - \partial_y^2 \right) \psi_- \\
& - \phi \left(\psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- \right) + \frac{1}{2g^2} \left(\partial_y \phi \right)^2
\end{aligned}
$$

Simple scaling argument for $z = 3/2$.

Quantum criticality of Ising-nematic ordering in a metal

$$\begin{aligned}\mathcal{L} &= \psi_+^\dagger \left(\cancel{\partial_\tau} - i\partial_x - \partial_y^2\right)\psi_+ + \psi_-^\dagger \left(\cancel{\partial_\tau} + i\partial_x - \partial_y^2\right)\psi_- \\ &\quad - \phi\left(\psi_+^\dagger\psi_+ + \psi_-^\dagger\psi_-\right) + \frac{1}{2g^2}\left(\partial_y\phi\right)^2\end{aligned}$$

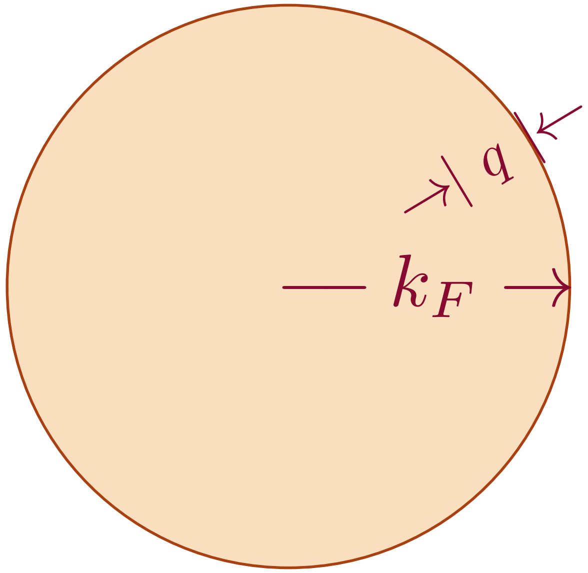
Simple scaling argument for $z = 3/2$.

Under the rescaling $x \to x/s$, $y \to y/s^{1/2}$, and $\tau \to \tau/s^z$, we find invariance provided

$$\begin{aligned}\phi &\to \phi\, s \\ \psi &\to \psi\, s^{(2z+1)/4} \\ g &\to g\, s^{(3-2z)/4}\end{aligned}$$

So the action is invariant provided $z = 3/2$.

FL
Fermi
liquid

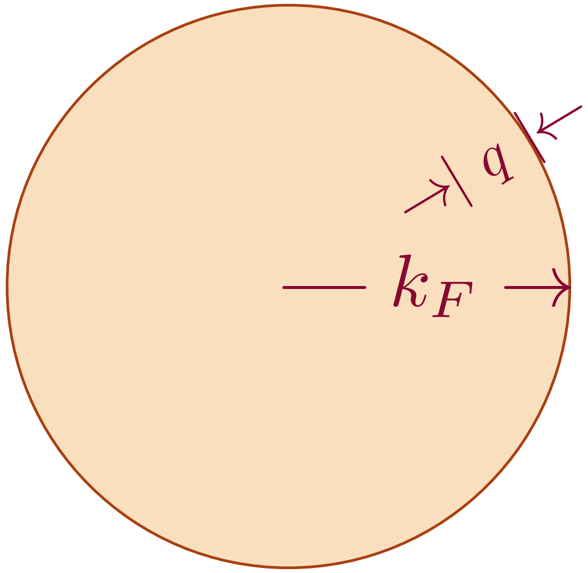


- $k_F^d \sim \mathcal{Q}$, the fermion density

- Sharp fermionic excitations near Fermi surface with $\omega \sim |q|^z$, and $z = 1$.

- Entropy density $S \sim T^{(d-\theta)/z}$ with violation of hyperscaling exponent $\theta = d - 1$.

FL Fermi liquid



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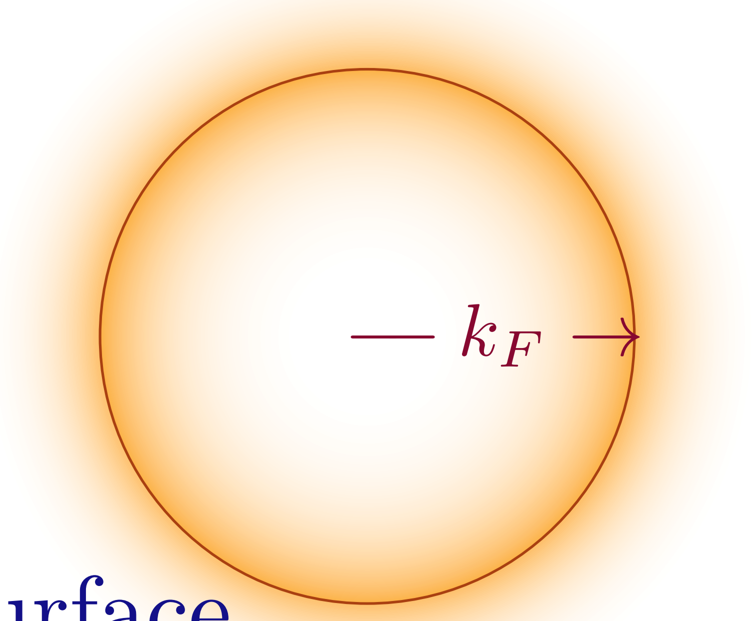
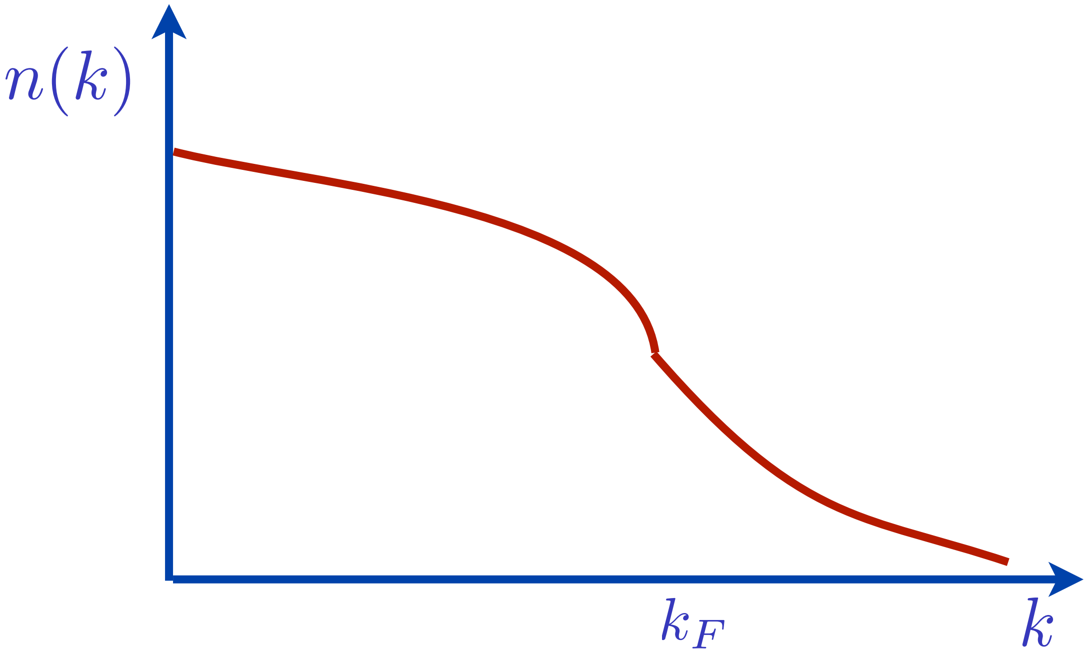
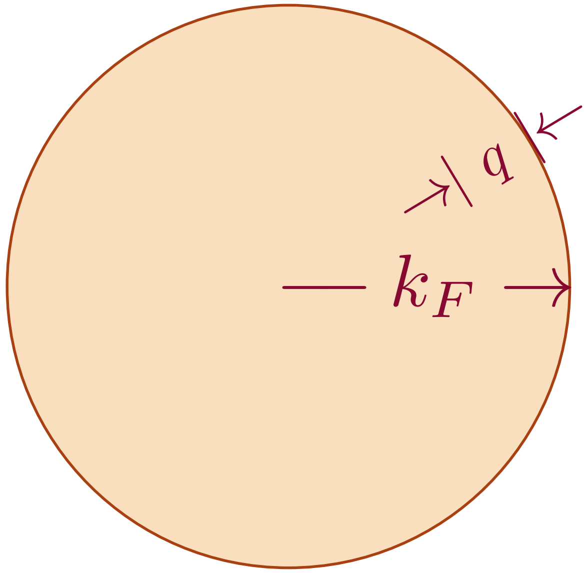
NFL Nematic QCP



- Fermi surface with $k_F^d \sim \mathcal{Q}$.



FL
Fermi liquid



- $k_F^d \sim \mathcal{Q}$, the fermion density

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NFL
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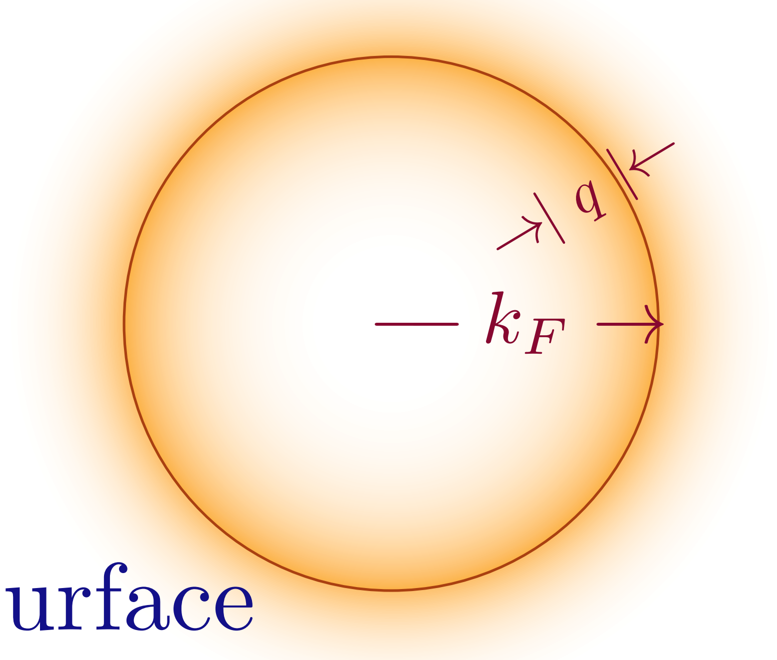

- Fermi surface with $k_F^d \sim \mathcal{Q}$.

- Diffuse fermionic excitations with $z = 3/2$ to three loops.

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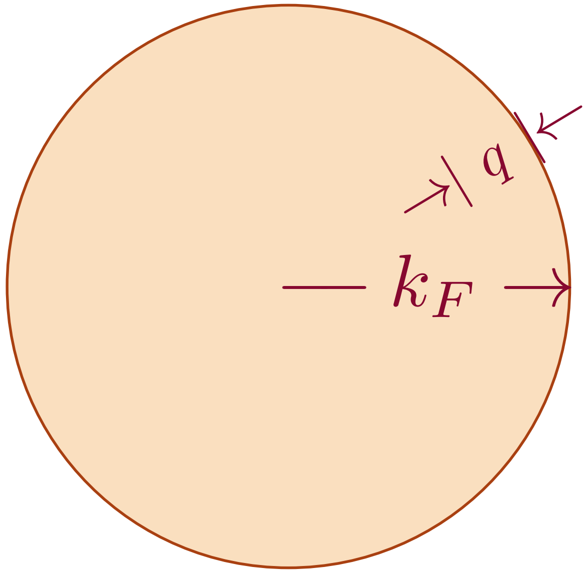
FL Fermi liquid



- $k_F^d \sim \mathcal{Q}$, the fermion density
- Sharp fermionic excitations near Fermi surface with $\omega \sim |q|^z$, and $z = 1$.
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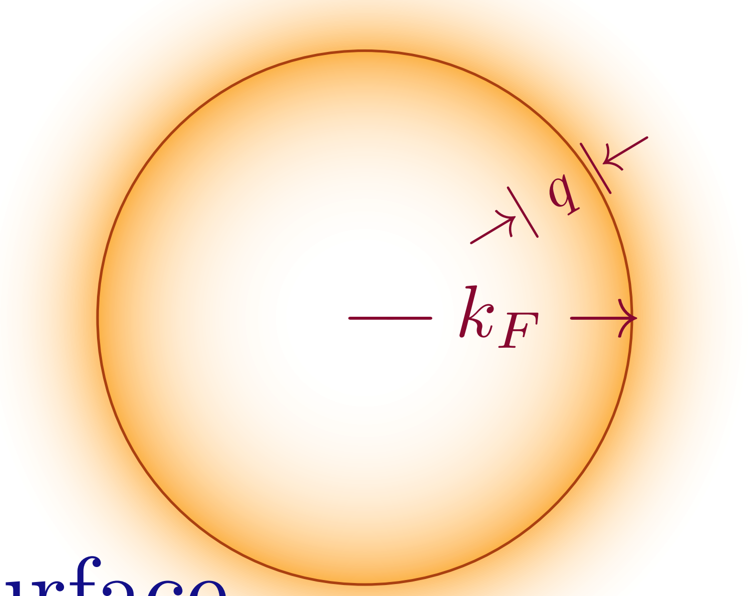
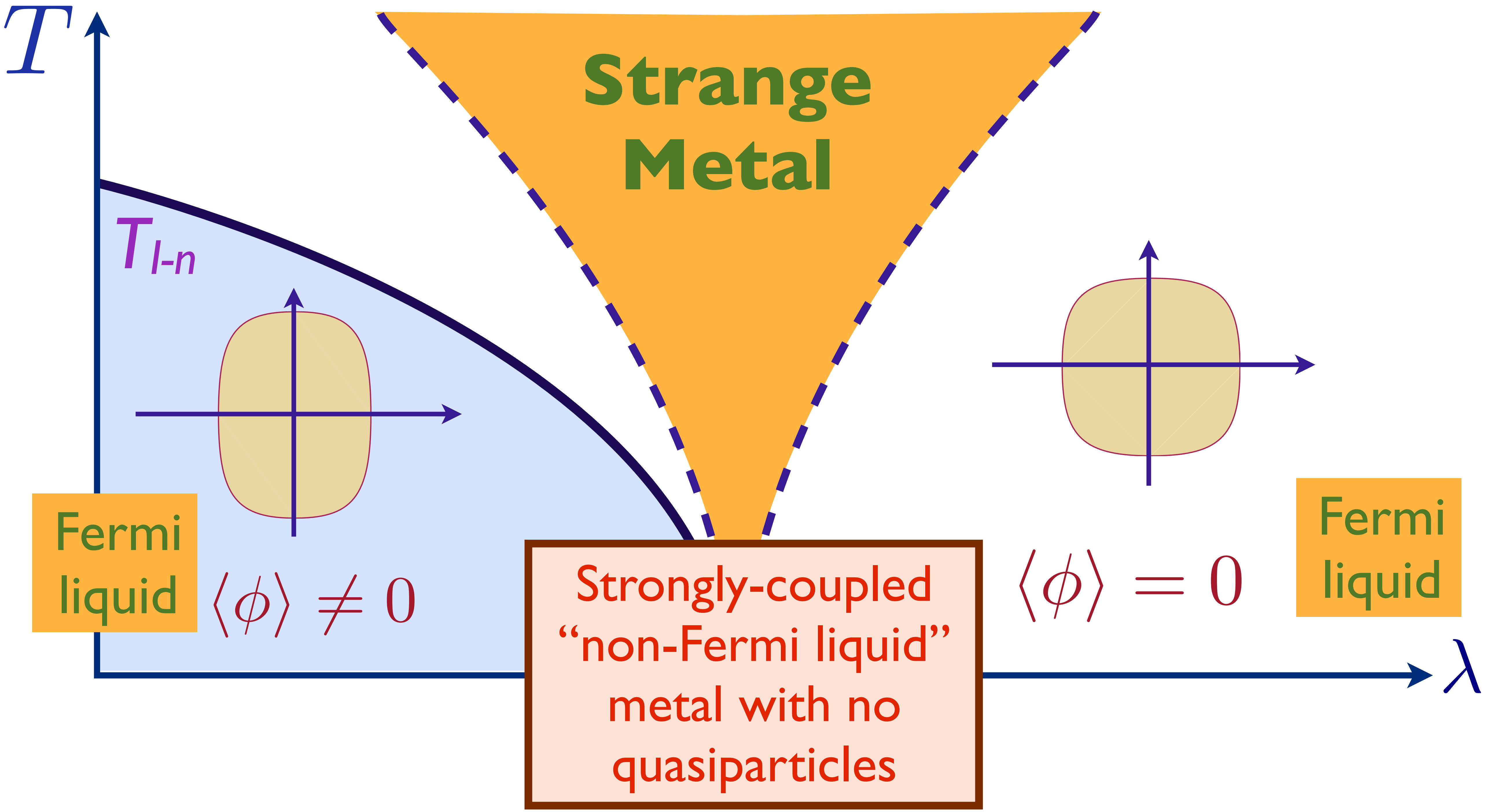
NFL Nematic QCP



- Fermi surface with $k_F^d \sim \mathcal{Q}$.
- Diffuse fermionic excitations with $z = 3/2$ to three loops.
- $S \sim T^{(d-\theta)/z}$ with $\theta = d - 1$.

Quantum criticality of Ising-nematic ordering in a metal



Strongly-coupled “non-Fermi liquid” metal with no quasiparticles

Phase diagram as a function of T and λ