

Challenges and Recent Advances in Mathematical Physics 2021

Wednesday 20 January 2021, 14:00 – 14:50

**A simple model of entangled qubits: how it describes
superconductors and black holes**

Subir Sachdev, *Harvard University*

Chaired by Mike Blake, *University of Bristol*

A simple model of entangled qubits, how it describes black holes and superconductors

Challenges and Recent Advances in Mathematical Physics
Heilbronn Institute for Mathematical Research

University of Bristol
January 20, 2021

Subir Sachdev

Talk online: sachdev.physics.harvard.edu



PHYSICS



HARVARD

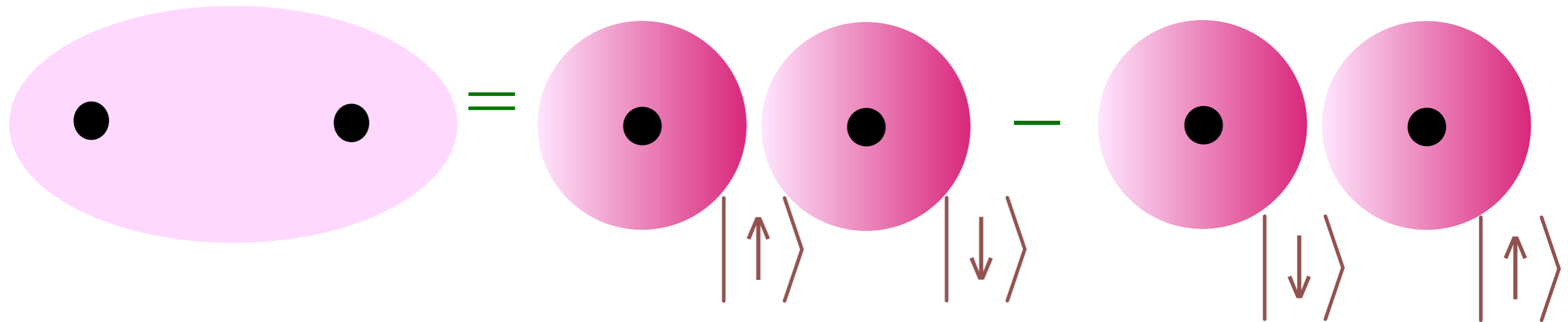
Quantum entanglement

Principles of Quantum Mechanics: II. Quantum Entanglement

Quantum Entanglement: quantum superposition with more than one particle



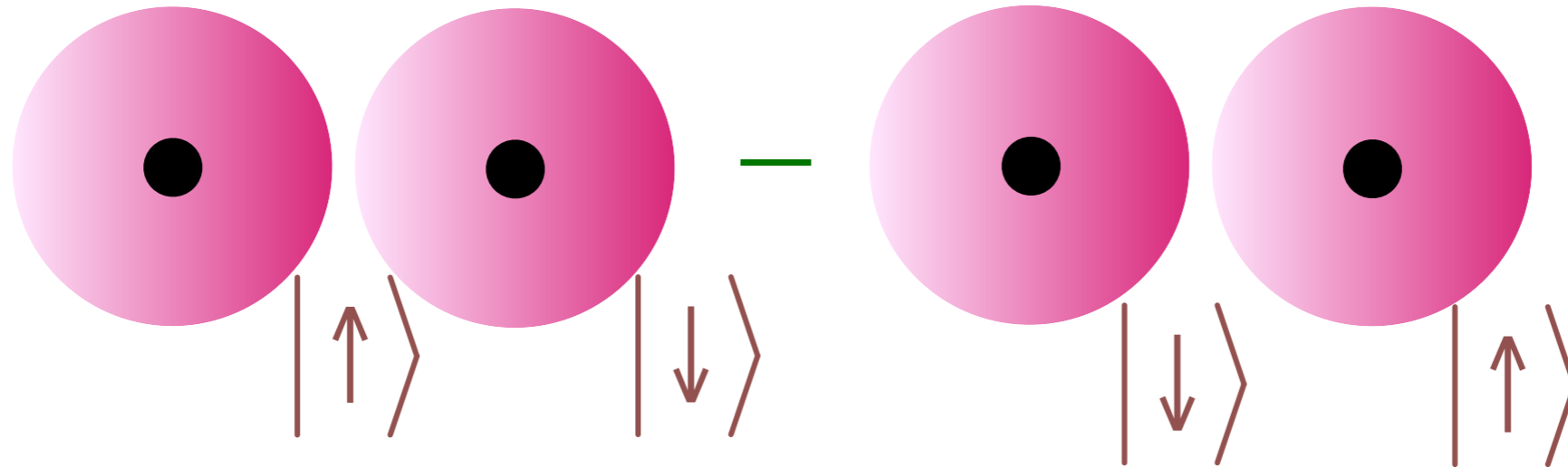
Hydrogen molecule:



$$= \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

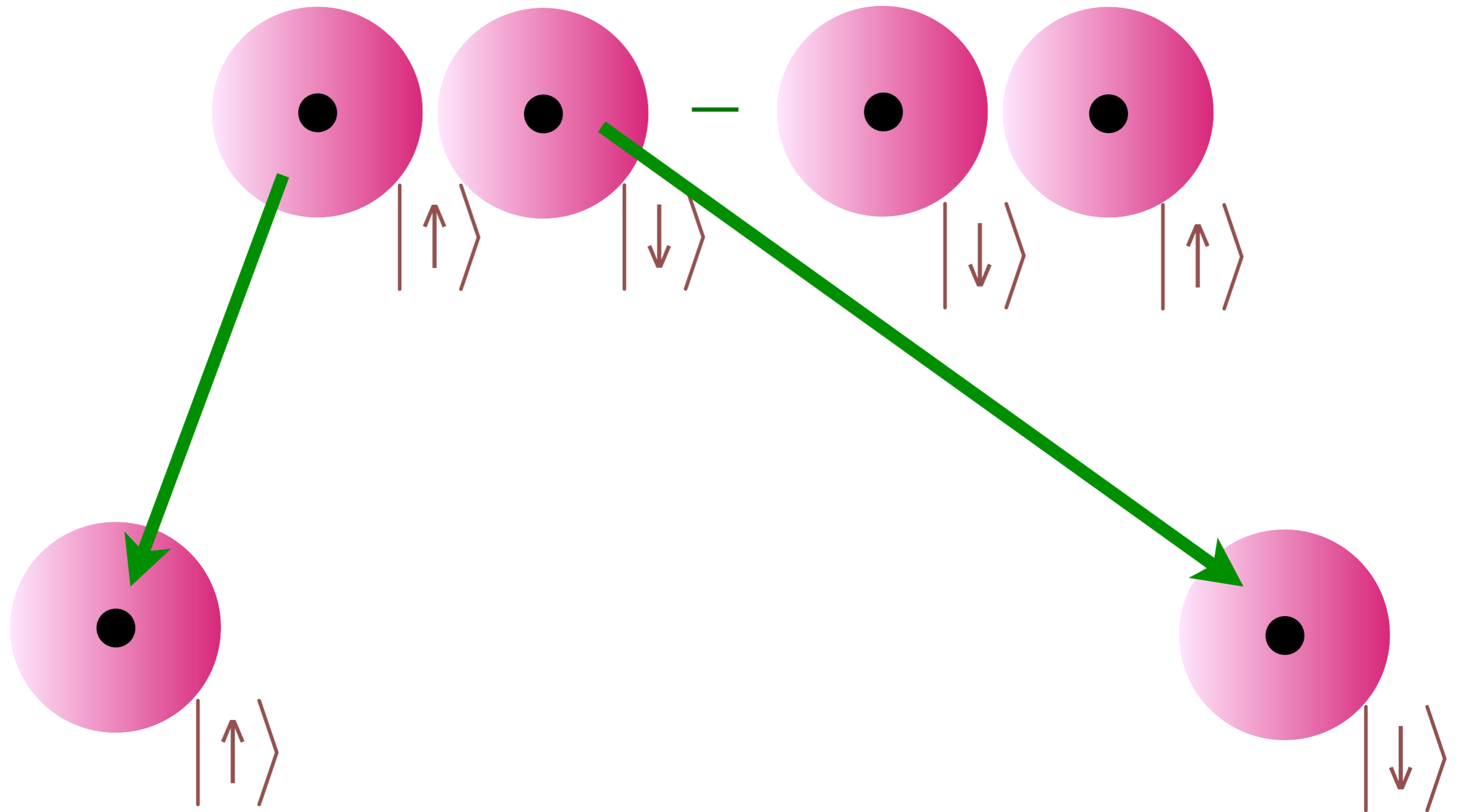
Principles of Quantum Mechanics: II. Quantum Entanglement

Quantum Entanglement: quantum superposition with more than one particle



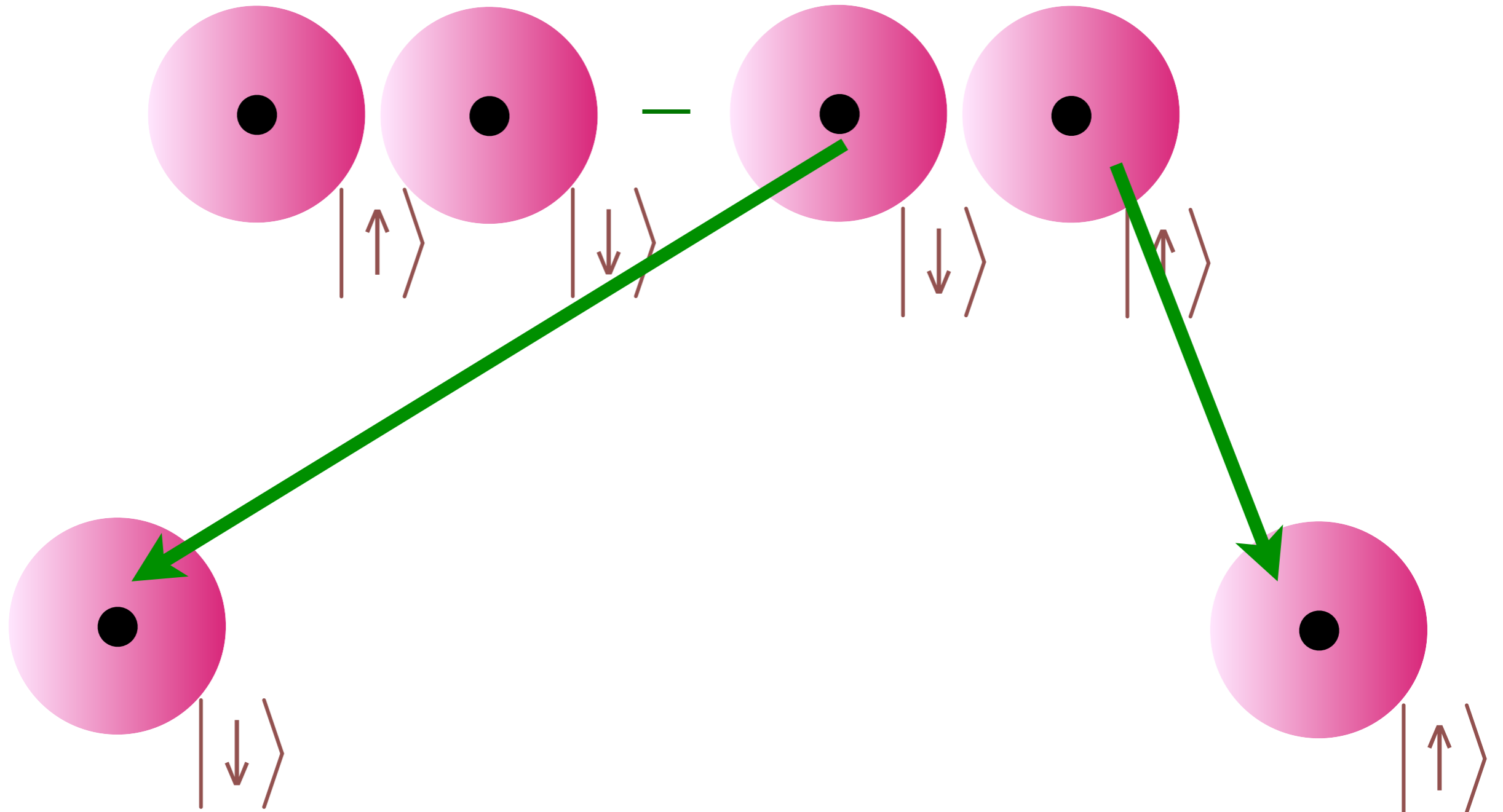
Principles of Quantum Mechanics: II. Quantum Entanglement

Quantum Entanglement: quantum superposition with more than one particle



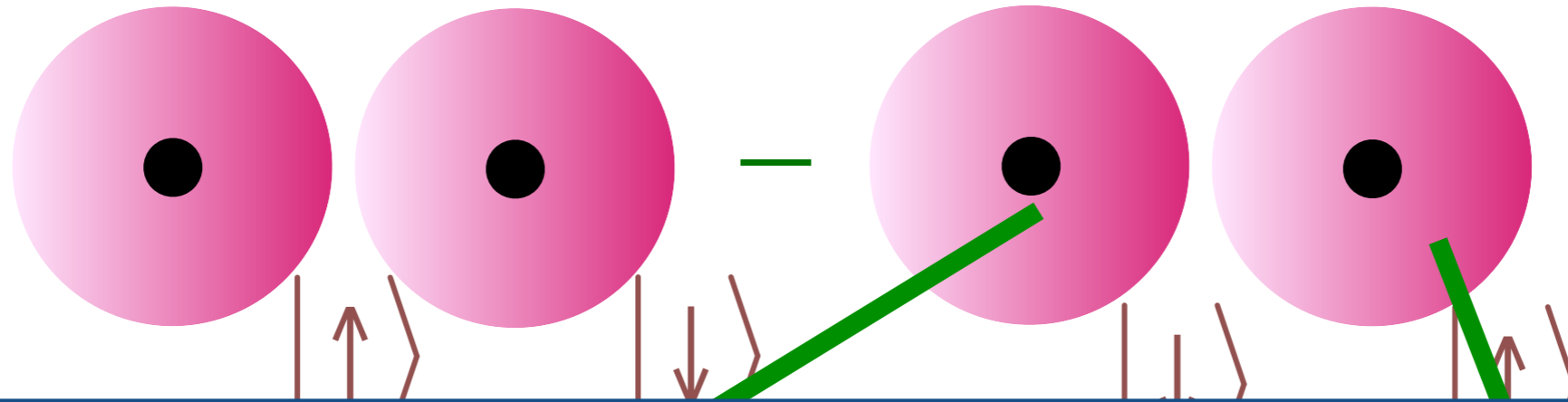
Principles of Quantum Mechanics: II. Quantum Entanglement

Quantum Entanglement: quantum superposition with more than one particle

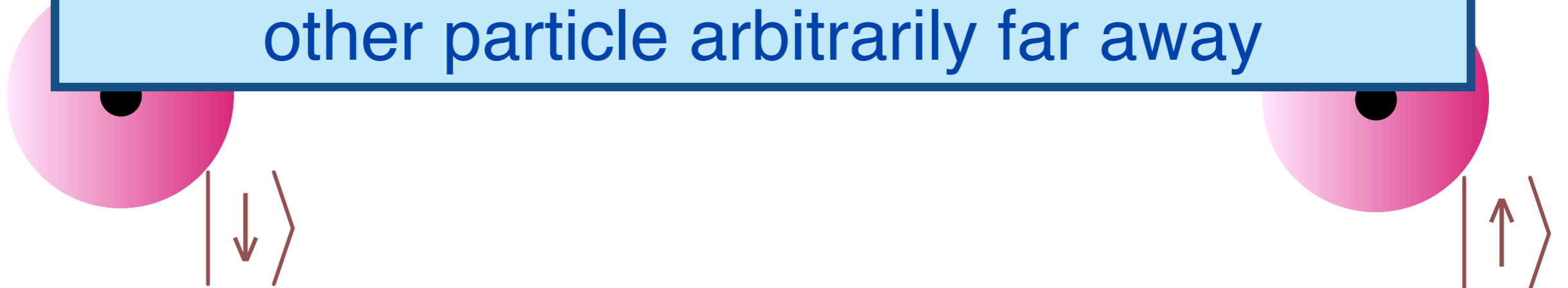


Principles of Quantum Mechanics: II. Quantum Entanglement

Quantum Entanglement: quantum superposition with more than one particle



Einstein-Podolsky-Rosen “paradox” (1935):
Measurement of one particle
instantaneously determines the state of the
other particle arbitrarily far away



Quantum entanglement

**Quantum
entanglement**

**A simple
qubit
model**

A qubit: 2 states $|\uparrow\rangle, |\downarrow\rangle$.

Pauli gates:

$$\begin{aligned} X|\uparrow\rangle &= |\downarrow\rangle & , & & X|\downarrow\rangle &= |\uparrow\rangle \\ Y|\uparrow\rangle &= i|\downarrow\rangle & , & & Y|\downarrow\rangle &= -i|\uparrow\rangle \\ Z|\uparrow\rangle &= |\uparrow\rangle & , & & Z|\downarrow\rangle &= -|\downarrow\rangle \end{aligned}$$

A qubit: 2 states $|\uparrow\rangle, |\downarrow\rangle$.

Pauli gates:

$$\begin{aligned} X|\uparrow\rangle &= |\downarrow\rangle & , & & X|\downarrow\rangle &= |\uparrow\rangle \\ Y|\uparrow\rangle &= i|\downarrow\rangle & , & & Y|\downarrow\rangle &= -i|\uparrow\rangle \\ Z|\uparrow\rangle &= |\uparrow\rangle & , & & Z|\downarrow\rangle &= -|\downarrow\rangle \end{aligned}$$

A 2-qubit Hamiltonian: $\mathcal{H} = J(X_1X_2 + Y_1Y_2 + Z_1Z_2)$

$$\begin{aligned} \text{Ground state: } & \frac{1}{\sqrt{2}} (|\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2) \\ \text{Energy} &= -3J \end{aligned}$$

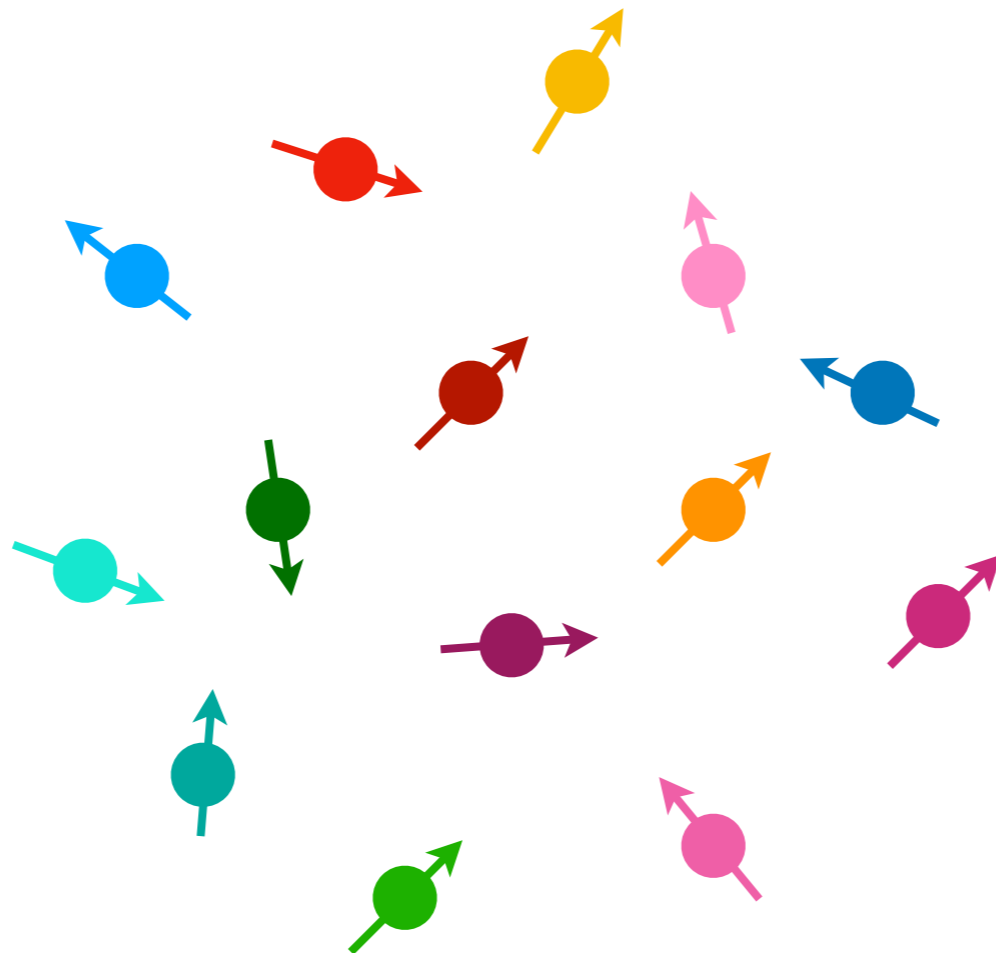
$$\begin{aligned} \text{Excited states: } & |\uparrow\rangle_1 |\uparrow\rangle_2, \quad |\downarrow\rangle_1 |\downarrow\rangle_2, \quad \frac{1}{\sqrt{2}} (|\uparrow\rangle_1 |\downarrow\rangle_2 + |\downarrow\rangle_1 |\uparrow\rangle_2) \\ \text{Energy} &= J \end{aligned}$$

The simple model

$$\mathcal{H} = \sum_{i < j=1}^N J_{ij} (X_i X_j + Y_i Y_j + Z_i Z_j)$$

J_{ij} are random numbers with zero mean, and variance $\sim J^2/N$

(Technical comment: the solvable model has states in the self-conjugate representation of $SU(M)$, and we take the limit $N \rightarrow \infty$ followed by the limit $M \rightarrow \infty$)

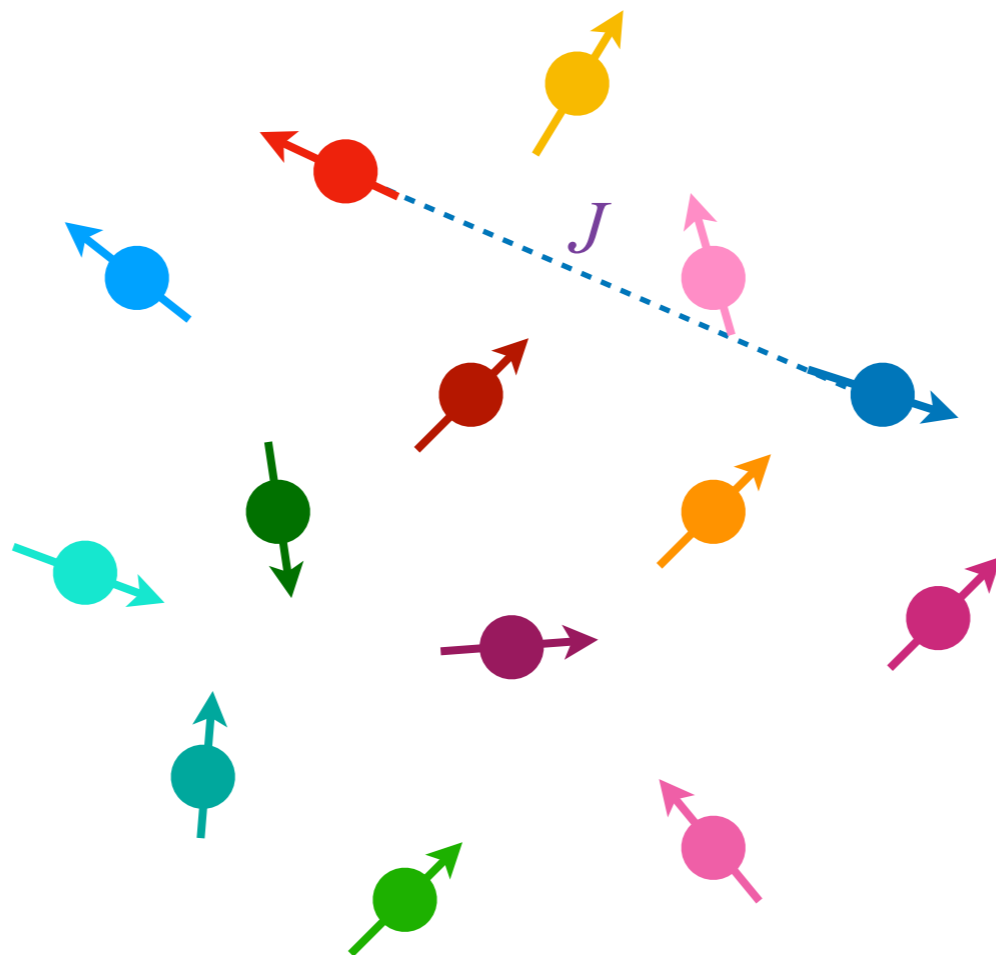


The simple model

$$\mathcal{H} = \sum_{i < j=1}^N J_{ij} (X_i X_j + Y_i Y_j + Z_i Z_j)$$

J_{ij} are random numbers with zero mean, and variance $\sim J^2/N$

(Technical comment: the solvable model has states in the self-conjugate representation of $SU(M)$, and we take the limit $N \rightarrow \infty$ followed by the limit $M \rightarrow \infty$)

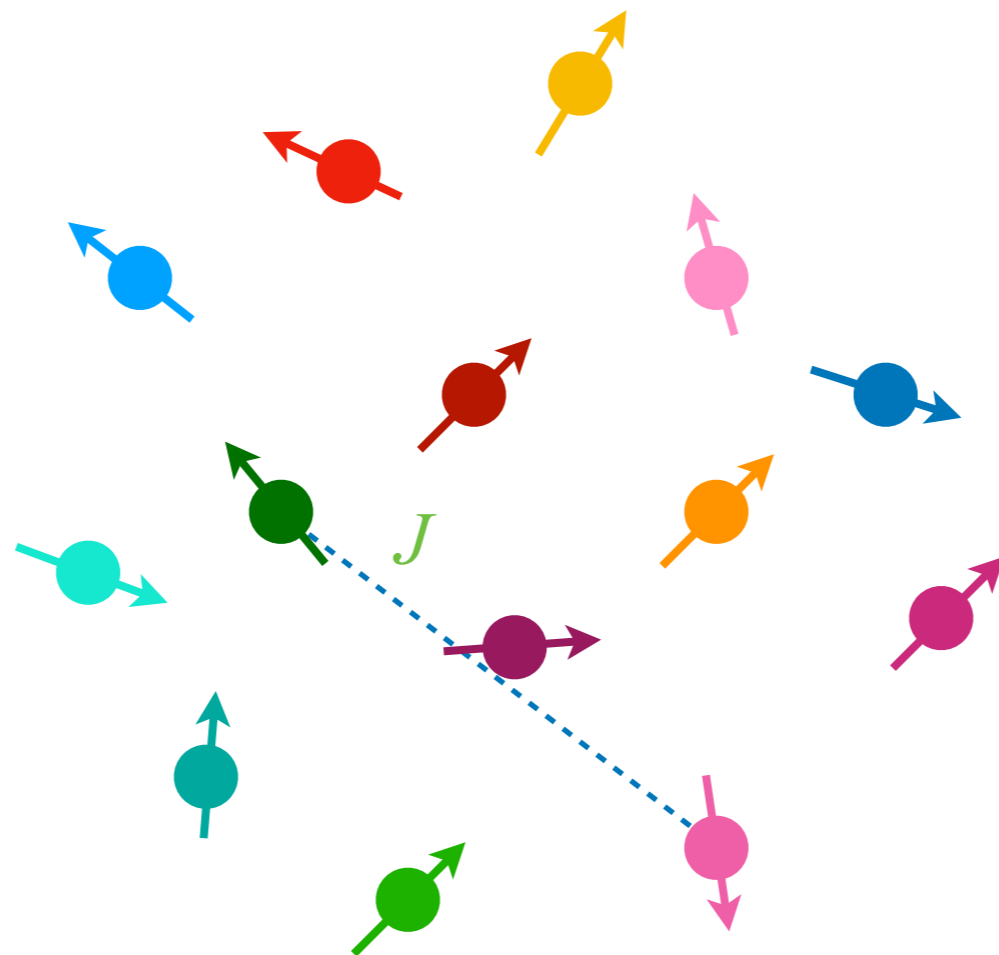


The simple model

$$\mathcal{H} = \sum_{i < j=1}^N J_{ij} (X_i X_j + Y_i Y_j + Z_i Z_j)$$

J_{ij} are random numbers with zero mean, and variance $\sim J^2/N$

(Technical comment: the solvable model has states in the self-conjugate representation of $SU(M)$, and we take the limit $N \rightarrow \infty$ followed by the limit $M \rightarrow \infty$)

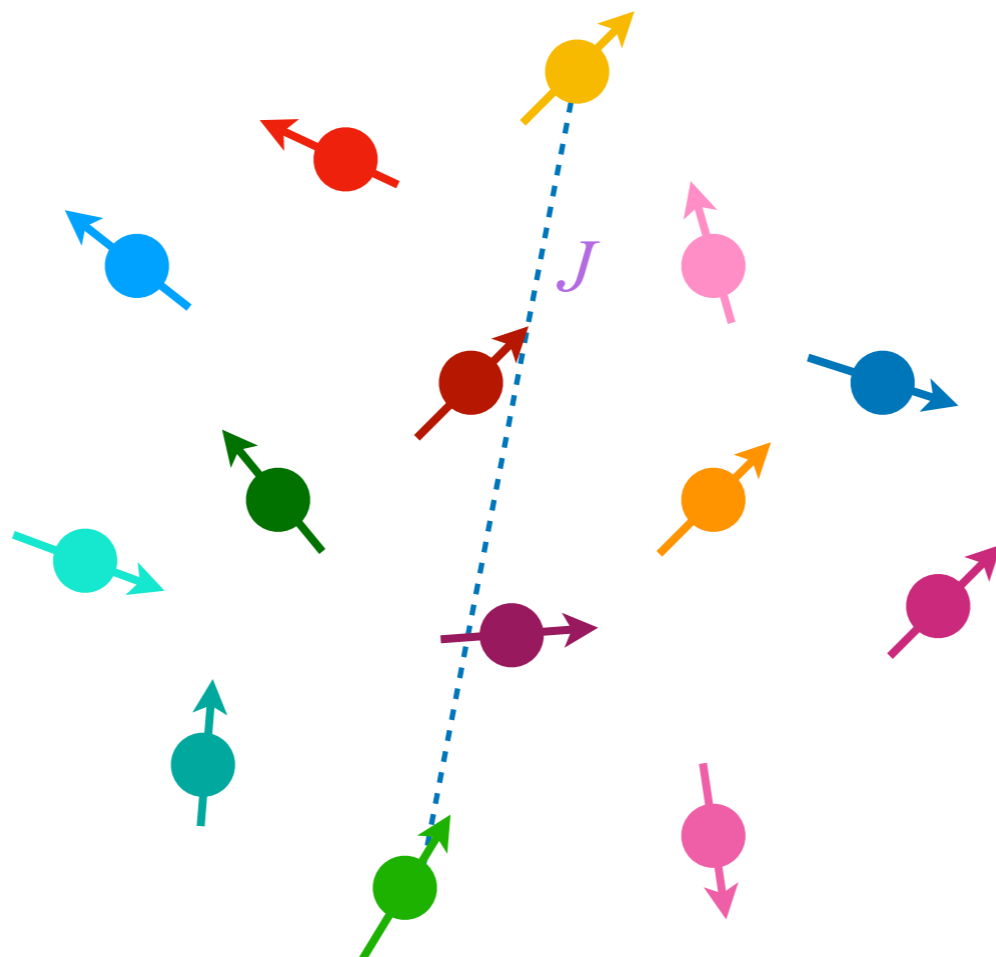


The simple model

$$\mathcal{H} = \sum_{i < j=1}^N J_{ij} (X_i X_j + Y_i Y_j + Z_i Z_j)$$

J_{ij} are random numbers with zero mean, and variance $\sim J^2/N$

(Technical comment: the solvable model has states in the self-conjugate representation of $SU(M)$, and we take the limit $N \rightarrow \infty$ followed by the limit $M \rightarrow \infty$)

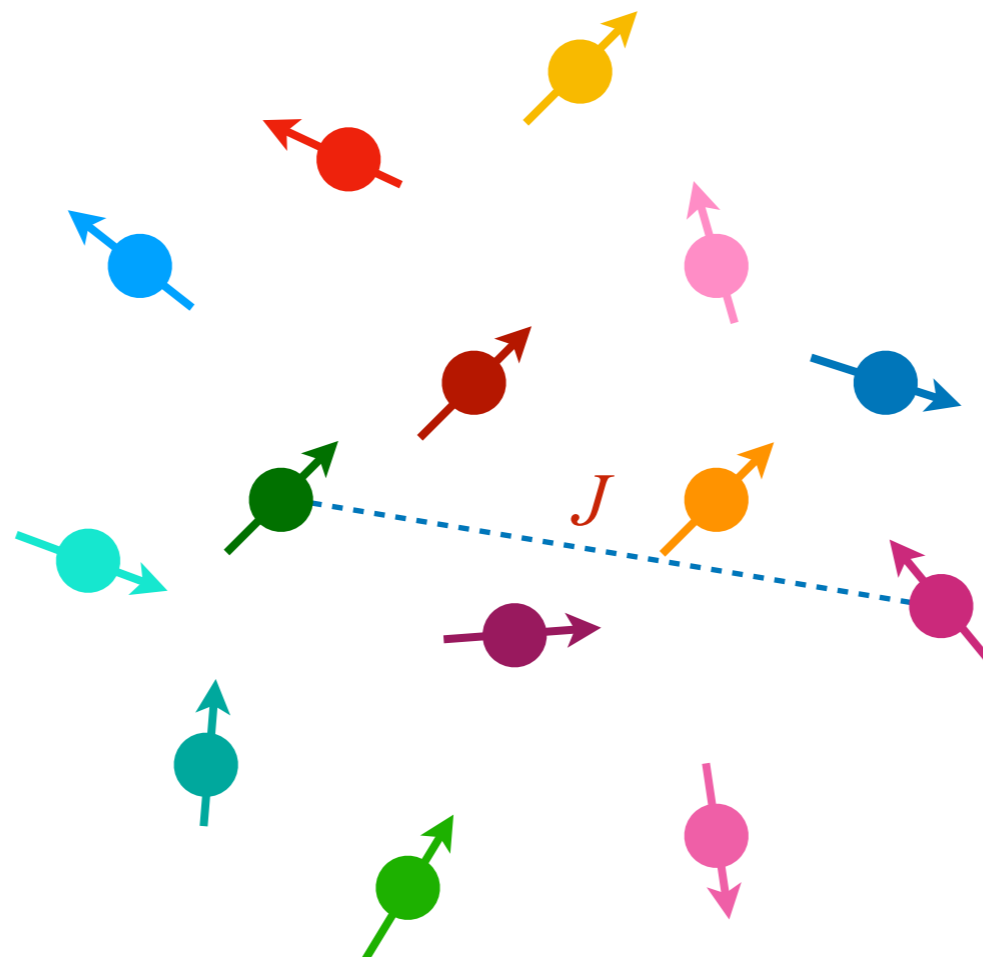


The simple model

$$\mathcal{H} = \sum_{i < j=1}^N J_{ij} (X_i X_j + Y_i Y_j + Z_i Z_j)$$

J_{ij} are random numbers with zero mean, and variance $\sim J^2/N$

(Technical comment: the solvable model has states in the self-conjugate representation of $SU(M)$, and we take the limit $N \rightarrow \infty$ followed by the limit $M \rightarrow \infty$)



Main result I

For $k_B T \ll J$

$$\begin{aligned} \mathcal{Z} &= \text{Tr} \exp \left(-\frac{\mathcal{H}}{k_B T} \right) \\ &= \exp \left(N \frac{S_0}{k_B} \right) \int \mathcal{D}f(\tau) \exp \left(-\frac{1}{\hbar} \mathcal{S}_{2\text{D-gravity}} [f(\tau)] \right) \end{aligned}$$

S. Sachdev, Phys. Rev. Lett. **105**, 151602 (2010)

A. Kitaev (2015)

J. Maldacena and D. Stanford, Phys. Rev. D **94**, 106002 (2016)

Main result I

For $k_B T \ll J$

$$\begin{aligned} \mathcal{Z} &= \text{Tr} \exp \left(-\frac{\mathcal{H}}{k_B T} \right) \\ &= \exp \left(N \frac{S_0}{k_B} \right) \int \mathcal{D}f(\tau) \exp \left(-\frac{1}{\hbar} \mathcal{S}_{2\text{D-gravity}} [f(\tau)] \right) \end{aligned}$$

S_0 is the $T \rightarrow 0$ entropy of the qubit model.

A. Georges, O. Parcollet, and S. Sachdev, Phys. Rev. B **63**, 134406 (2001)

It maps on to the Bekenstein-Hawking
entropy of charged black holes

S. Sachdev, Phys. Rev. Lett. **105**, 151602 (2010)

A. Kitaev (2015)

J. Maldacena and D. Stanford, Phys. Rev. D **94**, 106002 (2016)

Main result I

For $k_B T \ll J$

$$\begin{aligned} \mathcal{Z} &= \text{Tr} \exp \left(-\frac{\mathcal{H}}{k_B T} \right) \\ &= \exp \left(N \frac{S_0}{k_B} \right) \int \mathcal{D}f(\tau) \exp \left(-\frac{1}{\hbar} \mathcal{S}_{2\text{D-gravity}} [f(\tau)] \right) \end{aligned}$$

- $f(\tau)$ is the reparameterization of the imaginary time of the qubit model: τ on a circle of circumference $\hbar/(k_B T)$.

S. Sachdev, Phys. Rev. Lett. **105**, 151602 (2010)

A. Kitaev (2015)

J. Maldacena and D. Stanford, Phys. Rev. D **94**, 106002 (2016)

Main result I

For $k_B T \ll J$

$$\begin{aligned} \mathcal{Z} &= \text{Tr} \exp \left(-\frac{\mathcal{H}}{k_B T} \right) \\ &= \exp \left(N \frac{S_0}{k_B} \right) \int \mathcal{D}f(\tau) \exp \left(-\frac{1}{\hbar} \mathcal{S}_{\text{2D-gravity}} [f(\tau)] \right) \end{aligned}$$

- $f(\tau)$ is the reparameterization of the imaginary time of the qubit model: τ on a circle of circumference $\hbar/(k_B T)$.
- $f(\tau)$ is also the fluctuation of the boundary of a theory of 2D-gravity in 1+1 spacetime dimensions: a ‘boundary graviton’.

S. Sachdev, Phys. Rev. Lett. **105**, 151602 (2010)

A. Kitaev (2015)

J. Maldacena and D. Stanford, Phys. Rev. D **94**, 106002 (2016)

Quantum
entanglement

A simple
qubit
model

Low temperatures

Quantum gravity in
1+1 dimensions

**Quantum
entanglement**

**Black
holes**

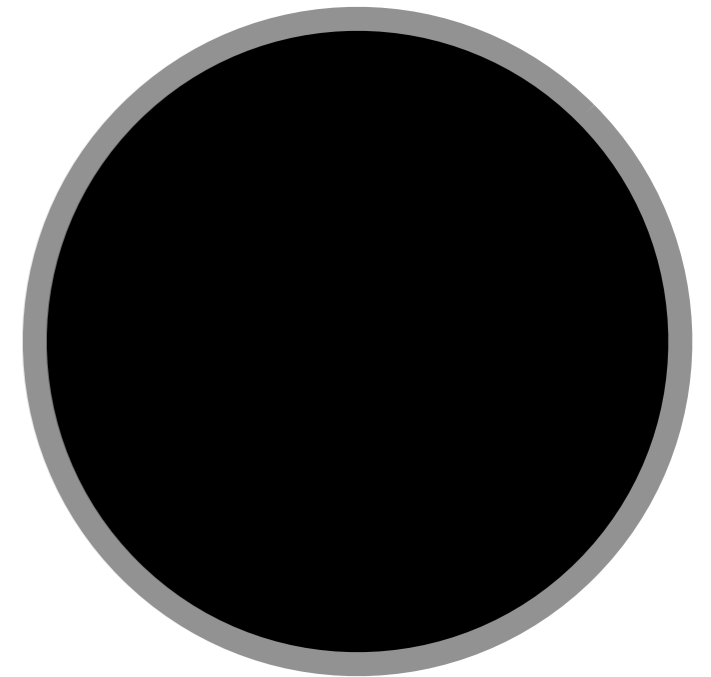
**A simple
qubit
model**

Black Holes

Objects so dense that light is gravitationally bound to them.

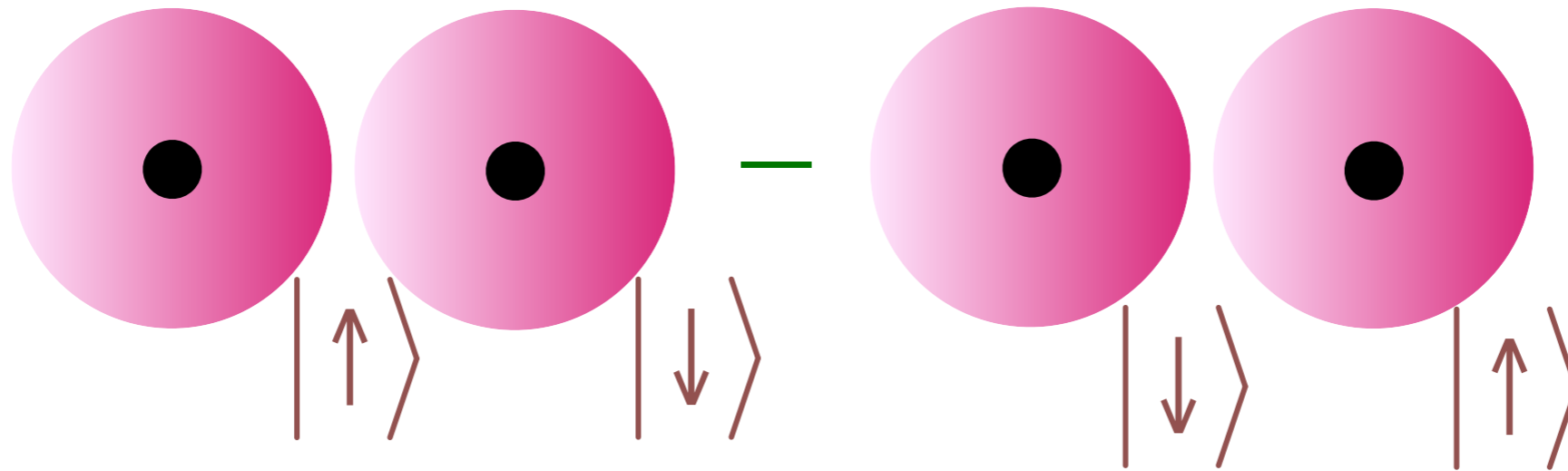
In Einstein's theory, the region inside the black hole **horizon** is disconnected from the rest of the universe.

$$\text{Horizon radius } R = \frac{2GM}{c^2}$$

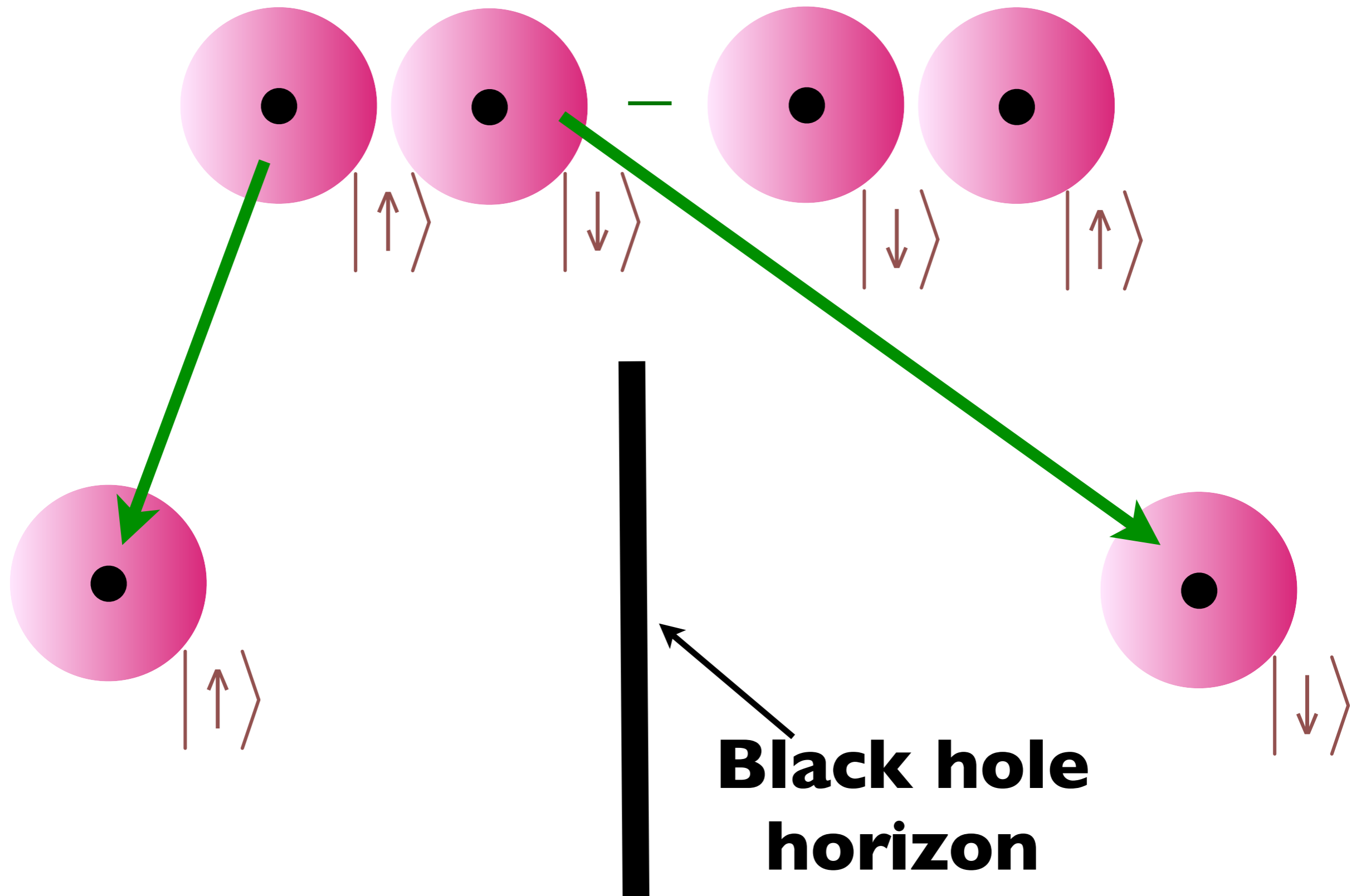


G Newton's constant, c velocity of light, M mass of black hole

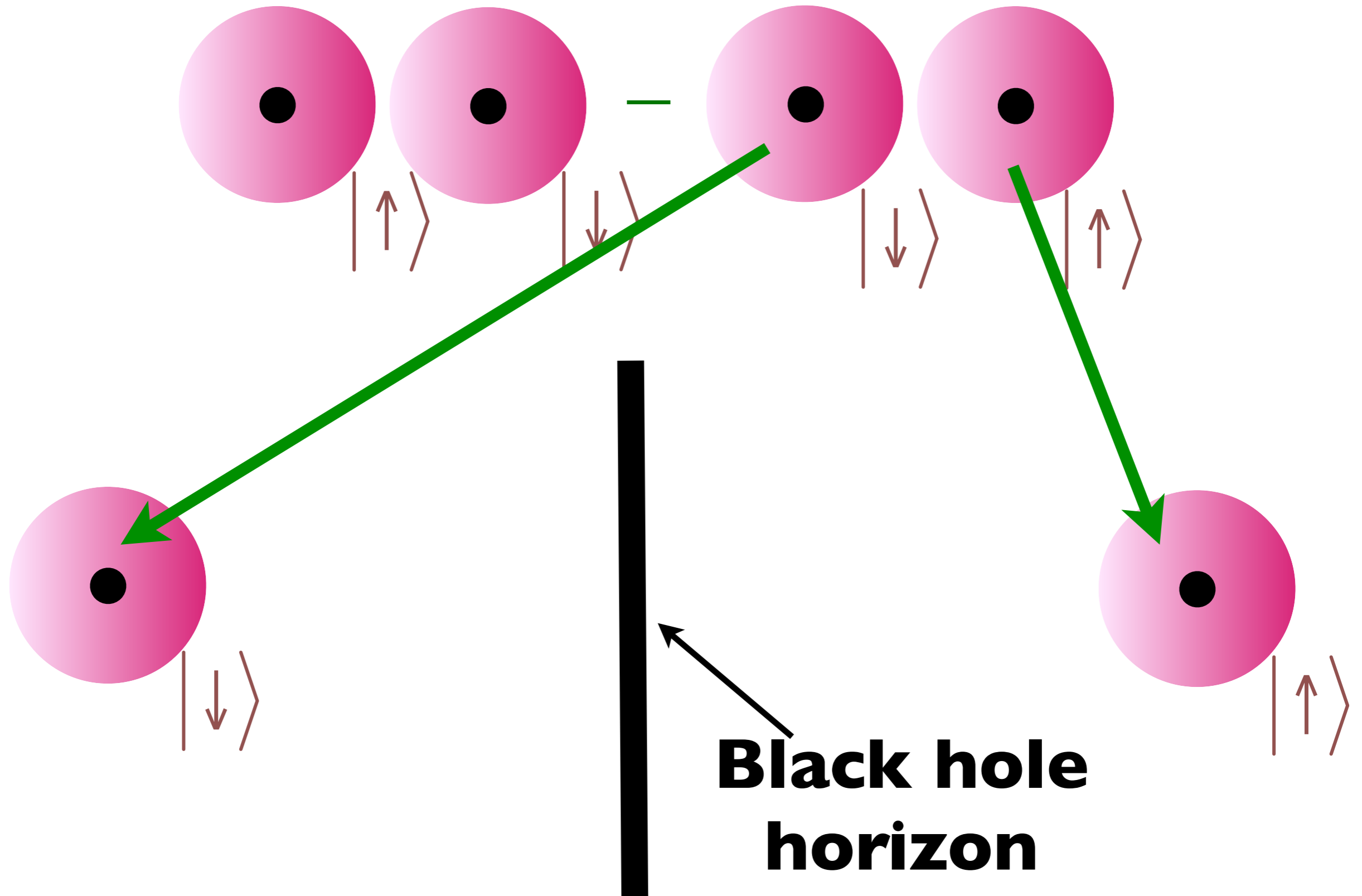
Quantum Entanglement across a black hole horizon



Quantum Entanglement across a black hole horizon

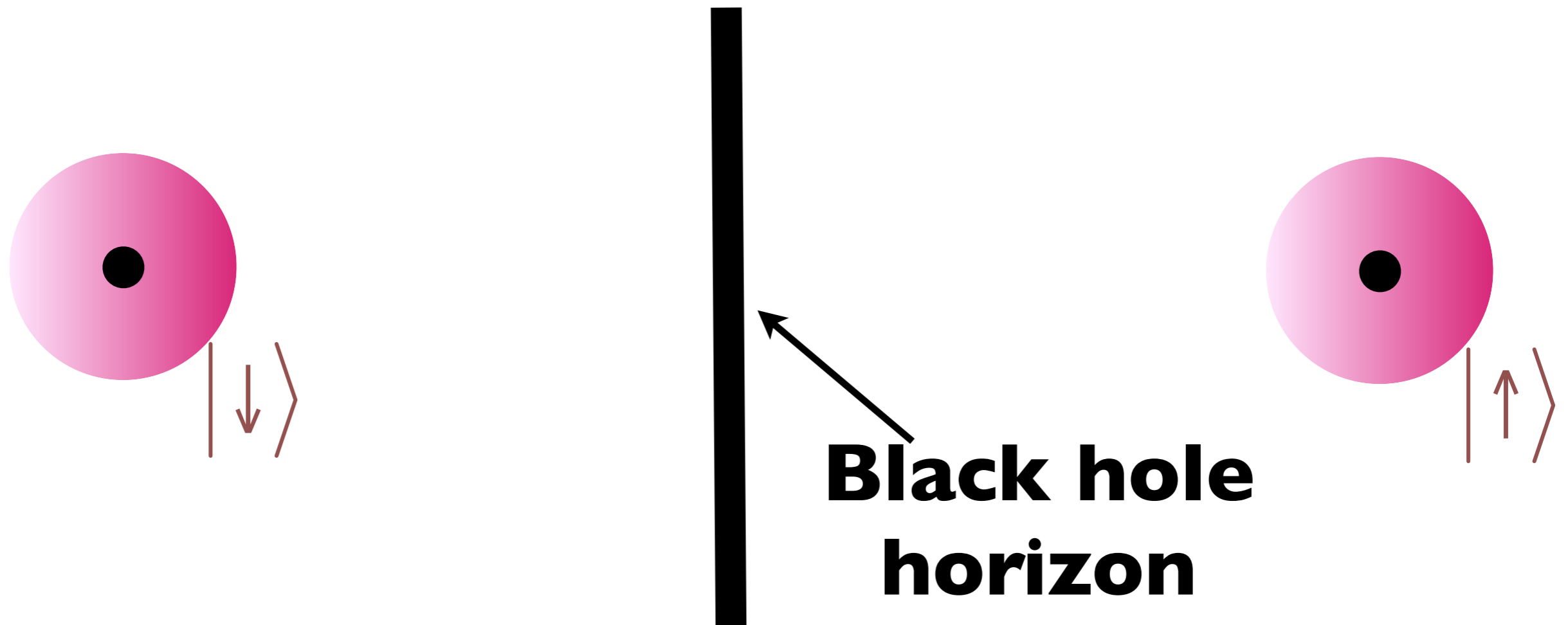


Quantum Entanglement across a black hole horizon



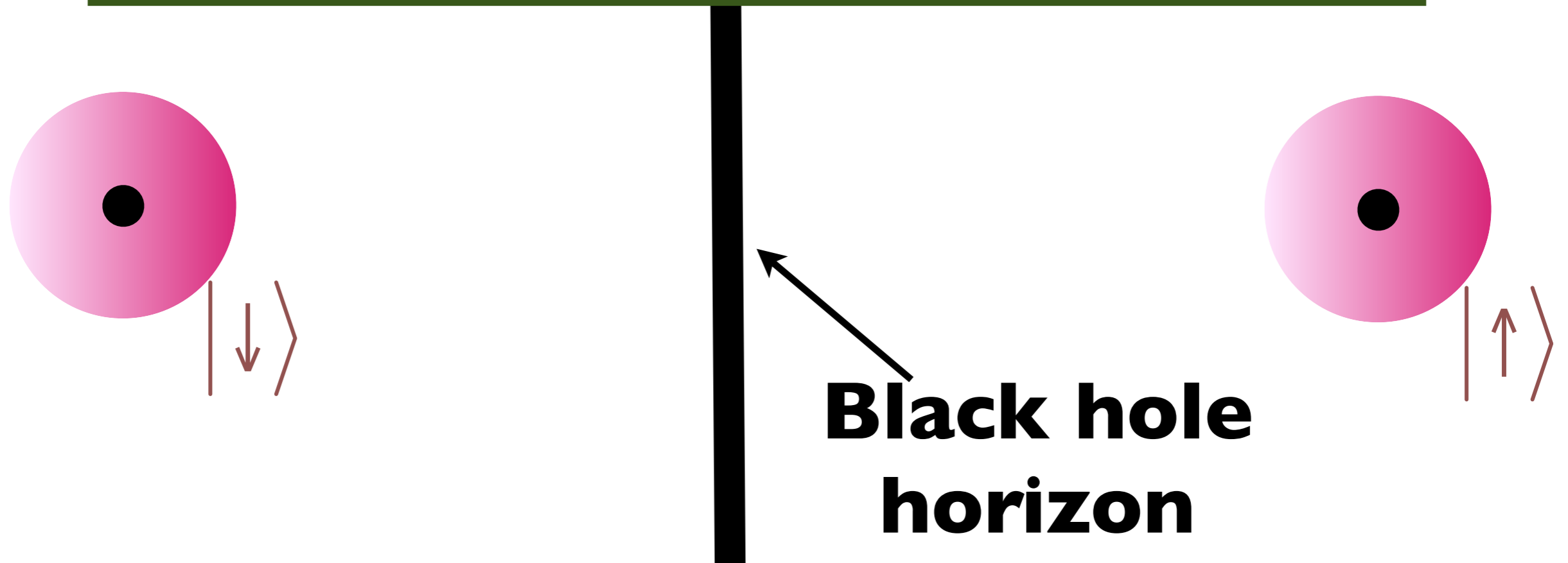
Quantum Entanglement across a black hole horizon

There is quantum entanglement between the inside and outside of a black hole



Quantum Entanglement across a black hole horizon

Hawking used this to show that black hole horizons have an entropy and a temperature
(because to an outside observer, the state of the electron inside the black hole is an unknown)



Quantum Black holes

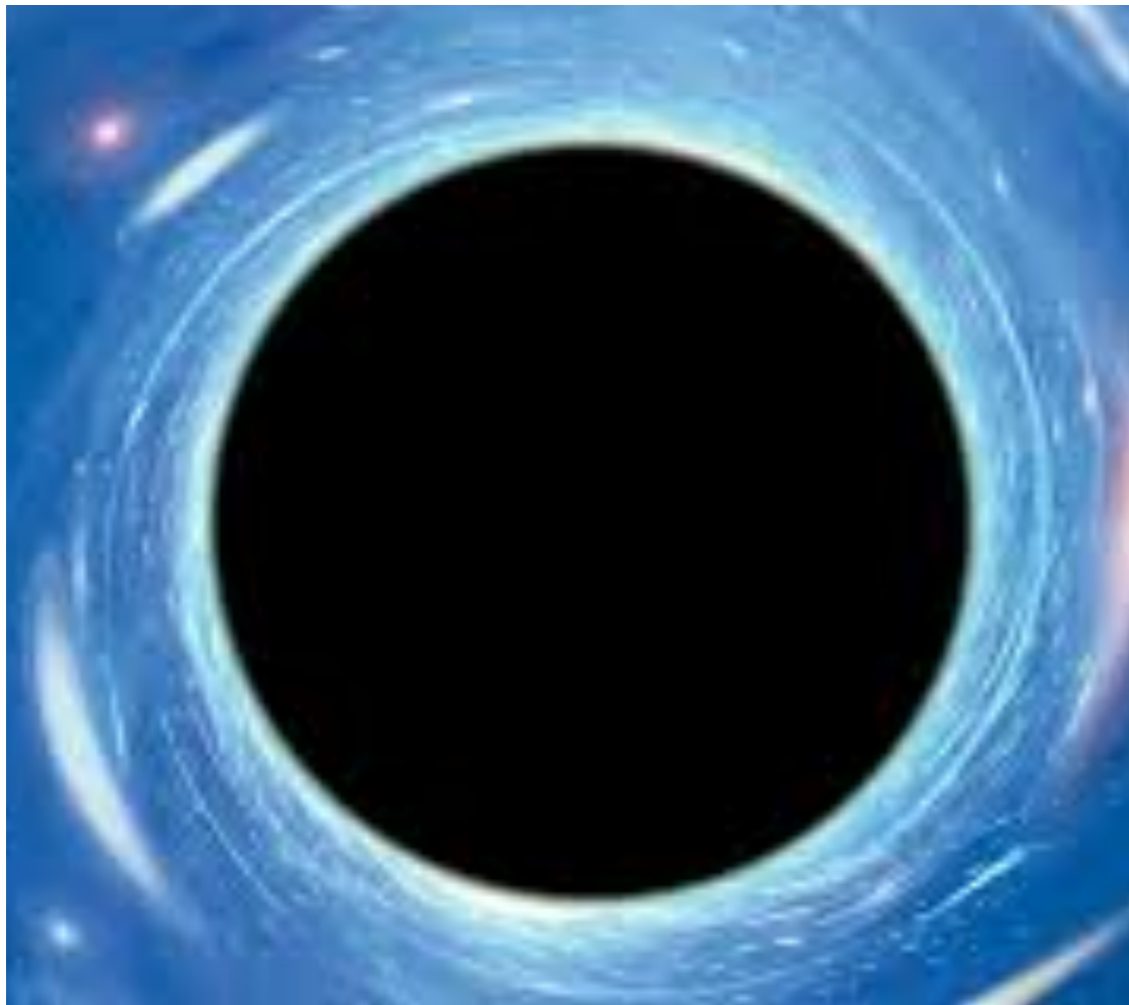
- Black holes have an entropy and a temperature, T_H .
- The entropy, S_{BH} is proportional to their surface area.

J. D. Bekenstein, PRD **7**, 2333 (1973)
S.W. Hawking, Nature **248**, 30 (1974)



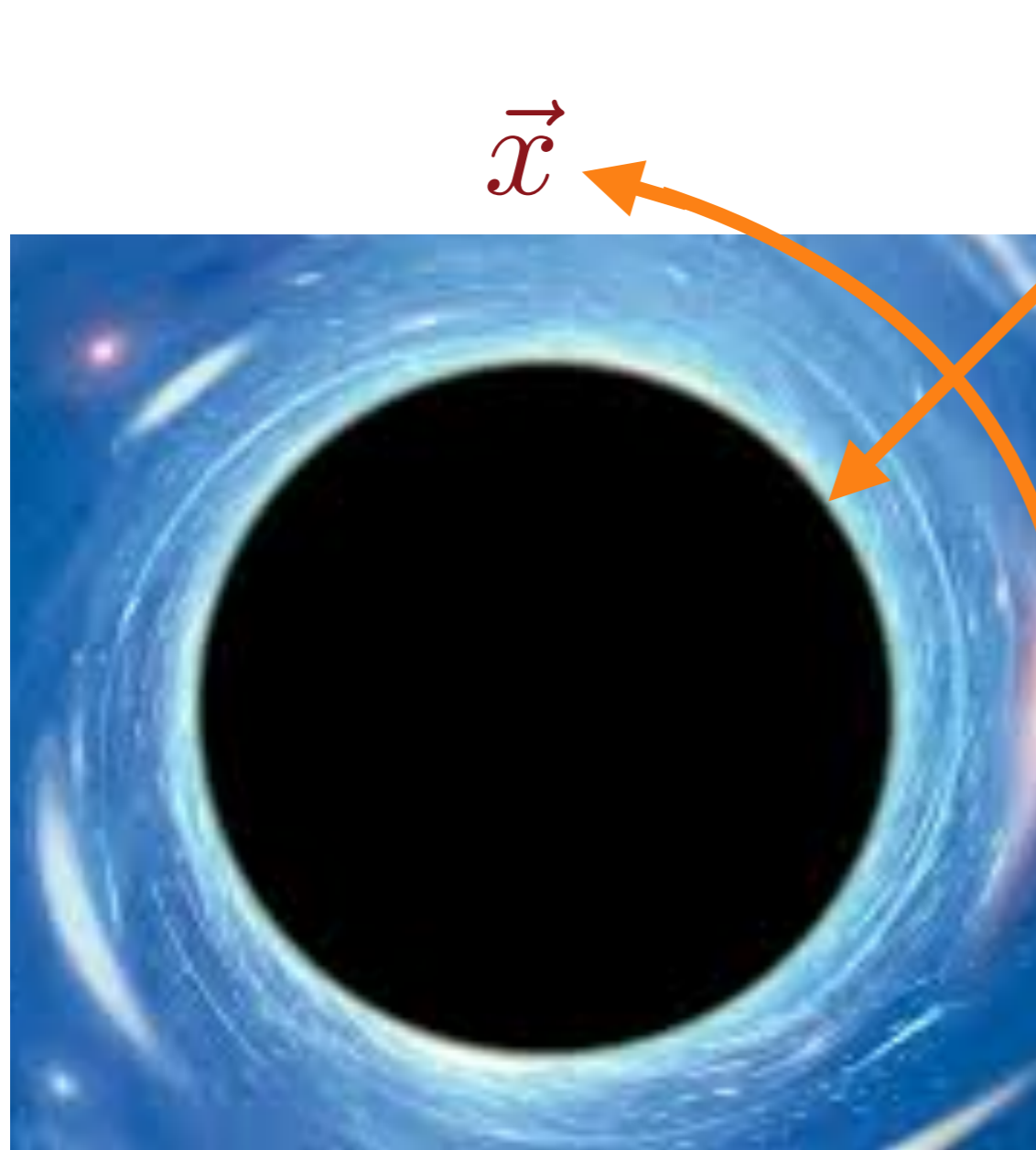


Maxwell's electromagnetism
and Einstein's general relativity
allow black hole solutions with a net charge





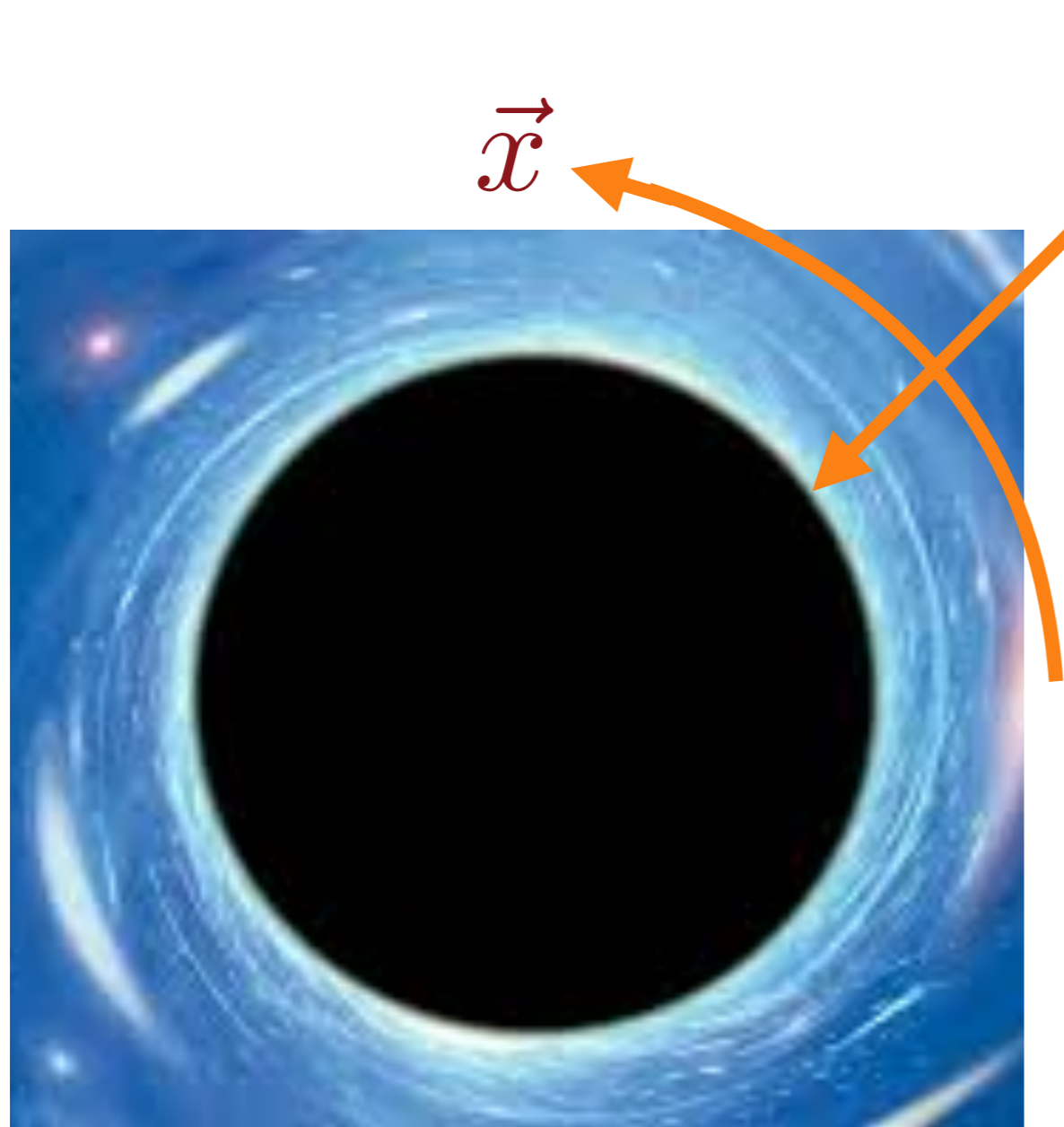
Maxwell's electromagnetism
and Einstein's general relativity
allow black hole solutions with a net charge



Zooming into the near-horizon region of a charged black hole at low temperature, yields a gravitational theory in one space (ζ) and one time dimension



Maxwell's electromagnetism
and Einstein's general relativity
allow black hole solutions with a net charge



This 2D-gravity theory
is precisely that
appearing in the low T
limit of the
Sachdev-Ye-Kitaev
(SYK) models
(including the qubit
model)!



Maxwell's electromagnetism and Einstein's general relativity allow black hole solutions with a net charge

$$I_{EM} = \int d^{d+2}x \sqrt{g} \left[-\frac{1}{2\kappa^2} \left(\mathcal{R}_{d+2} + \frac{d(d+1)}{L^2} \right) + \frac{1}{4g_F^2} F^2 \right]$$

Metric $g_{\mu\nu}$

Ricci scalar in $d+2$ dimensions, \mathcal{R}_{d+2}

Cosmological constant $\Lambda = -d(d+1)/L^2$

U(1) gauge field A_μ

Electromagnetic field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

Boundary conditions at spatial infinity:

Metric $\rightarrow \text{AdS}_{d+2}$

Electric field $\rightarrow Q/(4\pi r^2)$



Maxwell's electromagnetism
and Einstein's general relativity
allow black hole solutions with a net charge

$$I_{EM} = \int d^{d+2}x \sqrt{g} \left[-\frac{1}{2\kappa^2} \left(\mathcal{R}_{d+2} + \frac{d(d+1)}{L^2} \right) + \frac{1}{4g_F^2} F^2 \right]$$

Quantum gravity is 'defined' by the path integral

$$\mathcal{Z}_{\text{gravity}} = \int \mathcal{D}g \mathcal{D}A \exp(-I_{EM}/\hbar)$$

We will evaluate this integral exactly in a certain
low temperature limit for charged black holes.

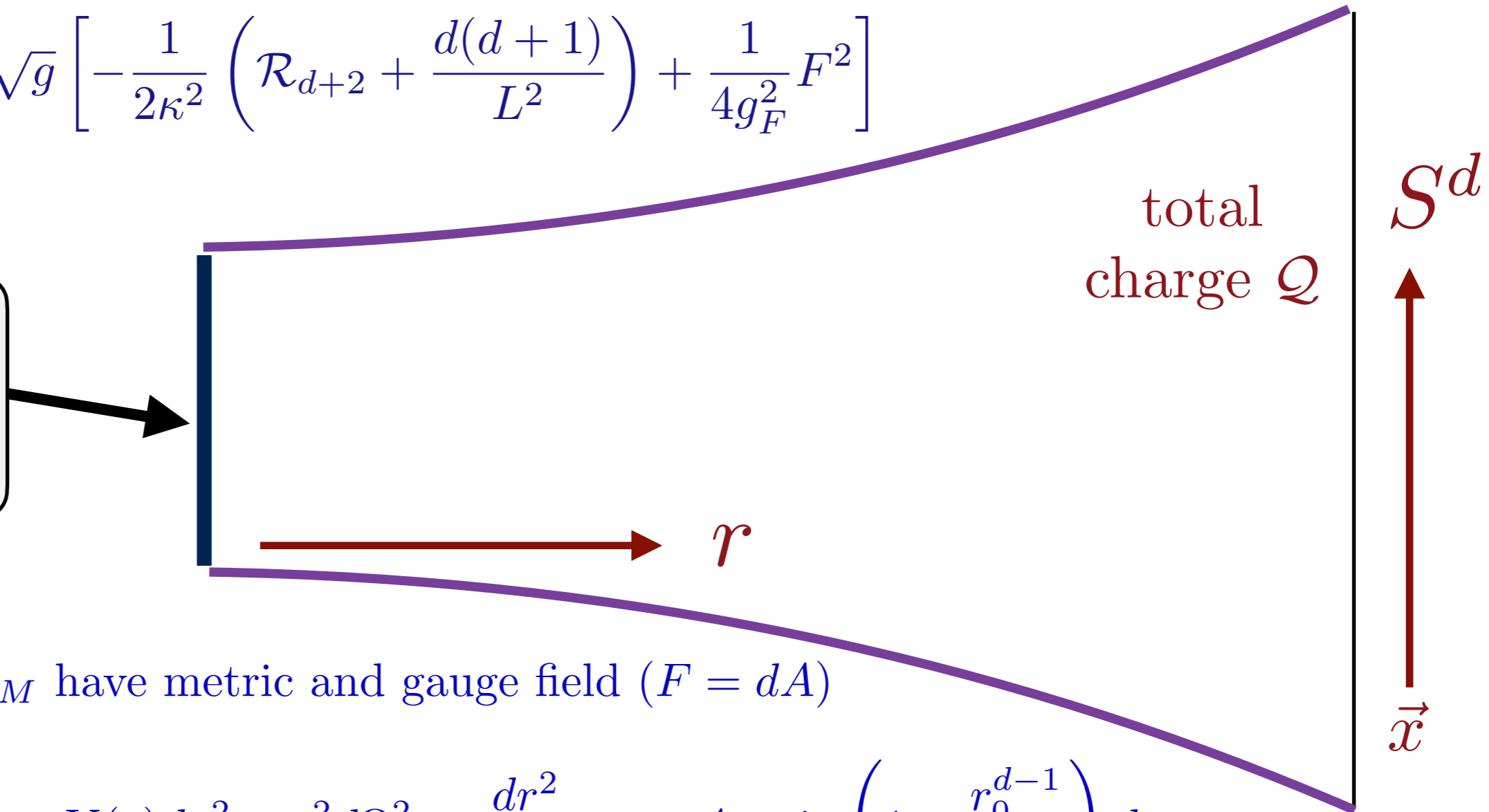
Charged black holes

$$I_{EM} = \int d^{d+2}x \sqrt{g} \left[-\frac{1}{2\kappa^2} \left(\mathcal{R}_{d+2} + \frac{d(d+1)}{L^2} \right) + \frac{1}{4g_F^2} F^2 \right]$$

Charged black holes

$$I_{EM} = \int d^{d+2}x \sqrt{g} \left[-\frac{1}{2\kappa^2} \left(\mathcal{R}_{d+2} + \frac{d(d+1)}{L^2} \right) + \frac{1}{4g_F^2} F^2 \right]$$

Black hole horizon of radius r_0



Solutions of I_{EM} have metric and gauge field ($F = dA$)

$$ds^2 = V(r)d\tau^2 + r^2 d\Omega_d^2 + \frac{dr^2}{V(r)} \quad , \quad A = i\mu \left(1 - \frac{r_0^{d-1}}{r^{d-1}} \right) d\tau$$

$$V(r) = 1 + \frac{r^2}{L^2} + \frac{\Theta^2}{r^{2d-2}} - \frac{M}{r^{d-1}}.$$

where $d\Omega_d^2$ is the metric of the d -sphere. All parameters of the solution are determined in terms of the chemical potential μ (related to the charge Q), and the Hawking temperature of horizon, T_H (related to the mass M).

Charged black holes

In the $T \rightarrow 0$ limit, at fixed μ , we obtain a charged black hole solution with radius $r_0(T \rightarrow 0, \mu) = R_h$. All properties of this black hole can be expressed in terms of R_h

- In the near-horizon region, we change co-ordinates from r to ζ so that

$$r - R_h = \frac{R_2^2}{\zeta} \quad , \quad R_2 = \frac{LR_h}{\sqrt{d(d+1)R_h^2 + (d-1)^2L^2}}.$$

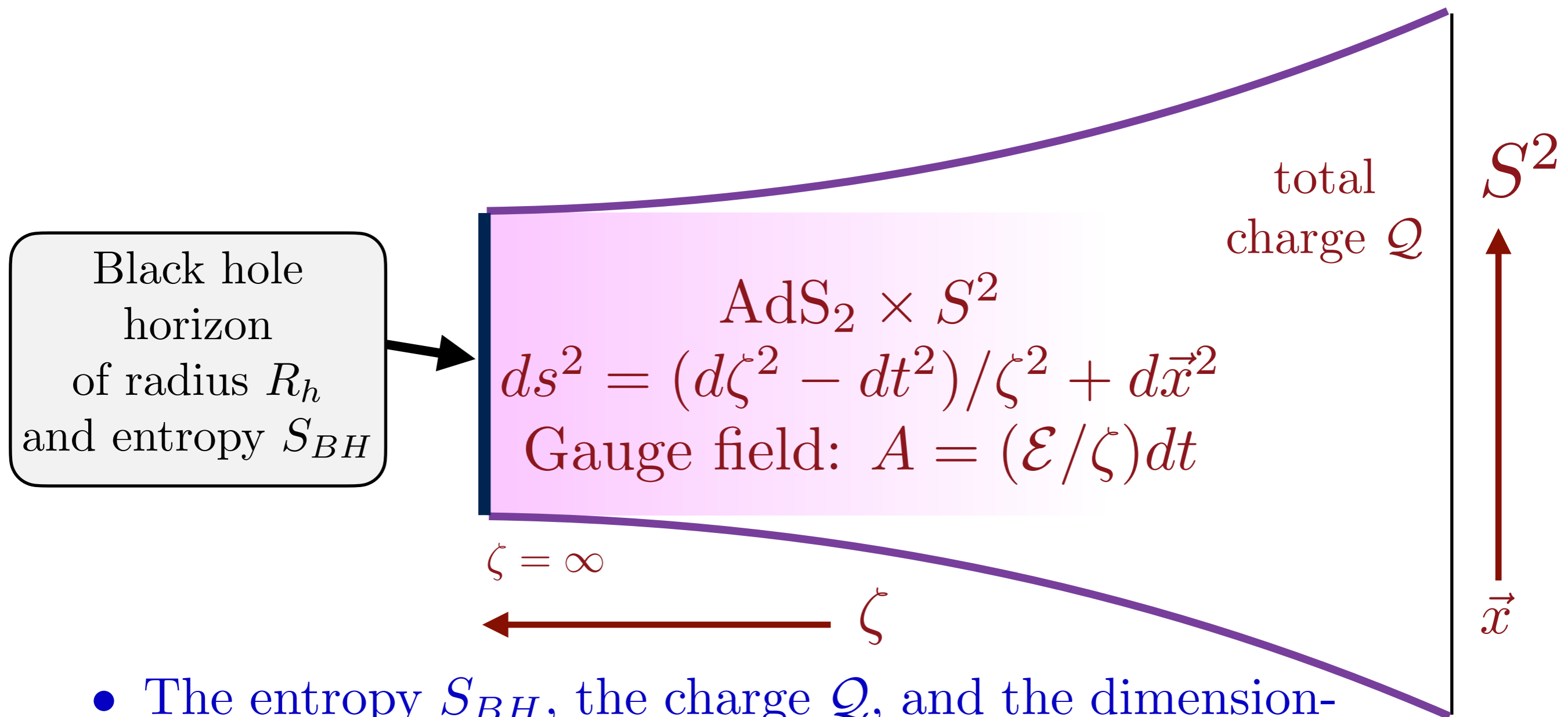
Then the near-horizon metric becomes $\text{AdS}_2 \times S_d$, with

$$ds^2 = R_2^2 \left[\frac{-dt^2 + d\zeta^2}{\zeta^2} \right] + R_h^2 d\Omega_d^2 \quad , \quad A = \frac{\mathcal{E}}{\zeta} dt.$$

where the dimensionless electric field \mathcal{E} is

$$\mathcal{E} = \frac{g_F R_h \sqrt{2d [(d+1)R_h^2 + (d-1)L^2]}}{2 [d(d+1)R_h^2 + (d-1)^2L^2]}.$$

Charged black holes



- The entropy S_{BH} , the charge Q , and the dimensionless electric field \mathcal{E} obey the same thermodynamic relation as the SYK model

$$\frac{dS_{BH}}{dQ} = 2\pi\mathcal{E}$$

2D gravity and black holes

- In imaginary time, AdS_2 is the homogeneous hyperbolic space: two-dimensional surface of constant negative curvature. Its metric is invariant under $\text{SL}(2, \mathbb{R})$

$ds^2 = (d\tau^2 + d\zeta^2)/\zeta^2$ is invariant under

$$\tau' + i\zeta' = \frac{a(\tau + i\zeta) + b}{c(\tau + i\zeta) + d} \text{ with } ad - bc = 1.$$



Euclidean AdS_2

2D gravity and black holes

- In imaginary time, AdS_2 is the homogeneous hyperbolic space: two-dimensional surface of constant negative curvature. Its metric is invariant under $\text{SL}(2, \mathbb{R})$
- The time reparameterization invariance of general relativity is broken down to $\text{SL}(2, \mathbb{R})$ by the AdS_2 saddle point.

$ds^2 = (d\tau^2 + d\zeta^2)/\zeta^2$ is invariant under

$$\tau' + i\zeta' = \frac{a(\tau + i\zeta) + b}{c(\tau + i\zeta) + d} \quad \text{with } ad - bc = 1.$$



Euclidean AdS_2

SYK model and charged black holes



Horizon

$\text{AdS}_2 \times S^2$

$$ds^2 = R_2^2 \frac{(d\zeta^2 - dt^2)}{\zeta^2} + R_h^2 d\Omega_2^2$$

$$\text{Gauge field: } A = \frac{\mathcal{E}}{\zeta} dt$$

total
charge Q

AdS_4

S^2

ζ

Solution of Euler-Lagrange equations of the action of Einstein gravity and Maxwell electromagnetism

SYK model and charged black holes



Horizon

$\text{AdS}_2 \times S^2$

$$ds^2 = R_2^2 \frac{(d\zeta^2 - dt^2)}{\zeta^2} + R_h^2 d\Omega_2^2$$

$$\text{Gauge field: } A = \frac{\mathcal{E}}{\zeta} dt$$

Boundary graviton

total charge Q

AdS_4

S^2



Fluctuations about the path integral saddle

Main result II

For $T \ll 1/R_h$

$\mathcal{Z}_{\text{charged black hole in EM theory}} =$

$$\exp\left(\frac{S_{BH}}{k_B}\right) \int \mathcal{D}f(\tau) \exp\left(-\frac{1}{\hbar} \mathcal{S}_{2D\text{-gravity}}[f(\tau)]\right)$$

Main result II

For $T \ll 1/R_h$

$$\mathcal{Z}_{\text{charged black hole in EM theory}} = \exp\left(\frac{S_{BH}}{k_B}\right) \int \mathcal{D}f(\tau) \exp\left(-\frac{1}{\hbar} \mathcal{S}_{2D\text{-gravity}}[f(\tau)]\right)$$

$$\mathcal{S}_{2D\text{-gravity}}[f(\tau)] = -\frac{\gamma}{4\pi^2} \int_0^{1/T} d\tau \{ \tan(\pi T f(\tau)), \tau \},$$

where $f(\tau)$ is a monotonic map from $[0, 1/T]$ to $[0, 1/T]$, and we have used the *Schwarzian*:

$$\{g, \tau\} \equiv \frac{d^3 g/d\tau^3}{dg/d\tau} - \frac{3}{2} \left(\frac{d^2 g/d\tau^2}{dg/d\tau} \right)^2.$$

The defining property of the Schwarzian is its invariance under $SL(2, \mathbb{R})$ transformations

$$\left\{ \frac{ag(\tau) + b}{cg(\tau) + d}, \tau \right\} = \{g(\tau), \tau\}$$

Remarkably, this path integral can be evaluated exactly, using the Duistermaat–Heckman formula (Stanford, Witten, arXiv:1703.04612).

Derivation of main result II

We write the $(d+2)$ -dimensional metric g of I_{EM} in terms of a two-dimensional metric h and a scalar field Φ :

$$ds^2 = \frac{ds_2^2}{\Phi^{d-1}} + \Phi^2 d\Omega_d^2.$$

The Einstein-Maxwell and Gibbons-Hawking actions reduce to and extension of Jackiw-Tietelbaum gravity ($x \equiv (\tau, \zeta)$)

$$I_{EM} = \int d^2x \sqrt{h} \left[-\frac{s_d}{2\kappa^2} \Phi^d \mathcal{R}_2 + U(\Phi) + \frac{Z(\Phi)}{4g_F^2} F^2 \right]$$
$$I_{GH} = -\frac{s_d}{\kappa^2} \int_{\partial} dx \sqrt{h_b} \Phi^d \mathcal{K}_1$$

The explicit forms of the potentials $U(\Phi)$ and $Z(\Phi)$ are,

$$U(\Phi) = -\frac{s_d}{2\kappa^2} \left(\frac{d(d-1)}{\Phi} + \frac{d(d+1)\Phi}{L^2} \right), \quad Z(\Phi) = s_d \Phi^{2d-1}.$$



Derivation of main result II

The exact saddle point of Φ relates to R_h the horizon radius at $T = 0$

$$\Phi(\zeta) = R_h + \frac{R_2^2}{\zeta}, \quad R_h \equiv \frac{L}{g_F} \left[\frac{(d-1)(\mu_0^2 \kappa^2 (d-1) - dg_F^2)}{d(d+1)} \right]^{1/2},$$

while the near-horizon, low $T \ll 1/R_h$ metric is AdS_2

$$ds_2^2 = \frac{R_2^2 R_h^{d-1}}{\zeta^2} \left[(1 - 4\pi^2 T^2 \zeta^2) d\tau^2 + \frac{d\zeta^2}{1 - 4\pi^2 T^2 \zeta^2} \right],$$

where

$$R_2 = \frac{LR_h}{\sqrt{d(d+1)R_h^2 + (d-1)^2 L^2}}$$

The field coupling to \mathcal{R}_2 is Φ^d

$$[\Phi(\zeta)]^d = R_h^d + \frac{\Phi_1}{\zeta} + \dots, \quad \Phi_1 = dR_h^{d-1} R_2^2,$$



Derivation of main result II

The field coupling to \mathcal{R}_2 is Φ^d

$$[\Phi(\zeta)]^d = R_h^d + \frac{\Phi_1}{\zeta} + \dots \quad , \quad \Phi_1 = dR_h^{d-1}R_2^2,$$

We choose the boundary of the AdS_2 region at bulk co-ordinates $(f(\tau), \zeta(\tau))$ with the induced boundary metric fixed at $(R_2^2 R_h^{d-1} / \zeta_b^2) d\tau^2$ by choosing

$$\zeta(\tau) = \zeta_b f'(\tau) + \zeta_b^3 \left(\frac{[f''(\tau)]^2}{2f'(\tau)} - 2\pi^2 T^2 [f'(\tau)]^3 \right) + \dots$$

Finally, we evaluate I_{GH} along this boundary curve

$$I_1[f] = -\frac{\gamma}{4\pi^2} \int_0^{1/T} d\tau \{ \tan(\pi T f(\tau)), \tau \},$$

where

$$\gamma = \frac{4\pi^2 s_d \Phi_1}{\kappa^2},$$



matches the linear-in- T co-efficient of the specific heat of the full Reissner-Nördstorm solution in $d + 2$ dimensions.

Main result II

For $T \ll 1/R_h$

$$\mathcal{Z}_{\text{charged black hole in EM theory}} = \exp\left(\frac{S_{BH}}{k_B}\right) \int \mathcal{D}f(\tau) \exp\left(-\frac{1}{\hbar} \mathcal{S}_{2D\text{-gravity}}[f(\tau)]\right)$$

$$\mathcal{S}_{2D\text{-gravity}}[f(\tau)] = -\frac{\gamma}{4\pi^2} \int_0^{1/T} d\tau \{ \tan(\pi T f(\tau)), \tau \},$$

where $f(\tau)$ is a monotonic map from $[0, 1/T]$ to $[0, 1/T]$, and we have used the *Schwarzian*:

$$\{g, \tau\} \equiv \frac{d^3 g/d\tau^3}{dg/d\tau} - \frac{3}{2} \left(\frac{d^2 g/d\tau^2}{dg/d\tau} \right)^2.$$

The defining property of the Schwarzian is its invariance under $SL(2, \mathbb{R})$ transformations

$$\left\{ \frac{ag(\tau) + b}{cg(\tau) + d}, \tau \right\} = \{g(\tau), \tau\}$$

Remarkably, this path integral can be evaluated exactly, using the Duistermaat–Heckman formula (Stanford, Witten, arXiv:1703.04612).

Quantum entanglement

A simple qubit model

Black holes

Low temperatures

Quantum gravity in 1+1 dimensions

Quantum entanglement

A simple qubit model

Black holes

Low temperatures

Quantum gravity in 1+1 dimensions

Derivation of main result I

We write the spin operator $\vec{S}_i = (X_i, Y_i, Z_i)$ in terms of spin-1/2 fermions $\vec{S}_i = (1/2)c_{i\alpha}^\dagger \vec{\sigma}_{\alpha\beta} c_{i\beta}$, where $\vec{\sigma}$ are the Pauli matrices. After integrating out the disorder (neglecting replicas for simplicity), the partition function can be written as

$$Z = \int \mathcal{D}c_{i\alpha}(\tau) \exp \left[- \sum_{i\alpha} \int_0^\beta d\tau c_{i\alpha}^\dagger \frac{\partial}{\partial \tau} c_{i\alpha} - \frac{J^2}{4NM} \sum_{\alpha\beta\gamma\delta} \int_0^\beta d\tau d\tau' \left| \sum_i c_{i\alpha}^\dagger(\tau) c_{i\beta}(\tau) c_{i\gamma}^\dagger(\tau') c_{i\delta}(\tau') \right|^2 \right].$$

We now introduce the identity

$$1 = \int \mathcal{D}G(\tau_1, \tau_2) \mathcal{D}\Sigma(\tau_1, \tau_2) \exp \left[-NM \int_0^\beta d\tau_1 d\tau_2 \Sigma(\tau_1, \tau_2) \left(G(\tau_2, \tau_1) + \frac{1}{NM} \sum_i c_{i\alpha}(\tau_2) c_{i\alpha}^\dagger(\tau_1) \right) \right].$$



Derivation of main result I

Then the partition function can be written as a path integral with an action S analogous to a Luttinger-Ward functional

$$Z = \int \mathcal{D}G(\tau_1, \tau_2) \mathcal{D}\Sigma(\tau_1, \tau_2) \exp(-NMS)$$

$$S = \ln \det [\delta(\tau_1 - \tau_2) \partial_{\tau_1} - \Sigma(\tau_1, \tau_2)] \\ + \int d\tau_1 d\tau_2 \Sigma(\tau_1, \tau_2) [G(\tau_2, \tau_1) + (J^2/4)G^2(\tau_2, \tau_1)G^2(\tau_1, \tau_2)]$$

At frequencies $\ll J$, the time derivative in the determinant is less important, and without it the path integral is invariant under the reparametrizations

$$\tau = f(\sigma)$$

$$G(\tau_1, \tau_2) = [f'(\sigma_1)f'(\sigma_2)]^{-1/4} G(\sigma_1, \sigma_2)$$

$$\Sigma(\tau_1, \tau_2) = [f'(\sigma_1)f'(\sigma_2)]^{-3/4} \Sigma(\sigma_1, \sigma_2)$$

where $f(\sigma)$ is an arbitrary function.

A. Georges and O. Parcollet
PRB **59**, 5341 (1999)

A. Kitaev, 2015

S. Sachdev, PRX **5**, 041025 (2015)



Derivation of main result I

Reparametrization mode

We can write the path integral for the SYK model as

$$\mathcal{Z} = \int \mathcal{D}G(\tau_1, \tau_1) \mathcal{D}\Sigma(\tau_1, \tau_2) e^{-NMS[G, \Sigma]}$$

for a known action $S[G, \Sigma]$. We find the saddle point, G_s, Σ_s , and only focus on the “Nambu-Goldstone” modes associated with breaking reparameterization symmetry by writing

$$G(\tau_1, \tau_2) = [f'(\tau_1)f'(\tau_2)]^{1/4} G_s(f(\tau_1) - f(\tau_2))$$

(and similarly for Σ). Then the path integral is approximated by

$$\mathcal{Z} = \int \mathcal{D}f(\tau) e^{-NMS_{2D-gravity}[f(\tau)]}.$$

J. Maldacena and D. Stanford, arXiv:1604.07818;

R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, arXiv:1612.00849;

S. Sachdev, PRX **5**, 041025 (2015); J. Maldacena, D. Stanford, and Zhenbin Yang, arXiv:1606.01857;

K. Jensen, arXiv:1605.06098; J. Engelsoy, T.G. Mertens, and H. Verlinde, arXiv:1606.03438



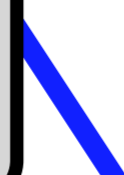
Quantum entanglement

A simple qubit model

Black holes

Low temperatures

Quantum gravity in 1+1 dimensions



Quantum entanglement

A simple qubit model

Black holes

Physical origin ?

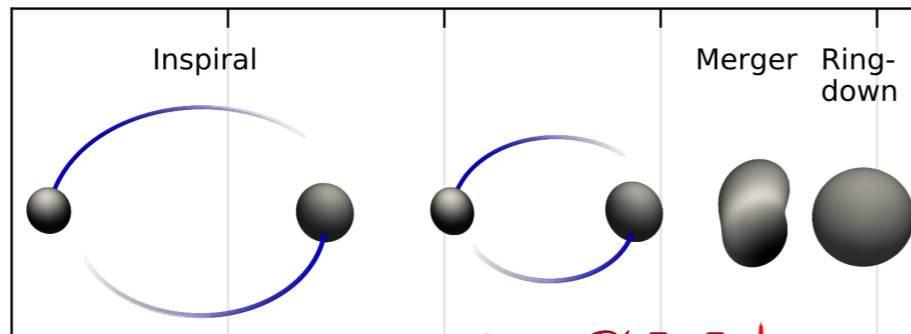
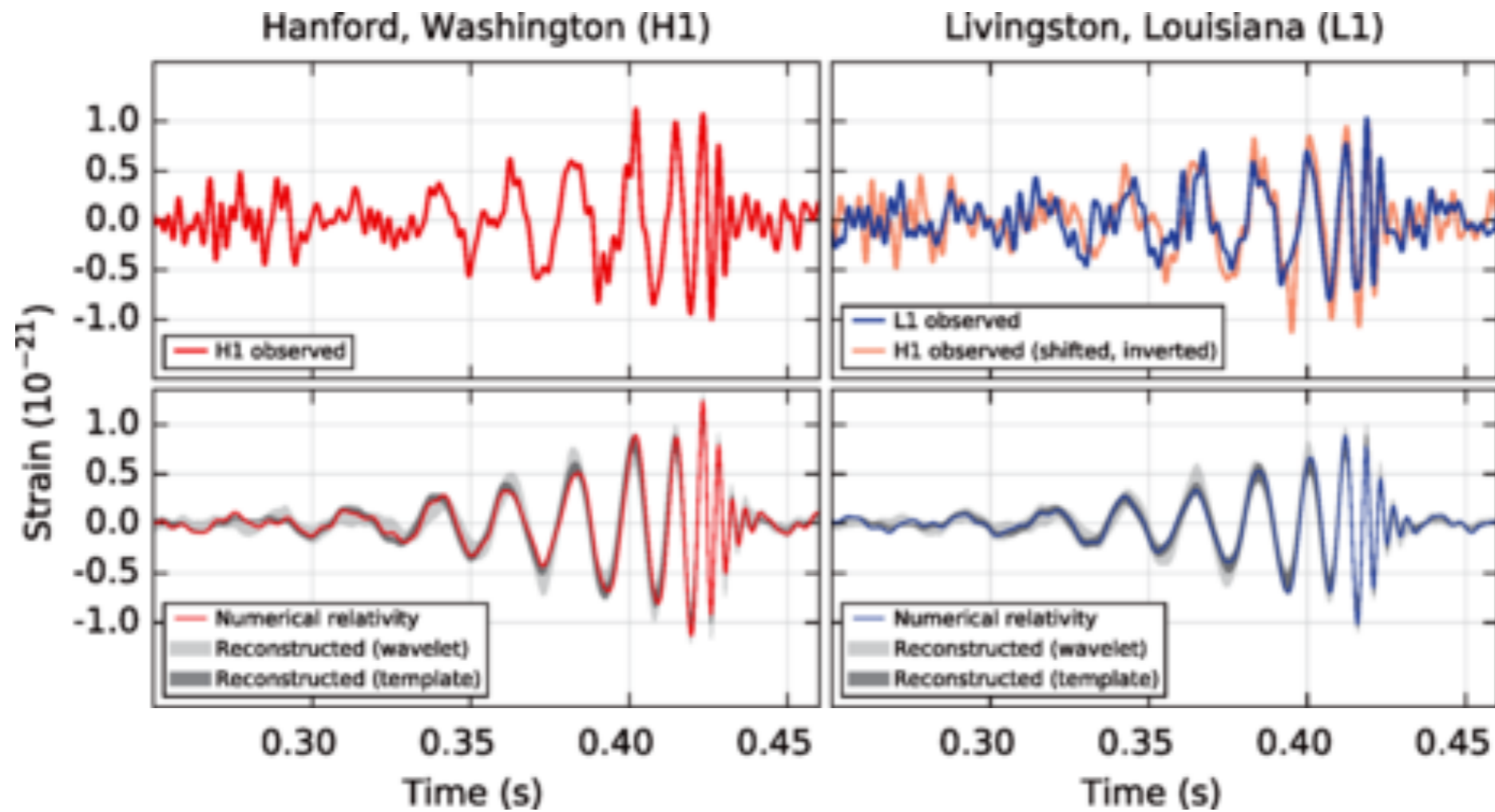
Low temperatures

Quantum gravity in 1+1 dimensions

Complex multi-particle entanglement
leads to quantum systems
without quasiparticle excitations.

Complex multi-particle entanglement
leads to quantum systems
without quasiparticle excitations.

Many-body chaos and
thermal equilibration
in the shortest possible
Planckian time $\sim \frac{\hbar}{k_B T}$.



LIGO
September 14, 2015

- The ring-down time $\frac{8\pi GM}{c^3} \sim 8$ milliseconds. Curiously, for essentially all types of black holes, the ring-down time equals

$$\frac{\hbar}{k_B T_H}$$

\hbar Planck's constant, k_B Boltzmann's constant

Quantum Black holes

- Black holes have an entropy and a temperature, T_H .
- The entropy, S_{BH} is proportional to their surface area.
- They relax to thermal equilibrium in a Planckian time $\sim \hbar/(k_B T_H)$.

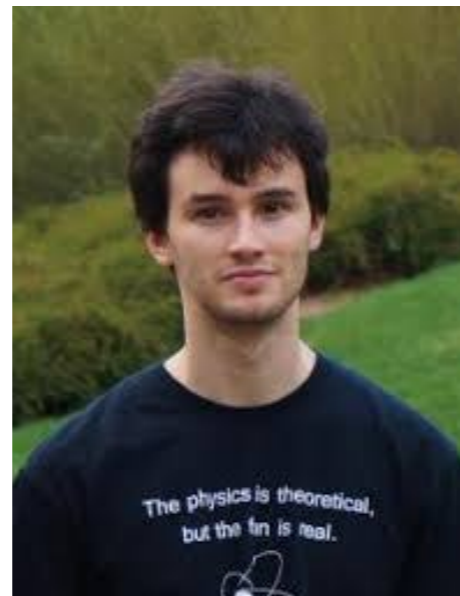




Maria Tikhanovskaya



Haoyu Guo



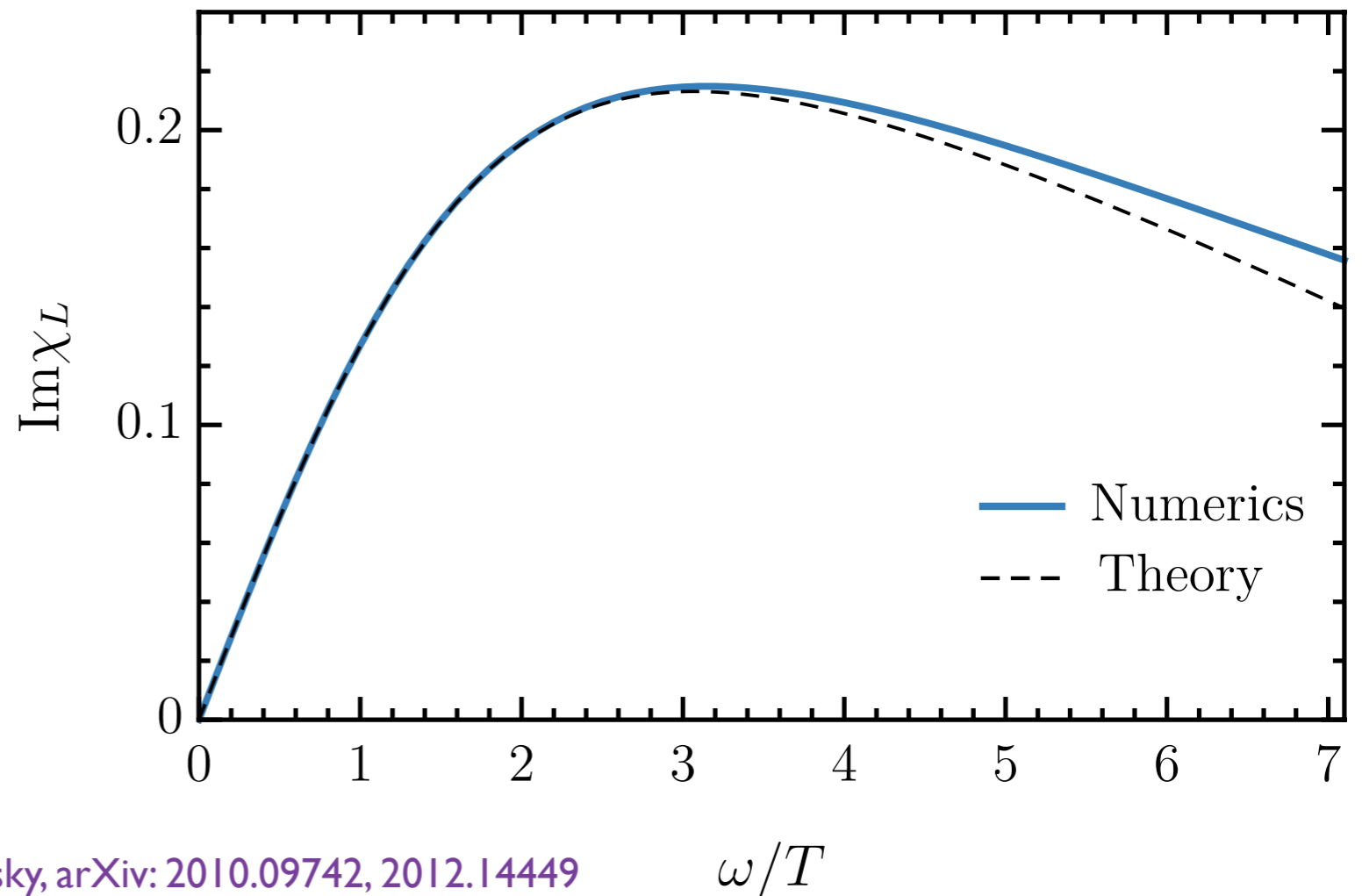
Grigory Tarnopolsky

arXiv:2010.09742
arXiv:2012.14449

Consequences of 2D-gravity for the dynamic spin susceptibility

$$\chi_L(\omega) = \sum_n |\langle 0 | X_i | n \rangle|^2 \delta(\hbar\omega - E_n + E_0), \text{ (at } T = 0)$$

$$\chi_L(\omega) \sim \tanh\left(\frac{\hbar\omega}{2k_B T}\right) \left[1 - C\gamma\omega \tanh\left(\frac{\hbar\omega}{2k_B T}\right) - \dots \right]$$



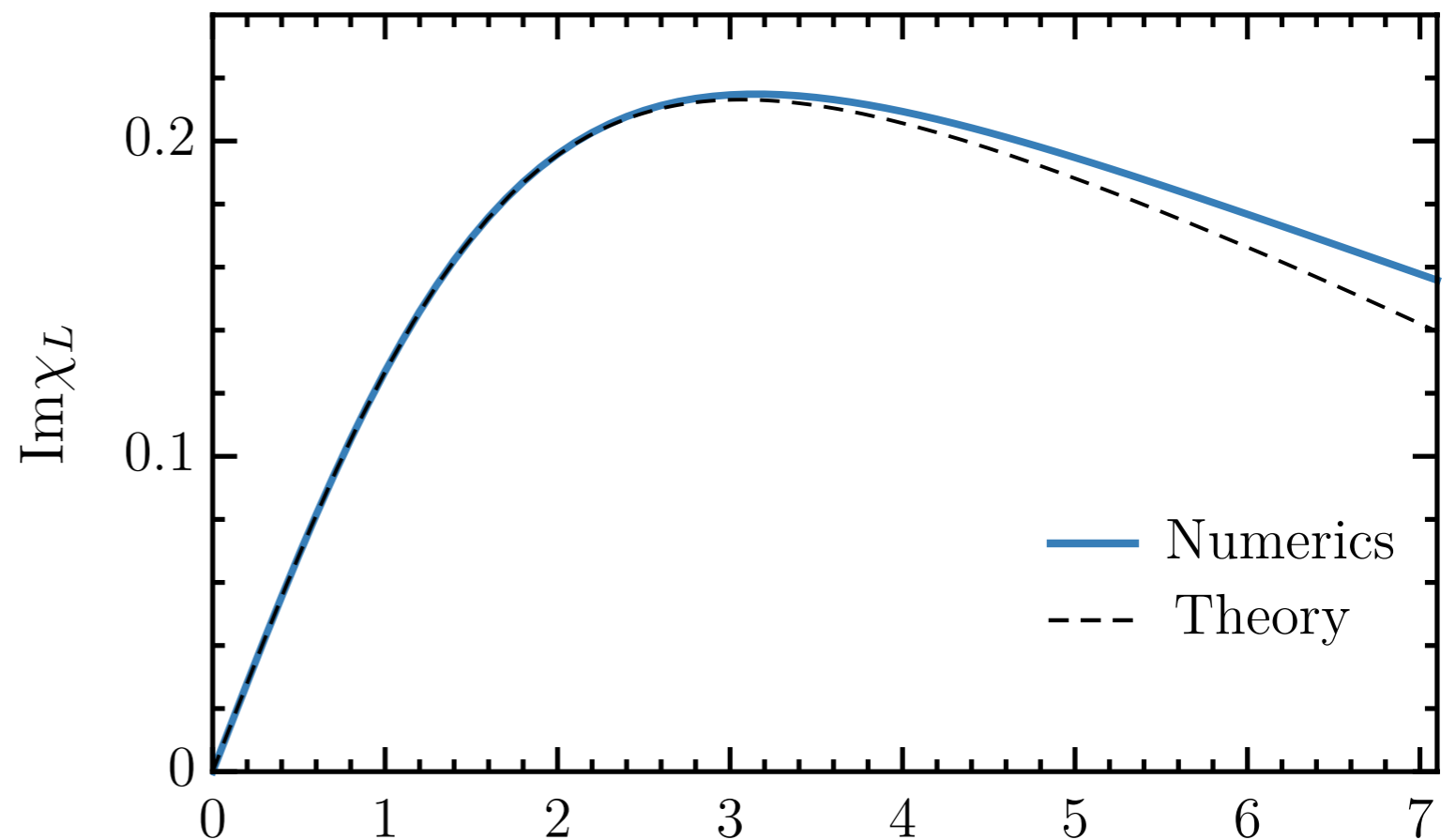
Consequences of 2D-gravity for the dynamic spin susceptibility

$$\chi_L(\omega) = \sum_n |\langle 0 | X_i | n \rangle|^2 \delta(\hbar\omega - E_n + E_0), \quad (\text{at } T = 0)$$

$$\chi_L(\omega) \sim \tanh\left(\frac{\hbar\omega}{2k_B T}\right) \left[1 - C\gamma\omega \tanh\left(\frac{\hbar\omega}{2k_B T}\right) - \dots \right]$$

Conformally (SL(2,R))
invariant result with
characteristic dissipative
time $\sim \hbar/(k_B T)$

A. Georges and O. Parcollet
PRB **59**, 5341 (1999)

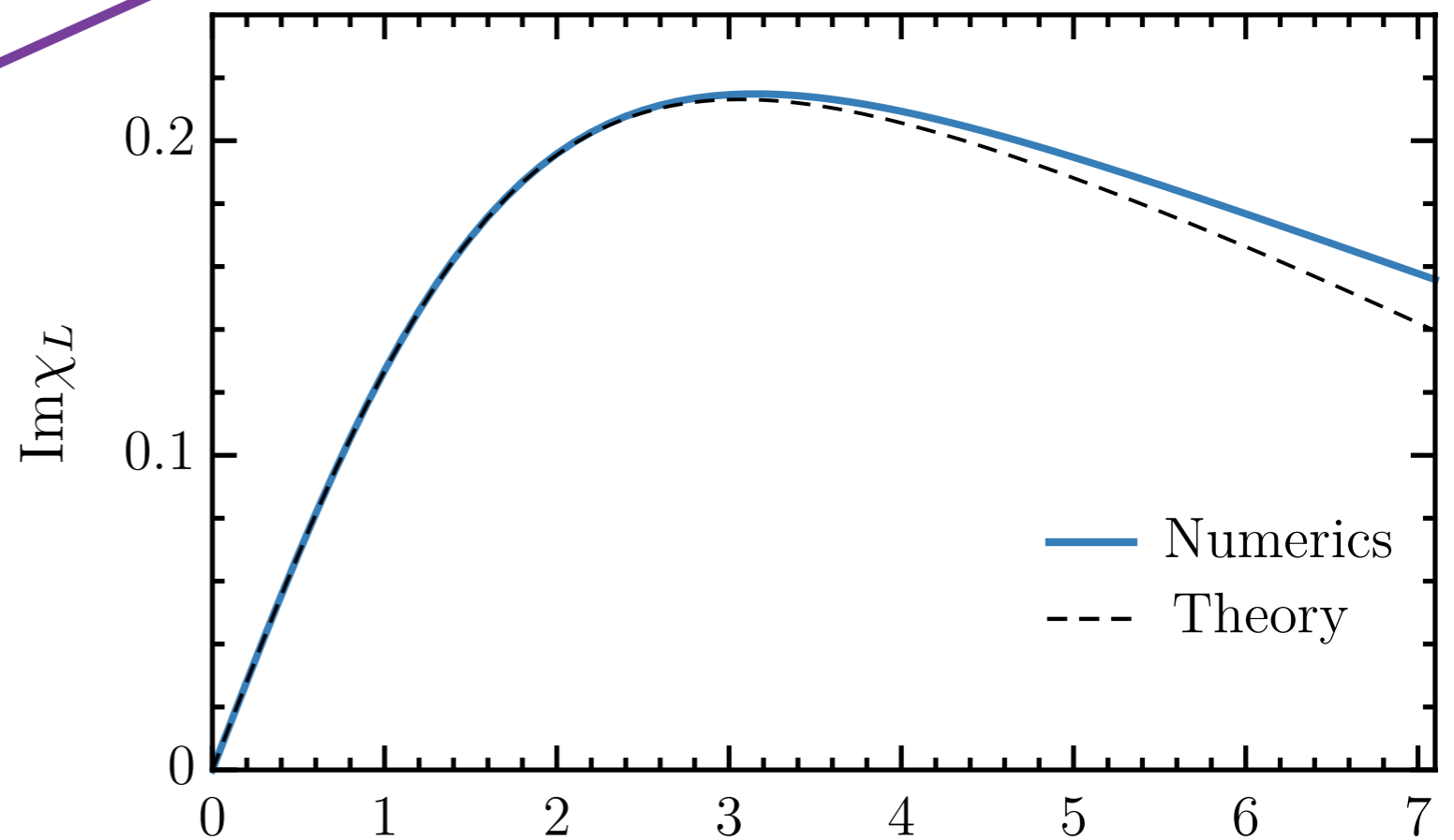


Consequences of 2D-gravity for the dynamic spin susceptibility

$$\chi_L(\omega) = \sum_n |\langle 0 | X_i | n \rangle|^2 \delta(\hbar\omega - E_n + E_0), \quad (\text{at } T = 0)$$

$$\chi_L(\omega) \sim \tanh\left(\frac{\hbar\omega}{2k_B T}\right) \left[1 - C\gamma\omega \tanh\left(\frac{\hbar\omega}{2k_B T}\right) - \dots \right]$$

Correction from
the boundary
graviton



Quantum entanglement

A simple qubit model

Black holes

Low temperatures

Quantum gravity in 1+1 dimensions

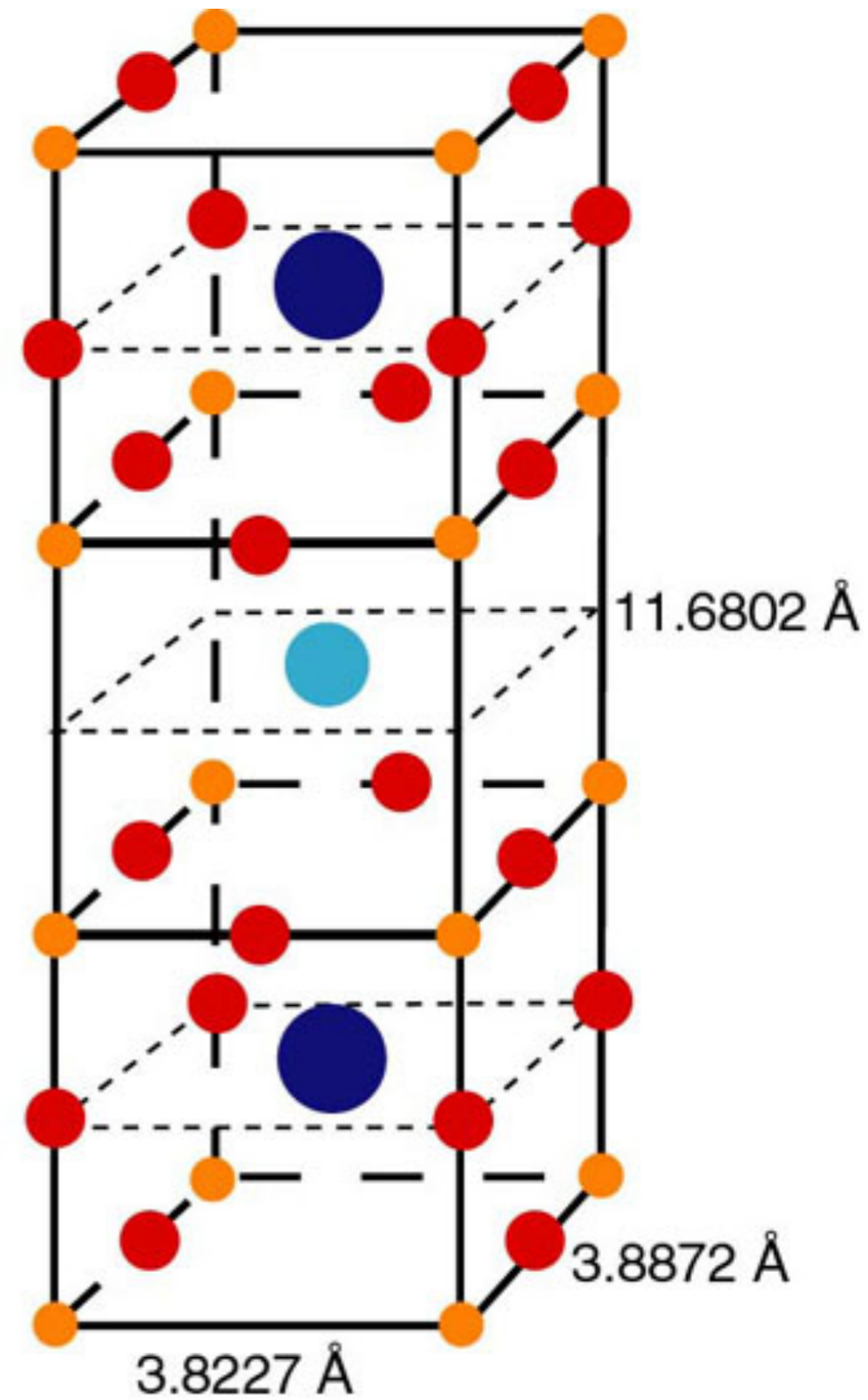
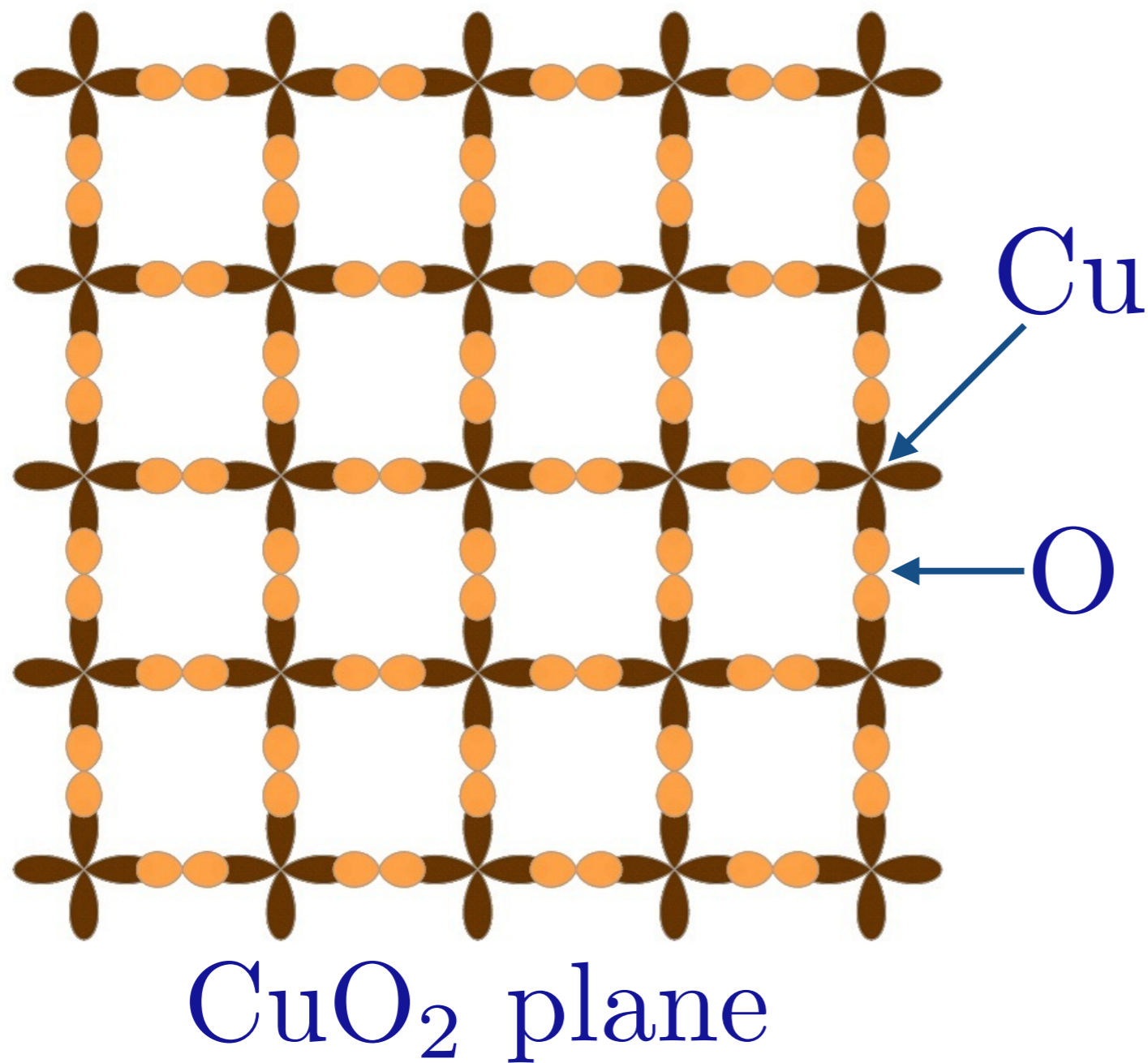
**Quantum
entanglement**

**Black
holes**

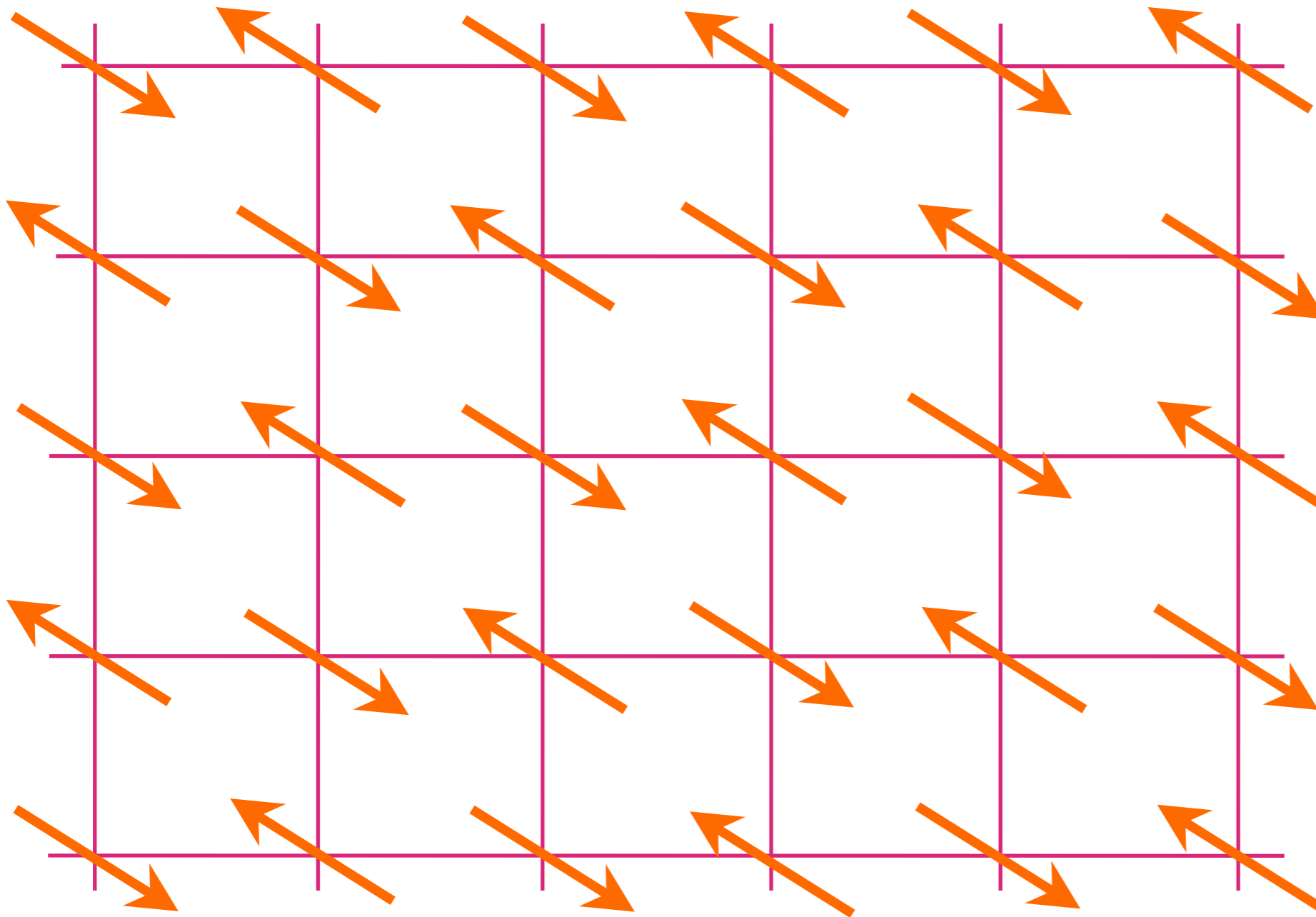
**A simple
qubit
model**

**Copper-based
superconductors**

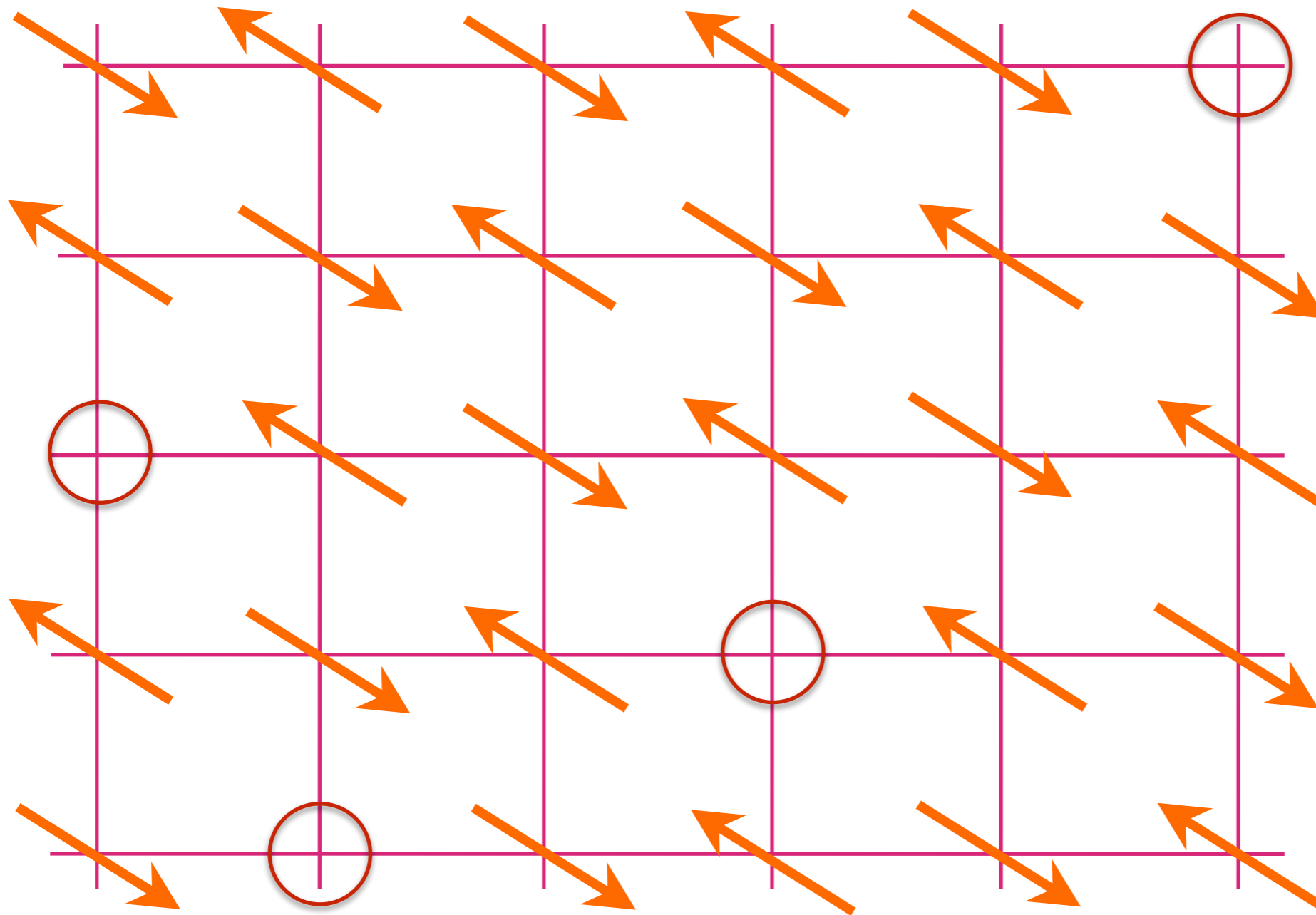
High temperature superconductors

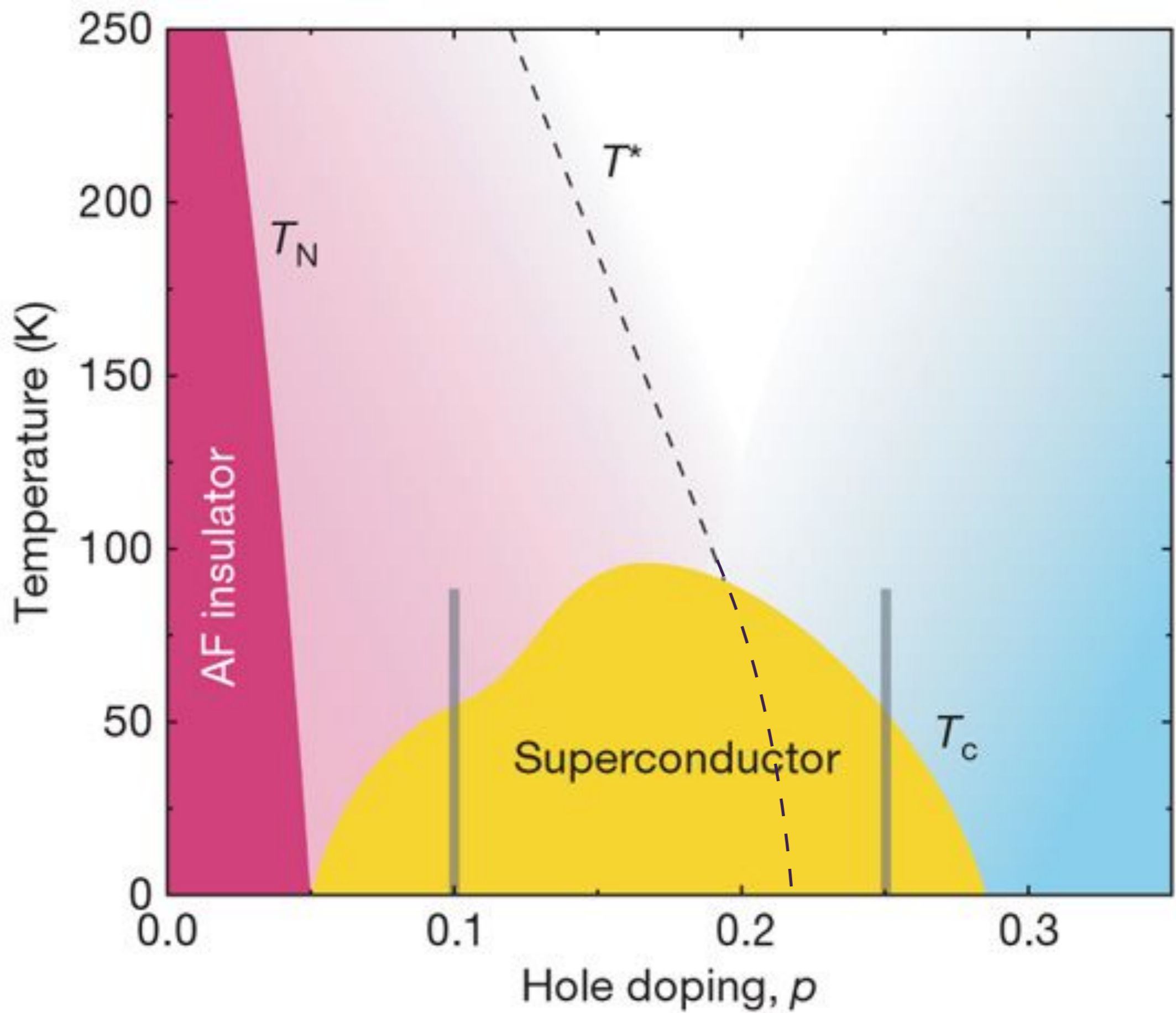


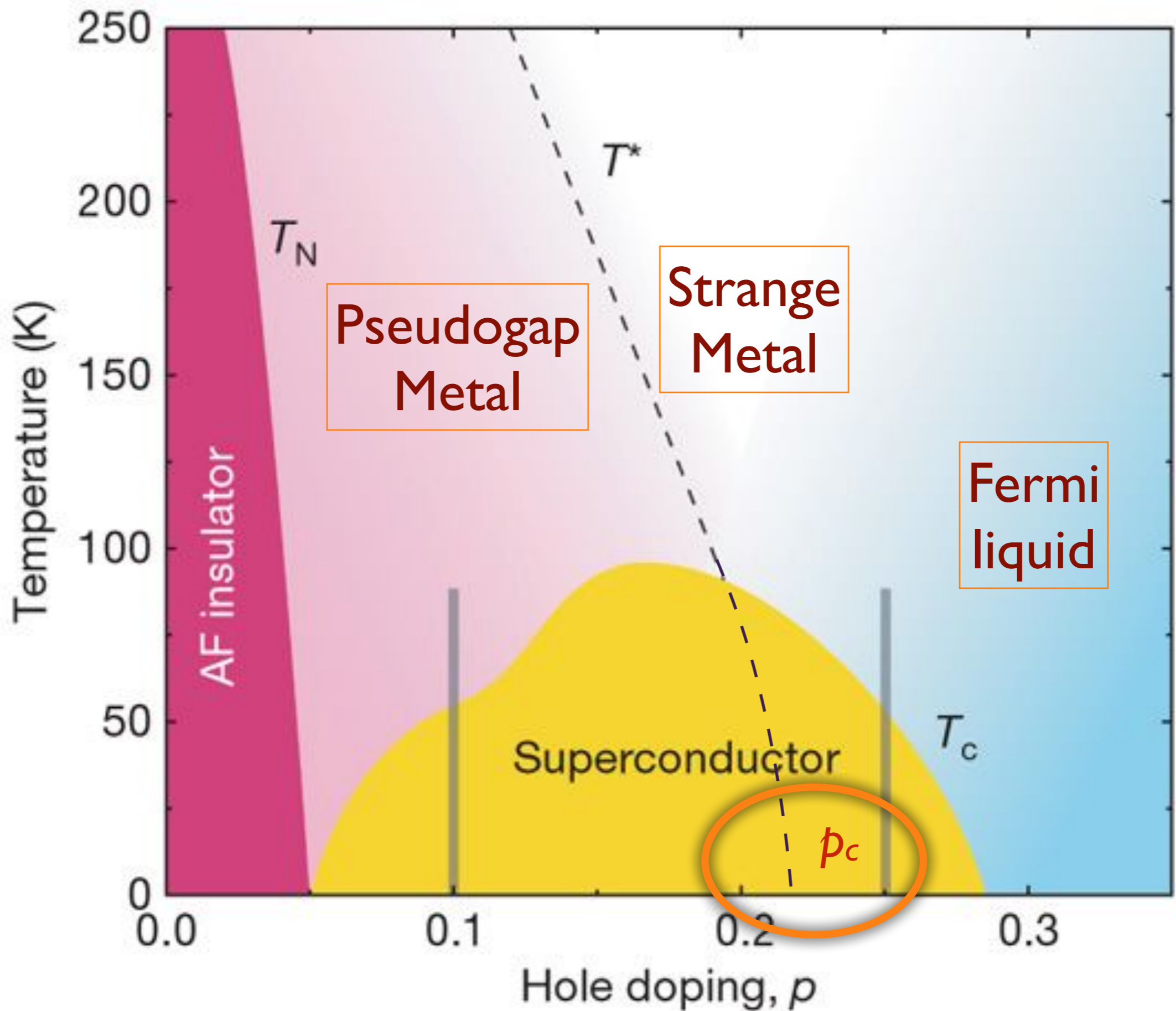
Insulating antiferromagnet



Antiferromagnet doped with hole density p







t-J model

$$H = -\frac{1}{\sqrt{N}} \sum_{i,j=1}^N t_{ij} c_{i\alpha}^\dagger c_{j\alpha} + \frac{1}{\sqrt{N}} \sum_{i<j=1}^N J_{ij} \vec{S}_i \cdot \vec{S}_j$$

We consider the hole-doped case, with no double occupancy.

$$\alpha = \uparrow, \downarrow, \quad \{c_{i\alpha}, c_{j\beta}^\dagger\} = \delta_{ij} \delta_{\alpha\beta}, \quad \{c_{i\alpha}, c_{j\beta}\} = 0$$

$$\vec{S}_i = \frac{1}{2} c_{i\alpha}^\dagger \vec{\sigma}_{\alpha\beta} c_{i\beta}, \quad \sum_{\alpha} c_{i\alpha}^\dagger c_{i\alpha} \leq 1, \quad \frac{1}{N} \sum_{i\alpha} c_{i\alpha}^\dagger c_{i\alpha} = 1 - p$$

$$\text{---} \\ |0\rangle$$

$$\text{---} \uparrow \\ c_{\uparrow}^\dagger |0\rangle$$

$$\text{---} \downarrow \\ c_{\downarrow}^\dagger |0\rangle$$

t-J model

$$H = -\frac{1}{\sqrt{N}} \sum_{i,j=1}^N t_{ij} c_{i\alpha}^\dagger c_{j\alpha} + \frac{1}{\sqrt{N}} \sum_{i<j=1}^N J_{ij} \vec{S}_i \cdot \vec{S}_j$$

We consider the hole-doped case, with no double occupancy.

$$\alpha = \uparrow, \downarrow, \quad \{c_{i\alpha}, c_{j\beta}^\dagger\} = \delta_{ij} \delta_{\alpha\beta}, \quad \{c_{i\alpha}, c_{j\beta}\} = 0$$

$$\vec{S}_i = \frac{1}{2} c_{i\alpha}^\dagger \vec{\sigma}_{\alpha\beta} c_{i\beta}, \quad \sum_{\alpha} c_{i\alpha}^\dagger c_{i\alpha} \leq 1, \quad \frac{1}{N} \sum_{i\alpha} c_{i\alpha}^\dagger c_{i\alpha} = 1 - p$$

$$\text{---} \\ |0\rangle$$

$$\text{---} \uparrow \\ c_{\uparrow}^\dagger |0\rangle$$

$$\text{---} \downarrow \\ c_{\downarrow}^\dagger |0\rangle$$

Random t - J model

$$H = -\frac{1}{\sqrt{N}} \sum_{i,j=1}^N t_{ij} c_{i\alpha}^\dagger c_{j\alpha} + \frac{1}{\sqrt{N}} \sum_{i<j=1}^N J_{ij} \vec{S}_i \cdot \vec{S}_j$$

We consider the hole-doped case, with no double occupancy.

$$\alpha = \uparrow, \downarrow, \quad \{c_{i\alpha}, c_{j\beta}^\dagger\} = \delta_{ij} \delta_{\alpha\beta}, \quad \{c_{i\alpha}, c_{j\beta}\} = 0$$

$$\vec{S}_i = \frac{1}{2} c_{i\alpha}^\dagger \vec{\sigma}_{\alpha\beta} c_{i\beta}, \quad \sum_{\alpha} c_{i\alpha}^\dagger c_{i\alpha} \leq 1, \quad \frac{1}{N} \sum_{i\alpha} c_{i\alpha}^\dagger c_{i\alpha} = 1 - p$$

$$J_{ij} \text{ random, } \overline{J_{ij}} = 0, \quad \overline{J_{ij}^2} = J^2$$

$$t_{ij} \text{ random, } \overline{t_{ij}} = 0, \quad \overline{t_{ij}^2} = t^2$$

$$\text{---} \\ |0\rangle$$

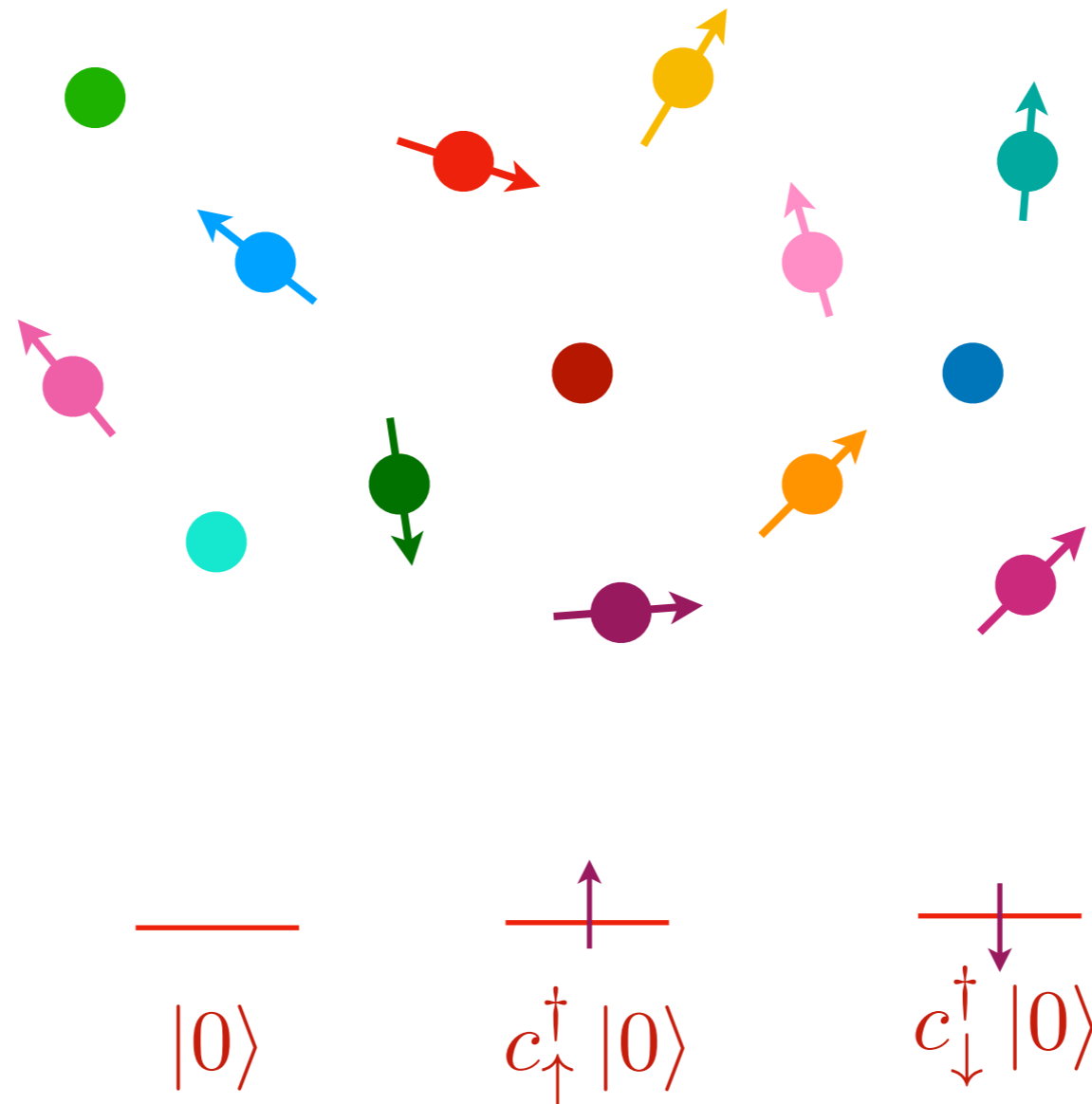
$$\text{---} \uparrow \\ c_{\uparrow}^\dagger |0\rangle$$

$$\text{---} \downarrow \\ c_{\downarrow}^\dagger |0\rangle$$

Random t - J model

$$H = -\frac{1}{\sqrt{N}} \sum_{i,j=1}^N t_{ij} c_{i\alpha}^\dagger c_{j\alpha} + \frac{1}{\sqrt{N}} \sum_{i<j=1}^N J_{ij} \vec{S}_i \cdot \vec{S}_j$$

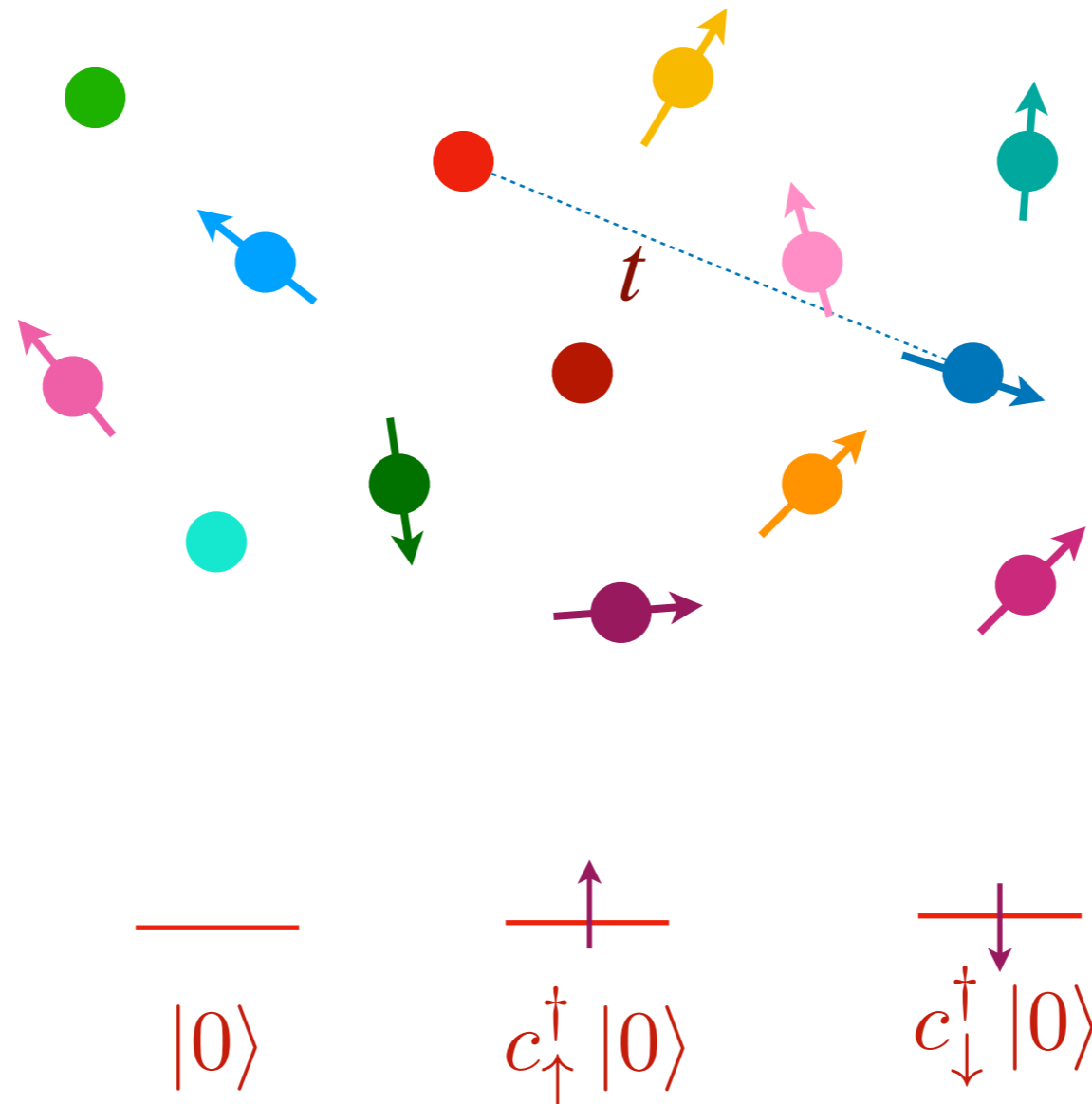
We consider the hole-doped case, with no double occupancy.



Random t - J model

$$H = -\frac{1}{\sqrt{N}} \sum_{i,j=1}^N t_{ij} c_{i\alpha}^\dagger c_{j\alpha} + \frac{1}{\sqrt{N}} \sum_{i<j=1}^N J_{ij} \vec{S}_i \cdot \vec{S}_j$$

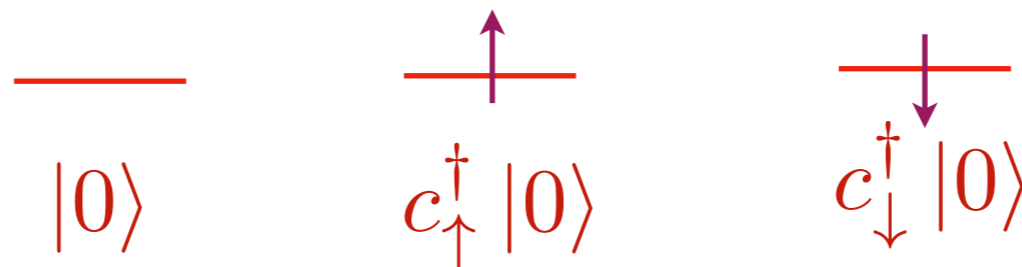
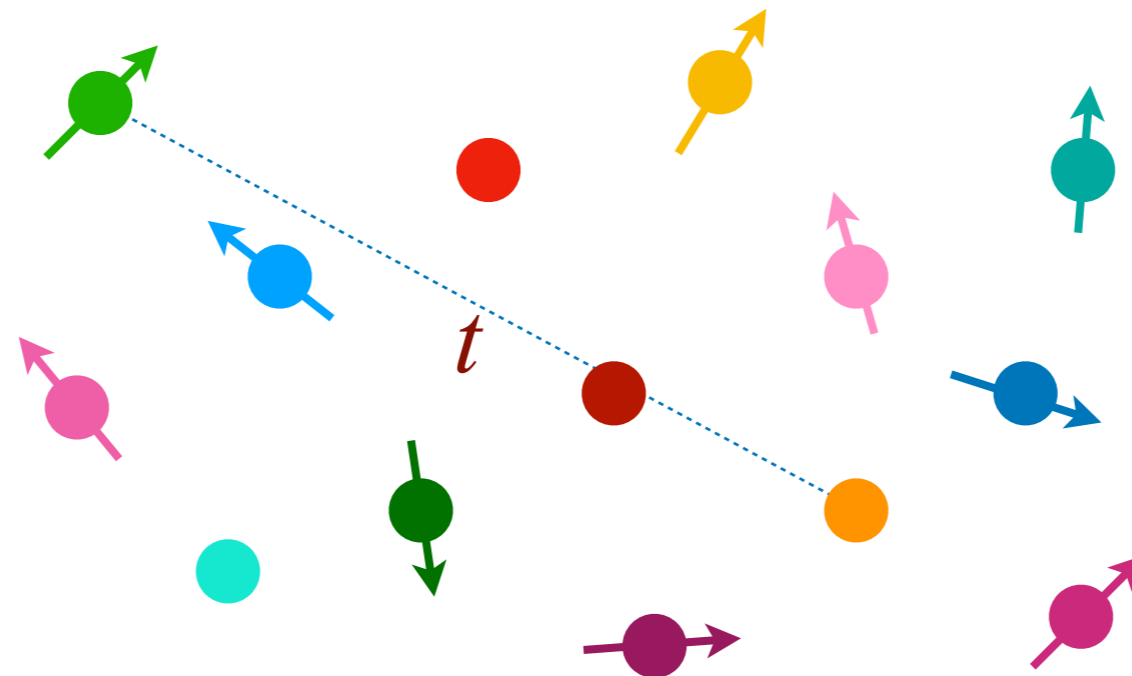
We consider the hole-doped case, with no double occupancy.



Random t - J model

$$H = -\frac{1}{\sqrt{N}} \sum_{i,j=1}^N t_{ij} c_{i\alpha}^\dagger c_{j\alpha} + \frac{1}{\sqrt{N}} \sum_{i<j=1}^N J_{ij} \vec{S}_i \cdot \vec{S}_j$$

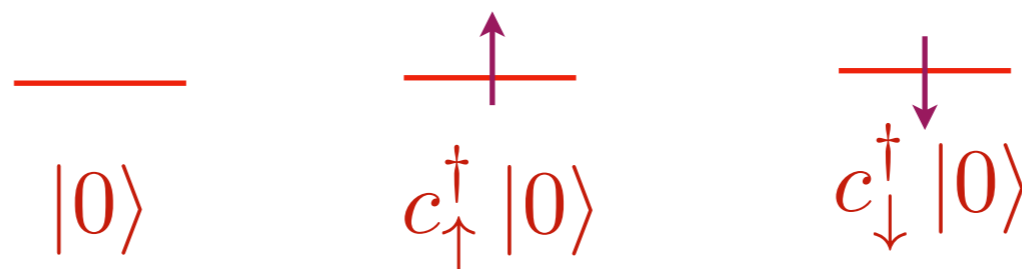
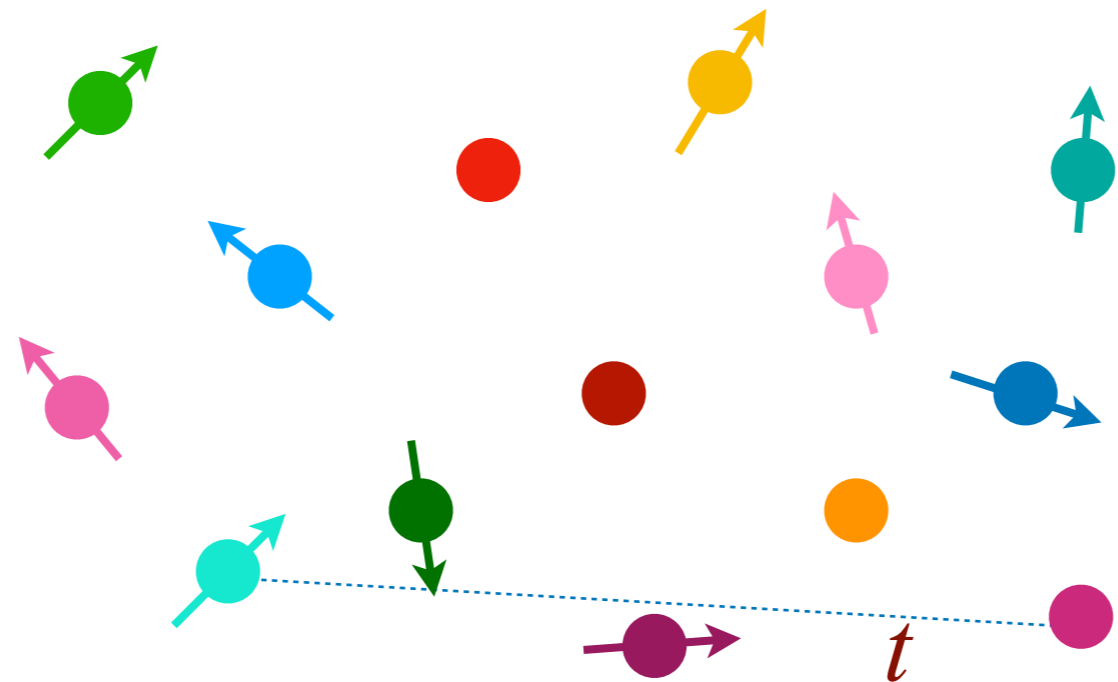
We consider the hole-doped case, with no double occupancy.



Random t - J model

$$H = -\frac{1}{\sqrt{N}} \sum_{i,j=1}^N t_{ij} c_{i\alpha}^\dagger c_{j\alpha} + \frac{1}{\sqrt{N}} \sum_{i<j=1}^N J_{ij} \vec{S}_i \cdot \vec{S}_j$$

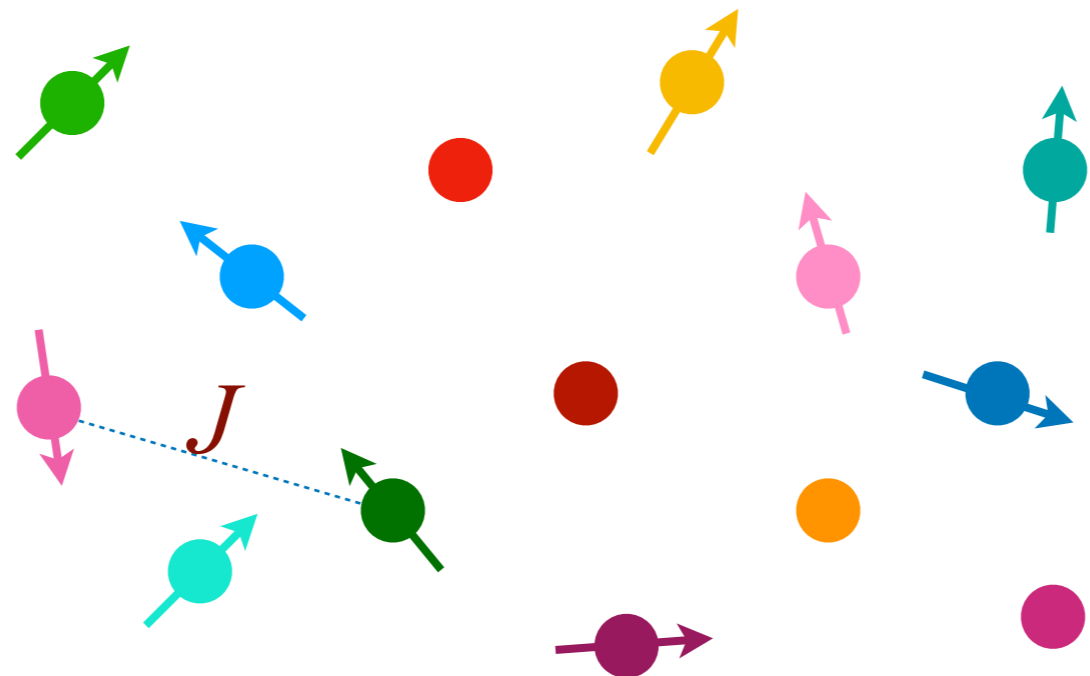
We consider the hole-doped case, with no double occupancy.



Random t - J model

$$H = -\frac{1}{\sqrt{N}} \sum_{i,j=1}^N t_{ij} c_{i\alpha}^\dagger c_{j\alpha} + \frac{1}{\sqrt{N}} \sum_{i<j=1}^N J_{ij} \vec{S}_i \cdot \vec{S}_j$$

We consider the hole-doped case, with no double occupancy.



$$\text{---} \\ |0\rangle$$

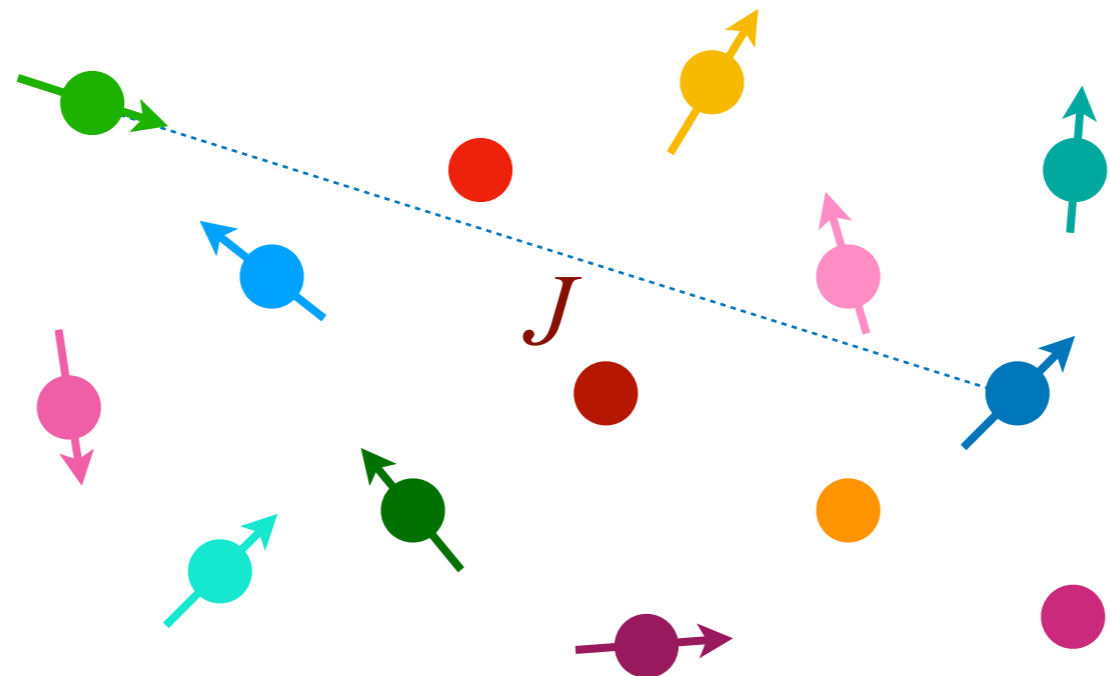
$$\text{---} \uparrow \\ c_{\uparrow}^\dagger |0\rangle$$

$$\text{---} \downarrow \\ c_{\downarrow}^\dagger |0\rangle$$

Random t - J model

$$H = -\frac{1}{\sqrt{N}} \sum_{i,j=1}^N t_{ij} c_{i\alpha}^\dagger c_{j\alpha} + \frac{1}{\sqrt{N}} \sum_{i<j=1}^N J_{ij} \vec{S}_i \cdot \vec{S}_j$$

We consider the hole-doped case, with no double occupancy.



$$\text{---} \\ |0\rangle$$

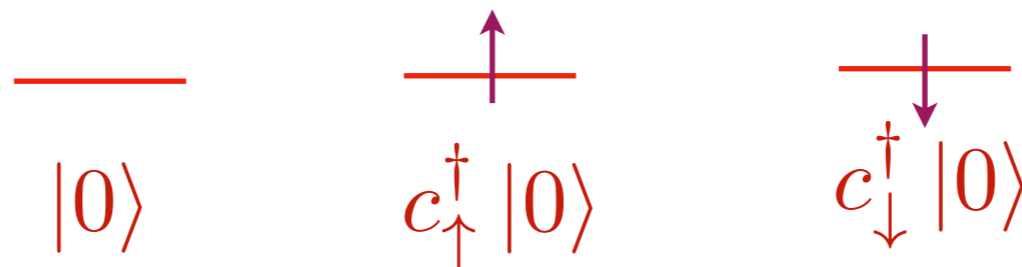
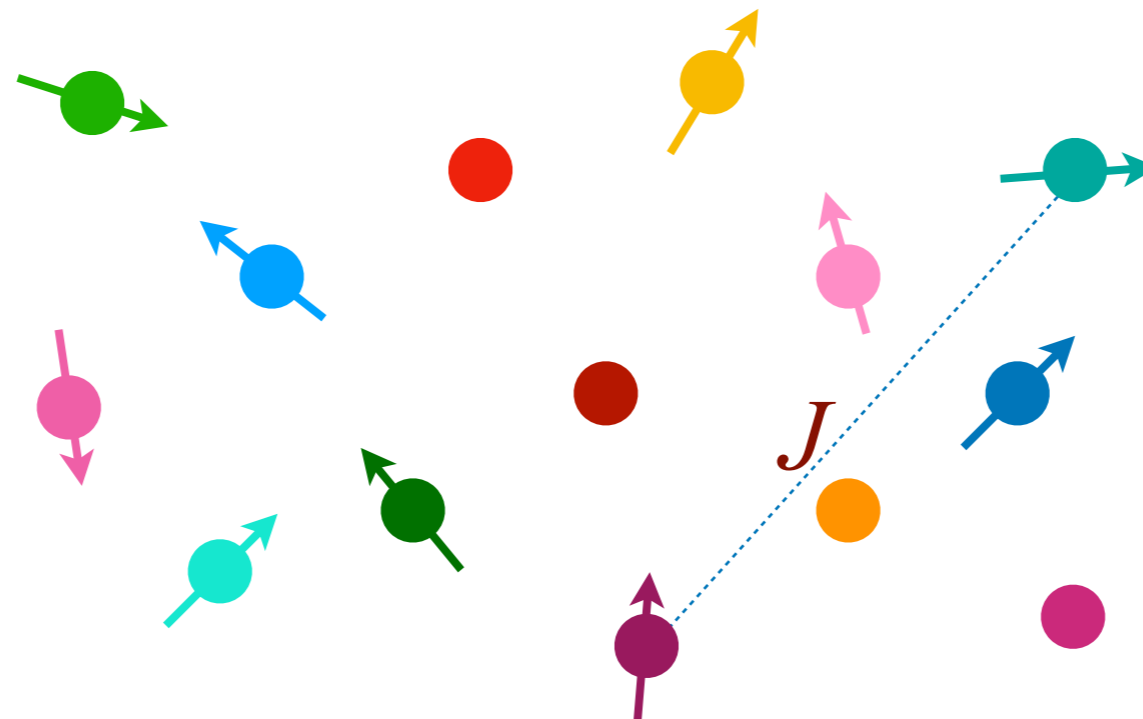
$$\text{---} \uparrow \\ c_{\uparrow}^\dagger |0\rangle$$

$$\text{---} \downarrow \\ c_{\downarrow}^\dagger |0\rangle$$

Random t - J model

$$H = -\frac{1}{\sqrt{N}} \sum_{i,j=1}^N t_{ij} c_{i\alpha}^\dagger c_{j\alpha} + \frac{1}{\sqrt{N}} \sum_{i<j=1}^N J_{ij} \vec{S}_i \cdot \vec{S}_j$$

We consider the hole-doped case, with no double occupancy.





Henry Shackleton



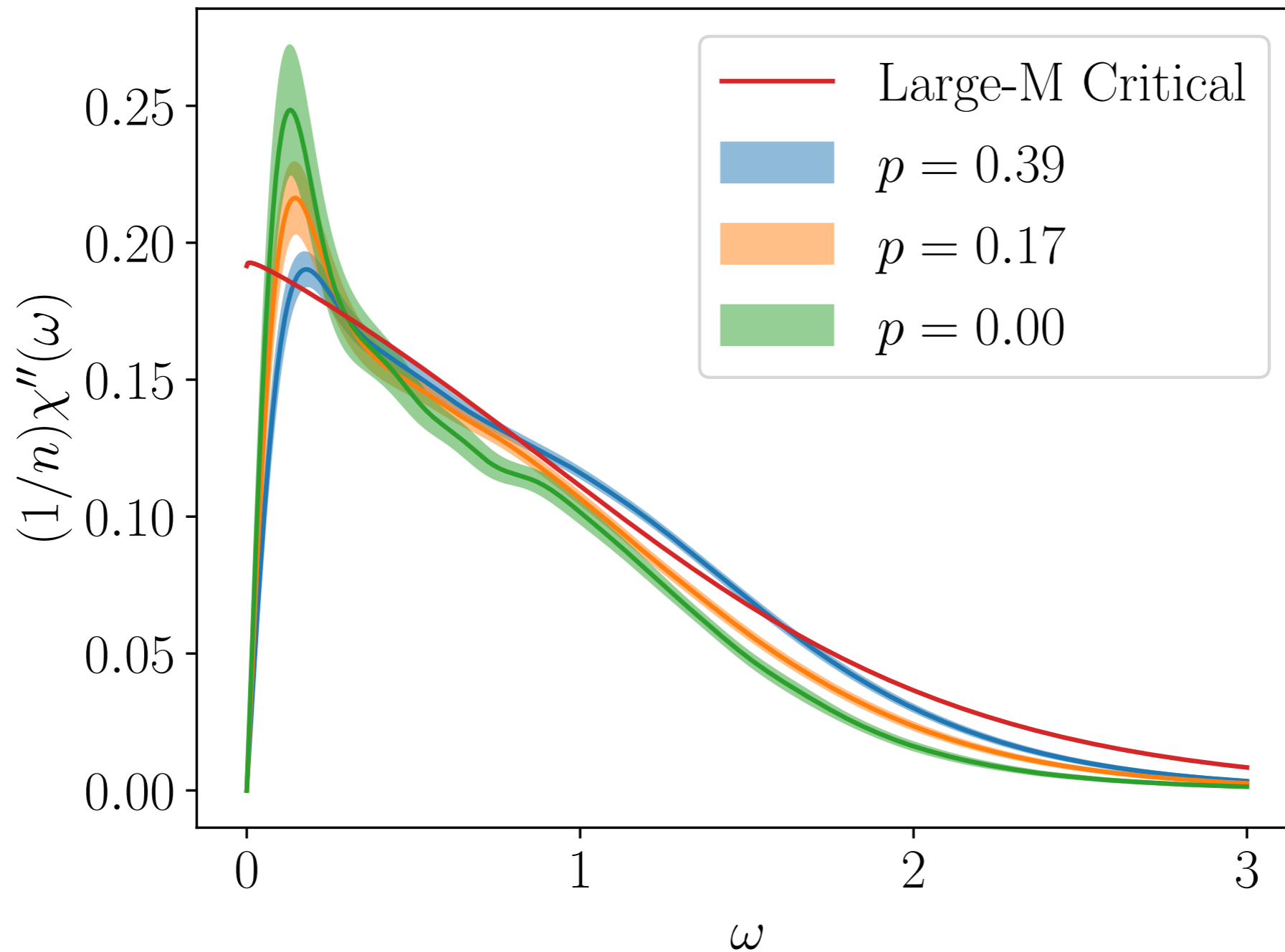
Alexander Wietek



Antoine Georges

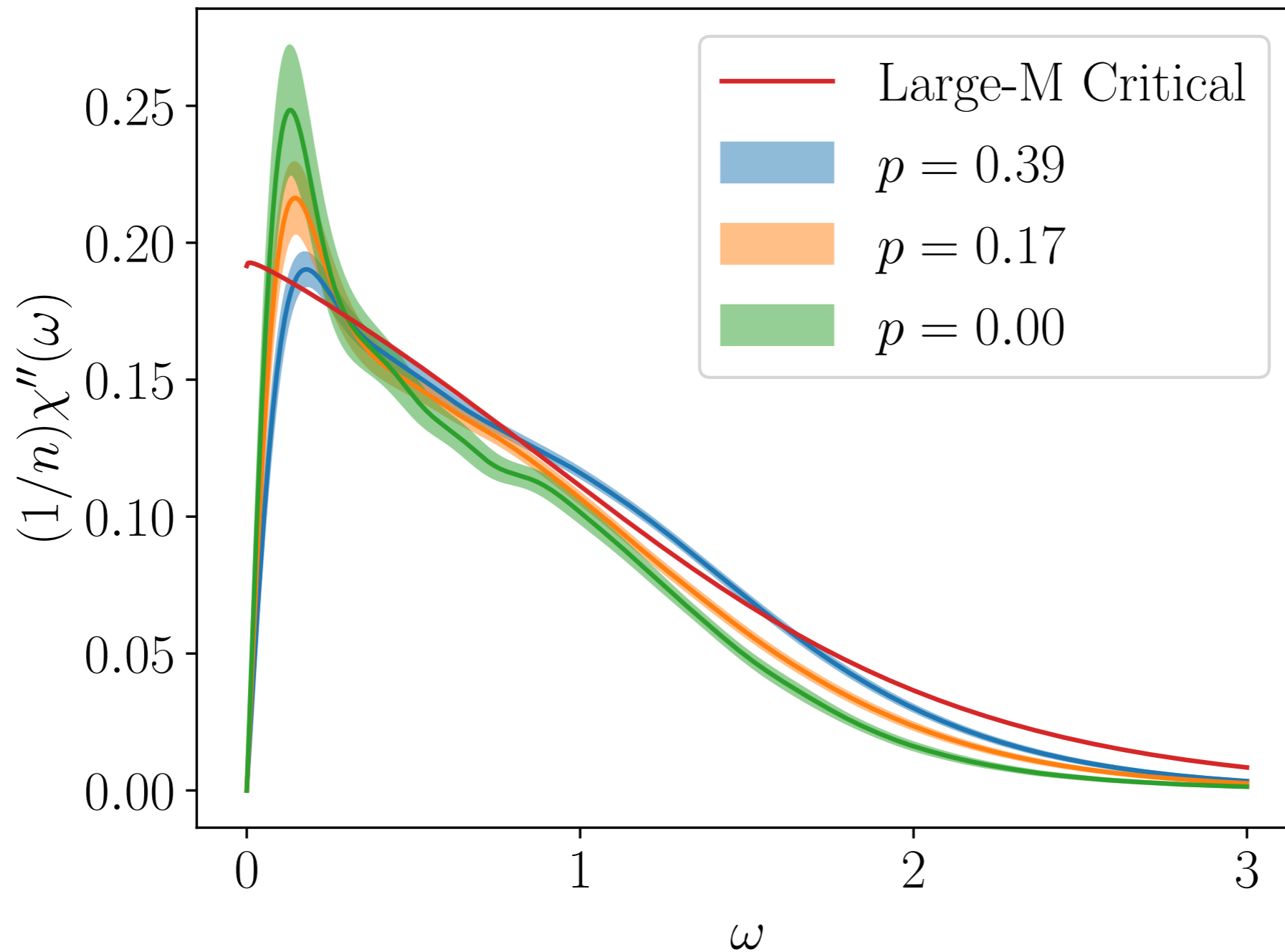
[arXiv:2012.06589](https://arxiv.org/abs/2012.06589)

Numerical exact diagonalization and SYK theory



Evidence for a quantum critical point at $p = p_c \approx 0.3$
with SYK criticality.
Spin glass order for $p < p_c$

Numerical exact diagonalization and SYK theory



Numerics matches many other observations, including the breakdown of the Luttinger-volume Fermi surface for $p < p_c$, and Planckian dissipation at scale $\hbar/(k_B T)$.

**Quantum
entanglement**

**Black
holes**

**A simple
qubit
model**

**Copper-based
superconductors**

**Quantum
entanglement**

**Black
holes**

2D-gravity theory
of charged black holes

**A simple
qubit
model**

**Copper-based
superconductors**

Quantum
entanglement

Black
holes

A simple
qubit
model

SYK criticality
near $p = p_c$

Copper-based
superconductors