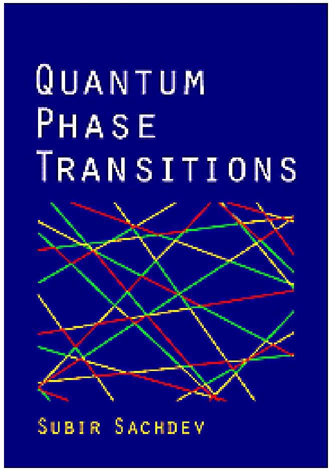
Quantum phase transitions of correlated electrons and atoms

Subir Sachdev

Harvard University

See also: *Quantum phase transitions of correlated electrons in two dimensions*, cond-mat/0109419.

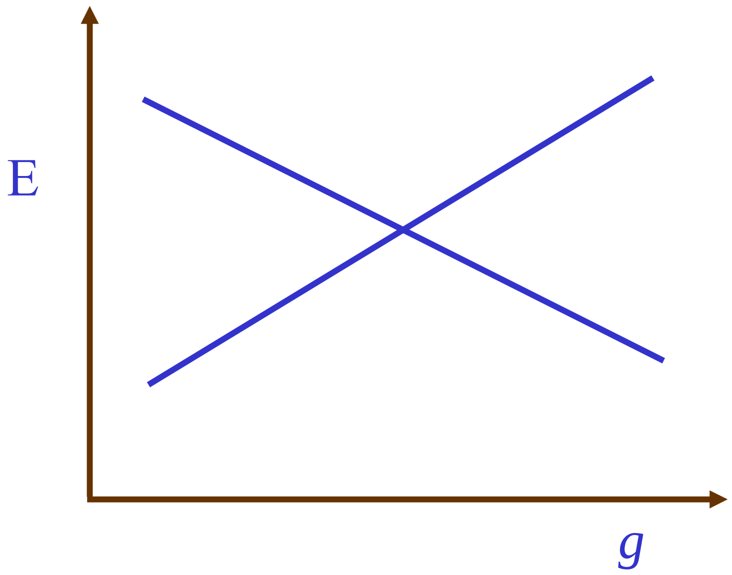

QUANTUM
PHASE
TRANSITIONS

SUBIR SACHDEV

Quantum Phase Transitions
Cambridge University Press

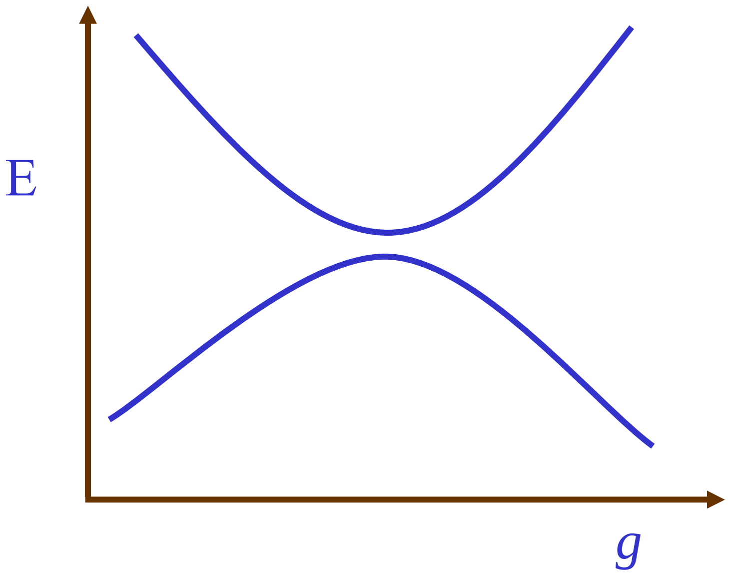
What is a quantum phase transition ?

Non-analyticity in ground state properties as a function of some control parameter *g*



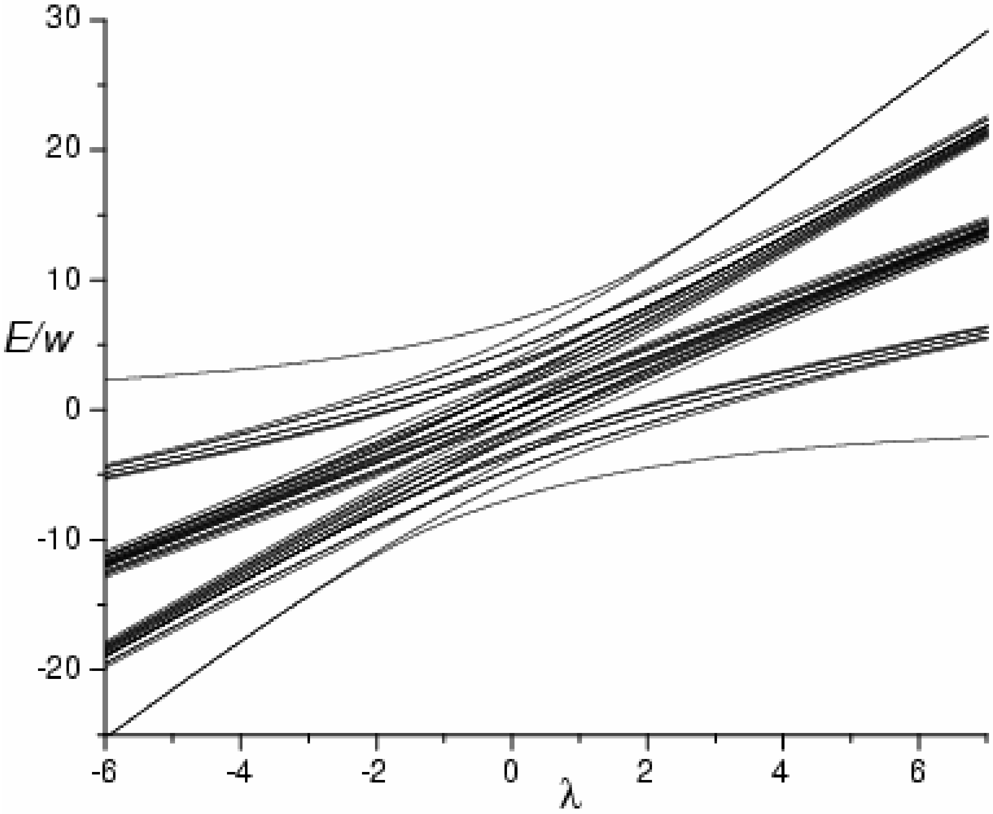
True level crossing:

Usually a *first*-order transition



Avoided level crossing which becomes sharp in the infinite volume limit:

second-order transition


30
20
10
0
-10
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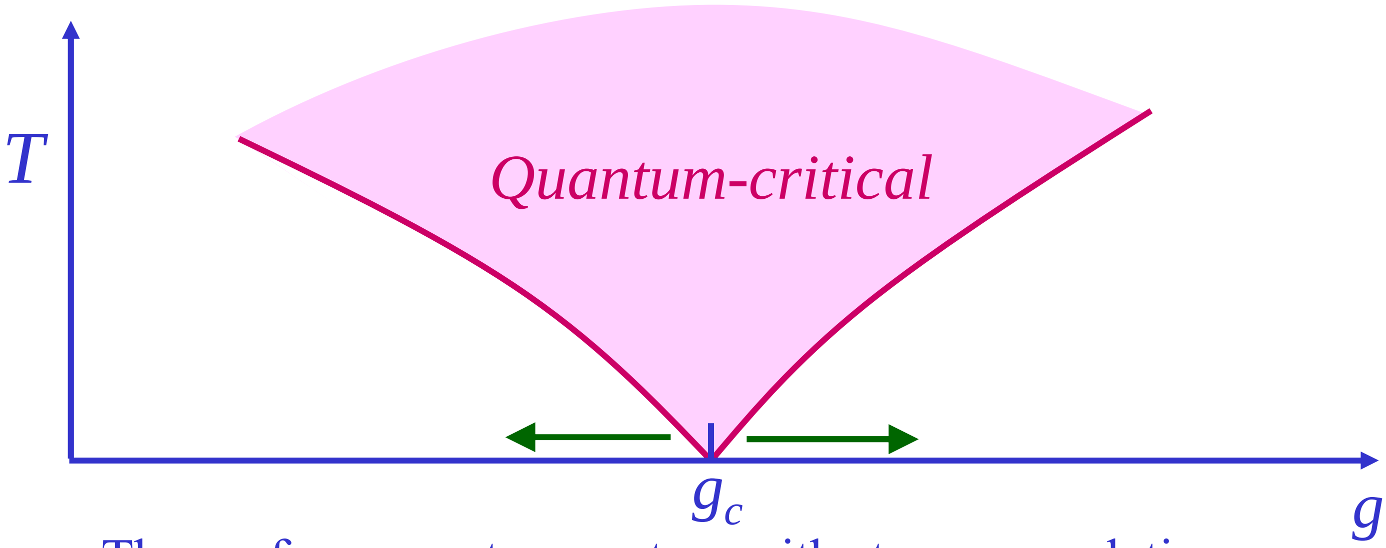
E/w
-6
-4
-2
0
2
4
6
λ

Why study quantum phase transitions ?



- Theory for a quantum system with strong correlations: describe phases on either side of g_c by expanding in deviation from the quantum critical point.

- Critical point is a novel state of matter without quasiparticle excitations

- Critical excitations control dynamics in the wide *quantum-critical* region at non-zero temperatures.

Important property of ground state at $g=g_c$:
temporal and spatial scale invariance;
characteristic energy scale at other values of g: $\Delta \sim |g - g_c|^{z\nu}$

Outline

I. Quantum Ising Chain

II. Landau-Ginzburg-Wilson theory
    Mean field theory and the evolution of the excitation spectrum.

III. Superfluid-insulator transition
    Boson Hubbard model at integer filling.

IV. Bosons at fractional filling
    Beyond the Landau-Ginzburg-Wilson paradigm.

V. Quantum phase transitions and the Luttinger theorem
    Depleting the Bose-Einstein condensate of trapped ultracold atoms – see talk by Stephen Powell

I. Quantum Ising Chain

I. Quantum Ising Chain

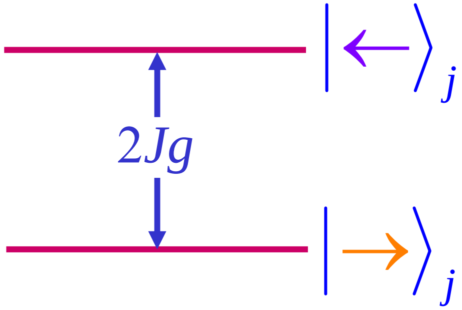
Degrees of freedom: $j = 1 \ldots N$ qubits, N "large"

$$|\uparrow\rangle_j , \; |\downarrow\rangle_j$$

or $$|\rightarrow\rangle_j = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_j + |\downarrow\rangle_j\right) , \quad |\leftarrow\rangle_j = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_j - |\downarrow\rangle_j\right)$$

Hamiltonian of decoupled qubits:

$$H_0 = -Jg \sum_j \sigma_j^x$$



Coupling between qubits:

$$H_1 = -J\sum_j \underbrace{\sigma_j^z\sigma_{j+1}^z}$$

$$\boxed{\Big(|\rightarrow\rangle_j\langle\leftarrow| + |\leftarrow\rangle_j\langle\rightarrow|\Big)\Big(|\rightarrow\rangle_{j+1}\langle\leftarrow| + |\leftarrow\rangle_{j+1}\langle\rightarrow|\Big)}$$

Prefers neighboring qubits

are *either* $|\uparrow\rangle_j|\uparrow\rangle_{j+1}$ *or* $|\downarrow\rangle_j|\downarrow\rangle_{j+1}$

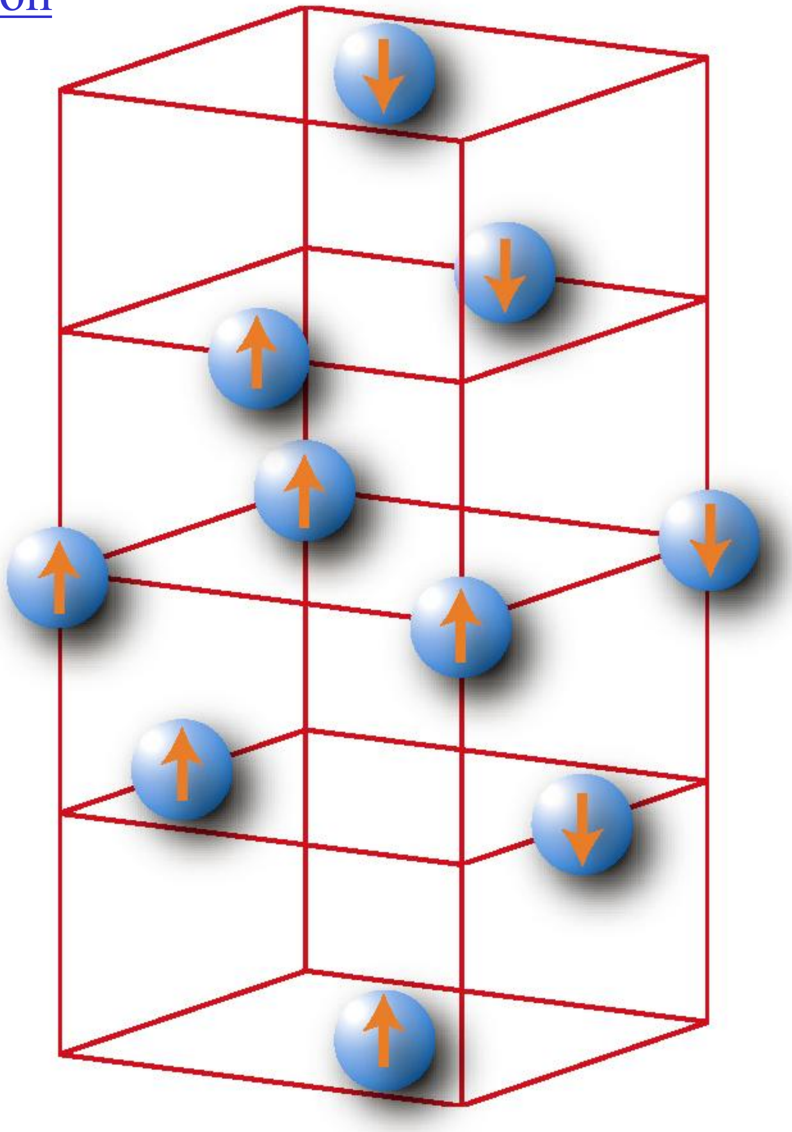
(not entangled)

Full Hamiltonian

$$\boxed{H = H_0 + H_1 = -J\sum_j\left(g\sigma_j^x + \sigma_j^z\sigma_{j+1}^z\right)}$$

leads to entangled states at *g* of order unity

Experimental realization



LiHoF$_4$

Weakly-coupled qubits $\left(g \gg 1\right)$

Ground state:

$$\left|G\right\rangle = \left|\cdots \rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow \cdots\right\rangle$$

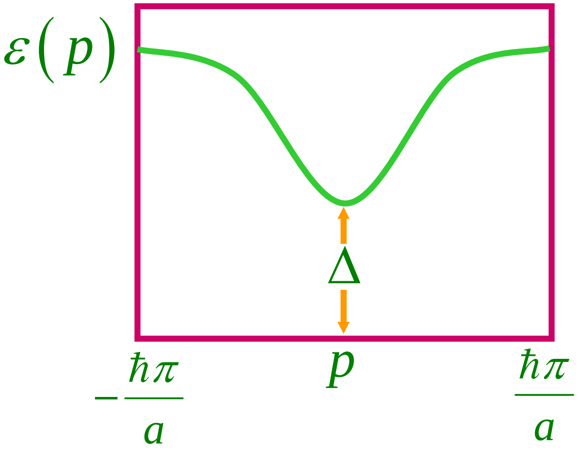
$$-\frac{1}{2g}\left|\cdots \rightarrow\rightarrow\rightarrow\rightarrow\leftarrow\leftarrow\rightarrow\rightarrow\rightarrow\rightarrow \cdots\right\rangle - \cdots$$

Lowest excited states:

$$\left|\ell_j\right\rangle = \left|\cdots \rightarrow\rightarrow\rightarrow\rightarrow\underset{j}{\leftarrow}\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow \cdots\right\rangle + \cdots$$

Coupling between qubits creates "flipped-spin" *quasiparticle* states at momentum p

$$\left|p\right\rangle = \sum_j e^{ipx_j/\hbar}\left|\ell_j\right\rangle$$



Excitation energy $\varepsilon(p) = \Delta + 4J\sin^2\left(\frac{pa}{2\hbar}\right) + O\left(g^{-1}\right)$

Excitation gap $\Delta = 2gJ - 2J + O\left(g^{-1}\right)$

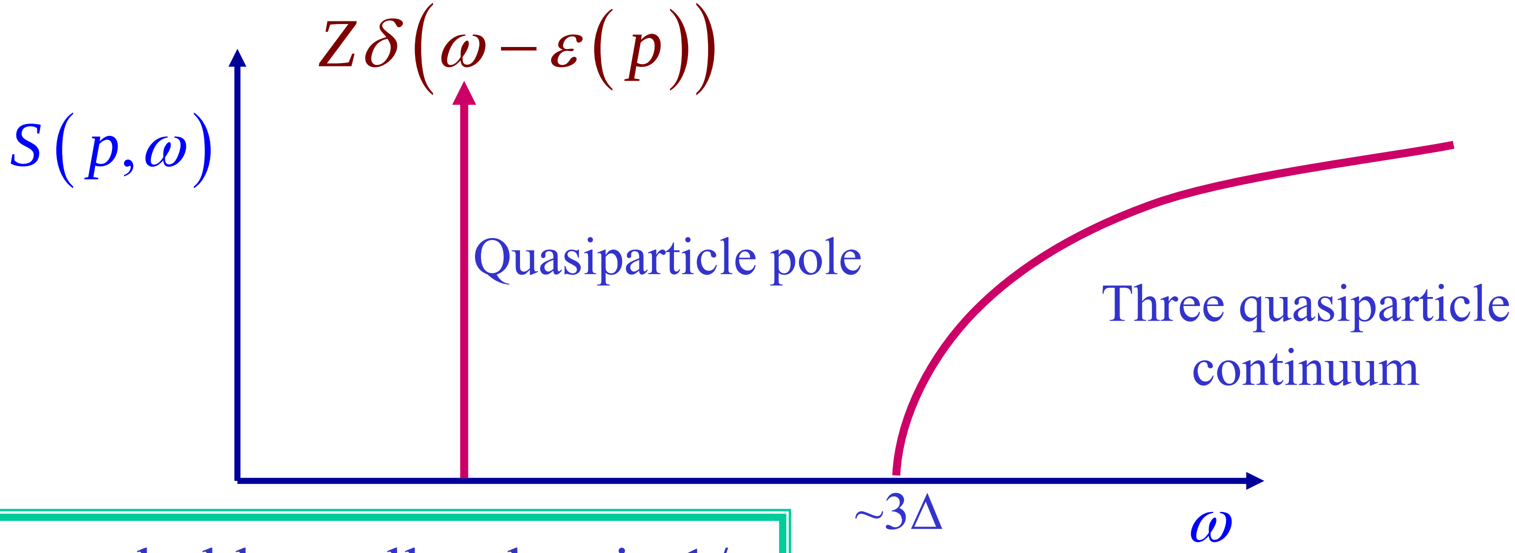
Entire spectrum can be constructed out of multi-quasiparticle states

Dynamic Structure Factor $S(p,\omega)$:

Weakly-coupled qubits $(g \gg 1)$

Cross-section to flip a $|\rightarrow\rangle$ to a $|\leftarrow\rangle$ (or vice versa)

while transferring energy $\hbar\omega$ and momentum p



Structure holds to all orders in $1/g$

At $T > 0$, collisions between quasiparticles broaden pole to a Lorentzian of width $1/\tau_\varphi$ where the ***phase coherence time*** τ_φ

is given by

$$\frac{1}{\tau_\varphi} = \frac{2k_BT}{\pi\hbar} e^{-\Delta/k_BT}$$

S. Sachdev and A.P. Young, *Phys. Rev. Lett.* **78**, 2220 (1997)

Strongly-coupled qubits $\left(g \ll 1\right)$

Ground states:

$$\left|G\uparrow\right\rangle = \left|\cdots\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\cdots\right\rangle$$

$$-\frac{g}{2}\left|\cdots\uparrow\uparrow\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\uparrow\uparrow\cdots\right\rangle - \cdots$$

Ferromagnetic moment

$$N_0 = \left\langle G\right|\sigma^z\left|G\right\rangle \neq 0$$

Second state $\left|G\downarrow\right\rangle$ obtained by $\uparrow \;\Leftrightarrow\; \downarrow$

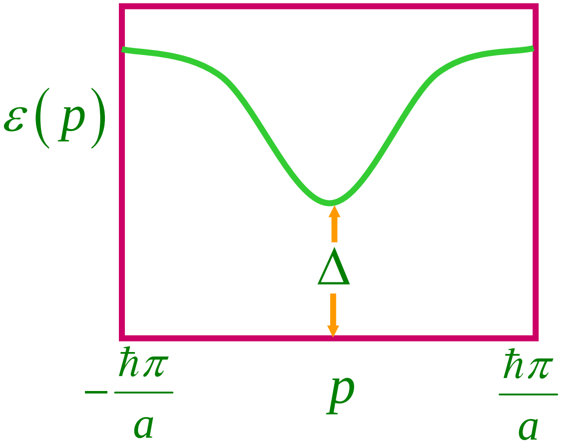
$\left|G\downarrow\right\rangle$ and $\left|G\uparrow\right\rangle$ mix only at order g^N

Lowest excited states: domain walls

$$\left|d_j\right\rangle = \left|\cdots\uparrow\uparrow\uparrow\uparrow\uparrow_j\downarrow\downarrow\downarrow\downarrow\downarrow\cdots\right\rangle + \cdots$$

Coupling between qubits creates new “domain-wall” *quasiparticle* states at momentum p

$$\left|p\right\rangle = \sum_j e^{ipx_j/\hbar}\left|d_j\right\rangle$$



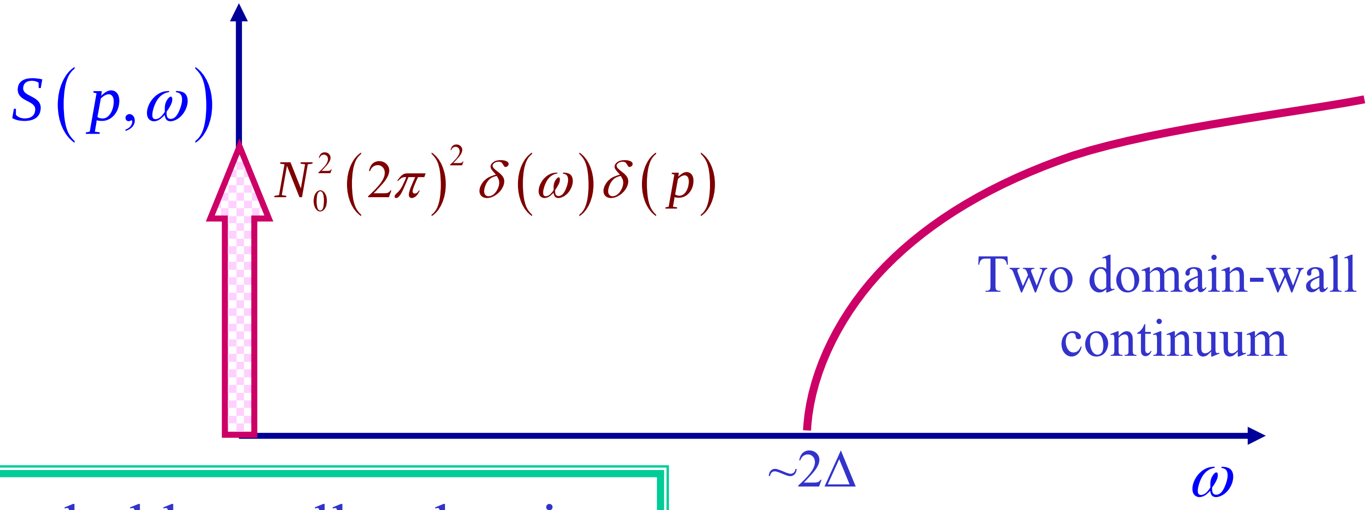
Excitation energy $\varepsilon\left(p\right) = \Delta + 4Jg\sin^2\left(\frac{pa}{2\hbar}\right) + O\left(g^2\right)$

Excitation gap $\Delta = 2J - 2gJ + O\left(g^2\right)$

Dynamic Structure Factor $S(p,\omega)$:

Strongly-coupled qubits $(g \ll 1)$

Cross-section to flip a $|\rightarrow\rangle$ to a $|\leftarrow\rangle$ (or vice versa) while transferring energy $\hbar\omega$ and momentum p



Structure holds to all orders in g

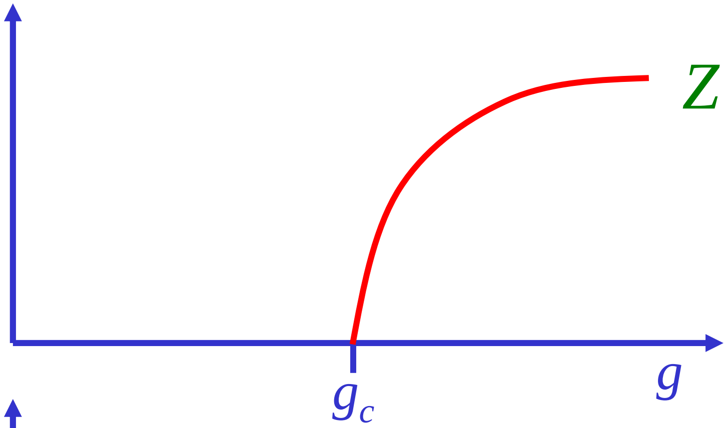
At $T > 0$, motion of domain walls leads to a finite ***phase coherence time*** τ_φ, and broadens coherent peak to a width $1/\tau_\varphi$ where

$$\frac{1}{\tau_\varphi} = \frac{2k_B T}{\pi\hbar} e^{-\Delta/k_B T}$$

S. Sachdev and A.P. Young, *Phys. Rev. Lett.* **78**, 2220 (1997)

Entangled states at g of order unity

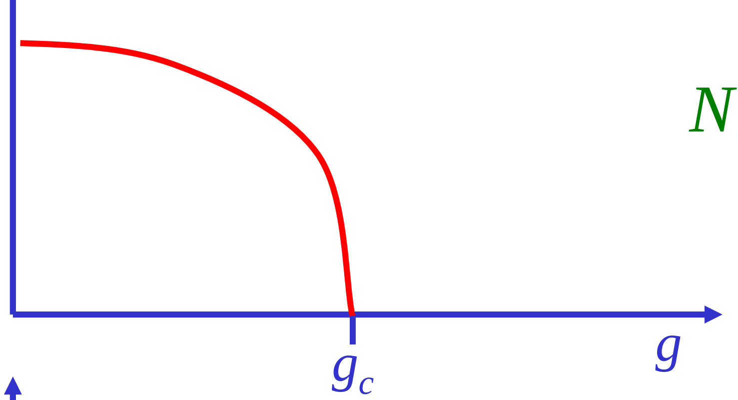
"Flipped-spin"
Quasiparticle
weight Z



$$Z \sim (g - g_c)^{1/4}$$

A.V. Chubukov, S. Sachdev, and J.Ye,
Phys. Rev. B **49**, 11919 (1994)

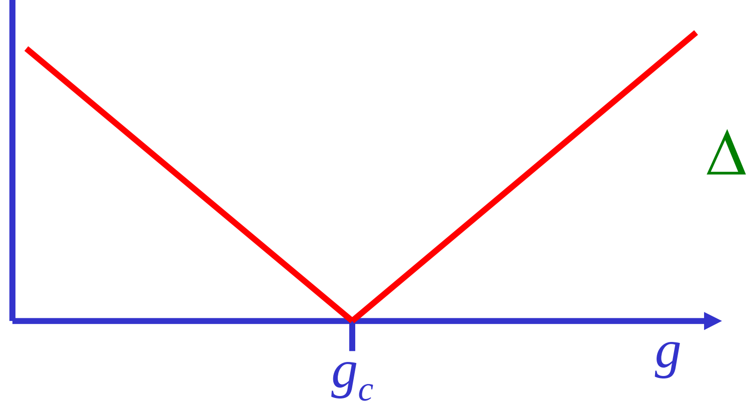
Ferromagnetic
moment N_0



$$N_0 \sim (g_c - g)^{1/8}$$

P. Pfeuty *Annals of Physics*, **57**, 79 (1970)

Excitation
energy gap Δ

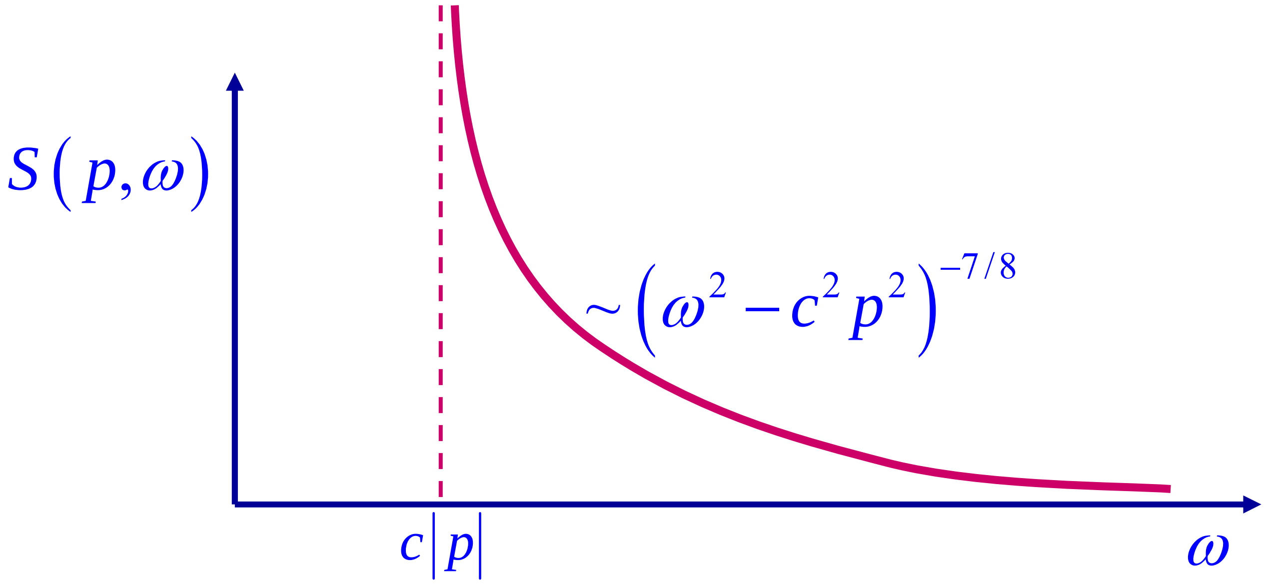


$$\Delta \sim |g - g_c|$$

Dynamic Structure Factor $S(p,\omega)$:

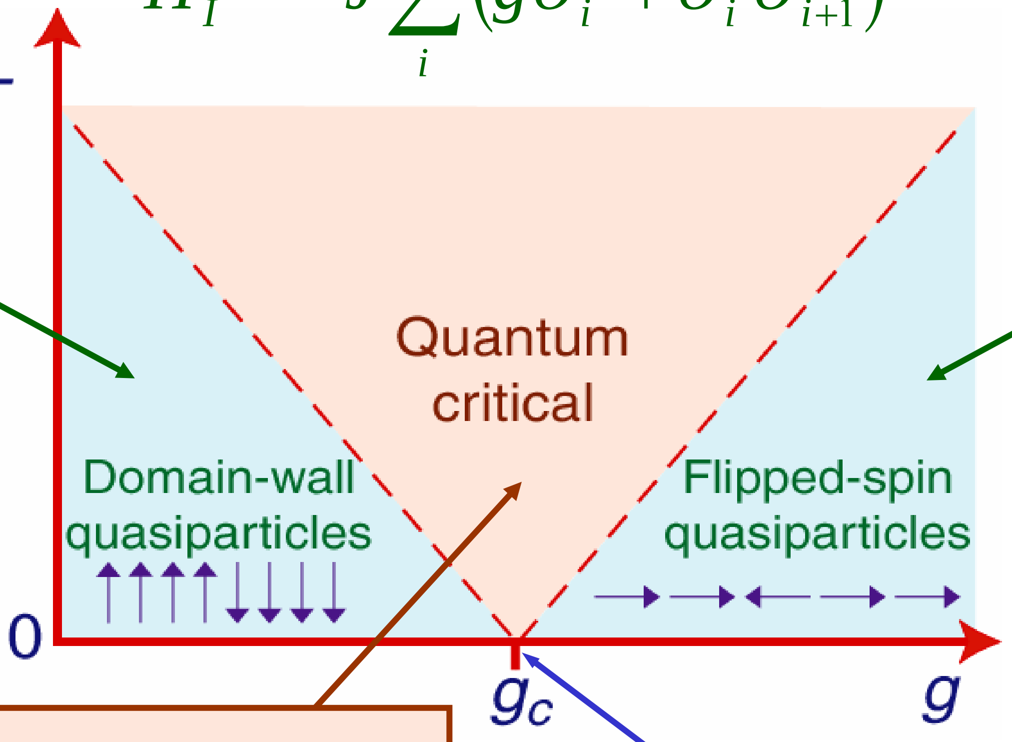
Critical coupling $(g = g_c)$

Cross-section to flip a $|\rightarrow\rangle$ to a $|\leftarrow\rangle$ (or vice versa) while transferring energy $\hbar\omega$ and momentum p



No quasiparticles --- dissipative critical continuum

$$H_I = -J\sum_i \left(g\sigma_i^x + \sigma_i^z \sigma_{i+1}^z \right)$$



Quasiclassical dynamics

Quasiclassical dynamics

$$\chi(\omega) = \frac{i}{\hbar}\sum_k \int_0^\infty dt \left\langle \left[\sigma_j^z(t), \sigma_k^z(0) \right] \right\rangle e^{i\omega t}$$

$$= \frac{A}{T^{7/4}\left(1 - i\omega/\Gamma_R + \ldots\right)}$$

$$\Gamma_R = \left(2\tan\frac{\pi}{16} \right) \frac{k_B T}{\hbar}$$

$$\left\langle \sigma_j^z \sigma_k^z \right\rangle \sim \frac{1}{\left| j-k \right|^{1/4}}$$

P. Pfeuty *Annals of Physics*, **57**, 79 (1970)

S. Sachdev and J. Ye, Phys. Rev. Lett. **69**, 2411 (1992).
S. Sachdev and A.P. Young, Phys. Rev. Lett. **78**, 2220 (1997).

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II. Landau-Ginzburg-Wilson theory

Mean field theory and the evolution of the excitation spectrum

- Identify order parameter $\phi(x,\tau) \sim \sigma_j^z$
- Symmetries:

$$\begin{aligned} \text{Spin inversion:} \quad & \phi \rightarrow -\phi \\ \text{Time reversal} \quad & \tau \rightarrow -\tau \\ \text{Spatial inversion} \quad & x \rightarrow -x \end{aligned}$$

- Write down most general Lagrangian consistent with symmetries

$$\begin{aligned} \mathcal{Z} &= \int \mathcal{D}\phi(x,\tau) \exp\left(- \int d^d x \int d\tau \mathcal{L}[\phi] \right) \\ \mathcal{L}[\phi] &= \frac{1}{2} (\partial_\tau \phi)^2 + \frac{c^2}{2} (\nabla_x \phi)^2 + \frac{r}{2}\phi^2 + \frac{u}{4}\phi^4 + \ldots \end{aligned}$$

- Identify phases at $r \gg 0$ and $r \ll 0$ with the paramagnet and the ferromagnet respectively.

Quantum field theory formally resembles the classical statistical mechanics of an Ising model in $d+1$ dimensions. Theory of second-order classical phase transitions implies that at the critical point the susceptibility depends on the $d+1$ dimensional momentum k as

$$\chi(k) \sim \frac{1}{k^{2-\eta}}$$

After analytic continuation, and using the “Lorentz invariance” of the critical theory, the quantum critical point therefore has the following dynamic susceptibility at $T=0$.

$$\chi(p,\omega) \sim \frac{1}{(c^2p^2-\omega^2)^{1-\eta/2}}$$

At $T>0$, we have to consider a classical statistical mechanics problem in finite geometry with a ‘temporal’ direction of extent $L_\tau = \hbar/(k_BT)$. *Finite size scaling* now implies that the susceptibility at the critical point obeys

$$\chi(k) \sim L_\tau^{2-\eta} F\left(kL_\tau\right)$$

After analytic continuation, the quantum system has the dynamic response (note: can no longer use ”Lorentz invariance”)

$$\chi''(0,\omega) \sim \frac{1}{T^{2-\eta}} \Phi\left(\frac{\hbar\omega}{k_BT}\right)$$

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III. Superfluid-insulator transition

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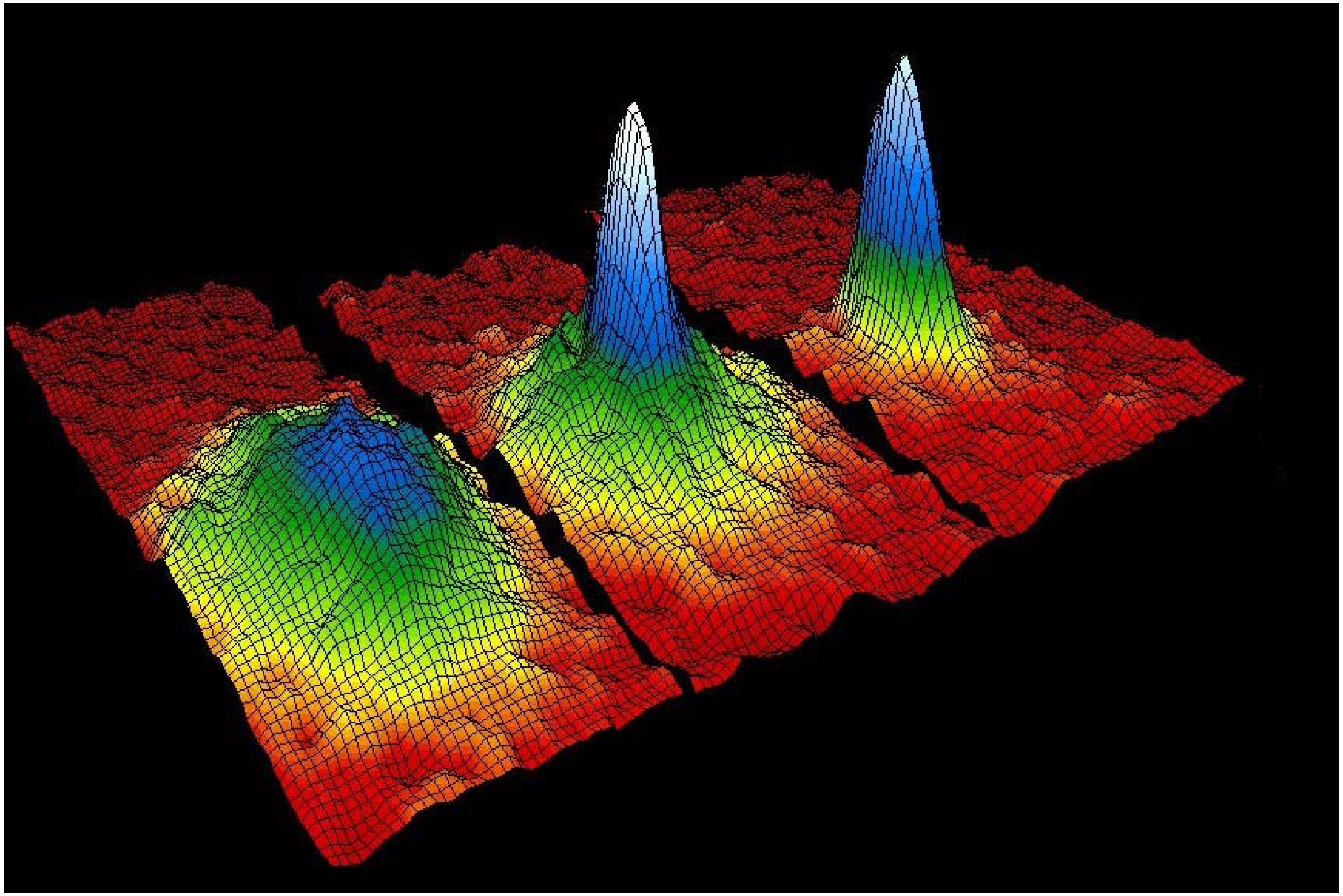
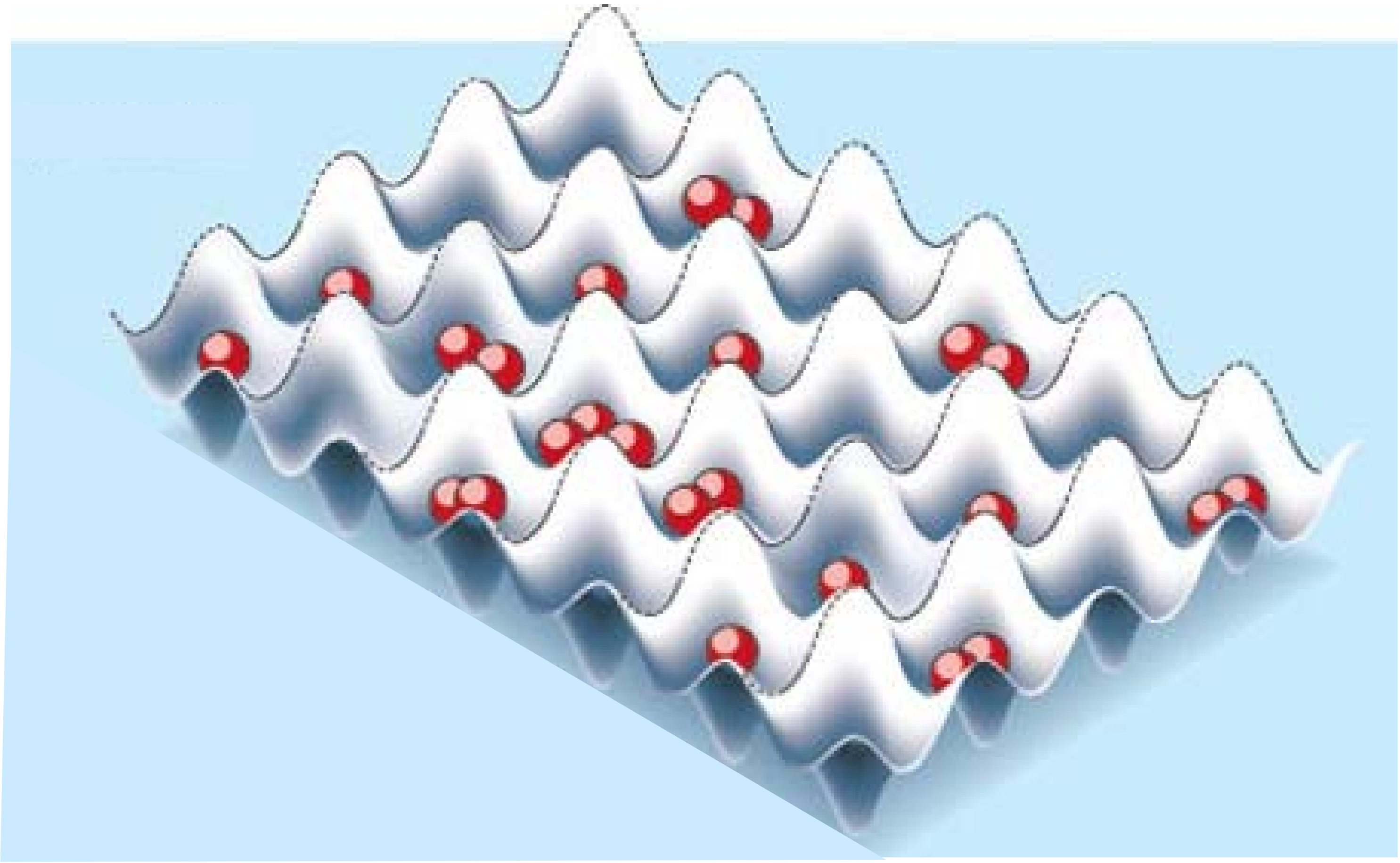
Bose condensation

Velocity distribution function of ultracold ^{87}Rb atoms

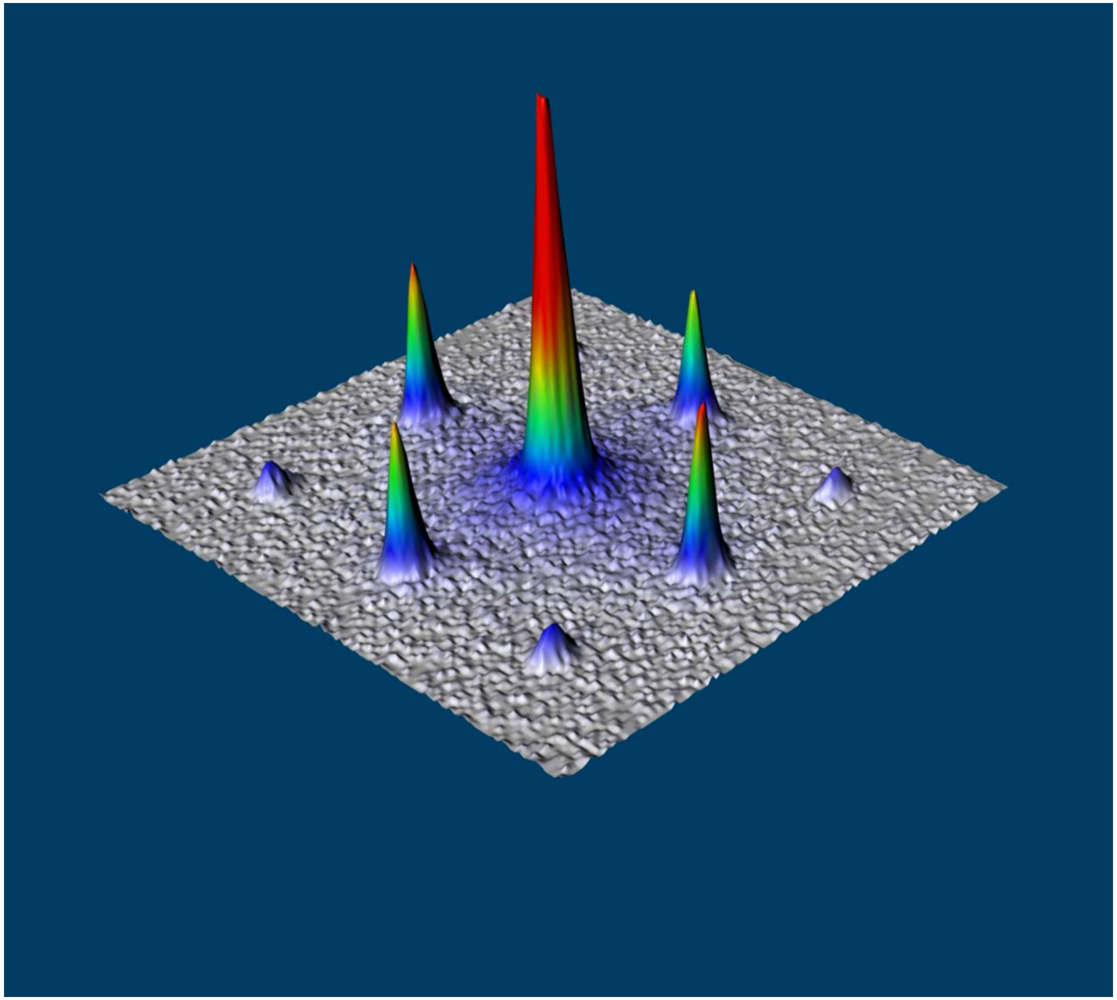


M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman and E. A. Cornell, *Science* **269**, 198 (1995)

Apply a periodic potential (standing laser beams) to trapped ultracold bosons (^{87}Rb)



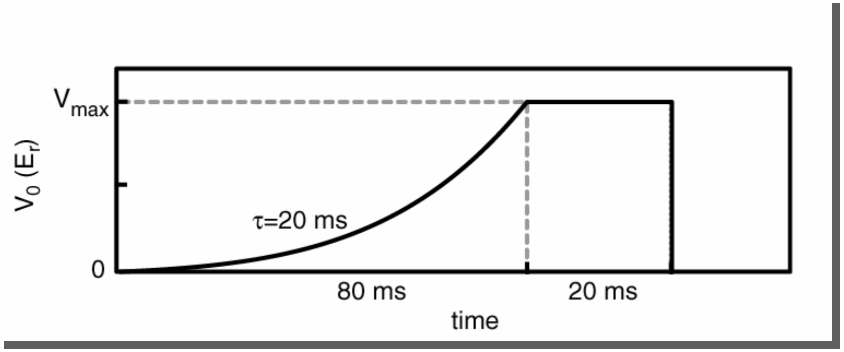
Momentum distribution function of bosons

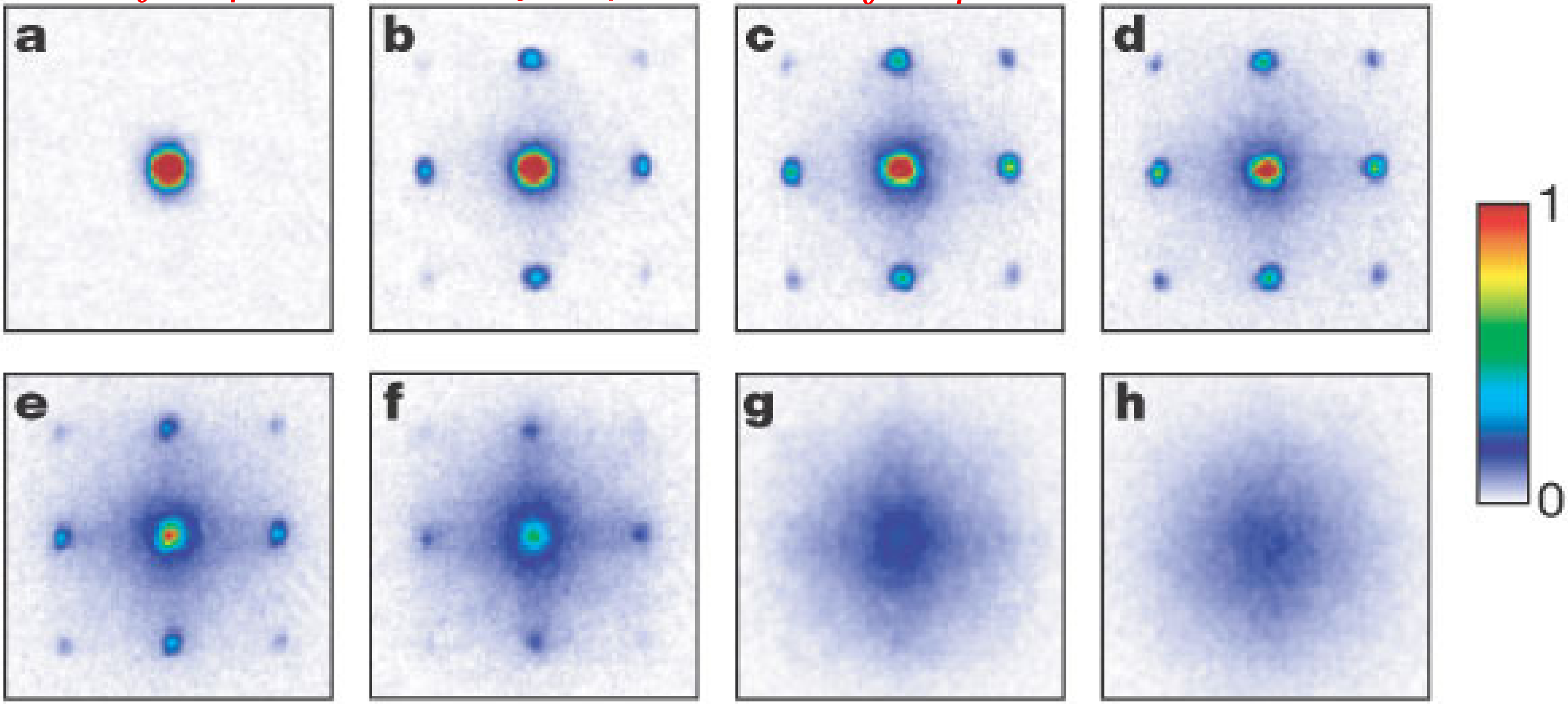


Bragg reflections of condensate at reciprocal lattice vectors

M. Greiner, O. Mandel, T. Esslinger, T. W. Hänsch, and I. Bloch, *Nature* **415**, 39 (2002).

Superfluid-insulator quantum phase transition at $T=0$





$V_0=0E_r$

$V_0=3E_r$

$V_0=7E_r$

$V_0=10E_r$

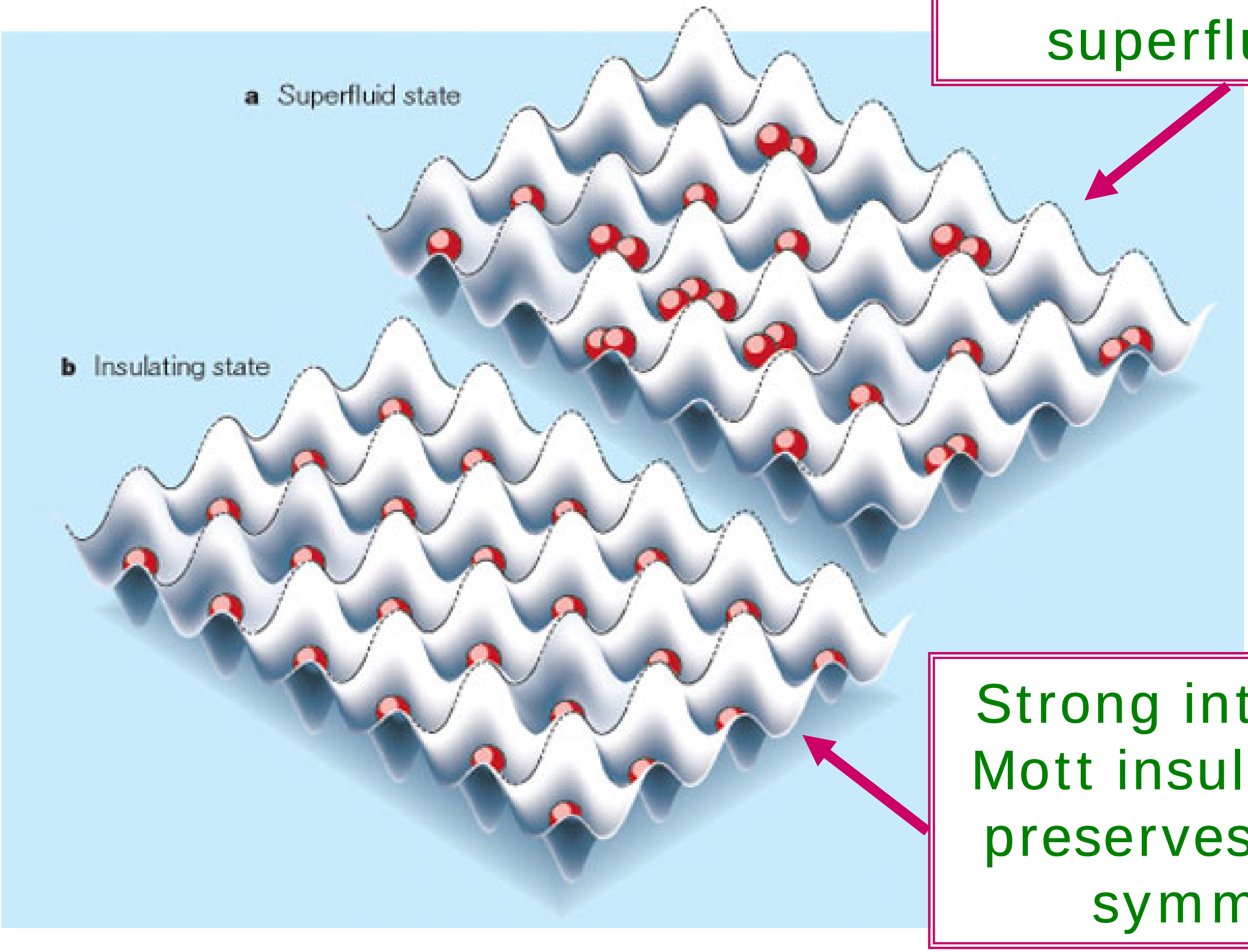
$V_0=13E_r$

$V_0=14E_r$

$V_0=16E_r$

$V_0=20E_r$

Bosons at filling fraction $f = 1$

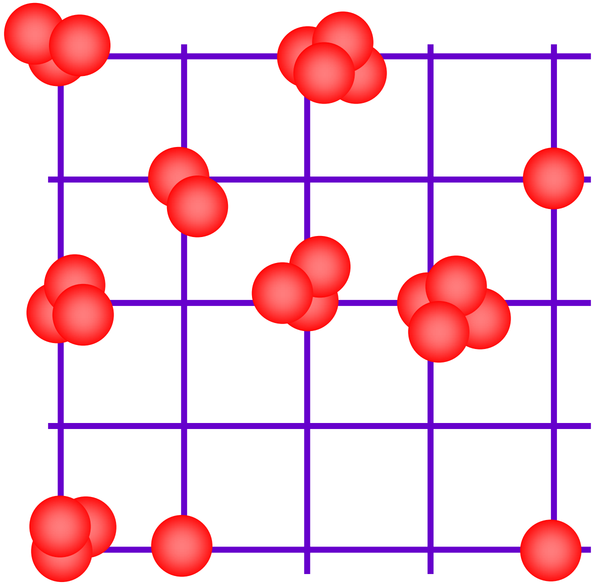


Weak interactions: superfluidity

Strong interactions: Mott insulator which preserves all lattice symmetries

M. Greiner, O. Mandel, T. Esslinger, T. W. Hänsch, and I. Bloch, *Nature* **415**, 39 (2002).

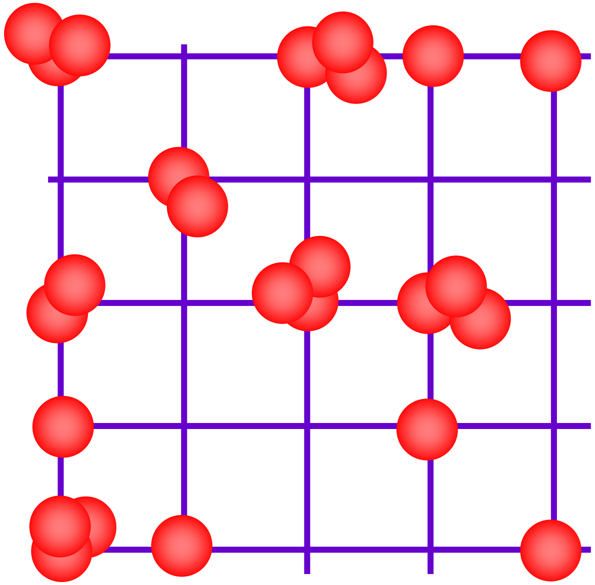
Bosons at filling fraction $f = 1$



$$\langle \Psi \rangle \neq 0$$

Weak interactions: superfluidity

Bosons at filling fraction $f = 1$



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Weak interactions: superfluidity

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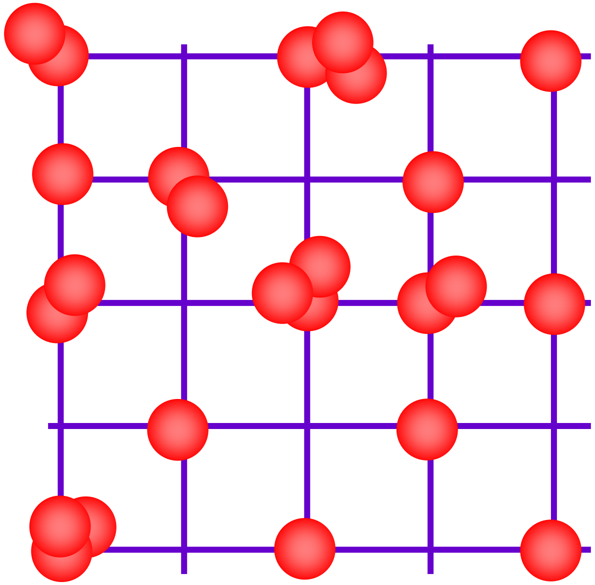

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Weak interactions: superfluidity

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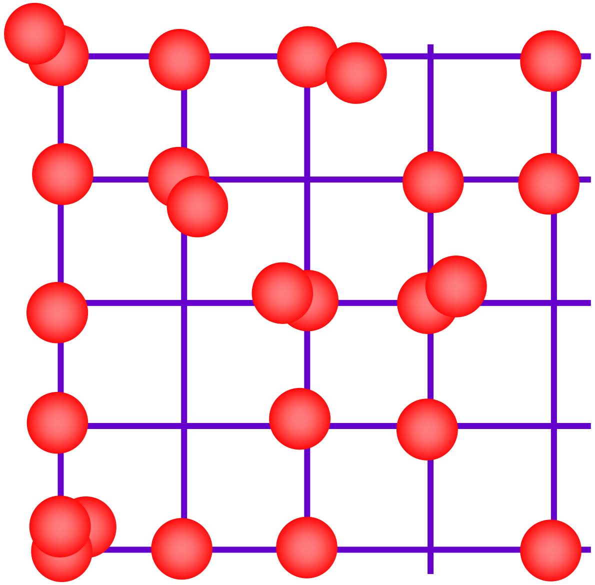

$$\langle \Psi \rangle \neq 0$$

Weak interactions: superfluidity

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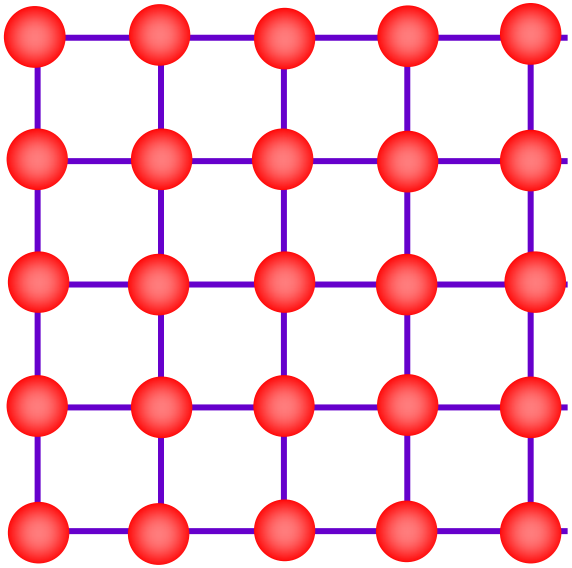


$$\langle \Psi \rangle = 0$$

Strong interactions: insulator

The Superfluid-Insulator transition

Boson Hubbard model

Degrees of freedom: Bosons, $b_j^\dagger$, hopping between the sites, j, of a lattice, with short-range repulsive interactions.

$$H = -t\sum_{\langle ij\rangle} b_i^\dagger b_j - \mu \sum_j n_j + \frac{U}{2}\sum_j n_j(n_j - 1) + \cdots$$

$$n_j \equiv b_j^\dagger b_j$$

M.PA. Fisher, P.B. Weichmann,
G. Grinstein, and D.S. Fisher
Phys. Rev. B **40**, 546 (1989).

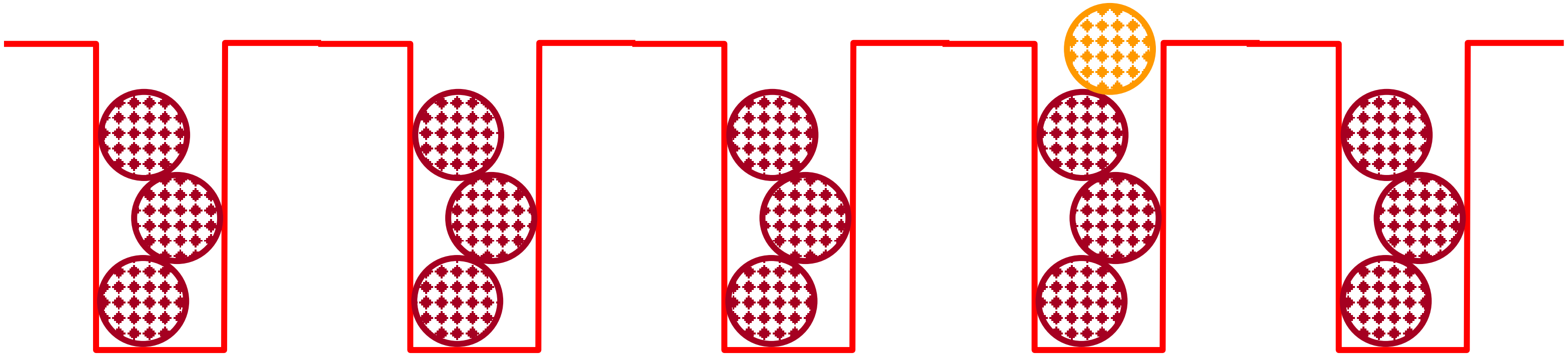
For small *U/t*, ground state is a superfluid BEC with

superfluid density $\approx$ density of bosons

What is the ground state for large U/t ?

Typically, the ground state remains a superfluid, but with

superfluid density $\ll$ density of bosons

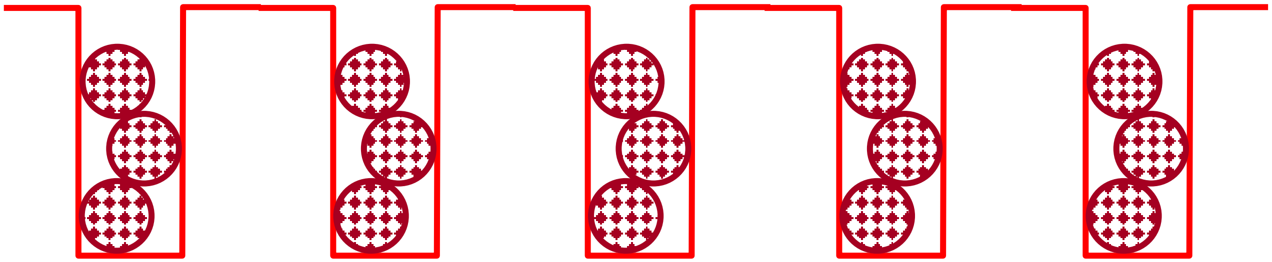


The superfluid density evolves smoothly from large values at small U/t, to small values at large U/t, and there is no quantum phase transition at any intermediate value of U/t.

(In systems with Galilean invariance and at zero temperature, superfluid density=density of bosons always, independent of the strength of the interactions)

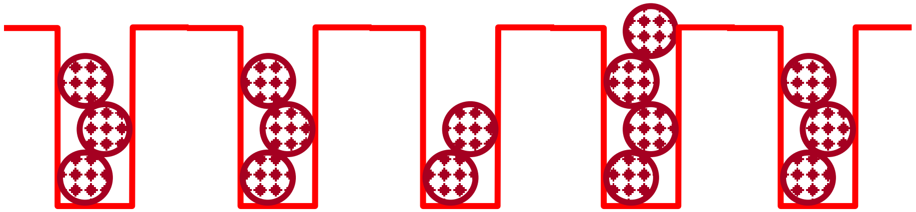
What is the ground state for large *U/t* ?

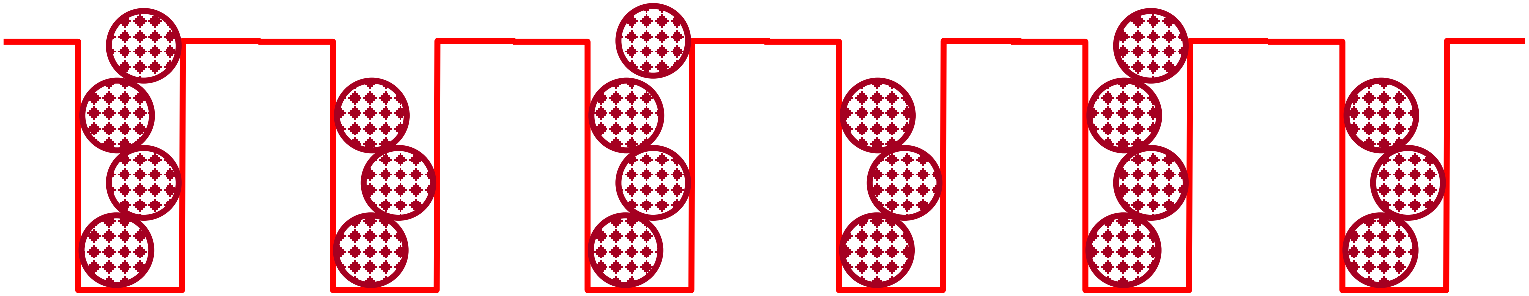
Incompressible, insulating ground states, with zero superfluid density, appear at special commensurate densities



$\langle n_j \rangle = 3$

$-\frac{t}{U}$

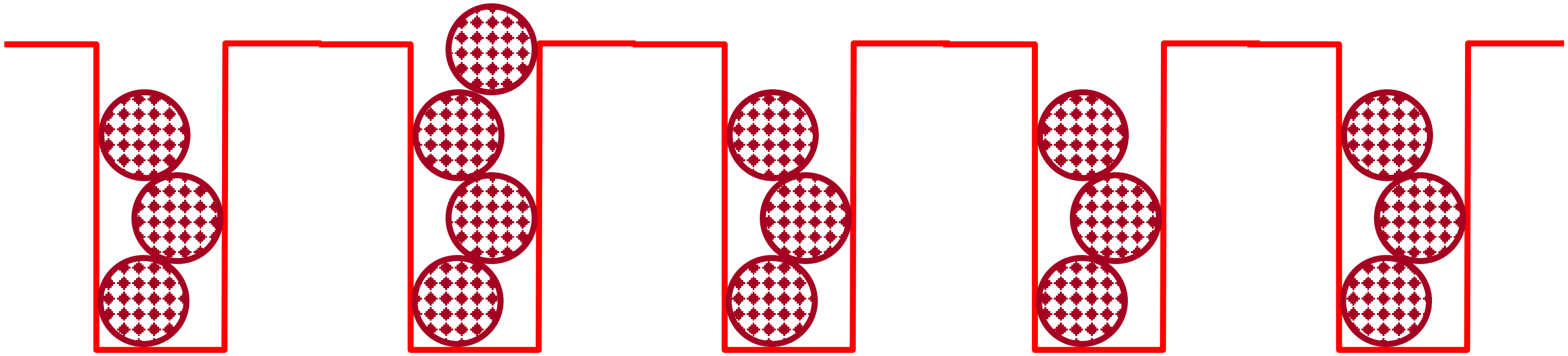




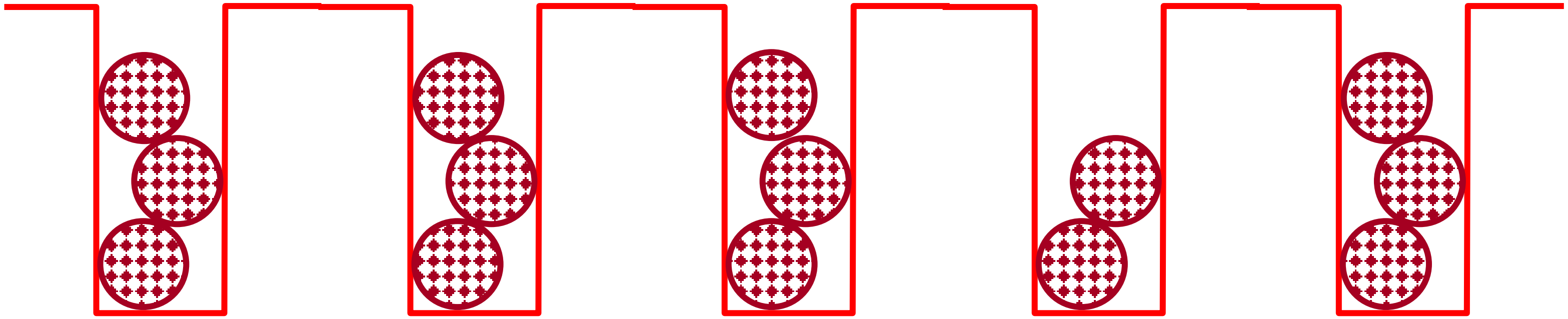
$\langle n_j \rangle = 7/2$

Ground state has "density wave" order, which spontaneously breaks lattice symmetries

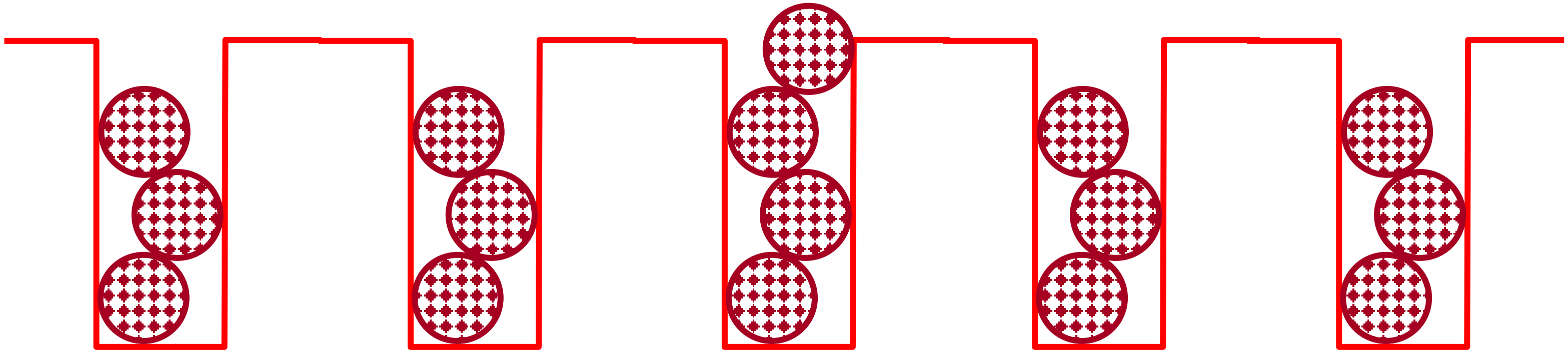
Excitations of the insulator: infinitely long-lived, finite energy *quasiparticles* and *quasiholes*



Energy of quasi-particles/holes: $\varepsilon_{p,h}(p) = \Delta_{p,h} + \frac{p^2}{2m^*_{p,h}}$



Excitations of the insulator: infinitely long-lived, finite energy *quasiparticles* and *quasiholes*



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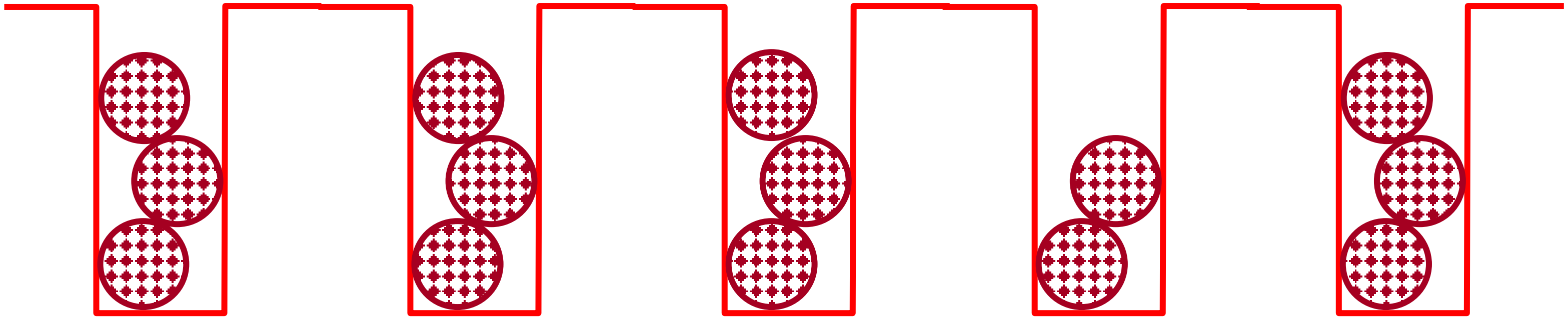
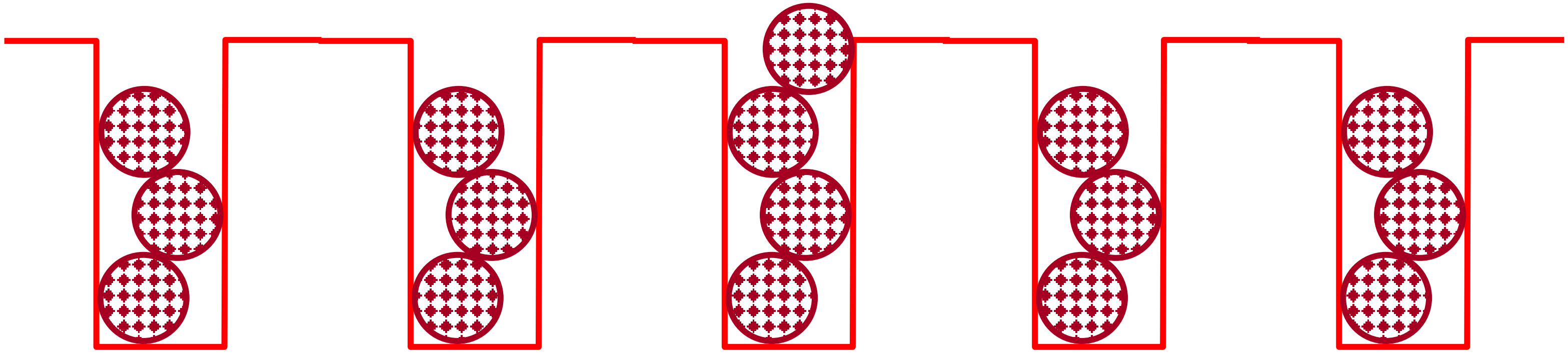
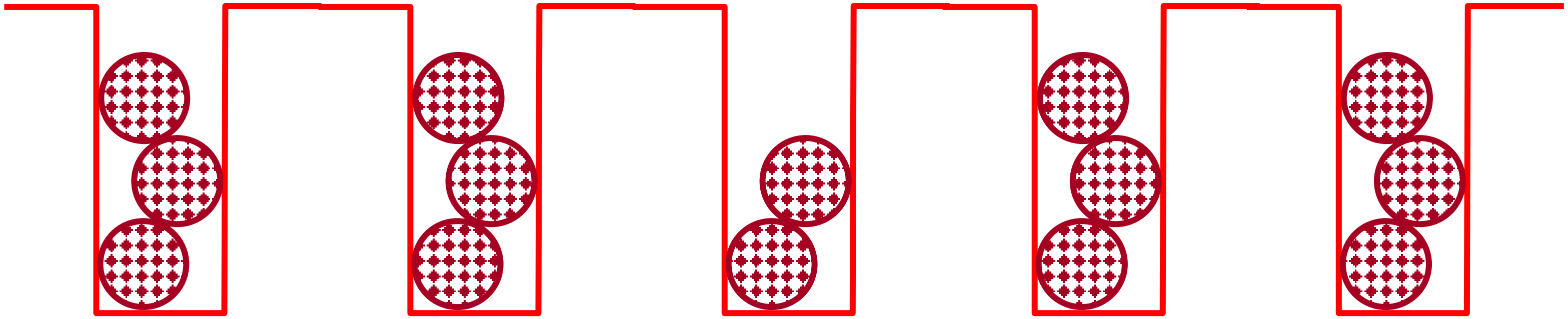


Excitations of the insulator: infinitely long-lived, finite energy *quasiparticles* and *quasiholes*



Energy of quasi-particles/holes: $\varepsilon_{p,h}(p) = \Delta_{p,h} + \frac{p^2}{2m^*_{p,h}}$



LGW theory of the superfluid insulator transition

- Identify order parameter $\Psi(x,\tau) \sim b_j^\dagger$
- Symmetries:

$$\begin{aligned} \text{Gauge invariance:} \quad & \Psi \rightarrow \Psi e^{i\theta} \\ \text{Time reversal} \quad & \tau \rightarrow -\tau \quad ; \quad \Psi \rightarrow \Psi^* \\ \text{Spatial inversion} \quad & x \rightarrow -x \end{aligned}$$

- Write down most general Lagrangian consistent with symmetries

$$\begin{aligned} \mathcal{Z} &= \int \mathcal{D}\Psi(x,\tau) \exp\left(- \int d^d x \int d\tau \mathcal{L}\left[\Psi\right] \right) \\ \mathcal{L}\left[\Psi\right] &= K \Psi^* \frac{\partial \Psi}{\partial \tau} + \left|\partial_\tau \Psi\right|^2 + c^2 \left|\nabla_x \Psi\right|^2 + r|\Psi|^2 + \frac{u}{2}|\Psi|^4 + \ldots \end{aligned}$$

- Identify phases at $r \gg 0$ and $r \ll 0$ with the insulator and the superfluid respectively.
- For $K \neq 0$, the particle and hole excitations have different energies.

- Gauge-invariance of the underlying boson Hamiltonian shows that

$$K = -\frac{\partial r}{\partial \mu}$$

- In mean-field theory, the ground state energy, E, across the superfluid-insulator transition has the non-analytic term

$$E = E_0 - \frac{r^2}{2u}\theta(-r)$$

(Beyond mean-field theory, the non-analytic term is $E \sim r^{(d+z)\nu}$).

- Because the density of bosons $= -\partial E/\partial\mu$, this implies a change in the boson density across the transition *unless* $\partial r/\partial\mu = 0$

- A superfluid-insulator transition at fixed boson density must have.

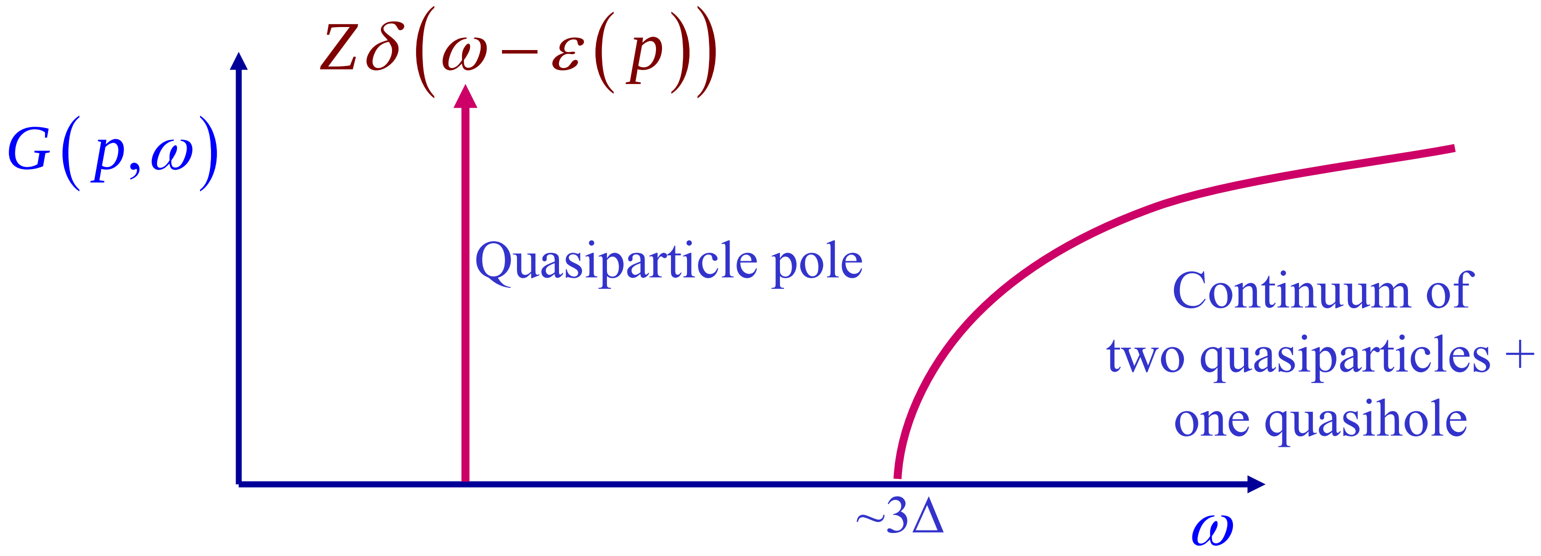
$$K = 0$$

Boson Green's function $G(p,\omega)$:

Insulating ground state

Cross-section to add a boson

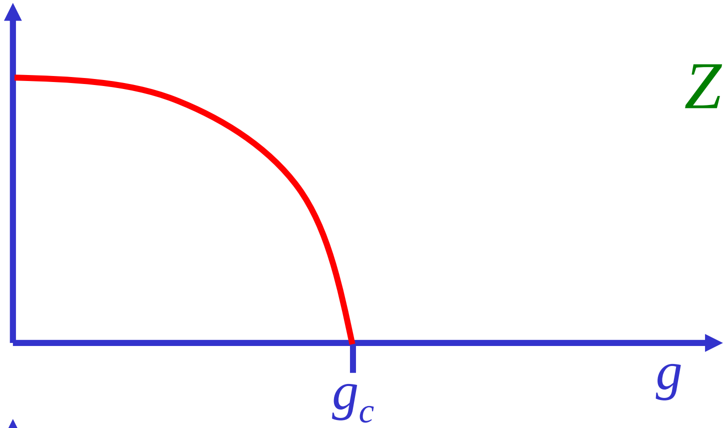
while transferring energy $\hbar\omega$ and momentum p



Similar result for quasi-hole excitations obtained by removing a boson

Entangled states at $g \equiv t/U$ of order unity

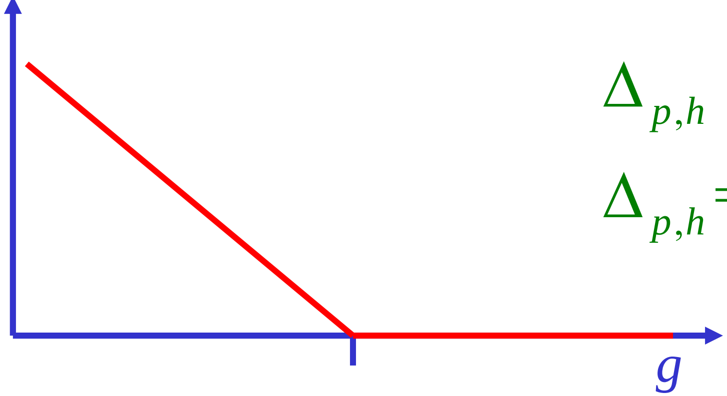
Quasiparticle weight Z



$$Z \sim \left(g_c - g\right)^{\eta\nu}$$

A.V. Chubukov, S. Sachdev, and J.Ye,
Phys. Rev. B **49**, 11919 (1994)

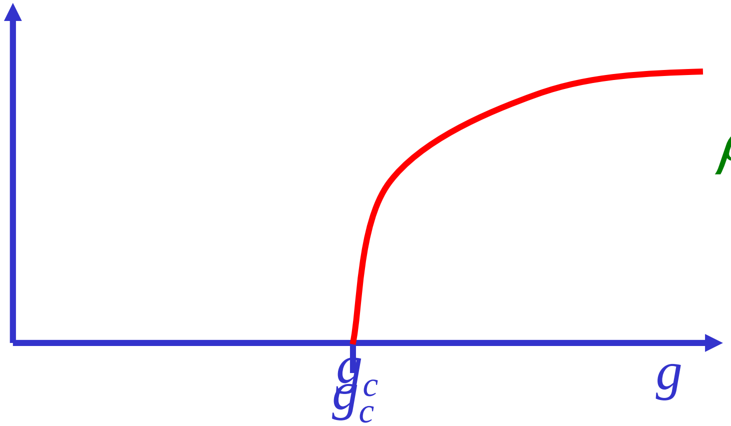
Excitation energy gap Δ



$$\Delta_{p,h} \sim \left(g_c - g\right)^{\nu} \text{ for } g < g_c$$

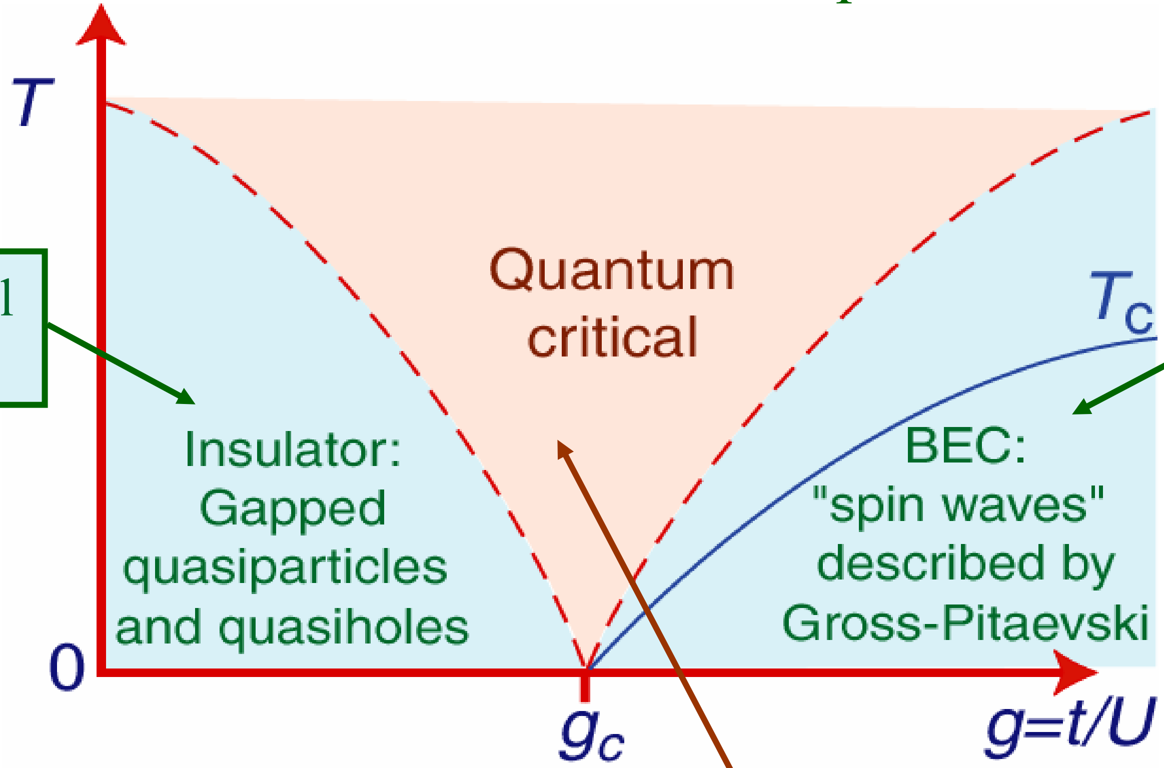
$$\Delta_{p,h} = 0 \text{ for } g > g_c$$

Superfluid density ρ_s



$$\rho_s \sim \left(g - g_c\right)^{(d+z-2)\nu}$$

Crossovers at nonzero temperature



Quasiclassical dynamics

Quasiclassical dynamics

Relaxational dynamics ("Bose molasses") with phase coherence/relaxation time τ_φ given by

$$\frac{1}{\tau_\varphi} = \left(\text{Universal number}\right)\frac{k_B T}{\hbar} \quad \left(1\mu\text{K} = 20.9\text{kHz}\right)$$

S. Sachdev and J. Ye,
Phys. Rev. Lett. **69**, 2411 (1992).
K. Damle and S. Sachdev
Phys. Rev. B **56**, 8714 (1997).

$$\text{Conductivity (in d=2)} = \frac{Q^2}{h}\Sigma\left(\frac{\hbar\omega}{k_B T}\right) \qquad \Sigma \rightarrow \text{universal function}$$

M.P.A. Fisher, G. Girvin, and G. Grinstein, *Phys. Rev. Lett*. **64**, 587 (1990).
K. Damle and S. Sachdev *Phys. Rev*. B **56**, 8714 (1997).

Outline

I. Quantum Ising Chain

II. Landau-Ginzburg-Wilson theory
  Mean field theory and the evolution of the excitation spectrum.

III. Superfluid-insulator transition
  Boson Hubbard model at integer filling.

IV. Bosons at fractional filling
  Beyond the Landau-Ginzburg-Wilson paradigm.

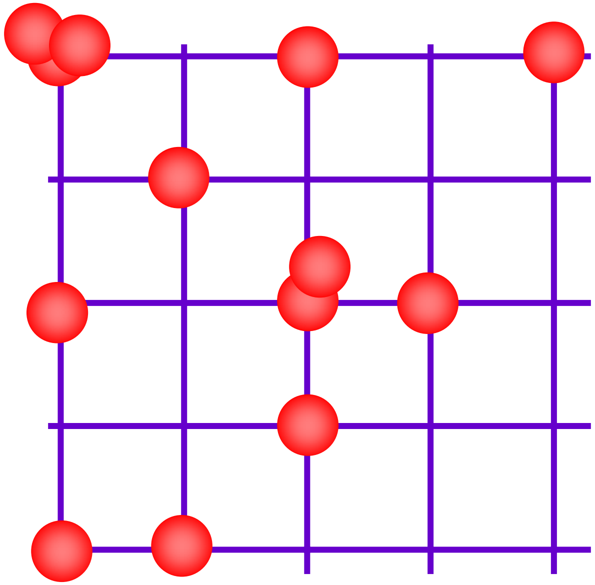
V. Quantum phase transitions and the Luttinger theorem
  Depleting the Bose-Einstein condensate of trapped ultracold atoms – see talk by Stephen Powell

IV. Bosons at fractional filling

Beyond the Landau-Ginzburg-Wilson paradigm

L. Balents, L. Bartosch, A. Burkov, S. Sachdev, K. Sengupta,
Physical Review B **71**, 144508 and 144509 (2005),
cond-mat/0502002, and cond-mat/0504692.

Bosons at filling fraction $f = 1/2$



$$\langle \Psi \rangle \neq 0$$

Weak interactions: superfluidity

C. Lannert, M.P.A. Fisher, and T. Senthil, *Phys. Rev.* B **63**, 134510 (2001)
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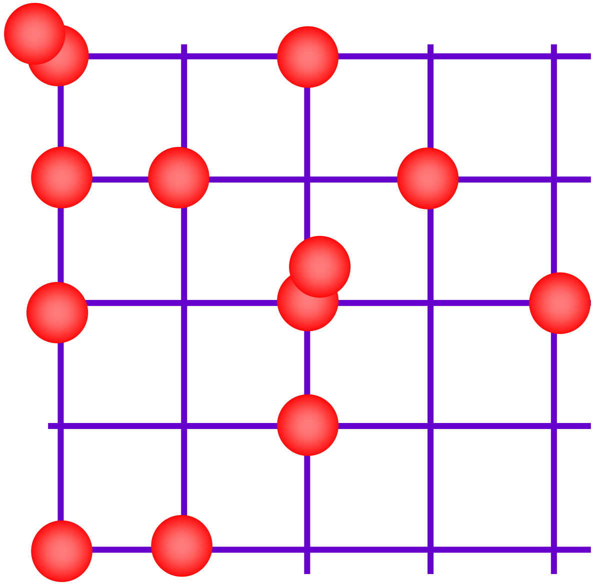

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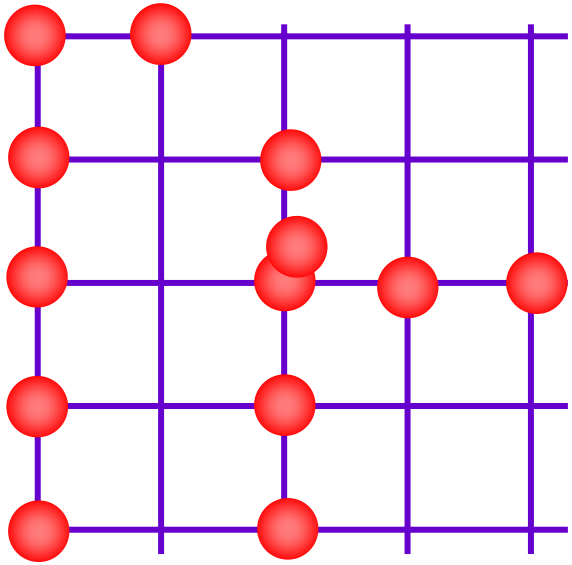

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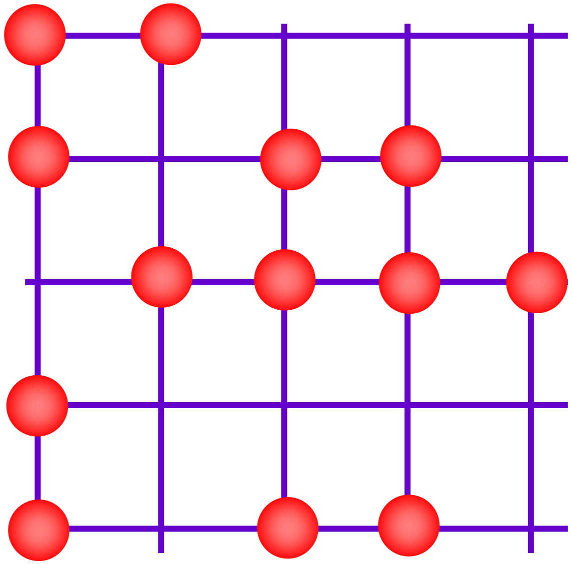

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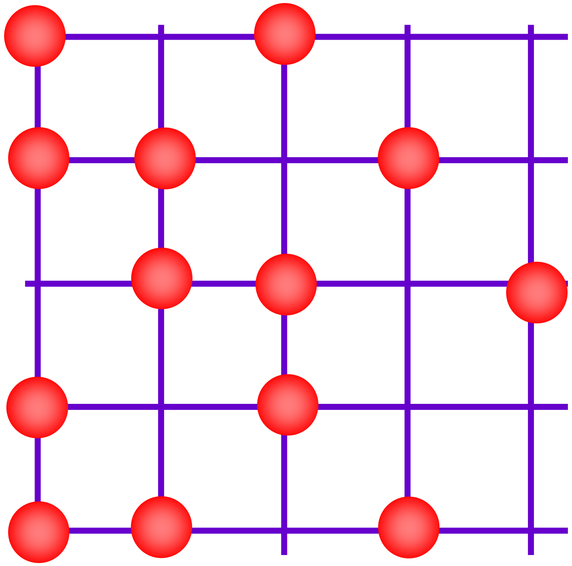

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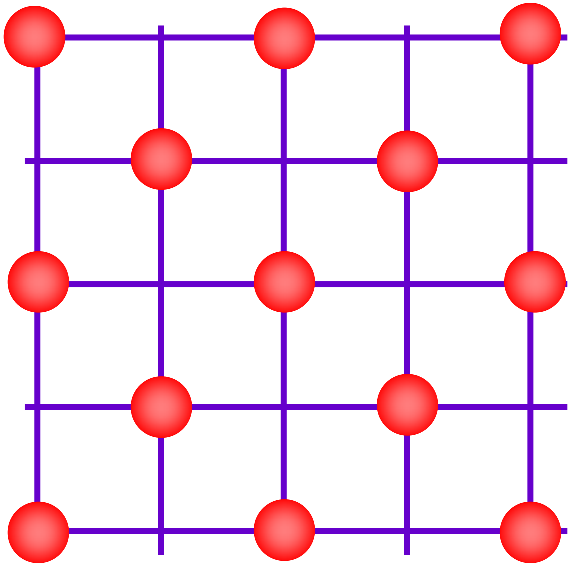

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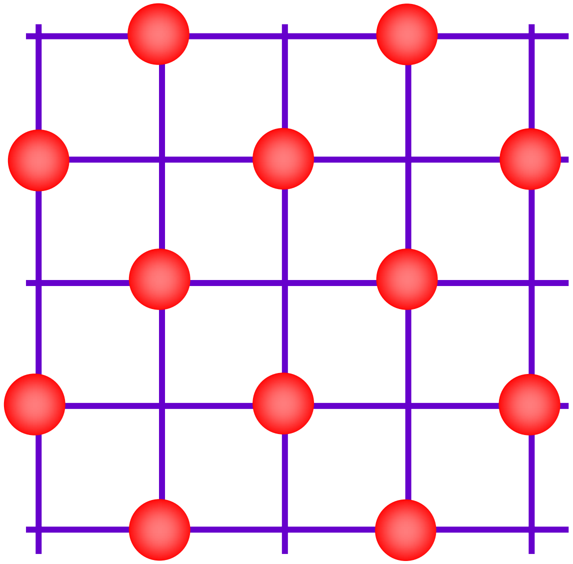

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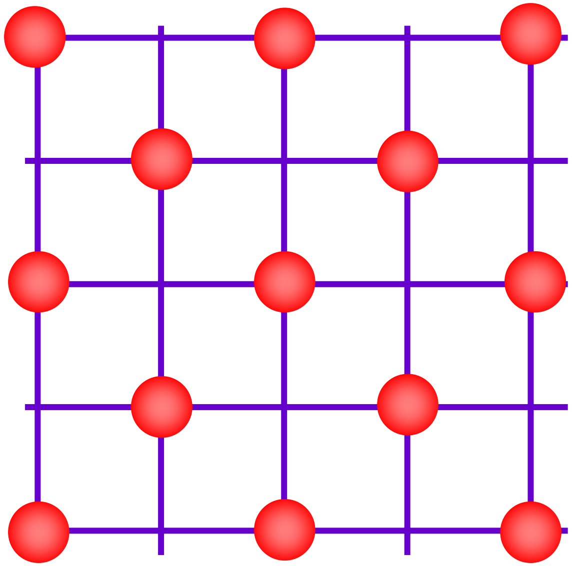

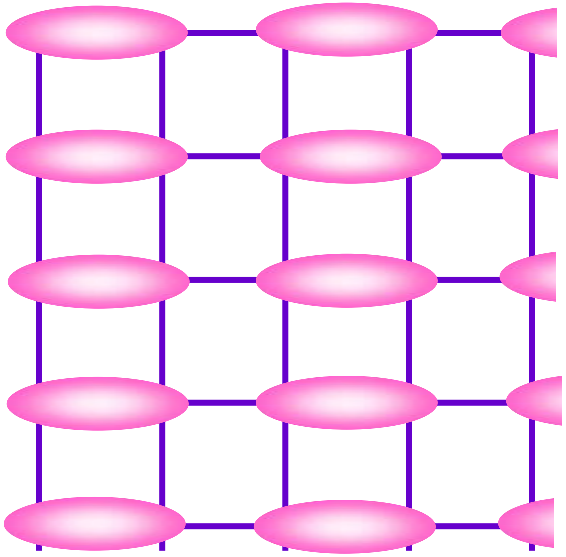

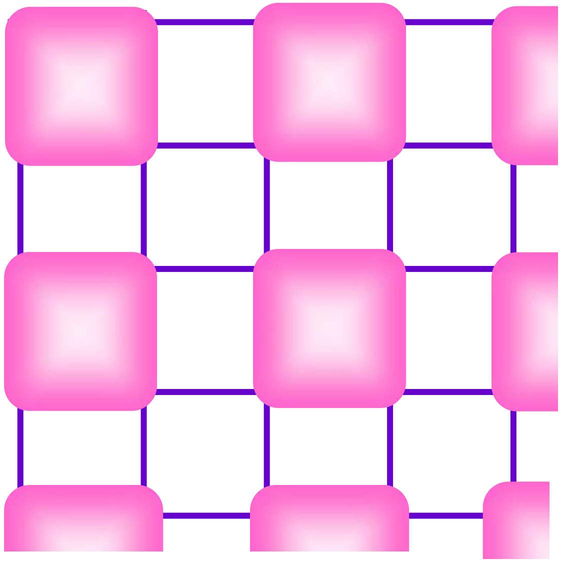

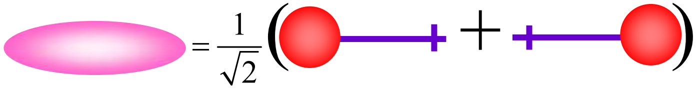

Charge density wave (CDW) order

Valence bond solid (VBS) order

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Can define a common CDW/VBS order using a generalized "density" $\rho(\mathbf{r}) = \sum_{\mathbf{Q}} \rho_{\mathbf{Q}} e^{i\mathbf{Q}.\mathbf{r}}$

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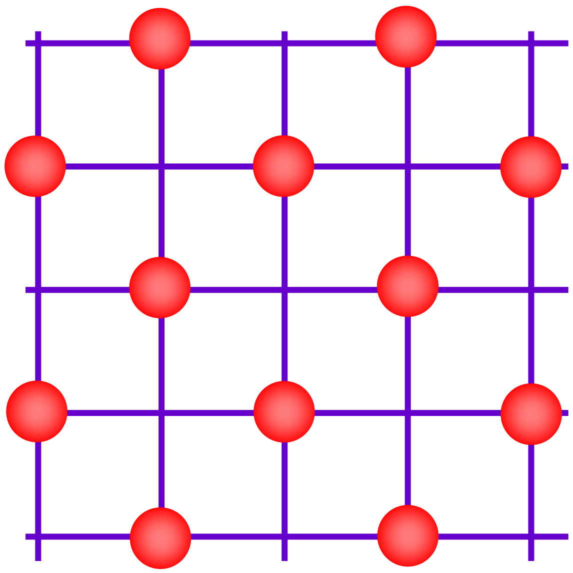

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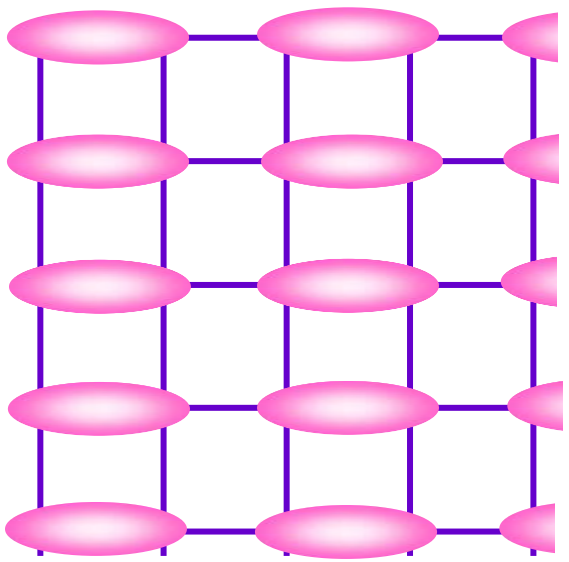

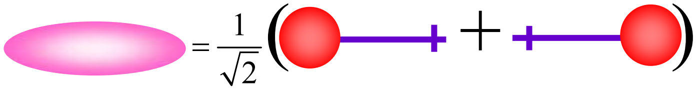

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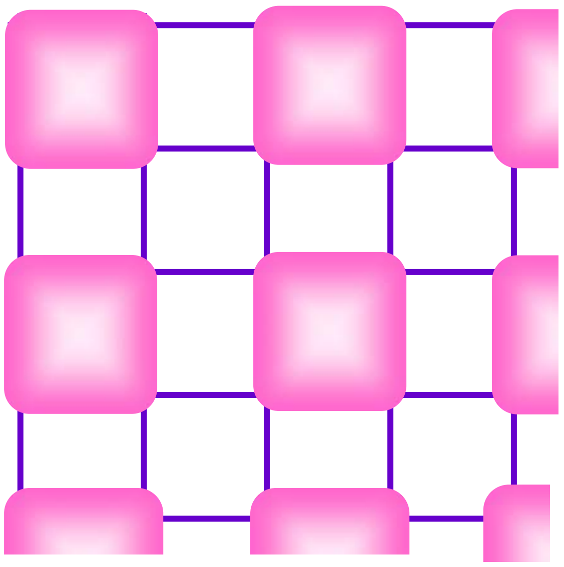

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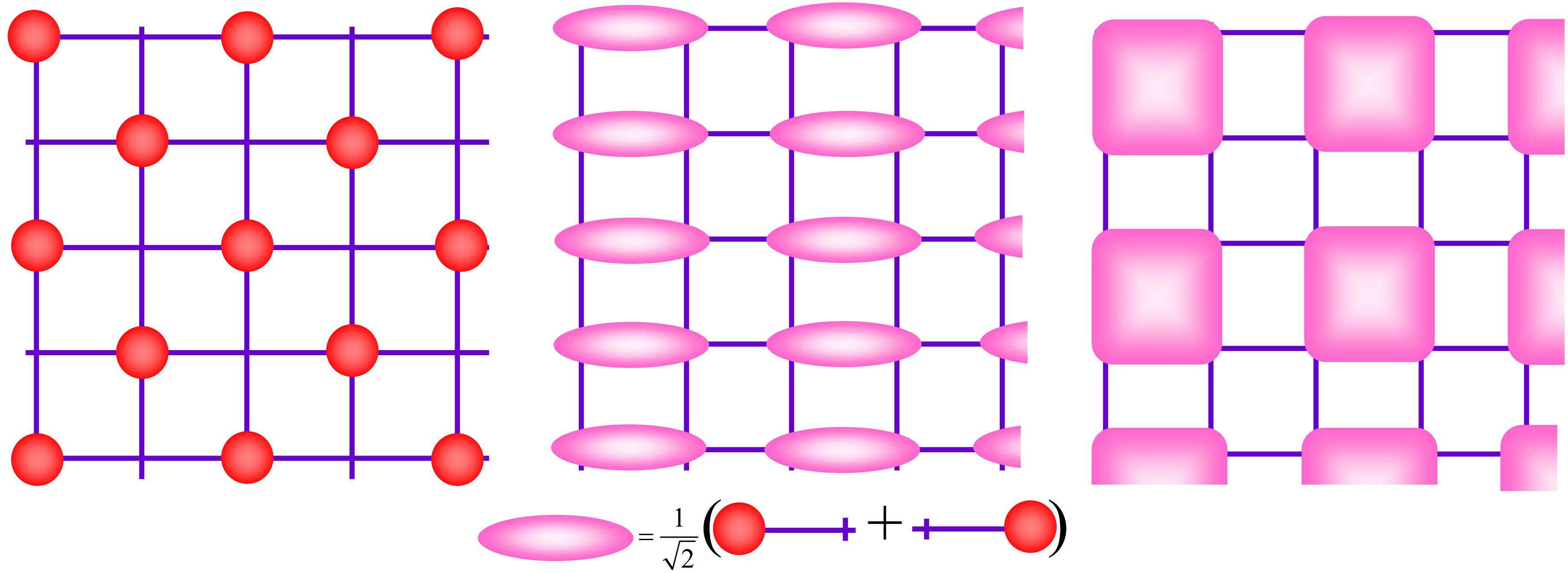

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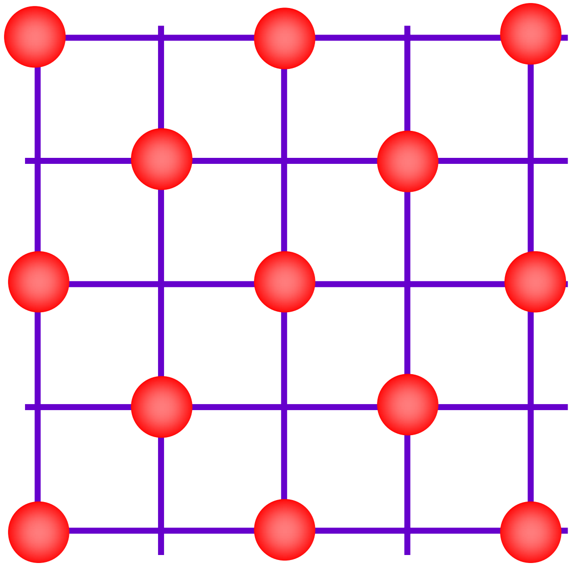

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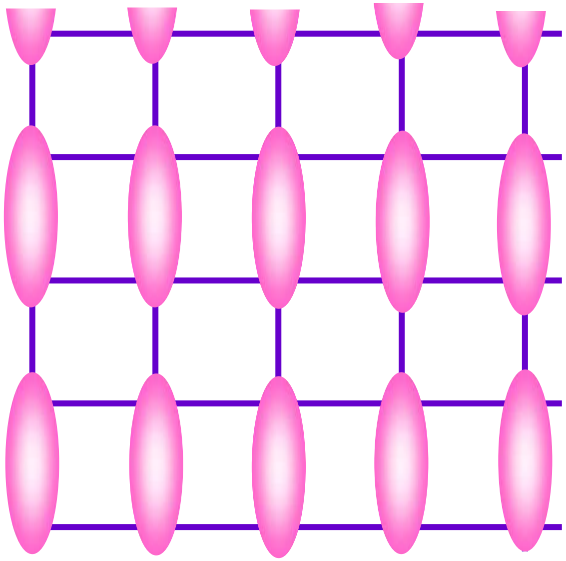

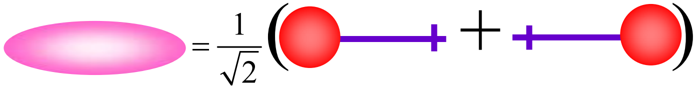

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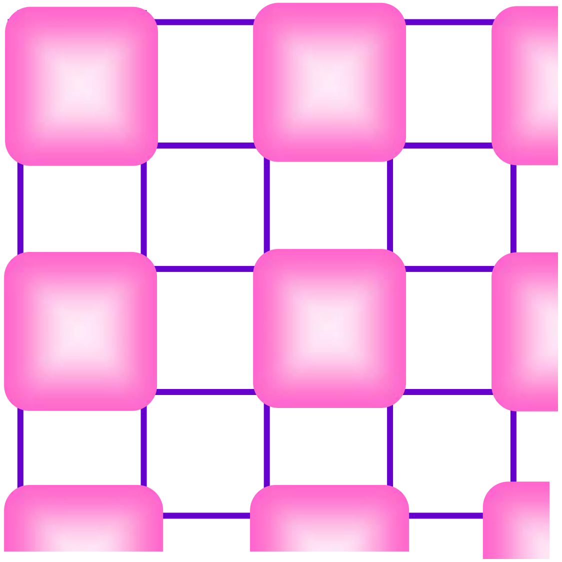

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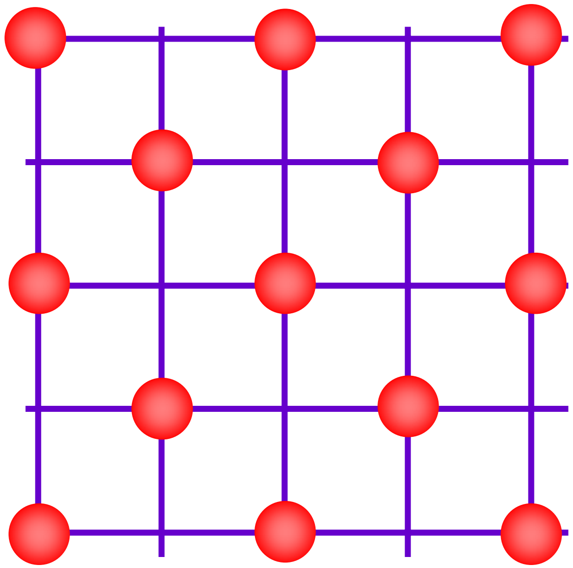

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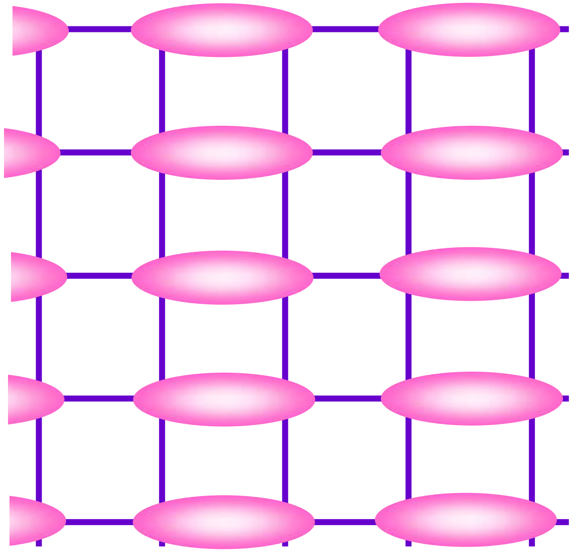

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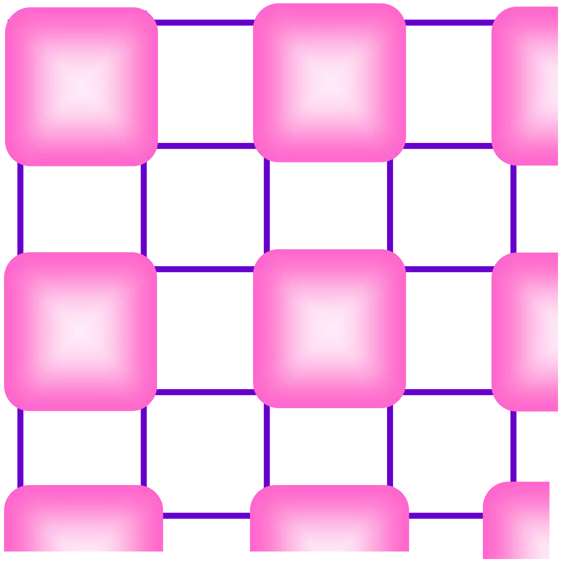

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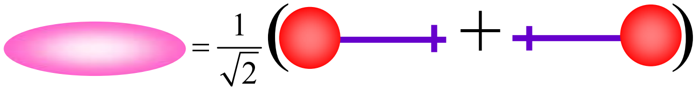

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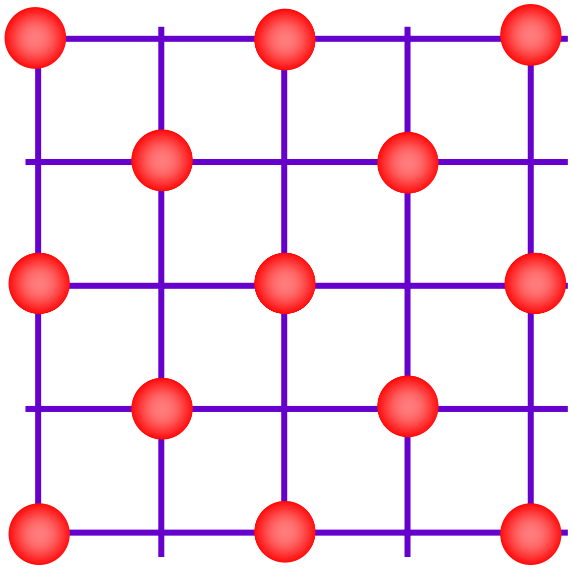

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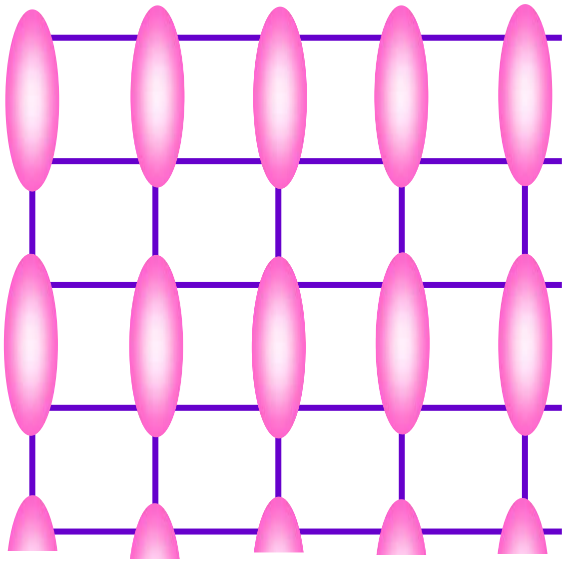

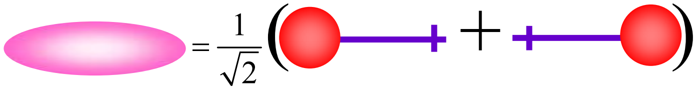

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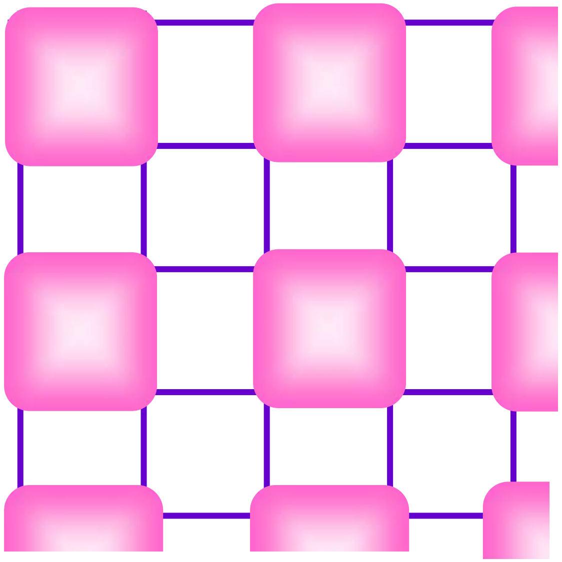

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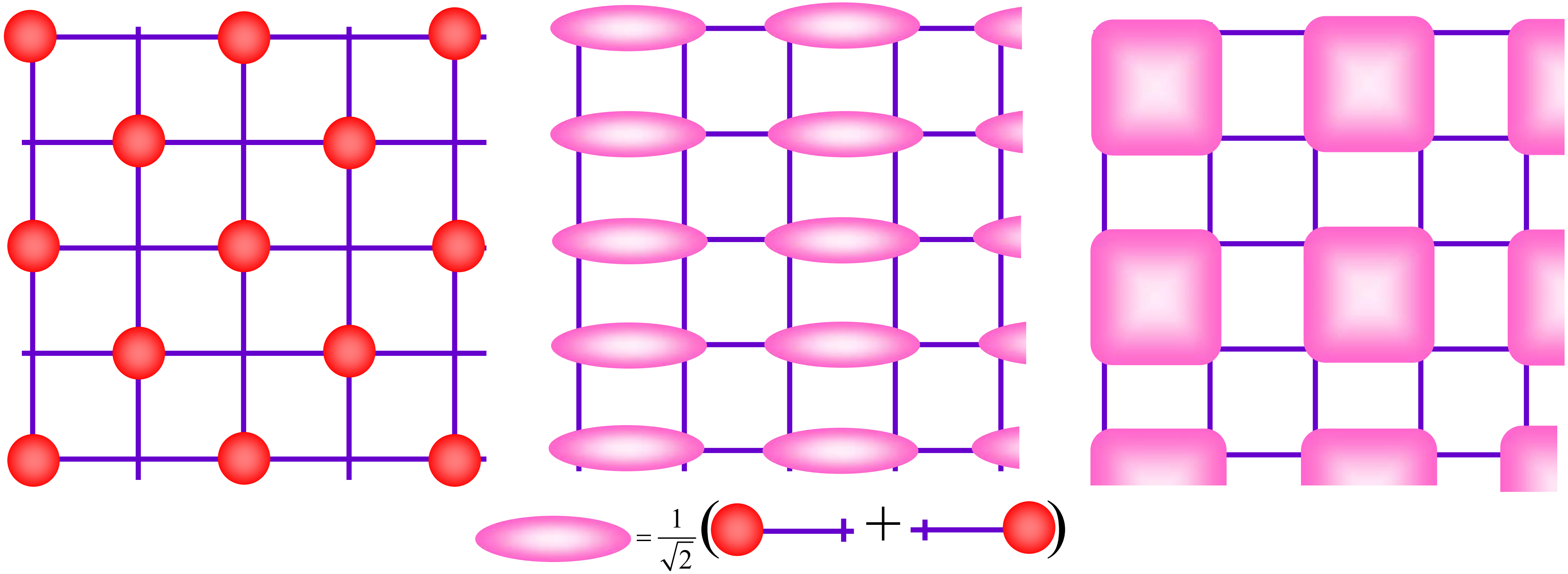

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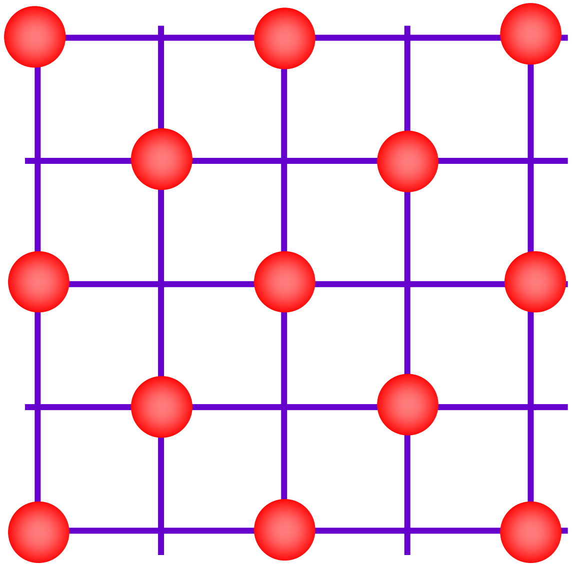

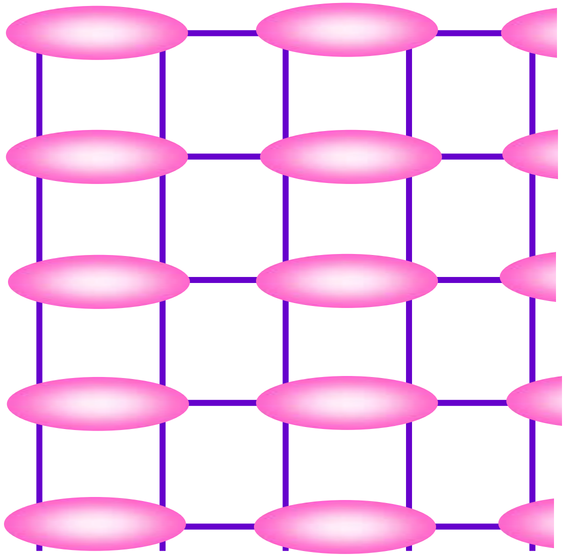

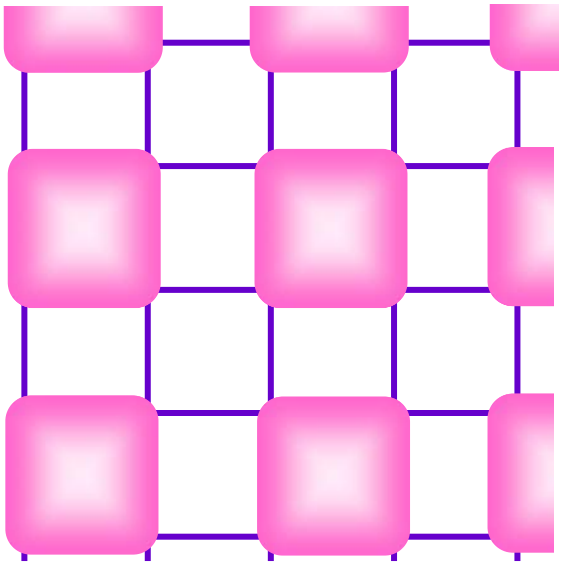

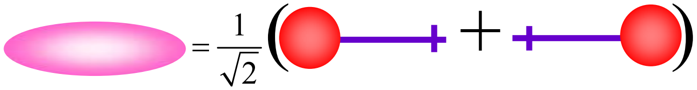

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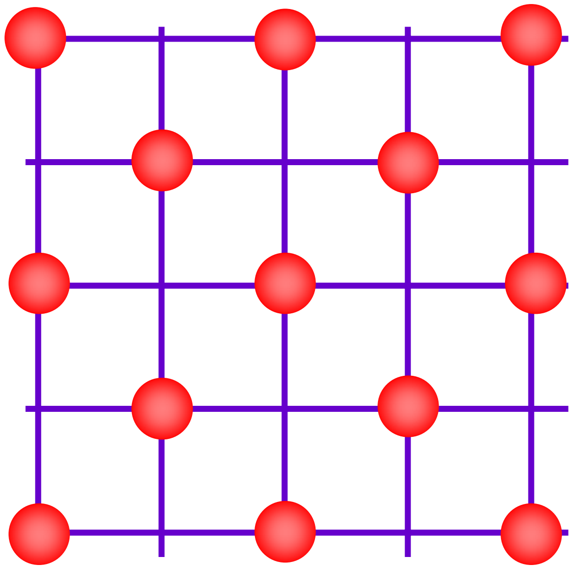
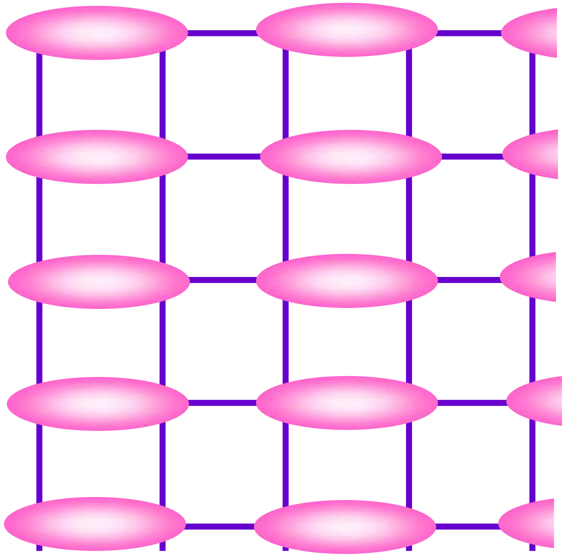
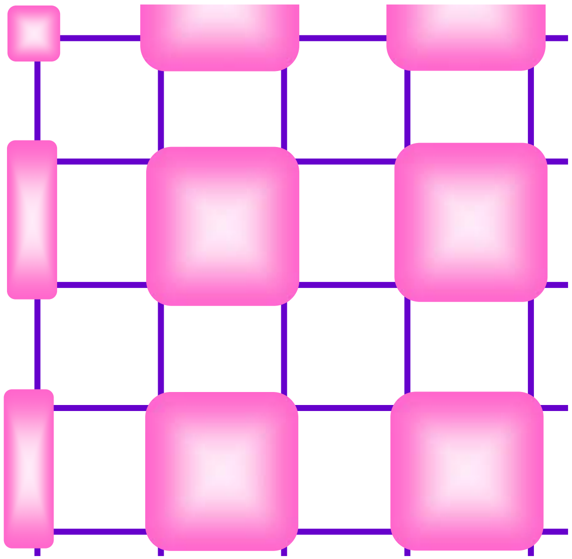
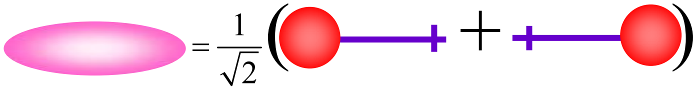








Charge density wave (CDW) order

Valence bond solid (VBS) order

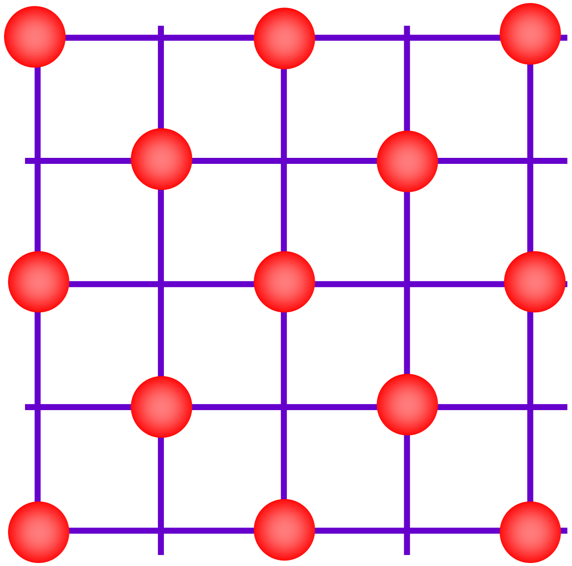
Valence bond solid (VBS) order

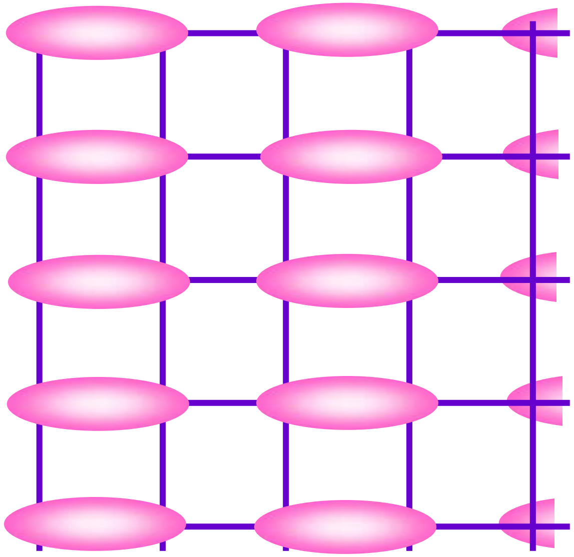
Can define a common CDW/VBS order using a generalized "density" $\rho(\mathbf{r}) = \sum_{\mathbf{Q}} \rho_{\mathbf{Q}} e^{i\mathbf{Q}\cdot\mathbf{r}}$

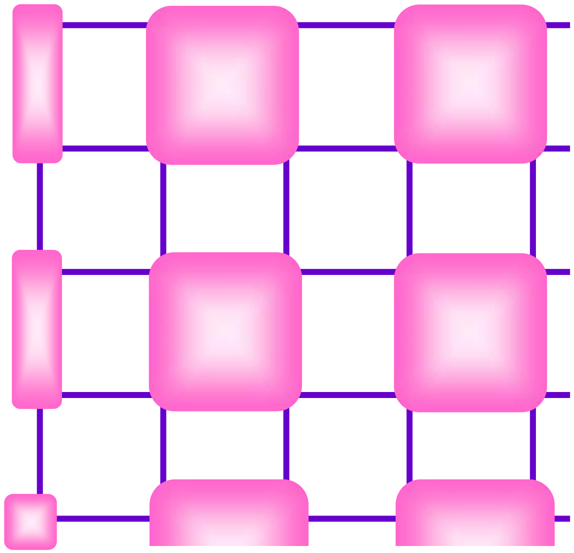
All insulators have $\langle \Psi \rangle = 0$ and $\langle \rho_{\mathbf{Q}} \rangle \neq 0$ for certain $\boldsymbol{Q}$

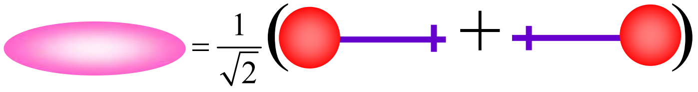
C. Lannert, M.P.A. Fisher, and T. Senthil, *Phys. Rev.* B **63**, 134510 (2001)
S. Sachdev and K. Park, *Annals of Physics*, **298**, 58 (2002)

Insulating phases of bosons at filling fraction $f = 1/2$









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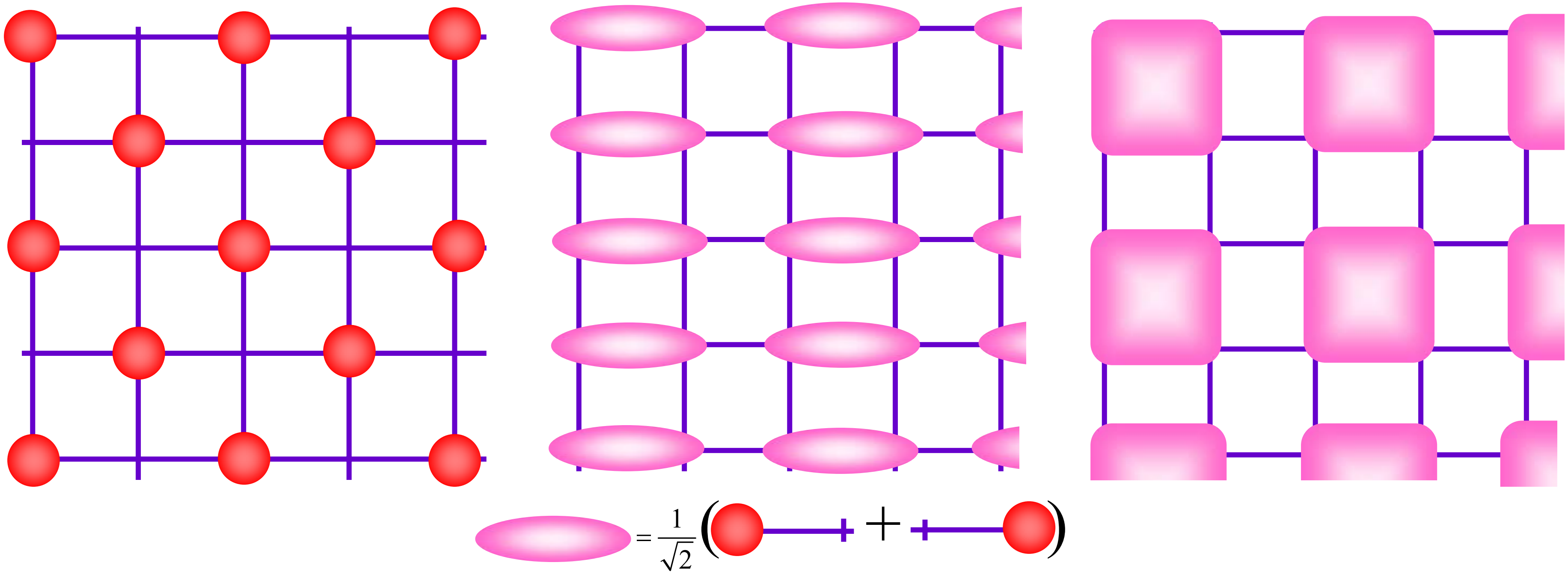

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S. Sachdev and K. Park, *Annals of Physics*, **298**, 58 (2002)

Ginzburg-Landau-Wilson approach to multiple order parameters:

$$F = F_{sc}\left[\Psi_{sc}\right] + F_{\text{charge}}\left[\rho_{\mathbf{Q}}\right] + F_{\text{int}}$$

$$F_{sc}\left[\Psi_{sc}\right] = r_1 \left|\Psi_{sc}\right|^2 + u_1 \left|\Psi_{sc}\right|^4 + \cdots$$

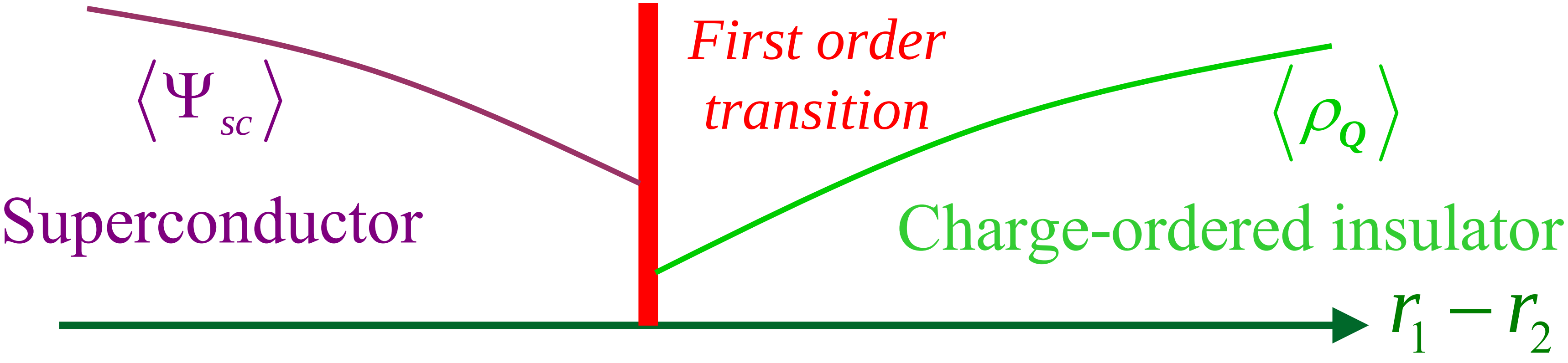
$$F_{\text{charge}}\left[\rho_{\mathbf{Q}}\right] = r_2 \left|\rho_{\mathbf{Q}}\right|^2 + u_2 \left|\rho_{\mathbf{Q}}\right|^4 + \cdots$$

$$F_{\text{int}} = v \left|\Psi_{sc}\right|^2 \left|\rho_{\mathbf{Q}}\right|^2 + \cdots$$

Distinct symmetries of order parameters permit couplings only between their energy densities

S. Sachdev and E. Demler, *Phys. Rev.* B **69**, 144504 (2004).

Predictions of LGW theory



$\langle \Psi_{sc} \rangle$

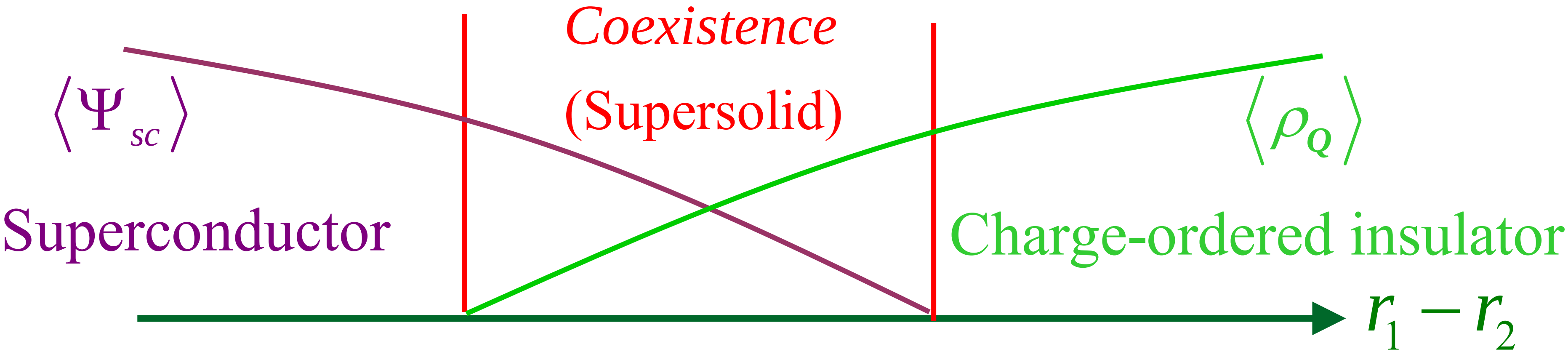
Superconductor

First order transition

$\langle \rho_{\mathbf{Q}} \rangle$

Charge-ordered insulator

$r_1 - r_2$



$\langle \Psi_{sc} \rangle$

Superconductor

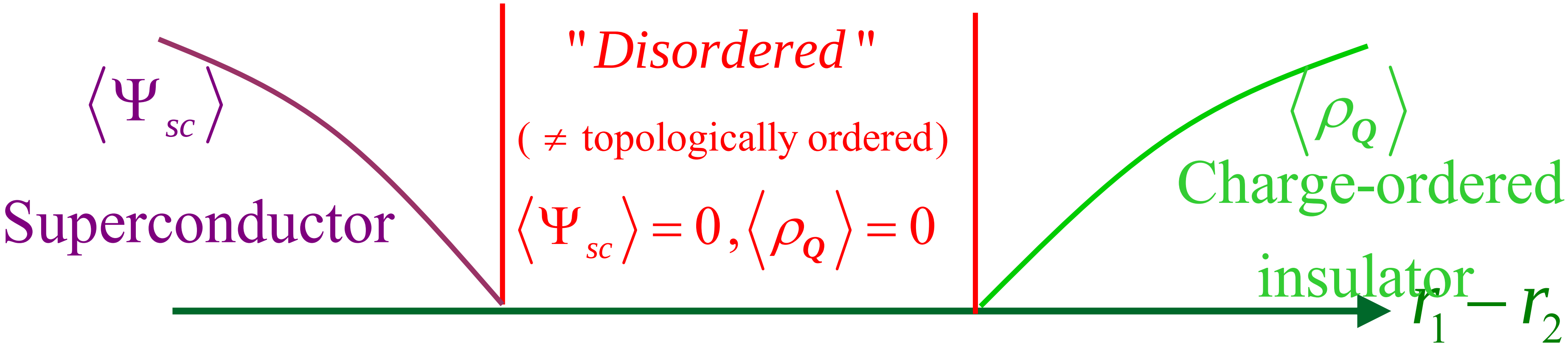
Coexistence

(Supersolid)

$\langle \rho_{\mathbf{Q}} \rangle$

Charge-ordered insulator

$r_1 - r_2$



$\langle \Psi_{sc} \rangle$

Superconductor

"*Disordered*"

(≠ topologically ordered)

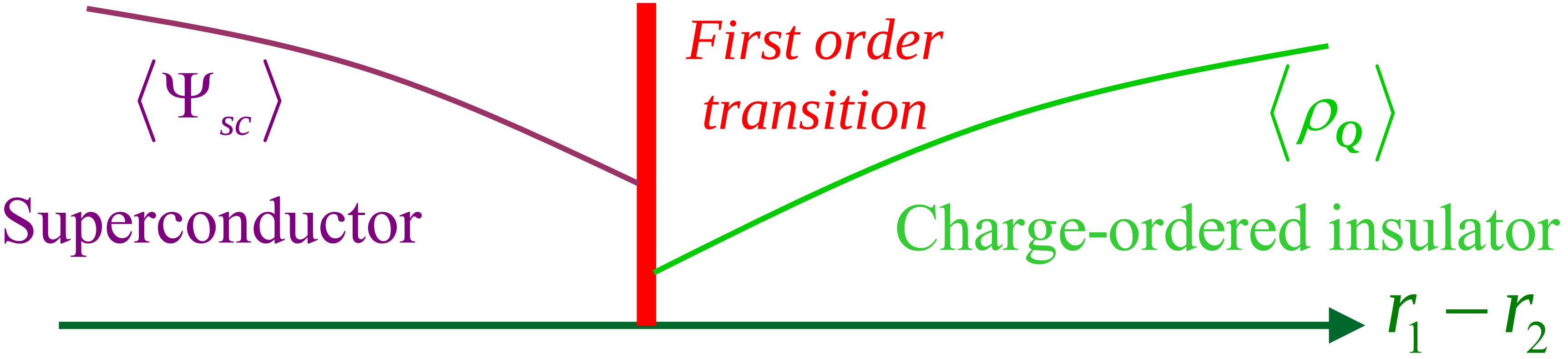
$\langle \Psi_{sc} \rangle = 0, \langle \rho_{\mathbf{Q}} \rangle = 0$

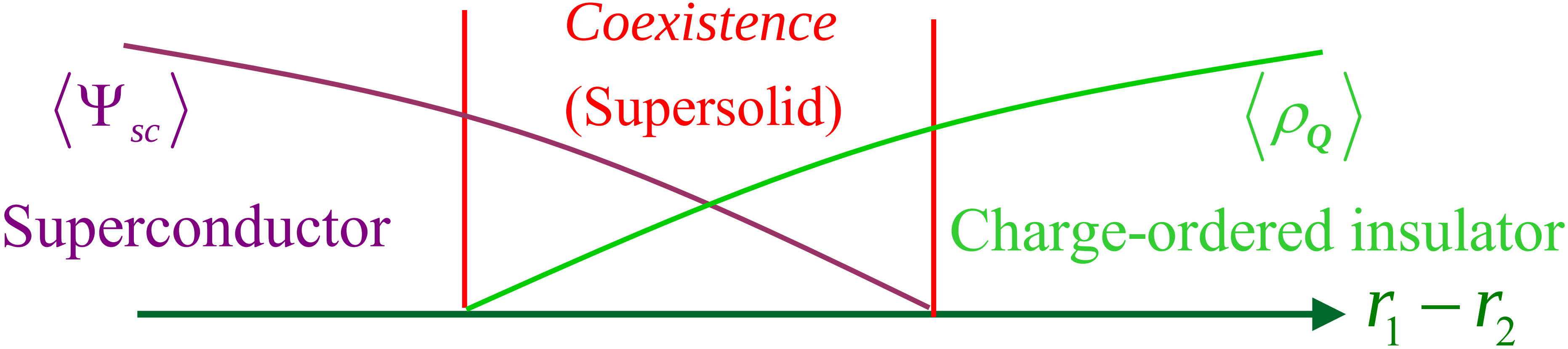
$\langle \rho_{\mathbf{Q}} \rangle$

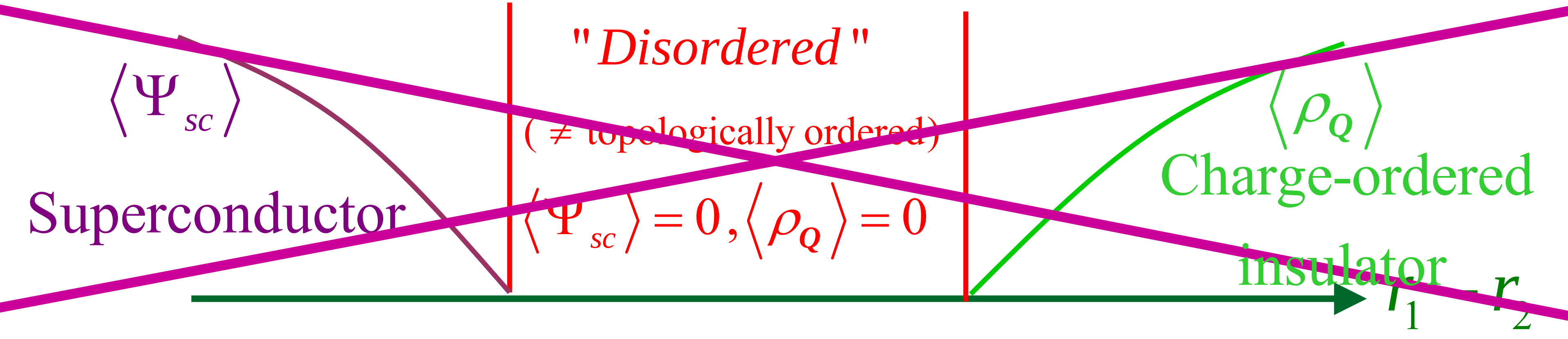
Charge-ordered insulator

$r_1 - r_2$

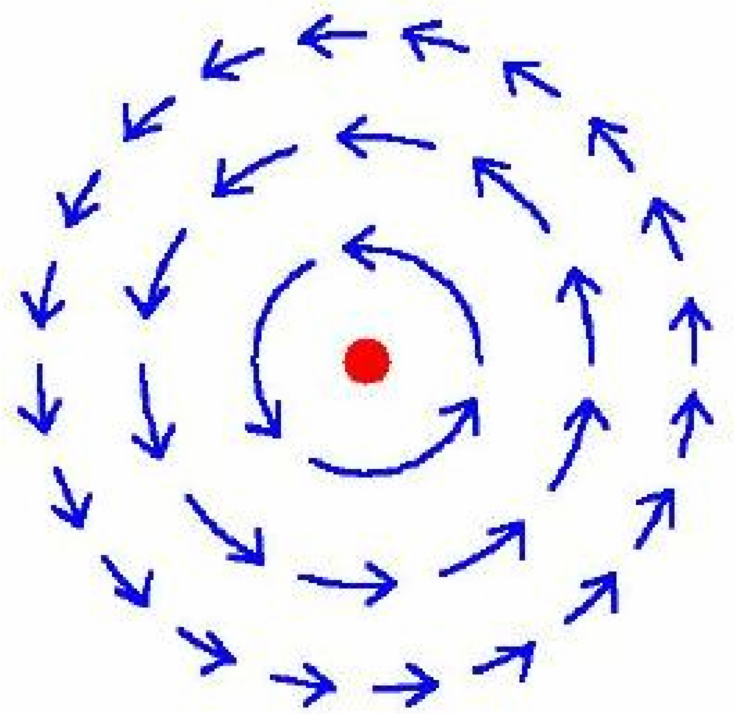
Predictions of LGW theory







Excitations of the superfluid: **Vortices**

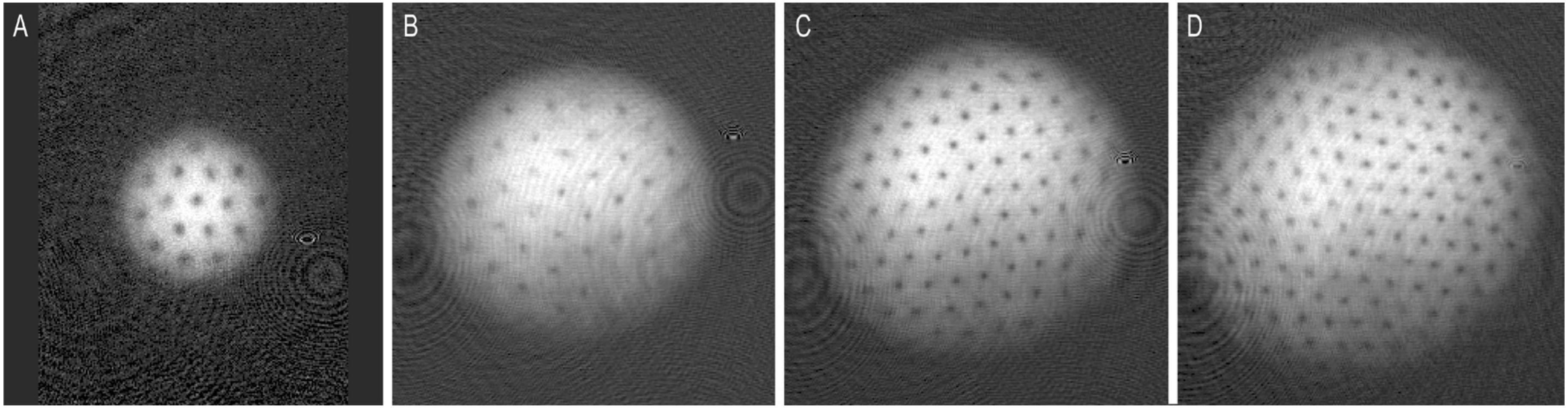


The circulation of a vortex is quantized:

$$\oint \mathbf{v}_s \cdot d\mathbf{r} = \frac{\hbar}{m} \oint \nabla\theta \cdot d\mathbf{r} = n\frac{h}{m}$$

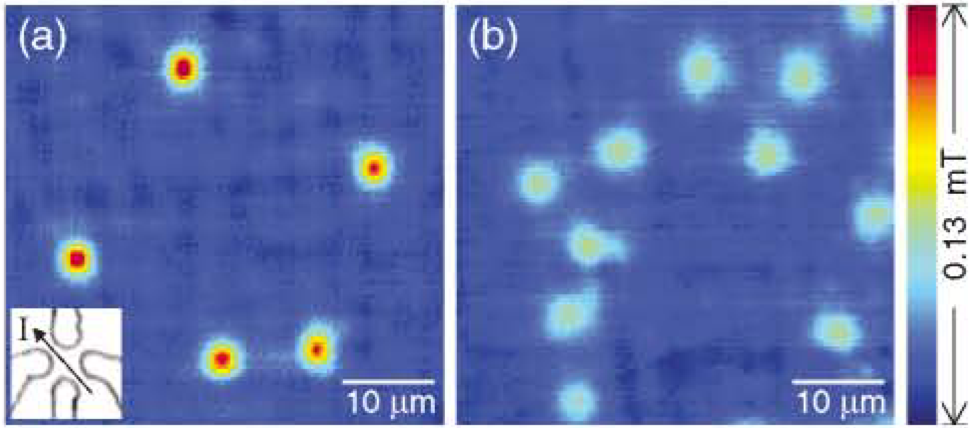
where n is an integer.

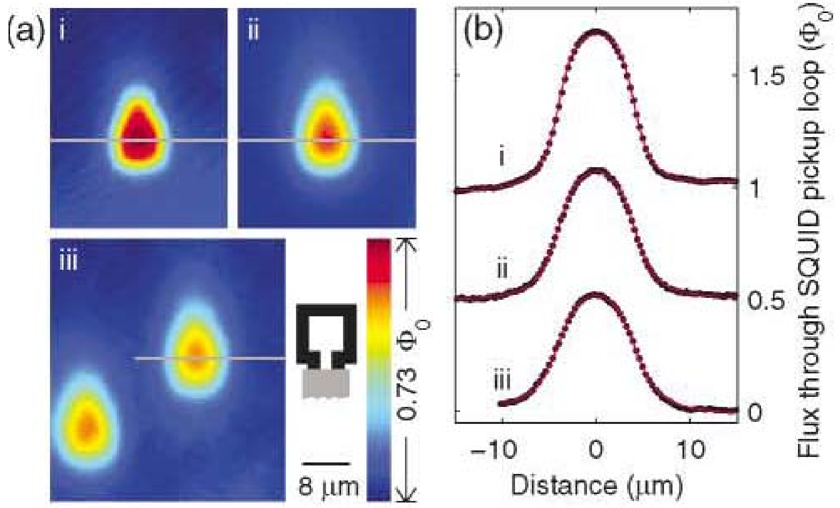
Observation of quantized vortices in rotating ultracold Na



J. R. Abo-Shaeer, C. Raman, J. M. Vogels, and W. Ketterle,
Observation of Vortex Lattices in Bose-Einstein Condensates,
Science **292**, 476 (2001).

Quantized fluxoids in $YBa_2Cu_3O_{6+y}$



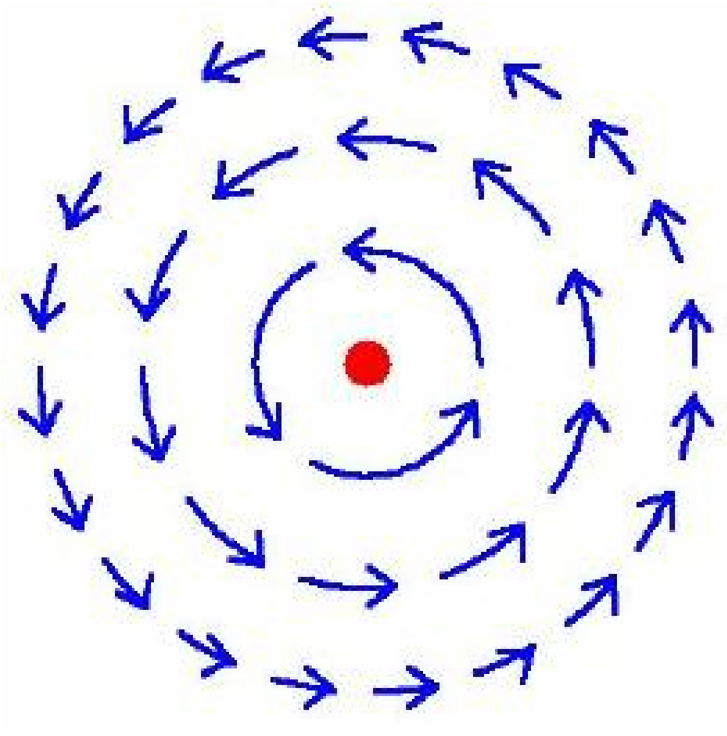


J. C. Wynn, D. A. Bonn, B.W. Gardner, Yu-Ju Lin, Ruixing Liang, W. N. Hardy, J. R. Kirtley, and K. A. Moler, *Phys. Rev. Lett*. **87**, 197002 (2001).

In superconductors, vortices carry quantized magnetic flux:

$$\int \mathbf{B}\cdot d\mathbf{S} = n\frac{hc}{2e}$$

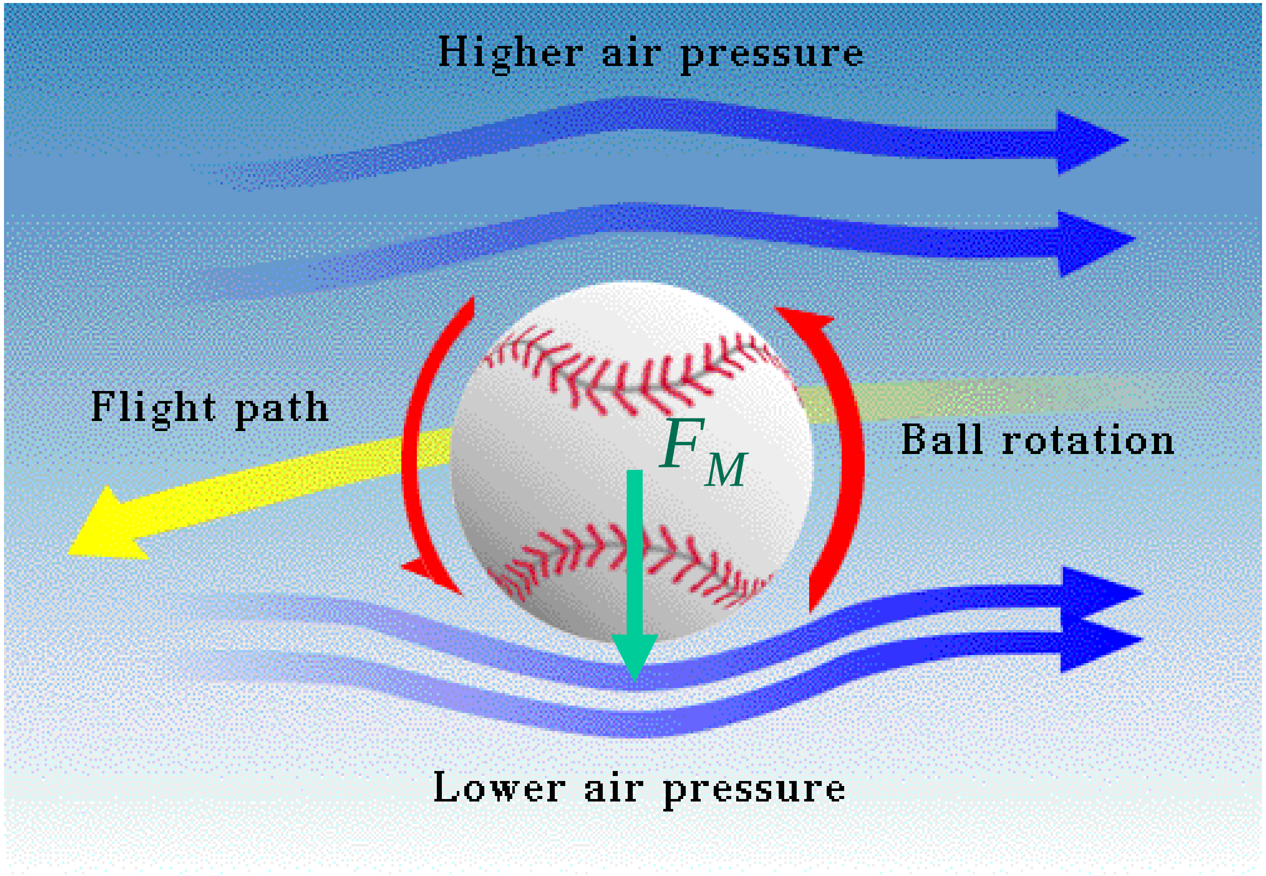
Excitations of the superfluid: **Vortices**



<u>Central question</u>:

In two dimensions, we can view the vortices as point particle excitations of the superfluid. What is the quantum mechanics of these “particles” ?

In ordinary fluids, vortices experience the Magnus Force



$$F_M = (\text{mass density of air}) \bullet (\text{velocity of ball}) \bullet (\text{circulation})$$

For a vortex in a superfluid, this is

$$\begin{aligned}\mathbf{F}_M &= (m\rho)\left(\left(\mathbf{v}_s - \frac{d\mathbf{r}_v}{dt}\right)\times\hat{\mathbf{z}}\right)\left(\oint \mathbf{v}_s\cdot d\mathbf{r}\right)\\ &= nh\rho\left(\mathbf{v}_s - \frac{d\mathbf{r}_v}{dt}\right)\times\hat{\mathbf{z}}\end{aligned}$$

where ρ = number density of bosons

$\mathbf{v}_s$ = local velocity of superfluid

$\mathbf{r}_v$ = position of vortex

For a vortex in a superfluid, this is

$$\begin{aligned}\mathbf{F}_M &= (m\rho)\left(\left(\mathbf{v}_s - \frac{d\mathbf{r}_v}{dt}\right)\times\hat{\mathbf{z}}\right)\left(\oint \mathbf{v}_s \cdot d\mathbf{r}\right)\\ &= nh\rho\left(\mathbf{v}_s - \frac{d\mathbf{r}_v}{dt}\right)\times\hat{\mathbf{z}}\\ &= n\left(\mathbf{E} + \frac{d\mathbf{r}_v}{dt}\times\mathbf{B}\right)\end{aligned}$$

where $\mathbf{E} = \rho\mathbf{v}_s \times \hat{\mathbf{z}}$ and $\mathbf{B} = -h\rho\hat{\mathbf{z}}$

Dual picture:

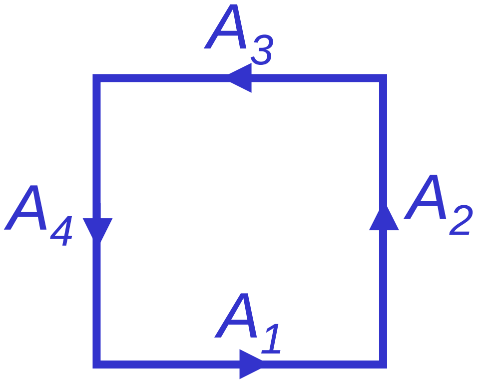
The vortex is a quantum particle with dual "electric" charge *n*, moving in a dual "magnetic" field of strength = *h*×(number density of Bose particles)

- The vortices are quantum particles moving in a periodic potential with the symmetry of the square lattice, and in the presence of a dual "magnetic" field of strength $= h\rho$, where ρ is the number density of bosons per unit cell.

- The vortex motion can be described by the effective Hofstadter Hamiltonian:

$$\mathcal{H}_v = -t \sum_{\langle ij \rangle} \left(e^{iA_{ij}} \varphi_i^* \varphi_j + \text{c.c.} \right)$$

where φ_i is an operator which annihilates a vortex particle at site i of a square lattice.



$$A_1 + A_2 + A_3 + A_4 = 2\pi f$$

where f is the boson filling fraction.

Bosons at filling fraction $f = 1$

- At f=1, the "magnetic" flux per unit cell is 2π, and the vortex does not pick up any phase from the boson density.

- The effective dual "magnetic" field acting on the vortex is zero, and the corresponding component of the Magnus force vanishes.

Bosons at rational filling fraction f=p/q

Quantum mechanics of the vortex "particle" in a periodic potential with *f* flux quanta per unit cell

Space group symmetries of Hofstadter Hamiltonian:

T_x, T_y : Translations by a lattice spacing in the x, y directions

R : Rotation by 90 degrees.

Magnetic space group:

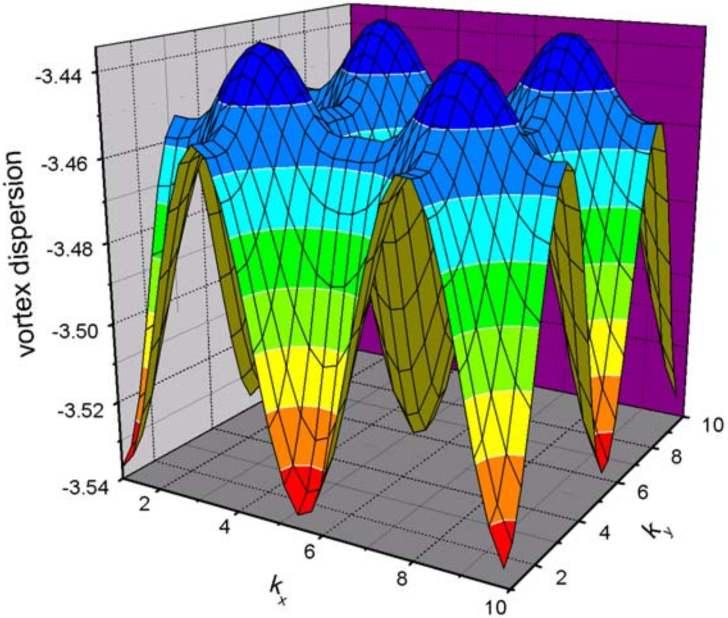
$$T_x T_y = e^{2\pi i f} T_y T_x \quad ;$$

$$R^{-1} T_y R = T_x \quad ; \quad R^{-1} T_x R = T_y^{-1} \quad ; \quad R^4 = 1$$

The low energy vortex states must form a representation of this algebra

Vortices in a superfluid near a Mott insulator at filling $f=p/q$

Hofstadter spectrum of the quantum vortex "particle" with field operator φ



At filling $f=p/q$, there are q species of vortices, φ_ℓ (with $\ell=1\ldots q$), associated with q degenerate minima in the vortex spectrum. These vortices realize the smallest, q-dimensional, representation of the magnetic algebra.

$$T_x : \varphi_\ell \to \varphi_{\ell+1} \quad ; \quad T_y : \varphi_\ell \to e^{2\pi i \ell f} \varphi_\ell$$

$$R : \varphi_\ell \to \frac{1}{\sqrt{q}} \sum_{m=1}^{q} \varphi_m e^{2\pi i \ell m f}$$

Vortices in a superfluid near a Mott insulator at filling $f=p/q$

The q φ_ℓ vortices characterize *both* superconducting and VBS/CDW orders

$$\underline{\text{Superconductor/insulator}} : \left\langle \varphi_\ell \right\rangle = 0 \big/ \left\langle \varphi_\ell \right\rangle \neq 0$$

Vortices in a superfluid near a Mott insulator at filling *f*=*p*/*q*

The $q \ \varphi_\ell$ vortices characterize *both* superconducting and VBS/CDW orders

VBS order:

Status of space group symmetry determined by density operators $\rho_{\mathbf{Q}}$ at wavevectors $\mathbf{Q}_{mn} = \frac{2\pi p}{q}(m,n)$

$$\rho_{mn} = e^{i\pi mnf} \sum_{\ell=1}^{q} \varphi_\ell^* \varphi_{\ell+n} e^{2\pi i \ell m f}$$

$$T_x : \rho_{\mathbf{Q}} \to \rho_{\mathbf{Q}} e^{i\mathbf{Q}\cdot\hat{x}} \quad ; \qquad T_y : \rho_{\mathbf{Q}} \to \rho_{\mathbf{Q}} e^{i\mathbf{Q}\cdot\hat{y}}$$

$$R : \rho(\mathbf{Q}) \to \rho(R\mathbf{Q})$$

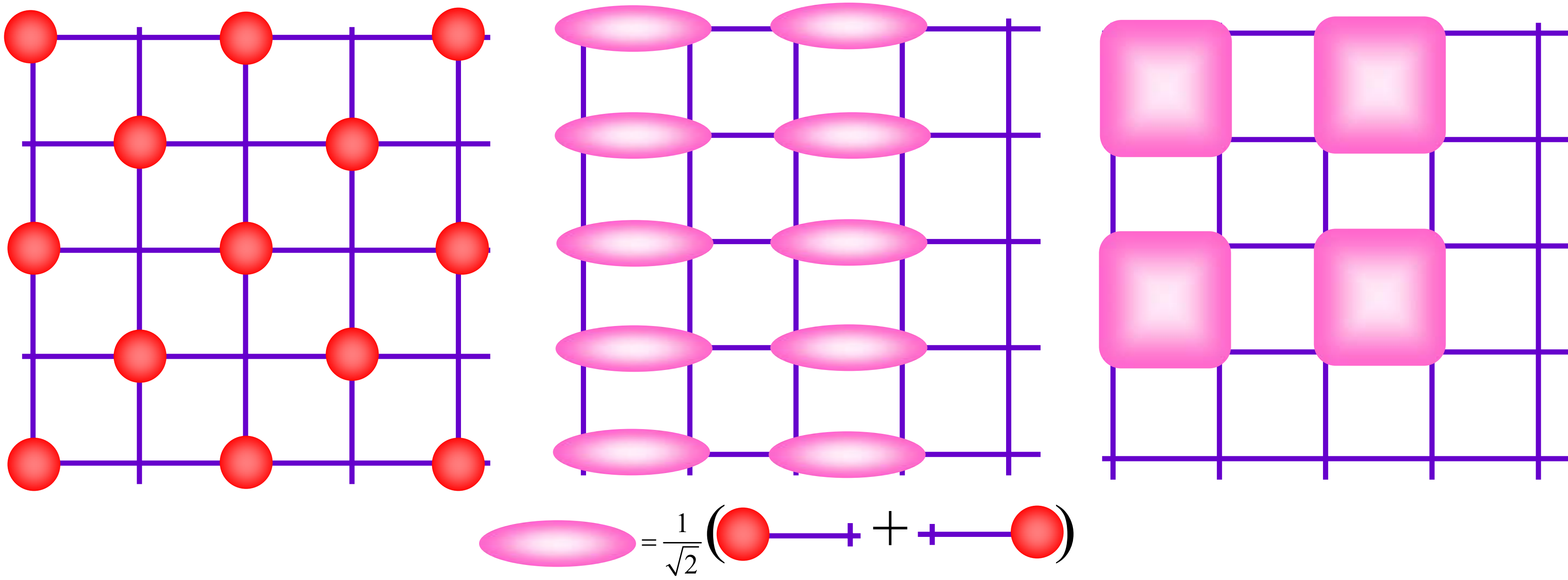
Vortices in a superfluid near a Mott insulator at filling $f=p/q$

- The excitations of the superfluid are described by the quantum mechanics of q flavors of low energy vortices moving in zero dual "magnetic" field.

- The orientation of the vortex in flavor space implies a particular configuration of VBS order in its vicinity.

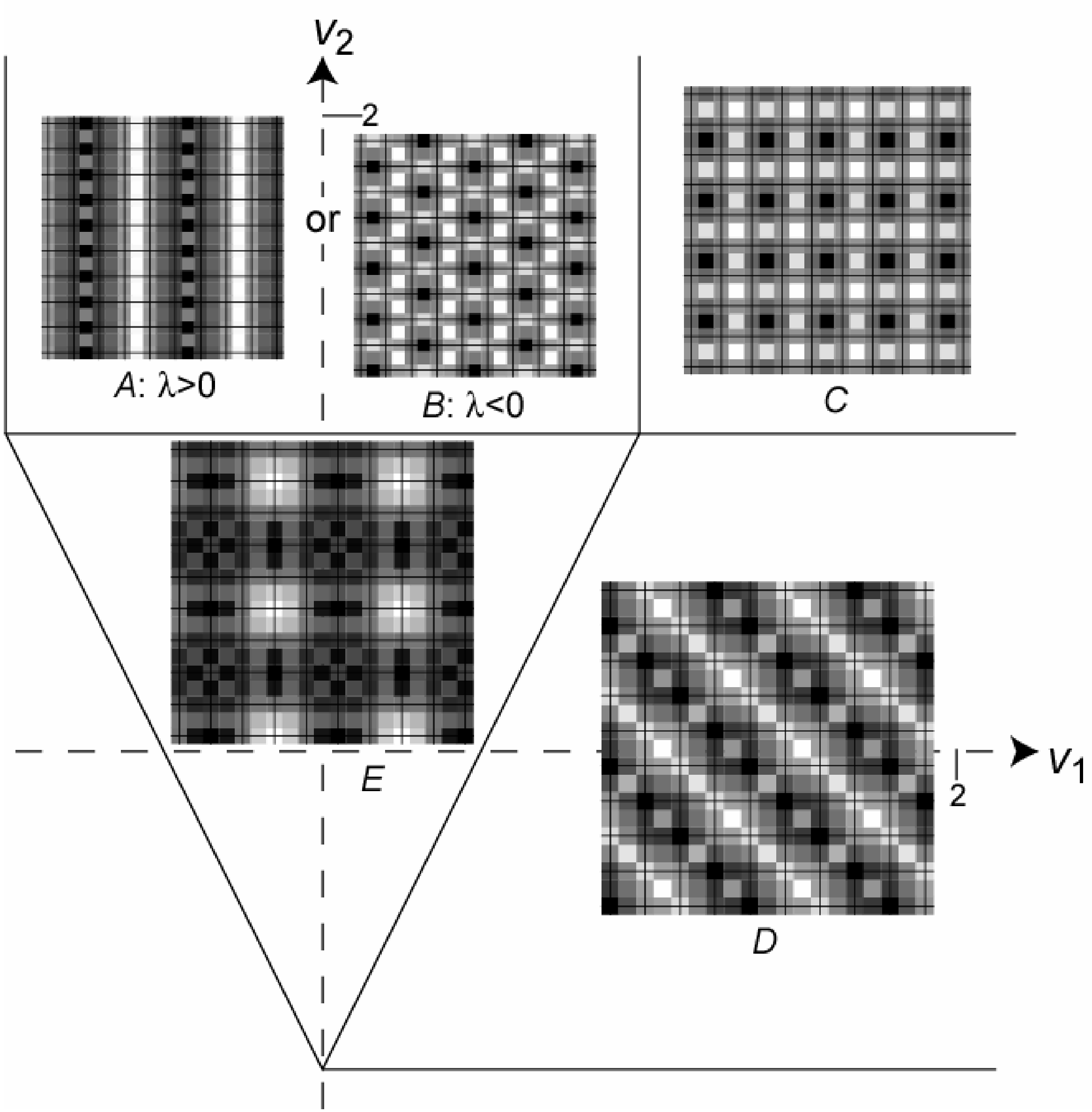
Mott insulators obtained by “condensing” vortices

Spatial structure of insulators for *q*=2 (*f*=1/2)



Field theory with projective symmetry

Spatial structure of insulators for q=4 (f=1/4 or 3/4)



$a \times b$ unit cells;

q/a, q/b, ab/q,

all integers

Vortices in a superfluid near a Mott insulator at filling $f=p/q$

- The excitations of the superfluid are described by the quantum mechanics of q flavors of low energy vortices moving in zero dual "magnetic" field.

- The orientation of the vortex in flavor space implies a particular configuration of VBS order in its vicinity.

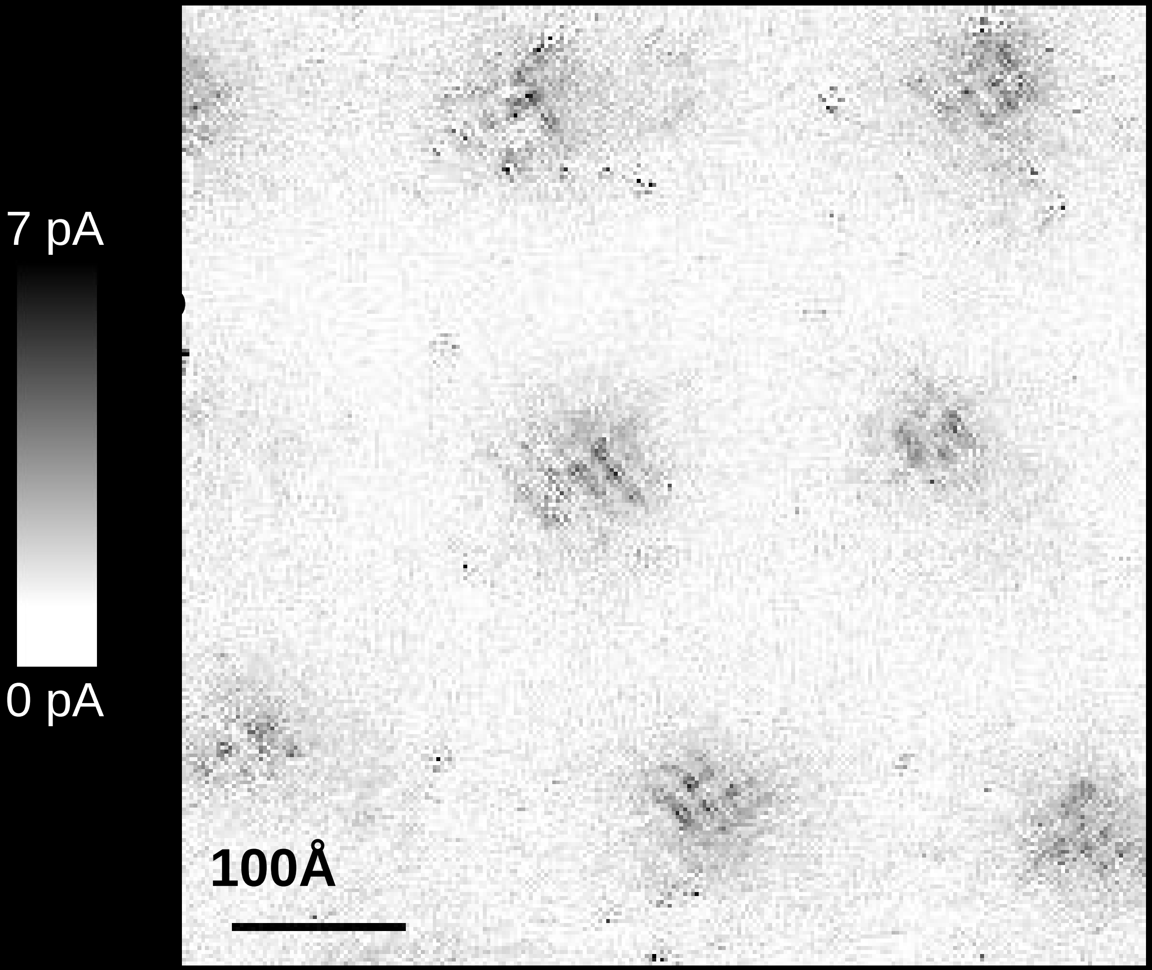
Vortices in a superfluid near a Mott insulator at filling $f=p/q$

- The excitations of the superfluid are described by the quantum mechanics of q flavors of low energy vortices moving in zero dual "magnetic" field.

- The orientation of the vortex in flavor space implies a particular configuration of VBS order in its vicinity.

- Any pinned vortex must pick an orientation in flavor space: this induces a halo of VBS order in its vicinity

Vortex-induced LDOS of $Bi_2Sr_2CaCu_2O_{8+\delta}$ integrated from 1meV to 12meV at 4K



Vortices have halos with LDOS modulations at a period ≈ 4 lattice spacings

Prediction of VBS order near vortices: K. Park and S. Sachdev, Phys. Rev. B **64**, 184510 (2001).

J. Hoffman, E. W. Hudson, K. M. Lang, V. Madhavan, S. H. Pan, H. Eisaki, S. Uchida, and J. C. Davis, *Science* 295, 466 (2002).

Measuring the inertial mass of a vortex

The spatial extent of the LDOS modulations measures the region over which the vortex executes its zero-point motion. The size of this region can be determined by solving the equations of motion

$$m_v \frac{d^2\mathbf{r}}{dt^2} = F_M$$

and so is determined by the inertial vortex mass m_v.

Measuring the inertial mass of a vortex

Preliminary estimates for the BSCCO experiment:

Inertial vortex mass $m_v \approx 10 m_e$

Vortex magnetoplasmon frequency $\nu_{\mathrm{p}} \approx 1\ \mathrm{THz} = 4\ \mathrm{meV}$

Future experiments can directly detect vortex zero point motion by looking for resonant absorption at this frequency.

Vortex oscillations can also modify the electronic density of states.

Superfluids near Mott insulators

The Mott insulator has average Cooper pair density, $f = p/q$ per site, while the density of the superfluid is close (but need not be identical) to this value

- Vortices with flux *h/(2e)* come in multiple (usually q) “flavors”
- The lattice space group acts in a projective representation on the vortex flavor space.
- These flavor quantum numbers provide a distinction between superfluids: they constitute a “quantum order”
- Any pinned vortex must chose an orientation in flavor space. This necessarily leads to modulations in the local density of states over the spatial region where the vortex executes its quantum zero point motion.