Quantum criticality and the phase diagram of the cuprates

Talk online: sachdev.physics.harvard.edu

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Andreas Hackl, Cologne and Harvard
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Andreas Hackl

Max Metlitski

Outline

1. Proposed phase diagram as a function of temperature, doping, and magnetic field

   Interplay of spin density waves, d-wave superconductivity and Fermi surface change

2. Nernst effect in cuprates

   (a) Hydrodynamic theory
   (b) Quasiparticle theory in spin/charge density wave and nematic states

3. Theory of SDW quantum critical point

   Dominance of planar graphs

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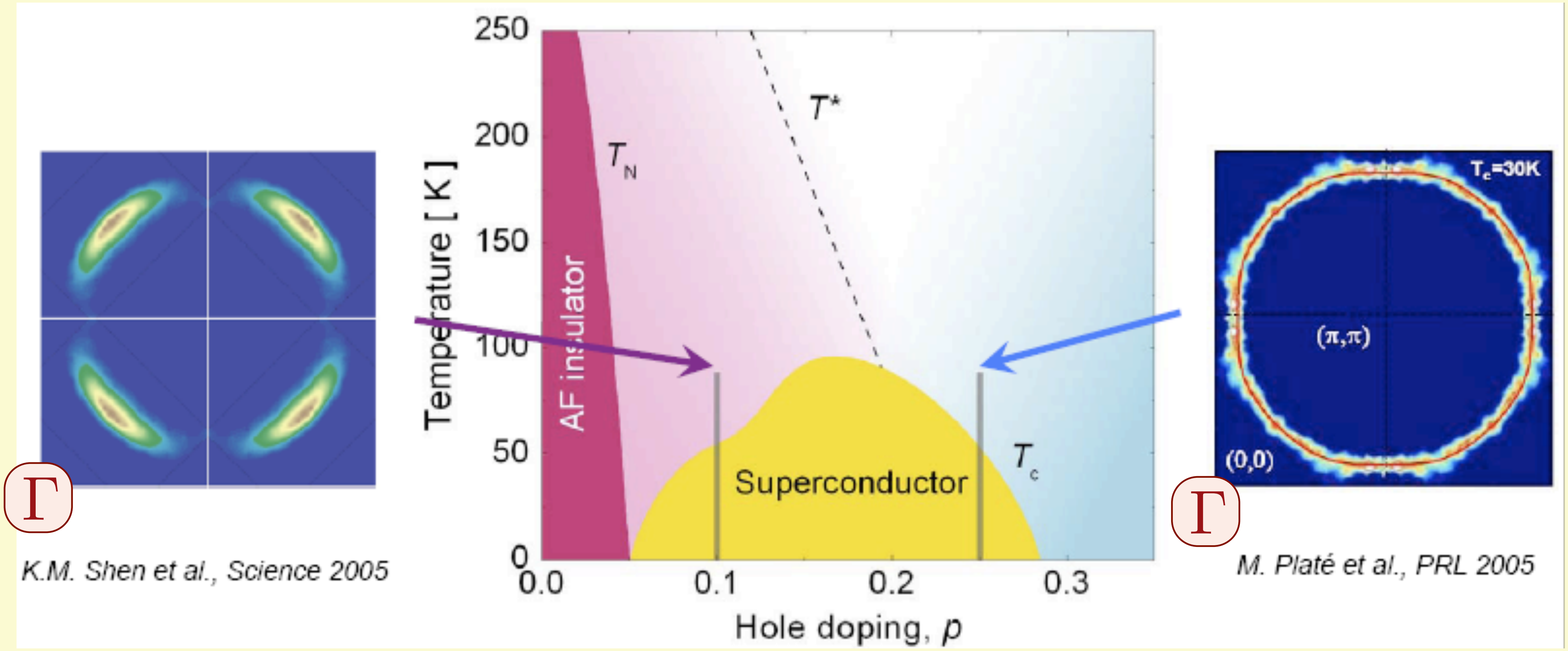
Central ingredients in cuprate phase diagram: antiferromagnetism, superconductivity, and change in Fermi surface



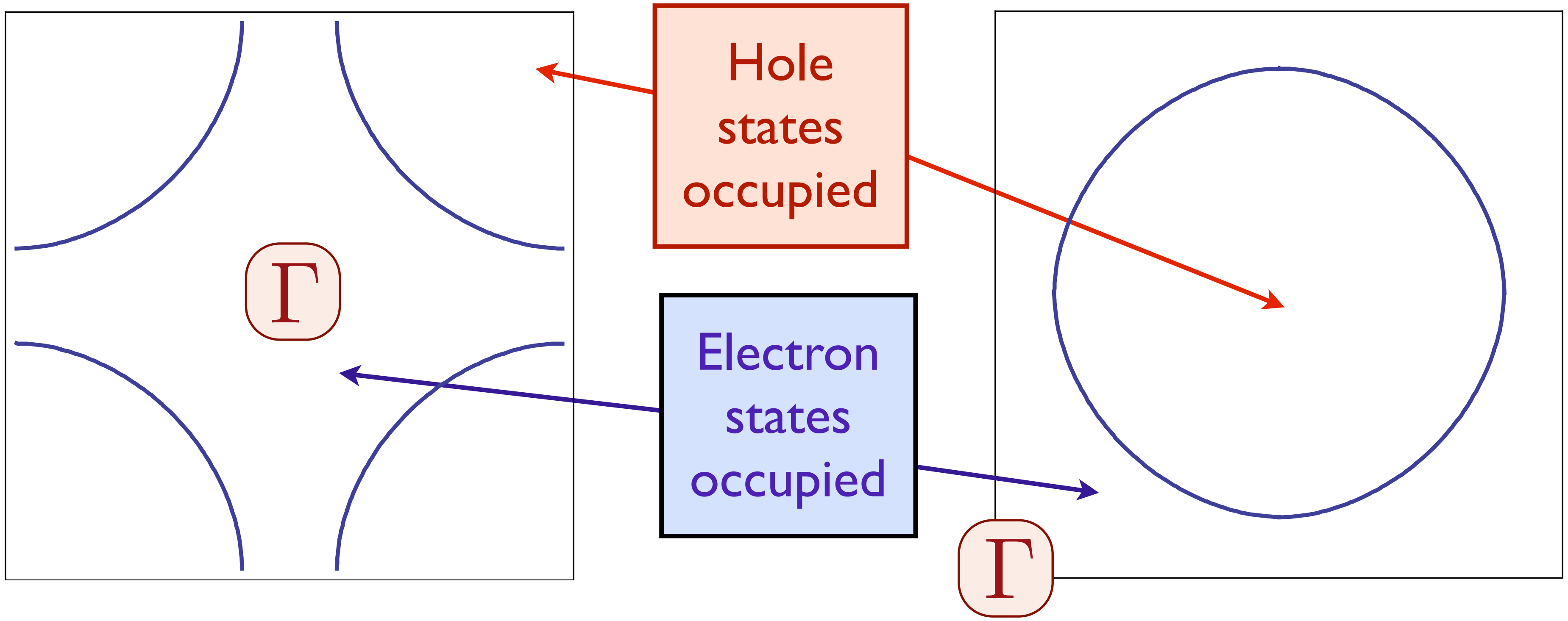
K.M. Shen et al., Science 2005

M. Platé et al., PRL 2005

Smaller hole
Fermi-pockets

Large hole
Fermi surface

“Large” Fermi surfaces in cuprates



Hole states occupied

Electron states occupied

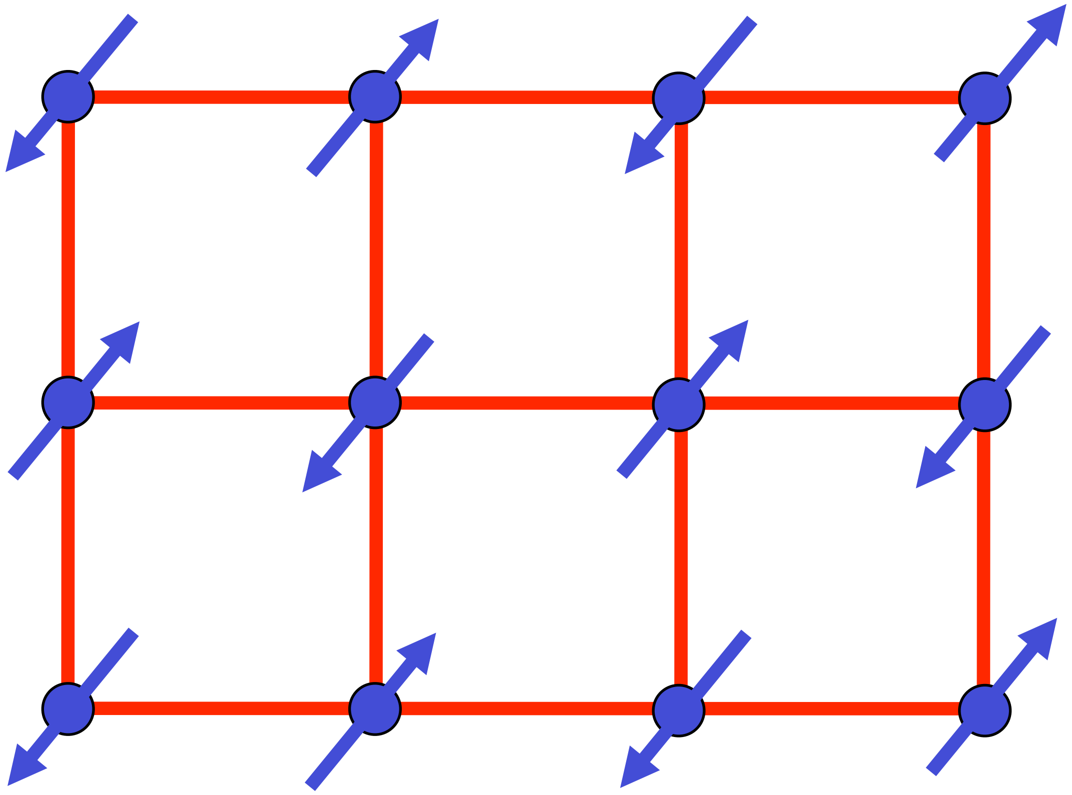
$$H_0 = -\sum_{i<j} t_{ij} c_{i\alpha}^{\dagger} c_{i\alpha} \equiv \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} c_{\mathbf{k}\alpha}^{\dagger} c_{\mathbf{k}\alpha}$$

The area of the occupied electron/hole states:

$$\mathcal{A}_e = \begin{cases} 2\pi^2(1-x) & \text{for hole-doping } x \\ 2\pi^2(1+p) & \text{for electron-doping } p \end{cases}$$

$$\mathcal{A}_h = 4\pi^2 - \mathcal{A}_e$$

Spin density wave theory

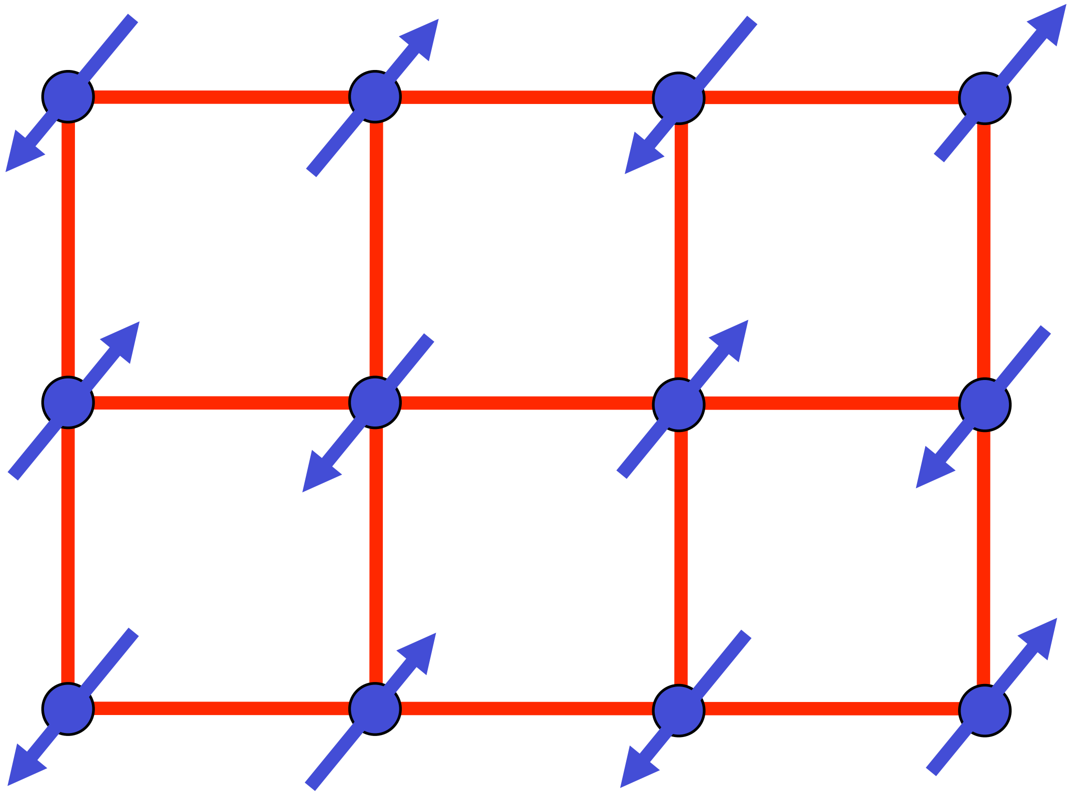


The electron spin polarization obeys

$$\left\langle \vec{S}(\mathbf{r},\tau) \right\rangle = \vec{\varphi}(\mathbf{r},\tau) e^{i\mathbf{K}\cdot\mathbf{r}}$$

where $\vec{\varphi}$ is the spin density wave (SDW) order parameter, and $\mathbf{K}$ is the ordering wavevector. For simplicity, we consider $\mathbf{K} = (\pi, \pi)$.

Spin density wave theory



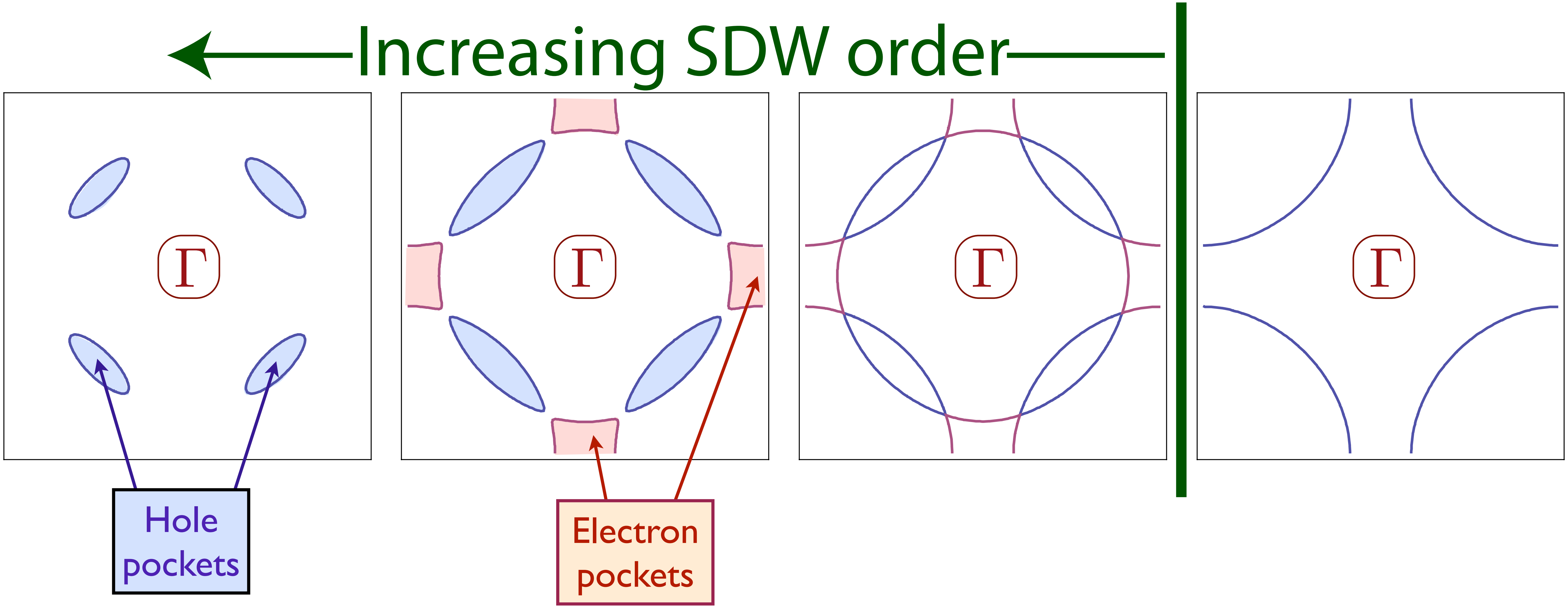
Spin density wave Hamiltonian

$$H_{\rm sdw} = \vec{\varphi} \cdot \sum_{\mathbf{k},\alpha,\beta} c^{\dagger}_{\mathbf{k},\alpha} \vec{\sigma}_{\alpha\beta} c_{\mathbf{k}+\mathbf{K},\beta}$$

Diagonalize $H_0 + H_{\rm sdw}$ for $\vec{\varphi} = (0, 0, \varphi)$

$$E_{\mathbf{k}\pm} = \frac{\varepsilon_{\mathbf{k}} + \varepsilon_{\mathbf{k}+\mathbf{K}}}{2} \pm \sqrt{\left(\frac{\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}+\mathbf{K}}}{2}\right) + \varphi^2}$$

Spin density wave theory



Increasing SDW order

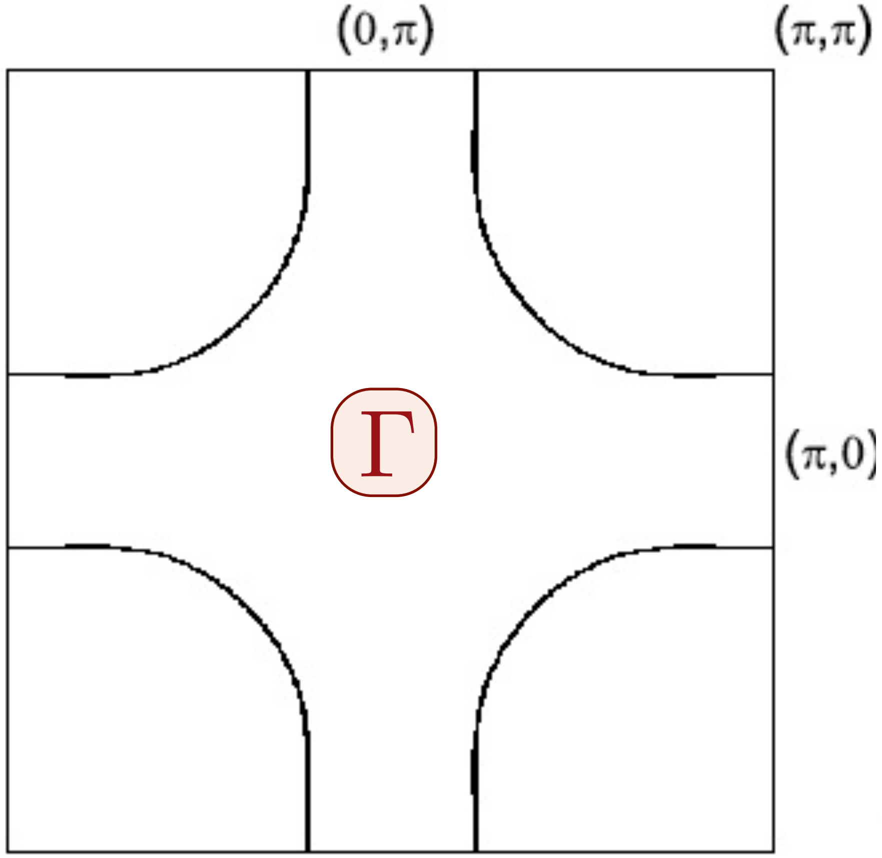
Hole pockets

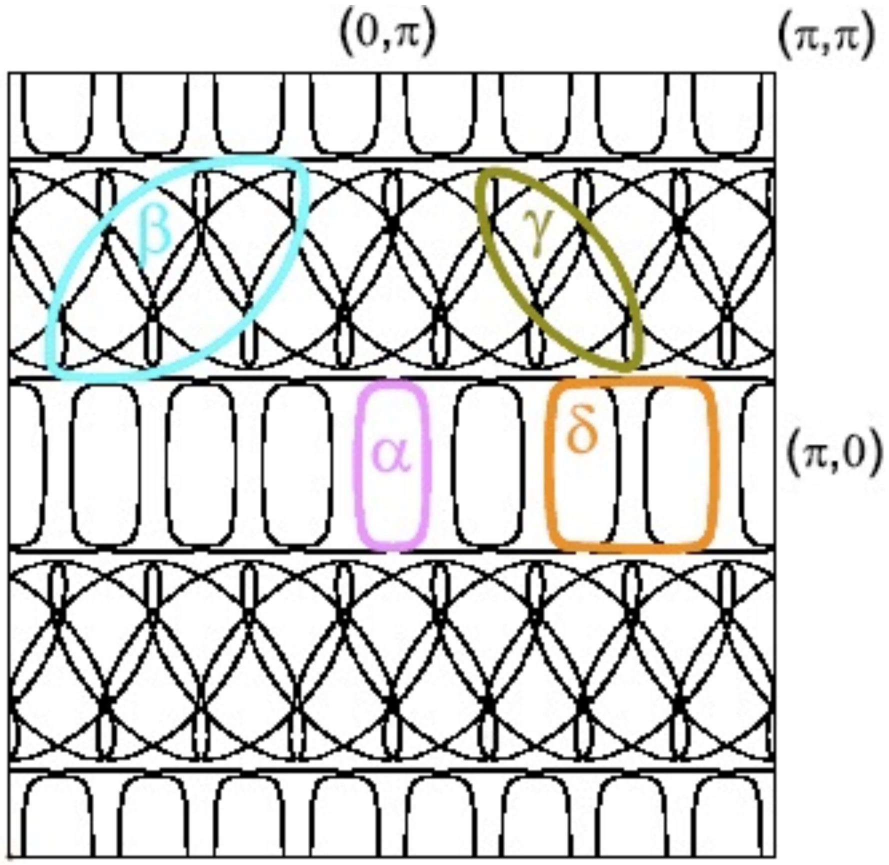
Electron pockets

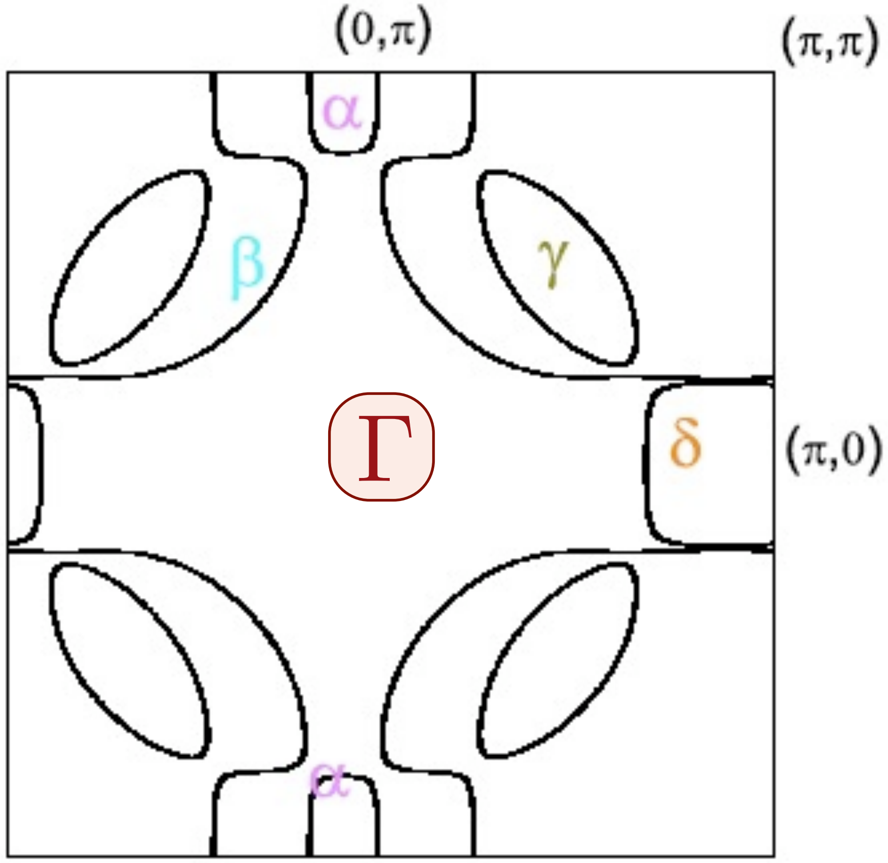
Large Fermi surface breaks up into electron and hole pockets

S. Sachdev, A. V. Chubukov, and A. Sokol, *Phys. Rev. B* **51**, 14874 (1995).
A. V. Chubukov and D. K. Morr, *Physics Reports* **288**, 355 (1997).
D. Senechal and A.-M. S. Tremblay, *Phys. Rev. Lett.* **92**, 126401 (2004)

Spin density wave theory in hole-doped cuprates



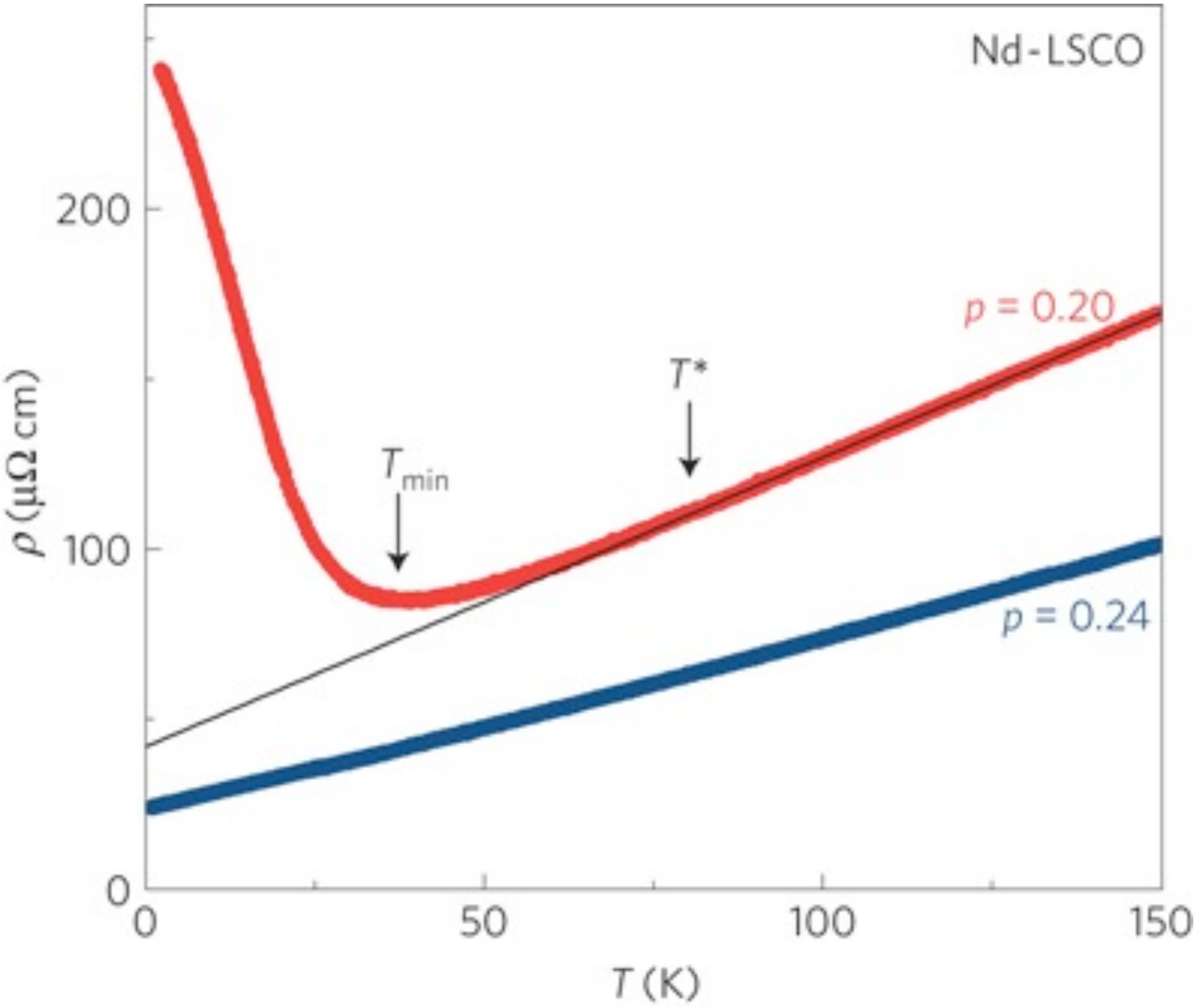




Incommensurate order in $YBa_2Cu_3O_{6+x}$

A. J. Millis and M. R. Norman, *Physical Review B* **76**, 220503 (2007).
N. Harrison, *Physical Review Letters* **102**, 206405 (2009).

Evidence for connection between linear resistivity and stripe-ordering in a cuprate with a low T_c

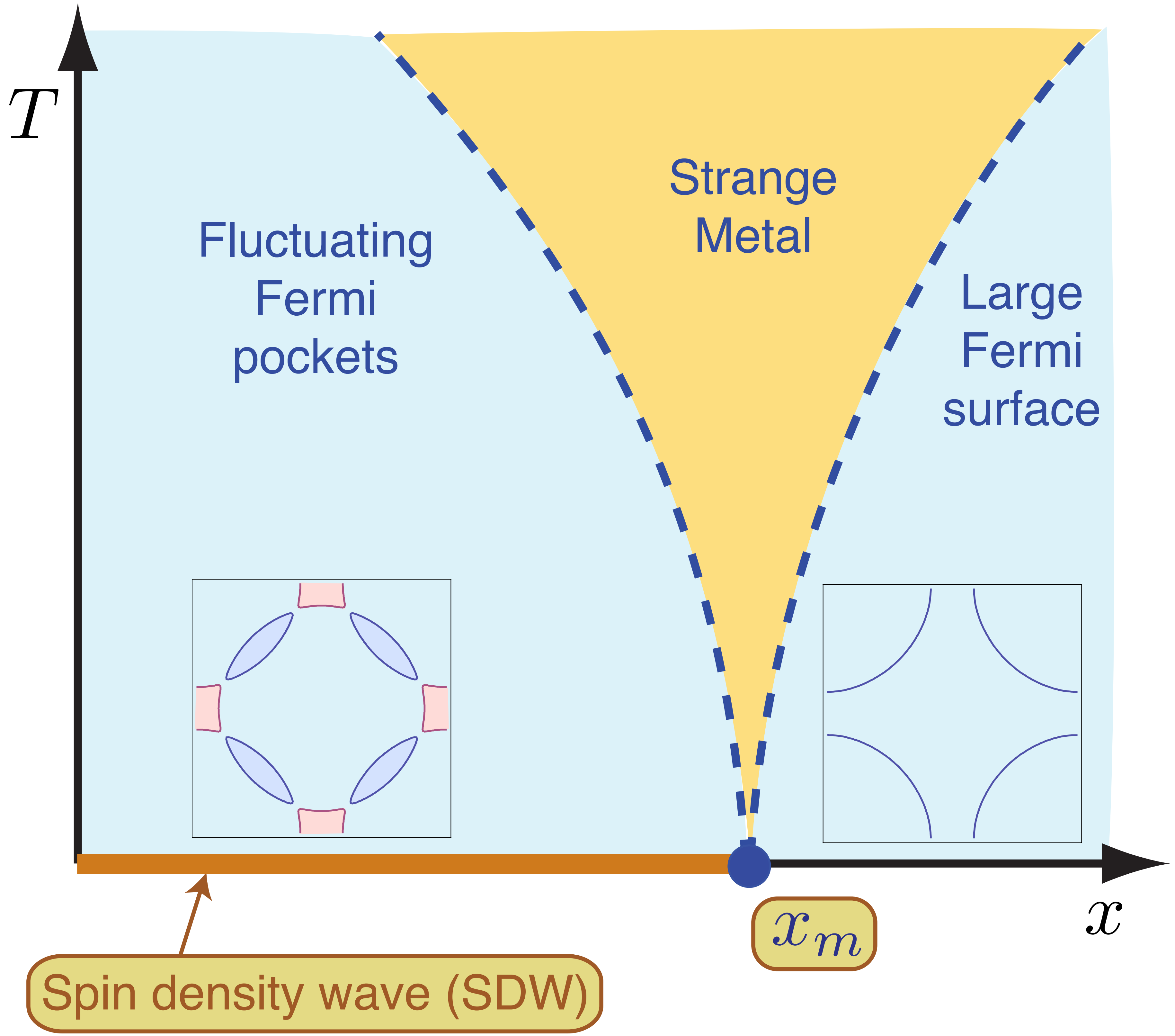


Magnetic field of upto 35 T used to suppress superconductivity

Linear temperature dependence of resistivity and change in the Fermi surface at the pseudogap critical point of a high-T_c superconductor

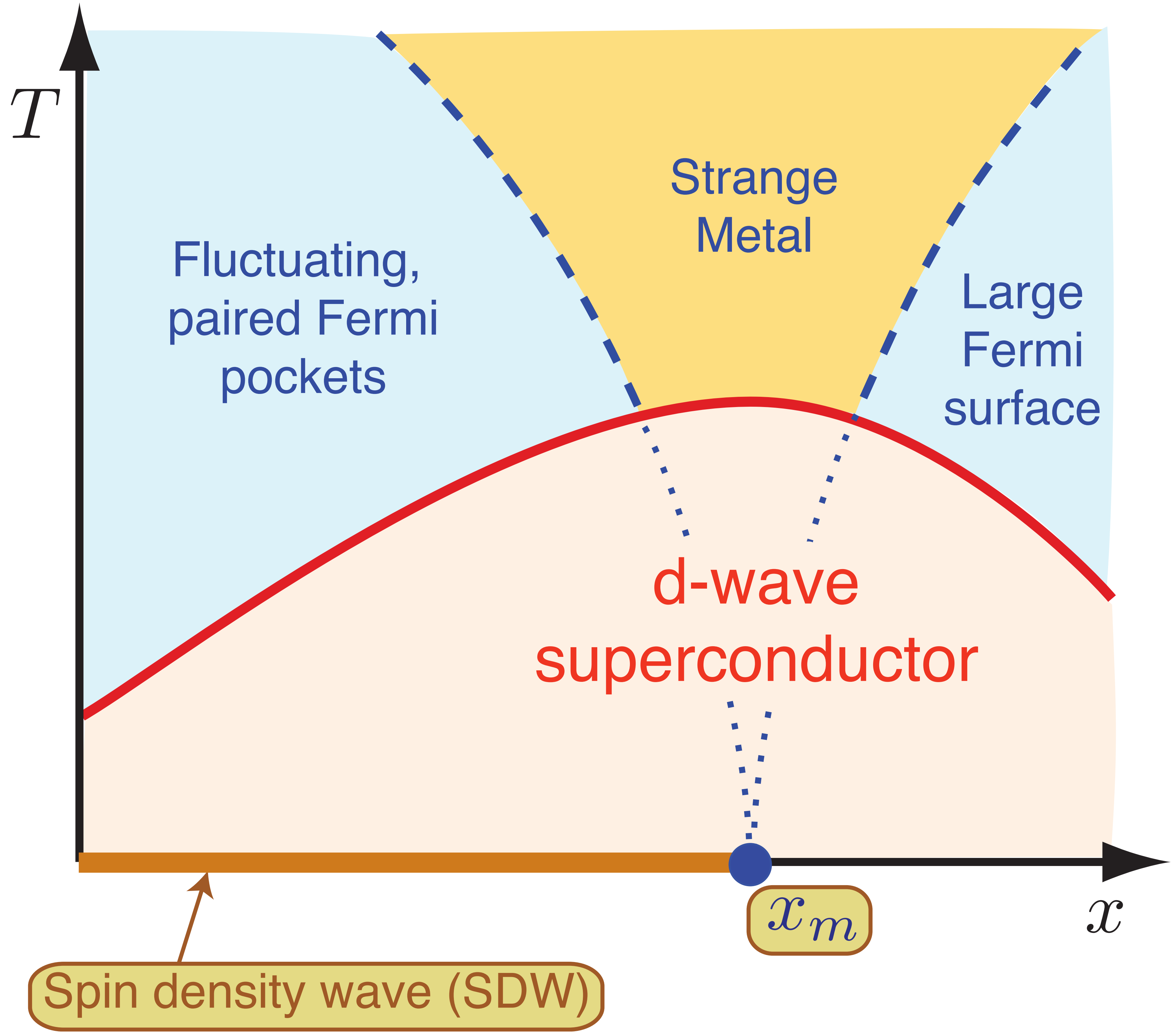
R. Daou, Nicolas Doiron-Leyraud, David LeBoeuf, S. Y. Li, Francis Laliberté, Olivier Cyr-Choinière, Y. J. Jo, L. Balicas, J.-Q. Yan, J.-S. Zhou, J. B. Goodenough & Louis Taillefer, *Nature Physics* **5**, 31 - 34 (2009)

Theory of quantum criticality in the cuprates



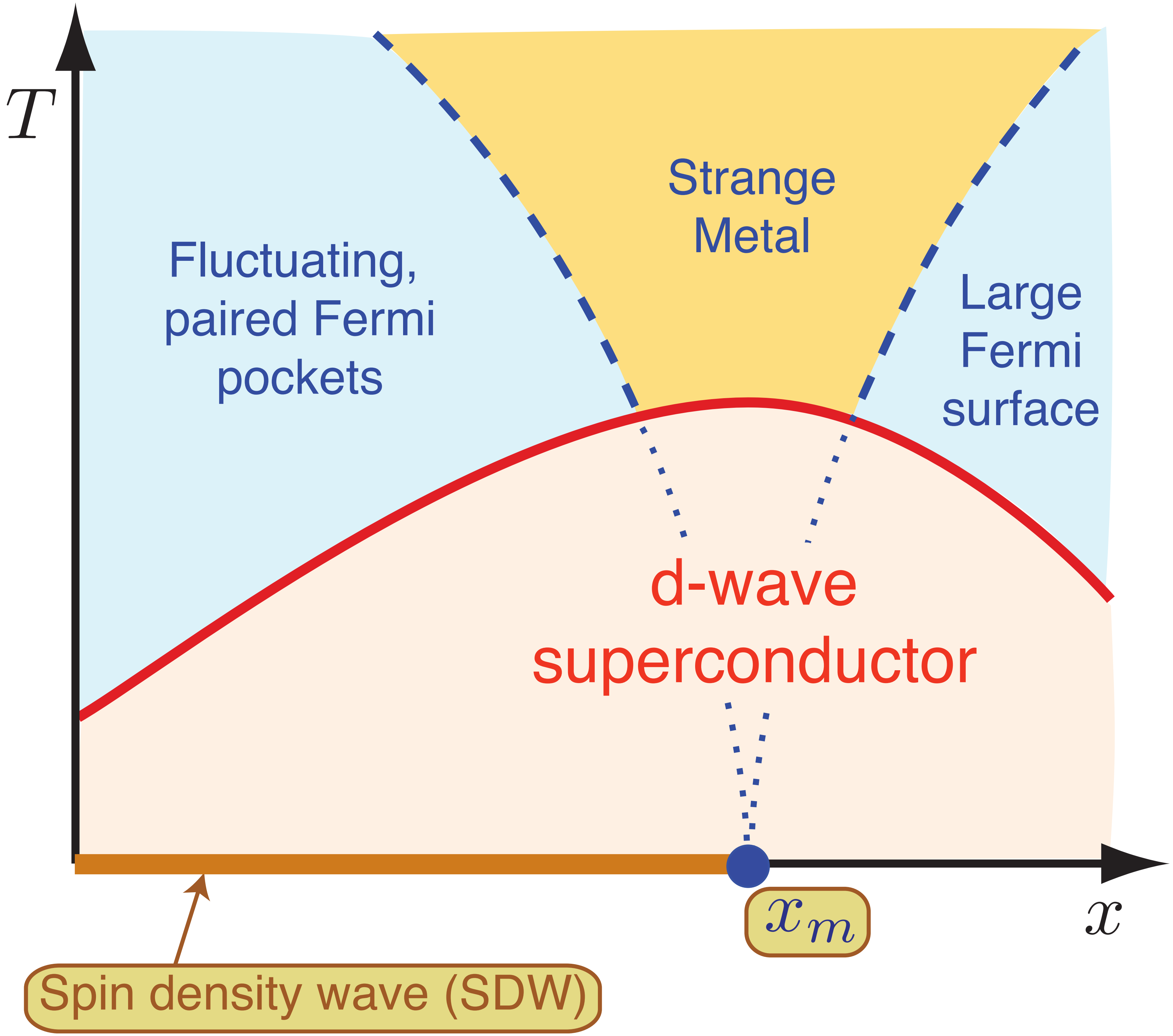
Underlying SDW ordering quantum critical point
in **metal** at $x = x_m$

Theory of quantum criticality in the cuprates



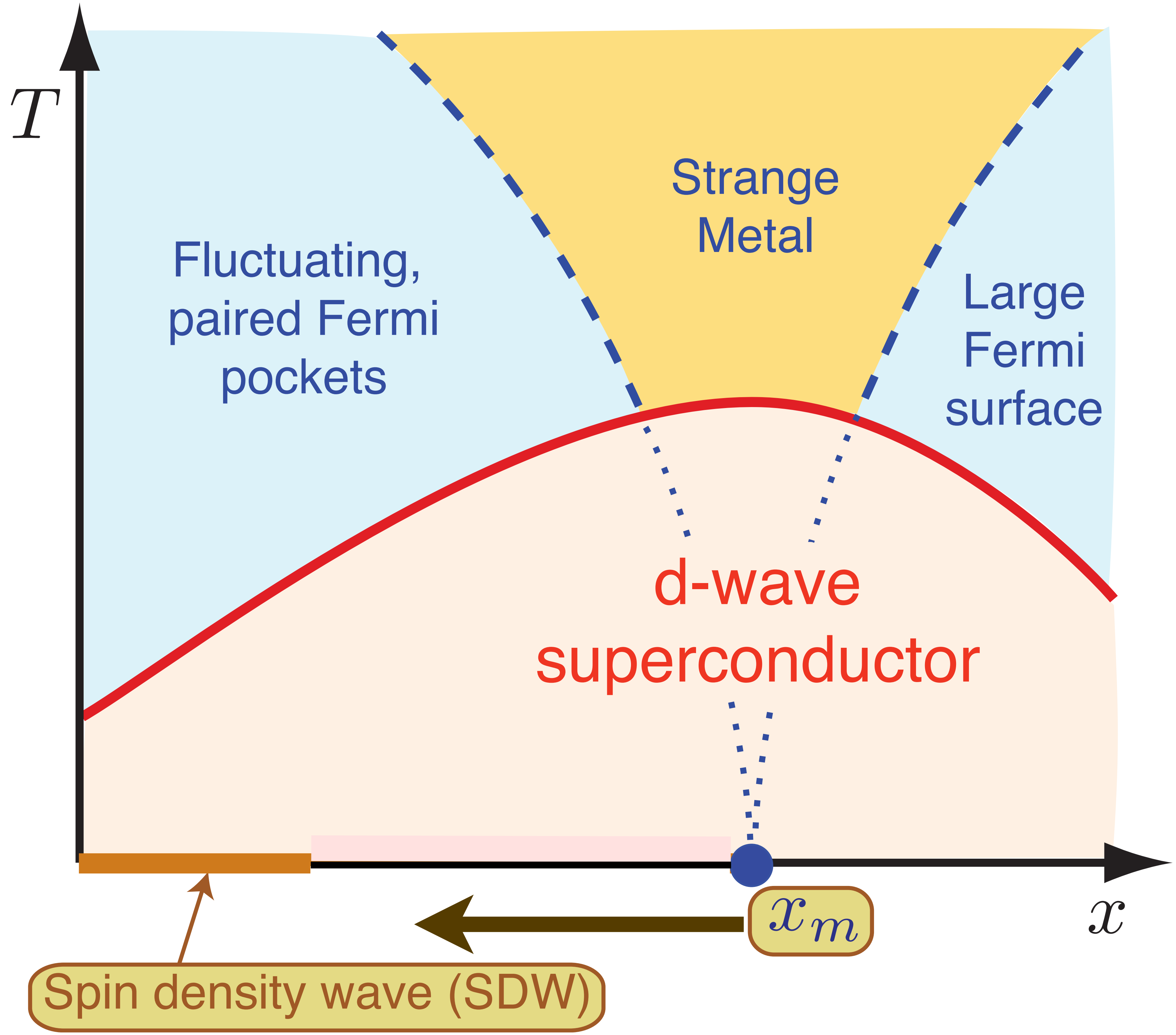
Onset of d-wave superconductivity hides the critical point $x = x_m$

Theory of quantum criticality in the cuprates



Competition between SDW order and superconductivity moves the actual quantum critical point to $x = x_s < x_m$.

Theory of quantum criticality in the cuprates



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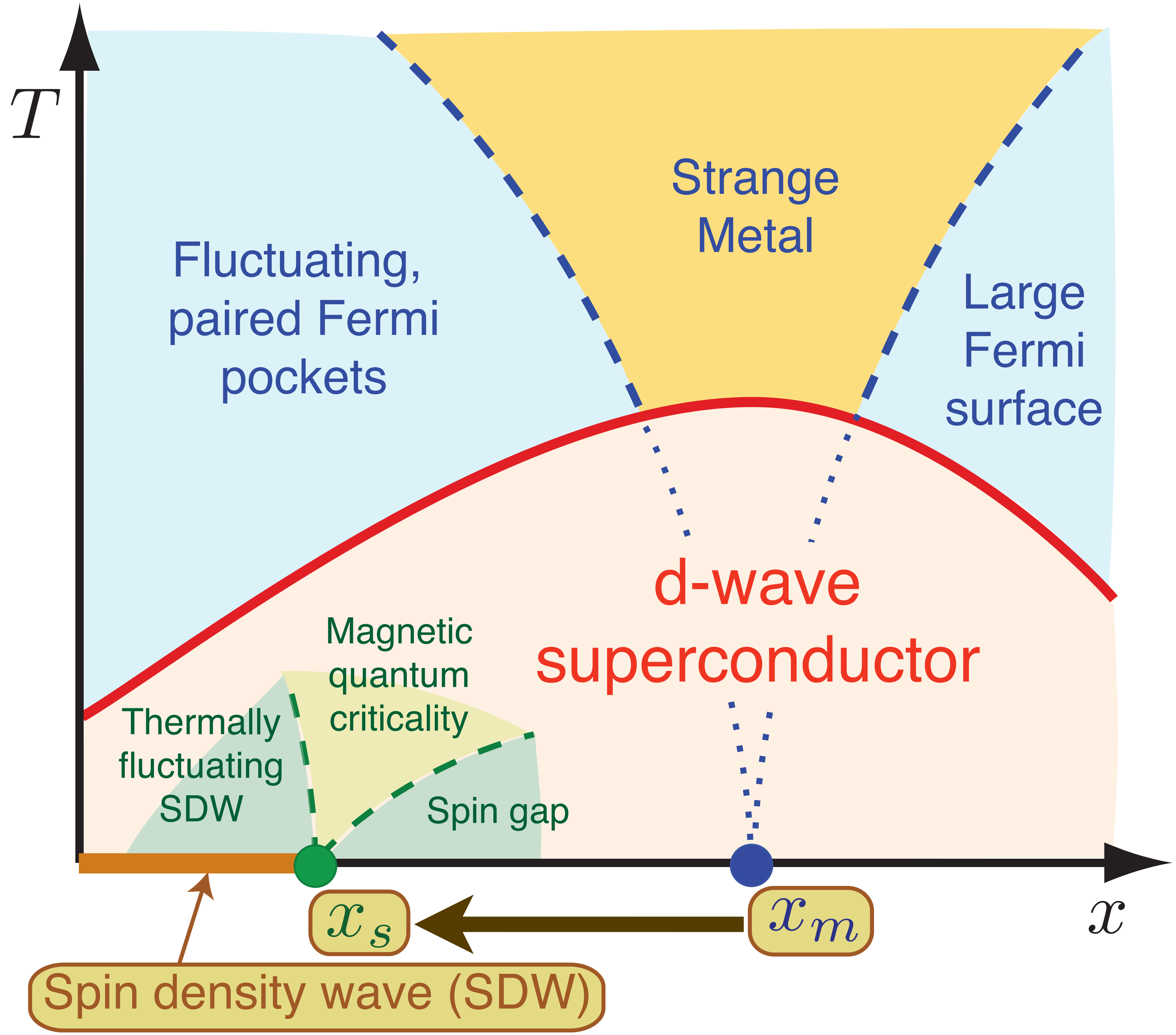

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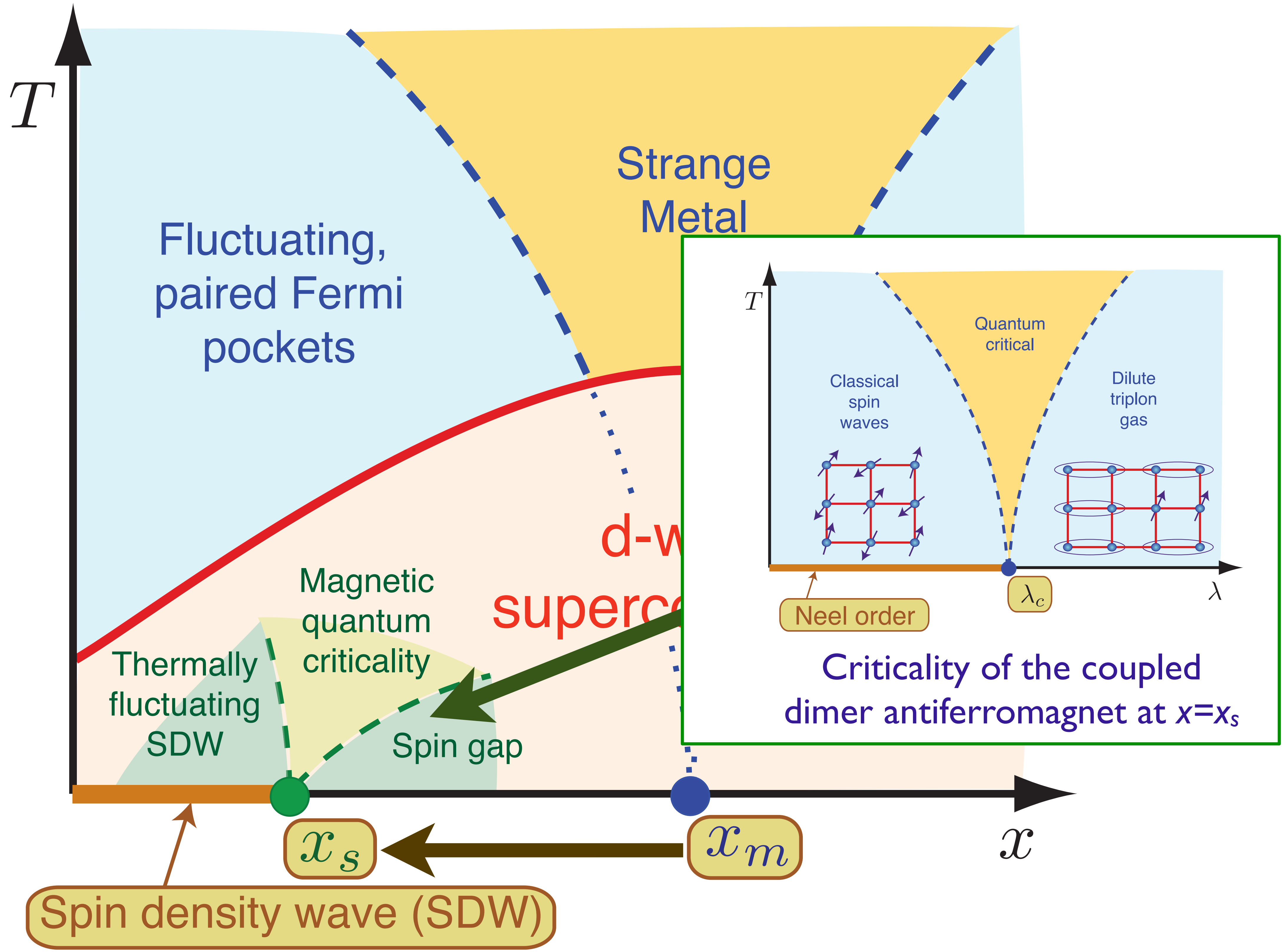
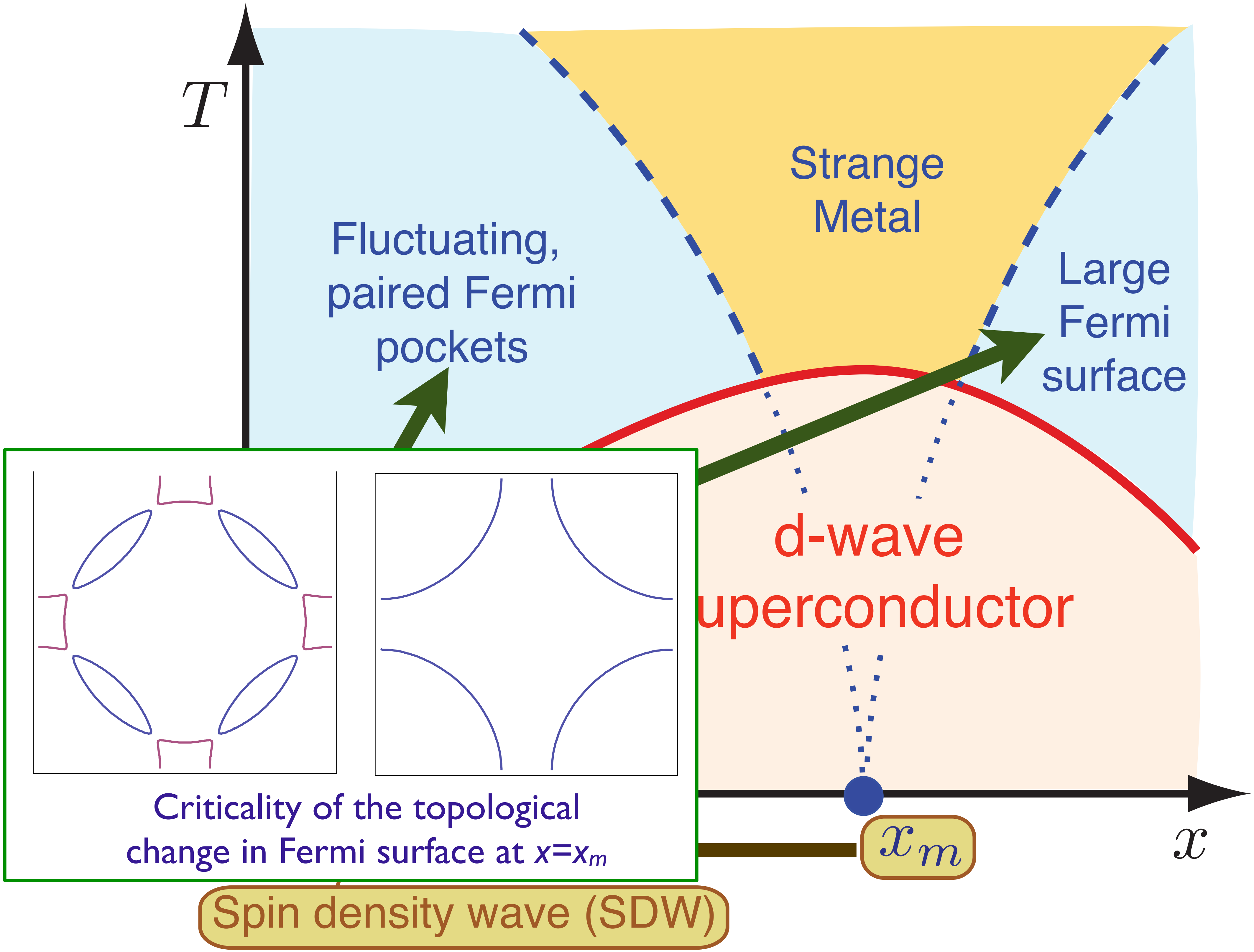


Criticality of the coupled dimer antiferromagnet at $x=x_s$

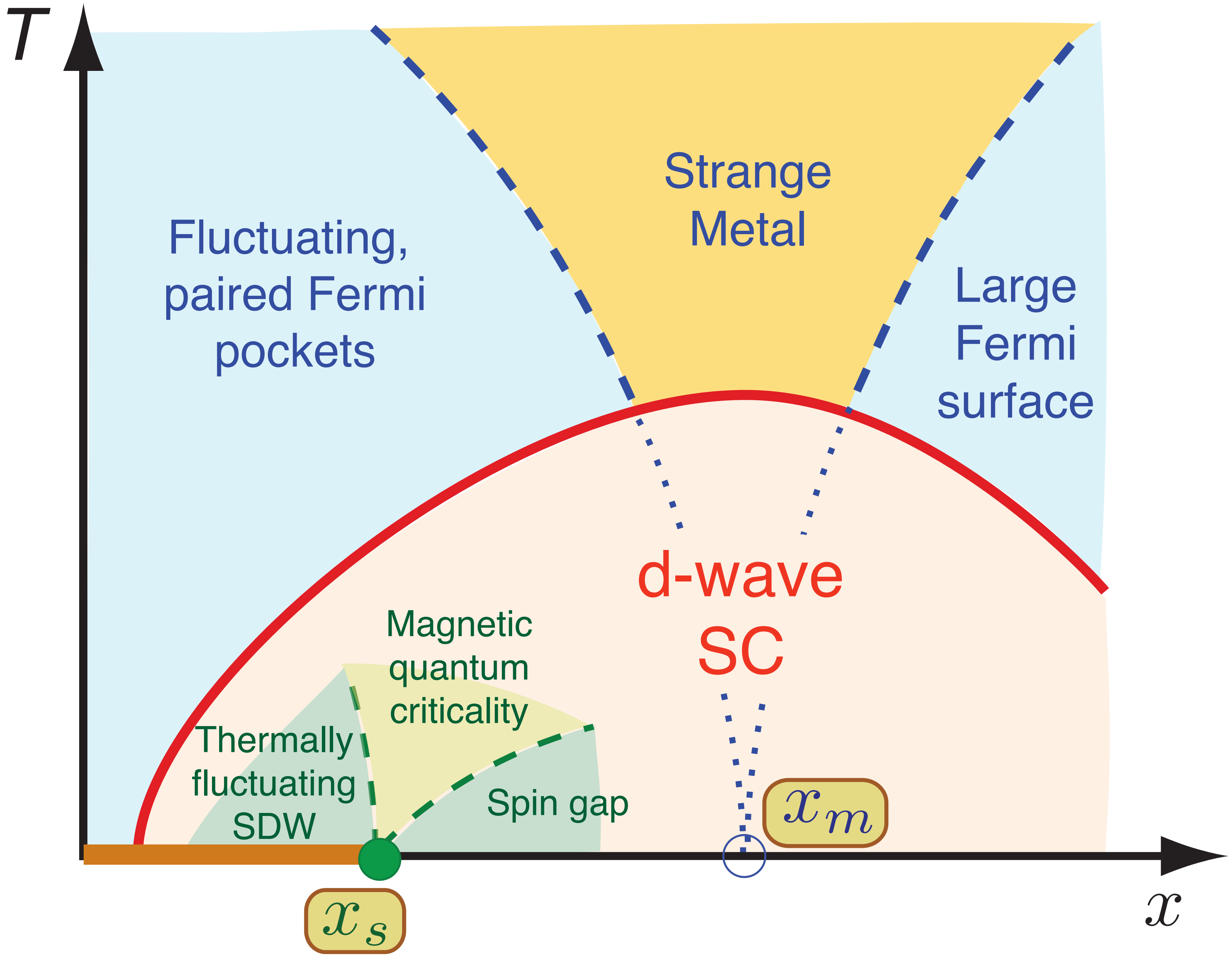
Competition between SDW order and superconductivity moves the actual quantum critical point to $x = x_s < x_m$.

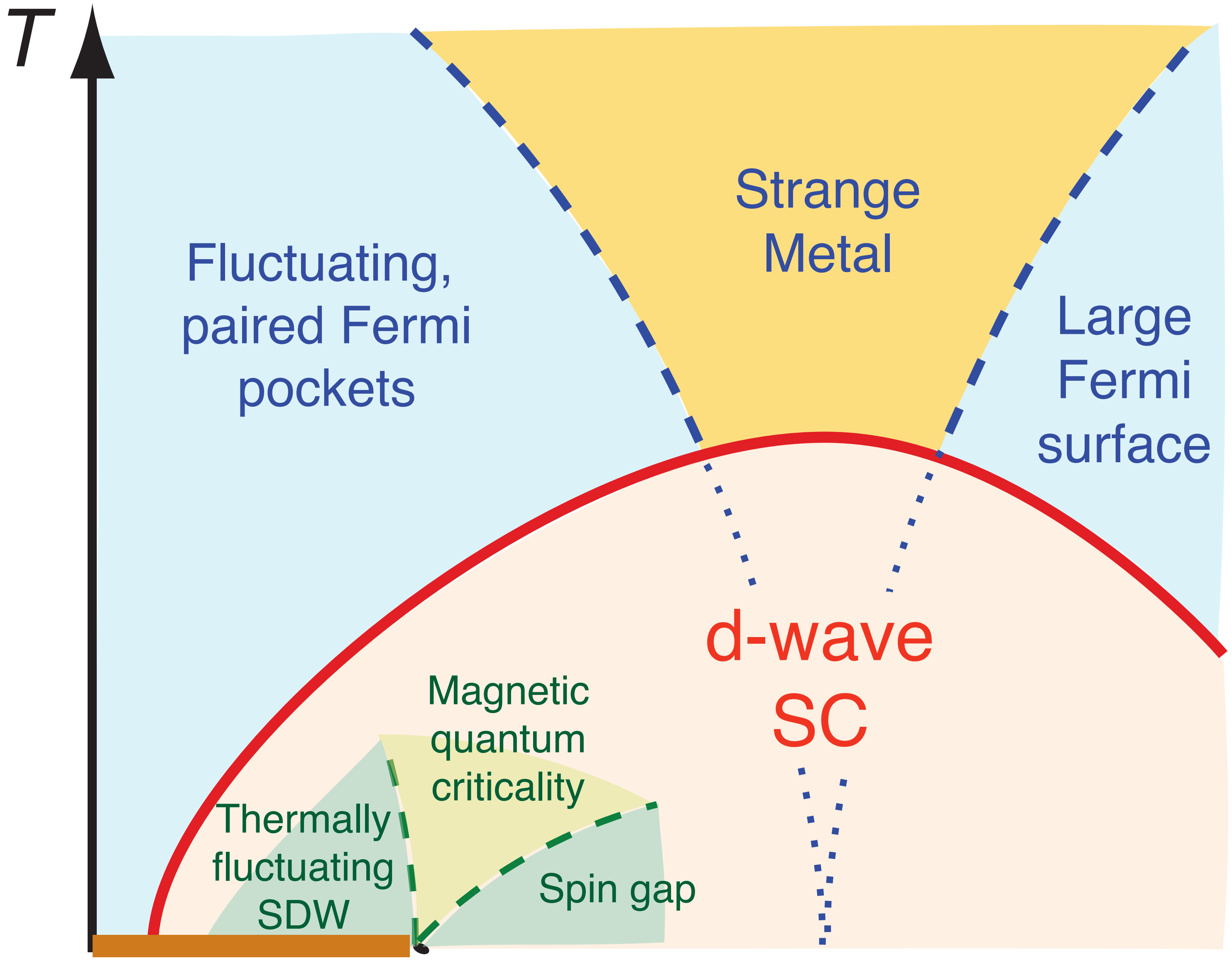
Theory of quantum criticality in the cuprates



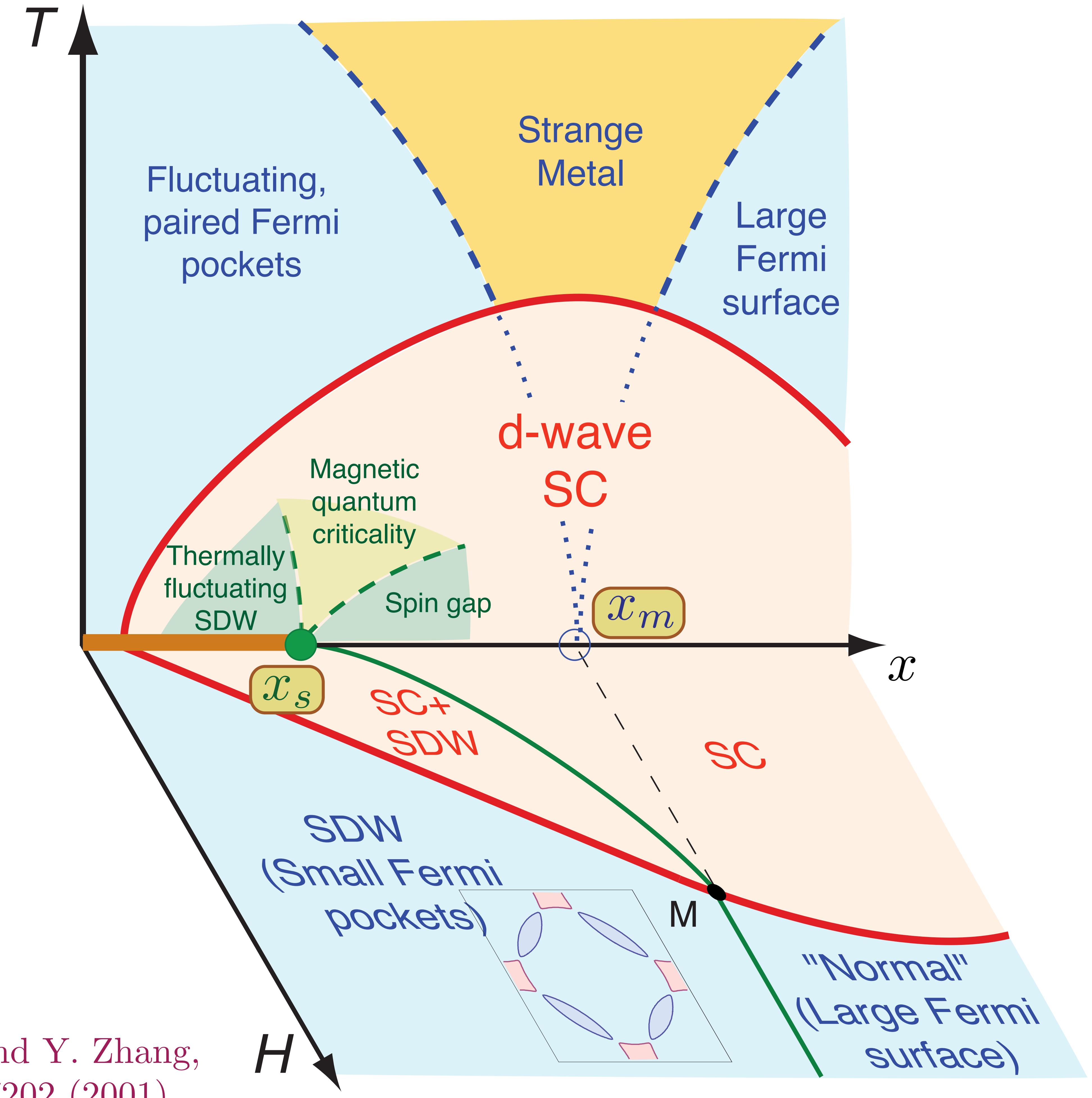
Criticality of the topological change in Fermi surface at $x = x_m$

Competition between SDW order and superconductivity moves the actual quantum critical point to $x = x_s < x_m$.


T
Fluctuating, paired Fermi pockets
Strange Metal
Large Fermi surface
d-wave SC
Magnetic quantum criticality
Thermally fluctuating SDW
Spin gap
x_m
x_s
x



E. Demler, S. Sachdev and Y. Zhang,
Phys. Rev. Lett. **87**, 067202 (2001).



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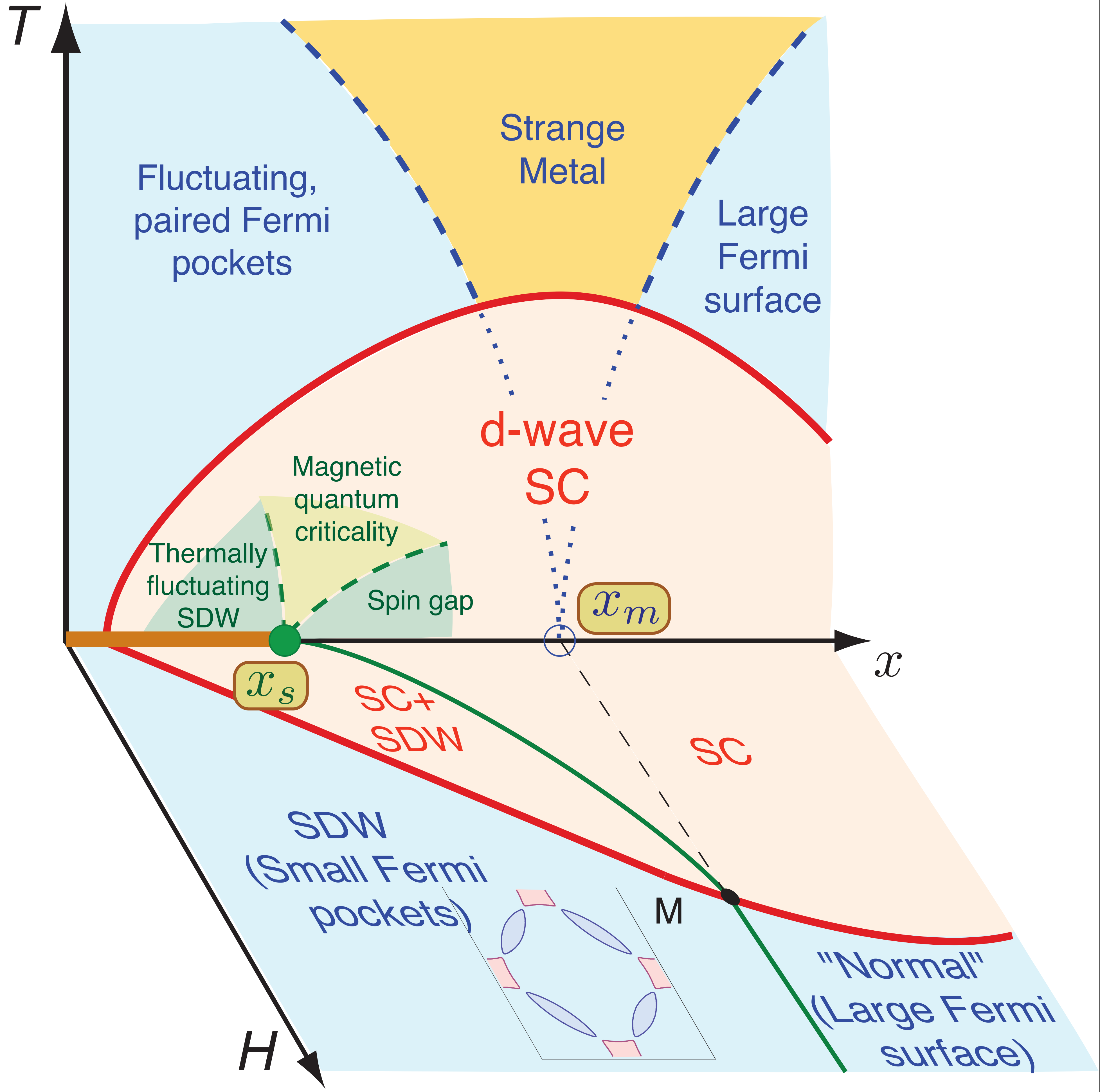
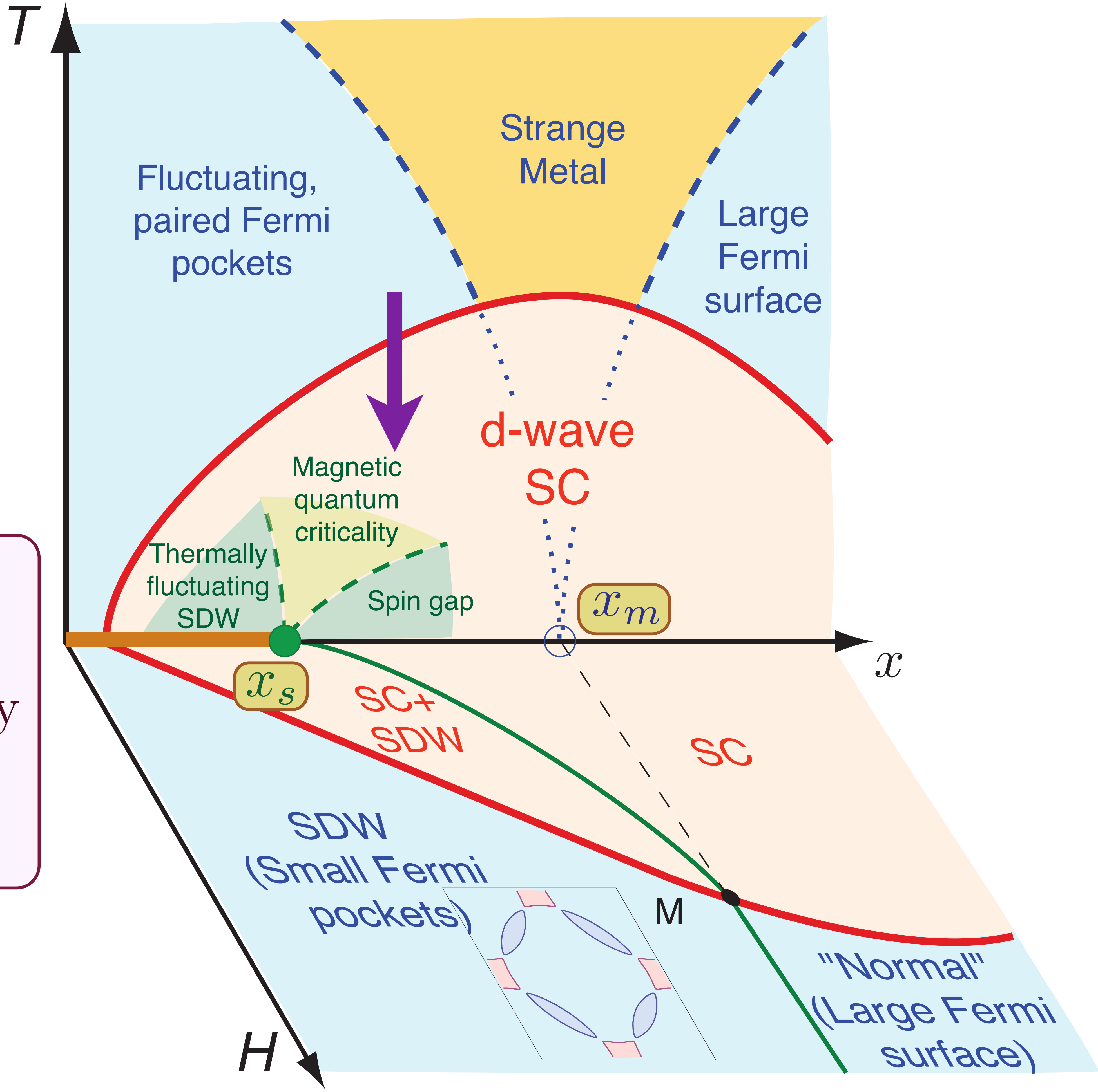
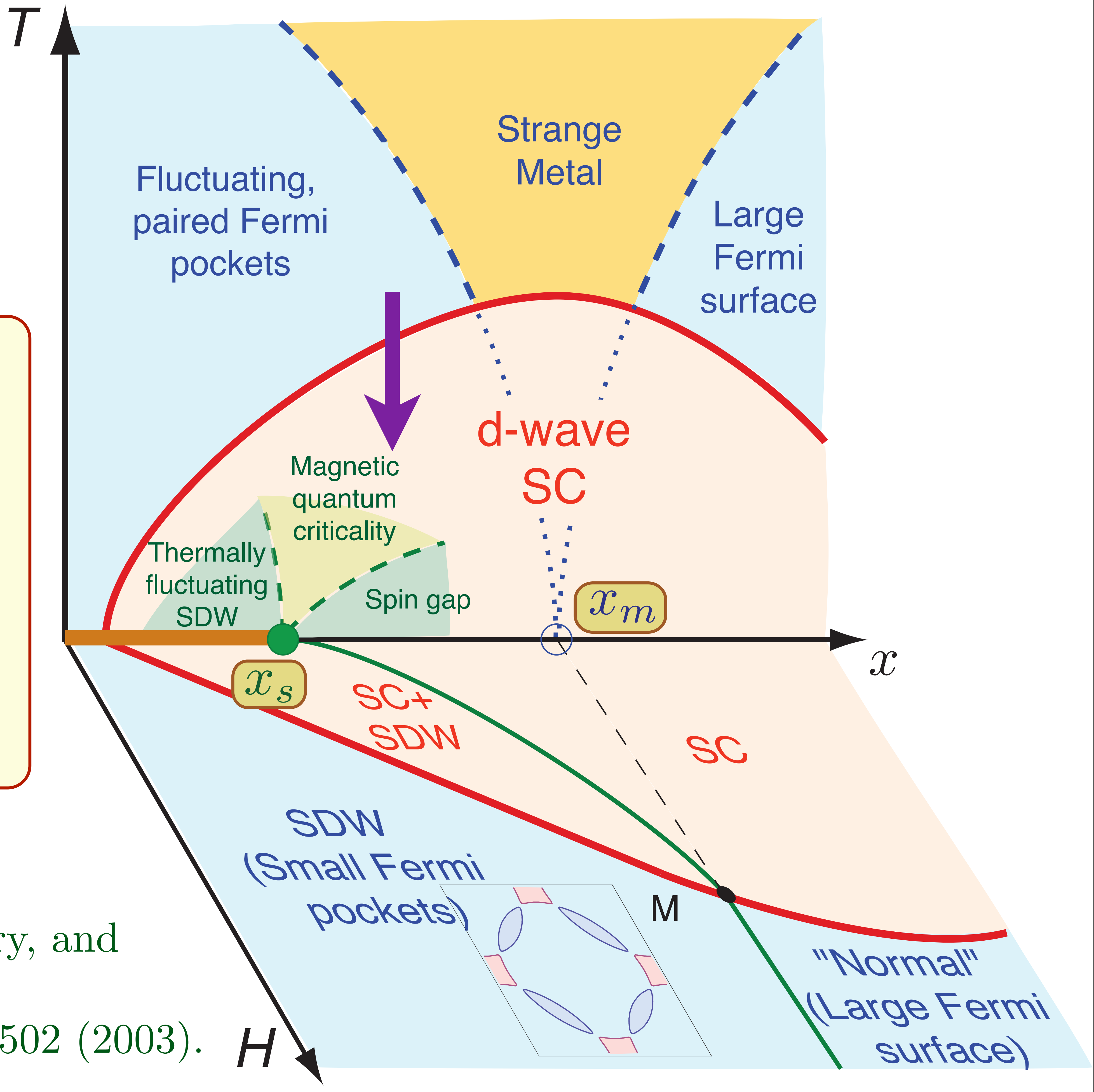

T
Strange
Metal
Fluctuating,
paired Fermi
pockets
Large
Fermi
surface
d-wave
SC
Magnetic
quantum
criticality
Thermally
fluctuating
SDW
Spin gap
x_m
x_s
x
SC+
SDW
SC
SDW
(Small Fermi
pockets)
M
"Normal"
(Large Fermi
surface)
H

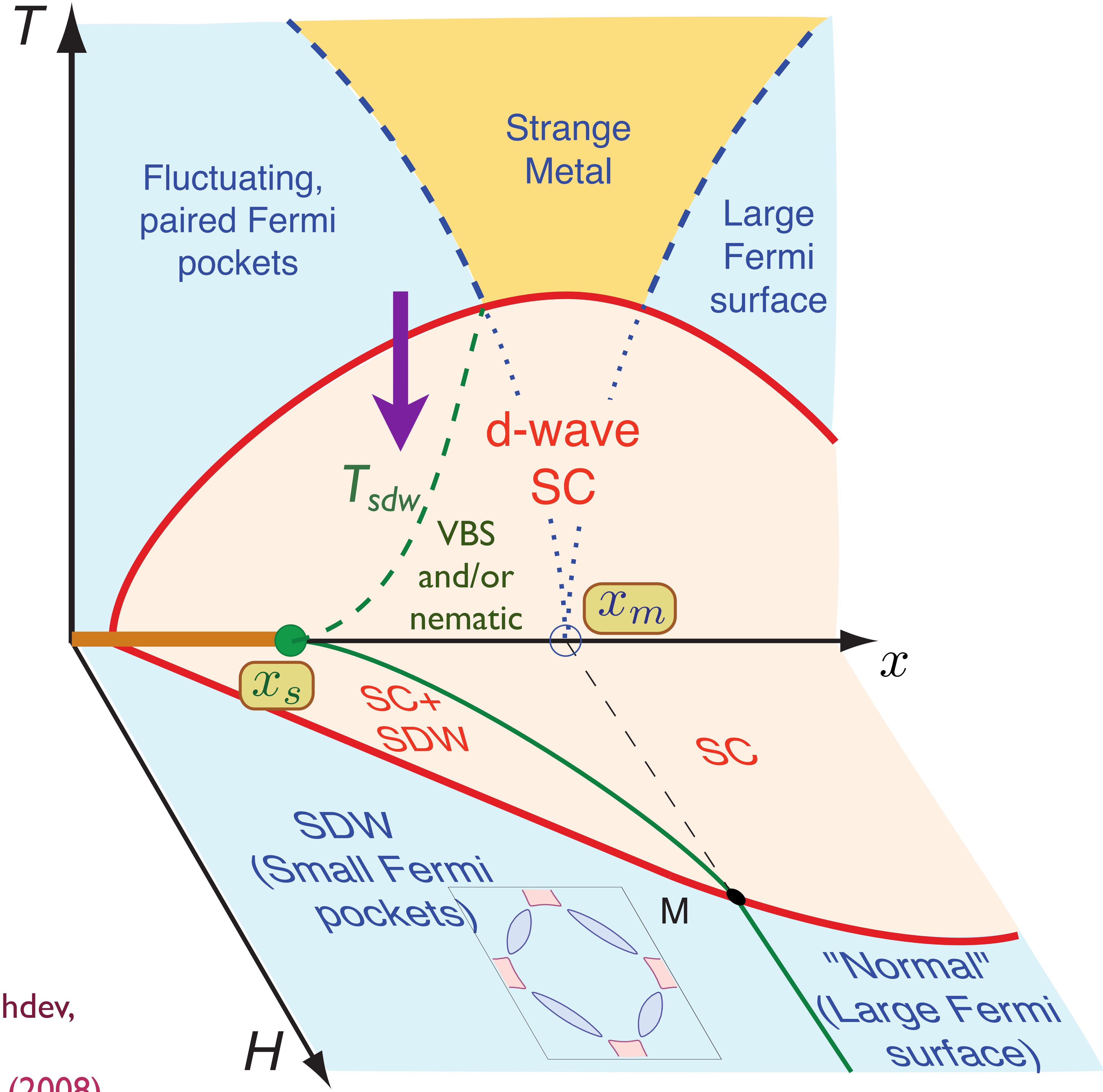
Theory of the onset of d-wave superconductivity from small Fermi pockets



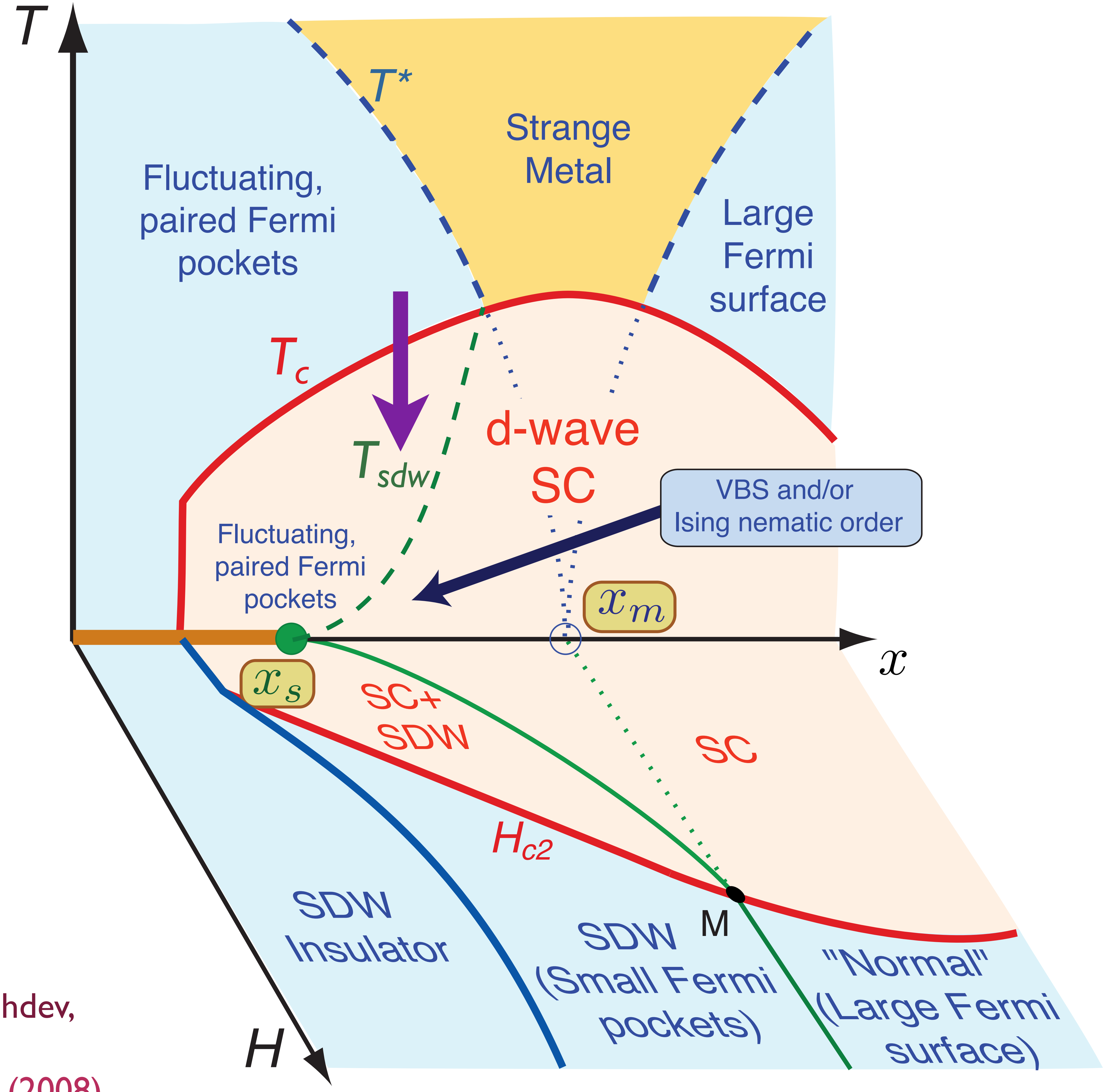


Physics of competition: *d*-wave SC and SDW "eat up" same pieces of the large Fermi surface.

B. Kyung, J.-S. Landry, and A.-M. S. Tremblay, Phys. Rev. B **68**, 174502 (2003).



R. K. Kaul, M. Metlitksi, S. Sachdev, and Cenke Xu, *Physical Review B* **78**, 045110 (2008).



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   Dominance of planar graphs

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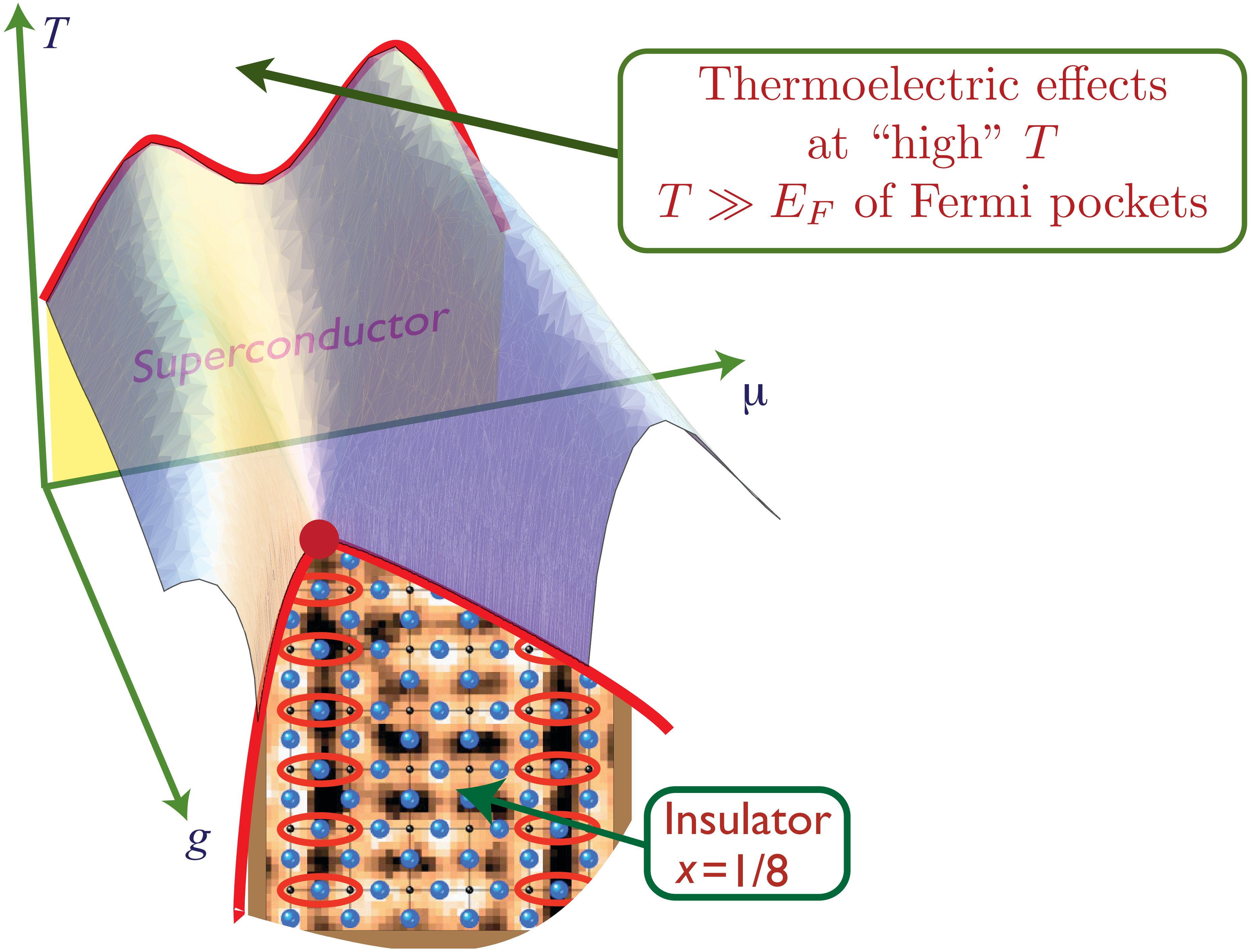
Hydrodynamic theory

- Assume proximity to a superfluid-insulator transition at $x = 1/8$



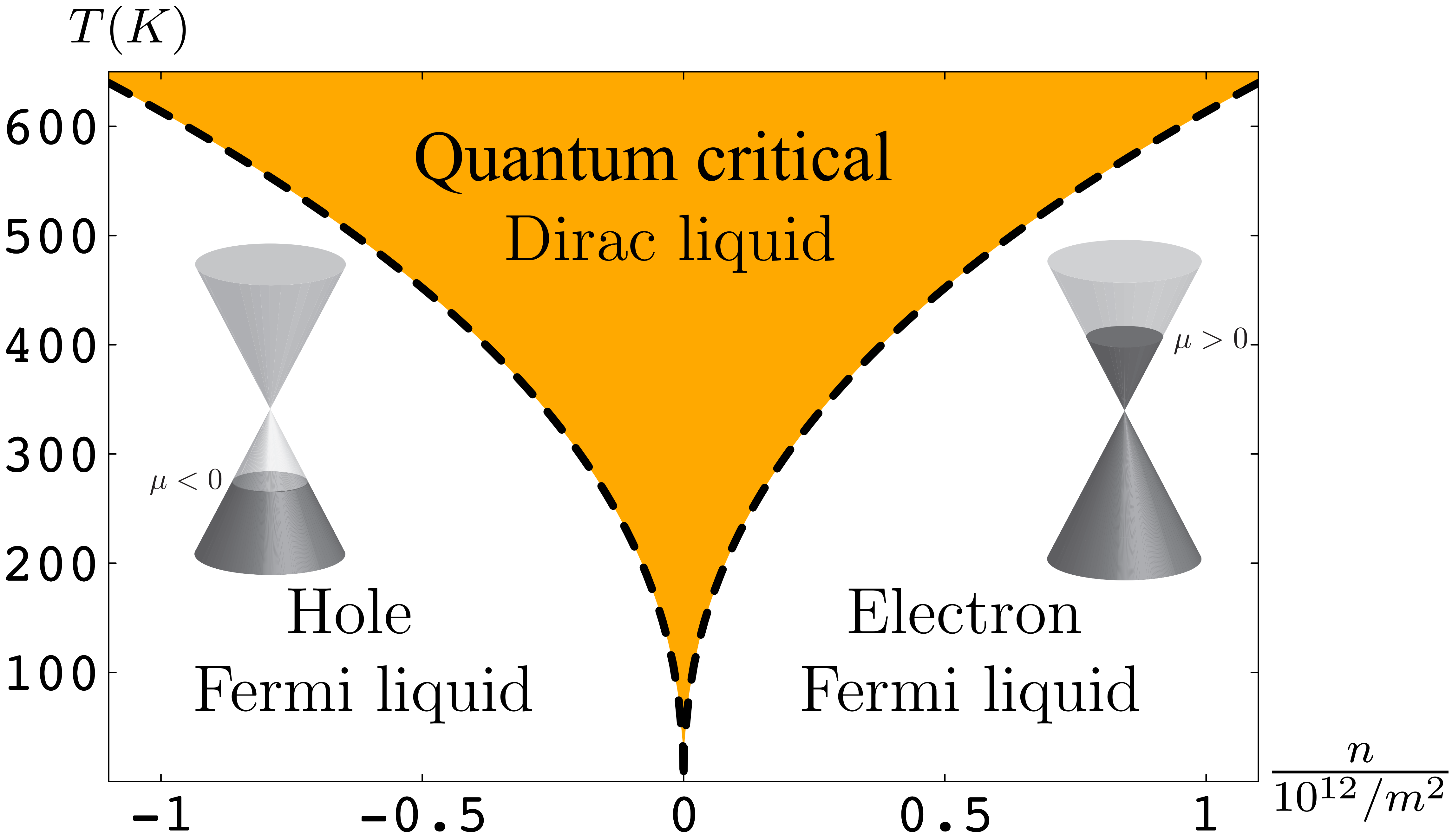
Thermoelectric effects
at “high” T
$T \gg E_F$ of Fermi pockets

Insulator
$x=1/8$

S.A. Hartnoll, P.K. Kovtun, M. Müller, and S. Sachdev, *Phys. Rev.* B **76** 144502 (2007)

Hydrodynamic theory

- Promising applications to graphene.



S.A. Hartnoll, P.K. Kovtun, M. Müller, and S. Sachdev, *Phys. Rev.* B **76** 144502 (2007)

Hydrodynamic theory

- Assume proximity to a superfluid-insulator transition at $x = 1/8$
- Perturb away from this quantum critical point with a small density ρ, small magnetic field B, and a small momentum relaxation rate $1/\tau_{\text{imp}}$ determined by impurities

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S.A. Hartnoll, P.K. Kovtun, M. Müller, and S. Sachdev, *Phys. Rev.* B **76** 144502 (2007)

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- Dominant scattering is due inelastic collisions, and dominant energy scale away from quantum critical point is $k_B T$.
- General hydrodynamic theory agrees precisely with results from a solvable supersymmetric theory which maps onto the classical Einstein-Maxwell theory of a dyonic black hole

S.A. Hartnoll, P.K. Kovtun, M. Müller, and S. Sachdev, *Phys. Rev.* B **76** 144502 (2007)

Hydrodynamic cyclotron resonance at a frequency

$$\omega_c = \frac{e^* B \rho v^2}{c(\varepsilon + P)}$$

and with width

$$\gamma = \sigma_Q \frac{B^2 v^2}{c^2(\varepsilon + P)}$$

where ε = energy density, P = pressure, v = velocity of "light" at quantum critical point, and $\sigma_Q e^2/h$ is the universal conductivity of the quantum critical point

S.A. Hartnoll, P.K. Kovtun, M. Müller, and S. Sachdev, *Phys. Rev.* B **76** 144502 (2007)

"Wiedemann-Franz"-like relation for thermal conductivity, κ at $B = 0$

$$\kappa = \sigma_Q \left(\frac{k_B^2 T}{e^{*2}} \right) \left(\frac{\varepsilon + P}{k_B T \rho} \right)^2 .$$

At $B \neq 0$ and $\rho = 0$ we have a "Wiedemann-Franz" relation for "vortices"

$$\kappa = \frac{1}{\sigma_Q} k_B^2 T \left(\frac{v(\varepsilon + P)}{k_B T B} \right)^2 .$$

S.A. Hartnoll, P.K. Kovtun, M. Müller, and S. Sachdev, *Phys. Rev.* B **76** 144502 (2007)

Nernst signal (transverse thermoelectric response)

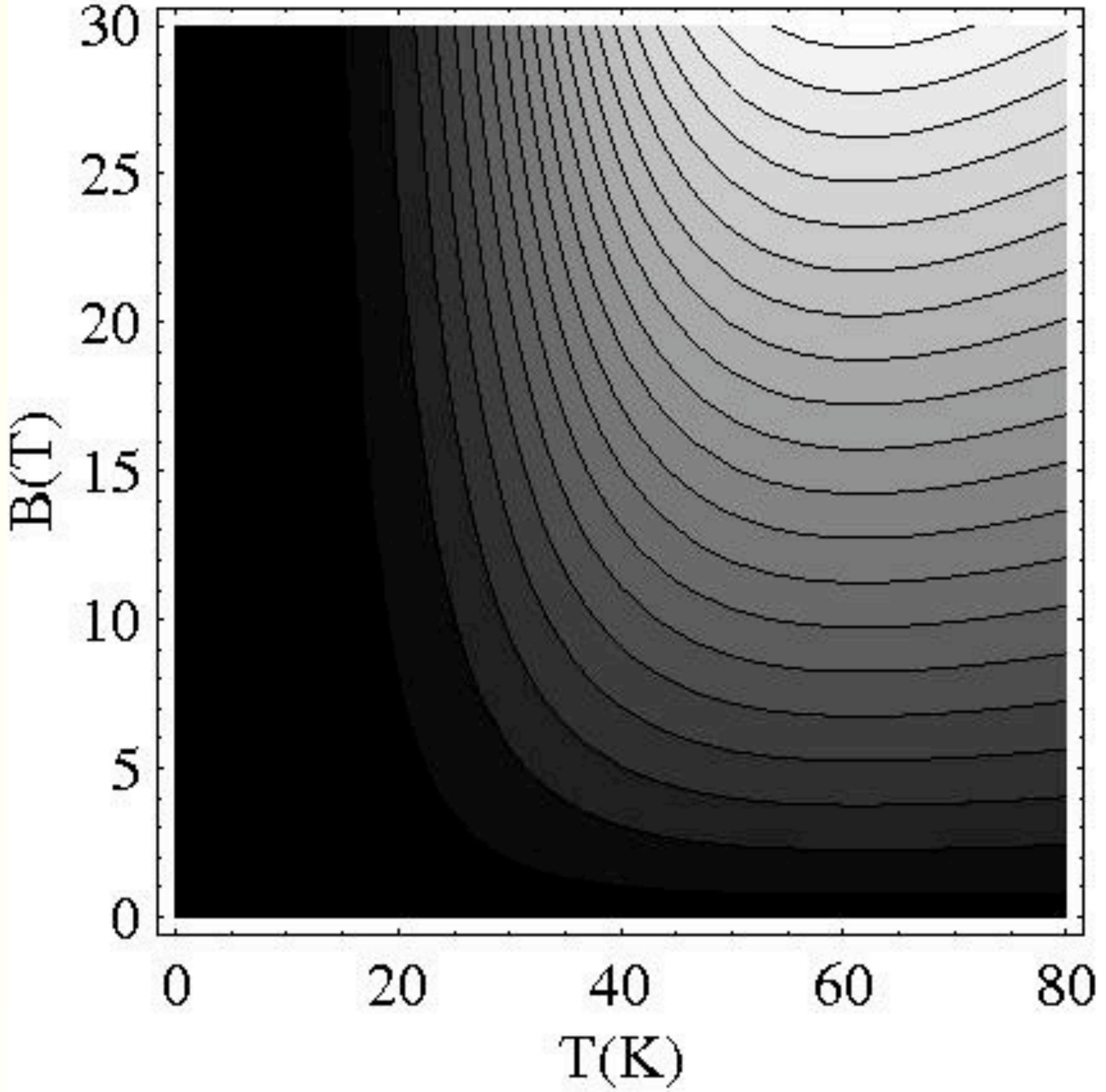
$$e_N = \left(\frac{k_B}{e^*}\right)\left(\frac{\varepsilon + P}{k_B T \rho}\right)\left[\frac{\omega_c/\tau_{\text{imp}}}{(\omega_c^2/\gamma + 1/\tau_{\text{imp}})^2 + \omega_c^2}\right]$$

where τ_{imp} is the momentum relaxation time due to impurities or umklapp scattering.

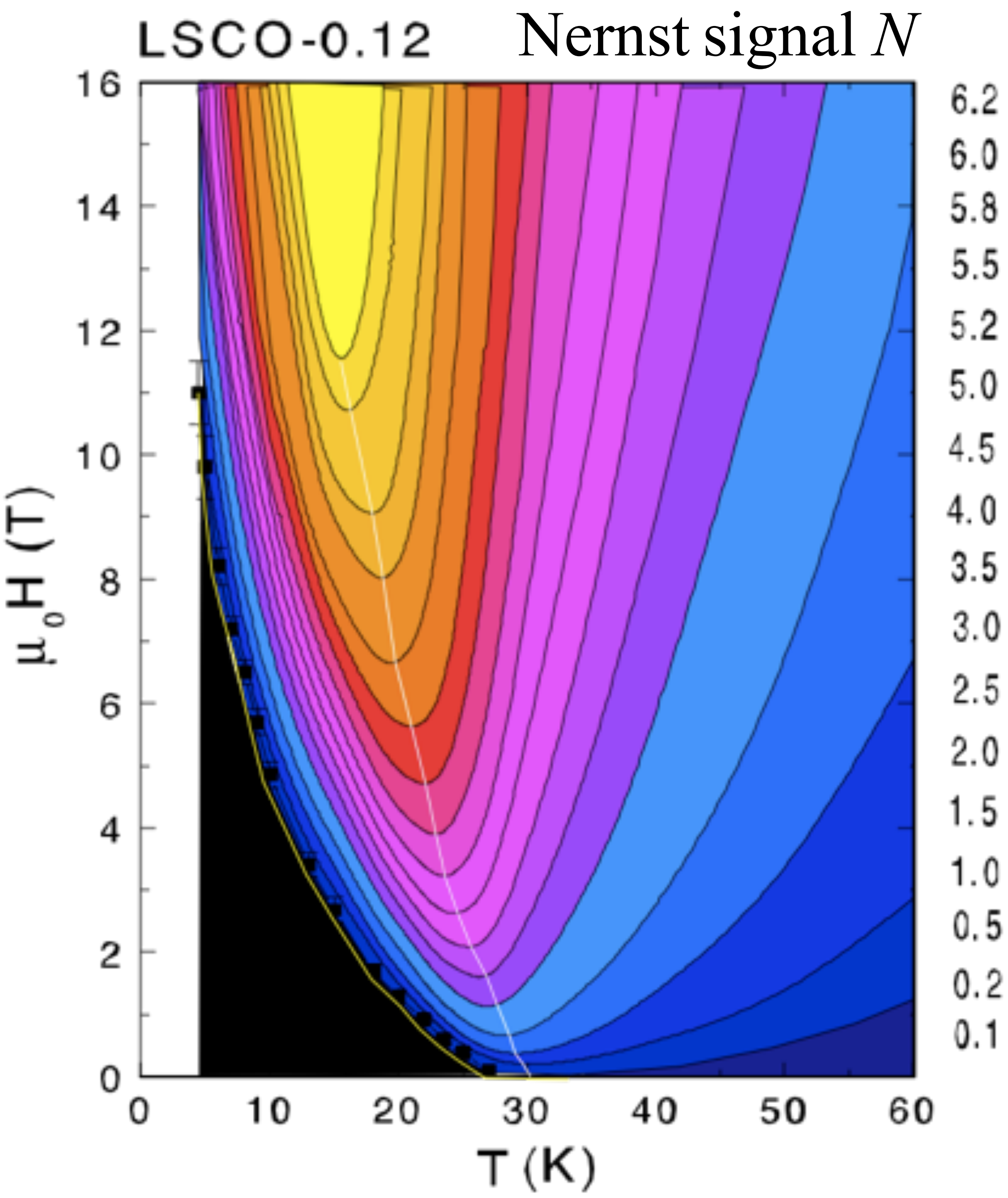
S.A. Hartnoll, P.K. Kovtun, M. Müller, and S. Sachdev, *Phys. Rev.* B **76** 144502 (2007)

LSCO Experiments

Theory N



Nernst signal N



B and T dependencies are in semi-quantitative agreement with observations on cuprates, with reasonable values for only 2 adjustable parameters, τ_{imp} and v.

Y. Wang, L. Li, and N. P. Ong, *Phys. Rev.* B **73**, 024510 (2006).
S.A. Hartnoll, P.K. Kovtun, M. Müller, and S. Sachdev, *Phys. Rev.* B **76** 144502 (2007)

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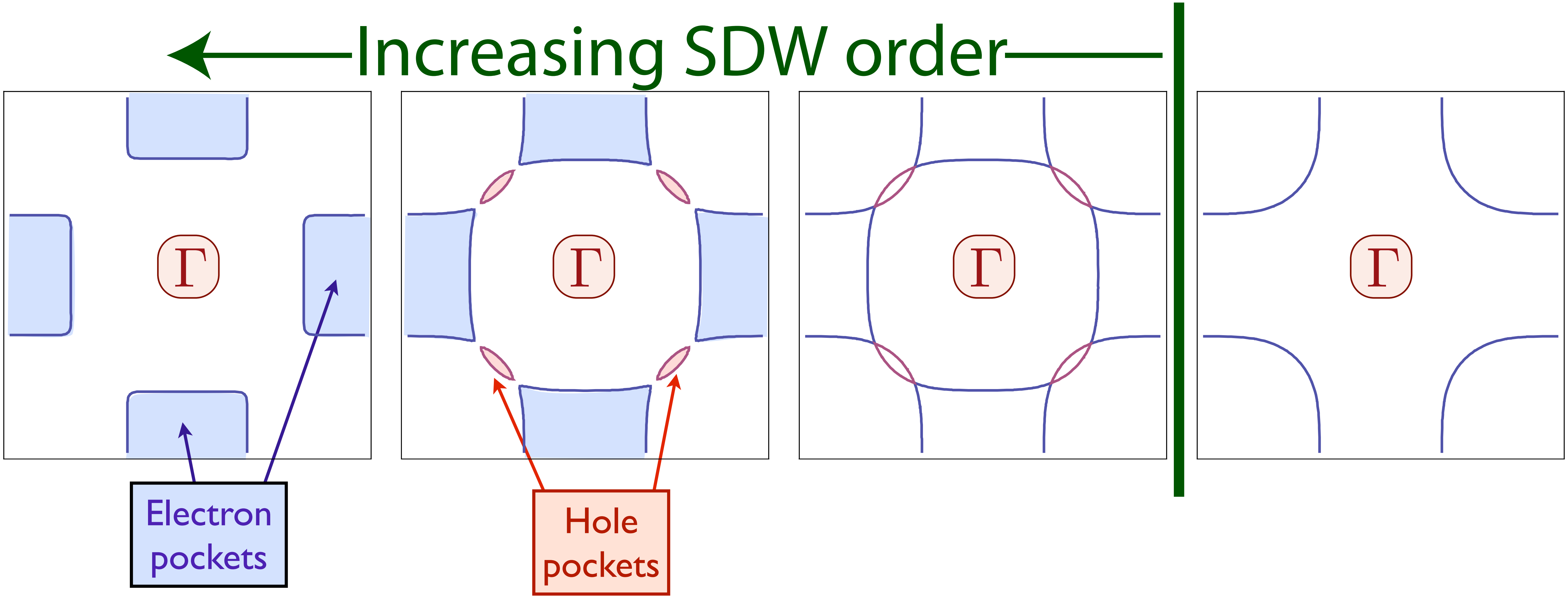
Nernst effect in electron-doped cuprates



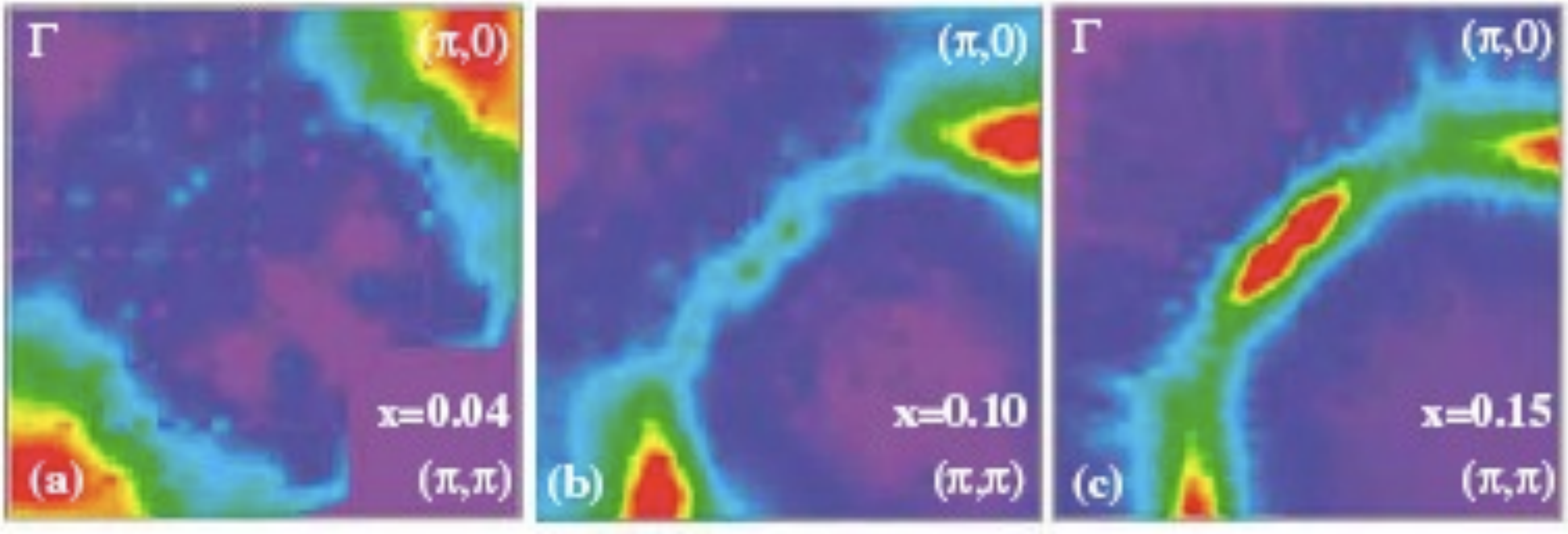
Increasing SDW order

Electron pockets

Hole pockets

Andreas Hackl and S. Sachdev, *Phys. Rev. B* **79**, 235124 (2009).

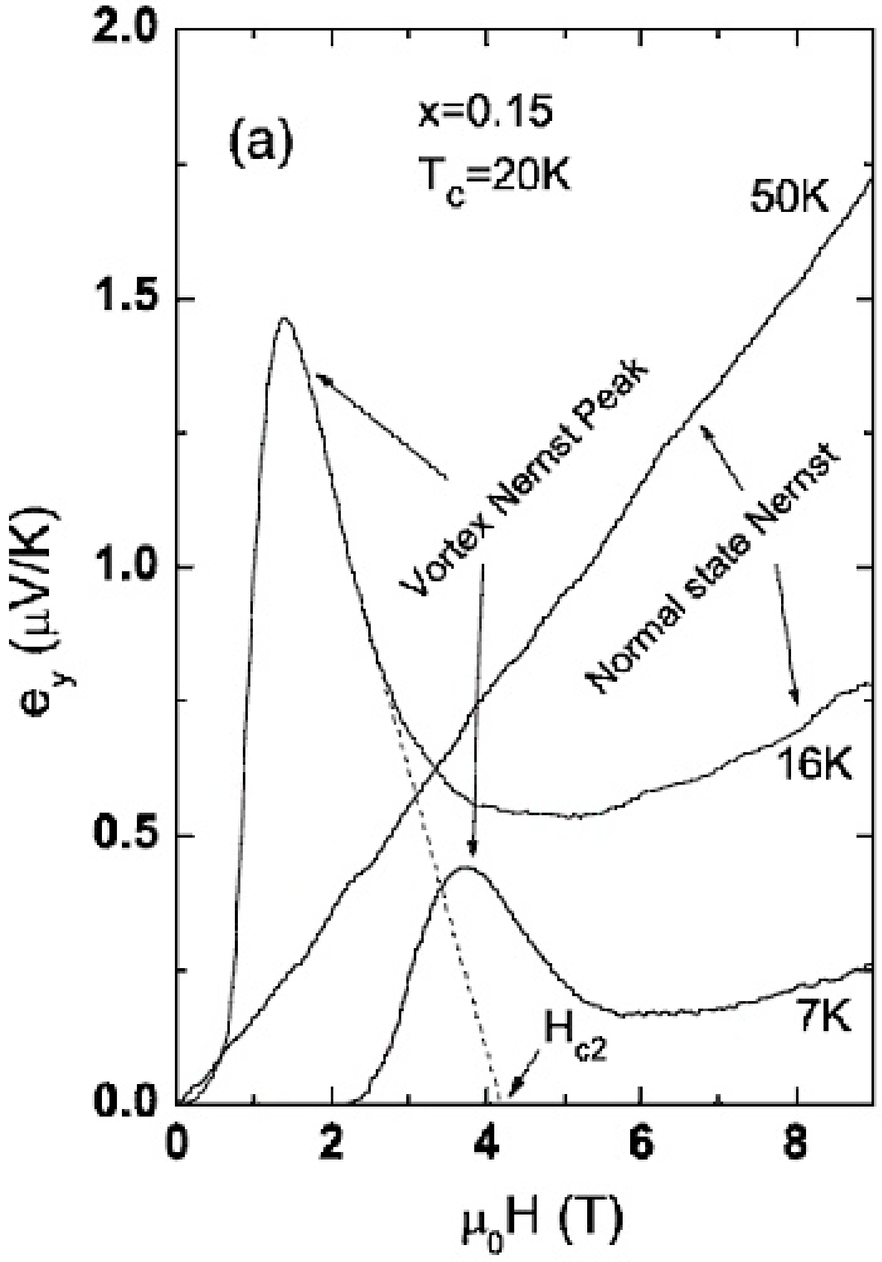
Photoemission in NCCO



N. P. Armitage *et al.*, Phys. Rev. Lett. **88**, 257001 (2002).

Normal state Nernst effect in PCCO

P. Li and R.L. Greene PRB 76, 174512 (2007)



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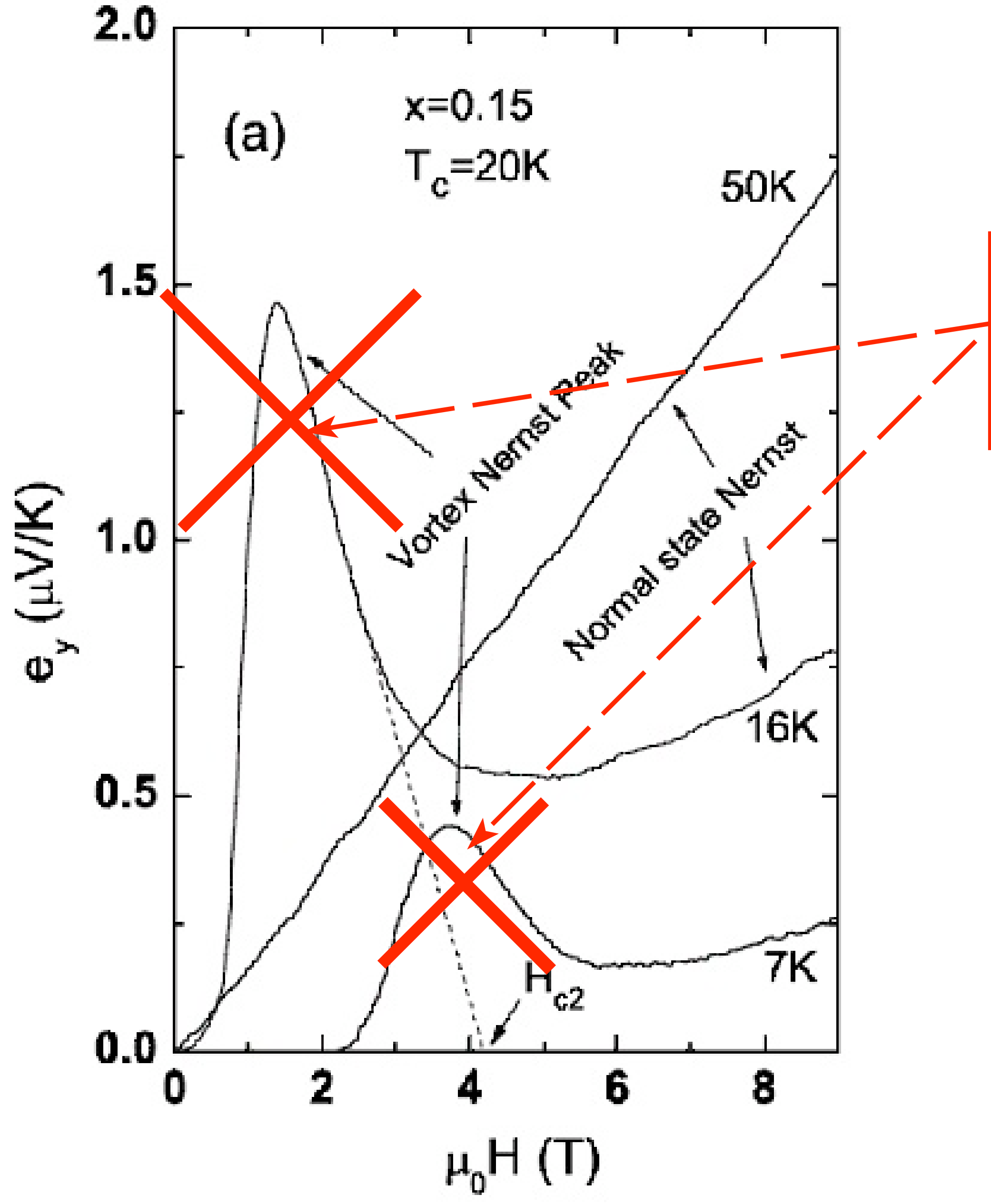
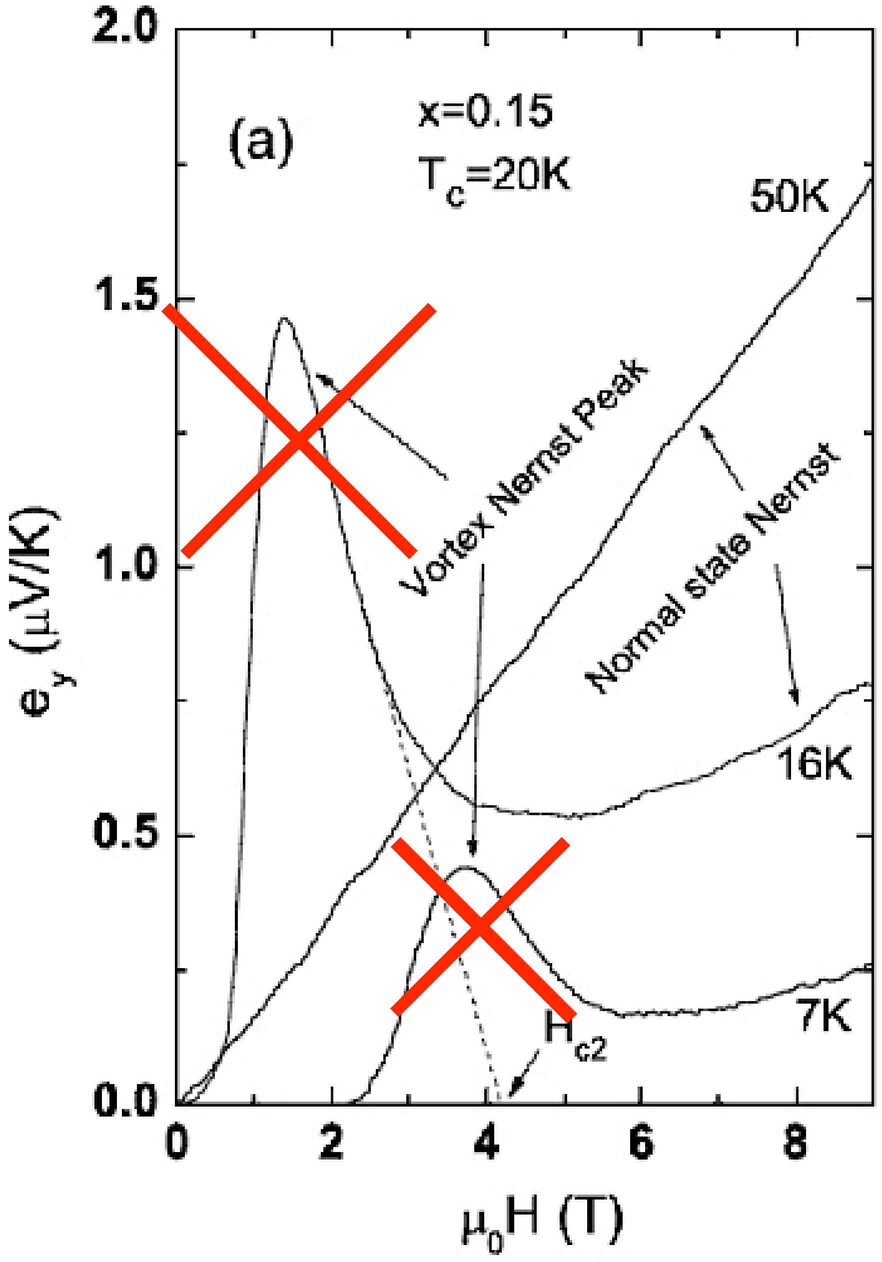
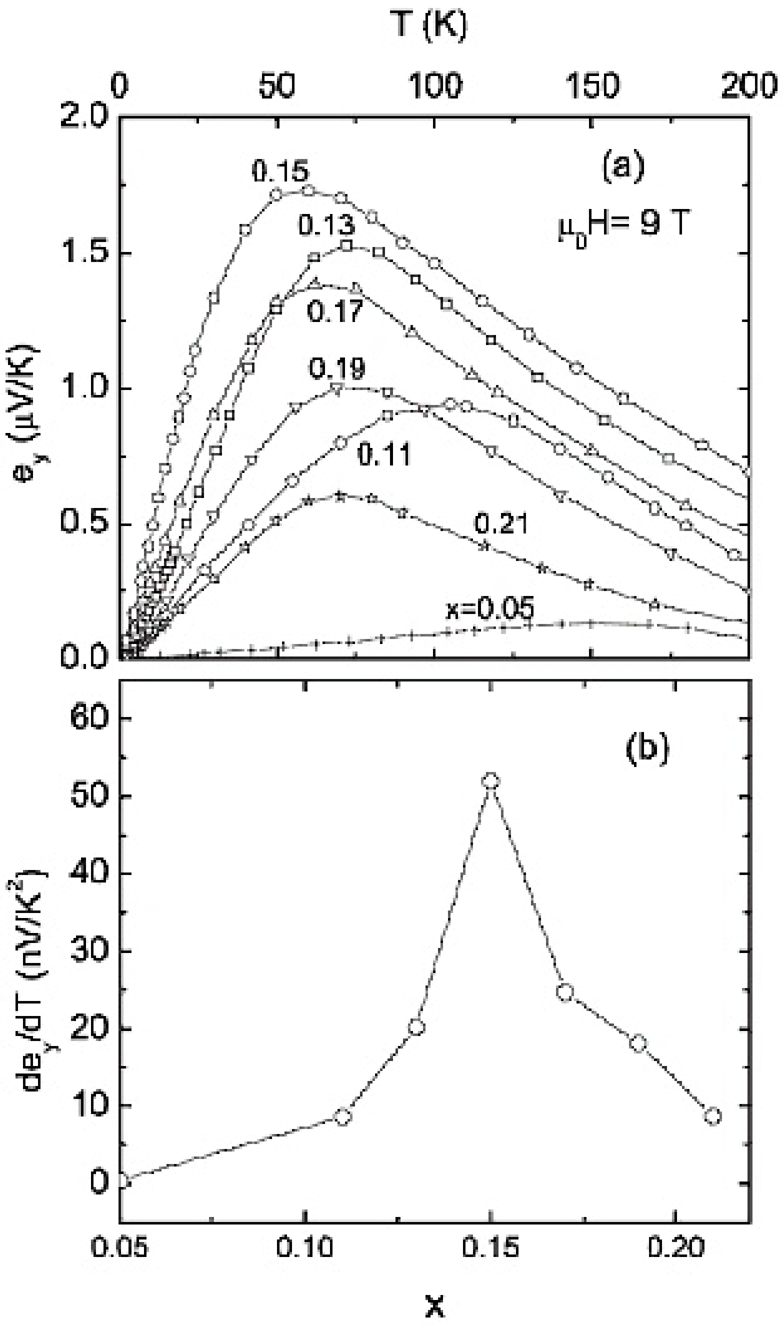


Neglect vortex Nernst effect (considering B=9T)

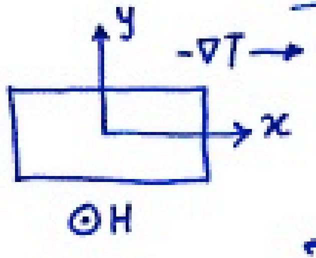
Normal state Nernst effect in PCCO

P. Li and R.L. Greene PRB 76, 174512 (2007)





Nernst effect of charge carriers



Peltier tensor

$$\vec{J} = \overleftrightarrow{\sigma}\cdot\vec{E} + \overleftrightarrow{\alpha}\cdot(-\vec{\nabla}T).$$

$$\overleftrightarrow{\sigma} = \begin{bmatrix}\sigma & \sigma_{xy}\\ \sigma_{yx} & \sigma\end{bmatrix}, \quad \overleftrightarrow{\alpha} = \begin{bmatrix}\alpha & \alpha_{xy}\\ \alpha_{yx} & \alpha\end{bmatrix}.$$

b.c. $J_x = 0 \quad \rightarrow \quad E_x = -S(-\partial_x T).$

Thermopower $\boxed{S = \frac{\alpha}{\sigma}}$

b.c. $0 = J_y = \sigma_{yx}E_x + \sigma E_y + \alpha_{yx}(-\partial_x T).$

opposite in sign
& equal in size
if $\tau \neq \tau(\varepsilon)$!

Nernst
signal

Nernst coef. $\nu_N = \frac{E_y}{|\nabla T|}\frac{1}{H}.$

$$\boxed{\nu_N = -S\frac{\tan\theta}{H} + \frac{\alpha_{xy}}{\sigma H}}$$

Hall angle term

Peltier Hall term

Transport coefficients

$$\alpha_{xx} = \frac{2e}{T} \sum_{\mathbf{k},\alpha=\pm} \frac{\partial f^0_{\mathbf{k}}}{\partial E^\alpha_{\mathbf{k}}} (E^\alpha_{\mathbf{k}} - \mu) \tau_{\mathbf{k}} (v^x_{\mathbf{k}})^2$$

$$\alpha_{xy} = \frac{2e^2 B}{T\hbar c} \sum_{\mathbf{k},\pm} \frac{\partial f^0_{\mathbf{k}}}{\partial E^\alpha_{\mathbf{k}}} (E^\alpha_{\mathbf{k}} - \mu) \tau^2_{\mathbf{k}} v^x_{\mathbf{k}} \left[v^y_{\mathbf{k}} \frac{\partial v^y_{\mathbf{k}}}{\partial k_x} - v^x_{\mathbf{k}} \frac{\partial v^y_{\mathbf{k}}}{\partial k_y} \right]$$

$$\sigma_{xx} = -2e^2 \sum_{\mathbf{k},\pm} \frac{\partial f^0_{\mathbf{k}}}{\partial E^\alpha_{\mathbf{k}}} \tau_{\mathbf{k}} (v^x_{\mathbf{k}})^2$$

$$\sigma_{xy} = -2\frac{e^3 B}{\hbar c} \sum_{\mathbf{k},\pm} \frac{\partial f^0_{\mathbf{k}}}{\partial E^\alpha_{\mathbf{k}}} \tau^2_{\mathbf{k}} v^x_{\mathbf{k}} \left[v^y_{\mathbf{k}} \frac{\partial v^y_{\mathbf{k}}}{\partial k_x} - v^x_{\mathbf{k}} \frac{\partial v^y_{\mathbf{k}}}{\partial k_y} \right]$$

Nernst signal: $\vec{E} = -\hat{\vartheta} \nabla T$ (no charge current, *B* field || z)

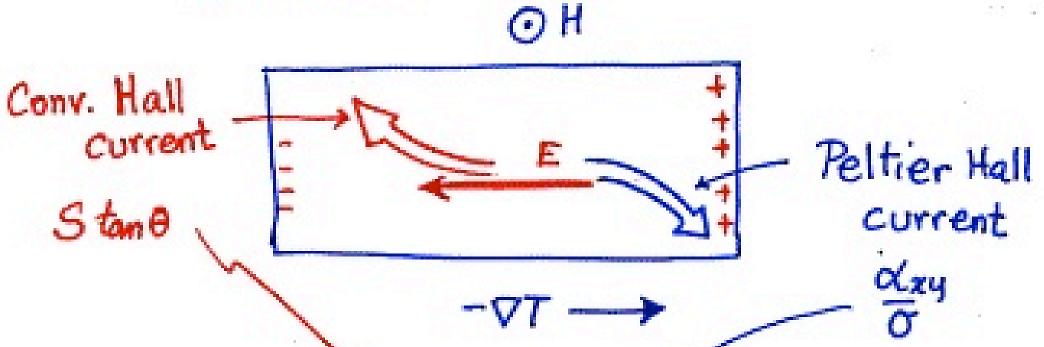
$$\vartheta_{yx} = -\frac{\sigma_{xx}\alpha_{yx} - \sigma_{yx}\alpha_{xx}}{\sigma_{xx}\sigma_{yy} - \sigma_{xy}\sigma_{yx}}$$

Nernst coefficient: $\nu = \vartheta_{yx}/B$ (~*T* at low *T*)

SONDHEIMER CANCELLATION

Why 2 "large" terms cancel

For holes



$$0 = J_y = [-S\sigma_{xy} + \alpha_{yx}](-\partial_x T) + \underset{\text{residual Nernst signal}}{\sigma E_y}$$

Why 2 large terms are equal if $\tau \neq \tau(\epsilon)$

Boltzmann theory

$$\alpha_{xy} = \frac{2e^2 B}{T\hbar}\sum_k \left(-\frac{\partial f}{\partial \varepsilon}\right)(\varepsilon_k-\mu)\,\ell_y \frac{\partial \ell_x}{\partial s}$$

$$\simeq \frac{k_B^2 T}{e}\left(\frac{\partial \sigma_{xy}}{\partial \varepsilon}\right)_\mu$$

Analogous to $\alpha \sim \frac{k_B^2 T}{e}\left(\frac{\partial \sigma}{\partial \varepsilon}\right)_\mu$

$\tau \neq \tau(\varepsilon)$ $\quad S\sigma_{xy} = \alpha_{yx}$ (Sondheimer) 1949

$$\hookrightarrow \left(\frac{\partial \ln\sigma}{\partial \varepsilon}\right)_\mu = \left(\frac{\partial \ln\sigma_{xy}}{\partial \varepsilon}\right)_\mu$$

Assumptions for magnetic field and disorder

- Relaxation time due to weak dilute disorder

$$\tau \sim 1/\rho_F$$

- Weak field regime, no Landau quantization and magnetic breakdown
- Neglect interference effect between scattering events

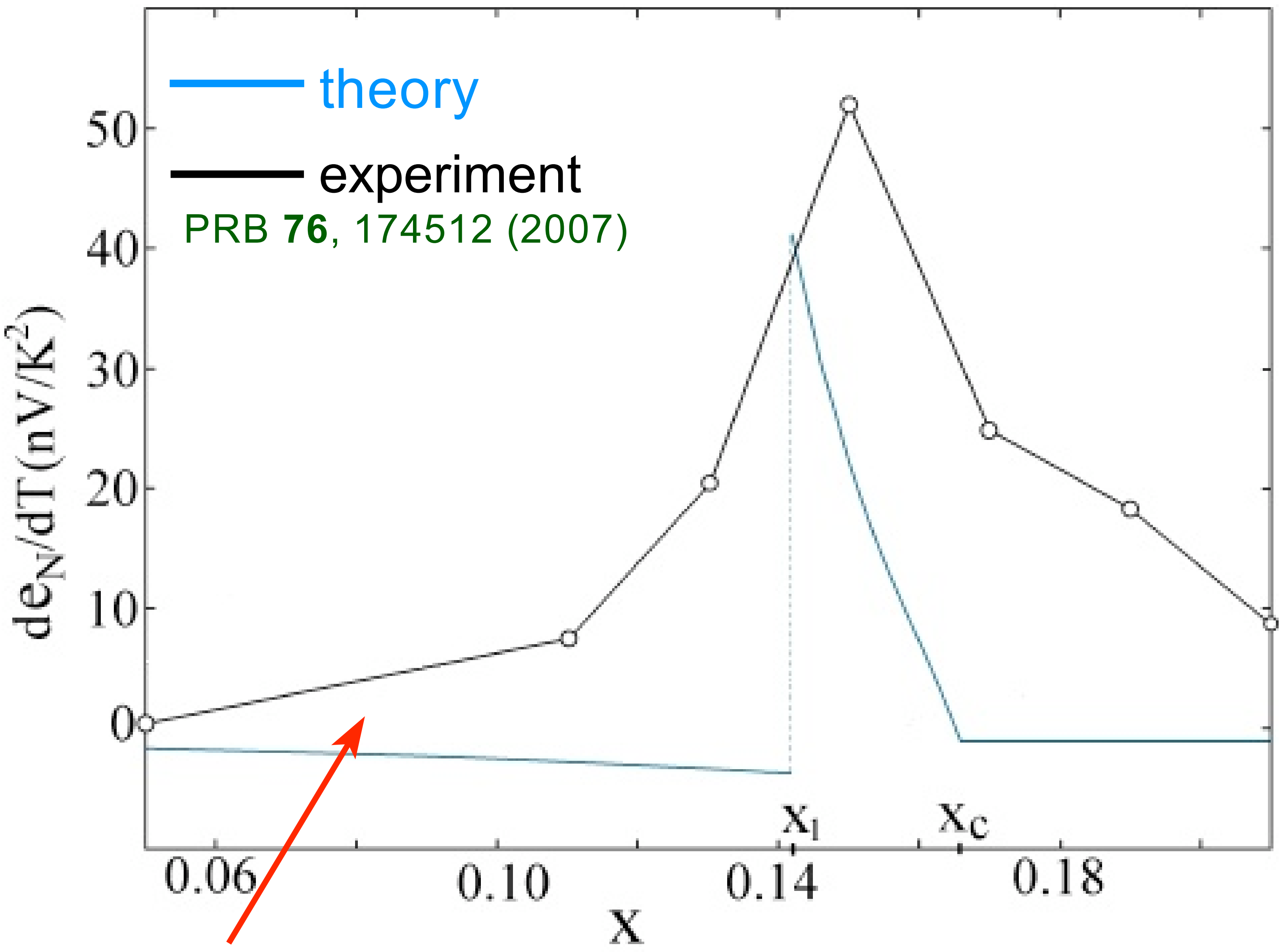
⇒ assume $l_S < l$ and $k_F l > 1$

scattering length on SDW order: $l_S \sim (1/\Delta)\, v_F$

Fit parameters

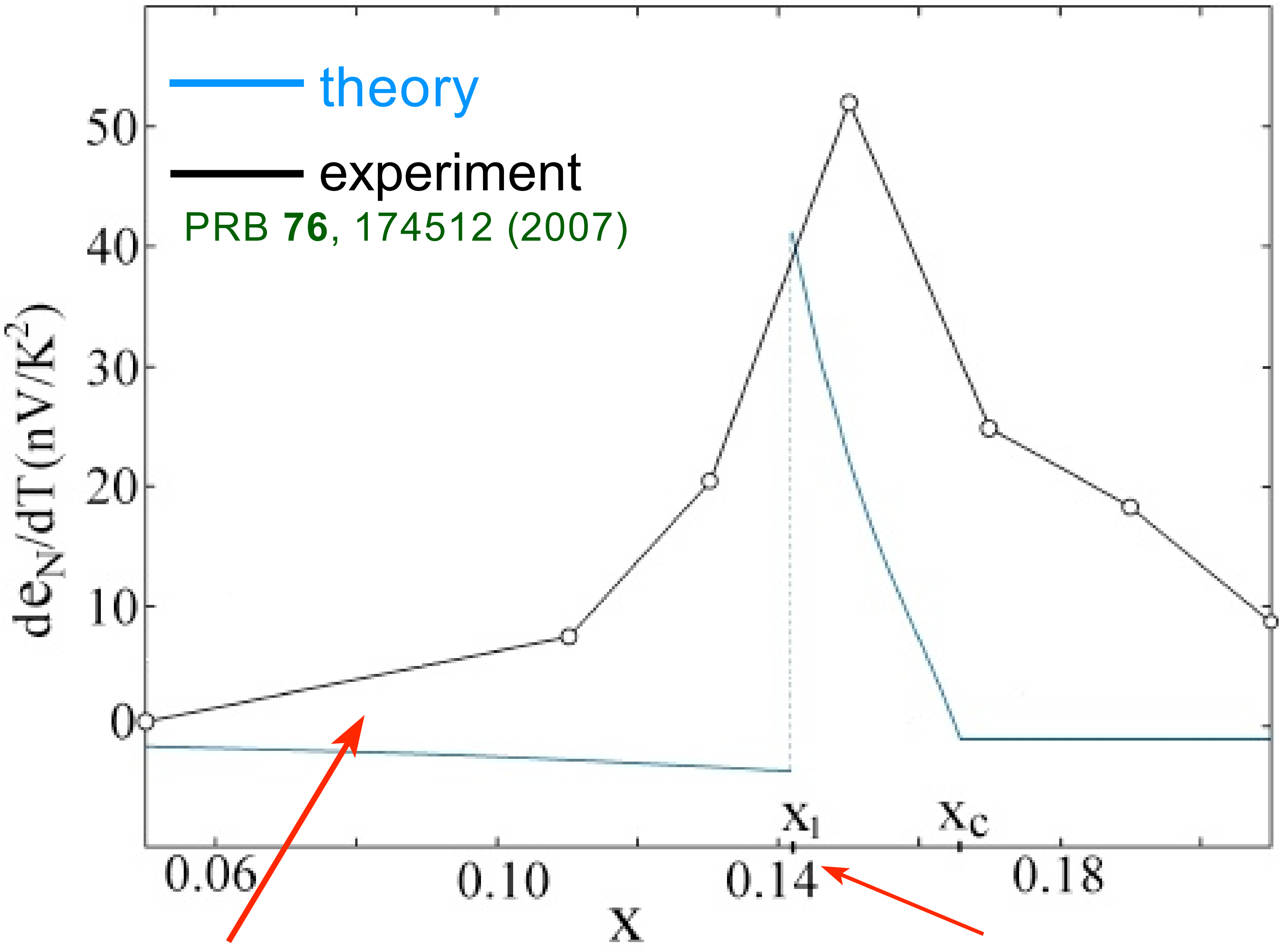
- Relaxation time $\tau = 3 \cdot 10^{-14}$ s
- order parameter $\Delta = 0.7[\mathrm{eV}](1-x/x_c)^{1/2}$

Comparison to experiment



Underdoped region, only electron-like carriers

Comparison to experiment

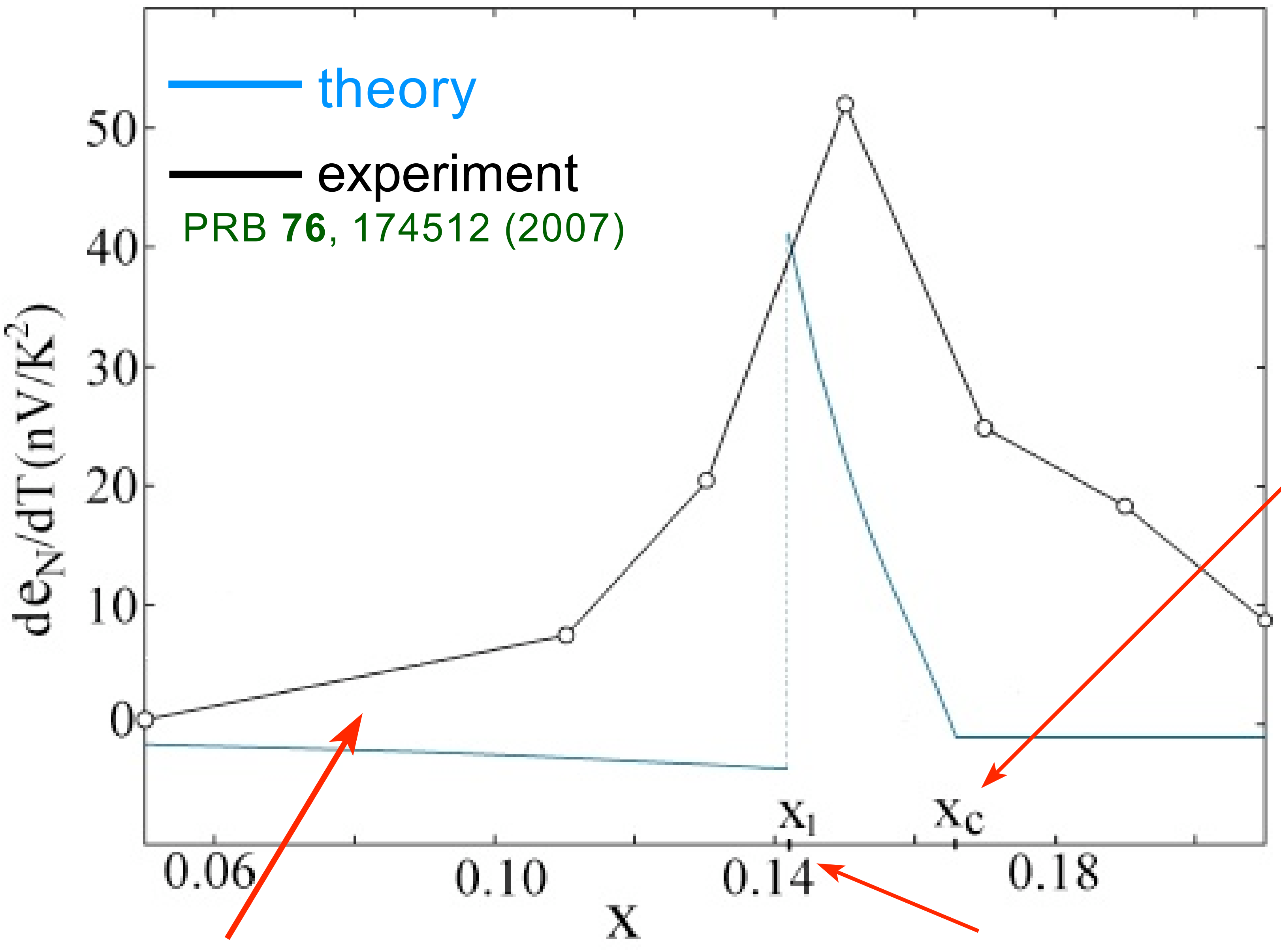


PRB **76**, 174512 (2007)

Underdoped region, only electron-like carriers

Opening of hole pocket, singularity in Nernst signal

Comparison to experiment



PRB **76**, 174512 (2007)

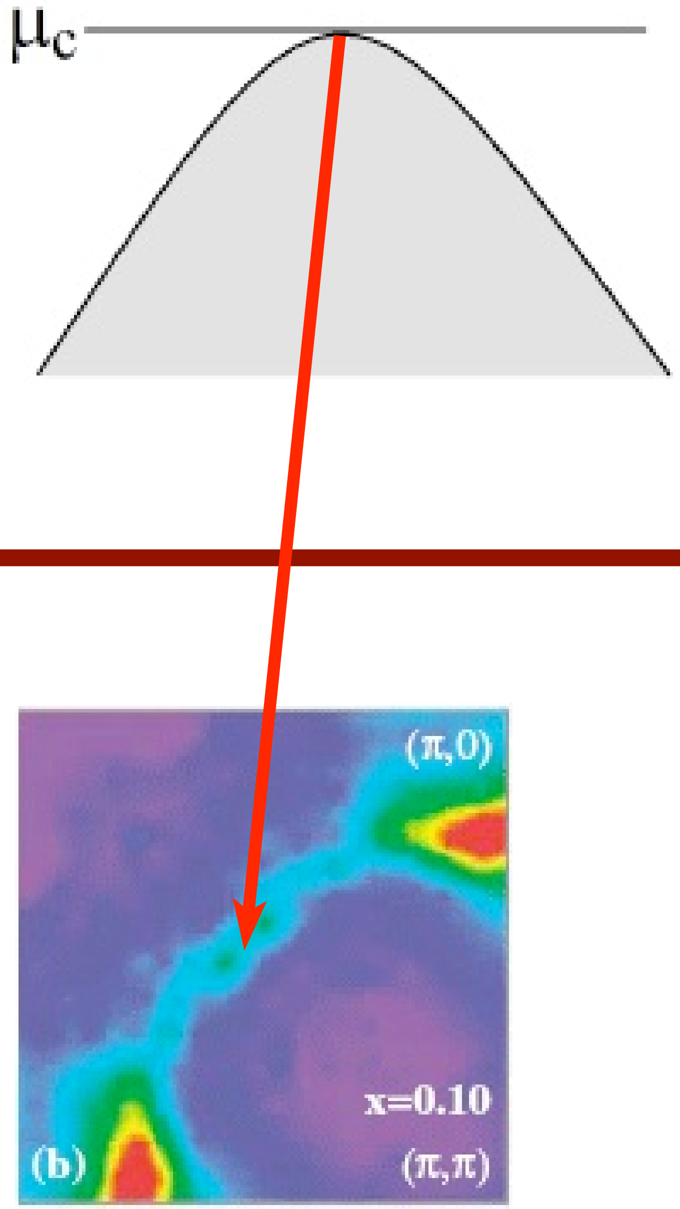
SDW quantum critical point, large hole like Fermi surface at larger doping

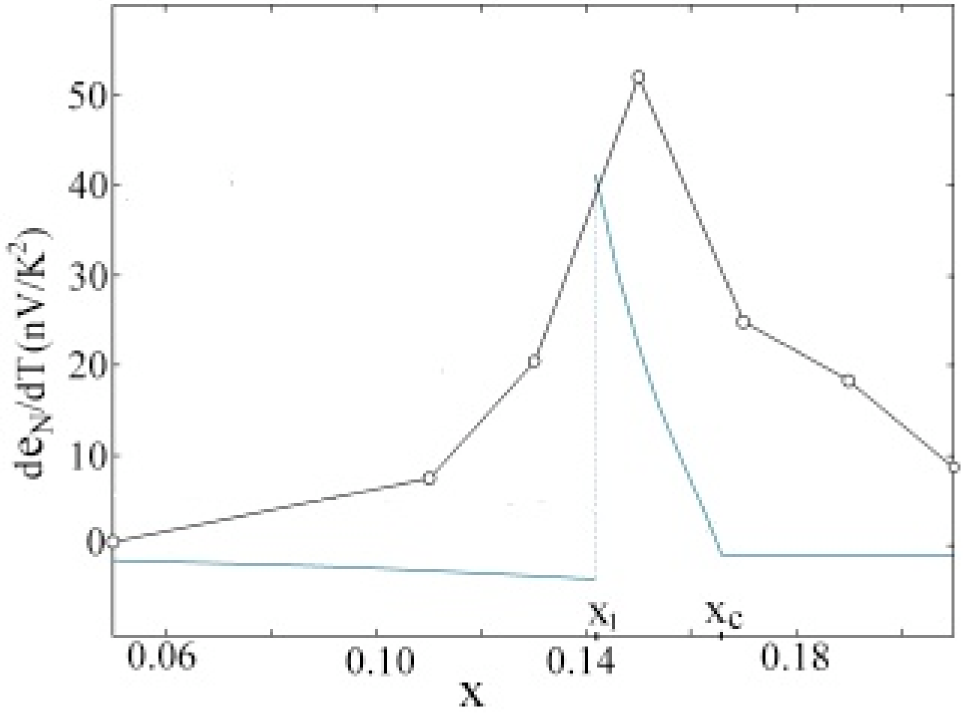
Underdoped region, only electron-like carriers

Opening of hole pocket, singularity in Nernst signal

Jump in Nernst signal

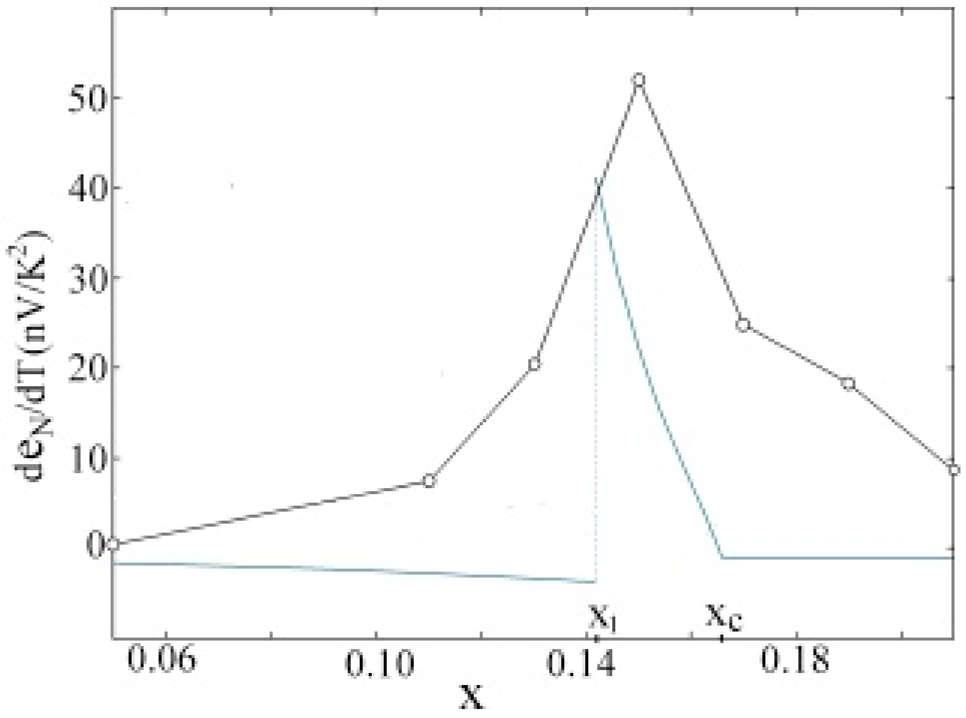
Singularity at doping x_1

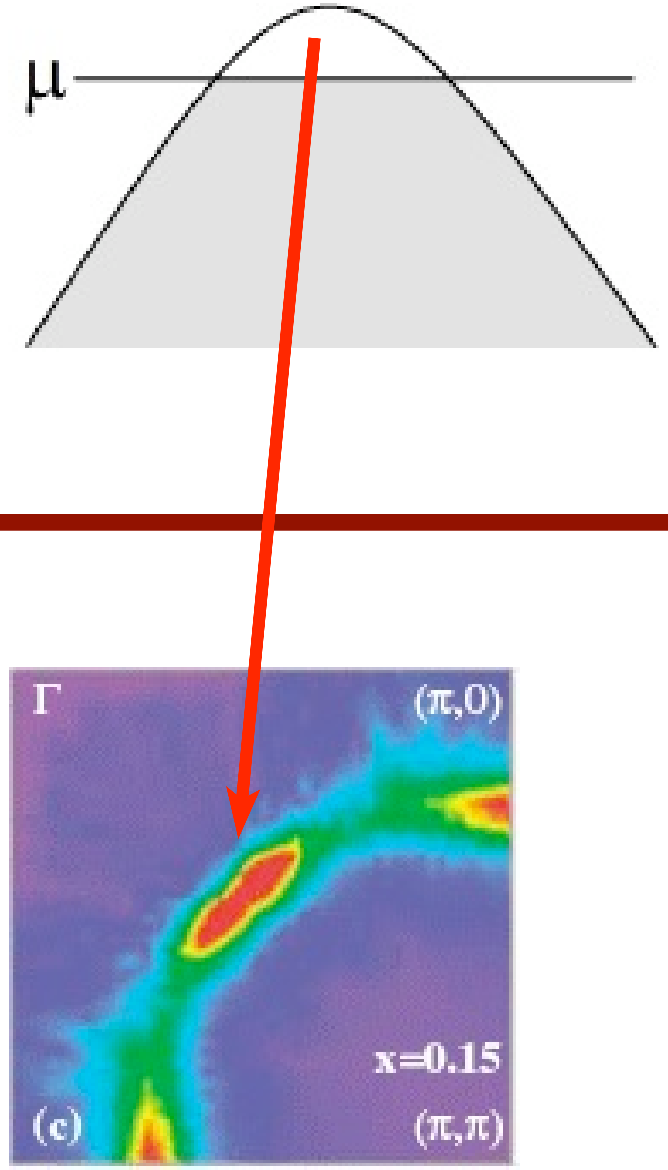




Jump in Nernst signal

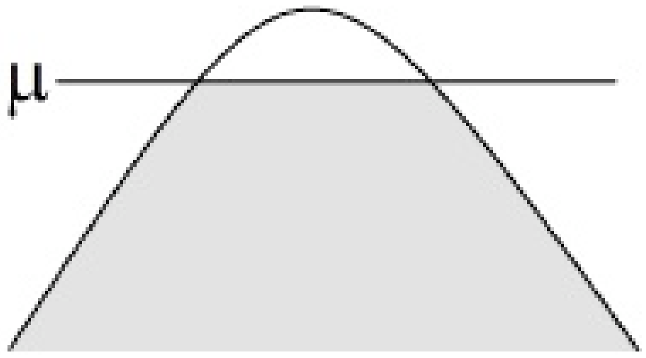
Singularity at doping x_1

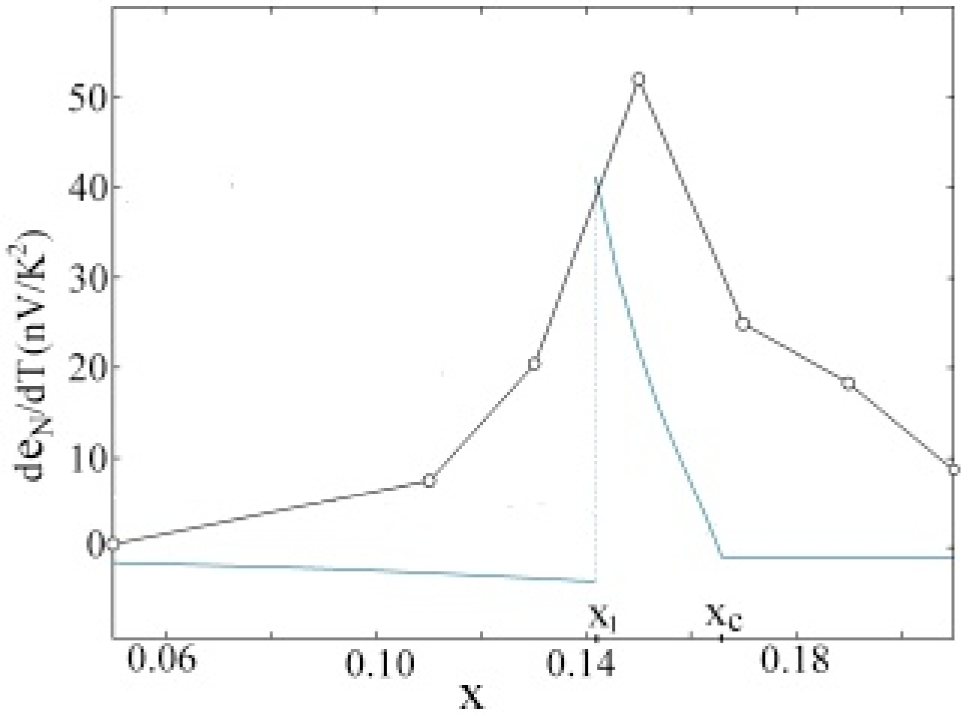




Jump in Nernst signal

Singularity at doping x_1





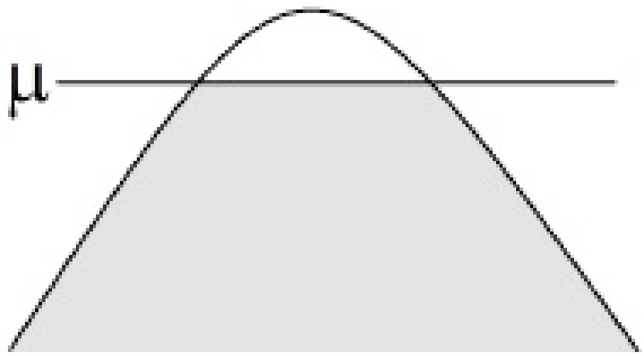
Thermoelectric coefficients

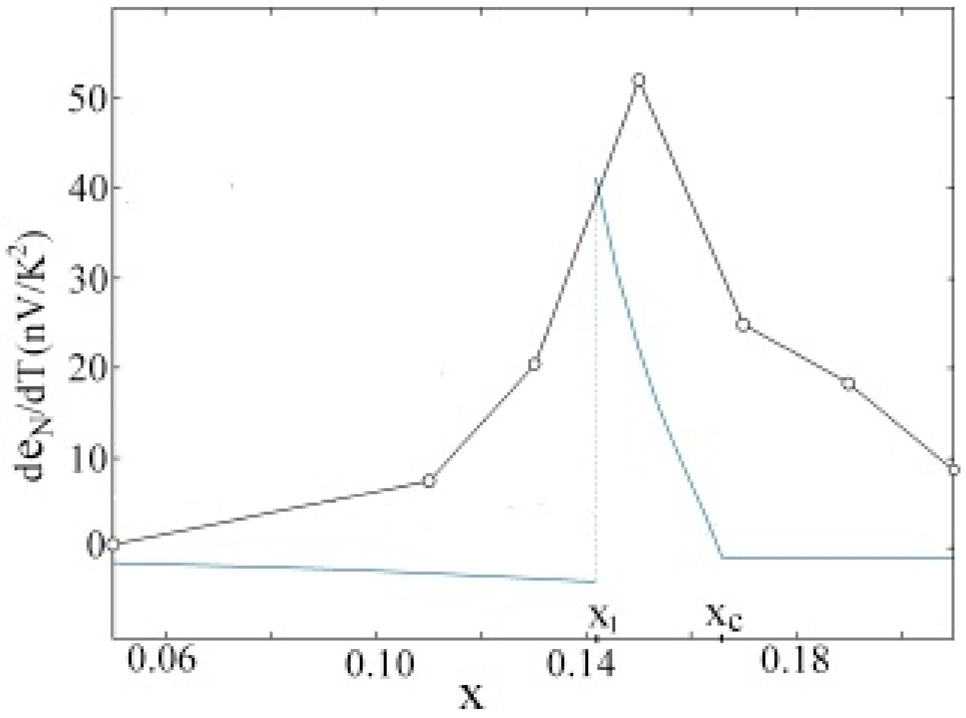
$$\alpha \simeq \frac{\mathbf{k_B^2 T}}{\mathbf{e}} \left(\frac{\partial \sigma}{\partial \varepsilon} \right)_{\mu}$$

jump at $\mu = \mu_c$

Jump in Nernst signal

Singularity at doping x_1





Thermoelectric coefficients

$$\alpha \simeq \frac{\mathbf{k_B^2 T}}{\mathbf{e}} \left(\frac{\partial \sigma}{\partial \varepsilon} \right)_{\mu}$$

jump at $\mu = \mu_c$

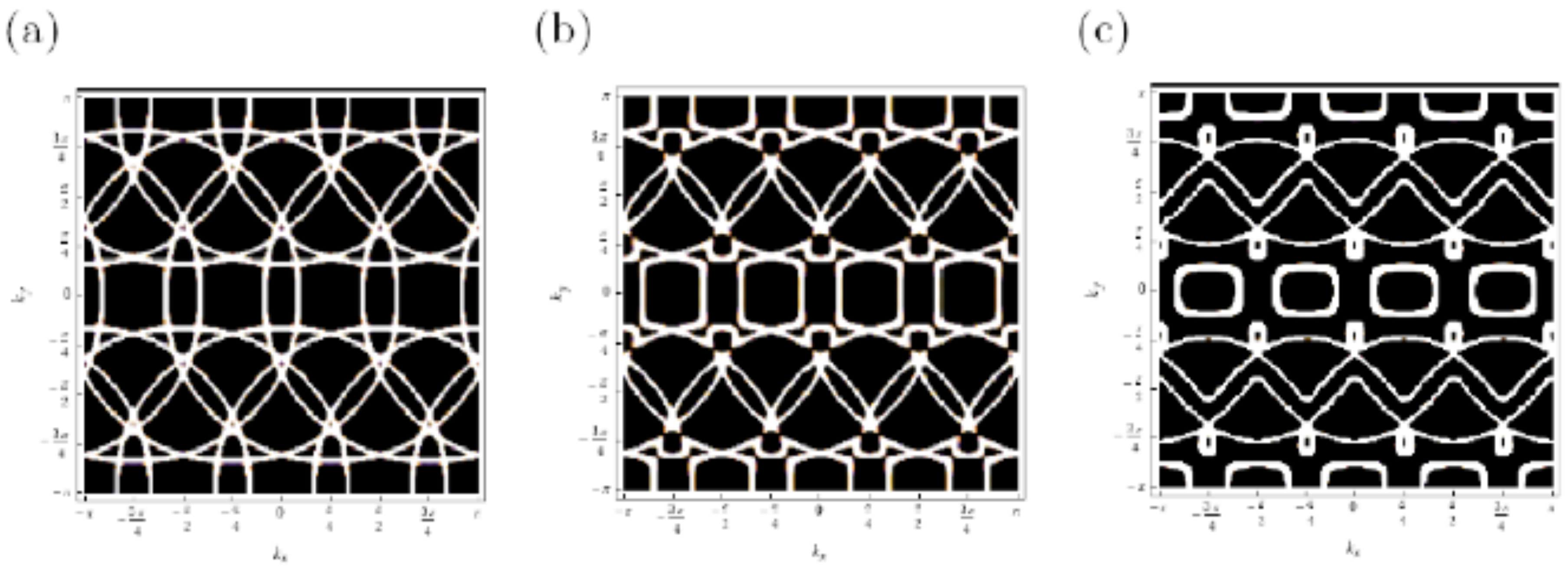
Jump in Nernst signal:

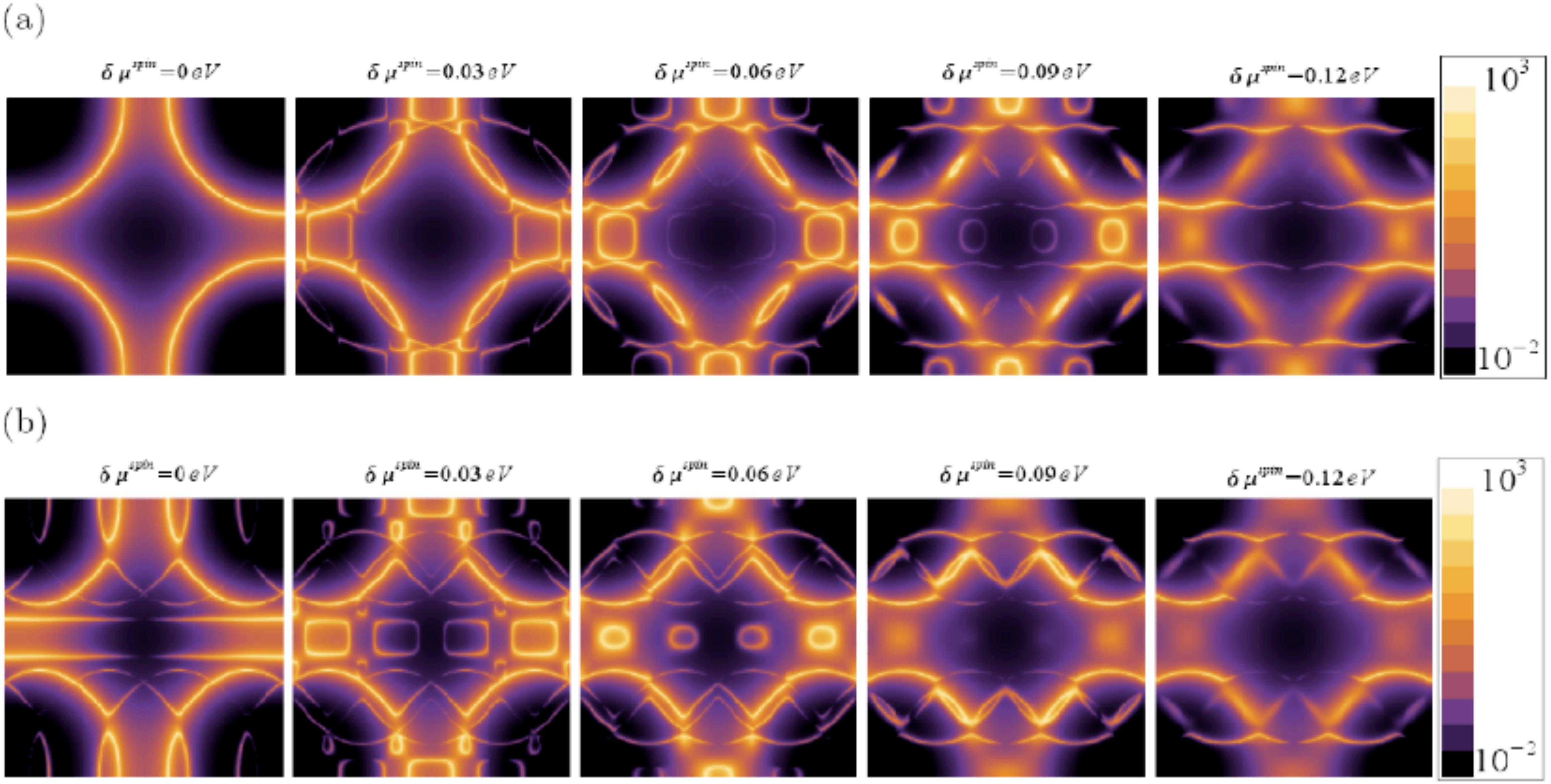
$$\mathbf{\Delta N_{yx}} = \left[\frac{\sigma_{\mathbf{e}} \alpha^{\mathbf{h}}_{\mathbf{xy}} - \sigma^{\mathbf{e}}_{\mathbf{xy}} \alpha^{\mathbf{h}}_{\mathbf{xx}}}{\sigma^{\mathbf{2}}_{\mathbf{e}}} \right]_{\mu_c}$$

Quasiparticle Nernst effect in hole-doped cuprates

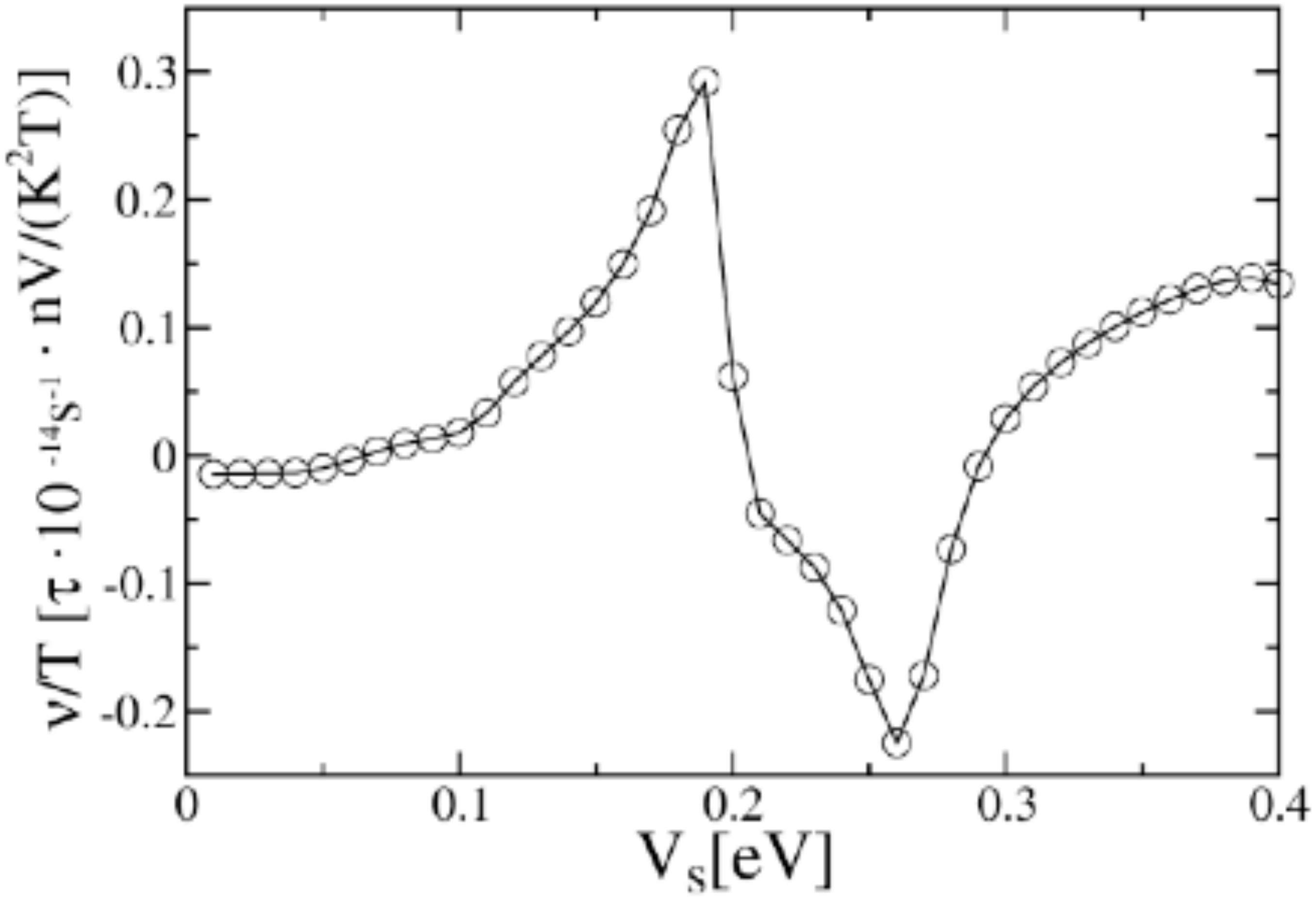
A. Hackl, M. Vojta, and S. Sachdev, arXiv:0908.1088

Fermi surface reconstruction: CDW + SDW (period 8)





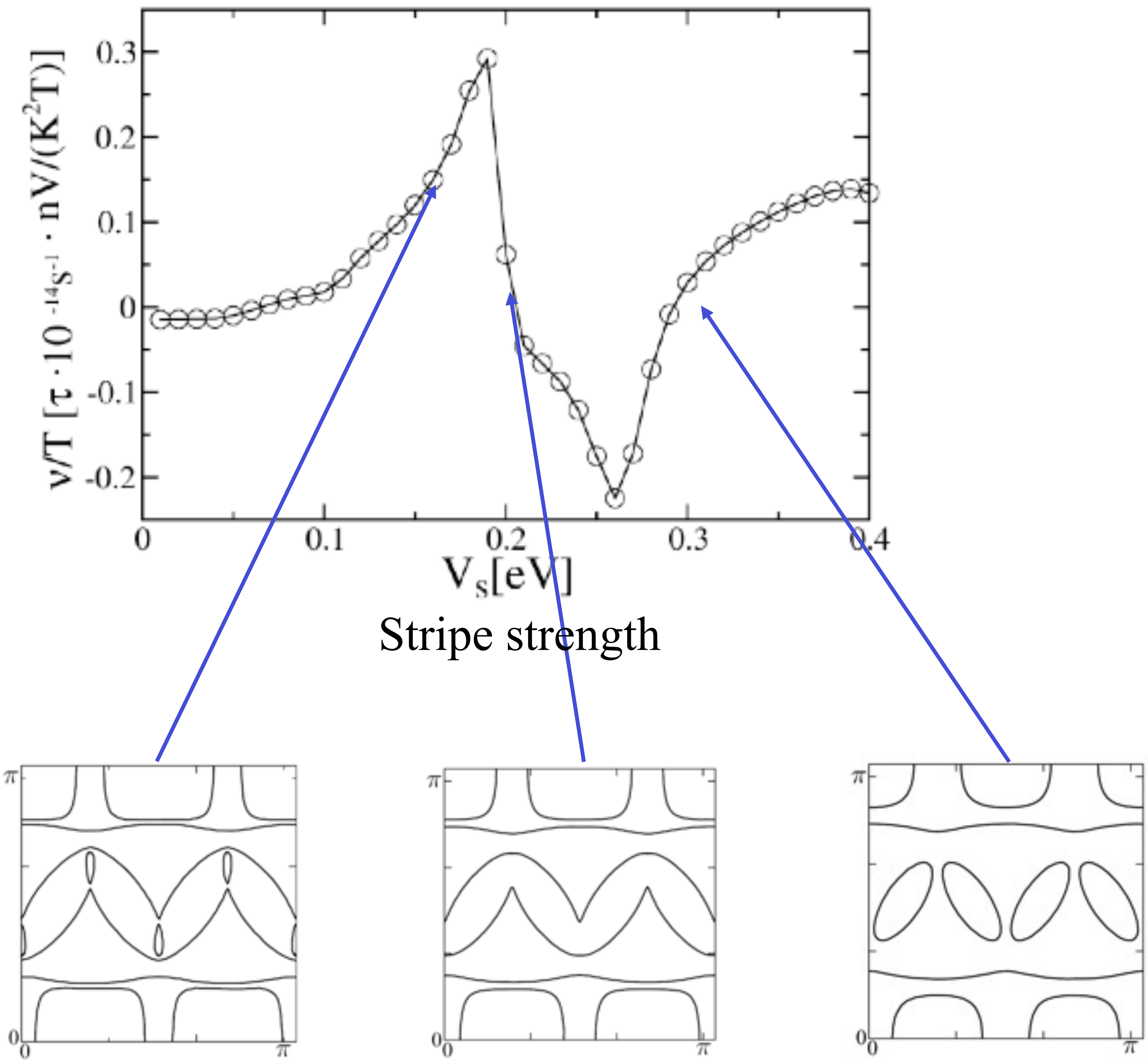
Nernst signal for CDW+SDW (period 8)



Stripe strength

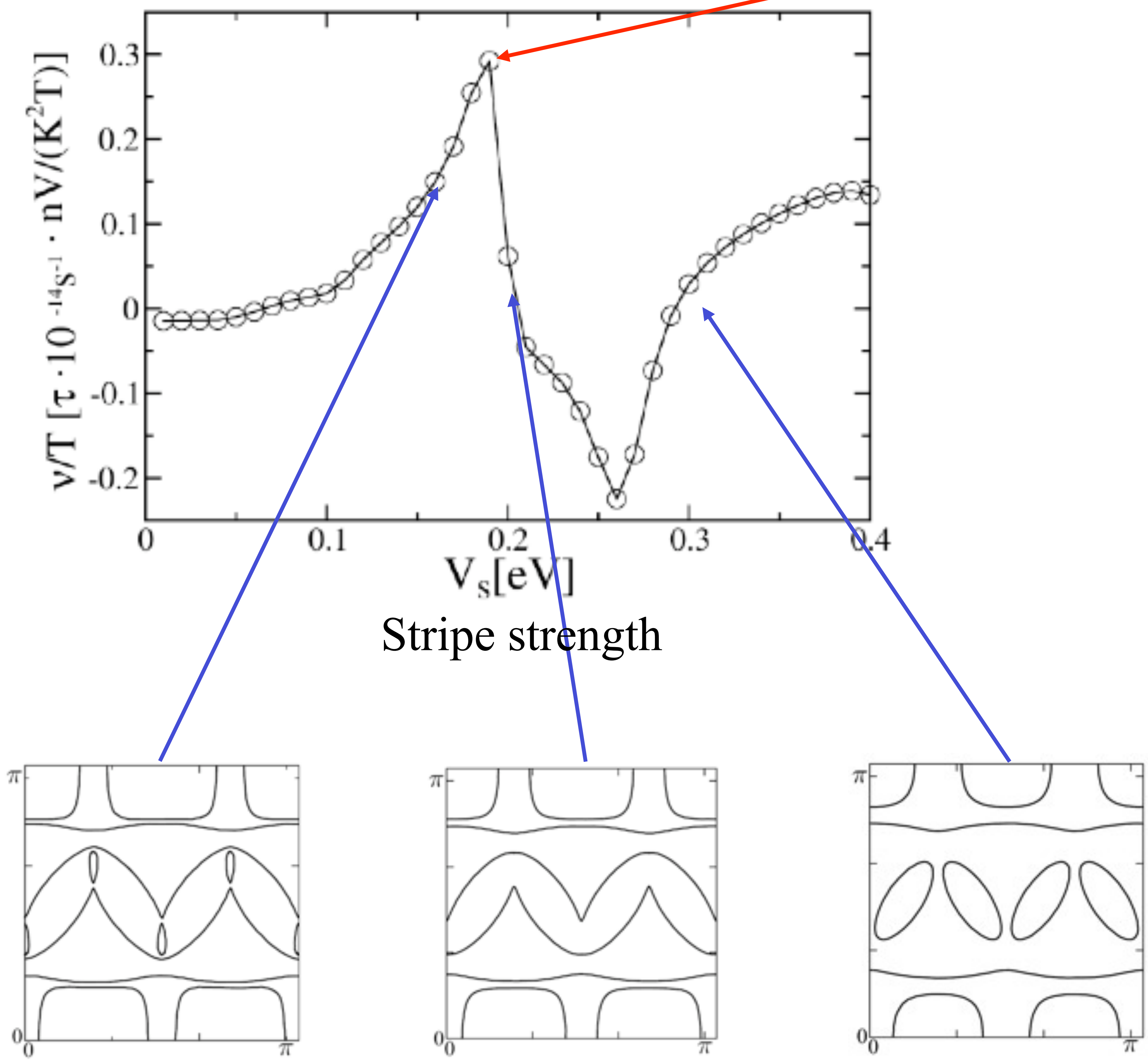
A. Hackl, M. Vojta, and S. Sachdev, arXiv:0908.1088

Nernst signal for CDW+SDW (period 8)



A. Hackl, M. Vojta, and S. Sachdev, arXiv:0908.1088

Nernst signal for CDW+SDW (period 8)



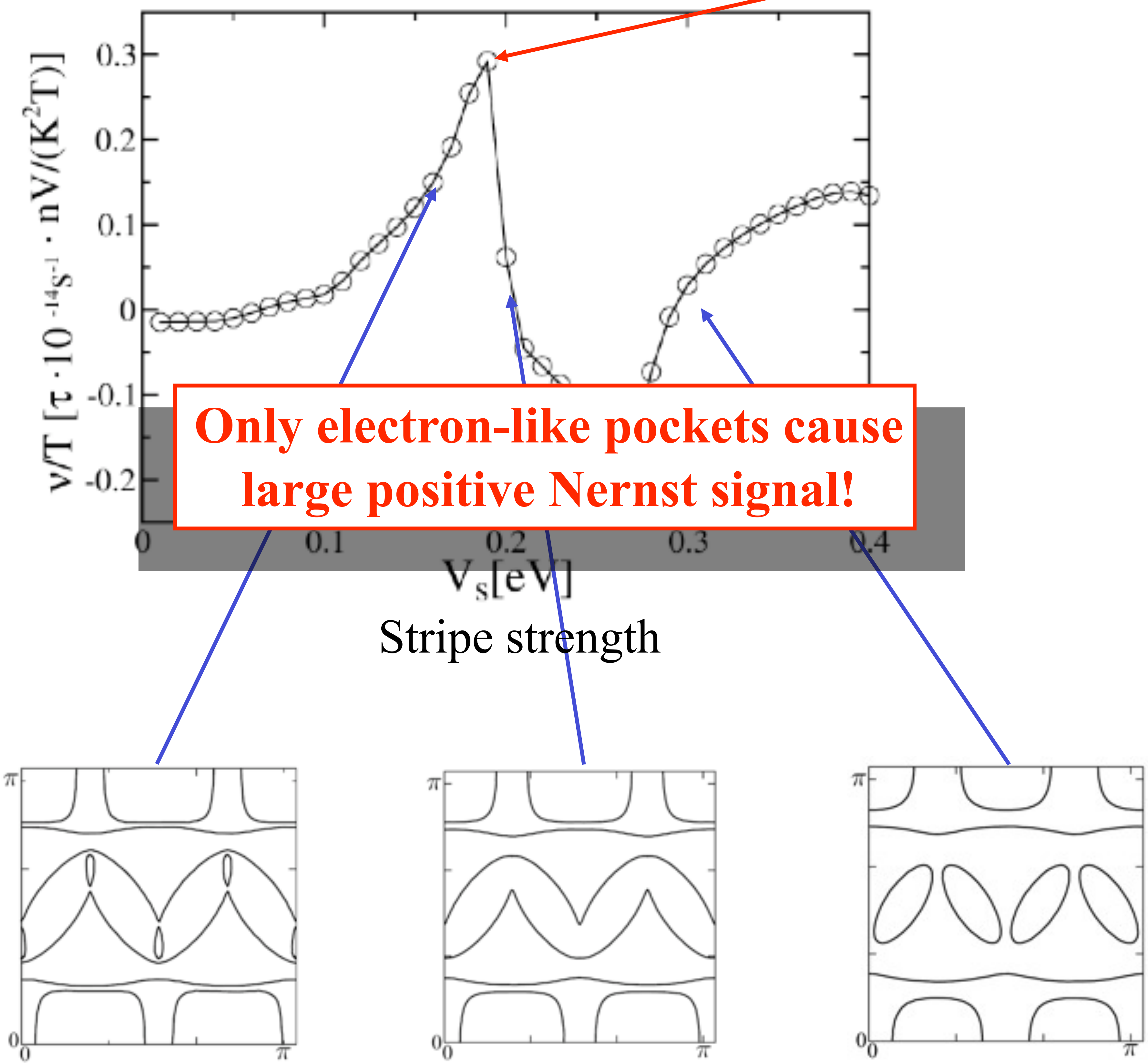
$<S_{max}> \sim 0.15$

Stripe strength

A. Hackl, M. Vojta, and S. Sachdev, arXiv:0908.1088

Nernst signal for CDW+SDW (period 8)

$<S_{\text{max}}> \sim 0.15$



Only electron-like pockets cause large positive Nernst signal!

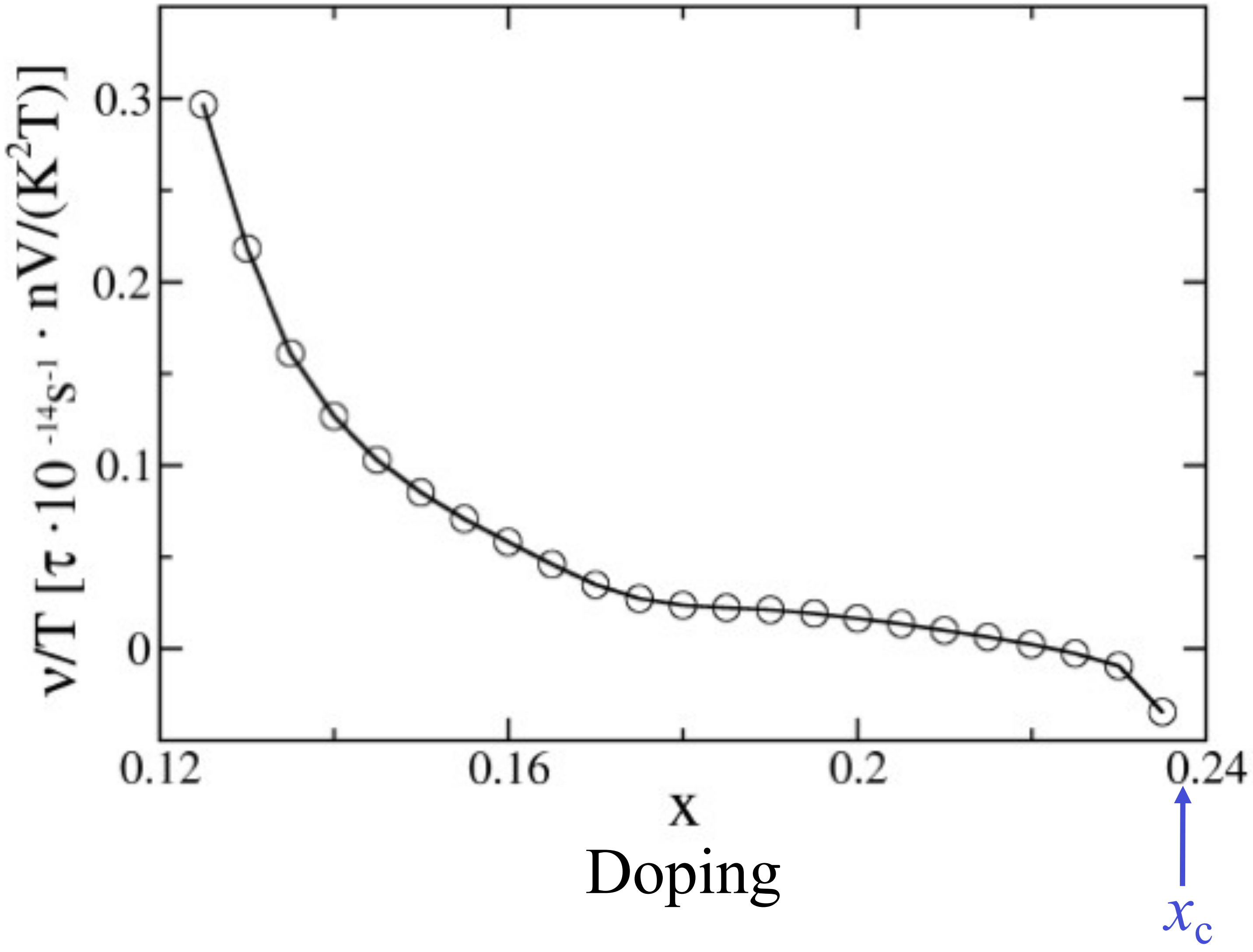
Stripe strength

A. Hackl, M. Vojta, and S. Sachdev, arXiv:0908.1088

Nernst signal for CDW+SDW (period 8)

Assuming a mean-field dependence
of the stripe order parameter on doping

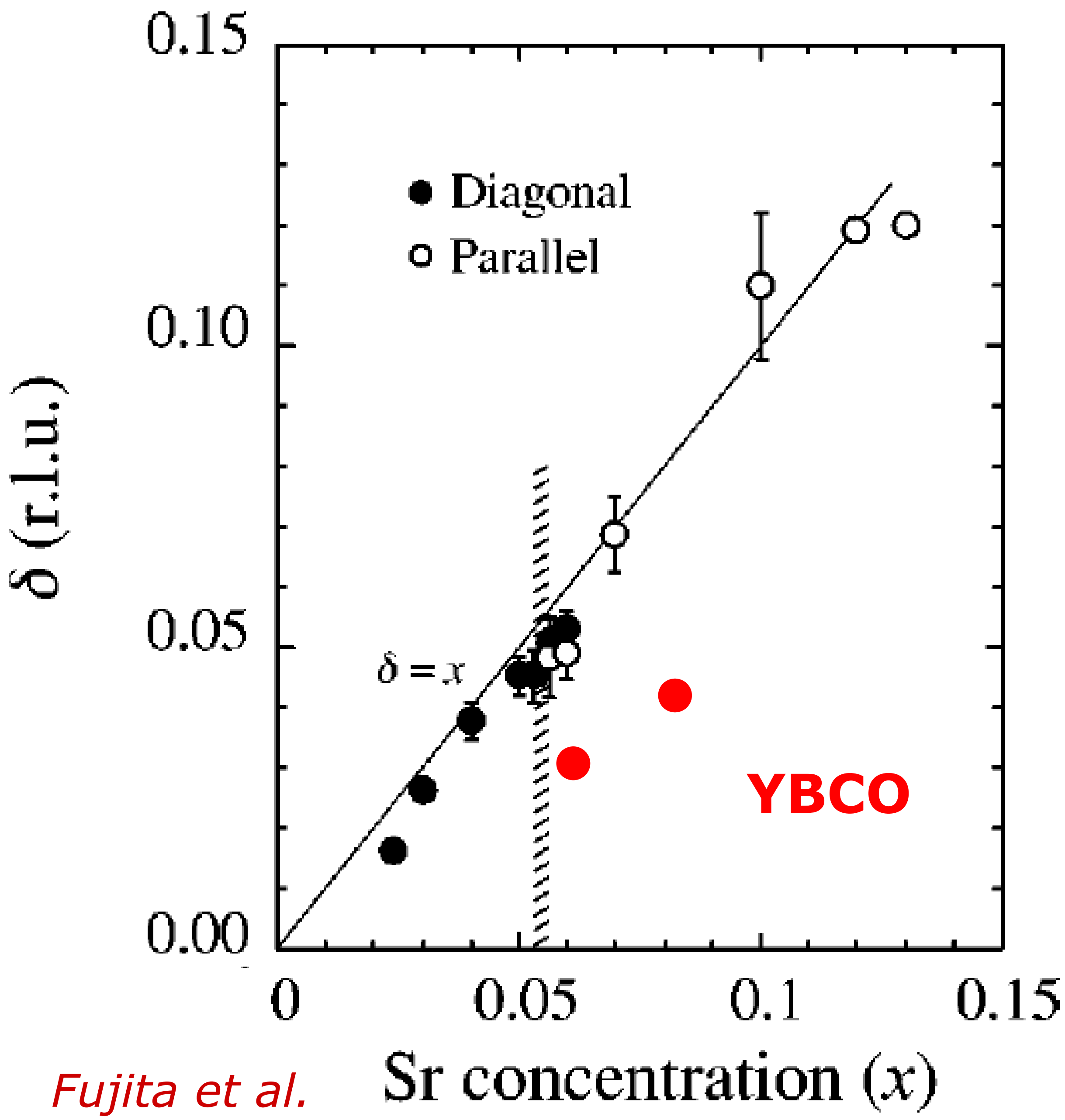
$$V_s(x) = V_0 \sqrt{1 - x/x_c}$$



A. Hackl, M. Vojta, and S. Sachdev, arXiv:0908.1088

“Yamada plot”

LSCO



Fujita et al.
PRB 2002

YBCO

- only “parallel” incommensurate order observed (so far)
- incommensurability only ~½ of that in LSCO at same doping level

Nernst signal for SDW (period 16)

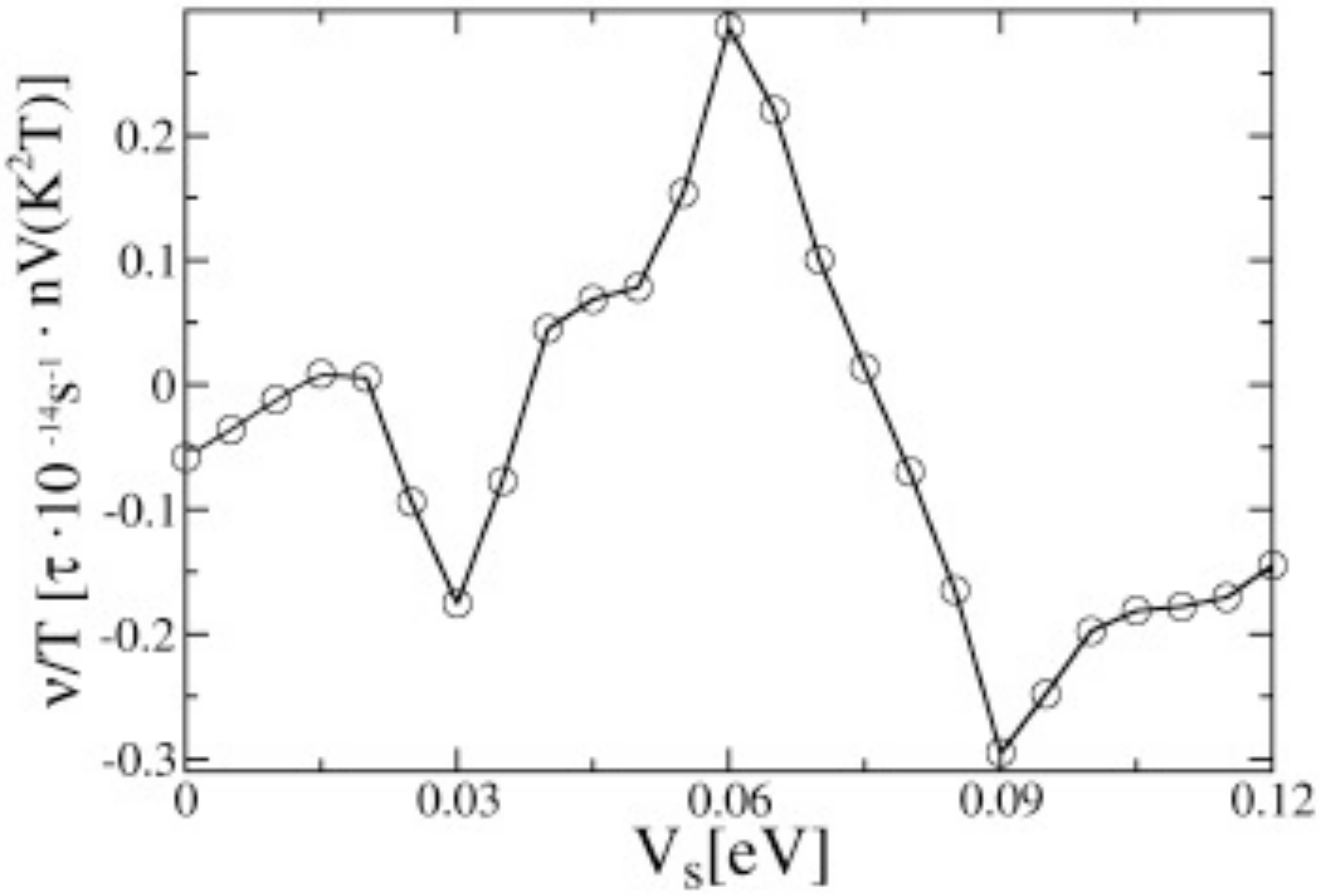
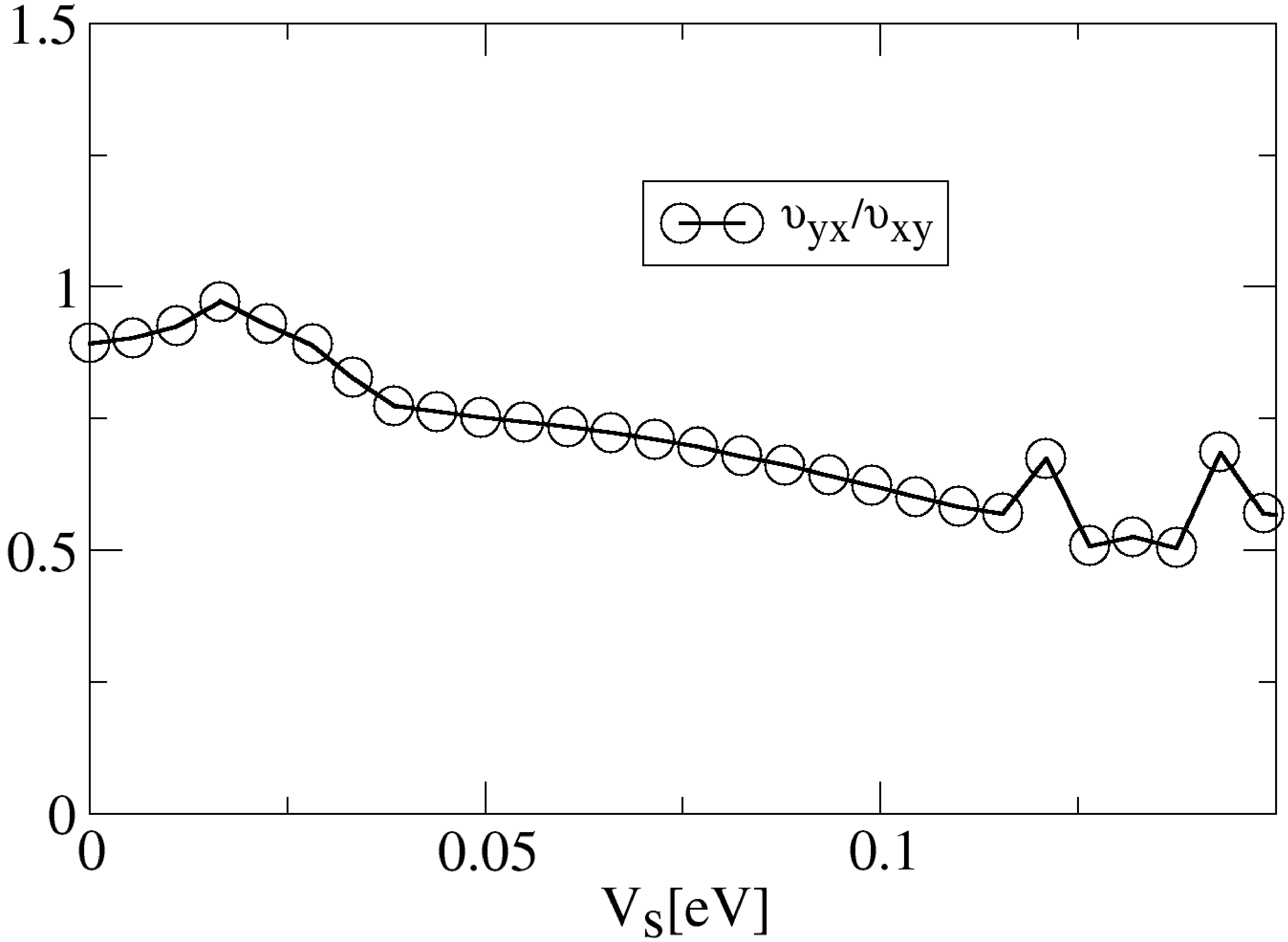


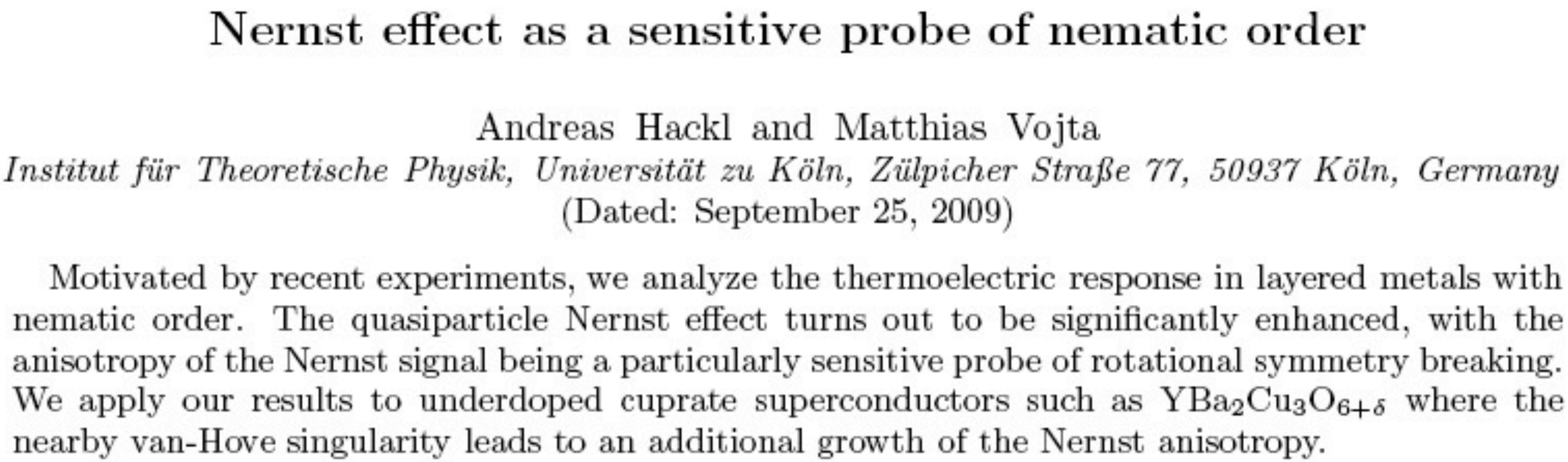
FIG. 11: Nernst effect for a period-16 SDW order as a function of V_s with $x = 0.1$. For $V_s \gtrsim 0.07\,\mathrm{eV}$ (corresponding to a maximal local moment of $m \gtrsim 0.20\mu_B$) the Nernst coefficient turns negative with an enhanced amplitude in comparison to the non-ordered state.

A. Hackl, M. Vojta, and S. Sachdev, arXiv:0908.1088

Nernst anisotropy for SDW (period 8)



A. Hackl, M. Vojta, and S. Sachdev, arXiv:0908.1088



Nernst effect as a sensitive probe of nematic order

Andreas Hackl and Matthias Vojta
Institut für Theoretische Physik, Universität zu Köln, Zülpicher Straße 77, 50937 Köln, Germany

(Dated: September 25, 2009)

Motivated by recent experiments, we analyze the thermoelectric response in layered metals with nematic order. The quasiparticle Nernst effect turns out to be significantly enhanced, with the anisotropy of the Nernst signal being a particularly sensitive probe of rotational symmetry breaking. We apply our results to underdoped cuprate superconductors such as $YBa_2Cu_3O_{6+\delta}$ where the nearby van-Hove singularity leads to an additional growth of the Nernst anisotropy.

arXiv:0909.4534

Outline

1. Proposed phase diagram as a function of temperature, doping, and magnetic field

   Interplay of spin density waves, d-wave superconductivity and Fermi surface change

2. Nernst effect in cuprates

   (a) Hydrodynamic theory

   (b) Quasiparticle theory in spin/charge density wave and nematic states

3. Theory of SDW quantum critical point

   Dominance of planar graphs

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3. Theory of SDW quantum critical point

   Dominance of planar graphs

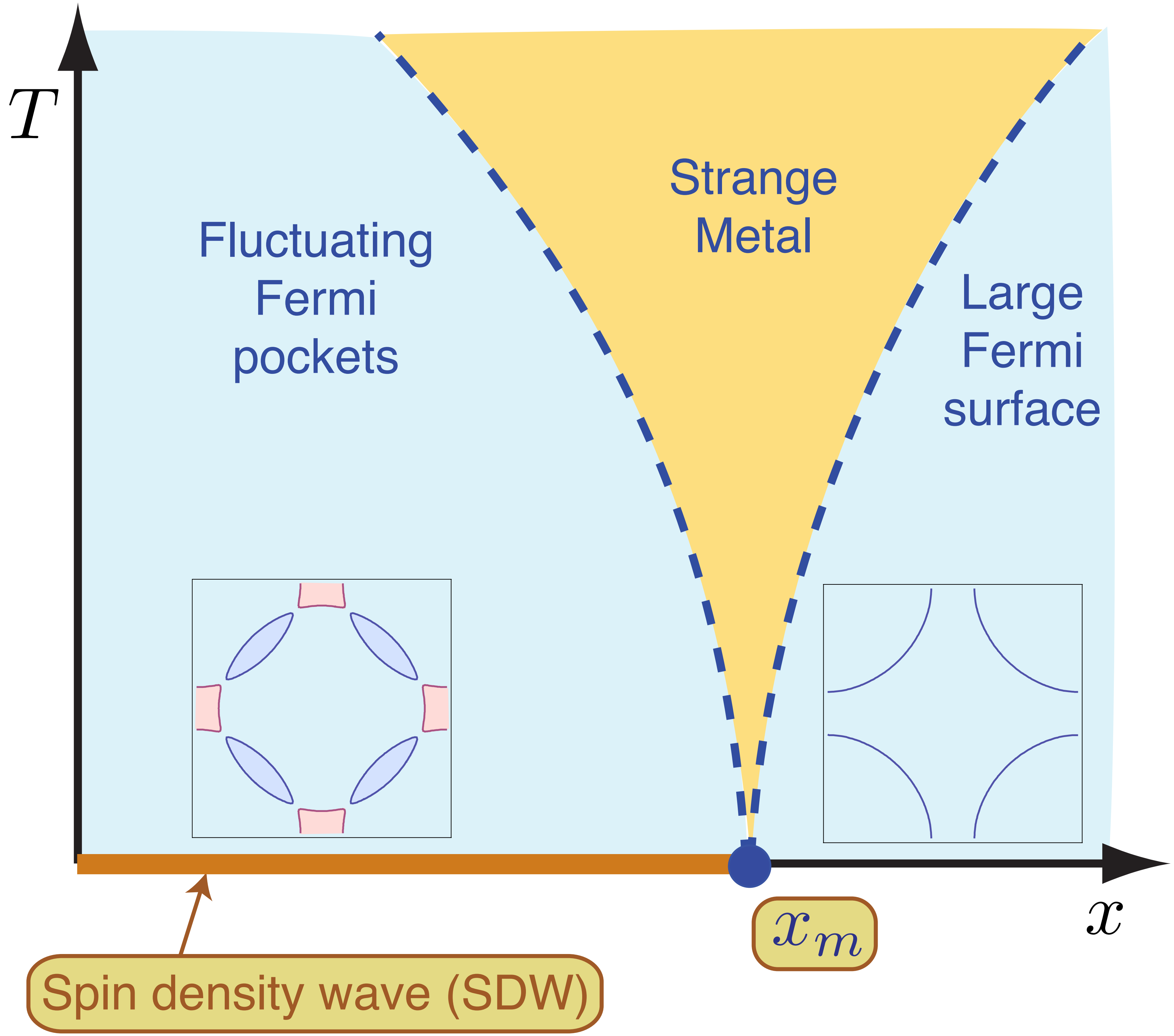
Max Metlitski

M. Metlitski and S. Sachdev, *to appear*

Ar. Abanov, A.V. Chubukov, and J. Schmalian,
Advances in Physics **52**, 119 (2003)

Sung-Sik Lee, arXiv:0905.4532.

Theory of quantum criticality in the cuprates



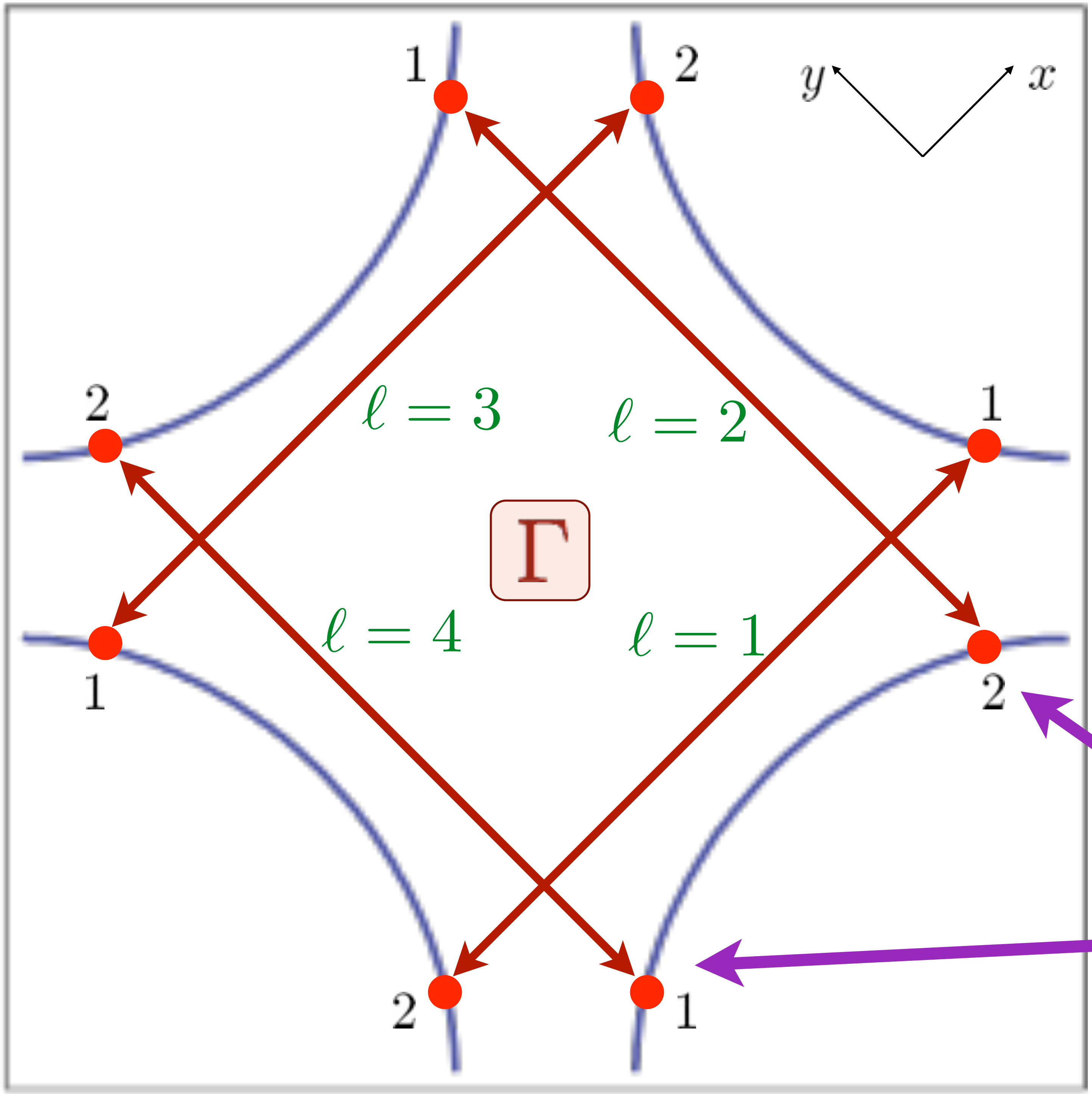
Underlying SDW ordering quantum critical point in **metal** at $x = x_m$

Hertz-Millis-Moriya (HMM) theory:
mean field theory
+
Gaussian fluctuations of overdamped paramagnons.

Theory for the onset of spin density wave order in metals is *strongly* coupled in two dimensions

Start from the "spin-fermion" model

$$
\begin{aligned}
\mathcal{Z} &= \int \mathcal{D}c_\alpha \mathcal{D}\vec{\varphi} \exp\left(-\mathcal{S}\right) \\
\mathcal{S} &= \int d\tau \sum_{\mathbf{k}} c^\dagger_{\mathbf{k}\alpha} \left(\frac{\partial}{\partial\tau} - \varepsilon_{\mathbf{k}} \right) c_{\mathbf{k}\alpha} \\
&\quad -\lambda \int d\tau \sum_i c^\dagger_{i\alpha} \vec{\varphi}_i \cdot \vec{\sigma}_{\alpha\beta} c_{i\beta} e^{i\mathbf{K}\cdot\mathbf{r}_i} \\
&\quad + \int d\tau d^2 r \left[\left(\partial_r \vec{\varphi}\right)^2 + \frac{1}{c^2} \left(\partial_\tau \vec{\varphi}\right)^2 \right]
\end{aligned}
$$

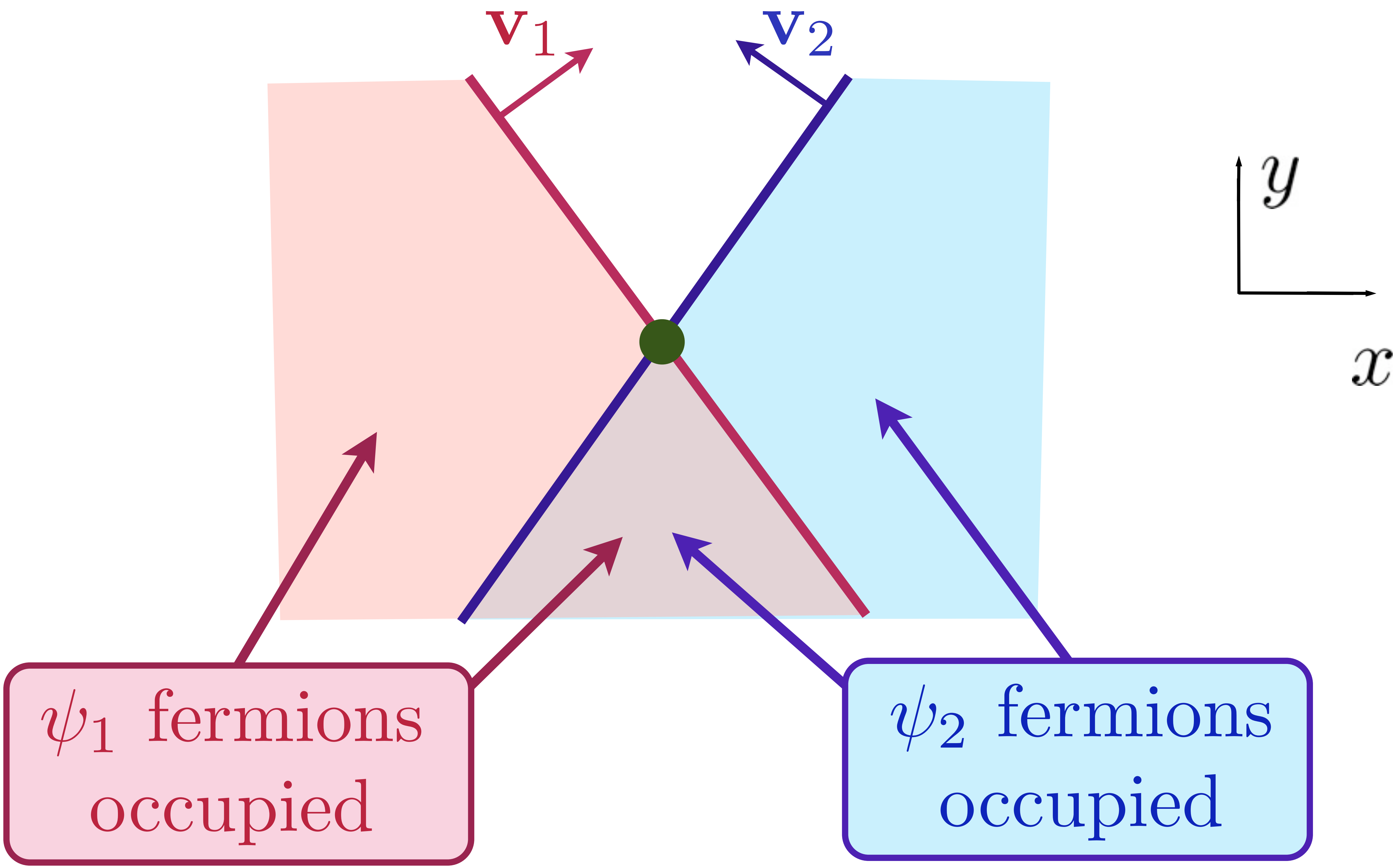


Low energy fermions
$\psi^{\ell}_{1\alpha},\, \psi^{\ell}_{2\alpha}$
$\ell = 1, \ldots, 4$

$$\mathcal{L}_f = \psi^{\ell\dagger}_{1\alpha}\left(\zeta\partial_\tau - i\mathbf{v}^{\ell}_{1}\cdot\boldsymbol{\nabla}_r\right)\psi^{\ell}_{1\alpha} + \psi^{\ell\dagger}_{2\alpha}\left(\zeta\partial_\tau - i\mathbf{v}^{\ell}_{2}\cdot\boldsymbol{\nabla}_r\right)\psi^{\ell}_{2\alpha}$$

$$\mathbf{v}^{\ell=1}_{1} = (v_x, v_y),\; \mathbf{v}^{\ell=1}_{2} = (-v_x, v_y)$$

$$\mathcal{L}_f = \psi_{1\alpha}^{\ell\dagger}\left(\zeta\partial_\tau - i\mathbf{v}_1^\ell \cdot \boldsymbol{\nabla}_r\right)\psi_{1\alpha}^\ell + \psi_{2\alpha}^{\ell\dagger}\left(\zeta\partial_\tau - i\mathbf{v}_2^\ell \cdot \boldsymbol{\nabla}_r\right)\psi_{2\alpha}^\ell$$



ψ_1 fermions occupied

ψ_2 fermions occupied

$$\mathcal{L}_f = \psi_{1\alpha}^{\ell\dagger}\left(\zeta\partial_\tau - i\mathbf{v}_1^\ell \cdot \boldsymbol{\nabla}_r\right)\psi_{1\alpha}^\ell + \psi_{2\alpha}^{\ell\dagger}\left(\zeta\partial_\tau - i\mathbf{v}_2^\ell \cdot \boldsymbol{\nabla}_r\right)\psi_{2\alpha}^\ell$$

Order parameter: $\mathcal{L}_\varphi = \frac{1}{2}\left(\boldsymbol{\nabla}_r \vec{\varphi}\right)^2 + \frac{s}{2}\vec{\varphi}^2 + \frac{u}{4}\vec{\varphi}^4$

$$\mathcal{L}_f = \psi_{1\alpha}^{\ell\dagger}\left(\zeta\partial_\tau - i\mathbf{v}_1^\ell\cdot\boldsymbol{\nabla}_r\right)\psi_{1\alpha}^\ell + \psi_{2\alpha}^{\ell\dagger}\left(\zeta\partial_\tau - i\mathbf{v}_2^\ell\cdot\boldsymbol{\nabla}_r\right)\psi_{2\alpha}^\ell$$

Order parameter: $\mathcal{L}_\varphi = \frac{1}{2}\left(\boldsymbol{\nabla}_r\vec{\varphi}\right)^2 + \frac{s}{2}\vec{\varphi}^2 + \frac{u}{4}\vec{\varphi}^4$

"Yukawa" coupling: $\mathcal{L}_c = -\vec{\varphi}\cdot\left(\psi_{1\alpha}^{\ell\dagger}\vec{\sigma}_{\alpha\beta}\psi_{2\beta}^\ell + \psi_{2\alpha}^{\ell\dagger}\vec{\sigma}_{\alpha\beta}\psi_{1\beta}^\ell\right)$

$$\mathcal{L}_f = \psi^{\ell\dagger}_{1\alpha}\left(\zeta\partial_\tau - i\mathbf{v}^\ell_1\cdot\boldsymbol{\nabla}_r\right)\psi^\ell_{1\alpha} + \psi^{\ell\dagger}_{2\alpha}\left(\zeta\partial_\tau - i\mathbf{v}^\ell_2\cdot\boldsymbol{\nabla}_r\right)\psi^\ell_{2\alpha}$$

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HMM theory

Integrate out fermions and obtain non-local corrections to $\mathcal{L}_\varphi$

$$\mathcal{L}_\varphi = \frac{1}{2}\vec{\varphi}^2\left[\mathbf{q}^2 + \gamma|\omega|\right]/2 \quad ; \quad \gamma = \frac{2}{\pi v_x v_y}$$

Exponent $z = 2$ and mean-field criticality (upto logarithms)

$$\mathcal{L}_f = \psi^{\ell\dagger}_{1\alpha}\left(\zeta\partial_\tau - i\mathbf{v}^\ell_1\cdot\boldsymbol{\nabla}_r\right)\psi^\ell_{1\alpha} + \psi^{\ell\dagger}_{2\alpha}\left(\zeta\partial_\tau - i\mathbf{v}^\ell_2\cdot\boldsymbol{\nabla}_r\right)\psi^\ell_{2\alpha}$$

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"Yukawa" coupling: $\mathcal{L}_c = -\vec{\varphi}\cdot\left(\psi^{\ell\dagger}_{1\alpha}\vec{\sigma}_{\alpha\beta}\psi^\ell_{2\beta} + \psi^{\ell\dagger}_{2\alpha}\vec{\sigma}_{\alpha\beta}\psi^\ell_{1\beta}\right)$

HMM theory

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$$\mathcal{L}_\varphi = \frac{1}{2}\vec{\varphi}^2\left[\mathbf{q}^2 + \gamma|\omega|\right]/2 \quad ; \quad \gamma = \frac{2}{\pi v_x v_y}$$

Exponent $z = 2$ and mean-field criticality (upto logarithms)

But, higher order terms contain an infinite number of marginal couplings ......

Ar. Abanov and A.V. Chubukov, *Phys. Rev. Lett.* **93**, 255702 (2004)

$$\mathcal{L}_f = \psi_{1\alpha}^{\ell\dagger}\left(\zeta\partial_\tau - i\mathbf{v}_1^\ell\cdot\boldsymbol{\nabla}_r\right)\psi_{1\alpha}^\ell + \psi_{2\alpha}^{\ell\dagger}\left(\zeta\partial_\tau - i\mathbf{v}_2^\ell\cdot\boldsymbol{\nabla}_r\right)\psi_{2\alpha}^\ell$$

Order parameter: $\mathcal{L}_\varphi = \frac{1}{2}\left(\boldsymbol{\nabla}_r\vec{\varphi}\right)^2 + \frac{s}{2}\vec{\varphi}^2 + \frac{u}{4}\vec{\varphi}^4$

“Yukawa” coupling: $\mathcal{L}_c = -\vec{\varphi}\cdot\left(\psi_{1\alpha}^{\ell\dagger}\vec{\sigma}_{\alpha\beta}\psi_{2\beta}^\ell + \psi_{2\alpha}^{\ell\dagger}\vec{\sigma}_{\alpha\beta}\psi_{1\beta}^\ell\right)$

Perform RG on both fermions and $\vec{\varphi}$,
using a *local* field theory.

$$\mathcal{L}_f = \psi^{\ell\dagger}_{1\alpha}\left(\zeta\partial_\tau - i\mathbf{v}^{\ell}_{1}\cdot\boldsymbol{\nabla}_r\right)\psi^{\ell}_{1\alpha} + \psi^{\ell\dagger}_{2\alpha}\left(\zeta\partial_\tau - i\mathbf{v}^{\ell}_{2}\cdot\boldsymbol{\nabla}_r\right)\psi^{\ell}_{2\alpha}$$

Order parameter: $$\mathcal{L}_\varphi = \frac{1}{2}\left(\boldsymbol{\nabla}_r\vec{\varphi}\right)^2 + \frac{s}{2}\vec{\varphi}^2 + \frac{u}{4}\vec{\varphi}^4$$

“Yukawa” coupling: $$\mathcal{L}_c = -\vec{\varphi}\cdot\left(\psi^{\ell\dagger}_{1\alpha}\vec{\sigma}_{\alpha\beta}\psi^{\ell}_{2\beta} + \psi^{\ell\dagger}_{2\alpha}\vec{\sigma}_{\alpha\beta}\psi^{\ell}_{1\beta}\right)$$

With $z = 2$ scaling, ζ is irrelevant.
So we take $\zeta \to 0$
(⚠ watch for dangerous irrelevancy).

$$\mathcal{L}_f = \psi^{\ell\dagger}_{1\alpha}\left(\zeta\partial_\tau - i\mathbf{v}^\ell_1\cdot\boldsymbol{\nabla}_r\right)\psi^\ell_{1\alpha} + \psi^{\ell\dagger}_{2\alpha}\left(\zeta\partial_\tau - i\mathbf{v}^\ell_2\cdot\boldsymbol{\nabla}_r\right)\psi^\ell_{2\alpha}$$

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“Yukawa” coupling: $$\mathcal{L}_c = -\vec{\varphi}\cdot\left(\psi^{\ell\dagger}_{1\alpha}\vec{\sigma}_{\alpha\beta}\psi^\ell_{2\beta} + \psi^{\ell\dagger}_{2\alpha}\vec{\sigma}_{\alpha\beta}\psi^\ell_{1\beta}\right)$$

Set $\vec{\varphi}$ wavefunction renormalization by keeping co-efficient of $(\boldsymbol{\nabla}_r\vec{\varphi})^2$ fixed (as usual).

$$\mathcal{L}_f = \psi_{1\alpha}^{\ell\dagger}\left(\zeta\partial_\tau - i\mathbf{v}_1^\ell \cdot \boldsymbol{\nabla}_r\right)\psi_{1\alpha}^\ell + \psi_{2\alpha}^{\ell\dagger}\left(\zeta\partial_\tau - i\mathbf{v}_2^\ell \cdot \boldsymbol{\nabla}_r\right)\psi_{2\alpha}^\ell$$

Order parameter: $\mathcal{L}_\varphi = \frac{1}{2}\left(\boldsymbol{\nabla}_r \vec{\varphi}\right)^2 + \frac{s}{2}\vec{\varphi}^2 + \frac{u}{4}\vec{\varphi}^4$

"Yukawa" coupling: $\mathcal{L}_c = -\vec{\varphi} \cdot \left(\psi_{1\alpha}^{\ell\dagger}\vec{\sigma}_{\alpha\beta}\psi_{2\beta}^\ell + \psi_{2\alpha}^{\ell\dagger}\vec{\sigma}_{\alpha\beta}\psi_{1\beta}^\ell\right)$

Set fermion wavefunction renormalization by keeping Yukawa coupling fixed.

Y. Huh and S. Sachdev, *Phys. Rev. B* **78**, 064512 (2008).

$$\mathcal{L}_f = \psi_{1\alpha}^{\ell\dagger}\left(\zeta\partial_\tau - i\mathbf{v}_1^\ell \cdot \boldsymbol{\nabla}_r\right)\psi_{1\alpha}^\ell + \psi_{2\alpha}^{\ell\dagger}\left(\zeta\partial_\tau - i\mathbf{v}_2^\ell \cdot \boldsymbol{\nabla}_r\right)\psi_{2\alpha}^\ell$$

Order parameter: $\mathcal{L}_\varphi = \frac{1}{2}\left(\boldsymbol{\nabla}_r\vec{\varphi}\right)^2 + \frac{s}{2}\vec{\varphi}^2 + \frac{u}{4}\vec{\varphi}^4$

"Yukawa" coupling: $\mathcal{L}_c = -\vec{\varphi}\cdot\left(\psi_{1\alpha}^{\ell\dagger}\vec{\sigma}_{\alpha\beta}\psi_{2\beta}^\ell + \psi_{2\alpha}^{\ell\dagger}\vec{\sigma}_{\alpha\beta}\psi_{1\beta}^\ell\right)$

We find consistent two-loop RG factors, as $\zeta \to 0$, for the velocities v_x, v_y, and the wavefunction renormalizations.

Consistency check: the expression for the boson damping constant, $\gamma = \dfrac{2}{\pi v_x v_y}$, is preserved under RG.

RG-improved Migdal-Eliashberg theory

RG flow can be computed a $1/N$ expansion (with N fermion species) in terms of a single dimensionless coupling $\alpha = v_y/v_x$ whose flow obeys

$$\boxed{\frac{d\alpha}{d\ell} = -\frac{3}{\pi N}\frac{\alpha^2}{1+\alpha^2}}$$

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The velocities flow as

$$\frac{1}{v_x}\frac{dv_x}{d\ell} = \frac{\mathcal{A}(\alpha)+\mathcal{B}(\alpha)}{2} \quad ; \quad \frac{1}{v_y}\frac{dv_y}{d\ell} = \frac{-\mathcal{A}(\alpha)+\mathcal{B}(\alpha)}{2}$$

$$\mathcal{A}(\alpha) \equiv \frac{3}{\pi N}\frac{\alpha}{1+\alpha^2}$$

$$\mathcal{B}(\alpha) \equiv \frac{1}{2\pi N}\left(\frac{1}{\alpha}-\alpha\right)\left(1+\left(\frac{1}{\alpha}-\alpha\right)\tan^{-1}\frac{1}{\alpha}\right)$$

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The anomalous dimensions of $\vec{\varphi}$ and ψ are

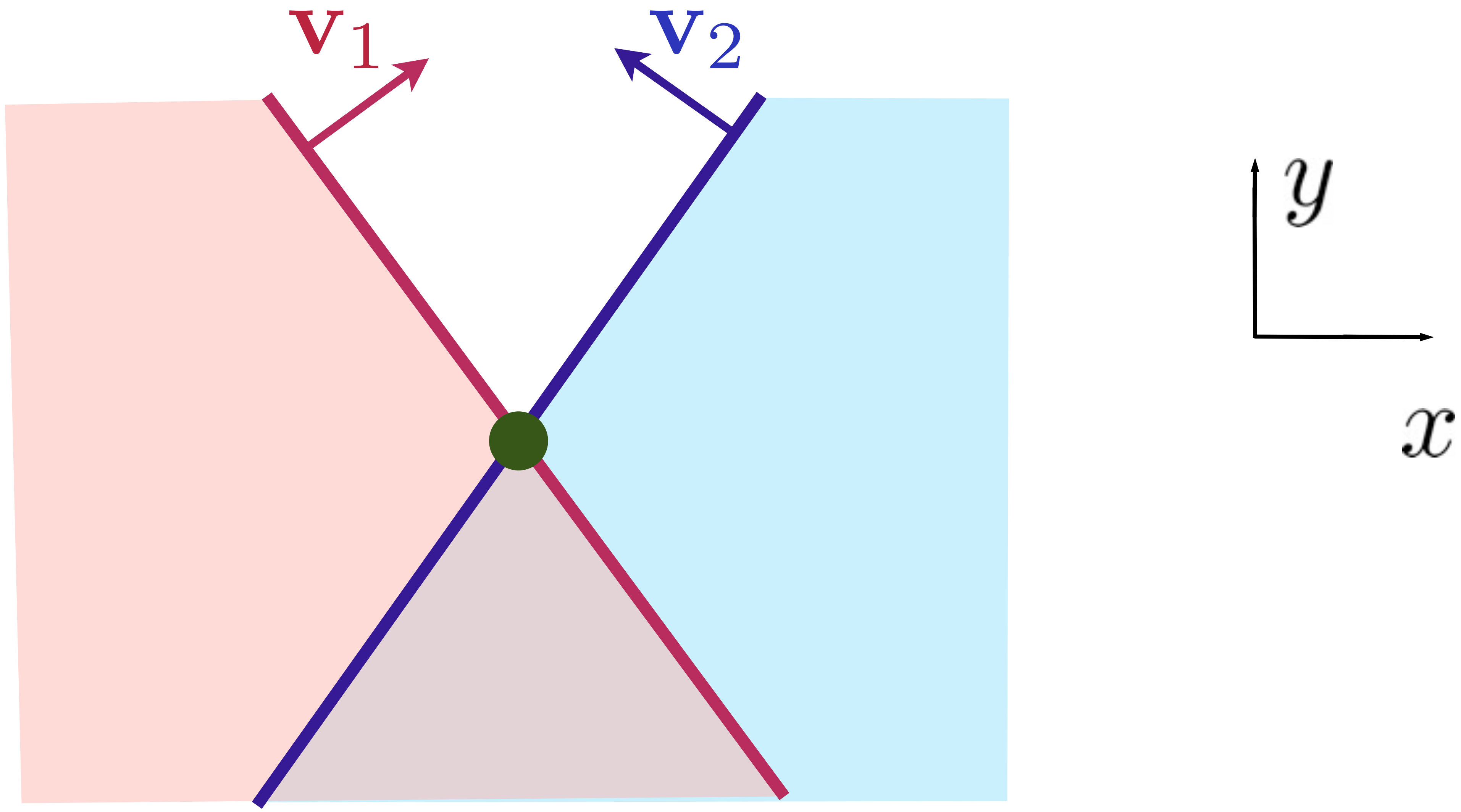
$$\eta_\varphi = \frac{1}{2\pi N}\left(\frac{1}{\alpha} - \alpha + \left(\frac{1}{\alpha^2} + \alpha^2\right)\tan^{-1}\frac{1}{\alpha}\right)$$

$$\eta_\psi = -\frac{1}{4\pi N}\left(\frac{1}{\alpha} - \alpha\right)\left(1 + \left(\frac{1}{\alpha} - \alpha\right)\tan^{-1}\frac{1}{\alpha}\right)$$

RG-improved Migdal-Eliashberg theory

$\alpha = v_y/v_x \to 0$ logarithmically in the infrared.

Dynamical Nesting



Bare Fermi surface

RG-improved Migdal-Eliashberg theory

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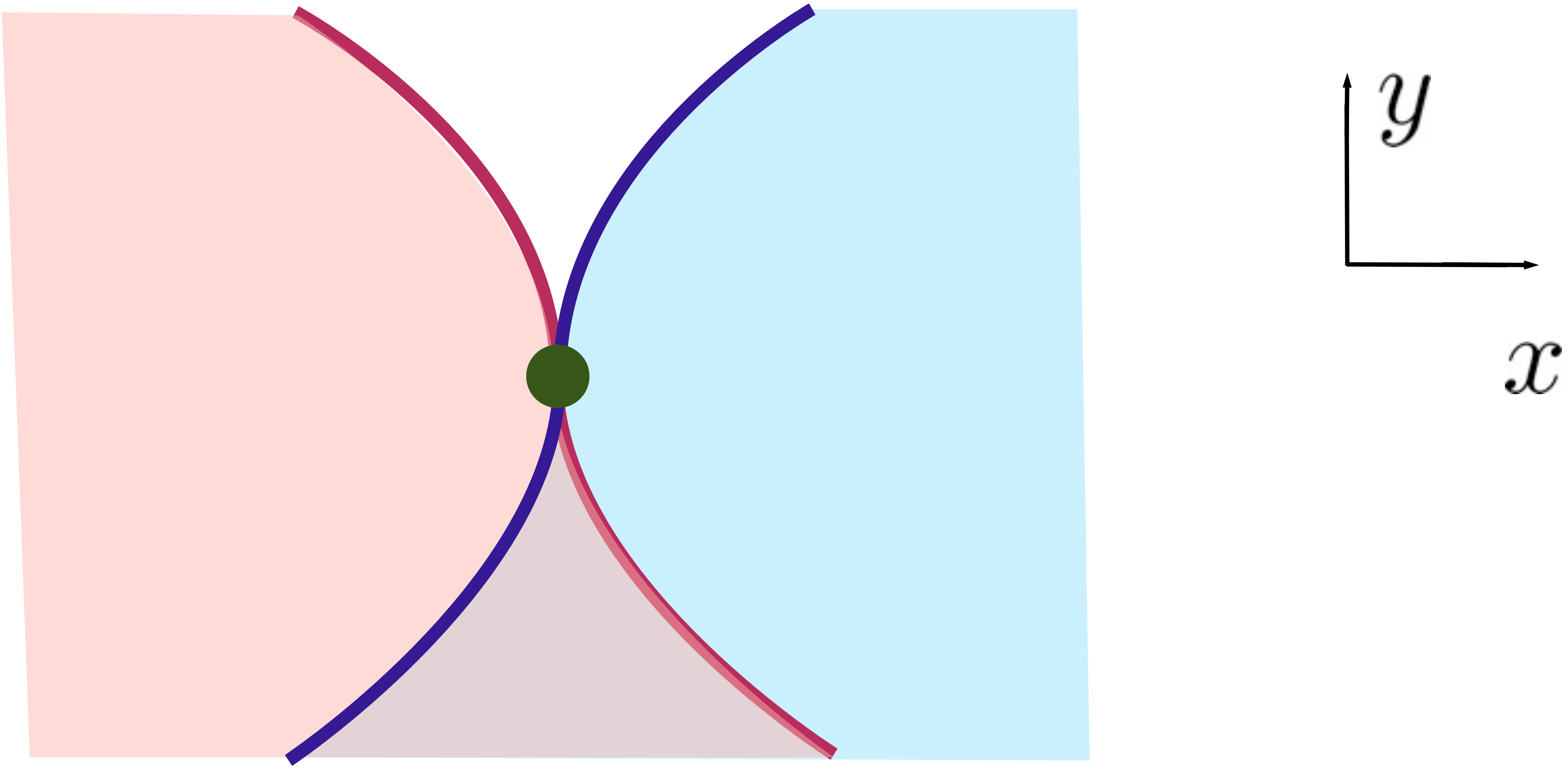
Dynamical Nesting



Dressed Fermi surface

RG-improved Migdal-Eliashberg theory

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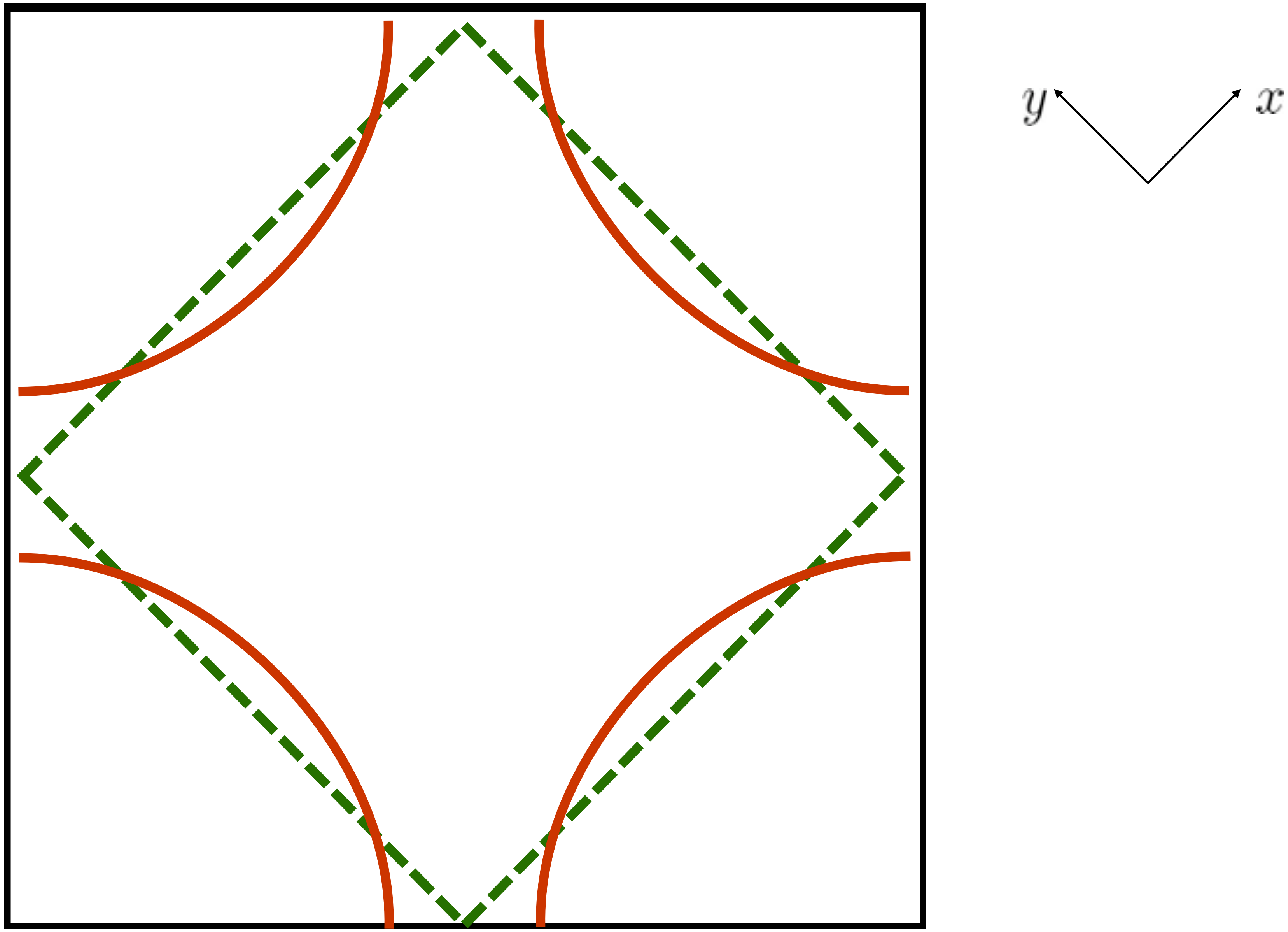
Dynamical Nesting



Bare Fermi surface

RG-improved Migdal-Eliashberg theory

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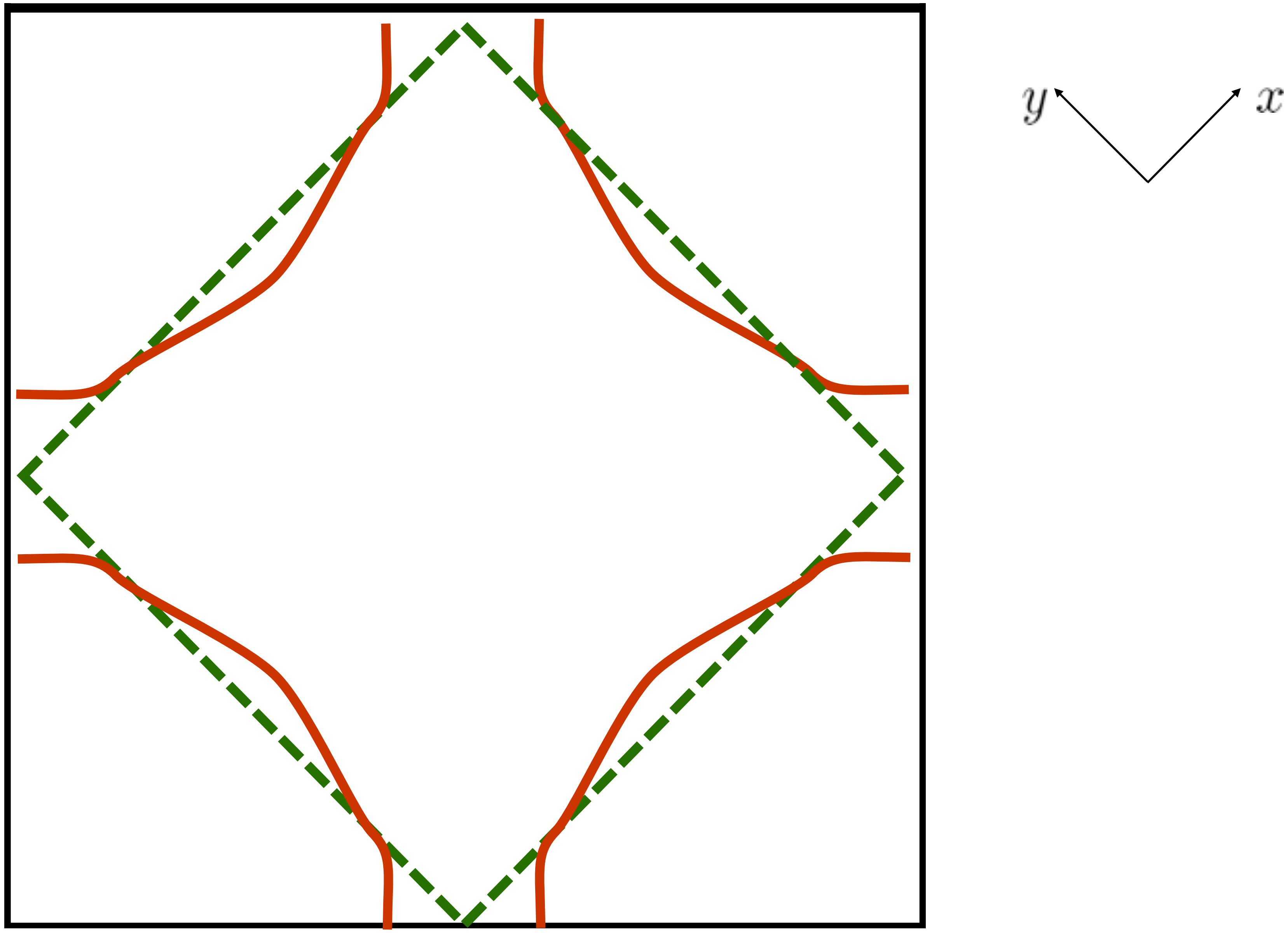
Dynamical Nesting



Dressed Fermi surface

RG-improved Migdal-Eliashberg theory

$\alpha = v_y/v_x \to 0$ logarithmically in the infrared.

In $\vec{\varphi}$ SDW fluctuations, characteristic q and ω scale as

$$q \sim \omega^{1/2} \exp\left(- \frac{3}{64\pi^2} \left(\frac{\ln(1/\omega)}{N} \right)^3 \right).$$

However, $1/N$ expansion cannot be trusted
in the asymptotic regime.

New infra-red singularities as $\zeta \to 0$ at higher loops (Breakdown of Migdal-Eliashberg)

$\vec{\varphi}$ propagator

$$\frac{1}{N}\frac{1}{\left(q^2+\gamma|\omega|\right)}$$

fermion propagator

$$\frac{1}{\mathbf{v}\cdot\mathbf{q}+i\zeta\omega+i\dfrac{1}{N\sqrt{\gamma}v}\sqrt{\omega}F\left(\dfrac{v^2q^2}{\omega}\right)}$$

New infra-red singularities as $\zeta \to 0$ at higher loops (Breakdown of Migdal-Eliashberg)

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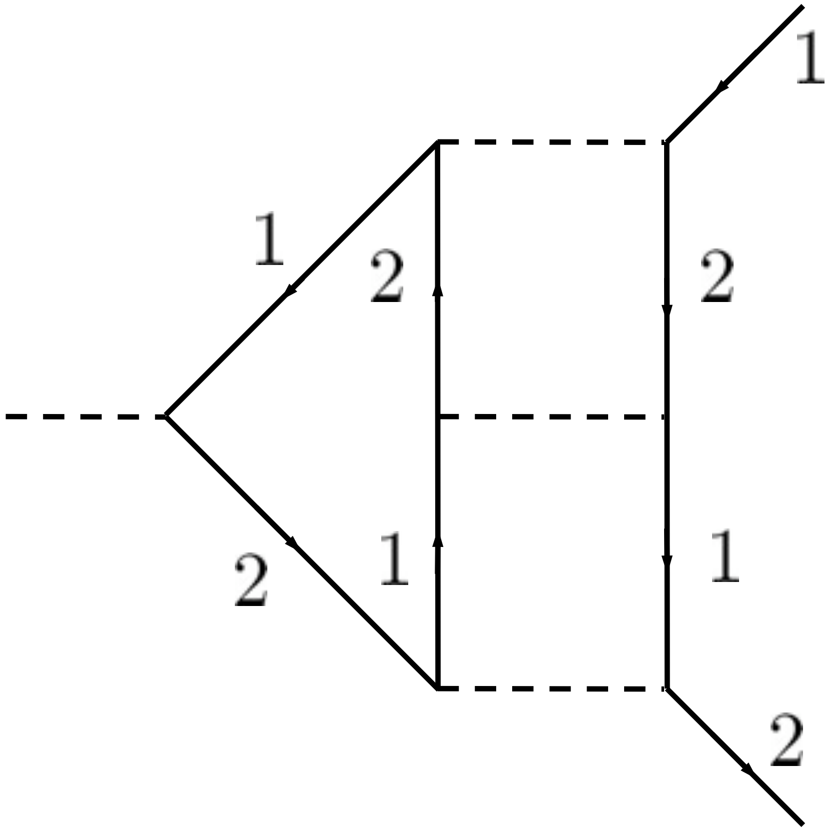
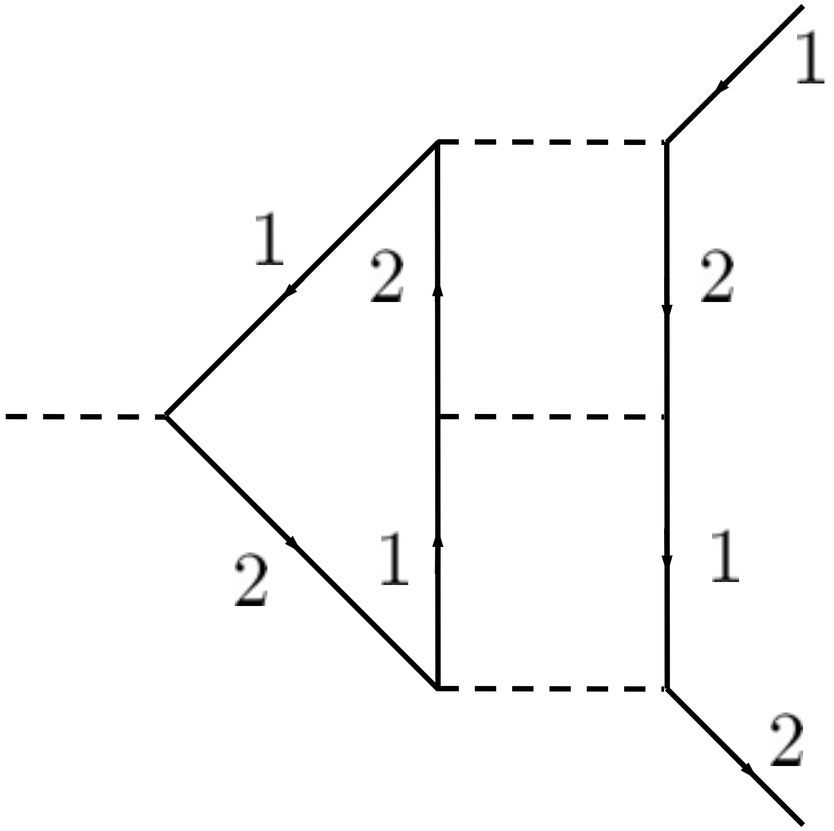
⚠ Dangerous

New infra-red singularities as $\zeta \to 0$ at higher loops (Breakdown of Migdal-Eliashberg)



Ignoring fermion self energy: $\sim \dfrac{1}{N^2} \times \dfrac{1}{\zeta^2} \times \dfrac{1}{\omega}$

New infra-red singularities as $\zeta \to 0$ at higher loops (Breakdown of Migdal-Eliashberg)



$$\text{Ignoring fermion self energy: } \sim \frac{1}{N^2} \times \frac{1}{\zeta^2} \times \frac{1}{\omega}$$

$$\text{Actual order } \sim \frac{1}{N^0}$$

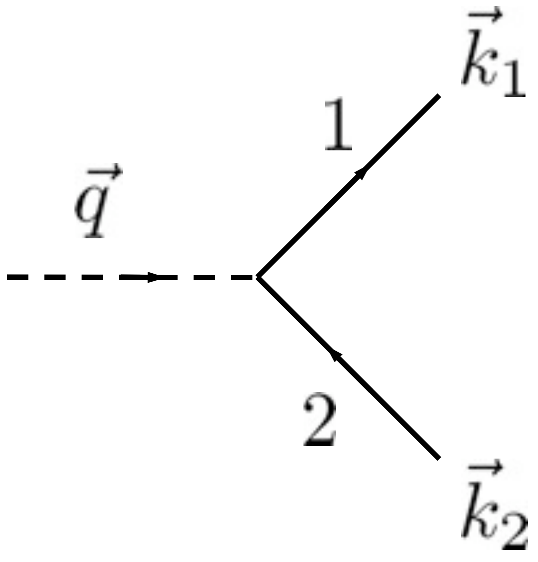
Double line representation

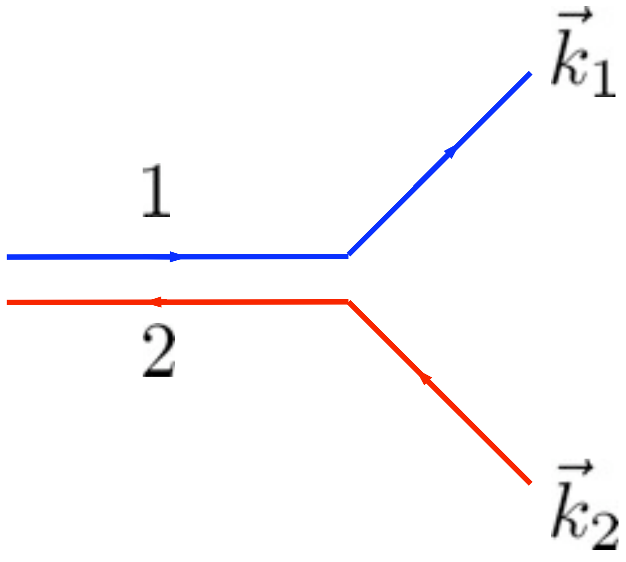
- A way to compute the order of a diagram.
- Extra powers of N come from the Fermi-surface

$$G(\omega,\vec{k}) = \frac{1}{-\Sigma_1(\omega,\vec{k}) - \vec{v}\cdot\vec{k}} \qquad \Sigma_1 \sim \frac{1}{N}$$

- What are the conditions for all propagators to be on the Fermi surface?
- Concentrate on diagrams involving a single pair of hot-spots
- Any bosonic momentum may be (uniquely) written as

$$\vec{q} = \vec{k}_1 - \vec{k}_2 \qquad \vec{k}_1 \in \text{FS of } \psi_1 \qquad \vec{k}_2 \in \text{FS of } \psi_2$$





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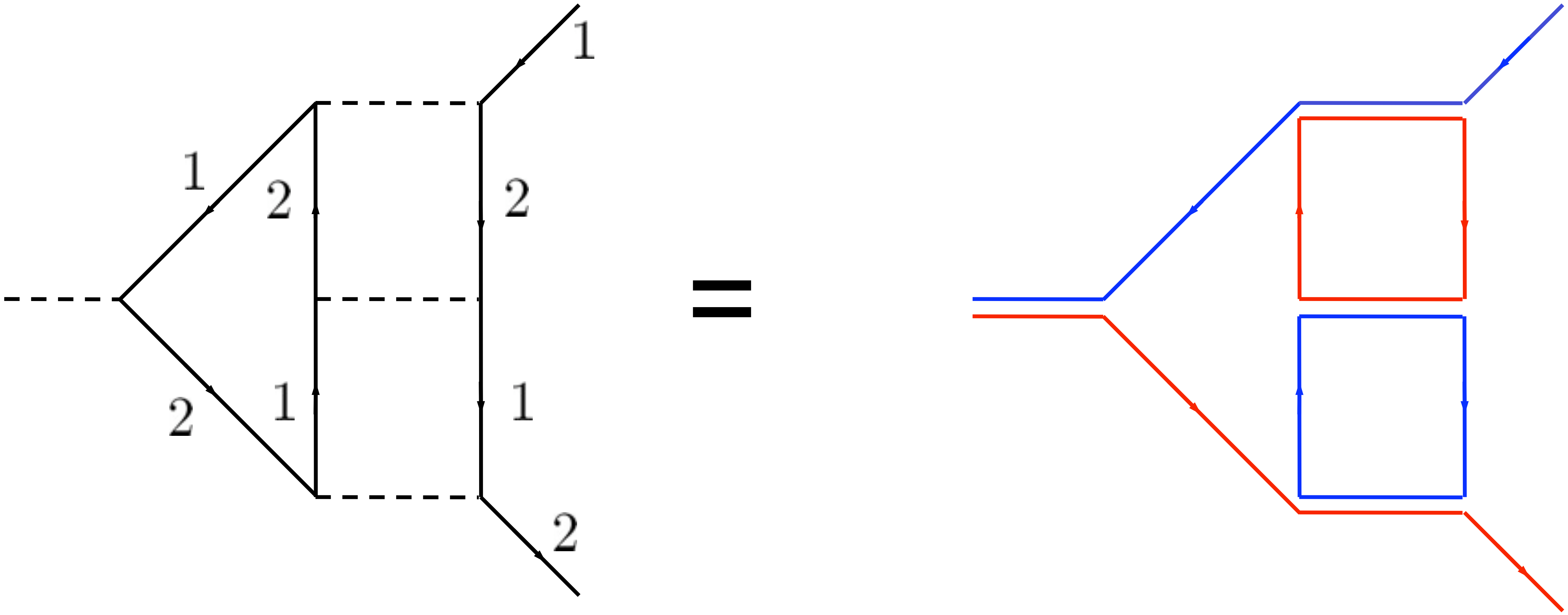
S. W. Tsai, A. H. Castro Neto, R. Shankar, and D. K. Campbell, Phys. Rev. B **72**, 054531 (2005).

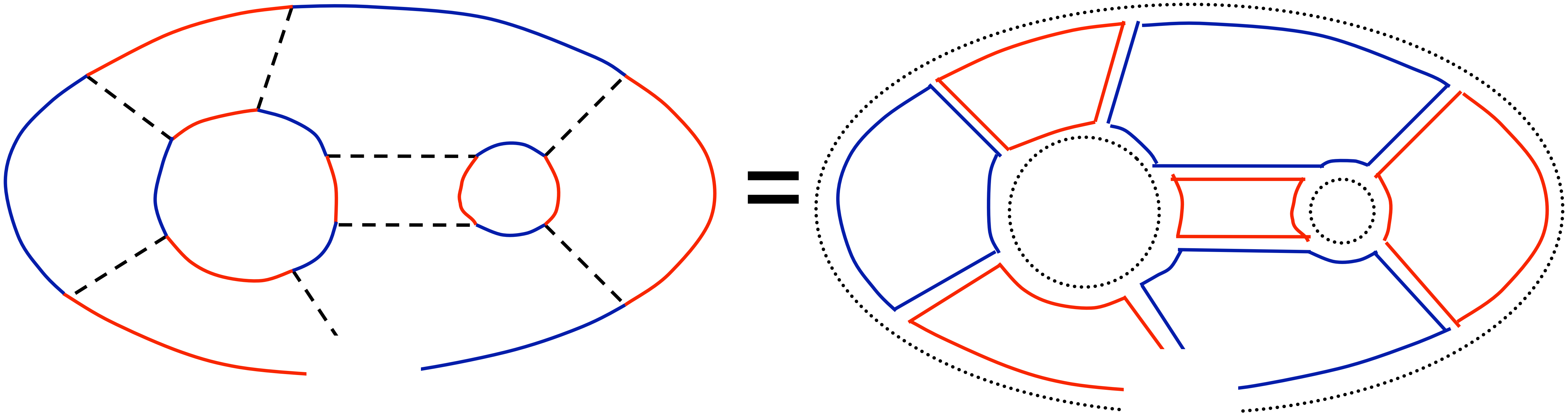
New infra-red singularities as $\zeta \to 0$ at higher loops (Breakdown of Migdal-Eliashberg)



Singularities as $\zeta \to 0$ appear when fermions in closed blue and red line loops are exactly on the Fermi surface

$$\text{Actual order} \sim \frac{1}{N^0}$$

New infra-red singularities as $\zeta \to 0$ at higher loops (Breakdown of Migdal-Eliashberg)

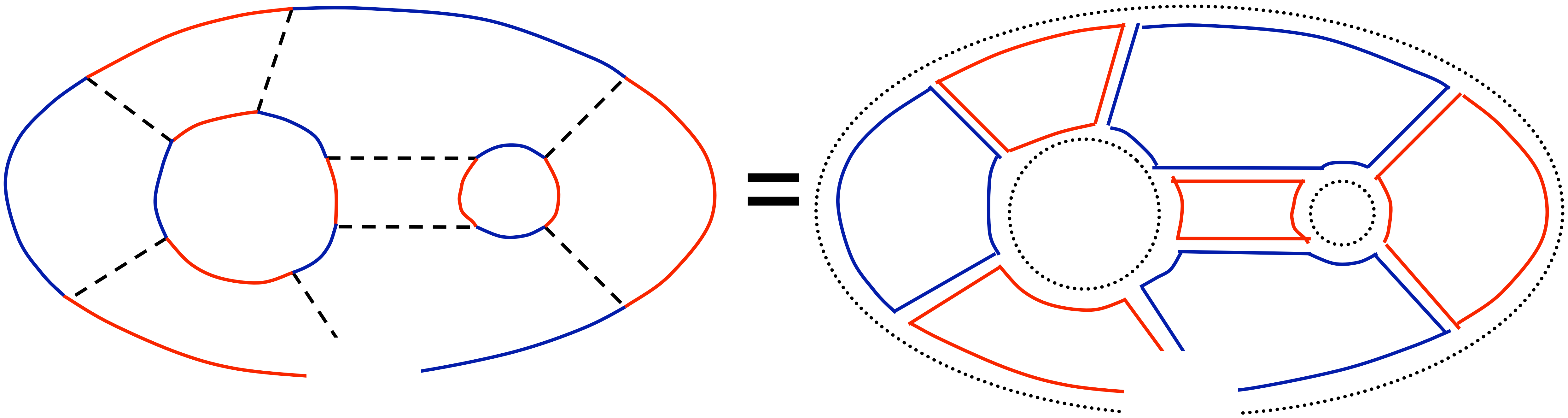


$$\text{Actual order} \sim \frac{1}{N^0}$$

Graph is **planar** after turning fermion propagators also into double lines by drawing additional dotted single line loops for each fermion loop

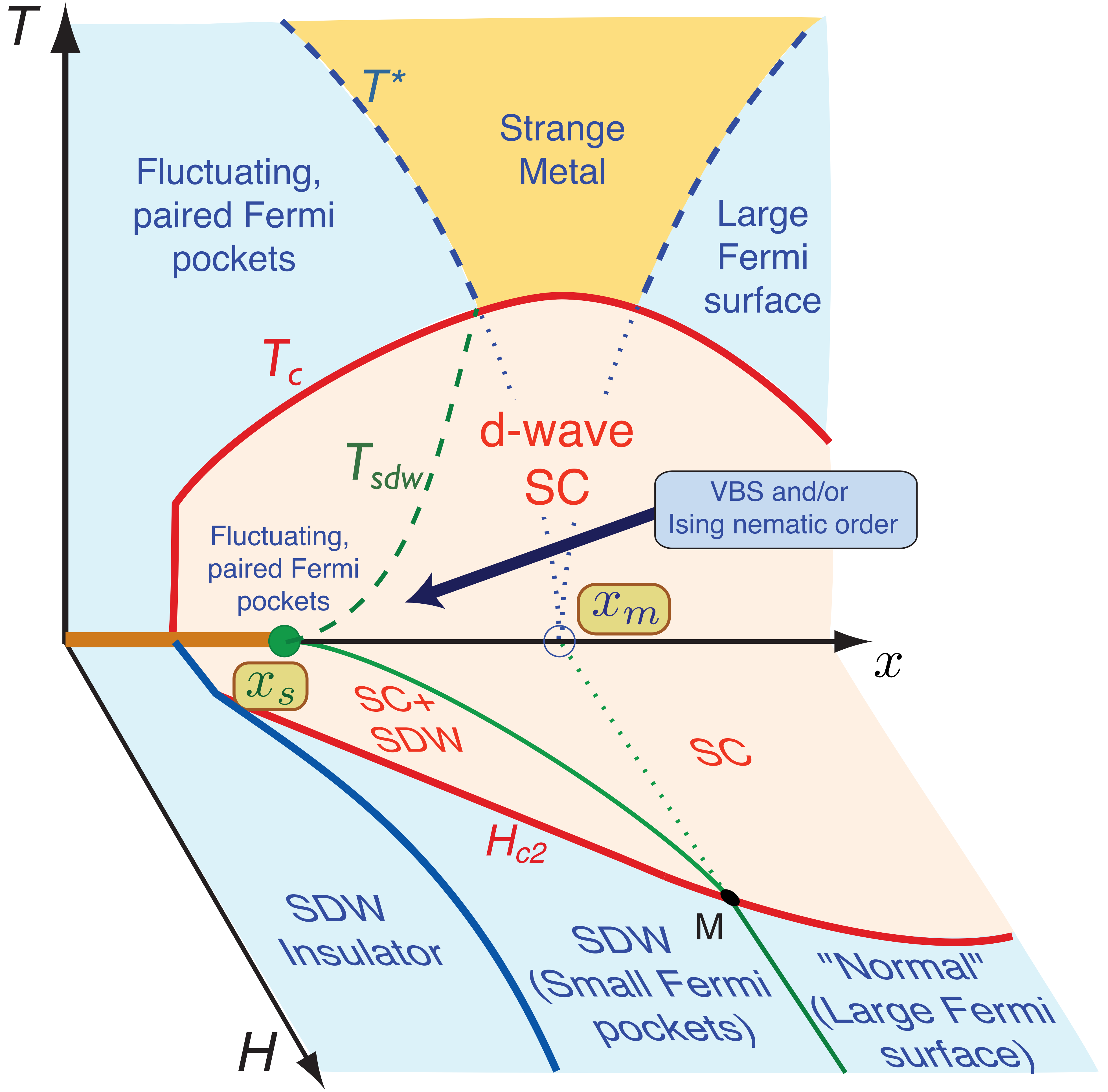
Sung-Sik Lee, arXiv:0905.4532

New infra-red singularities as $\zeta \to 0$ at higher loops (Breakdown of Migdal-Eliashberg)



$$\text{Actual order} \sim \frac{1}{N^0}$$

⚠ A consistent analysis requires resummation of all planar graphs ⚠


T
T^*
Strange Metal
Fluctuating, paired Fermi pockets
Large Fermi surface
T_c
T_{sdw}
d-wave SC
VBS and/or Ising nematic order
Fluctuating, paired Fermi pockets
x_m
x
x_s
SC+ SDW
SC
H_{c2}
M
SDW Insulator
SDW (Small Fermi pockets)
"Normal" (Large Fermi surface)
H

Conclusions

Identified quantum criticality in cuprate superconductors with a critical point at optimal doping associated with onset of spin density wave order in a metal

Elusive optimal doping quantum critical point has been "hiding in plain sight".

It is shifted to lower doping by the onset of superconductivity

Conclusions

Theory for the onset of spin density wave order in metals is *strongly* coupled in two dimensions