

# The quantum mechanics of two dimensional superfluids

*Physical Review B* **71**, 144508 and 144509 (2005),  
cond-mat/0502002

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Anton Burkov (UCSB)

Subir Sachdev (Yale)

Krishnendu Sengupta (Toronto)



Talk online: [Google](#) Sachdev

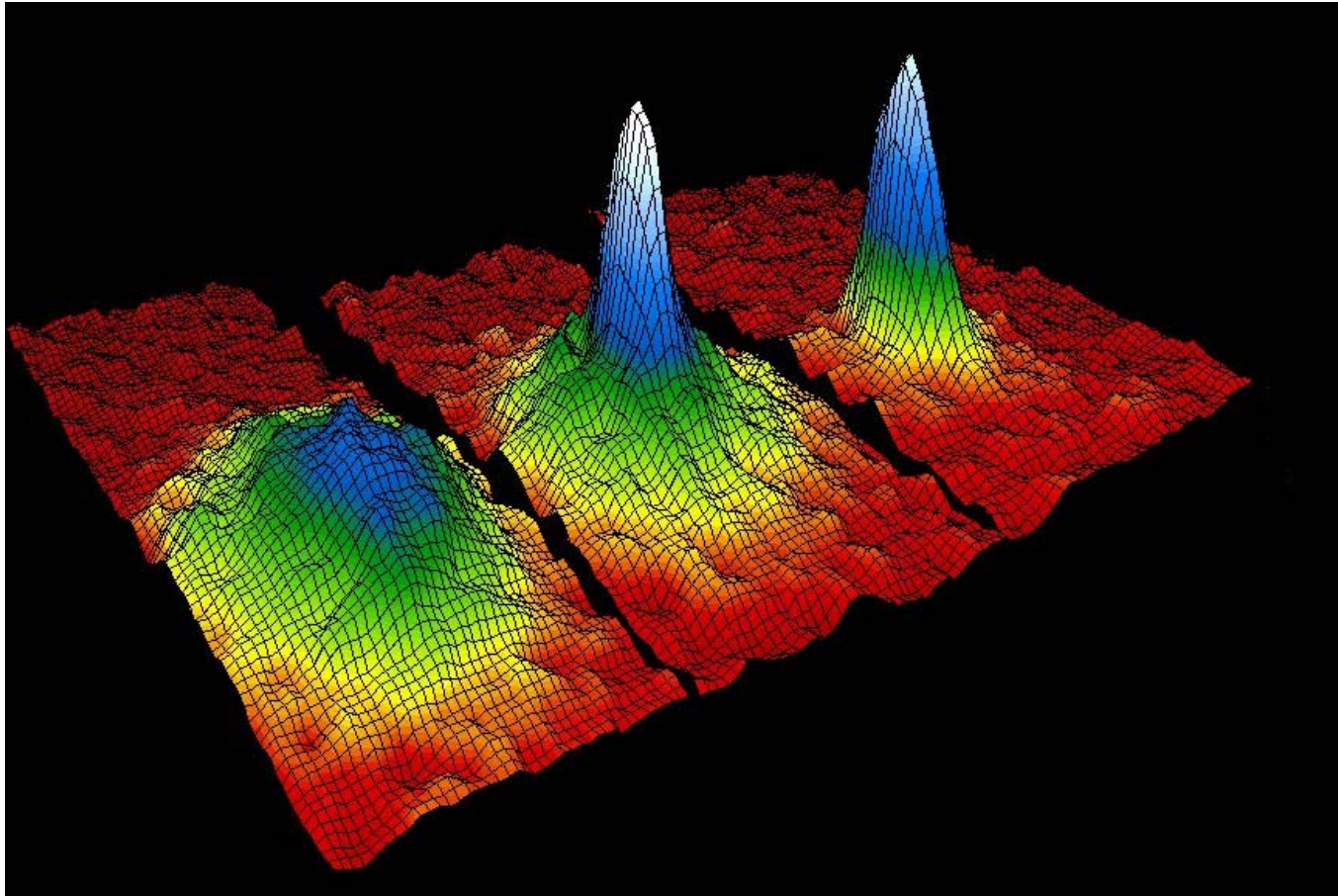
# Outline

- I. The superfluid-Mott insulator quantum phase transition
- II. The cuprate superconductors  
*Superfluids proximate to finite doping Mott insulators with VBS order ?*
- III. Vortices in the superfluid
- IV. Vortices in superfluids near the superfluid-insulator quantum phase transition  
*The “quantum order” of the superconducting state: evidence for vortex flavors*

# I. The superfluid-Mott insulator quantum phase transition

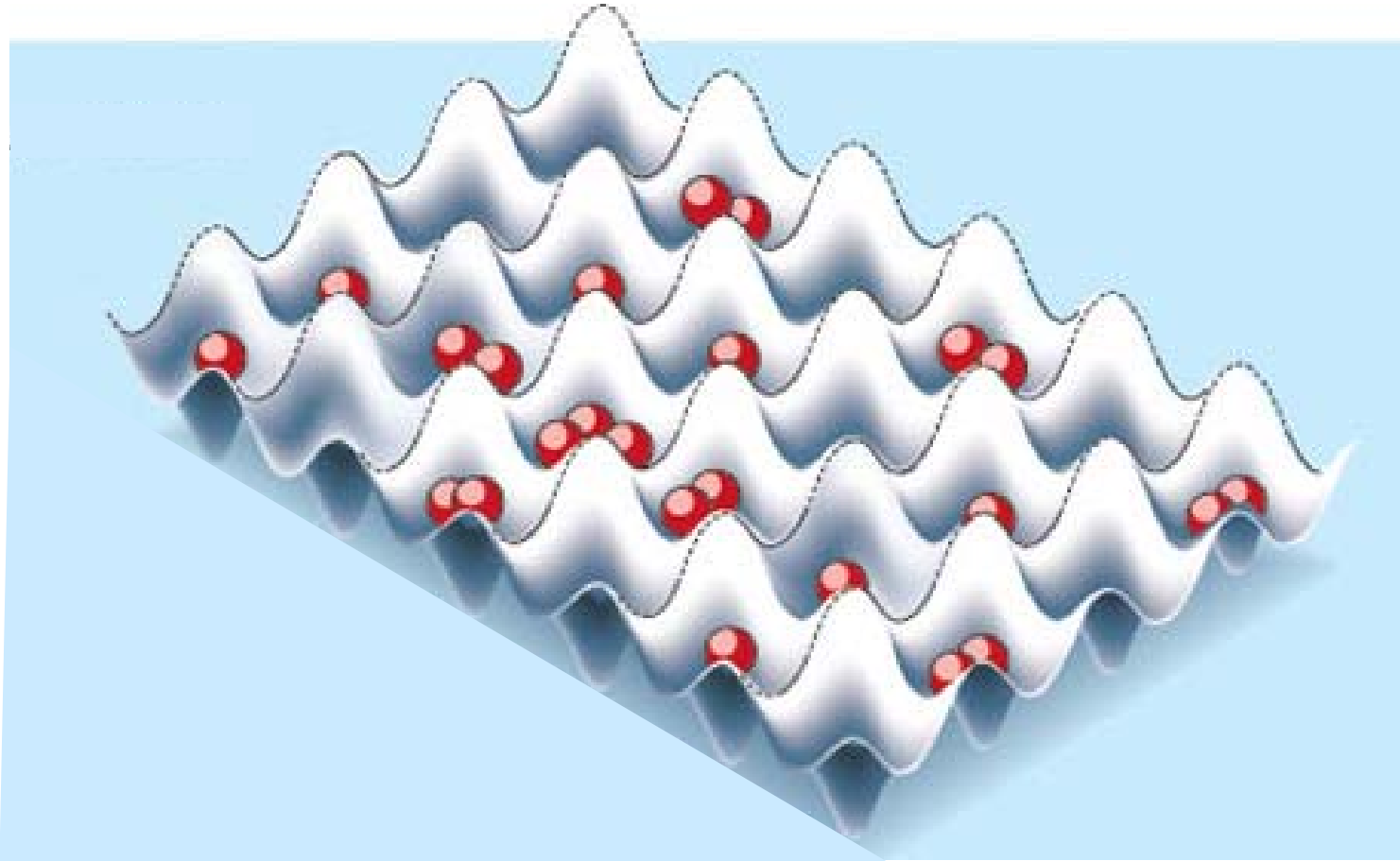
## Bose condensation

Velocity distribution function of ultracold  $^{87}\text{Rb}$  atoms

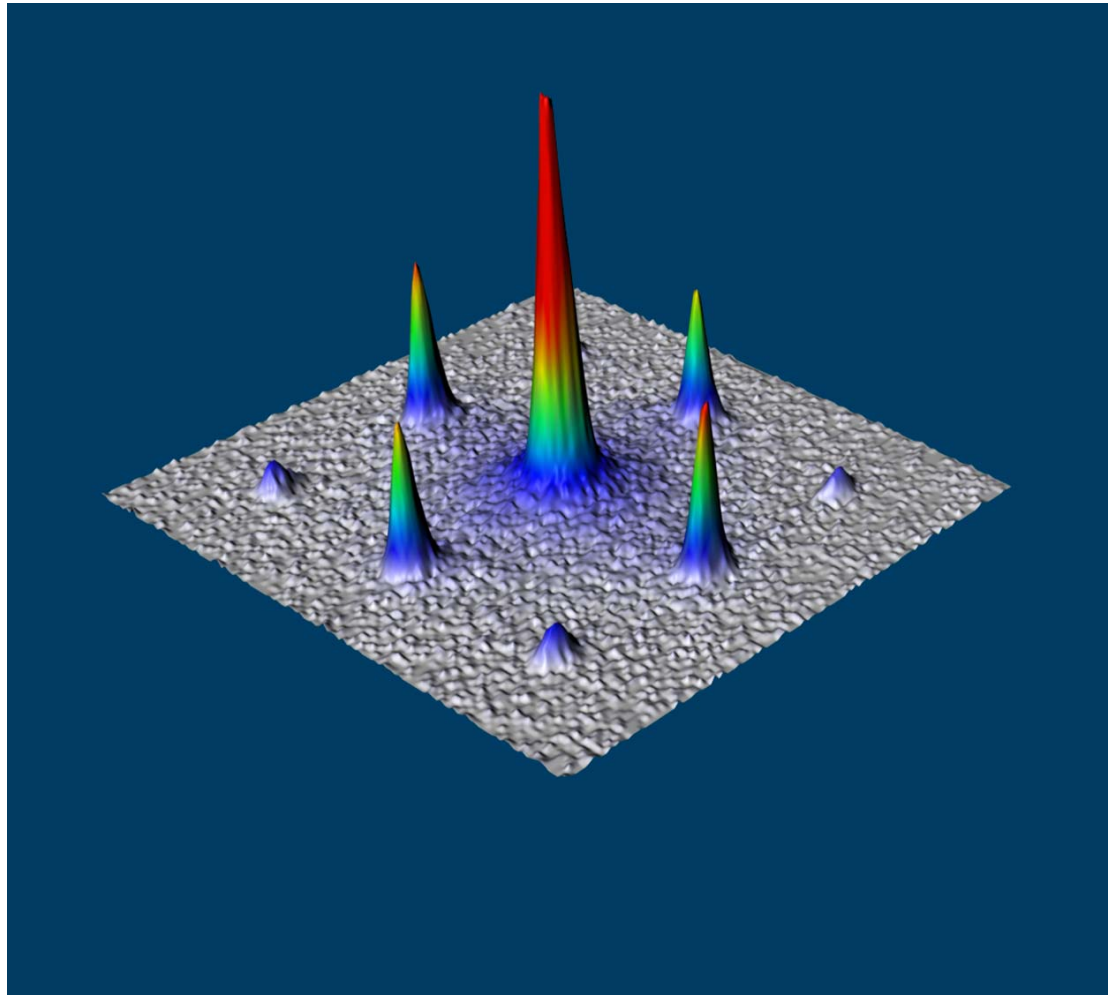


M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman  
and E. A. Cornell, *Science* **269**, 198 (1995)

Apply a periodic potential (standing laser beams)  
to trapped ultracold bosons ( $^{87}\text{Rb}$ )

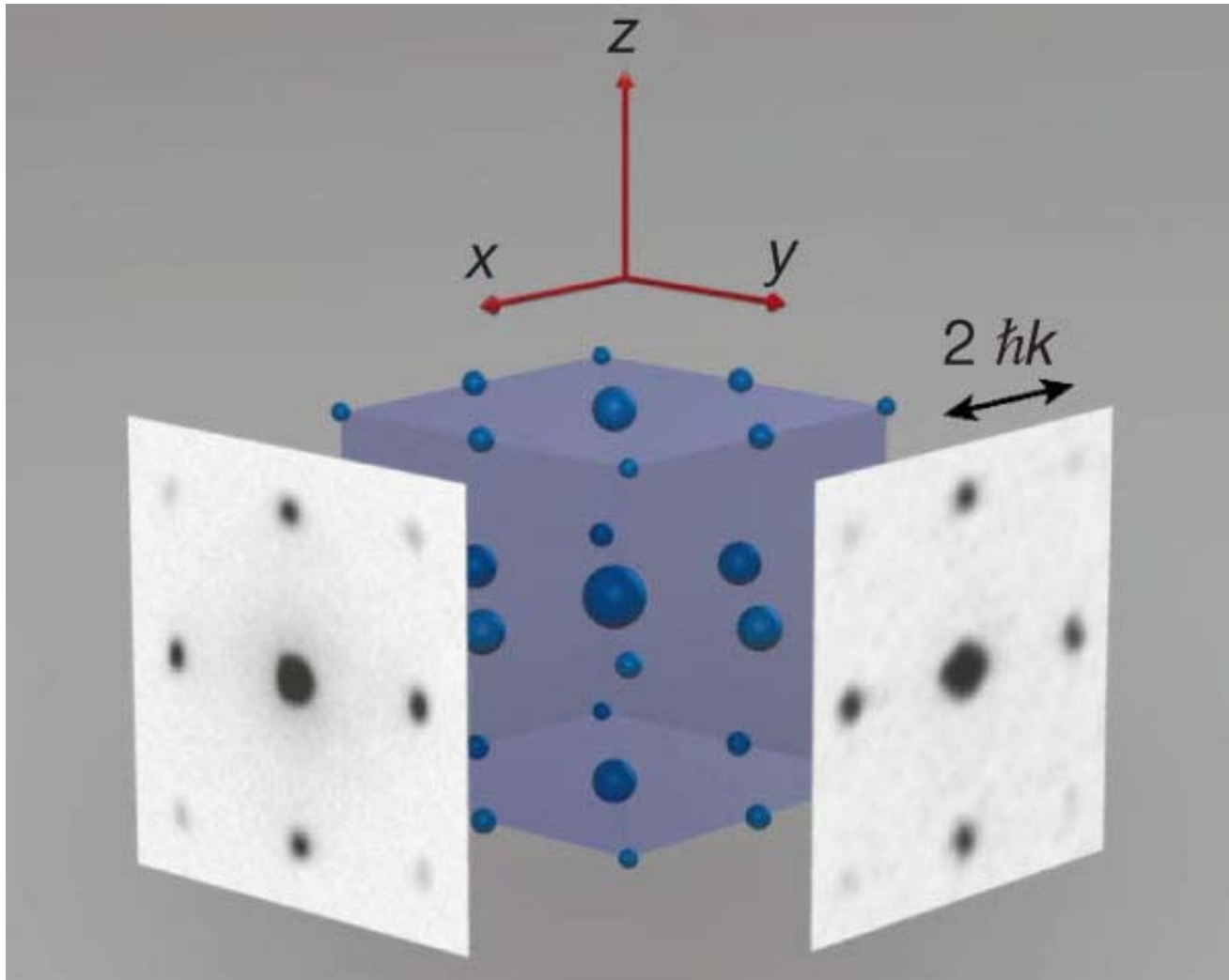


# Momentum distribution function of bosons



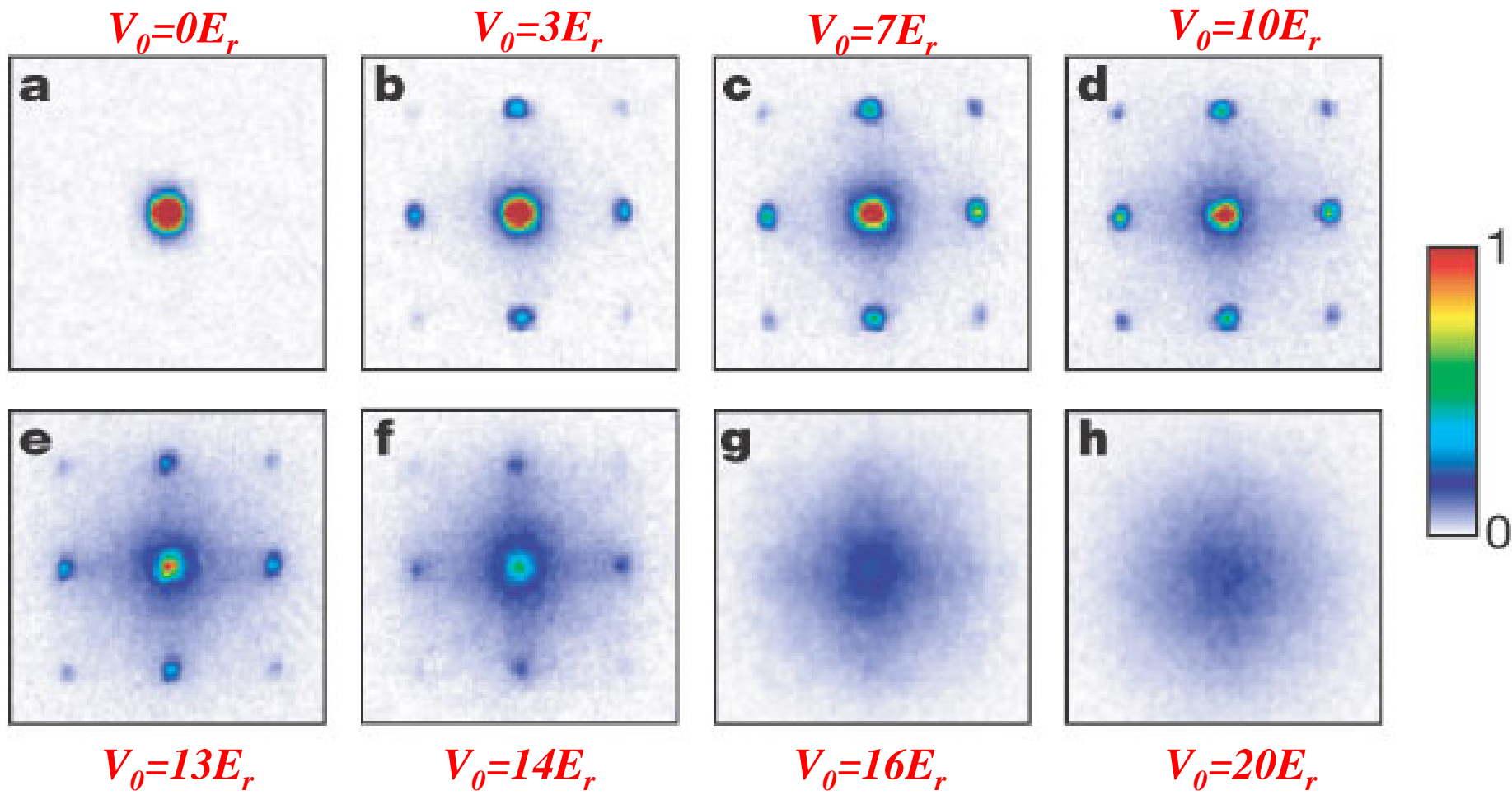
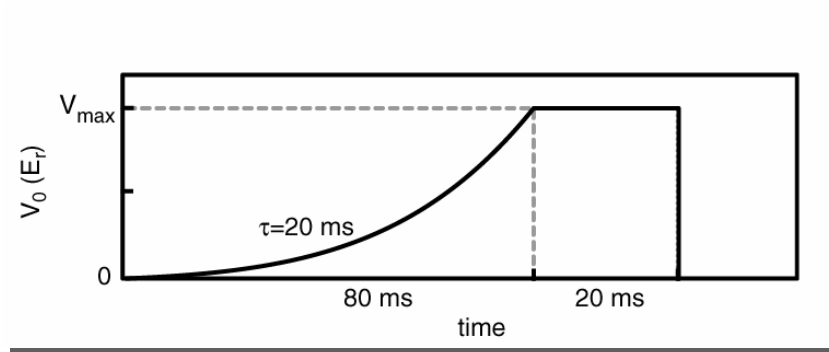
Bragg reflections of condensate at reciprocal lattice vectors

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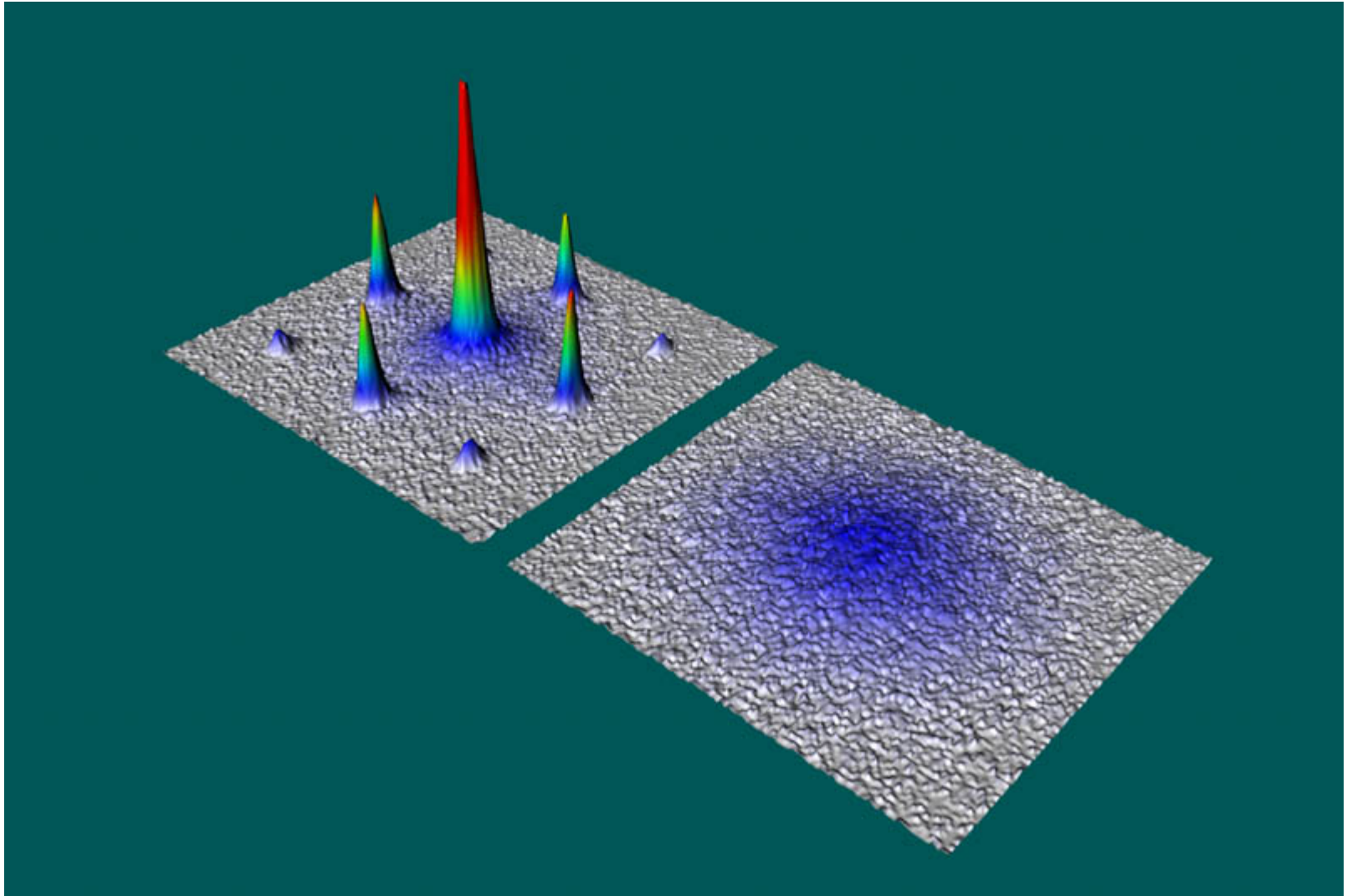


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# Superfluid-insulator quantum phase transition at $T=0$



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M. Greiner, O. Mandel, T. Esslinger, T. W. Hänsch, and I. Bloch, *Nature* **415**, 39 (2002).

# Bosons at filling fraction $f = 1$

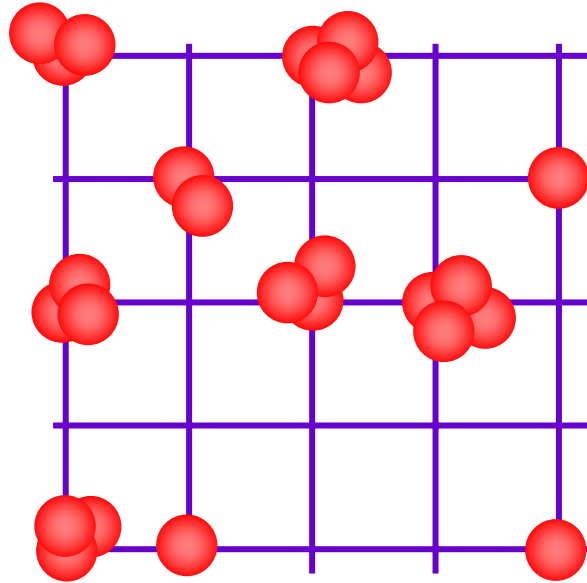
Weak interactions:  
superfluidity

**a** Superfluid state

**b** Insulating state

Strong interactions:  
Mott insulator which  
preserves all lattice  
symmetries

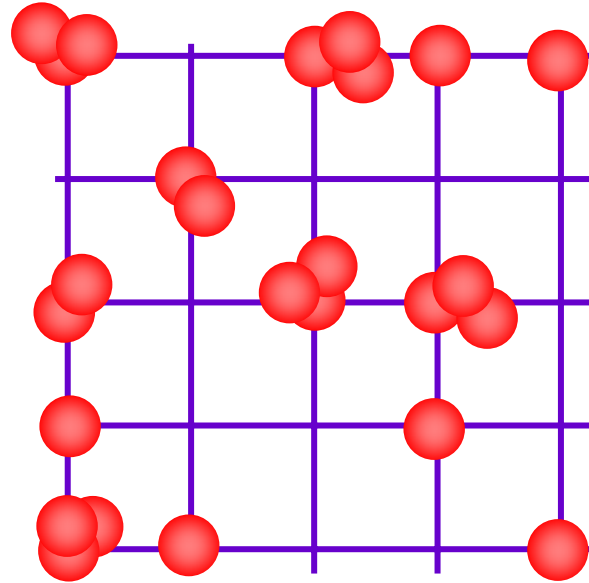
# Bosons at filling fraction $f = 1$



$$\langle \Psi \rangle \neq 0$$

Weak interactions: superfluidity

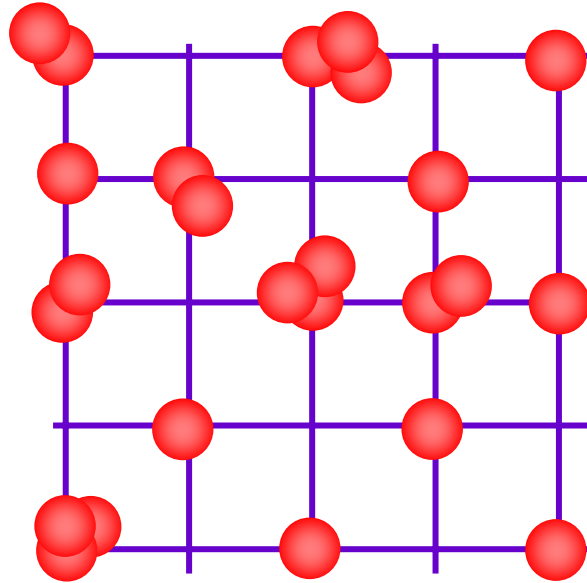
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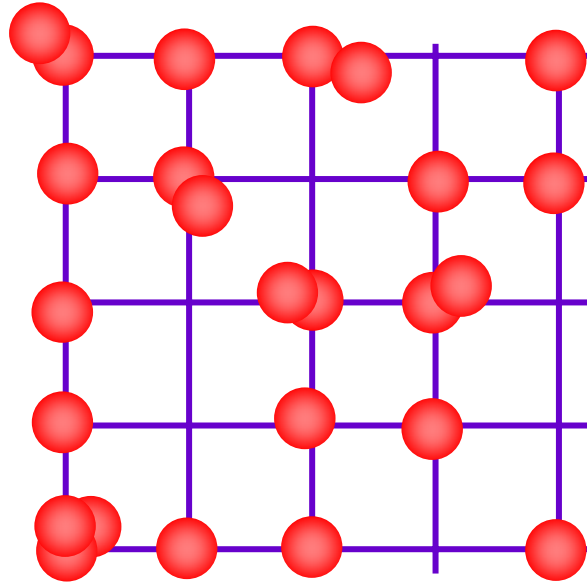
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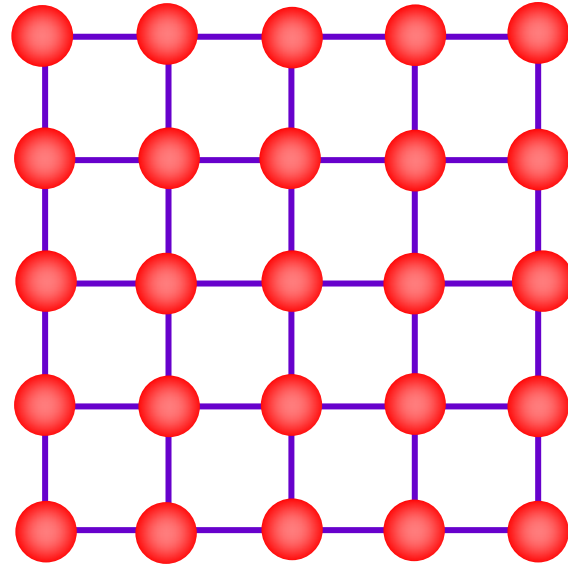
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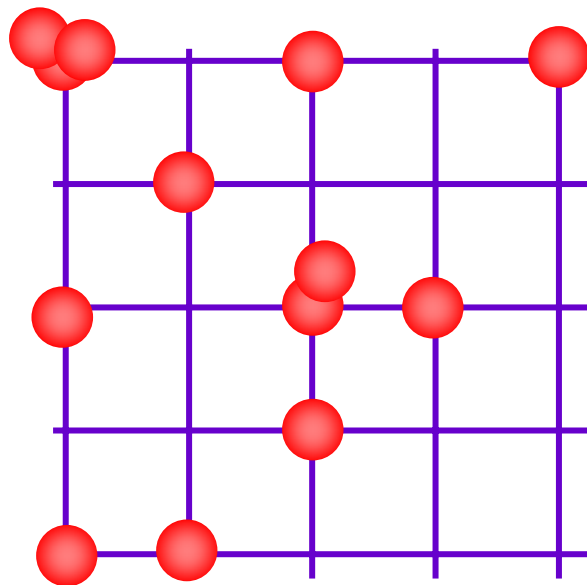
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Strong interactions: insulator

## Bosons at filling fraction $f = 1/2$



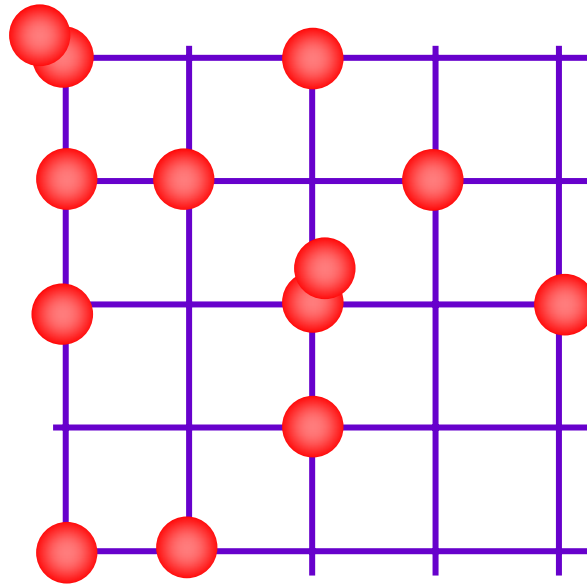
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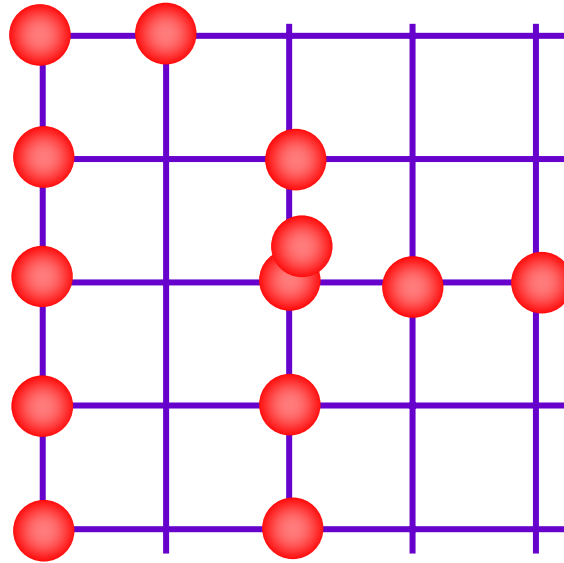
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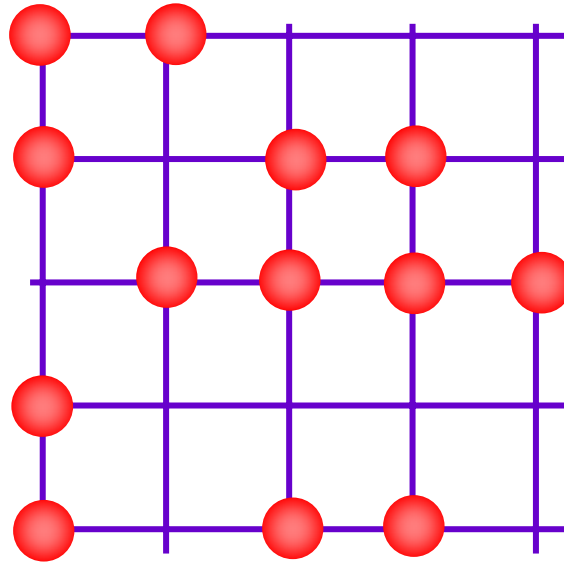
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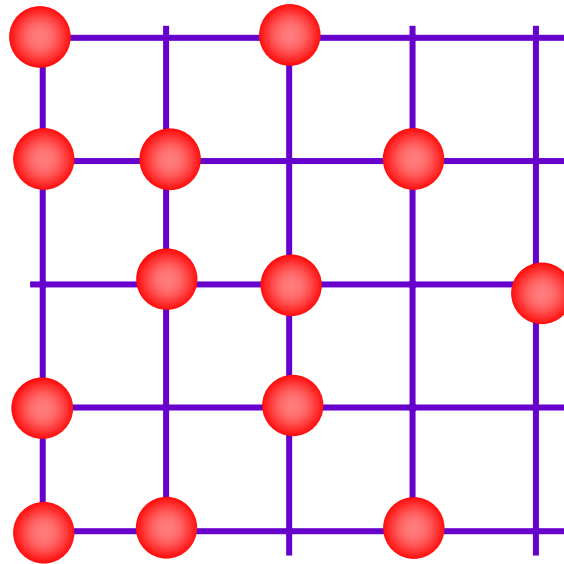
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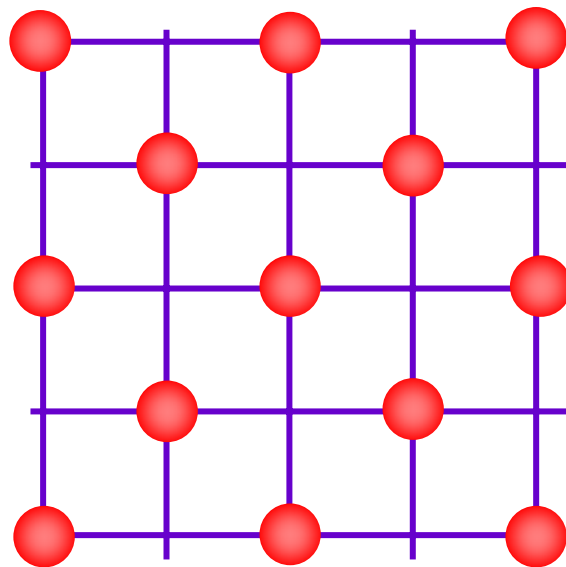
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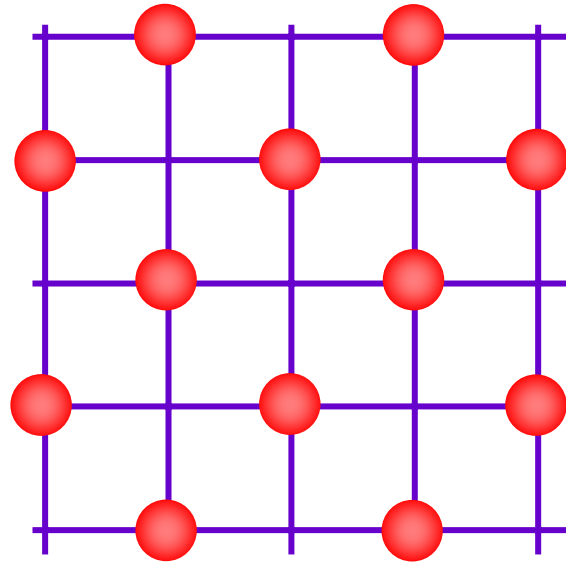
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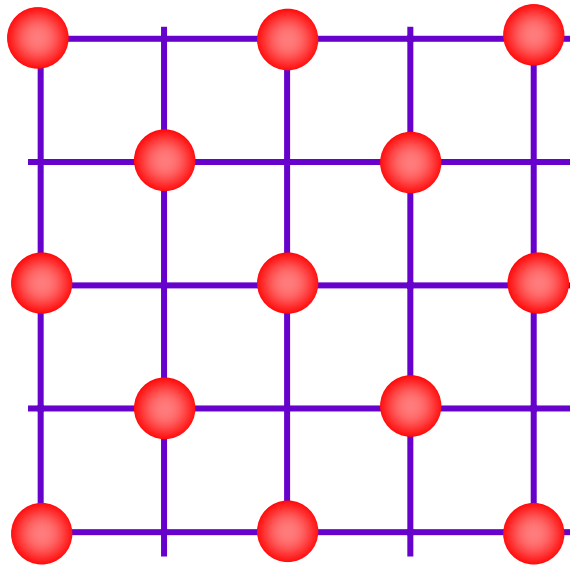
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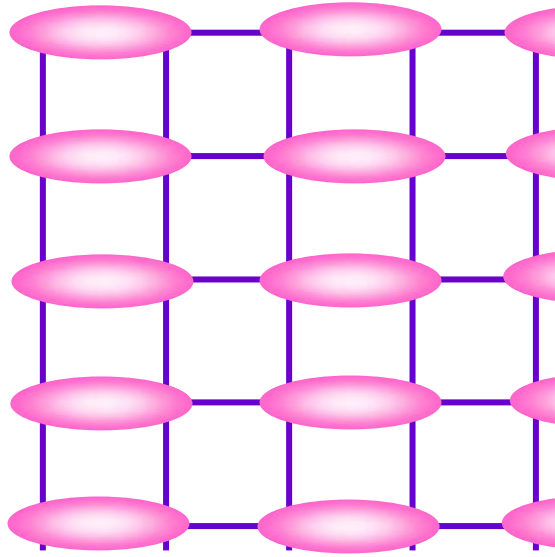
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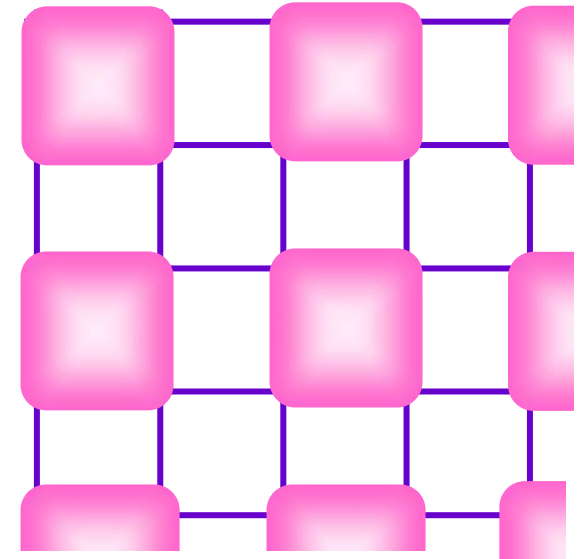
# Insulating phases of bosons at filling fraction $f = 1/2$



Charge density wave (CDW) order



Valence bond solid (VBS) order



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$$\text{pink oval} = \frac{1}{\sqrt{2}} \left( \text{red sphere} - \text{red sphere} \right)$$

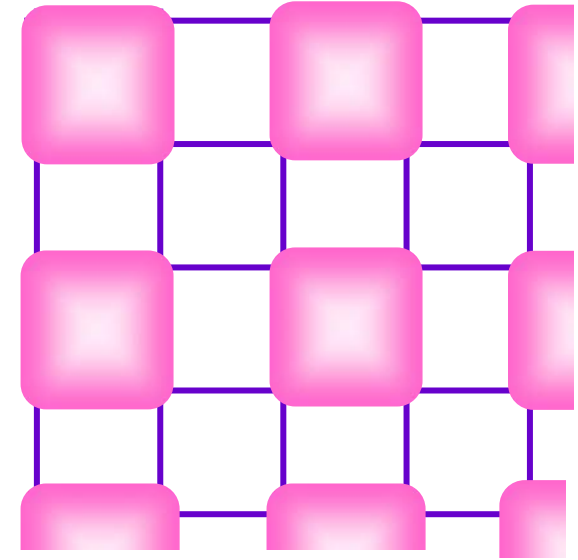
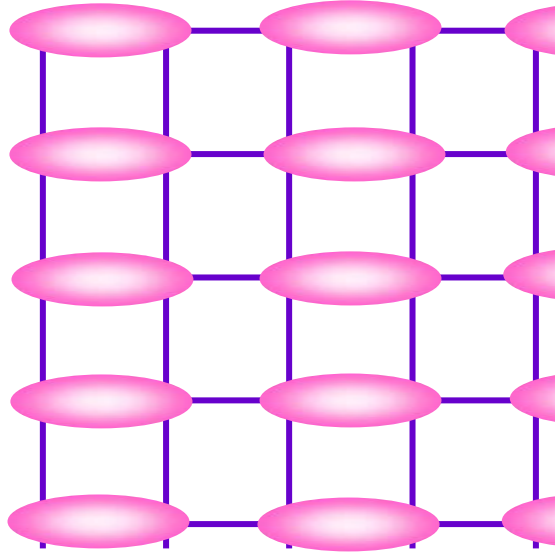
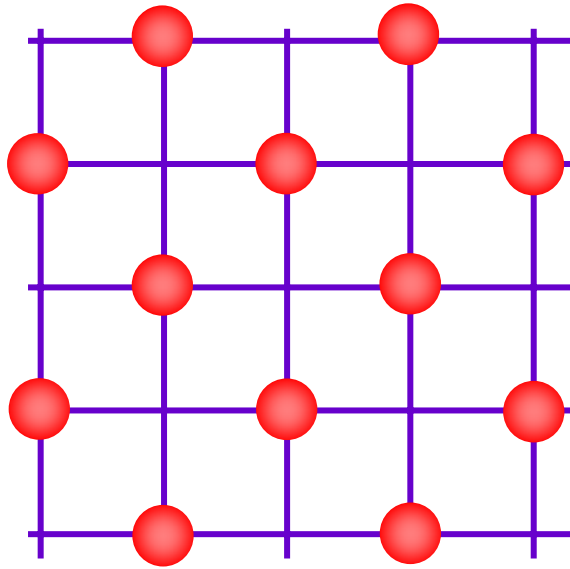
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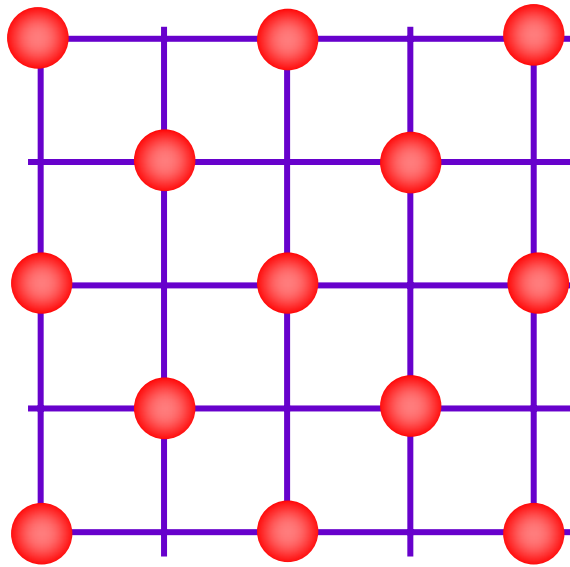
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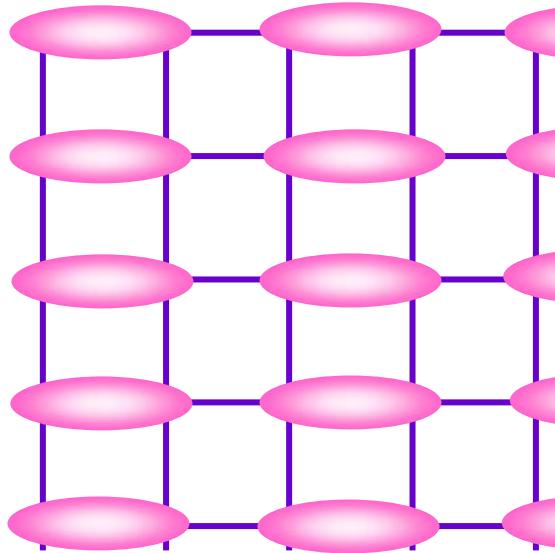
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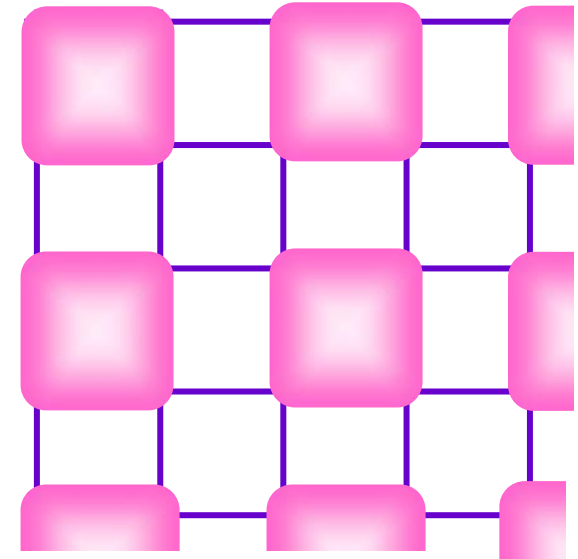
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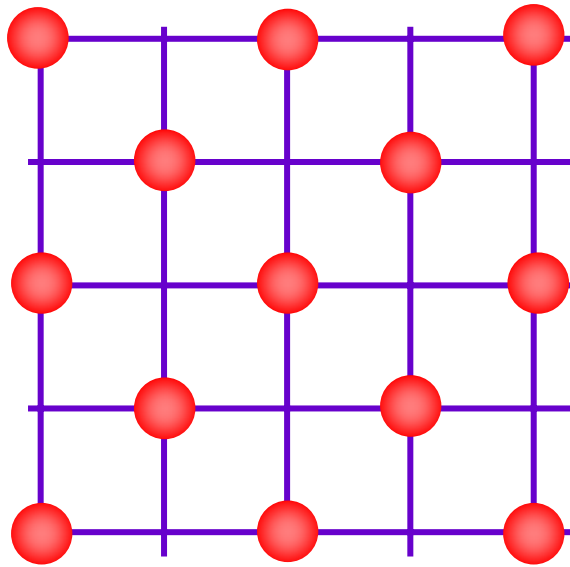
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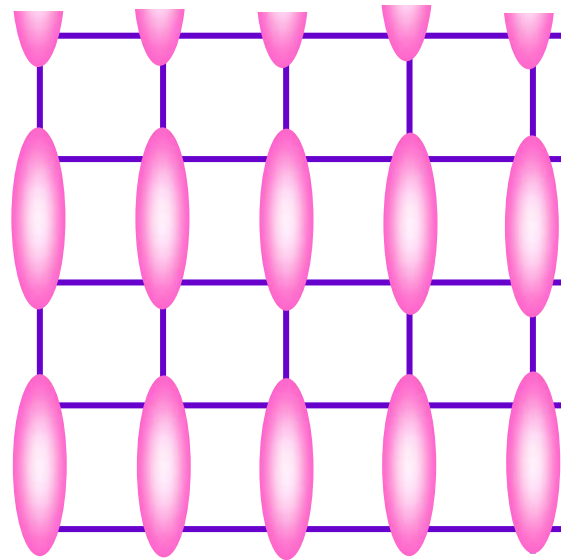
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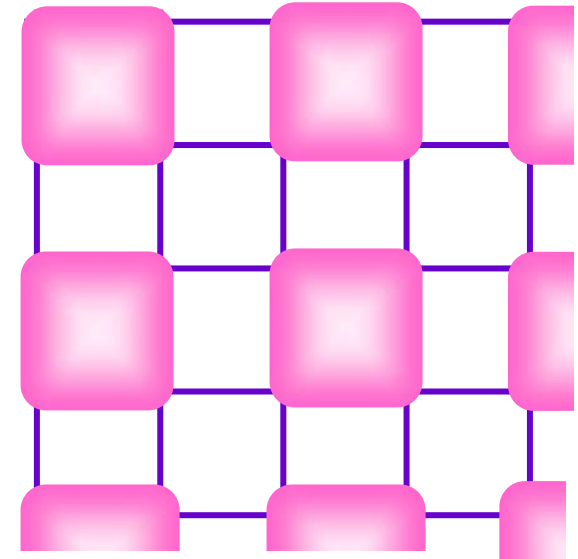
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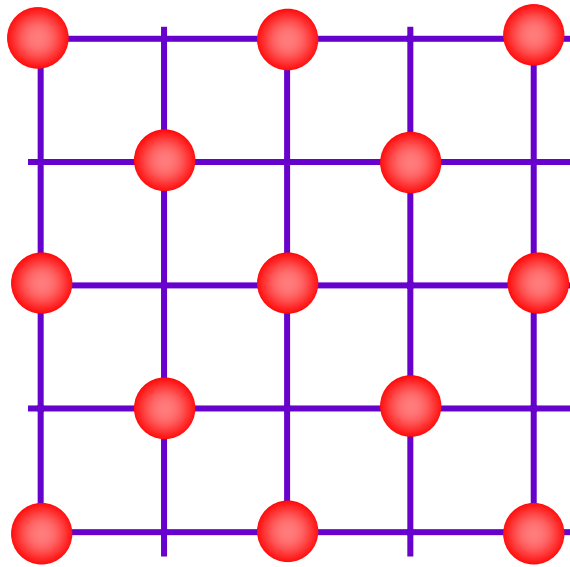
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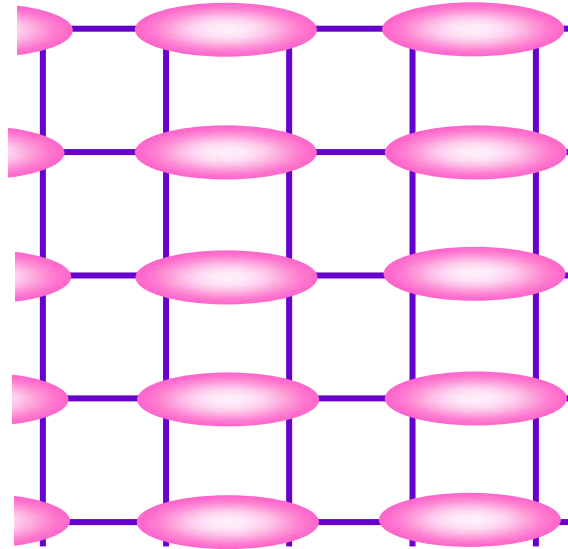
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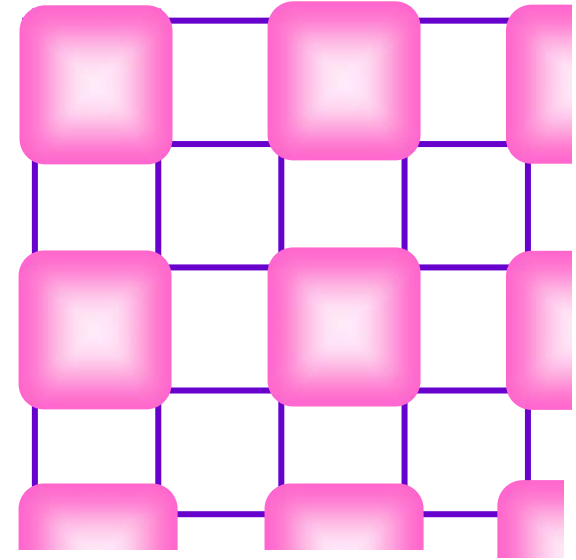
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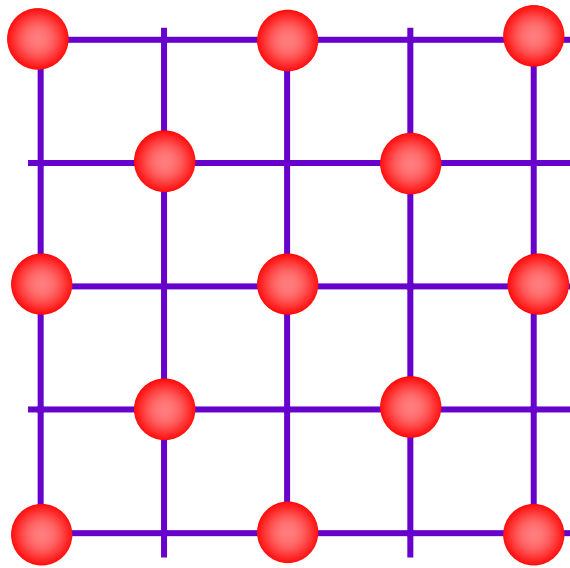
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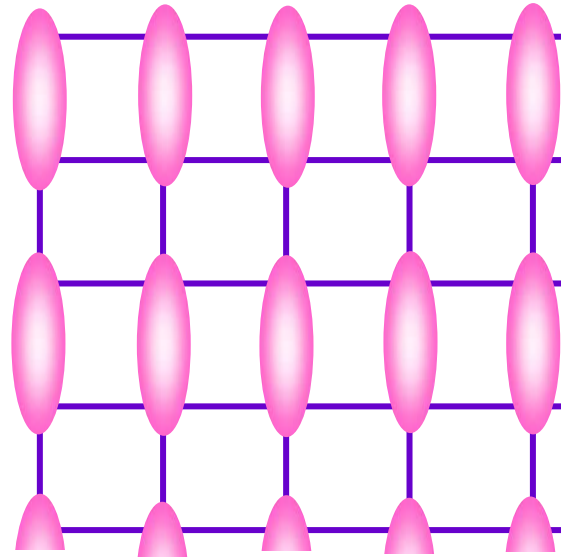
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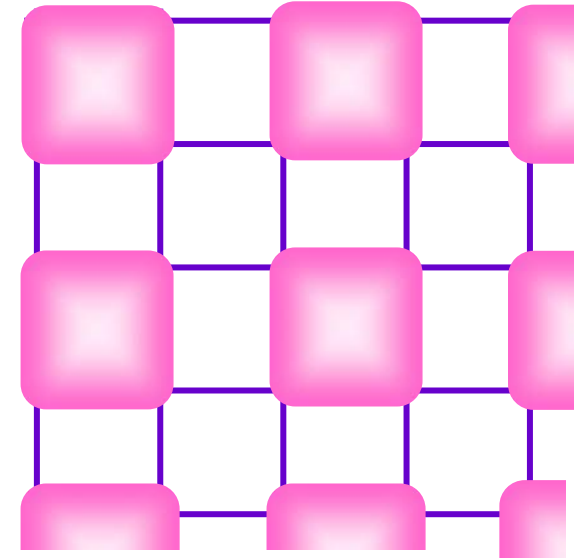
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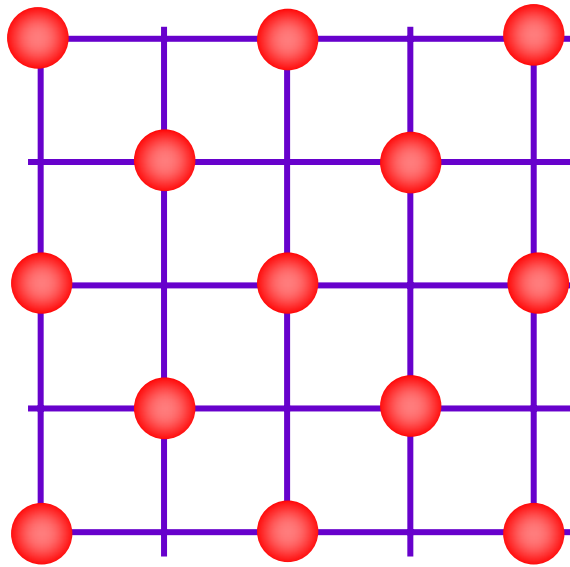
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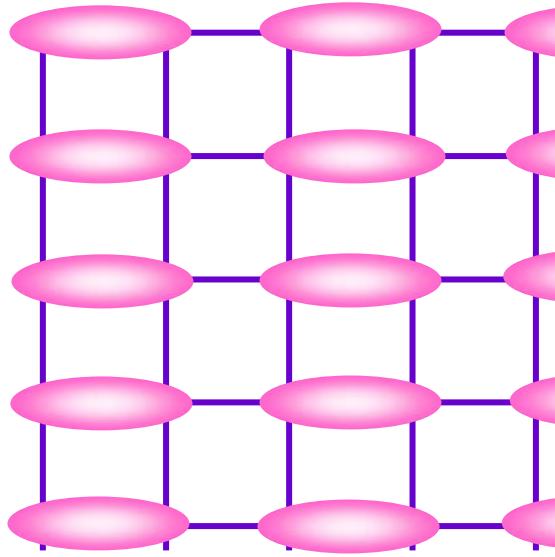
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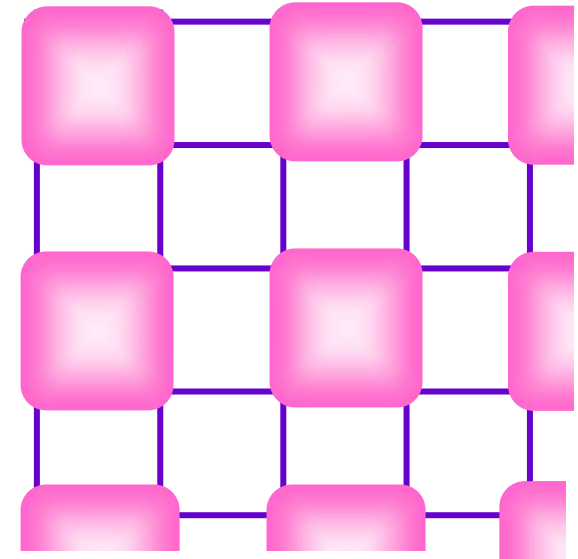
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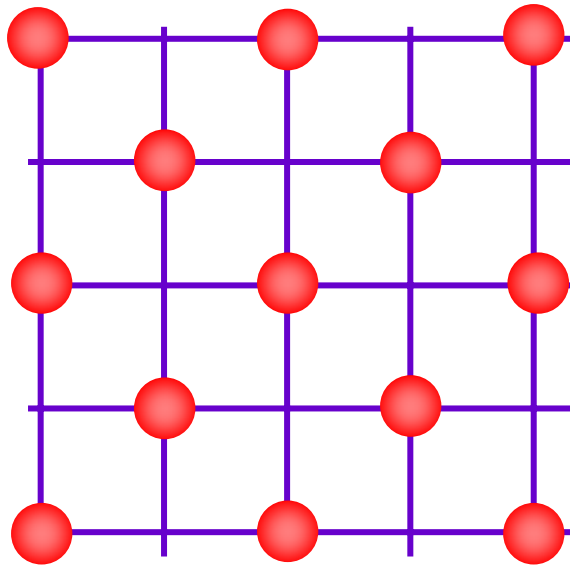
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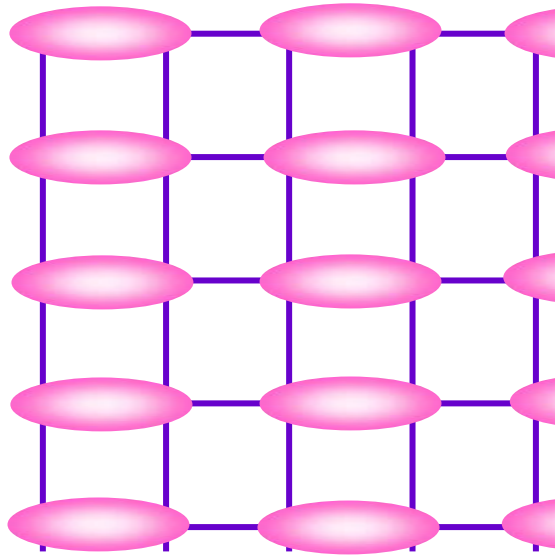
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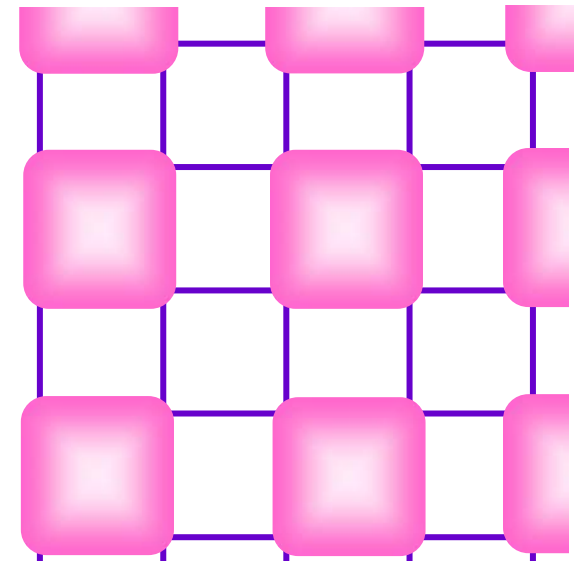
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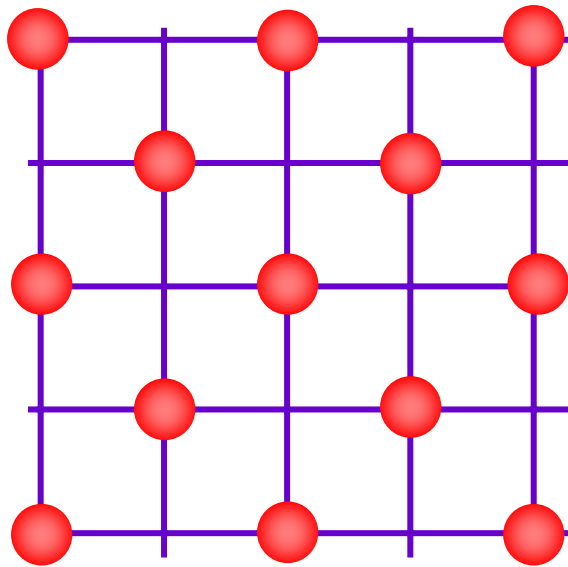
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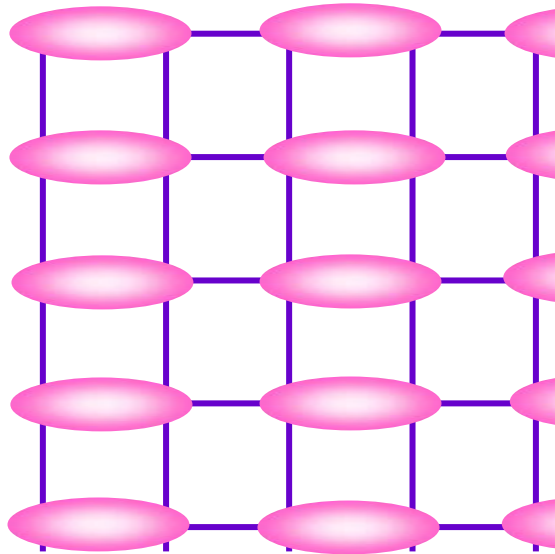
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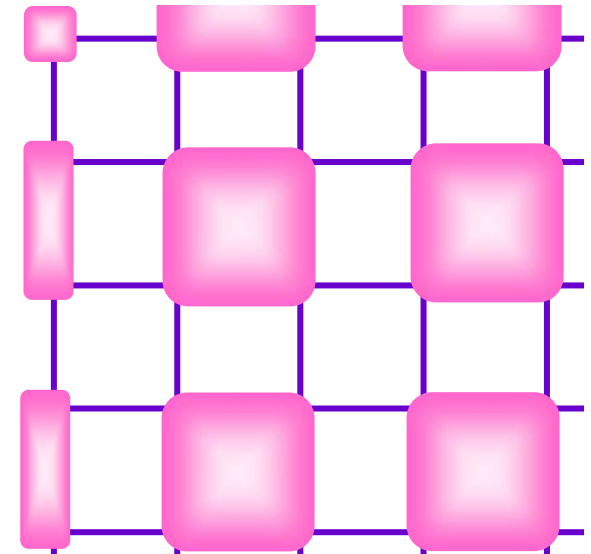
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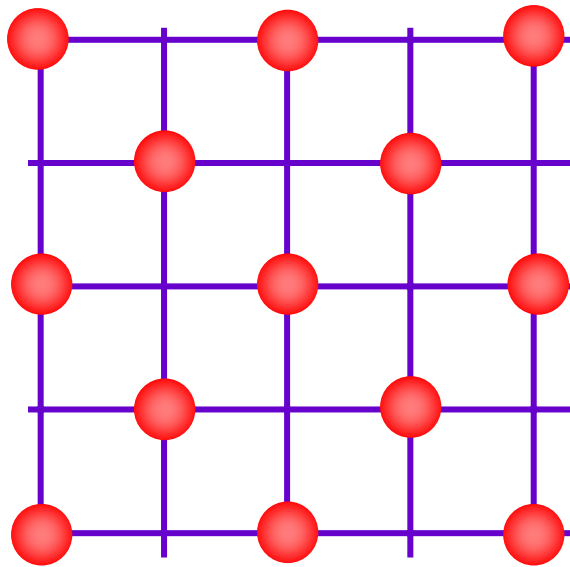
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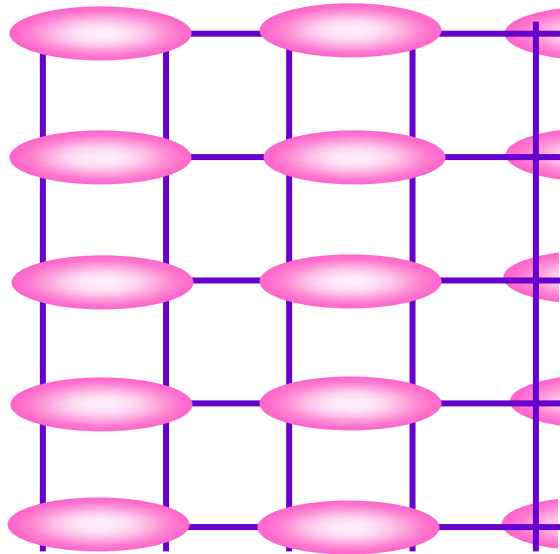
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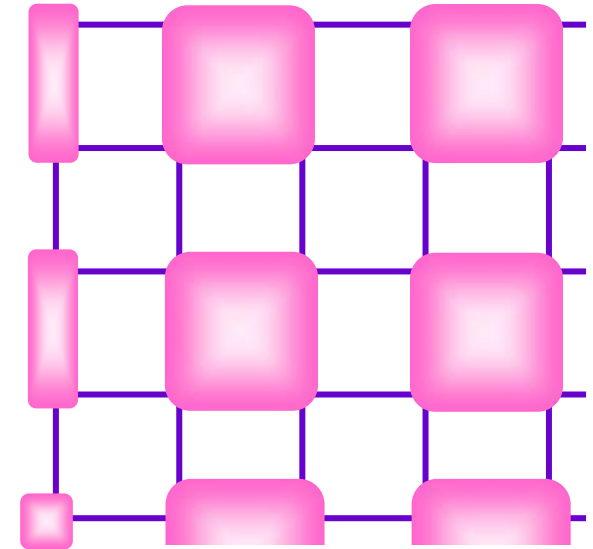
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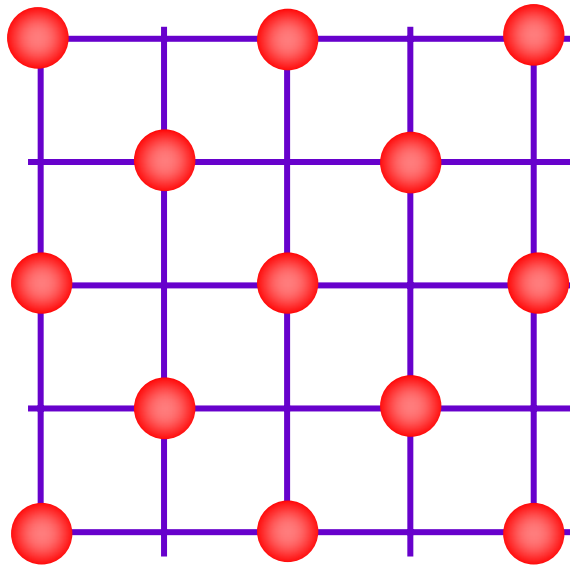
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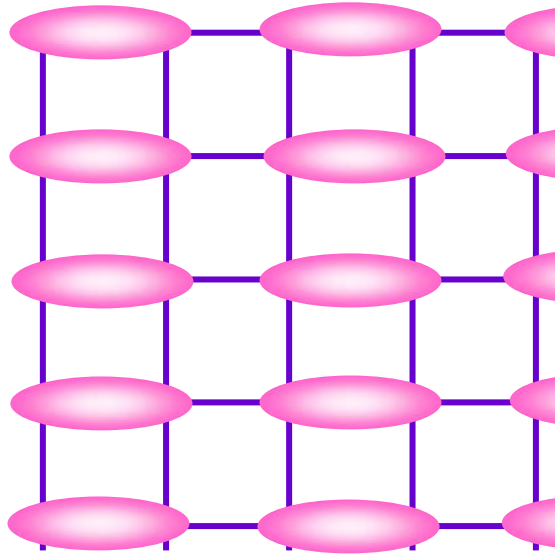
C. Lannert, M.P.A. Fisher, and T. Senthil, *Phys. Rev. B* **63**, 134510 (2001)

S. Sachdev and K. Park, *Annals of Physics*, **298**, 58 (2002)

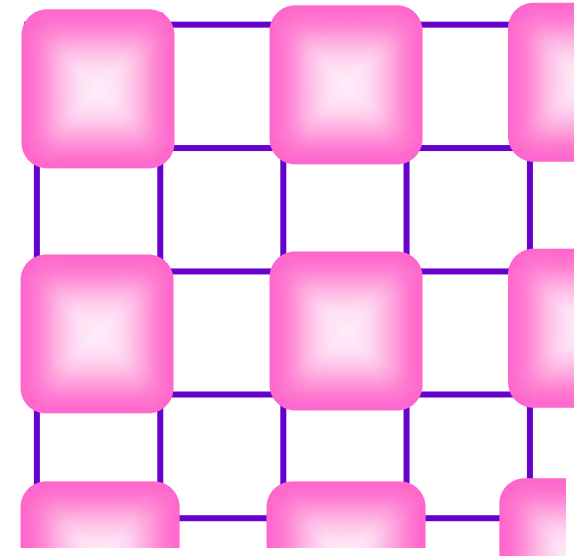
# Insulating phases of bosons at filling fraction $f = 1/2$



Charge density wave (CDW) order



Valence bond solid (VBS) order



Valence bond solid (VBS) order

$$\text{pink oval} = \frac{1}{\sqrt{2}} \left( \text{red sphere} - \text{red sphere} \right)$$

Can define a common CDW/VBS order using a generalized "density"  $\rho(\mathbf{r}) = \sum_{\mathbf{Q}} \rho_{\mathbf{Q}} e^{i\mathbf{Q} \cdot \mathbf{r}}$

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C. Lannert, M.P.A. Fisher, and T. Senthil, *Phys. Rev. B* **63**, 134510 (2001)

S. Sachdev and K. Park, *Annals of Physics*, **298**, 58 (2002)

*Superfluid-insulator transition of bosons at  
generic filling fraction  $f$*

The transition is characterized by multiple distinct order parameters (boson condensate, VBS/CDW order)

*Traditional (Landau-Ginzburg-Wilson) view:*

Such a transition is first order, and there are no precursor fluctuations of the order of the insulator in the superfluid.

## Superfluid-insulator transition of bosons at generic filling fraction $f$

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Such a transition is first order, and there are no precursor fluctuations of the order of the insulator in the superfluid.

### Recent theories:

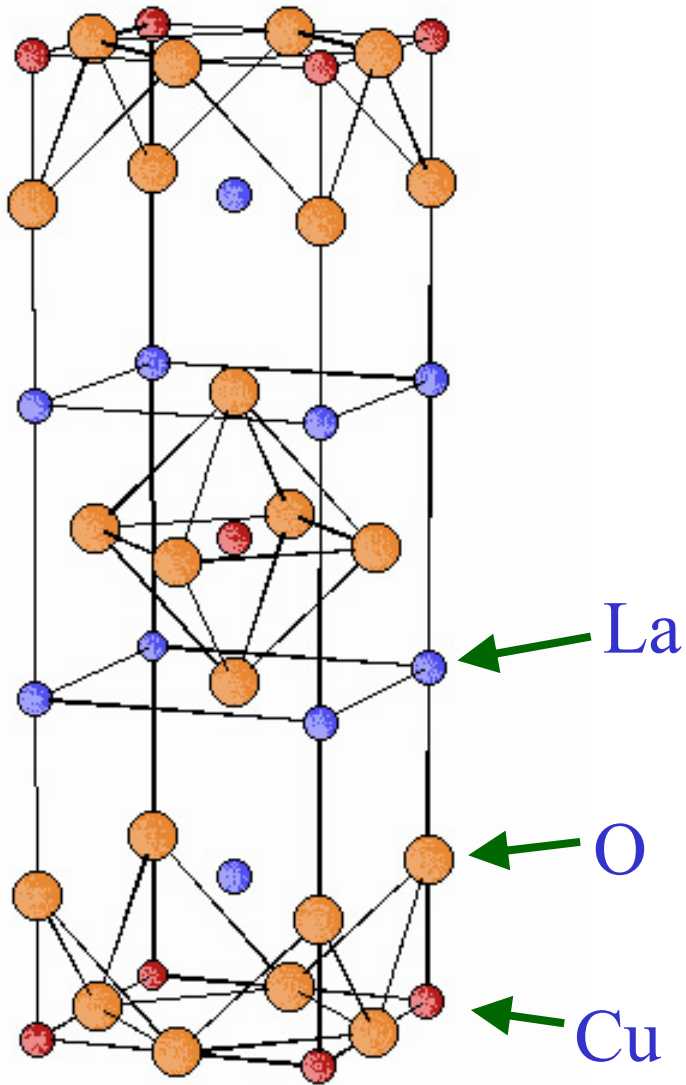
Quantum interference effects can render such transitions second order, and the superfluid does contain precursor VBS/CDW fluctuations.

N. Read and S. Sachdev, *Phys. Rev. Lett.* **62**, 1694 (1989).

T. Senthil, A. Vishwanath, L. Balents, S. Sachdev and M.P.A. Fisher, *Science* **303**, 1490 (2004).

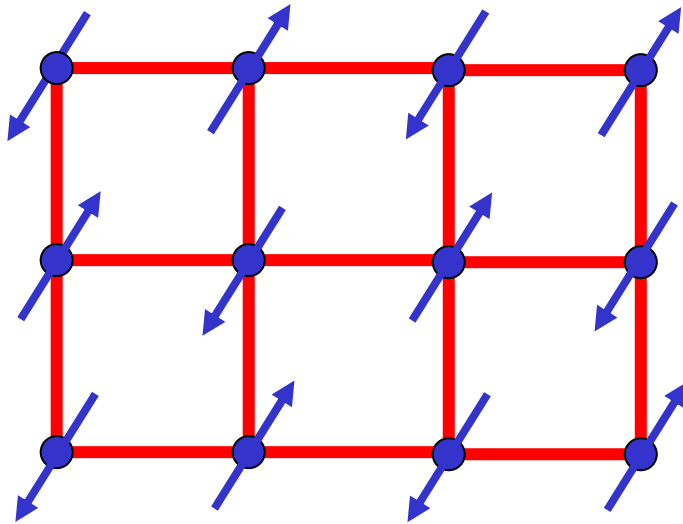
## II. The cuprate superconductors

*Superfluids proximate to finite doping*  
*Mott insulators with VBS order ?*





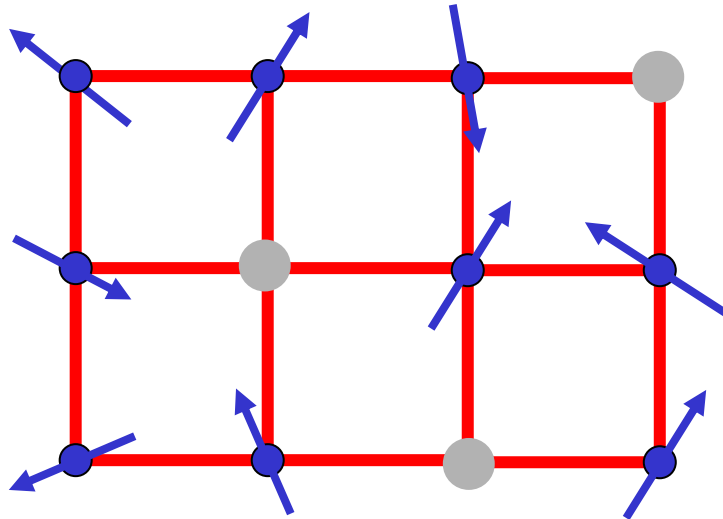
Mott insulator: square lattice antiferromagnet



$$H = \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j$$



Superfluid: condensate of paired holes

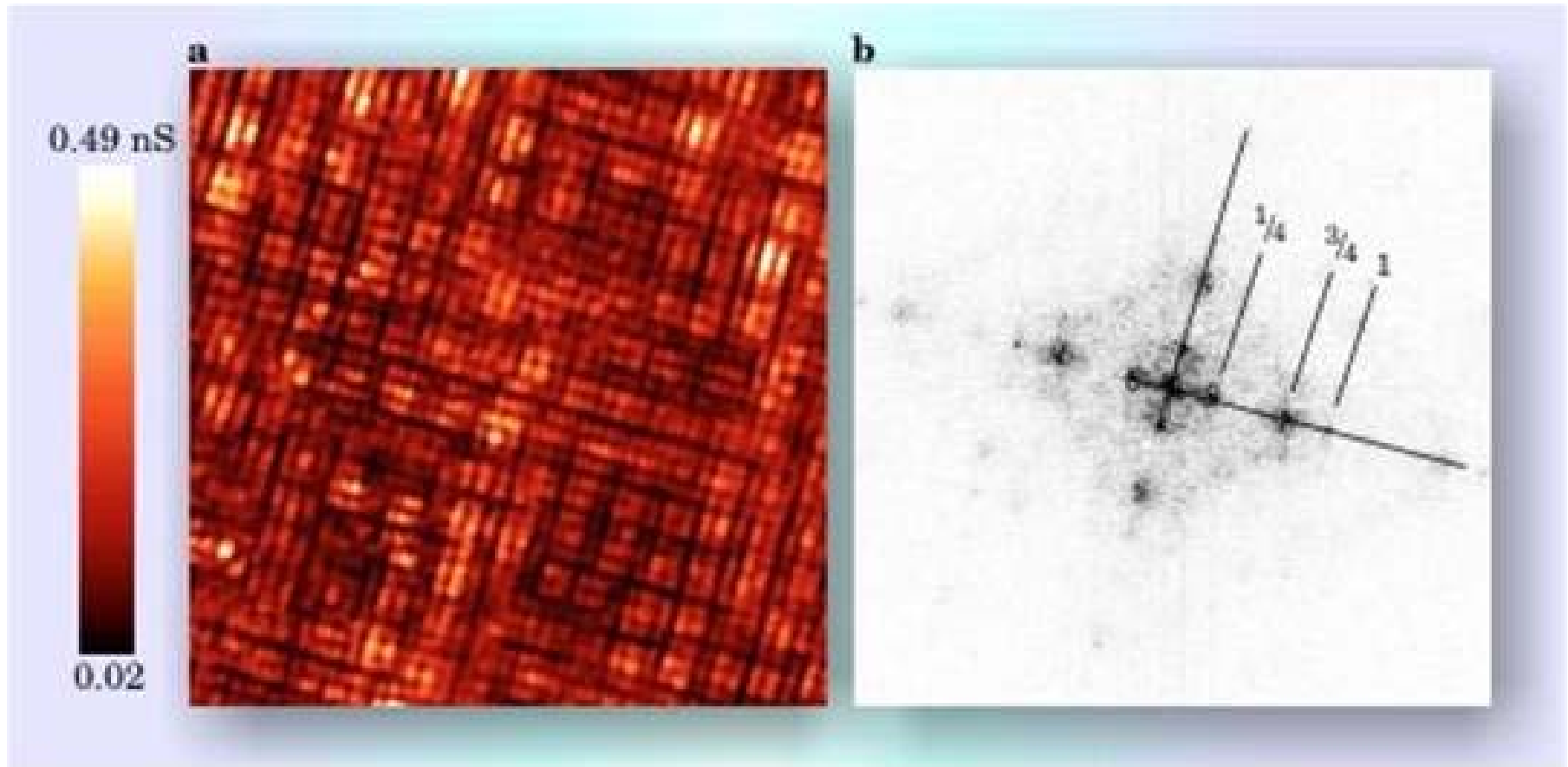


$$\langle \vec{S} \rangle = 0$$

Many experiments on the cuprate superconductors show:

- Tendency to produce modulations in spin singlet observables at wavevectors  $(2\pi/a)(1/4,0)$  and  $(2\pi/a)(0,1/4)$ .
- Proximity to a Mott insulator at hole density  $\delta=1/8$  with long-range charge modulations at wavevectors  $(2\pi/a)(1/4,0)$  and  $(2\pi/a)(0,1/4)$ .

# The cuprate superconductor $\text{Ca}_{2-x}\text{Na}_x\text{CuO}_2\text{Cl}_2$



T. Hanaguri, C. Lupien, Y. Kohsaka, D.-H. Lee, M. Azuma, M. Takano, H. Takagi, and J. C. Davis, *Nature* **430**, 1001 (2004).

Many experiments on the cuprate superconductors show:

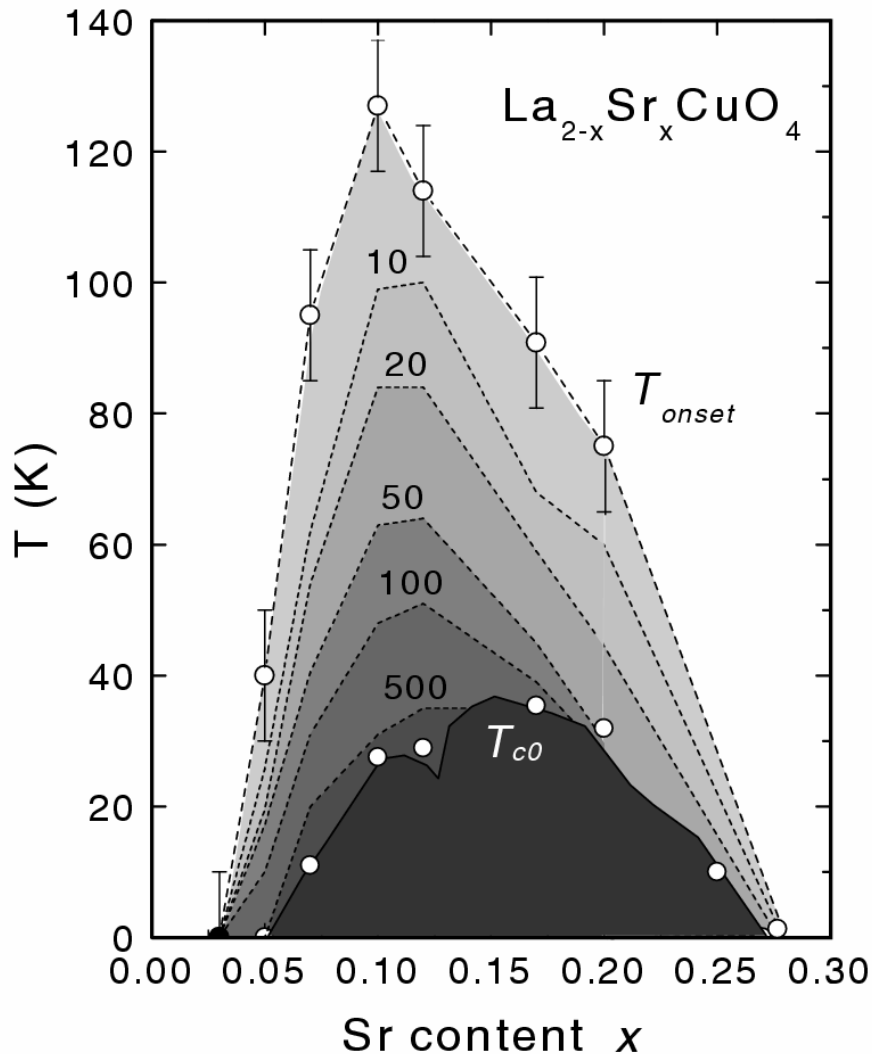
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*Superfluids proximate to finite doping  
Mott insulators with VBS order ?*

Experiments on the cuprate superconductors also show strong vortex fluctuations above  $T_c$



Measurements of Nernst effect are well explained by a model of a liquid of vortices and anti-vortices

N. P. Ong, Y. Wang, S. Ono, Y. Ando, and S. Uchida, *Annalen der Physik* **13**, 9 (2004).

Y. Wang, S. Ono, Y. Onose, G. Gu, Y. Ando, Y. Tokura, S. Uchida, and N. P. Ong, *Science* **299**, 86 (2003).

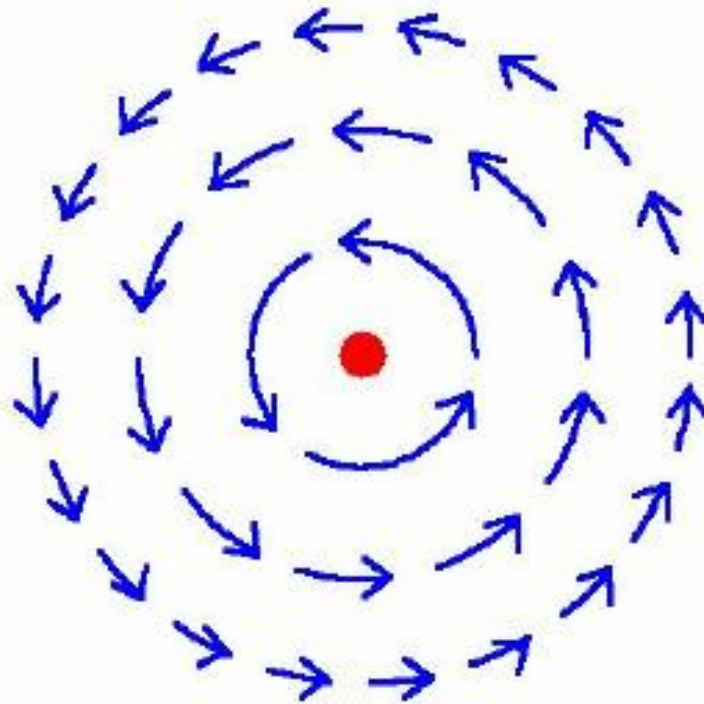
*Main claims:*

- *There are precursor fluctuations of VBS order in the superfluid.*
- *These fluctuations are intimately tied to the quantum theory of vortices in the superfluid*

### III. Vortices in the superfluid

*Magnus forces, duality, and point vortices as dual “electric” charges*

## Excitations of the superfluid: **Vortices**

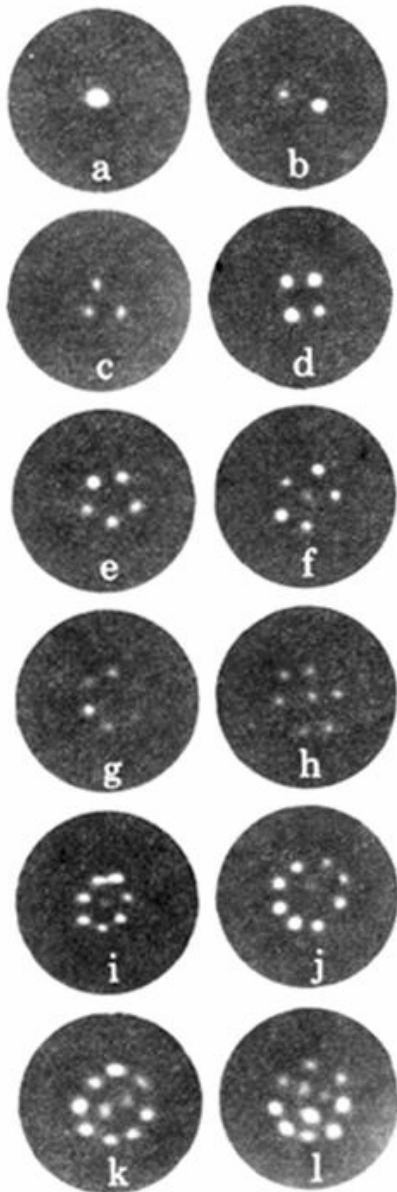


The circulation of a vortex is quantized:

$$\oint \mathbf{v}_s \cdot d\mathbf{r} = \frac{\hbar}{m} \oint \nabla \theta \cdot d\mathbf{r} = n \frac{h}{m}$$

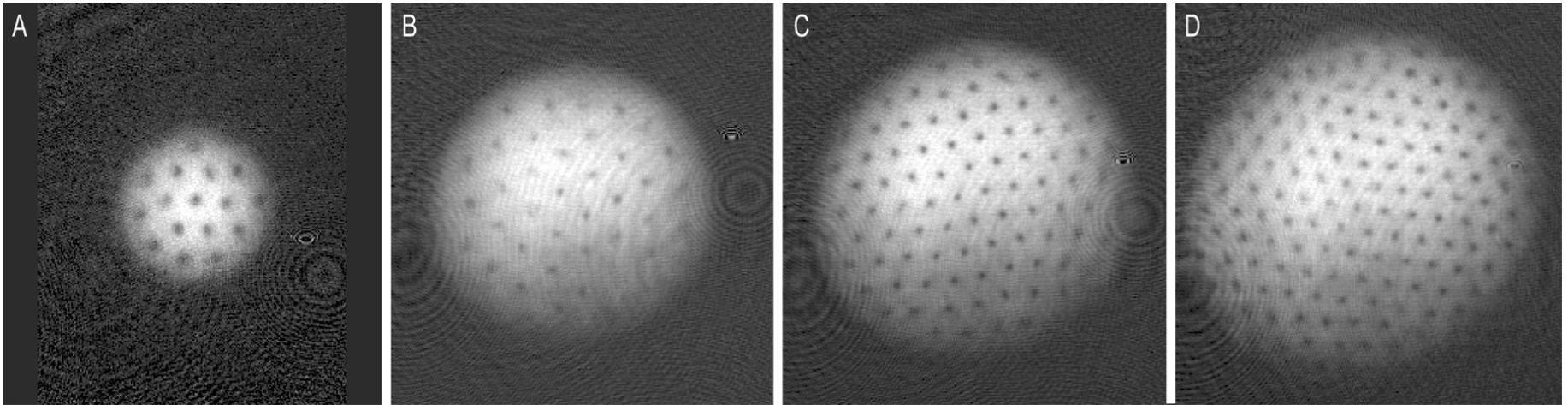
where  $n$  is an integer.

# Observation of quantized vortices in rotating $^4\text{He}$



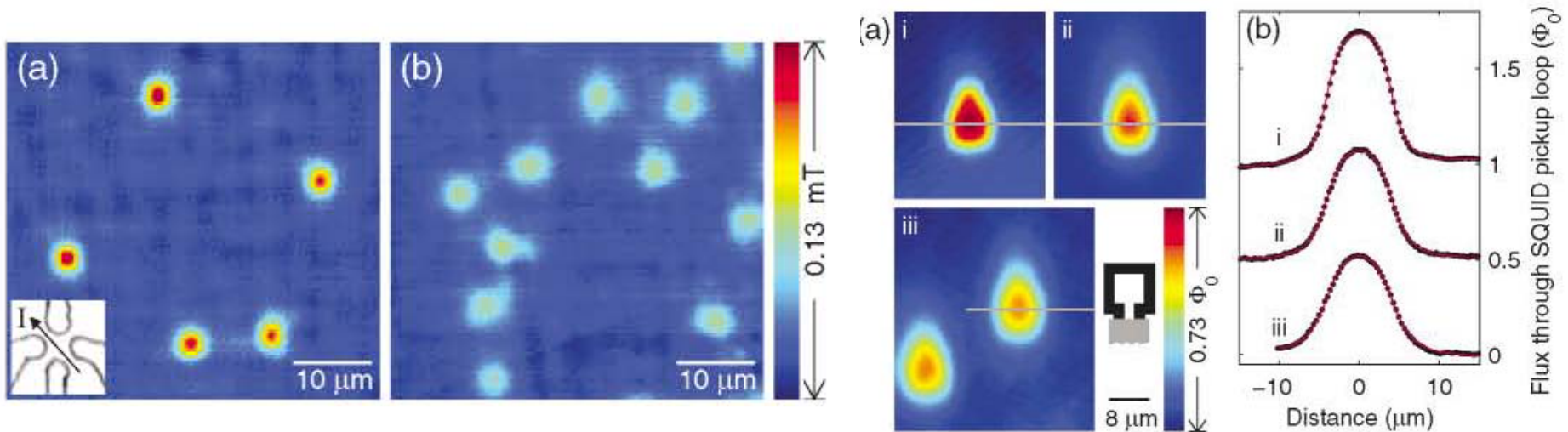
E.J. Yarmchuk, M.J.V. Gordon, and  
R.E. Packard,  
*Observation of Stationary Vortex  
Arrays in Rotating Superfluid Helium,*  
*Phys. Rev. Lett.* **43**, 214 (1979).

# Observation of quantized vortices in rotating ultracold Na



J. R. Abo-Shaeer, C. Raman, J. M. Vogels, and W. Ketterle,  
*Observation of Vortex Lattices in Bose-Einstein Condensates*,  
*Science* **292**, 476 (2001).

# Quantized fluxoids in $\text{YBa}_2\text{Cu}_3\text{O}_{6+y}$

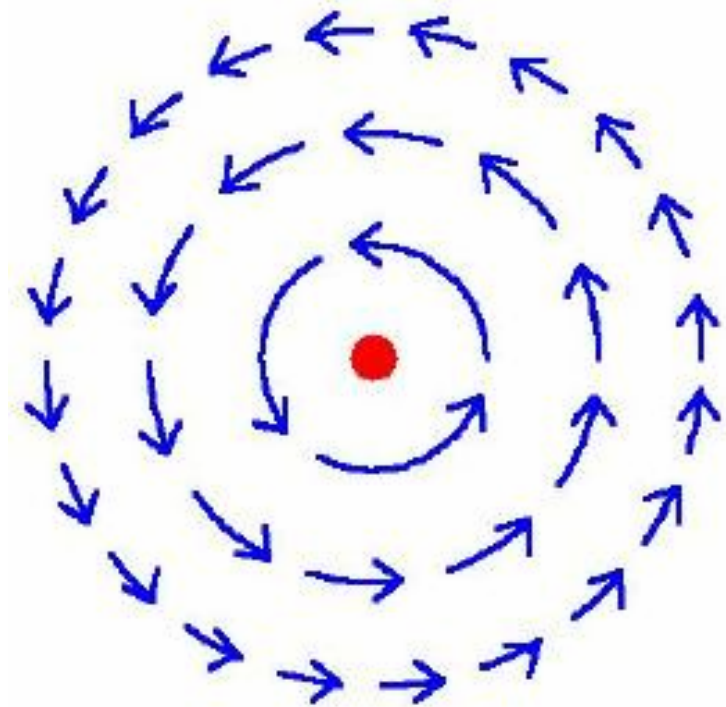


J. C. Wynn, D. A. Bonn, B.W. Gardner, Yu-Ju Lin, Ruixing Liang, W. N. Hardy, J. R. Kirtley, and K. A. Moler, *Phys. Rev. Lett.* **87**, 197002 (2001).

In superconductors, vortices carry quantized magnetic flux:

$$\int \mathbf{B} \cdot d\mathbf{S} = n \frac{hc}{2e}$$

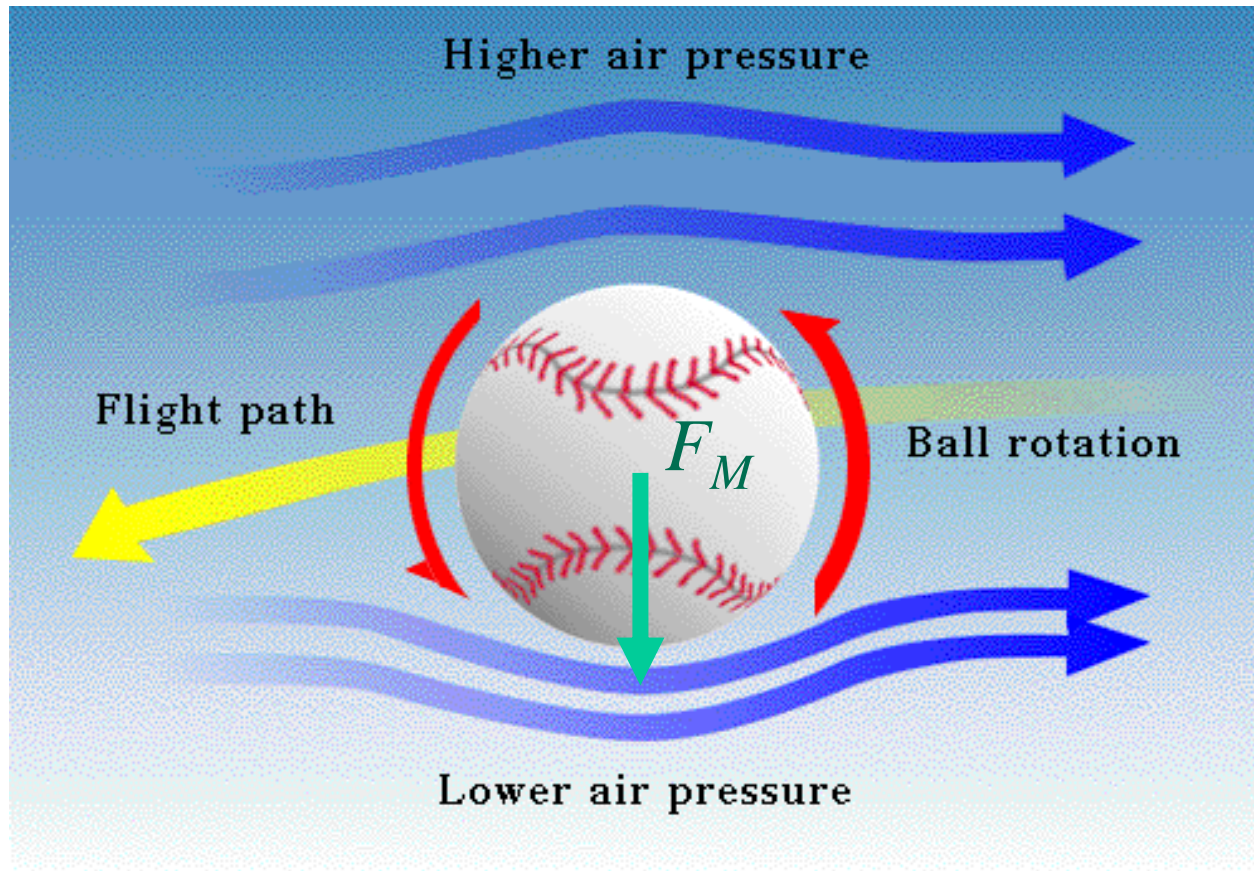
# Excitations of the superfluid: **Vortices**



## Central question:

In two dimensions, we can view the vortices as point particle excitations of the superfluid. What is the quantum mechanics of these “particles” ?

In ordinary fluids, vortices experience the Magnus Force



$$F_M = (\text{mass density of air}) \cdot (\text{velocity of ball}) \cdot (\text{circulation})$$

For a vortex in a superfluid, this is

$$\begin{aligned}\mathbf{F}_M &= (m\rho) \left( \left( \mathbf{v}_s - \frac{d\mathbf{r}_v}{dt} \right) \times \hat{\mathbf{z}} \right) \left( \oint \mathbf{v}_s \cdot d\mathbf{r} \right) \\ &= nh\rho \left( \mathbf{v}_s - \frac{d\mathbf{r}_v}{dt} \right) \times \hat{\mathbf{z}}\end{aligned}$$

where  $\rho$  = number density of bosons

$\mathbf{v}_s$  = local velocity of superfluid

$\mathbf{r}_v$  = position of vortex

For a vortex in a superfluid, this is

$$\begin{aligned}\mathbf{F}_M &= (m\rho) \left( \left( \mathbf{v}_s - \frac{d\mathbf{r}_v}{dt} \right) \times \hat{\mathbf{z}} \right) \left( \oint \mathbf{v}_s \cdot d\mathbf{r} \right) \\ &= nh\rho \left( \mathbf{v}_s - \frac{d\mathbf{r}_v}{dt} \right) \times \hat{\mathbf{z}} \\ &= n \left( \mathbf{E} + \frac{d\mathbf{r}_v}{dt} \times \mathbf{B} \right)\end{aligned}$$

where  $\mathbf{E} = \rho\mathbf{v}_s \times \hat{\mathbf{z}}$  and  $\mathbf{B} = -h\rho\hat{\mathbf{z}}$

### Dual picture:

The vortex is a quantum particle with dual “electric” charge  $n$ , moving in a dual “magnetic” field of strength =  $h \times$  (number density of Bose particles)

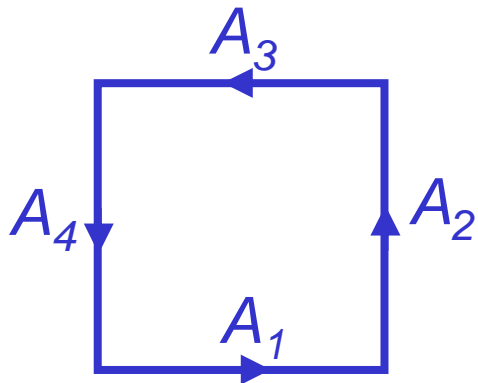
## IV. Vortices in superfluids near the superfluid-insulator quantum phase transition

*The “quantum order” of the superconducting state: evidence for vortex flavors*

- The vortices are quantum particles moving in a periodic potential with the symmetry of the square lattice, and in the presence of a dual “magnetic” field of strength  $= h\rho$ , where  $\rho$  is the number density of bosons per unit cell.
- The vortex motion can be described by the effective Hofstadter Hamiltonian:

$$\mathcal{H}_v = -t \sum_{\langle ij \rangle} (e^{iA_{ij}} \varphi_i^* \varphi_j + \text{c.c.})$$

where  $\varphi_i$  is an operator which annihilates a vortex particle at site  $i$  of a square lattice.



$$A_1 + A_2 + A_3 + A_4 = 2\pi f$$

where  $f$  is the boson filling fraction.

## Bosons at filling fraction $f = 1$

- At  $f=1$ , the “magnetic” flux per unit cell is  $2\pi$ , and the vortex does not pick up any phase from the boson density.
- The effective dual “magnetic” field acting on the vortex is zero, and the corresponding component of the Magnus force vanishes.

## Bosons at rational filling fraction $f=p/q$

Quantum mechanics of the vortex “particle” in a periodic potential with  $f$  flux quanta per unit cell

Space group symmetries of Hofstadter Hamiltonian:

$T_x, T_y$  : Translations by a lattice spacing in the  $x, y$  directions

$R$  : Rotation by 90 degrees.

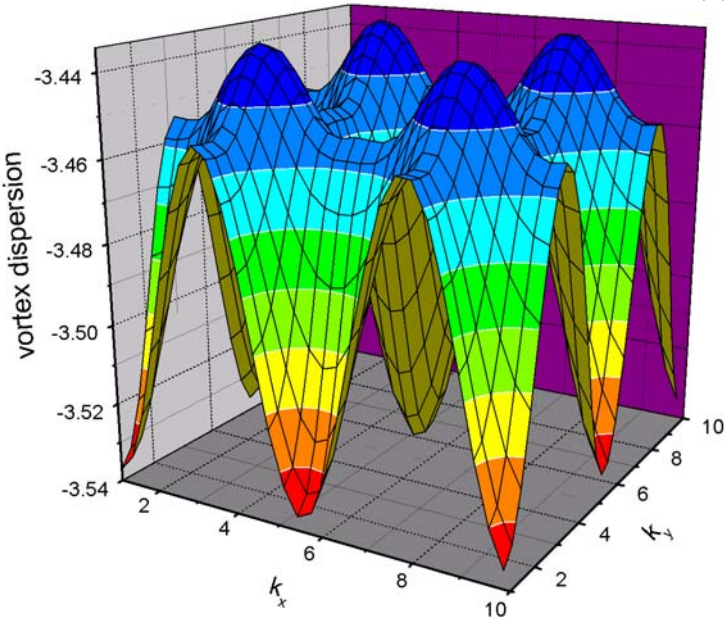
Magnetic space group:

$$T_x T_y = e^{2\pi i f} T_y T_x \ ;$$

$$R^{-1} T_y R = T_x \ ; \ R^{-1} T_x R = T_y^{-1} \ ; \ R^4 = 1$$

The low energy vortex states must form a representation of this algebra

# Vortices in a superfluid near a Mott insulator at filling $f=p/q$ Hofstadter spectrum of the quantum vortex “particle” with field operator $\varphi$



At filling  $f=p/q$ , there are  $q$  species of vortices,  $\varphi_\ell$  (with  $\ell=1\dots q$ ), associated with  $q$  degenerate minima in the vortex spectrum. These vortices realize the smallest,  $q$ -dimensional, representation of the magnetic algebra.

$$T_x : \varphi_\ell \rightarrow \varphi_{\ell+1} \quad ; \quad T_y : \varphi_\ell \rightarrow e^{2\pi i \ell f} \varphi_\ell$$

$$R : \varphi_\ell \rightarrow \frac{1}{\sqrt{q}} \sum_{m=1}^q \varphi_m e^{2\pi i \ell m f}$$

# Vortices in a superfluid near a Mott insulator at filling $f=p/q$

The  $q$   $\varphi_\ell$  vortices characterize *both* superconducting and VBS/CDW orders

VBS order:

Status of space group symmetry determined by

density operators  $\rho_{\mathbf{Q}}$  at wavevectors  $\mathbf{Q}_{mn} = \frac{2\pi p}{q}(m, n)$

$$\rho_{mn} = e^{i\pi mnf} \sum_{\ell=1}^q \varphi_\ell^* \varphi_{\ell+n} e^{2\pi i \ell m f}$$

$$T_x : \rho_{\mathbf{Q}} \rightarrow \rho_{\mathbf{Q}} e^{i\mathbf{Q} \cdot \hat{x}} \quad ; \quad T_y : \rho_{\mathbf{Q}} \rightarrow \rho_{\mathbf{Q}} e^{i\mathbf{Q} \cdot \hat{y}}$$

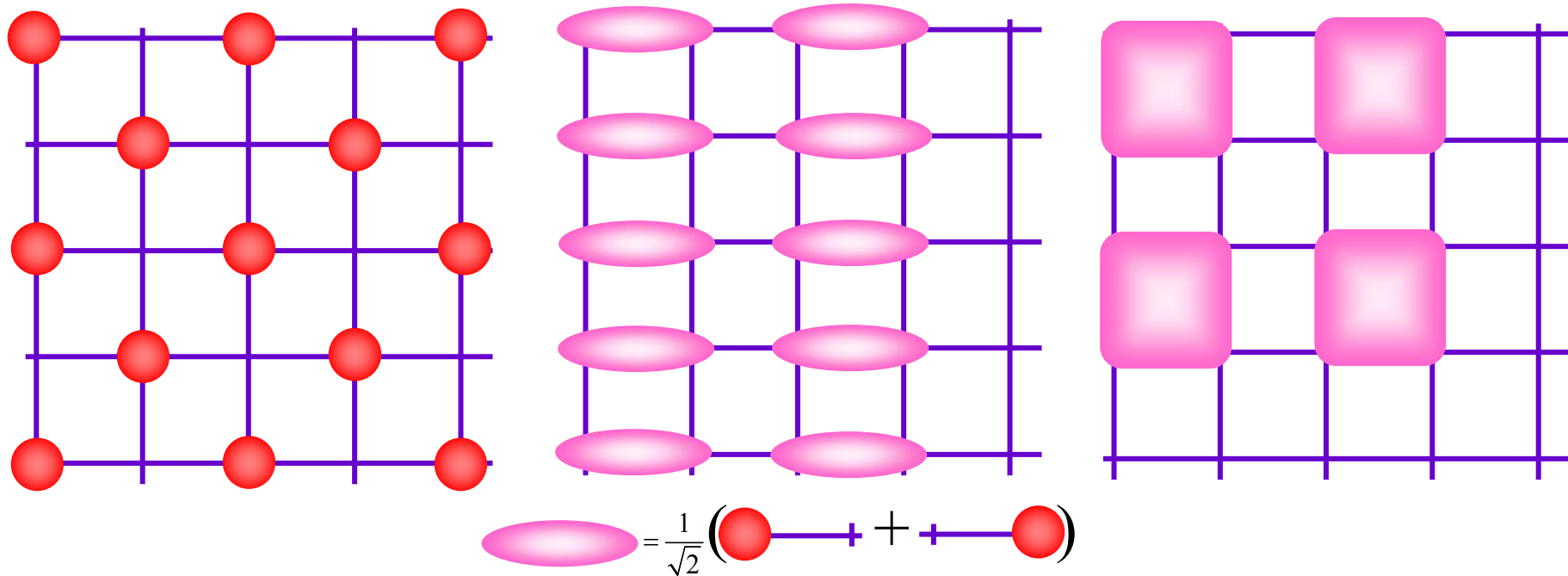
$$R : \rho(\mathbf{Q}) \rightarrow \rho(R\mathbf{Q})$$

## Vortices in a superfluid near a Mott insulator at filling $f=p/q$

- The excitations of the superfluid are described by the quantum mechanics of  $q$  flavors of low energy vortices moving in zero dual "magnetic" field.
- The orientation of the vortex in flavor space implies a particular configuration of VBS order in its vicinity.

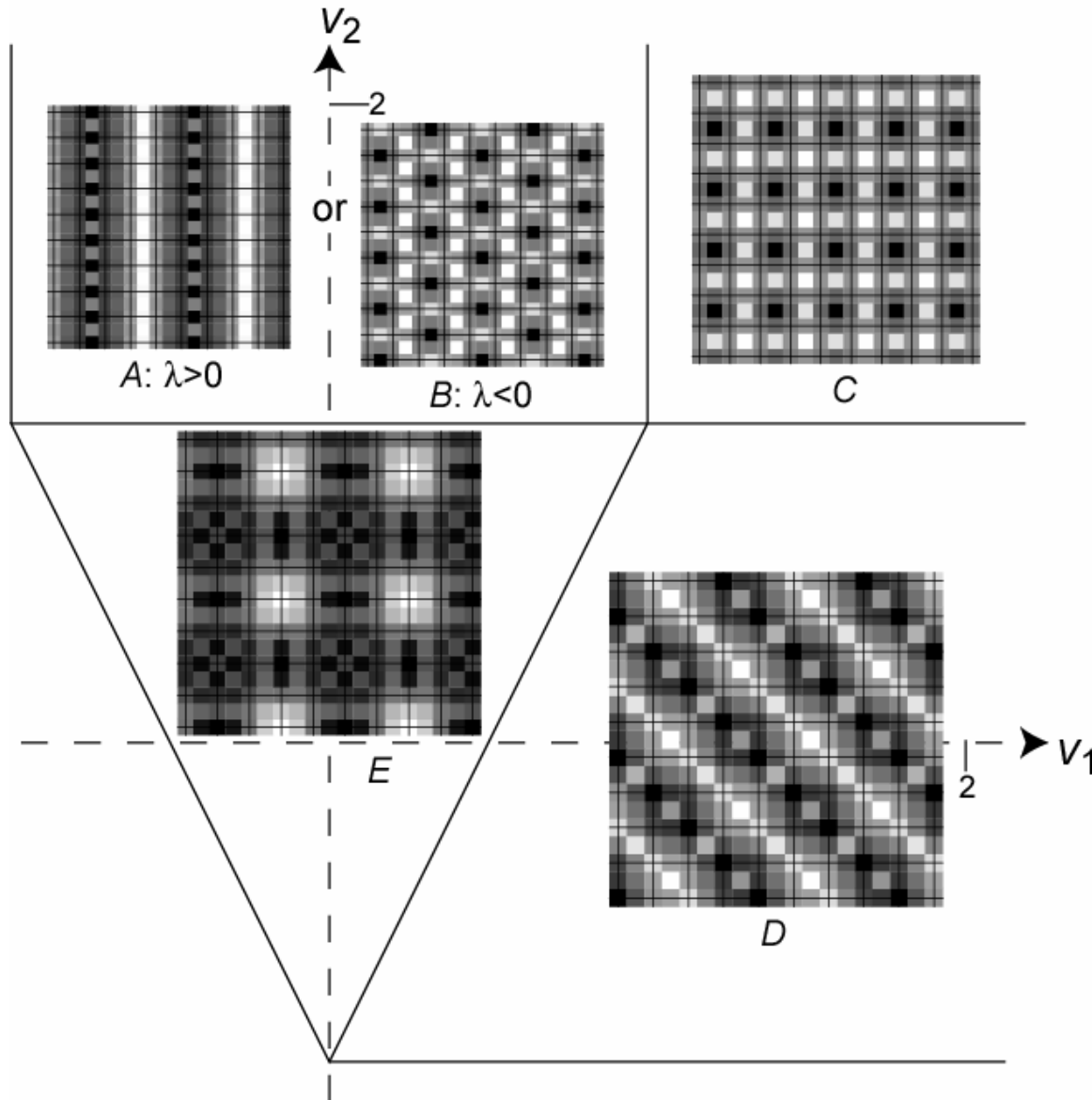
# Mott insulators obtained by “condensing” vortices

Spatial structure of insulators for  $q=2$  ( $f=1/2$ )



# Field theory with projective symmetry

Spatial structure of insulators for  $q=4$  ( $f=1/4$  or  $3/4$ )



$a \times b$  unit cells;  
 $q/a$ ,  $q/b$ ,  $ab/q$ ,  
all integers

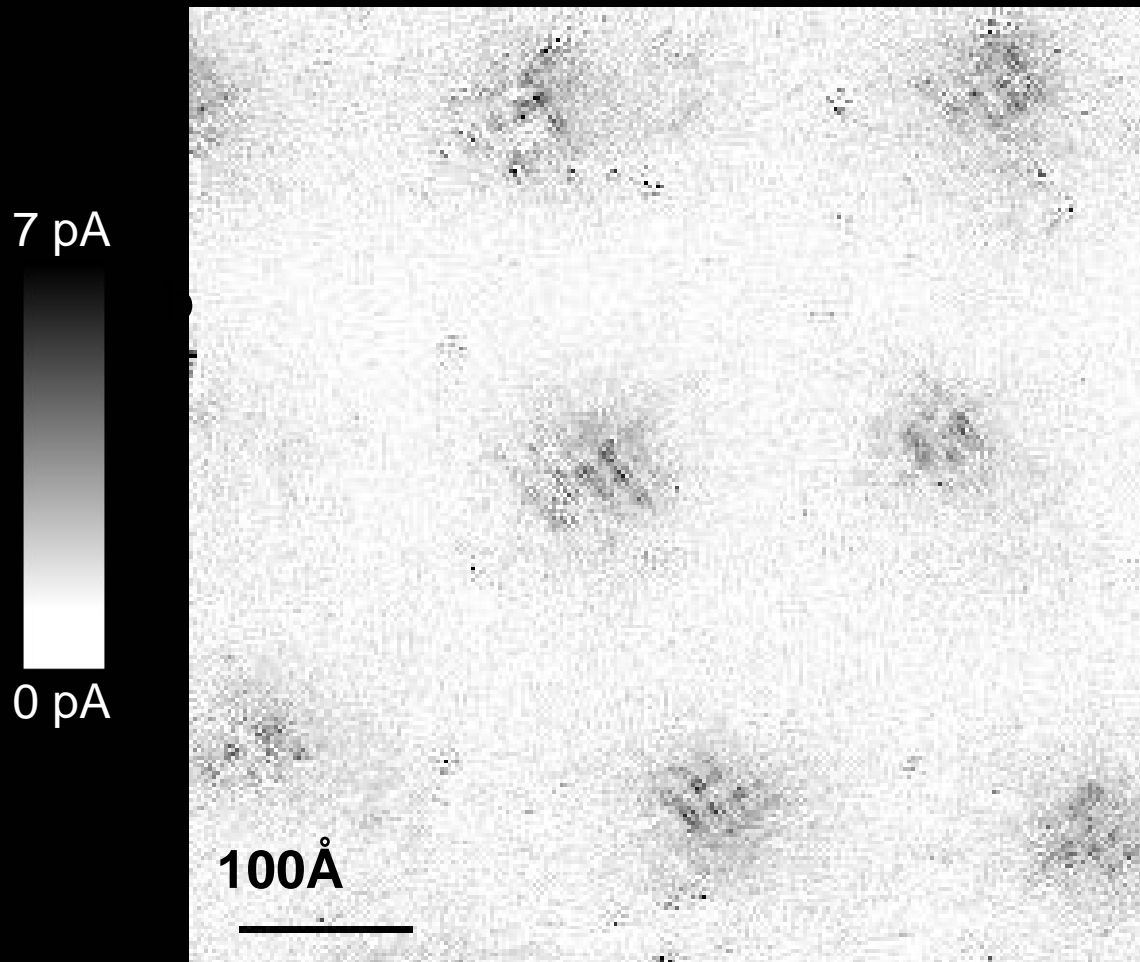
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- The orientation of the vortex in flavor space implies a particular configuration of VBS order in its vicinity.
- Any pinned vortex must pick an orientation in flavor space: this induces a halo of VBS order in its vicinity

# Vortex-induced LDOS of $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}$ integrated from 1meV to 12meV at 4K



Vortices have halos with LDOS modulations at a period  $\approx 4$  lattice spacings

Prediction of VBS order near vortices: K. Park and S. Sachdev, *Phys. Rev. B* **64**, 184510 (2001).

J. Hoffman, E. W. Hudson, K. M. Lang, V. Madhavan, S. H. Pan, H. Eisaki, S. Uchida, and J. C. Davis, *Science* **295**, 466 (2002).

## Measuring the inertial mass of a vortex

The spatial extent of the LDOS modulations measures the region over which the vortex executes its zero-point motion. The size of this region can be determined by solving the equations of motion

$$m_v \frac{d^2 \mathbf{r}}{dt^2} = F_M$$

and so is determined by the inertial vortex mass  $m_v$ .

# Measuring the inertial mass of a vortex

Preliminary estimates for the BSCCO experiment:

Inertial vortex mass  $m_v \approx 10m_e$

Vortex magnetoplasmon frequency  $\nu_p \approx 1 \text{ THz} = 4 \text{ meV}$

Future experiments can directly detect vortex zero point motion by looking for resonant absorption at this frequency.

Vortex oscillations can also modify the electronic density of states.

## Superfluids near Mott insulators

*The Mott insulator has average Cooper pair density,  $f = p/q$  per site, while the density of the superfluid is close (but need not be identical) to this value*

- Vortices with flux  $h/(2e)$  come in multiple (usually  $q$ ) “flavors”
- The lattice space group acts in a projective representation on the vortex flavor space.
- These flavor quantum numbers provide a distinction between superfluids: they constitute a “quantum order”
- Any pinned vortex must choose an orientation in flavor space. This necessarily leads to modulations in the local density of states over the spatial region where the vortex executes its quantum zero point motion.