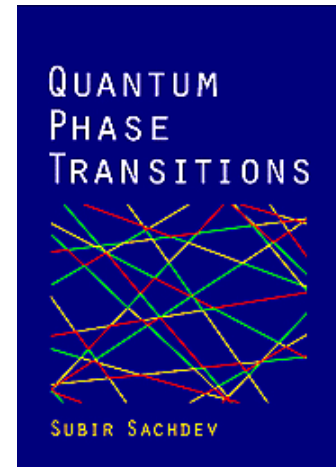


# Quantum phase transitions

- <http://onsager.physics.yale.edu/c41.pdf>
- cond-mat/0109419

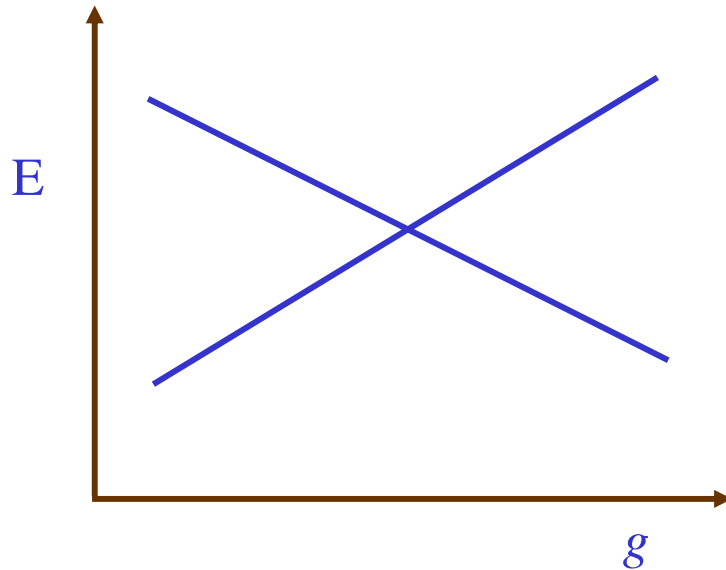


*Quantum Phase Transitions*  
Cambridge University Press

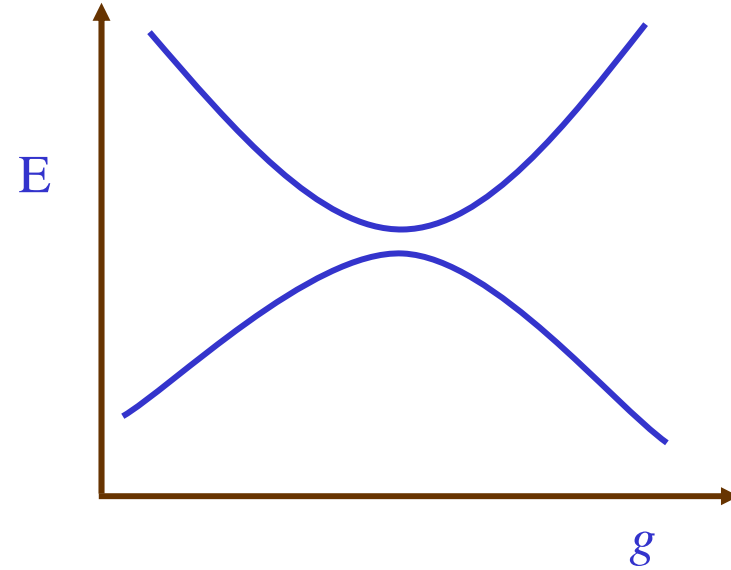


# What is a quantum phase transition ?

Non-analyticity in ground state properties as a function of some control parameter  $g$

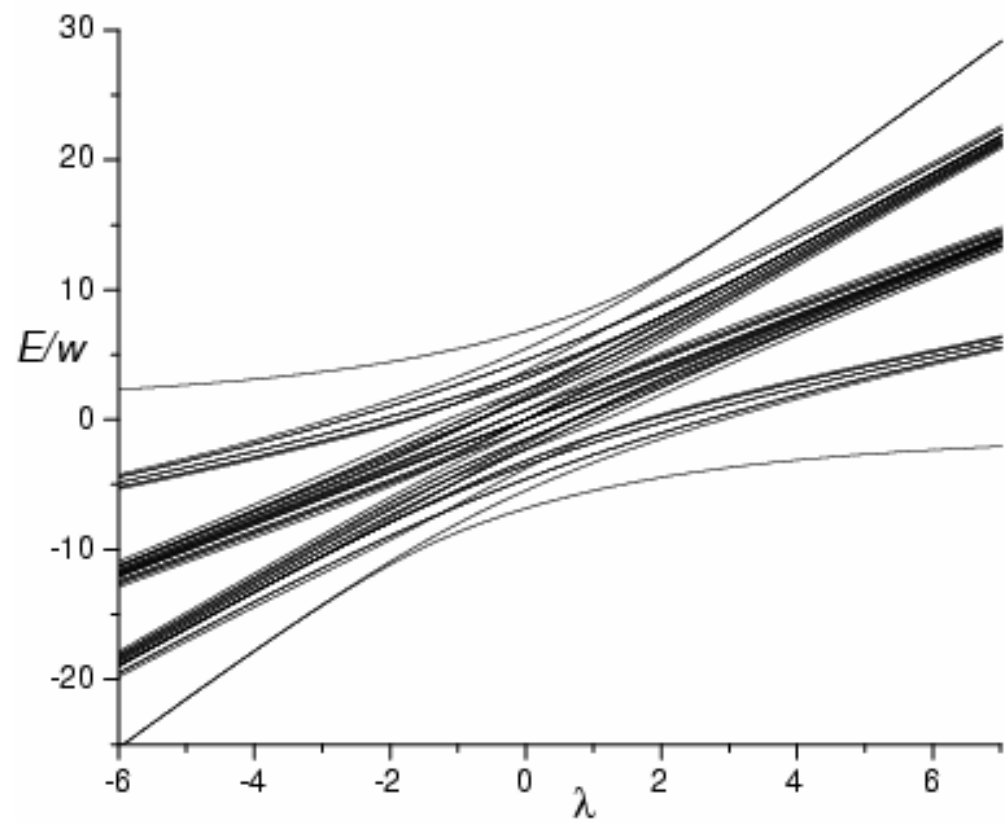


True level crossing:  
Usually a *first-order* transition

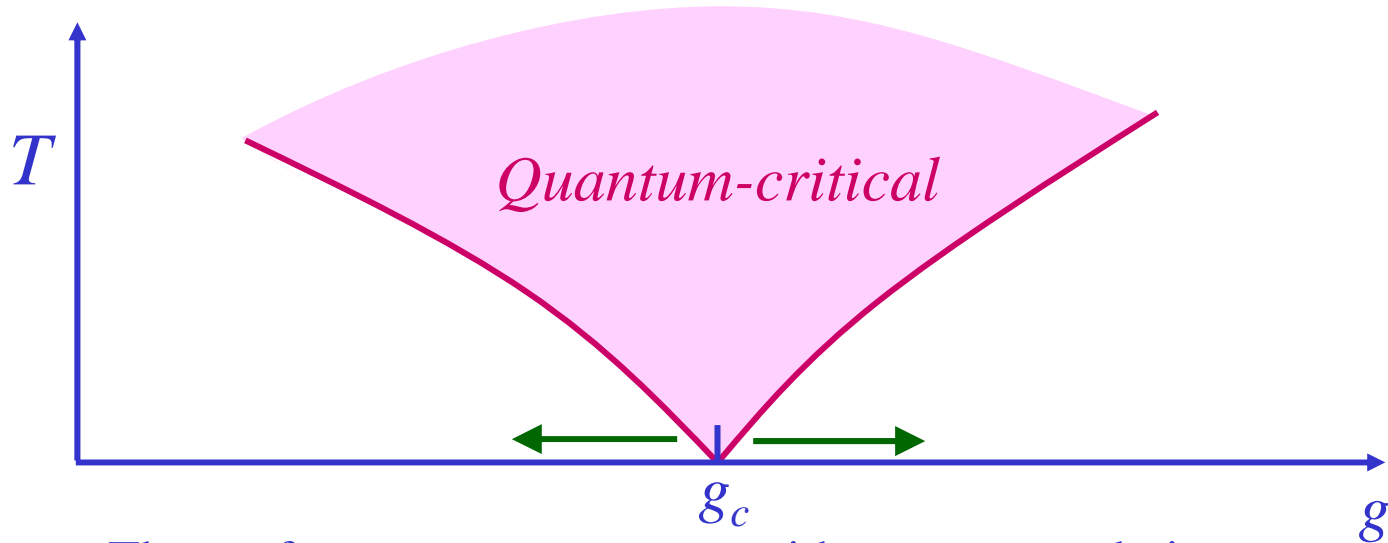


Avoided level crossing which becomes sharp in the infinite volume limit:

*second-order* transition



## Why study quantum phase transitions ?



- Theory for a quantum system with strong correlations: describe phases on either side of  $g_c$  by expanding in deviation from the quantum critical point.
- Critical point is a novel state of matter without quasiparticle excitations
- Critical excitations control dynamics in the wide *quantum-critical* region at non-zero temperatures.

Important property of ground state at  $g=g_c$  :  
temporal and spatial scale invariance;  
characteristic energy scale at other values of  $g$ :  $\Delta \sim |g - g_c|^{z\nu}$

# Outline

- I. Quantum Ising Chain
- II. Landau-Ginzburg-Wilson theory  
Mean field theory and the evolution of the excitation spectrum.
- III. Superfluid-insulator transition  
Boson Hubbard model at integer filling.
- IV. Tilting the Mott insulator  
Density wave order at an Ising transition.
- V. Bosons at fractional filling  
Beyond the Landau-Ginzburg-Wilson paradigm.

# I. Quantum Ising Chain

# I. Quantum Ising Chain

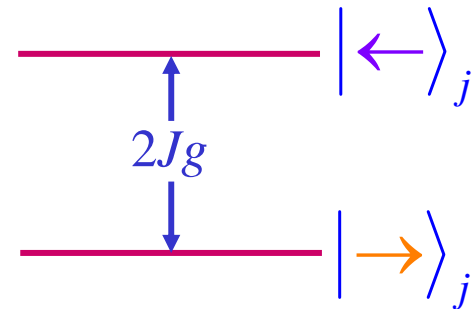
Degrees of freedom:  $j = 1 \dots N$  qubits,  $N$  "large"

$$|\uparrow\rangle_j, |\downarrow\rangle_j$$

$$\text{or } |\rightarrow\rangle_j = \frac{1}{\sqrt{2}} (|\uparrow\rangle_j + |\downarrow\rangle_j), \quad |\leftarrow\rangle_j = \frac{1}{\sqrt{2}} (|\uparrow\rangle_j - |\downarrow\rangle_j)$$

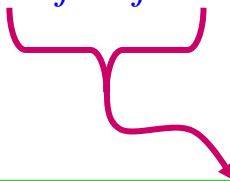
Hamiltonian of decoupled qubits:

$$H_0 = -Jg \sum_j \sigma_j^x$$



Coupling between qubits:

$$H_1 = -J \sum_j \sigma_j^z \sigma_{j+1}^z$$


$$\left( \left| \rightarrow \right\rangle_j \left\langle \leftarrow \right| + \left| \leftarrow \right\rangle_j \left\langle \rightarrow \right| \right) \left( \left| \rightarrow \right\rangle_{j+1} \left\langle \leftarrow \right| + \left| \leftarrow \right\rangle_{j+1} \left\langle \rightarrow \right| \right)$$

Prefers neighboring qubits

are *either*  $\left| \uparrow \right\rangle_j \left| \uparrow \right\rangle_{j+1}$  *or*  $\left| \downarrow \right\rangle_j \left| \downarrow \right\rangle_{j+1}$

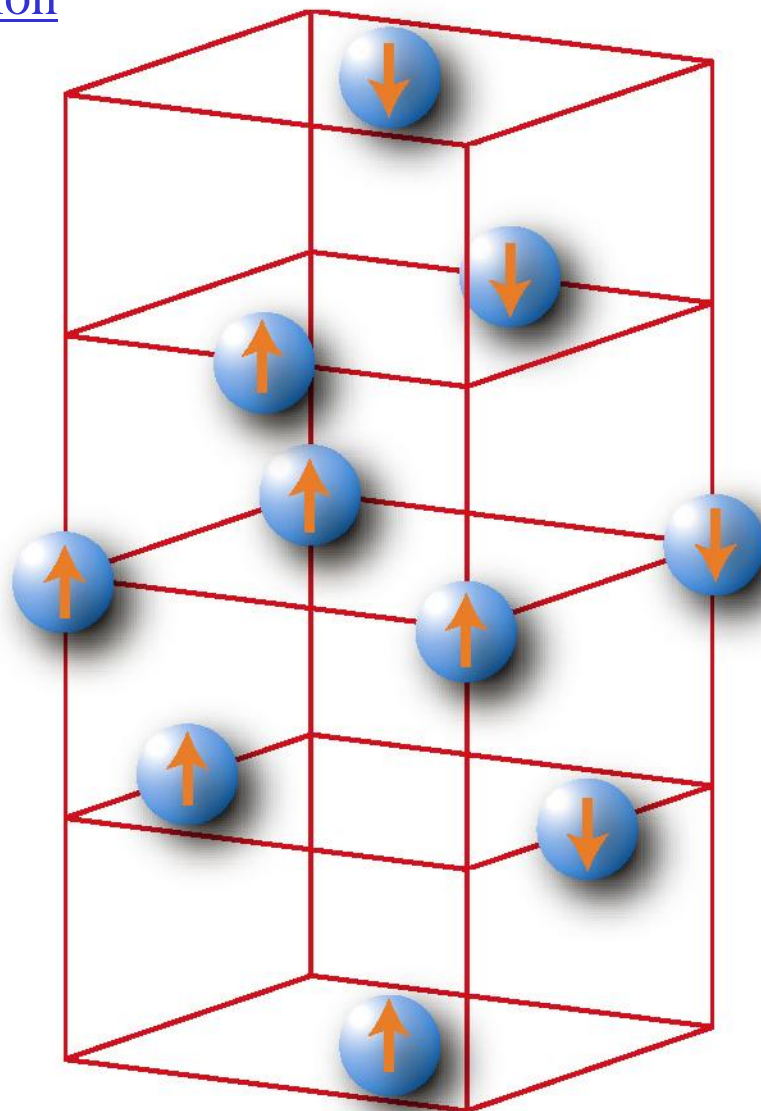
(not entangled)

Full Hamiltonian

$$H = H_0 + H_1 = -J \sum_j \left( g \sigma_j^x + \sigma_j^z \sigma_{j+1}^z \right)$$

leads to entangled states at  $g$  of order unity

Experimental realization



# Weakly-coupled qubits ( $g \gg 1$ )

Ground state:

$$|G\rangle = |\cdots \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \cdots\rangle$$

$$-\frac{1}{2g} |\cdots \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \leftarrow \leftarrow \rightarrow \rightarrow \rightarrow \rightarrow \cdots\rangle - \cdots$$

Lowest excited states:

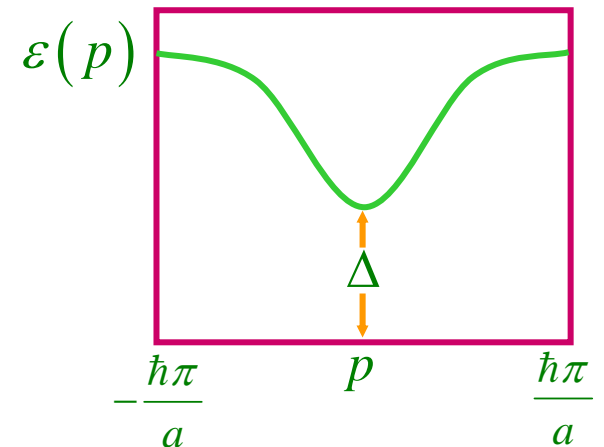
$$|\ell_j\rangle = |\cdots \rightarrow \rightarrow \rightarrow \rightarrow \leftarrow_j \rightarrow \rightarrow \rightarrow \rightarrow \cdots\rangle + \cdots$$

Coupling between qubits creates “flipped-spin” *quasiparticle* states at momentum  $p$

$$|p\rangle = \sum_j e^{ipx_j/\hbar} |\ell_j\rangle$$

$$\text{Excitation energy } \varepsilon(p) = \Delta + 4J \sin^2\left(\frac{pa}{2\hbar}\right) + O(g^{-1})$$

$$\text{Excitation gap } \Delta = 2gJ - 2J + O(g^{-1})$$

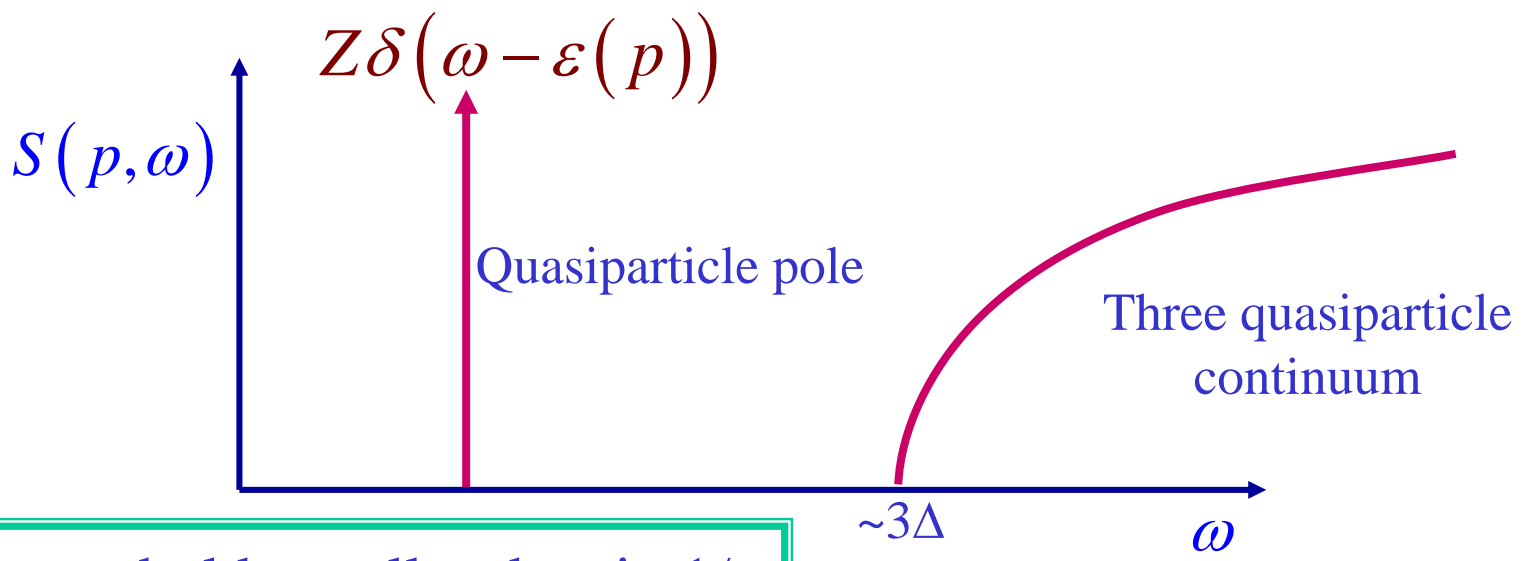


Entire spectrum can be constructed out of multi-quasiparticle states

Dynamic Structure Factor  $S(p, \omega)$ :

Weakly-coupled qubits ( $g \gg 1$ )

Cross-section to flip a  $|\rightarrow\rangle$  to a  $|\leftarrow\rangle$  (or vice versa) while transferring energy  $\hbar\omega$  and momentum  $p$



Structure holds to all orders in  $1/g$

At  $T > 0$ , collisions between quasiparticles broaden pole to a Lorentzian of width  $1/\tau_\phi$  where the **phase coherence time**  $\tau_\phi$

is given by

$$\frac{1}{\tau_\phi} = \frac{2k_B T}{\pi \hbar} e^{-\Delta/k_B T}$$

# Strongly-coupled qubits ( $g \ll 1$ )

Ground states:

$$|G \uparrow\rangle = |\dots \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \dots\rangle$$

$$-\frac{g}{2} |\dots \uparrow \uparrow \uparrow \uparrow \downarrow \uparrow \uparrow \uparrow \uparrow \uparrow \dots\rangle - \dots$$

Ferromagnetic moment

$$N_0 = \langle G | \sigma^z | G \rangle \neq 0$$

Second state  $|G \downarrow\rangle$  obtained by  $\uparrow \Leftrightarrow \downarrow$

$|G \downarrow\rangle$  and  $|G \uparrow\rangle$  mix only at order  $g^N$

Lowest excited states: domain walls

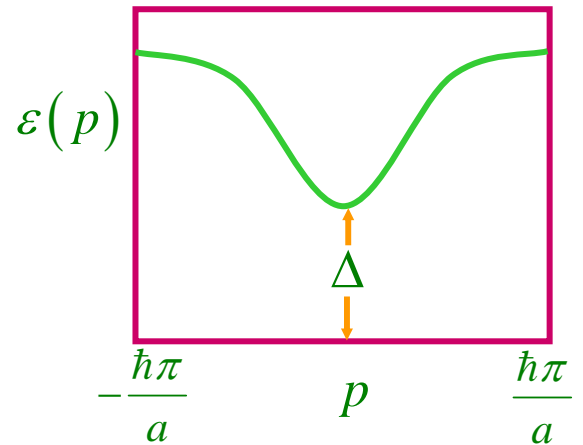
$$|d_j\rangle = |\dots \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow_j \downarrow \downarrow \downarrow \downarrow \downarrow \dots\rangle + \dots$$

Coupling between qubits creates new “domain-wall” *quasiparticle* states at momentum  $p$

$$|p\rangle = \sum_j e^{ipx_j/\hbar} |d_j\rangle$$

Excitation energy  $\varepsilon(p) = \Delta + 4Jg \sin^2\left(\frac{pa}{2\hbar}\right) + O(g^2)$

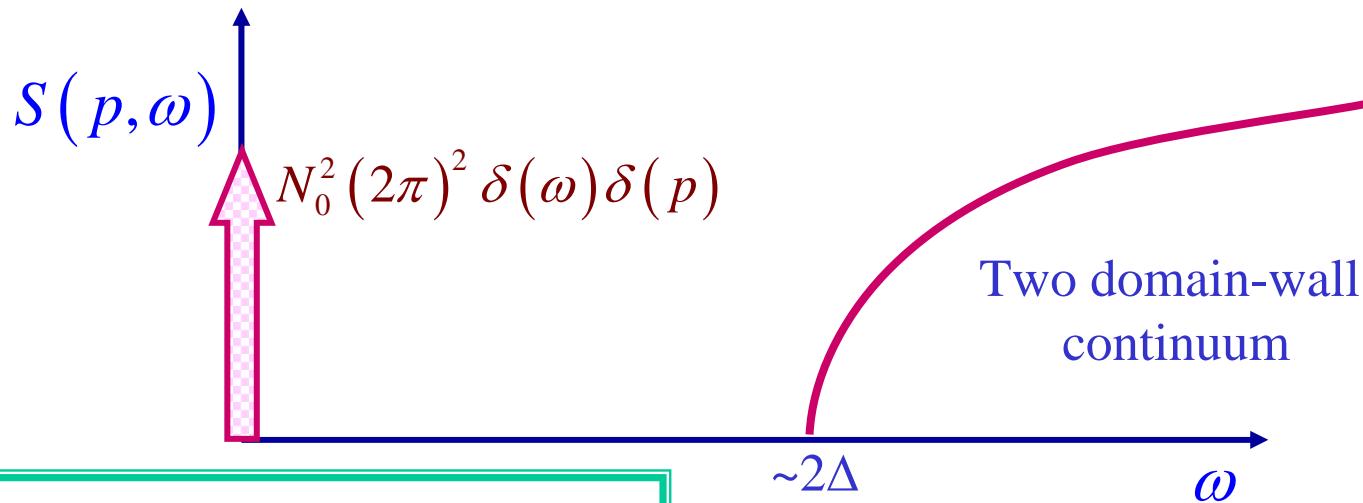
Excitation gap  $\Delta = 2J - 2gJ + O(g^2)$



Dynamic Structure Factor  $S(p, \omega)$ :

Strongly-coupled qubits ( $g \ll 1$ )

Cross-section to flip a  $|\rightarrow\rangle$  to a  $|\leftarrow\rangle$  (or vice versa)  
while transferring energy  $\hbar\omega$  and momentum  $p$



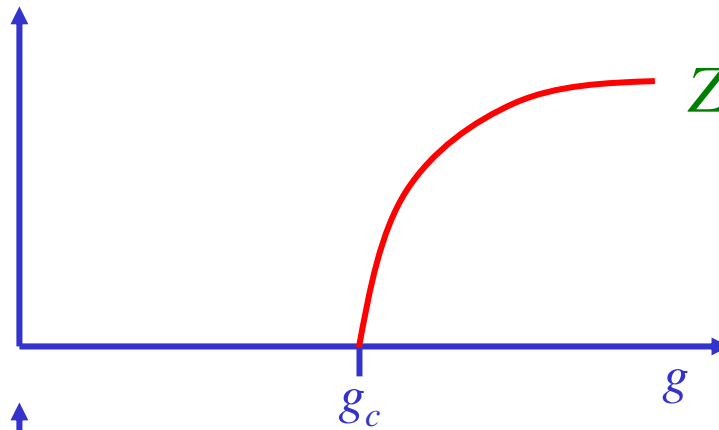
Structure holds to all orders in  $g$

At  $T > 0$ , motion of domain walls leads to a finite *phase coherence time*  $\tau_\phi$ ,

and broadens coherent peak to a width  $1/\tau_\phi$  where 
$$\frac{1}{\tau_\phi} = \frac{2k_B T}{\pi \hbar} e^{-\Delta/k_B T}$$

## Entangled states at $g$ of order unity

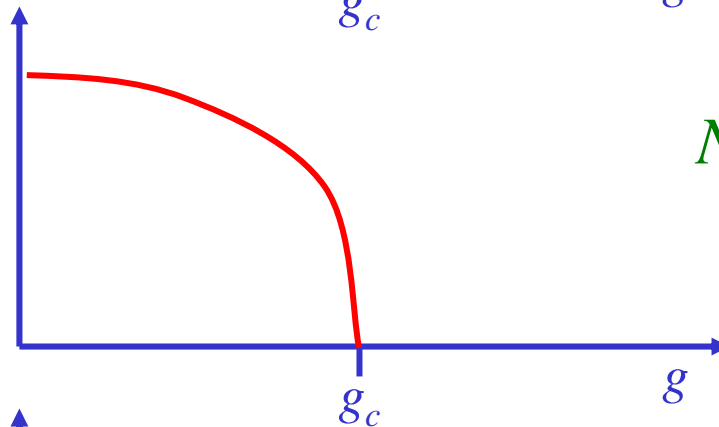
“Flipped-spin”  
Quasiparticle  
weight  $Z$



$$Z \sim (g - g_c)^{1/4}$$

A.V. Chubukov, S. Sachdev, and J. Ye,  
*Phys. Rev. B* **49**, 11919 (1994)

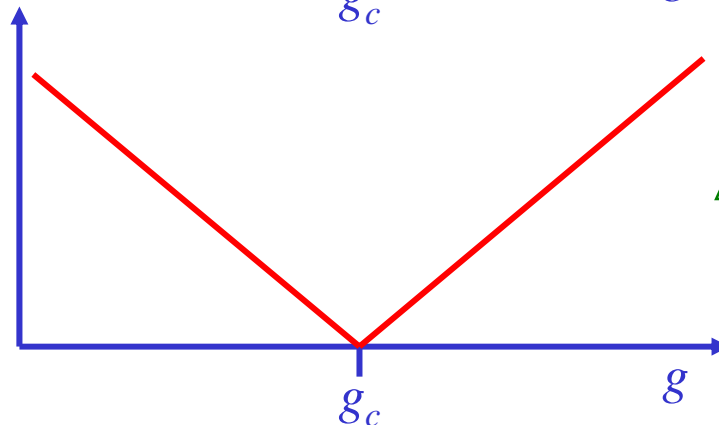
Ferromagnetic  
moment  $N_0$



$$N_0 \sim (g_c - g)^{1/8}$$

P. Pfeuty *Annals of Physics*, **57**, 79 (1970)

Excitation  
energy gap  $\Delta$

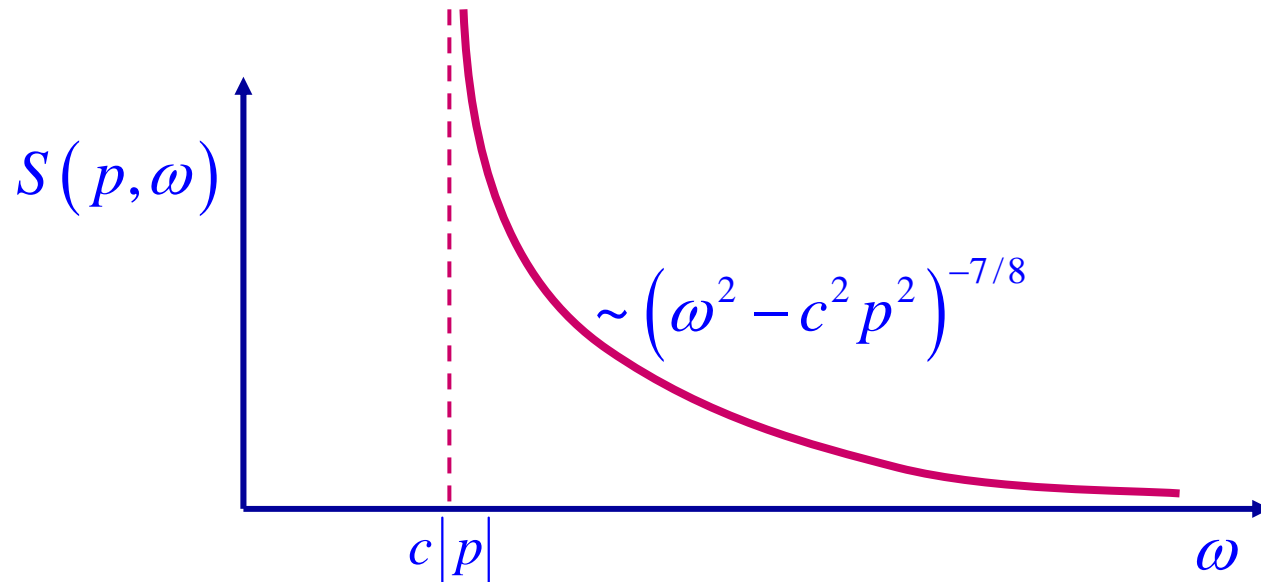


$$\Delta \sim |g - g_c|$$

Dynamic Structure Factor  $S(p, \omega)$ :

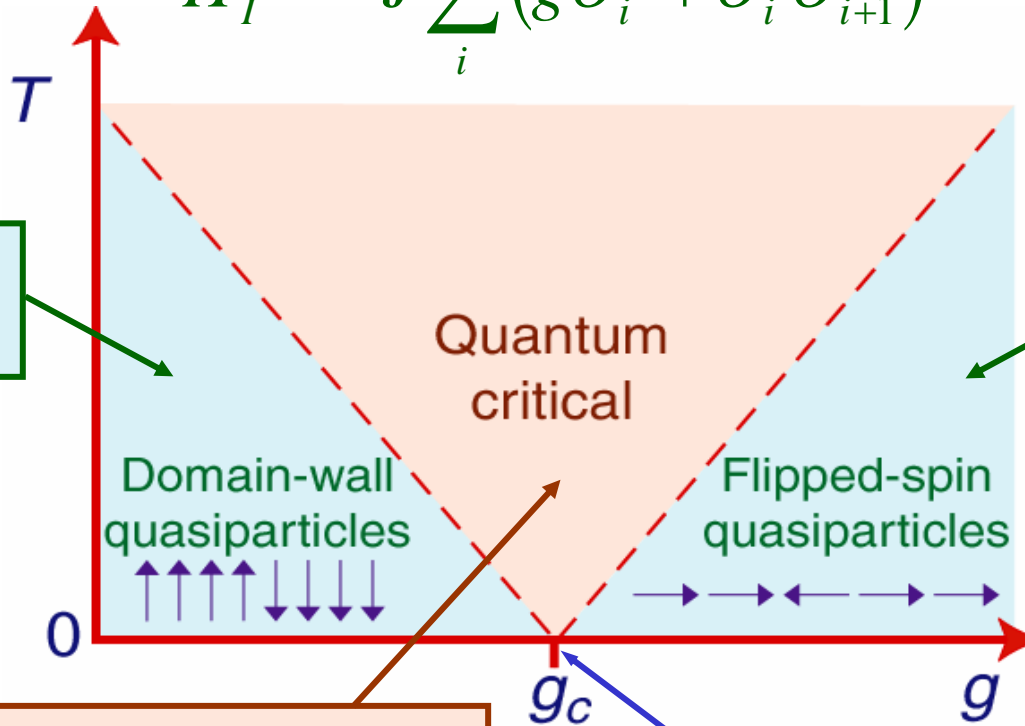
Critical coupling ( $g = g_c$ )

Cross-section to flip a  $|\rightarrow\rangle$  to a  $|\leftarrow\rangle$  (or vice versa)  
while transferring energy  $\hbar\omega$  and momentum  $p$



No quasiparticles --- dissipative critical continuum

$$H_I = -J \sum_i (g \sigma_i^x + \sigma_i^z \sigma_{i+1}^z)$$



Quasiclassical dynamics

Quasiclassical dynamics

$$\chi(\omega) = \frac{i}{\hbar} \sum_k \int_0^\infty dt \langle [\sigma_j^z(t), \sigma_k^z(0)] \rangle e^{i\omega t}$$

$$= \frac{A}{T^{7/4} (1 - i\omega/\Gamma_R + \dots)}$$

$$\Gamma_R = \left( 2 \tan \frac{\pi}{16} \right) \frac{k_B T}{\hbar}$$

$$\langle \sigma_j^z \sigma_k^z \rangle \sim \frac{1}{|j-k|^{1/4}}$$

P. Pfeuty *Annals of Physics*, **57**, 79 (1970)

S. Sachdev and J. Ye, *Phys. Rev. Lett.* **69**, 2411 (1992).  
 S. Sachdev and A.P. Young, *Phys. Rev. Lett.* **78**, 2220 (1997).

# Outline

- I. Quantum Ising Chain
- II. Landau-Ginzburg-Wilson theory  
Mean field theory and the evolution of the excitation spectrum.
- III. Superfluid-insulator transition  
Boson Hubbard model at integer filling.
- IV. Tilting the Mott insulator  
Density wave order at an Ising transition.
- V. Bosons at fractional filling  
Beyond the Landau-Ginzburg-Wilson paradigm.

## II. Landau-Ginzburg-Wilson theory

*Mean field theory and the evolution of the  
excitation spectrum*

- Identify order parameter  $\phi(x, \tau) \sim \sigma_j^z$
- Symmetries:

$$\text{Spin inversion:} \quad \phi \rightarrow -\phi$$

$$\text{Time reversal} \quad \tau \rightarrow -\tau$$

$$\text{Spatial inversion} \quad x \rightarrow -x$$

- Write down most general Lagrangian consistent with symmetries

$$\mathcal{Z} = \int \mathcal{D}\phi(x, \tau) \exp \left( - \int d^d x \int d\tau \mathcal{L}[\phi] \right)$$

$$\mathcal{L}[\phi] = \frac{1}{2} (\partial_\tau \phi)^2 + \frac{c^2}{2} (\nabla_x \phi)^2 + \frac{r}{2} \phi^2 + \frac{u}{4} \phi^4 + \dots$$

- Identify phases at  $r \gg 0$  and  $r \ll 0$  with the paramagnet and the ferromagnet respectively.

Quantum field theory formally resembles the classical statistical mechanics of an Ising model in  $d + 1$  dimensions. Theory of second-order classical phase transitions implies that at the critical point the susceptibility depends on the  $d + 1$  dimensional momentum  $k$  as

$$\chi(k) \sim \frac{1}{k^{2-\eta}}$$

After analytic continuation, and using the “Lorentz invariance” of the critical theory, the quantum critical point therefore has the following dynamic susceptibility at  $T = 0$ .

$$\chi(p, \omega) \sim \frac{1}{(c^2 p^2 - \omega^2)^{1-\eta/2}}$$

At  $T > 0$ , we have to consider a classical statistical mechanics problem in finite geometry with a ‘temporal’ direction of extent  $L_\tau = \hbar/(k_B T)$ . *Finite size scaling* now implies that the susceptibility at the critical point obeys

$$\chi(k) \sim L_\tau^{2-\eta} F(kL_\tau)$$

After analytic continuation, the quantum system has the dynamic response (note: can no longer use “Lorentz invariance”)

$$\chi''(0, \omega) \sim \frac{1}{T^{2-\eta}} \Phi\left(\frac{\hbar\omega}{k_B T}\right)$$

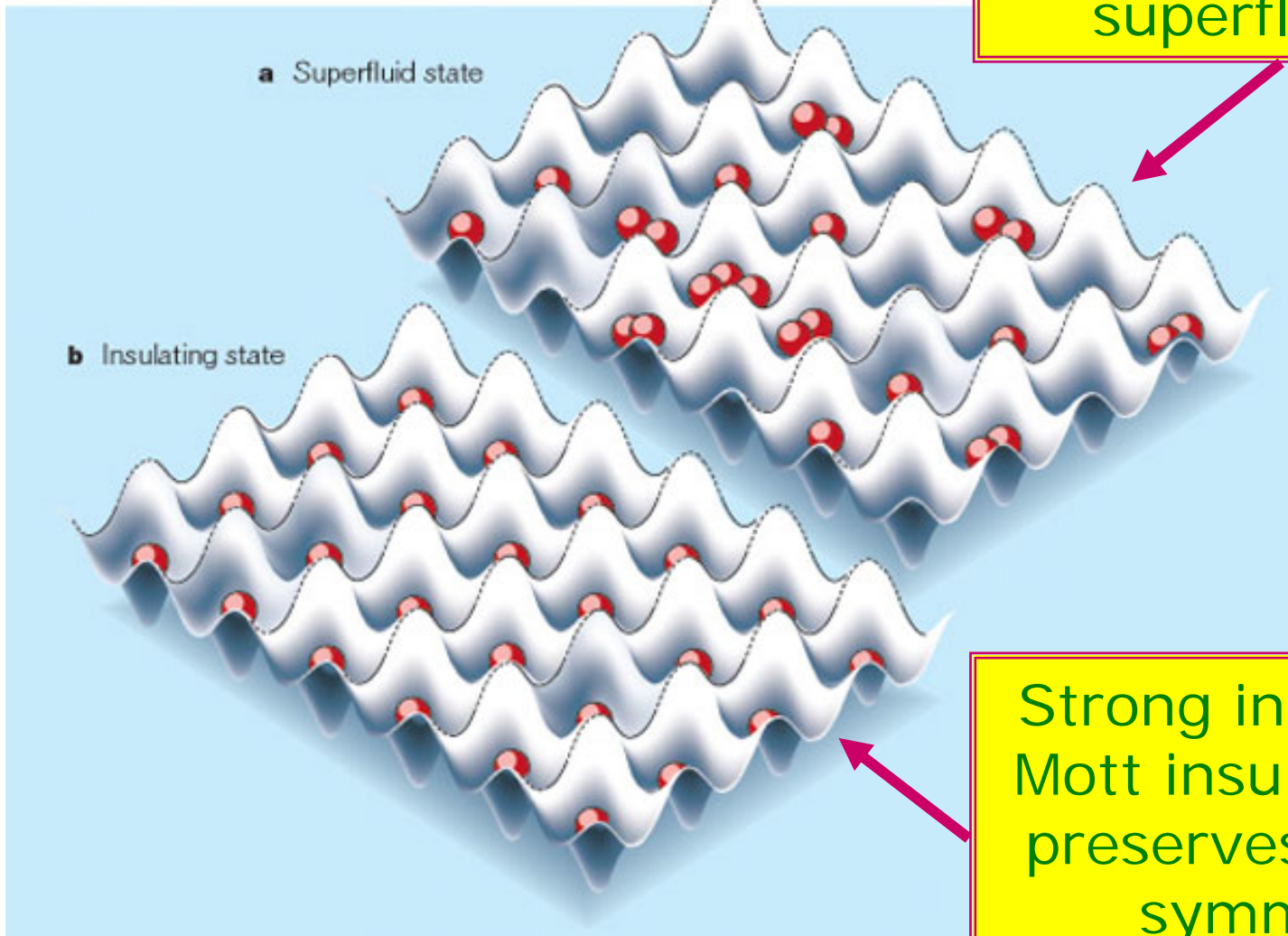
## Outline

- I. Quantum Ising Chain
- II. Landau-Ginzburg-Wilson theory  
Mean field theory and the evolution of the excitation spectrum.
- III. Superfluid-insulator transition  
Boson Hubbard model at integer filling.
- IV. Tilting the Mott insulator  
Density wave order at an Ising transition.
- V. Bosons at fractional filling  
Beyond the Landau-Ginzburg-Wilson paradigm.

### III. Superfluid-insulator transition

*Boson Hubbard model at integer filling*

# Bosons at density $f = 1$



Weak interactions:  
superfluidity

Strong interactions:  
Mott insulator which  
preserves all lattice  
symmetries

LGW theory: continuous quantum transitions between these states

M. Greiner, O. Mandel, T. Esslinger, T. W. Hänsch, and I. Bloch, *Nature* **415**, 39 (2002).

# I. The Superfluid-Insulator transition

## Boson Hubbard model

Degrees of freedom: Bosons,  $b_j^\dagger$ , hopping between the sites,  $j$ , of a lattice, with short-range repulsive interactions.

$$H = -t \sum_{\langle ij \rangle} b_i^\dagger b_j - \mu \sum_j n_j + \frac{U}{2} \sum_j n_j (n_j - 1) + \dots$$

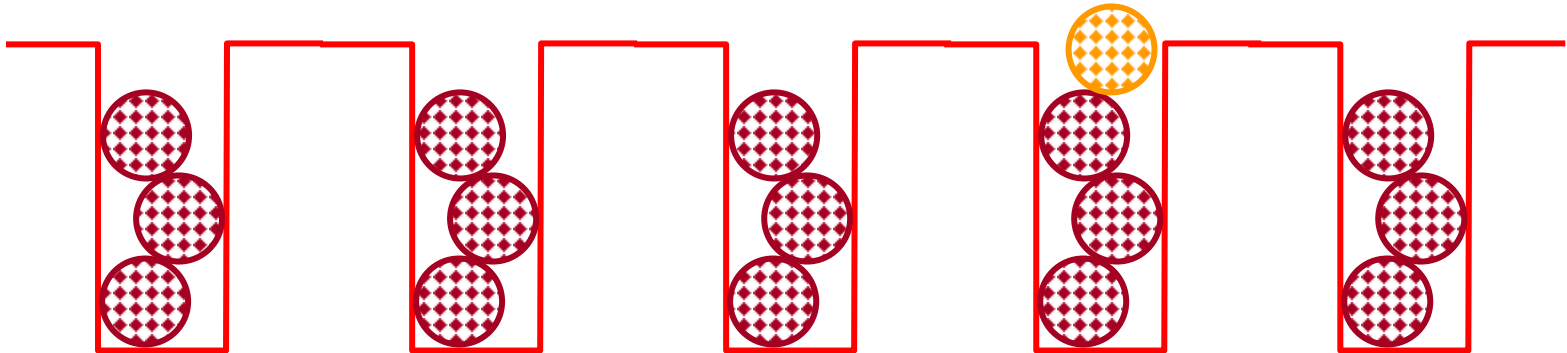
$$n_j \equiv b_j^\dagger b_j$$

M.P.A. Fisher, P.B. Weichmann,  
G. Grinstein, and D.S. Fisher  
*Phys. Rev. B* **40**, 546 (1989).

For small  $U/t$ , ground state is a superfluid BEC with  
superfluid density  $\approx$  density of bosons

## What is the ground state for large $U/t$ ?

Typically, the ground state remains a superfluid, but with  
superfluid density  $\ll$  density of bosons

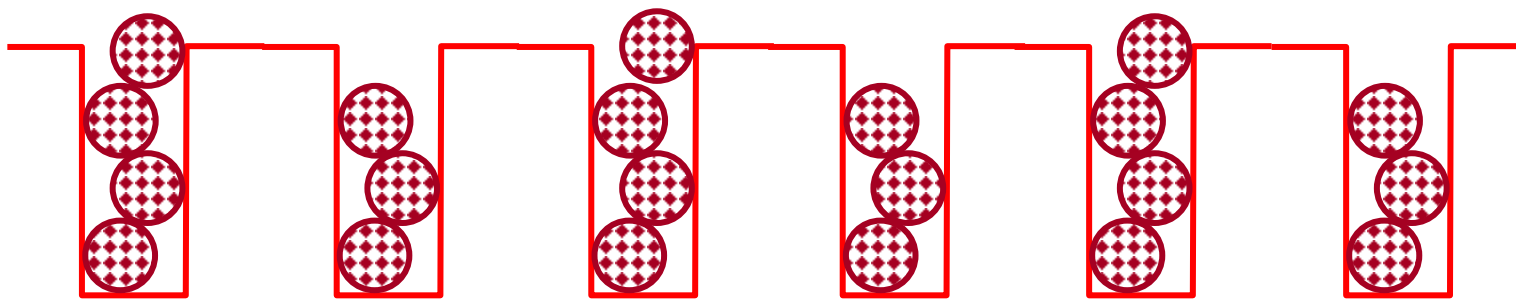
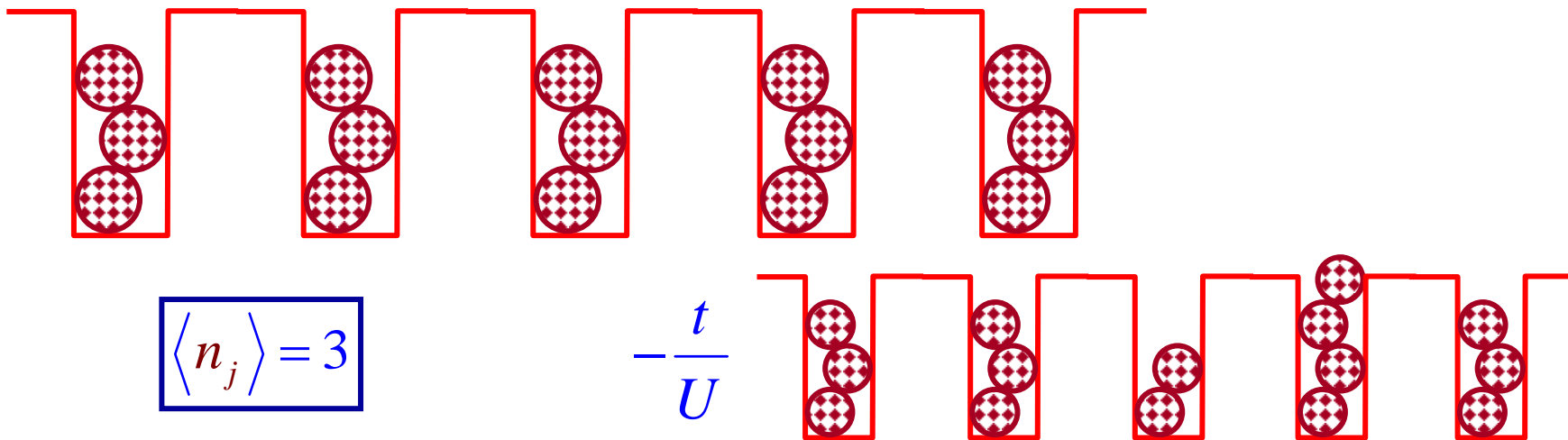


The superfluid density evolves smoothly from large values at small  $U/t$ , to small values at large  $U/t$ , and there is no quantum phase transition at any intermediate value of  $U/t$ .

(In systems with Galilean invariance and at zero temperature, superfluid density=density of bosons always, independent of the strength of the interactions)

## What is the ground state for large $U/t$ ?

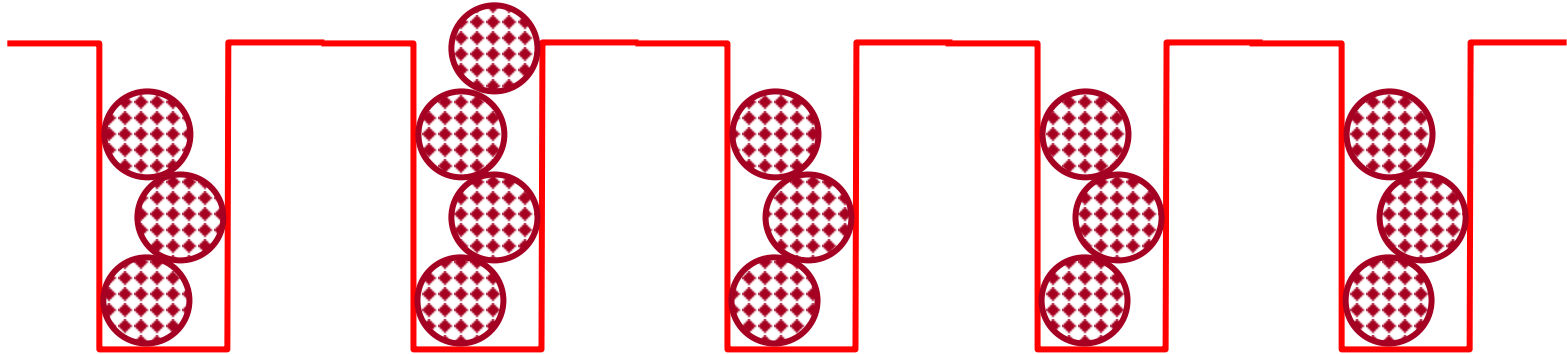
Incompressible, insulating ground states, with zero superfluid density, appear at special commensurate densities



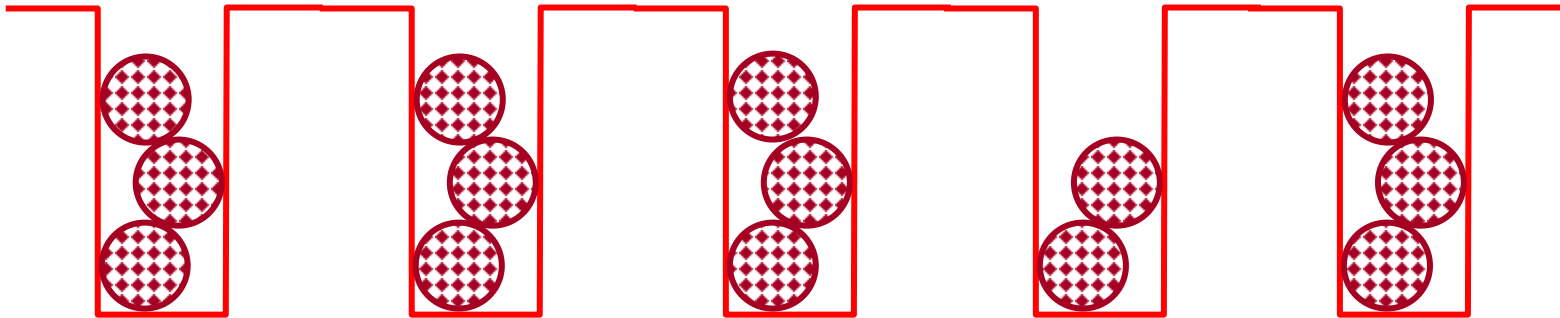
$$\langle n_j \rangle = 7/2$$

Ground state has “density wave” order, which spontaneously breaks lattice symmetries

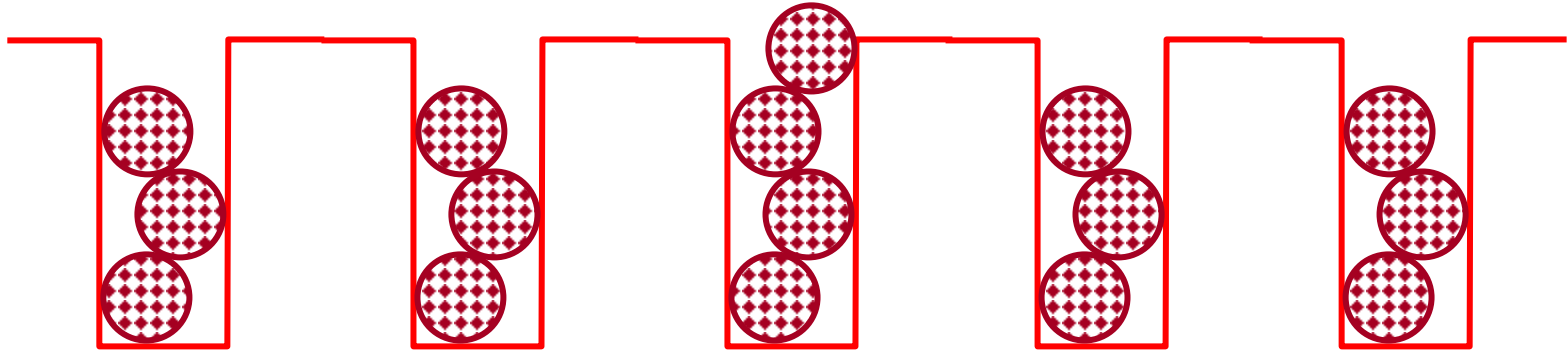
Excitations of the insulator: infinitely long-lived, finite energy  
*quasiparticles and quasiholes*



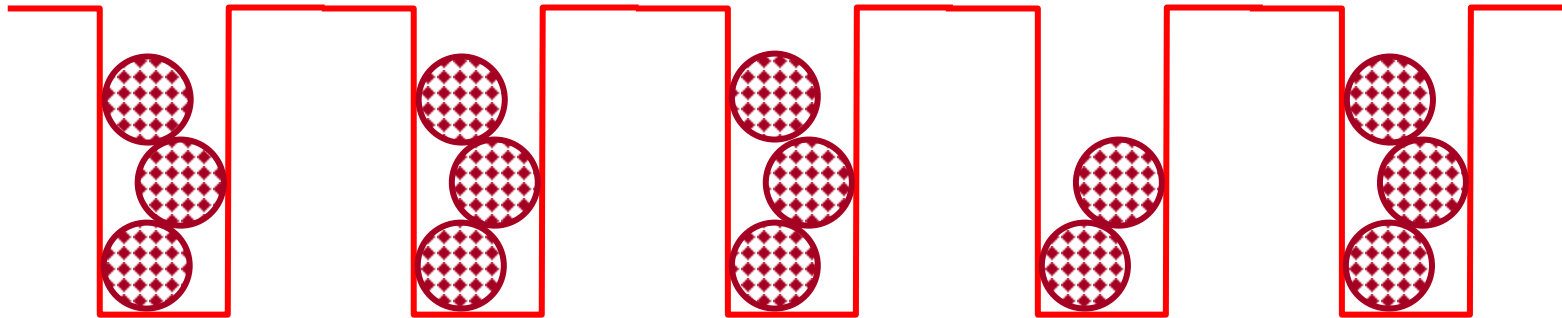
Energy of quasi-particles/holes:  $\varepsilon_{p,h}(p) = \Delta_{p,h} + \frac{p^2}{2m_{p,h}^*}$



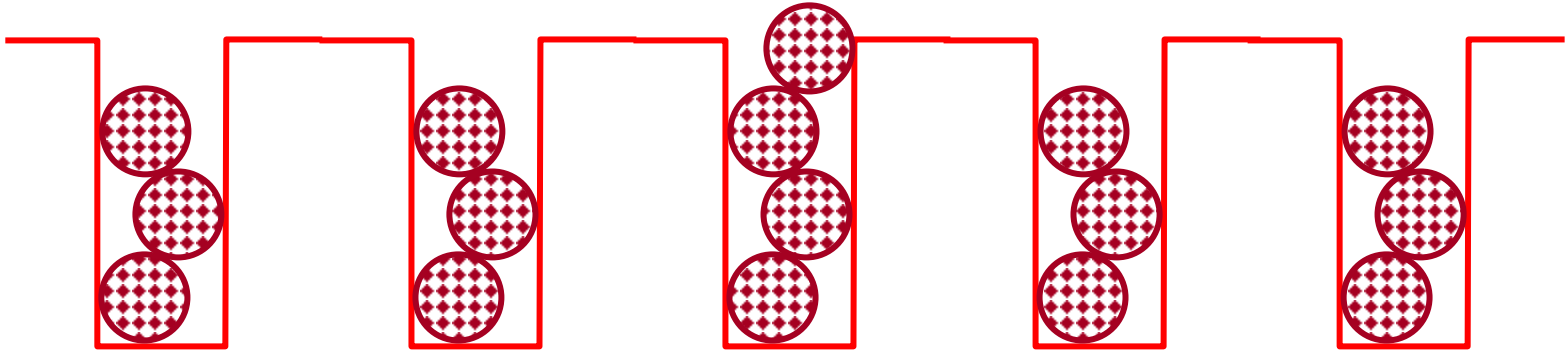
Excitations of the insulator: infinitely long-lived, finite energy  
*quasiparticles and quasiholes*



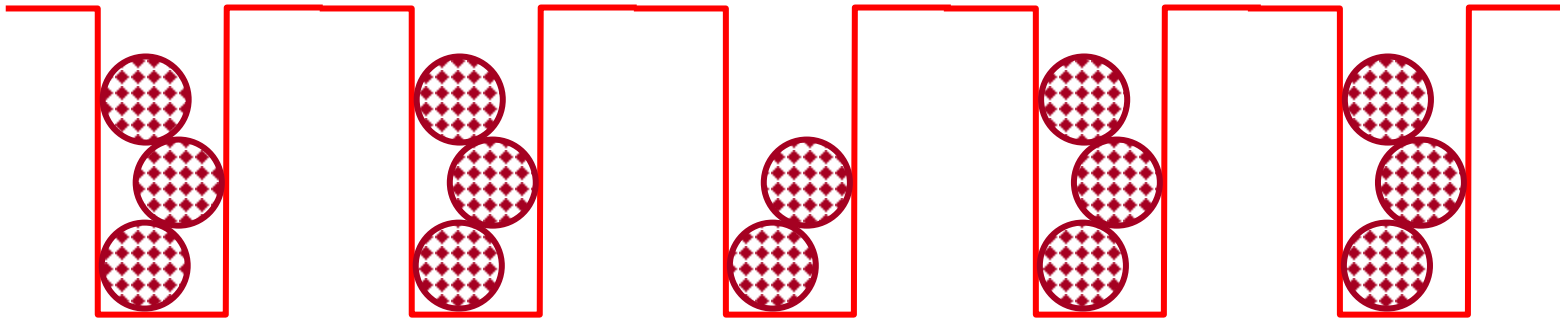
Energy of quasi-particles/holes:  $\varepsilon_{p,h}(p) = \Delta_{p,h} + \frac{p^2}{2m_{p,h}^*}$



Excitations of the insulator: infinitely long-lived, finite energy  
*quasiparticles and quasiholes*



Energy of quasi-particles/holes:  $\varepsilon_{p,h}(p) = \Delta_{p,h} + \frac{p^2}{2m_{p,h}^*}$



## LGW theory of the superfluid insulator transition

- Identify order parameter  $\Psi(x, \tau) \sim b_j^\dagger$
- Symmetries:

$$\text{Gauge invariance:} \quad \Psi \rightarrow \Psi e^{i\theta}$$

$$\text{Time reversal} \quad \tau \rightarrow -\tau \quad ; \quad \Psi \rightarrow \Psi^*$$

$$\text{Spatial inversion} \quad x \rightarrow -x$$

- Write down most general Lagrangian consistent with symmetries

$$\mathcal{Z} = \int \mathcal{D}\Psi(x, \tau) \exp \left( - \int d^d x \int d\tau \mathcal{L}[\Psi] \right)$$
$$\mathcal{L}[\Psi] = K \Psi^* \frac{\partial \Psi}{\partial \tau} + |\partial_\tau \Psi|^2 + c^2 |\nabla_x \Psi|^2 + r |\Psi|^2 + \frac{u}{2} |\Psi|^4 + \dots$$

- Identify phases at  $r \gg 0$  and  $r \ll 0$  with the insulator and the superfluid respectively.
- For  $K \neq 0$ , the particle and hole excitations have different energies.

- Gauge-invariance of the underlying boson Hamiltonian shows that

$$K = -\frac{\partial r}{\partial \mu}$$

- In mean-field theory, the ground state energy,  $E$ , across the superfluid-insulator transition has the non-analytic term

$$E = E_0 - \frac{r^2}{2u}\theta(-r)$$

(Beyond mean-field theory, the non-analytic term is  $E \sim r^{(d+z)\nu}$ ).

- Because the density of bosons  $= -\partial E/\partial \mu$ , this implies a change in the boson density across the transition *unless*  $\partial r/\partial \mu = 0$
- A superfluid-insulator transition at fixed boson density must have.

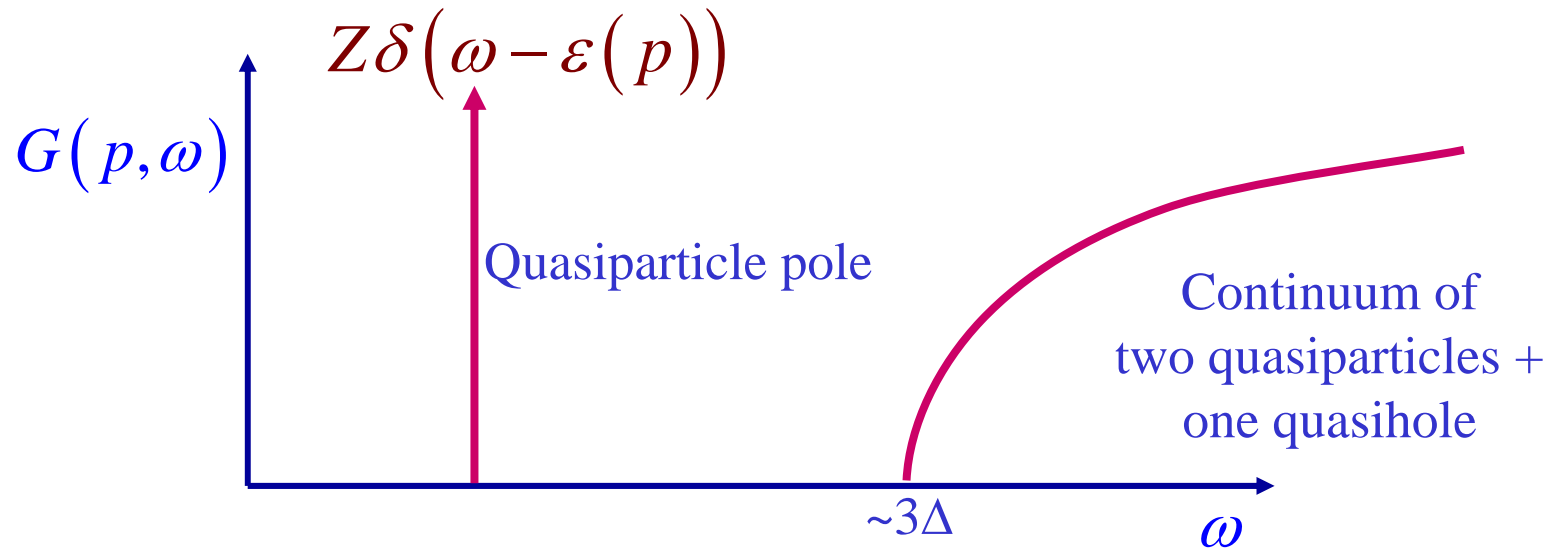
$$K = 0$$

Boson Green's function  $G(p, \omega)$ :

Insulating ground state

Cross-section to add a boson

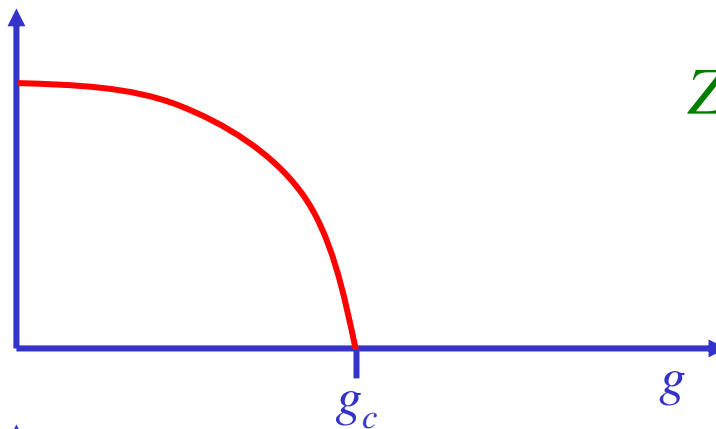
while transferring energy  $\hbar\omega$  and momentum  $p$



Similar result for quasi-hole excitations obtained by removing a boson

# Entangled states at $g \equiv t/U$ of order unity

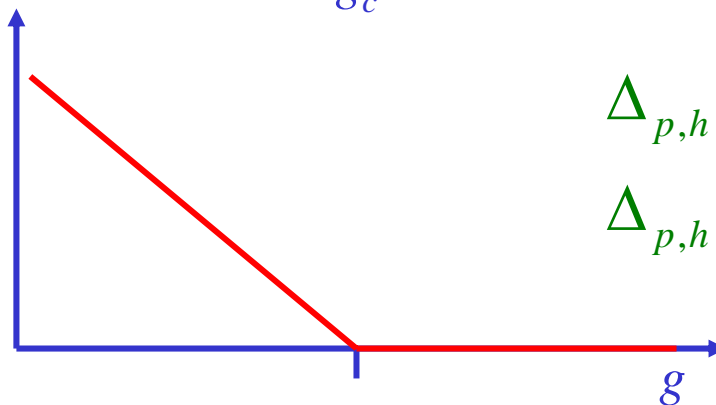
Quasiparticle weight  $Z$



$$Z \sim (g_c - g)^{\eta\nu}$$

A.V. Chubukov, S. Sachdev, and J. Ye,  
*Phys. Rev. B* **49**, 11919 (1994)

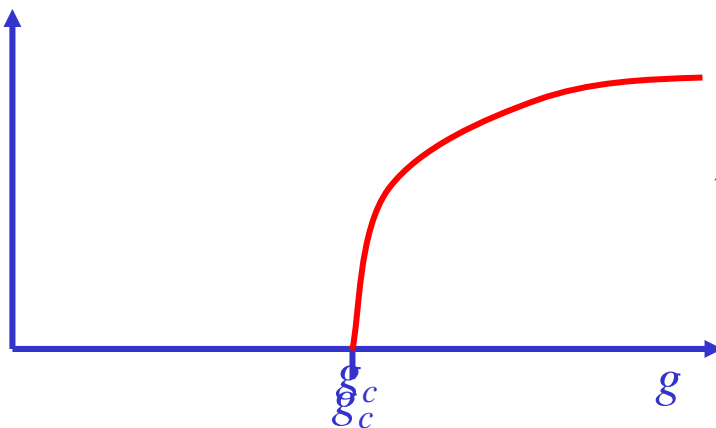
Excitation energy gap  $\Delta$



$$\Delta_{p,h} \sim (g_c - g)^\nu \text{ for } g < g_c$$

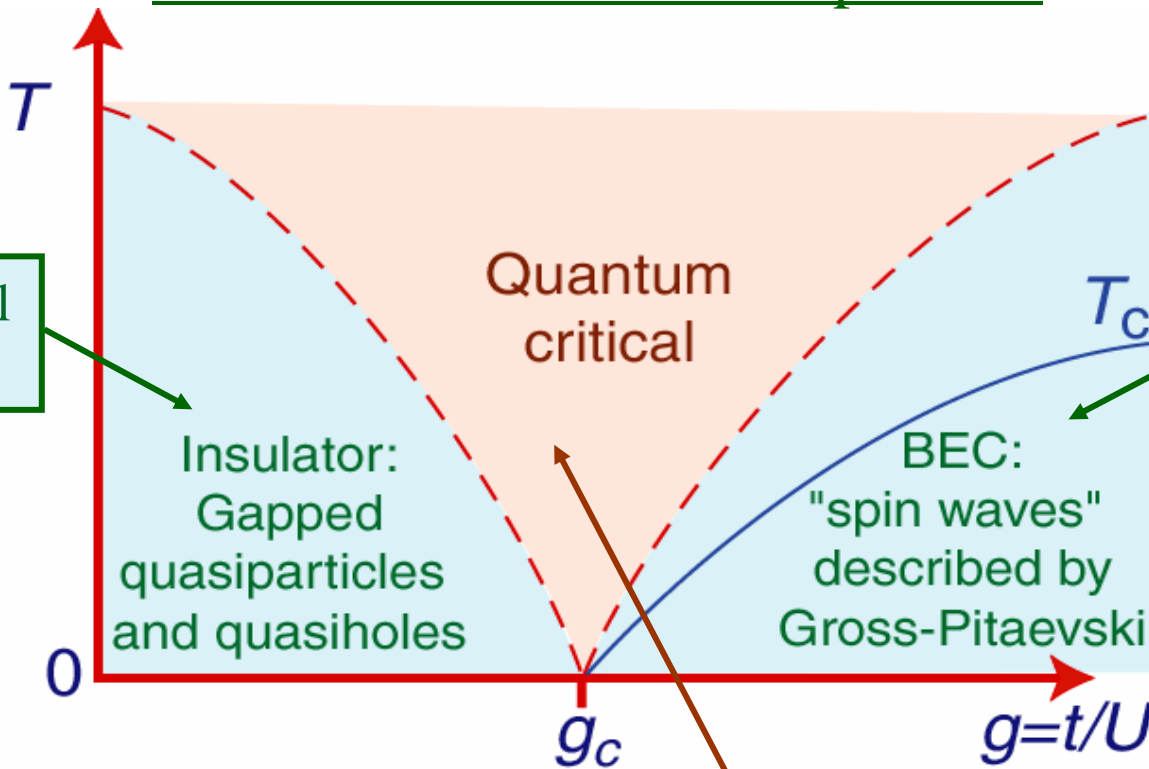
$$\Delta_{p,h} = 0 \text{ for } g > g_c$$

Superfluid density  $\rho_s$



$$\rho_s \sim (g - g_c)^{(d+z-2)\nu}$$

# Crossovers at nonzero temperature



Quasiclassical dynamics

Quasiclassical dynamics

Relaxational dynamics ("Bose molasses") with phase coherence/relaxation time  $\tau_\phi$  given by

$$\frac{1}{\tau_\phi} = (\text{Universal number}) \frac{k_B T}{\hbar} \quad (1\mu\text{K} = 20.9\text{kHz})$$

S. Sachdev and J. Ye, *Phys. Rev. Lett.* **69**, 2411 (1992).  
 K. Damle and S. Sachdev *Phys. Rev. B* **56**, 8714 (1997).

Conductivity (in d=2) =  $\frac{Q^2}{h} \Sigma \left( \frac{\hbar\omega}{k_B T} \right)$   $\Sigma \rightarrow$  universal function

M.P.A. Fisher, G. Girvin, and G. Grinstein, *Phys. Rev. Lett.* **64**, 587 (1990).  
 K. Damle and S. Sachdev *Phys. Rev. B* **56**, 8714 (1997).

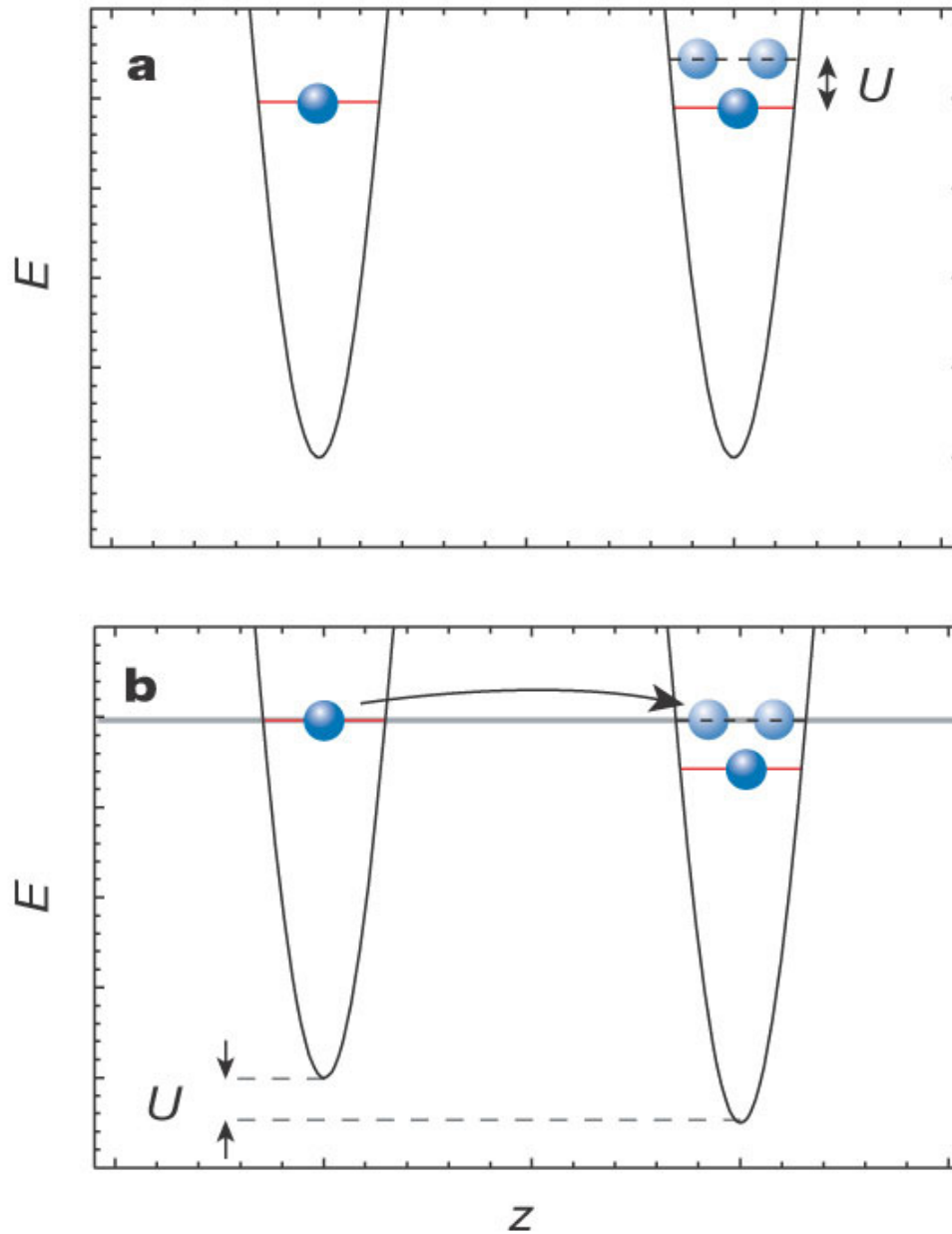
# Outline

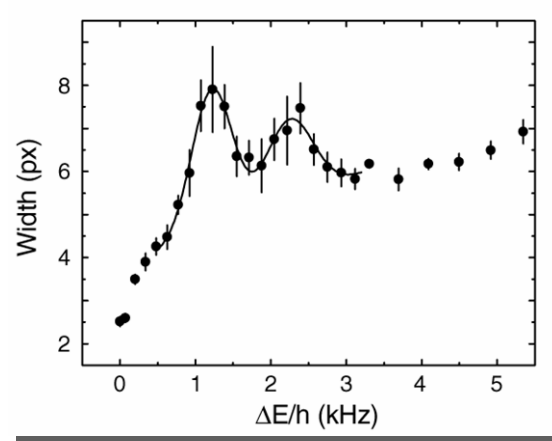
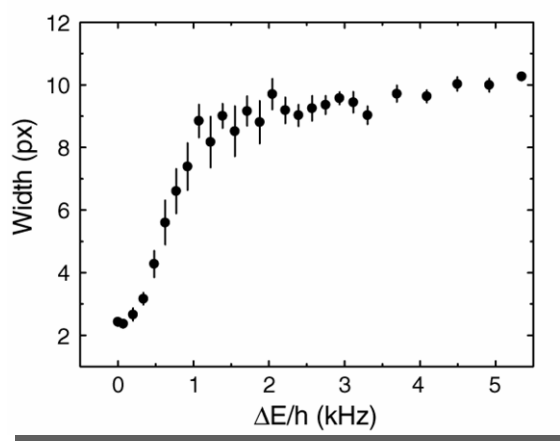
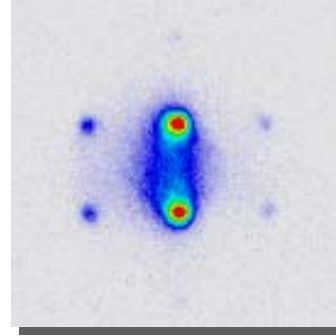
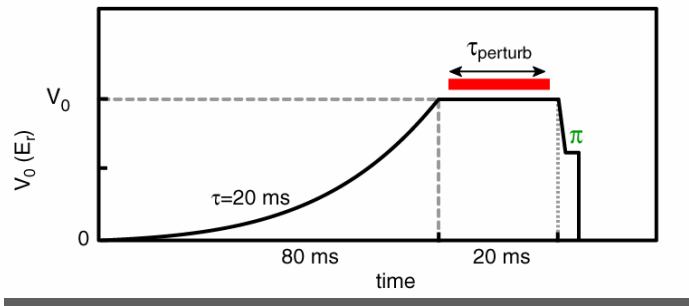
- I. Quantum Ising Chain
- II. Landau-Ginzburg-Wilson theory  
Mean field theory and the evolution of the excitation spectrum.
- III. Superfluid-insulator transition  
Boson Hubbard model at integer filling.
- IV. Tilting the Mott insulator  
Density wave order at an Ising transition.
- V. Bosons at fractional filling  
Beyond the Landau-Ginzburg-Wilson paradigm.

## IV. Tilting the Mott insulator

*Density wave order at an Ising transition*

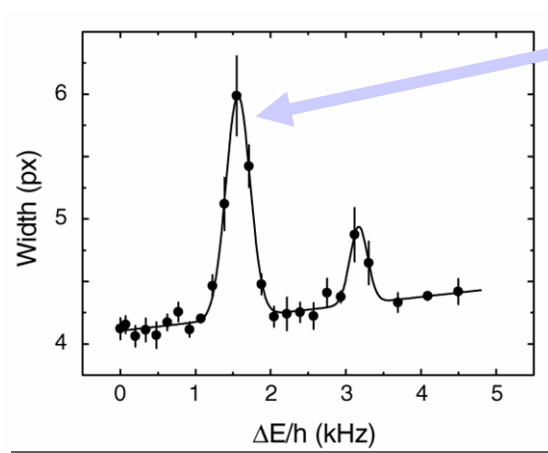
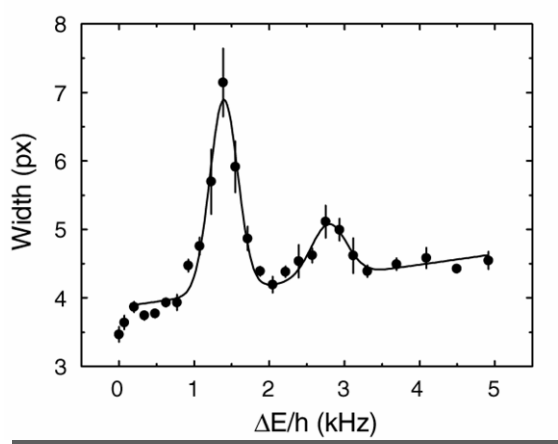
# Applying an “electric” field to the Mott insulator





$V_0 = 10 E_{\text{recoil}}$   $\tau_{\text{perturb}} = 2$  ms

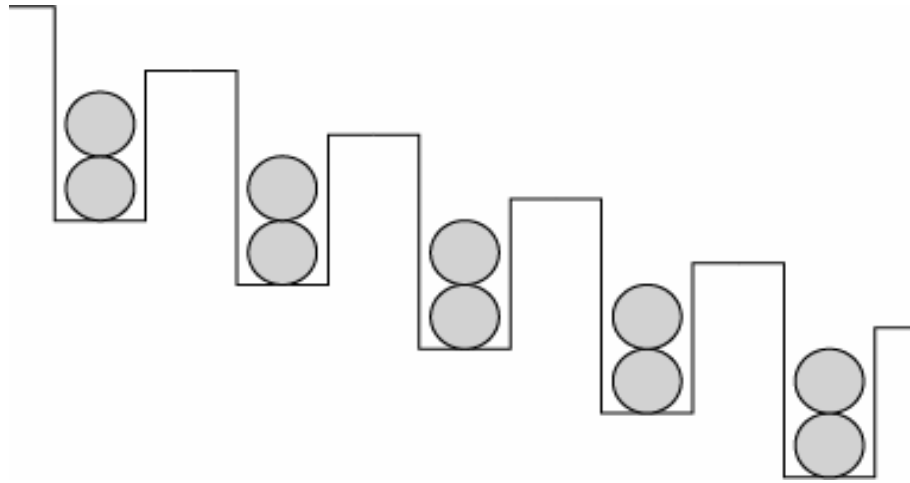
$V_0 = 13 E_{\text{recoil}}$   $\tau_{\text{perturb}} = 4$  ms



What is the quantum state here ?

$V_0 = 16 E_{\text{recoil}}$   $\tau_{\text{perturb}} = 9$  ms

$V_0 = 20 E_{\text{recoil}}$   $\tau_{\text{perturb}} = 20$  ms



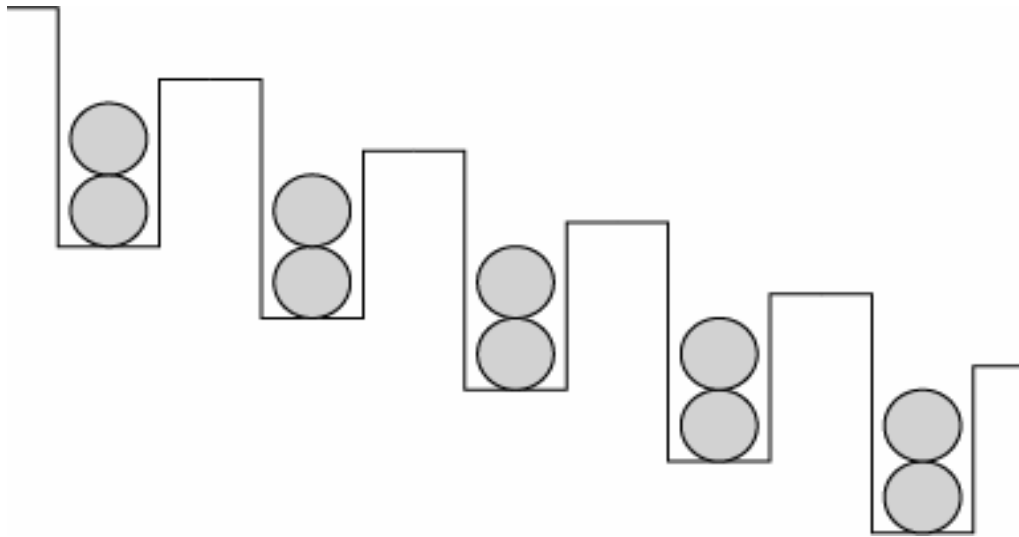
$$H = -t \sum_{\langle ij \rangle} (b_i^\dagger b_j + b_j^\dagger b_i) + \frac{U}{2} \sum_i n_i (n_i - 1) - \sum_i \mathbf{E} \cdot \mathbf{r}_i n_i$$

$$n_i = b_i^\dagger b_i$$

$$|U - E|, t \ll E, U$$

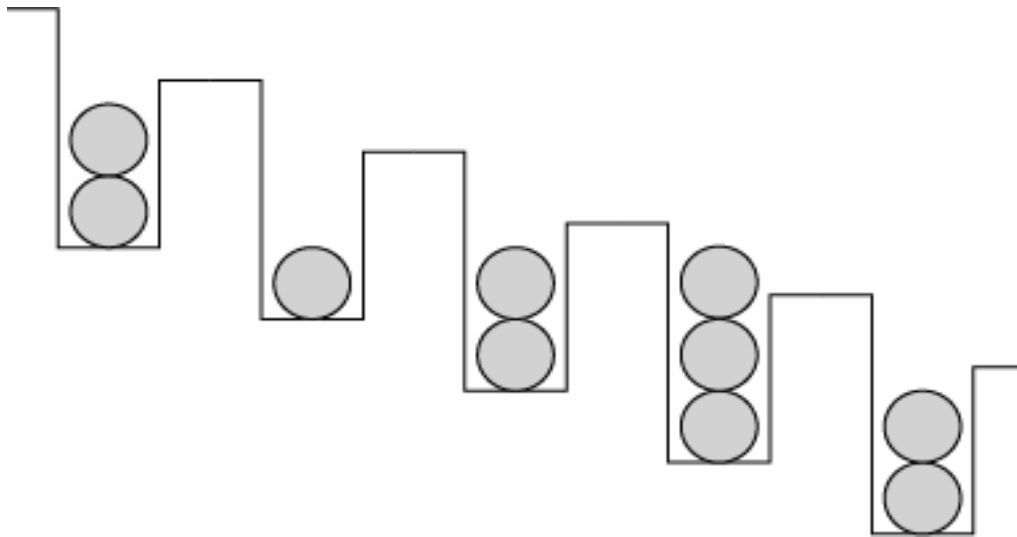
Describe spectrum in subspace of states resonantly coupled to the Mott insulator

# Important neutral excitations (in one dimension)



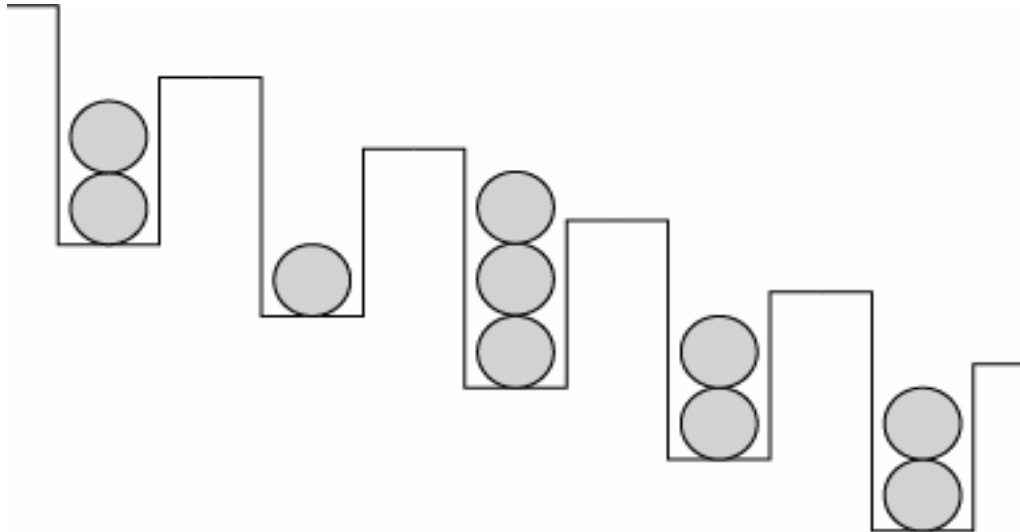


## Important neutral excitations (in one dimension)



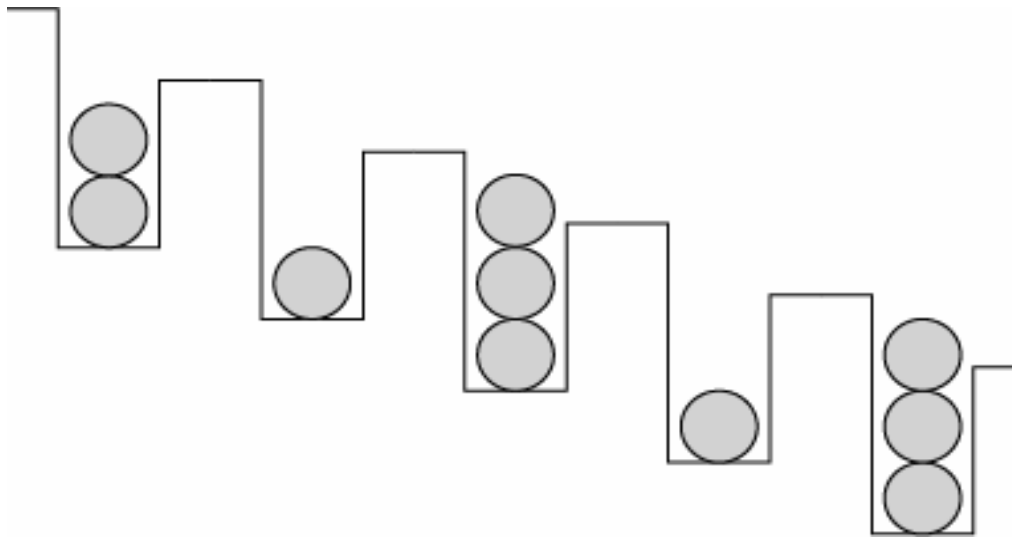
Creating dipoles on nearest neighbor links creates a state with relative energy  $U-2E$  ; such states are *not* part of the resonant manifold

Important neutral excitations (in one dimension)



Nearest neighbor dipole

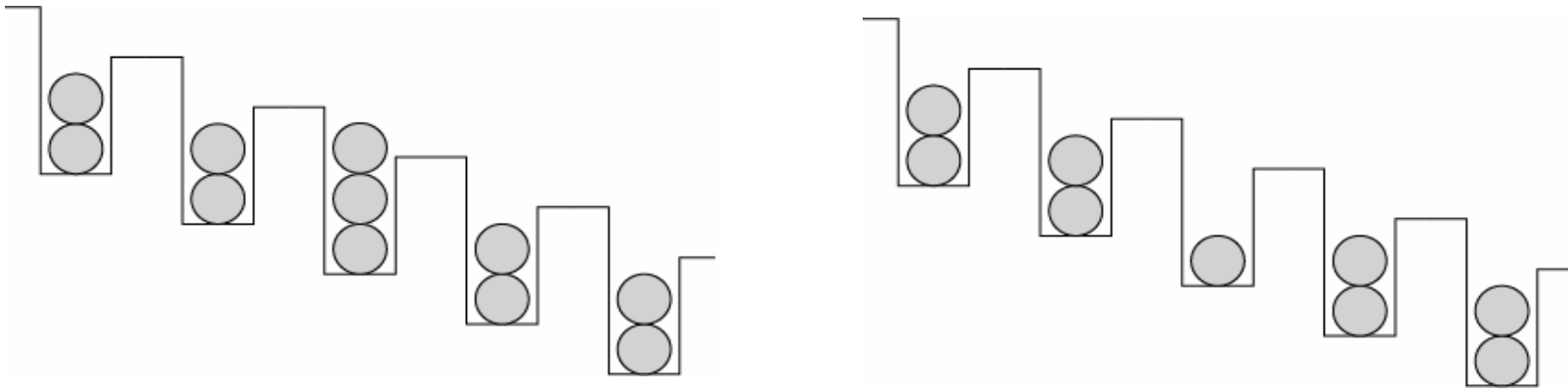
## Important neutral excitations (in one dimension)



Nearest-neighbor dipoles

Dipoles can appear resonantly on non-nearest-neighbor links.  
Within resonant manifold, dipoles have infinite on-link  
and nearest-link repulsion

## Charged excitations (in one dimension)



Effective Hamiltonian for a quasiparticle in one dimension (similar for a quasihole):

$$H_{\text{eff}} = -\sum_j \left[ 3t \left( b_j^\dagger b_{j+1} + b_{j+1}^\dagger b_j \right) + E j b_j^\dagger b_j \right]$$

Exact eigenvalues  $\varepsilon_m = Em$  ;  $m = -\infty \dots \infty$

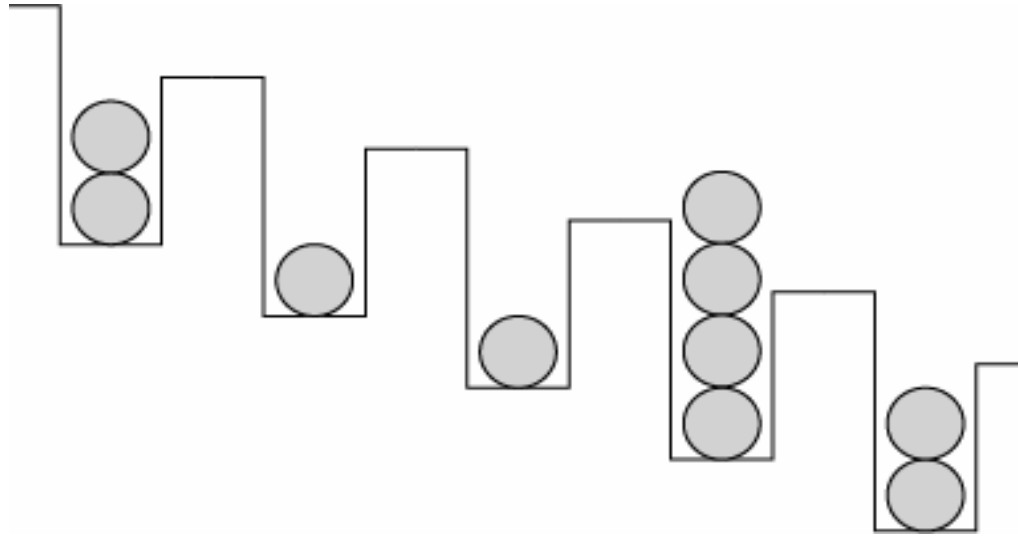
Exact eigenvectors  $\psi_m(j) = J_{j-m}(6t/E)$

All charged excitations are strongly localized in the plane perpendicular electric field.

Wavefunction is periodic in time, with period  $h/E$  (Bloch oscillations)

Quasiparticles and quasiholes are not accelerated out to infinity

## A non-dipole state



State has energy  $3(U-E)$  but is connected to resonant state by a matrix element smaller than  $t^2/U$

State is not part of resonant manifold

## Hamiltonian for resonant dipole states (in one dimension)

$d_\ell^\dagger \Rightarrow$  Creates dipole on link  $\ell$

$$H_d = -\sqrt{6}t \sum_\ell (d_\ell^\dagger + d_\ell) + (U - E) \sum_\ell d_\ell^\dagger d_\ell$$

$$\text{Constraints: } d_\ell^\dagger d_\ell \leq 1 \quad ; \quad d_{\ell+1}^\dagger d_{\ell+1} d_\ell^\dagger d_\ell = 0$$

Determine phase diagram of  $H_d$  as a function of  $(U-E)/t$

Note: there is no explicit dipole hopping term.

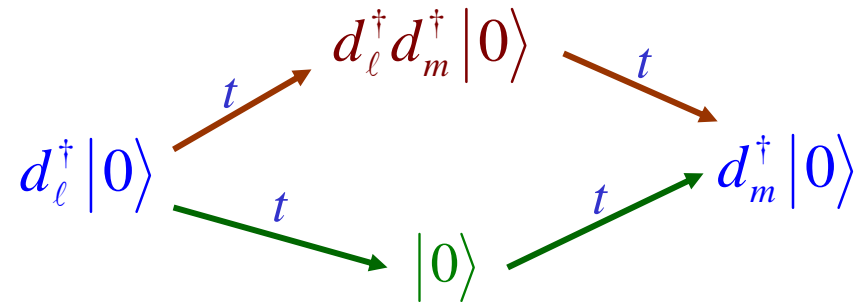
However, dipole hopping is generated by the interplay of terms in  $H_d$  and the constraints.

Weak electric fields:  $(U-E) \gg t$

Ground state is dipole vacuum (Mott insulator)  $|0\rangle$

First excited levels: single dipole states  $d_\ell^\dagger |0\rangle$

Effective hopping between dipole states



If both processes are permitted, they exactly cancel each other.

The top processes is blocked when  $\ell, m$  are nearest neighbors

$\Rightarrow$  A nearest-neighbor dipole hopping term  $\sim \frac{t^2}{U-E}$  is generated

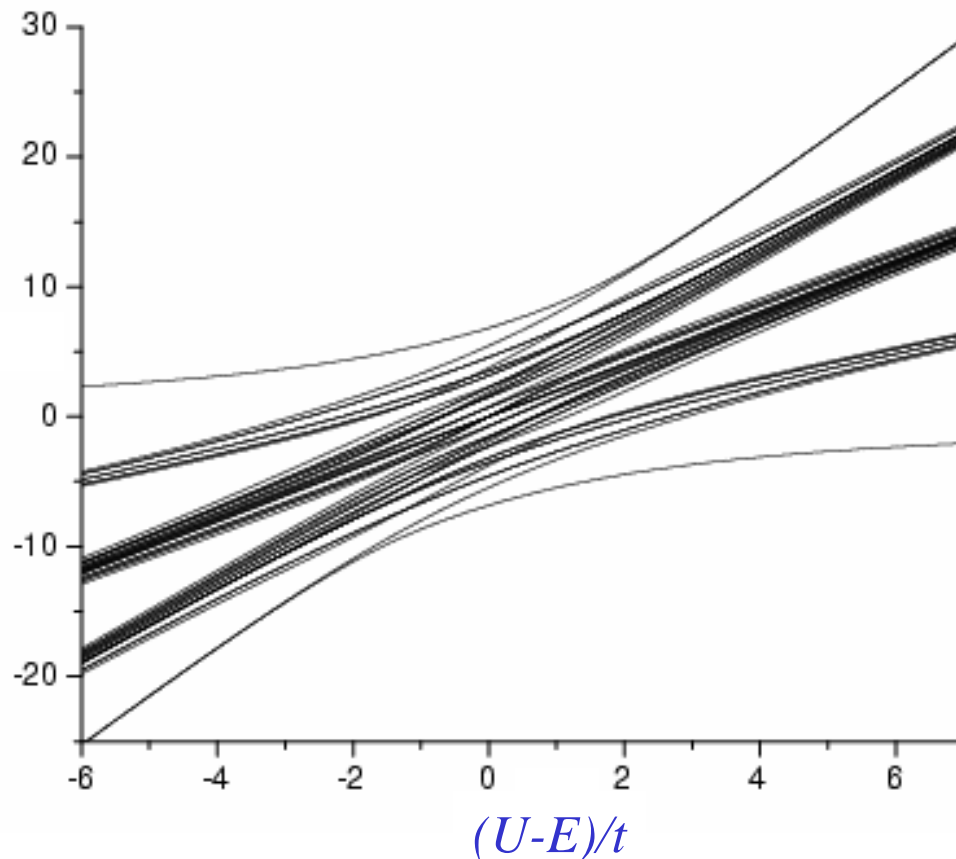
Strong electric fields:  $(E-U) \gg t$

Ground state has maximal dipole number.

Two-fold degeneracy associated with Ising density wave order:

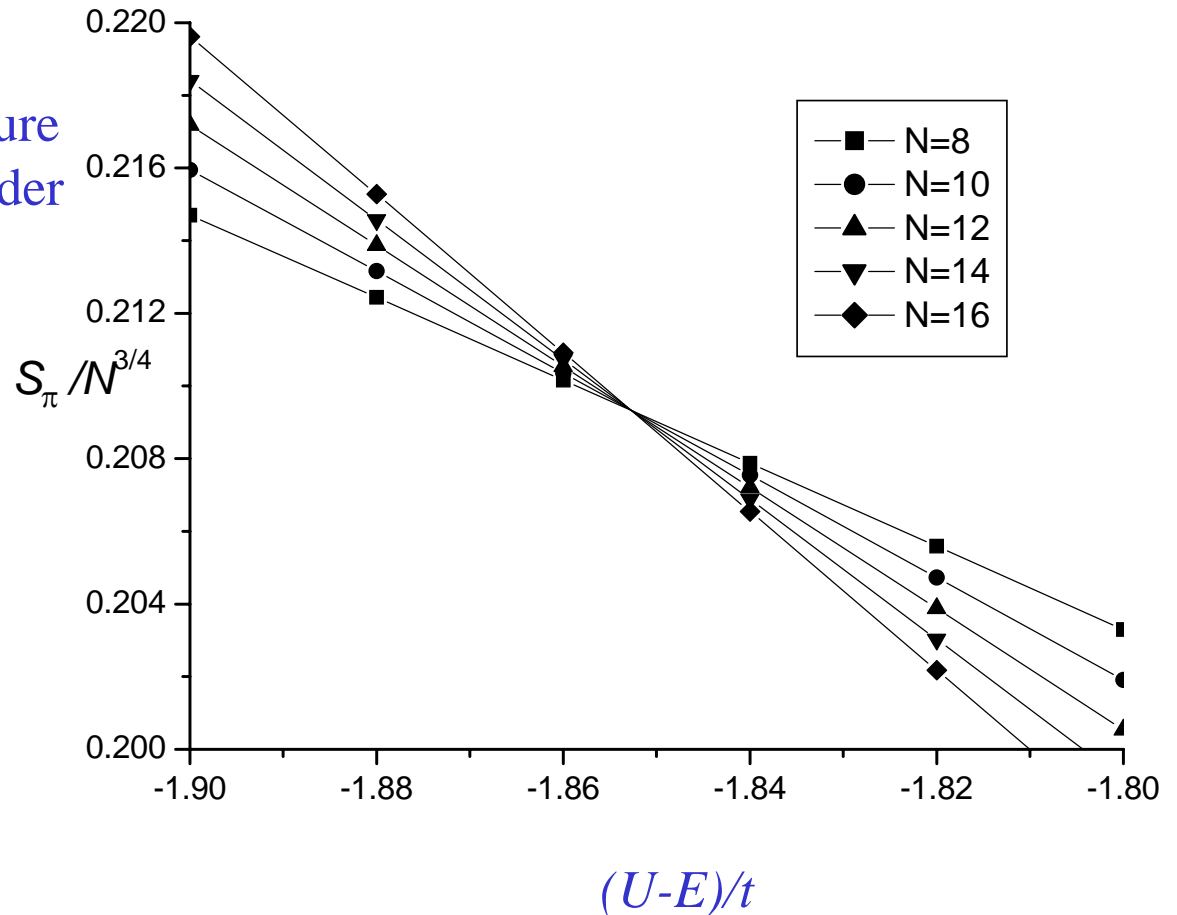
$$\cdots d_1^\dagger d_3^\dagger d_5^\dagger d_7^\dagger d_9^\dagger d_{11}^\dagger \cdots |0\rangle \quad \text{or} \quad \cdots d_2^\dagger d_4^\dagger d_6^\dagger d_8^\dagger d_{10}^\dagger d_{12}^\dagger \cdots |0\rangle$$

Eigenvalues



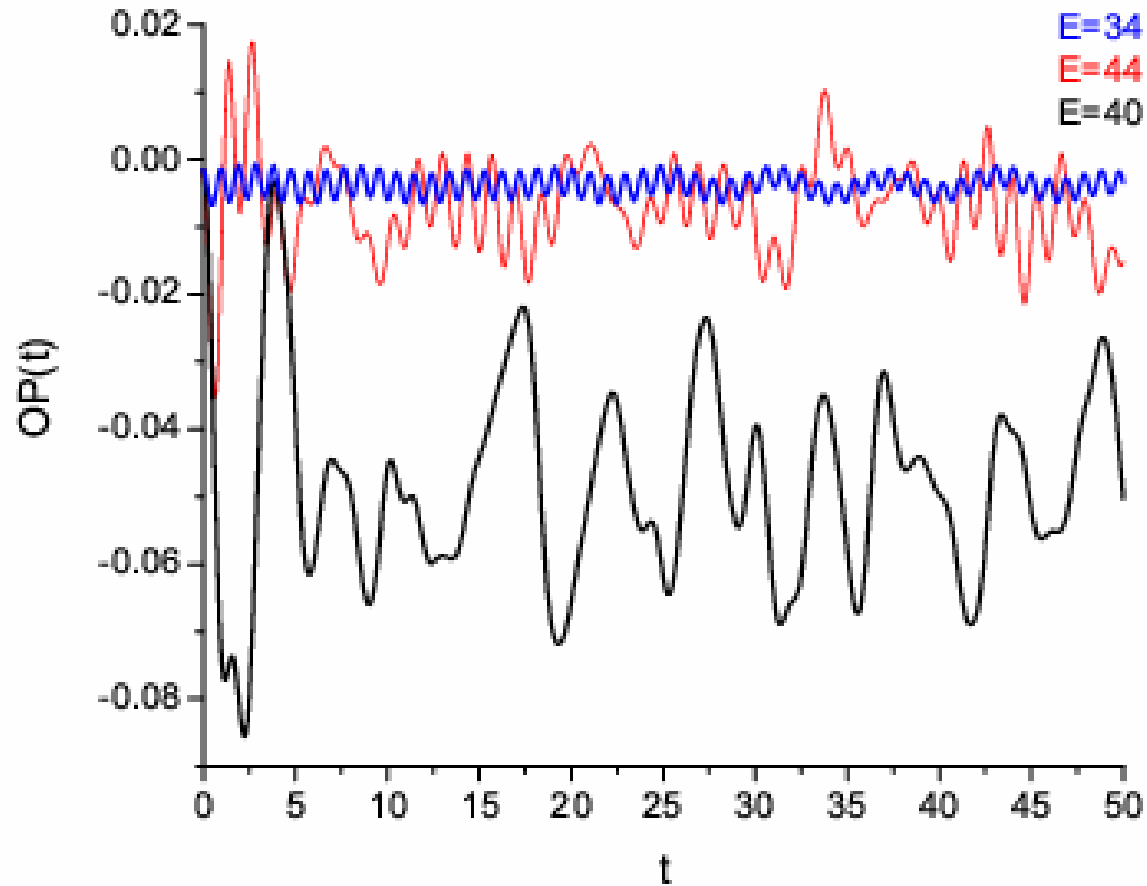
Ising quantum critical point at  $E-U=1.08 t$

Equal-time structure factor for Ising order parameter



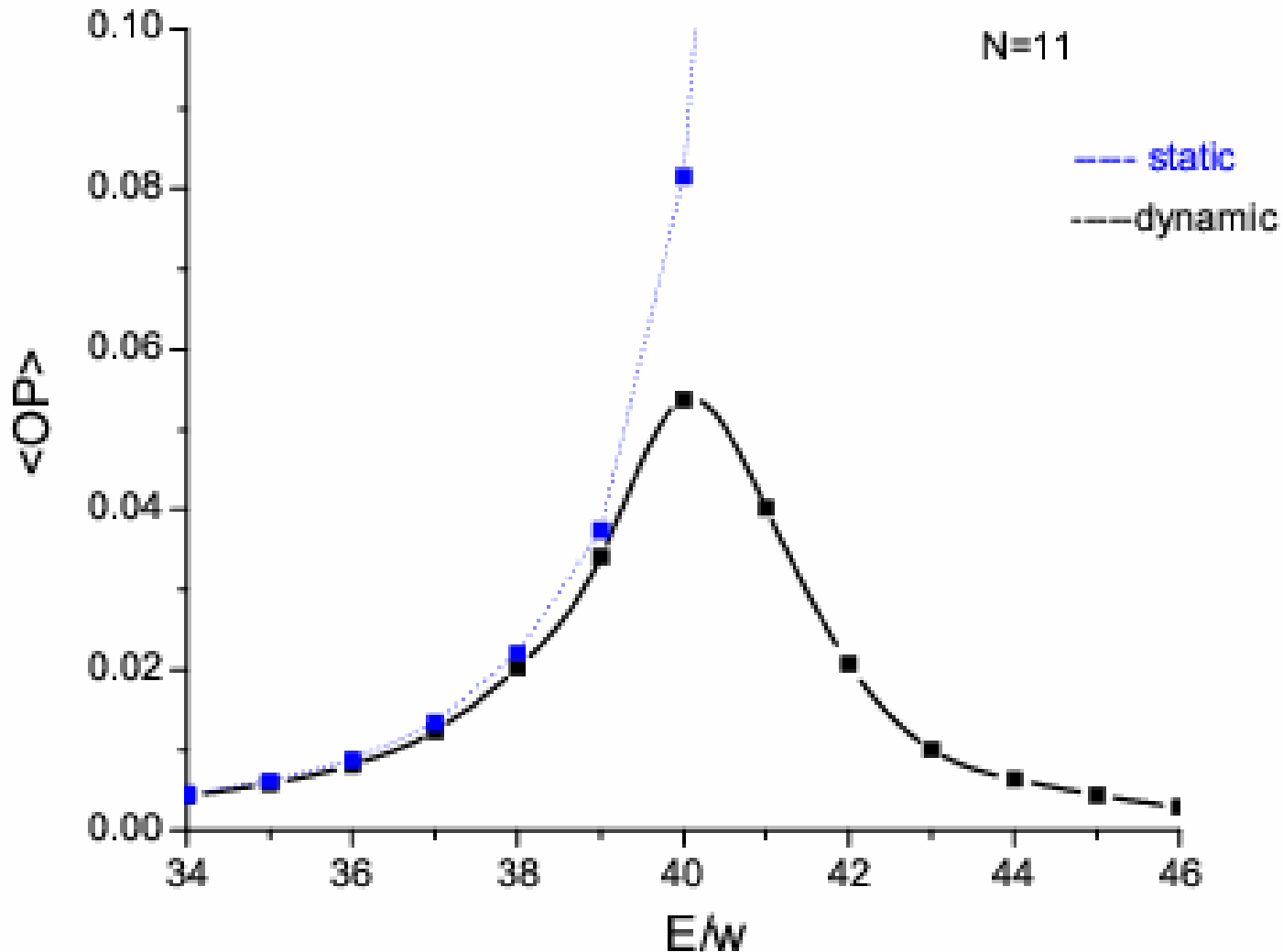
## Non-equilibrium dynamics in one dimension

Start with the ground state at  $E=32$  on a chain with open boundaries. Suddenly change the value of  $E$  and follow the evolution of the wavefunction



Critical point at  $E=41.85$

## Non-equilibrium dynamics in one dimension



Non-equilibrium response is maximal near the Ising critical point

K. Sengupta, S. Powell, and S. Sachdev, *Physical Review A* **69**, 053616 (2004)

# Outline

- I. Quantum Ising Chain
- II. Landau-Ginzburg-Wilson theory  
Mean field theory and the evolution of the excitation spectrum.
- III. Superfluid-insulator transition  
Boson Hubbard model at integer filling.
- IV. Tilting the Mott insulator  
Density wave order at an Ising transition.
- V. Bosons at fractional filling  
Beyond the Landau-Ginzburg-Wilson paradigm.

## V. Bosons at fractional filling

*Beyond the Landau-Ginzburg-Wilson  
paradigm*

L. Balents, L. Bartosch, A. Burkov, S. Sachdev, K. Sengupta, *to appear*.

# Bosons at density $f = 1$

**a** Superfluid state

Weak interactions:  
superfluidity

**b** Insulating state

Strong interactions:  
Mott insulator which  
preserves all lattice  
symmetries

LGW theory: continuous quantum transitions between these states

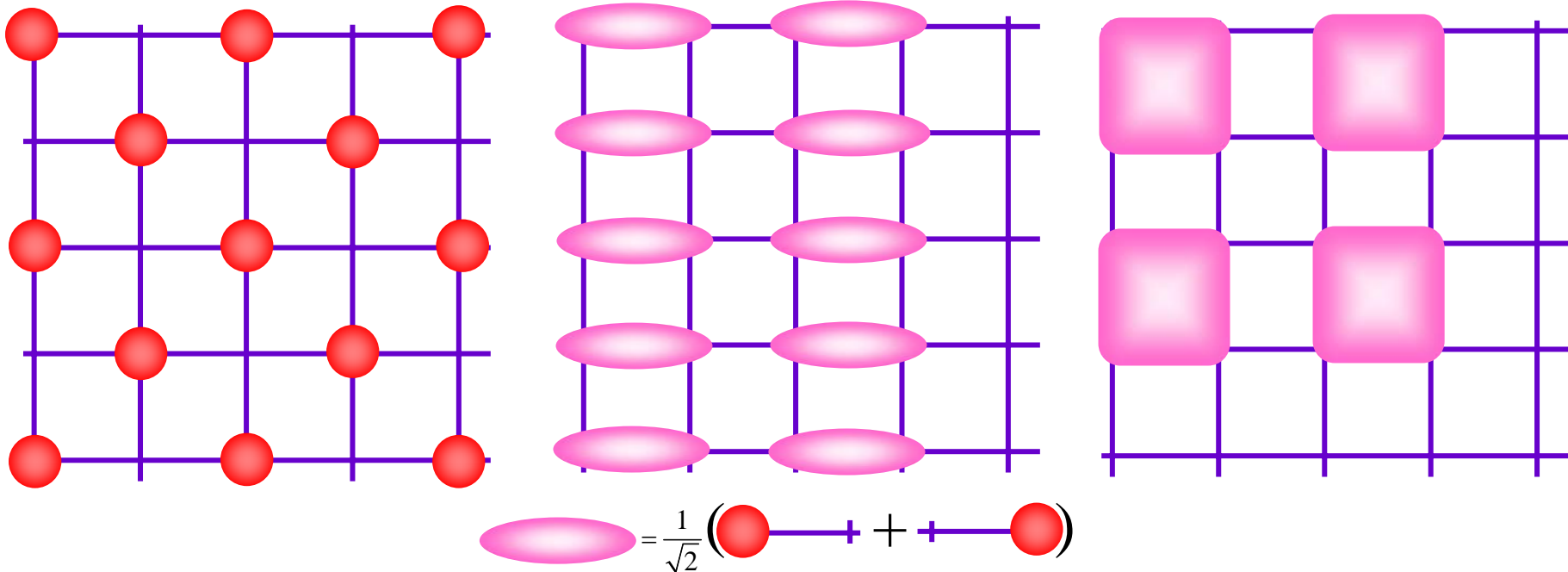
M. Greiner, O. Mandel, T. Esslinger, T. W. Hänsch, and I. Bloch, *Nature* **415**, 39 (2002).

# Bosons at density $f = 1/2$

Weak interactions: superfluidity

$$\langle \Psi_{sc} \rangle \neq 0$$

Strong interactions: Candidate insulating states



All insulating phases have density-wave order  $\rho(\mathbf{r}) = \sum_{\mathbf{q}} \rho_{\mathbf{q}} e^{i\mathbf{q} \cdot \mathbf{r}}$  with  $\langle \rho_{\mathbf{q}} \rangle \neq 0$

C. Lannert, M.P.A. Fisher, and T. Senthil, *Phys. Rev. B* **63**, 134510 (2001)

S. Sachdev and K. Park, *Annals of Physics*, **298**, 58 (2002)

Ginzburg-Landau-Wilson approach to multiple order parameters:

$$F = F_{sc} [\Psi_{sc}] + F_{\text{charge}} [\rho_Q] + F_{\text{int}}$$

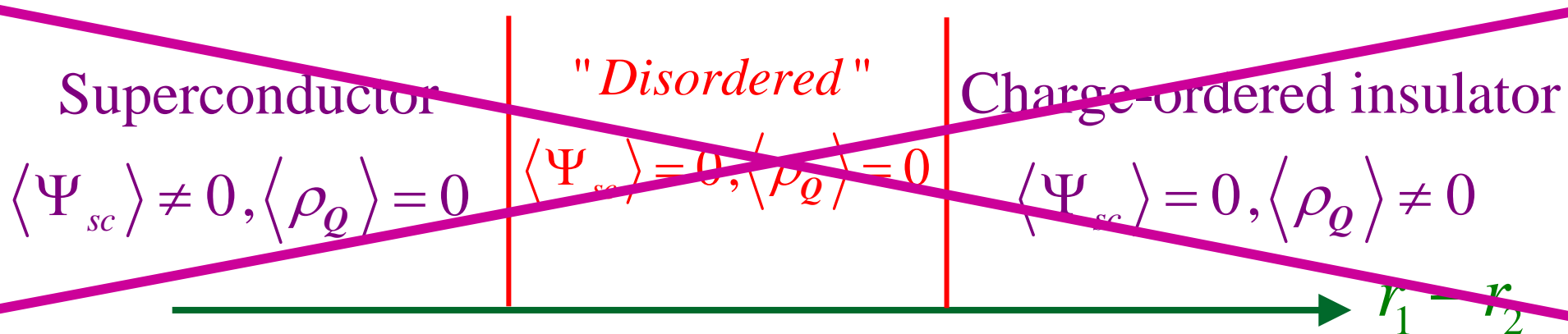
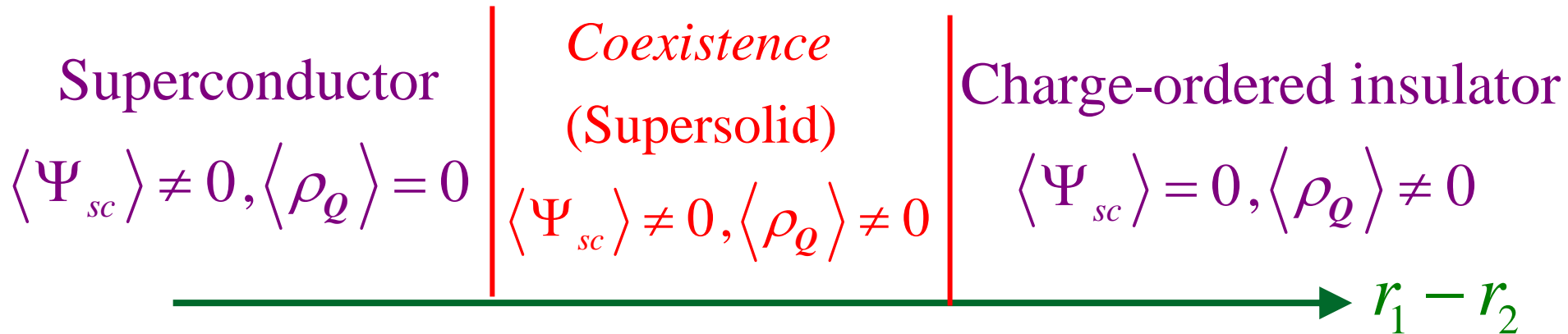
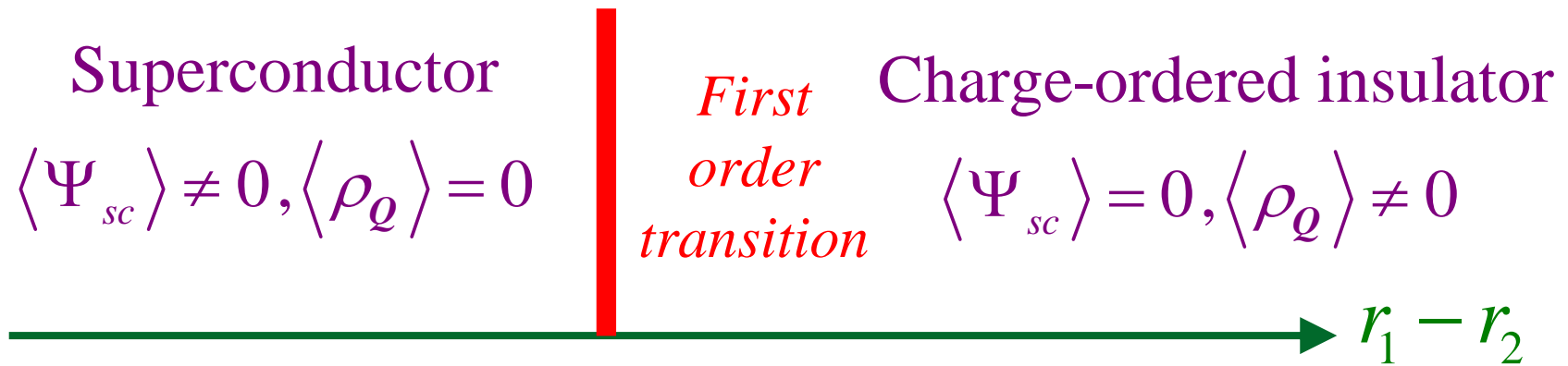
$$F_{sc} [\Psi_{sc}] = r_1 |\Psi_{sc}|^2 + u_1 |\Psi_{sc}|^4 + \dots$$

$$F_{\text{charge}} [\rho_Q] = r_2 |\rho_Q|^2 + u_2 |\rho_Q|^4 + \dots$$

$$F_{\text{int}} = v |\Psi_{sc}|^2 |\rho_Q|^2 + \dots$$

Distinct symmetries of order parameters permit couplings only between their energy densities

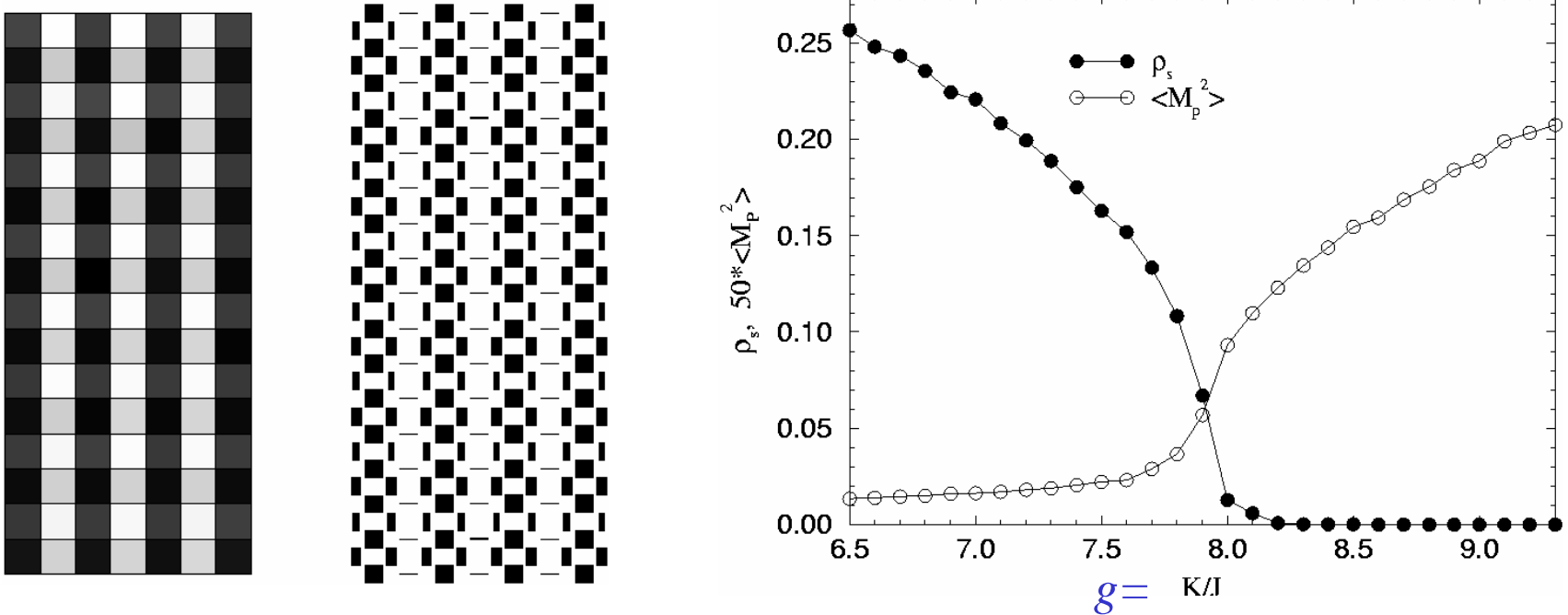
# Predictions of LGW theory



# Superfluid insulator transition of hard core bosons at $f=1/2$

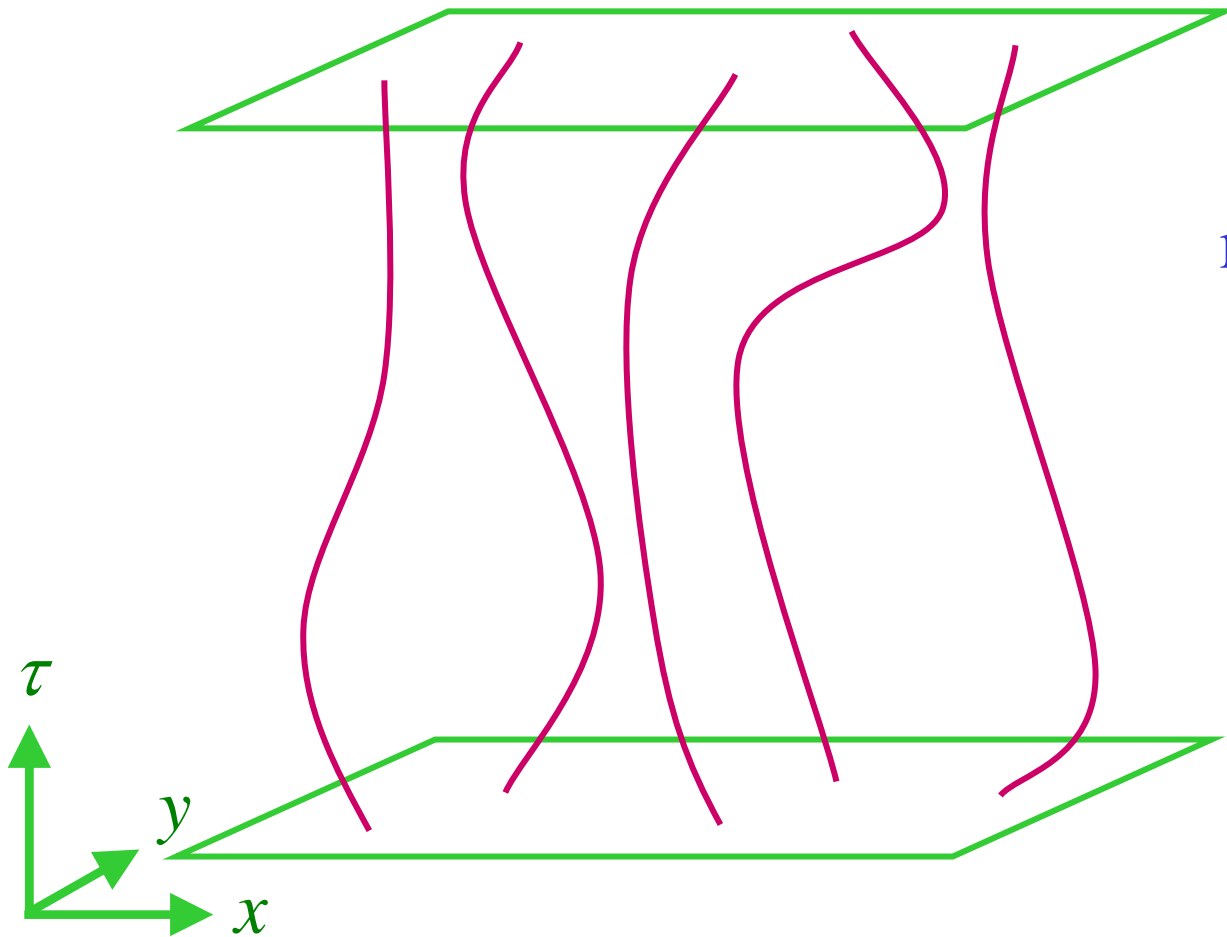
A. W. Sandvik, S. Daul, R. R. P. Singh, and D. J. Scalapino, *Phys. Rev. Lett.* **89**, 247201 (2002)

Large scale (> 8000 sites) numerical study of the destruction of superfluid order at half filling with full square lattice symmetry



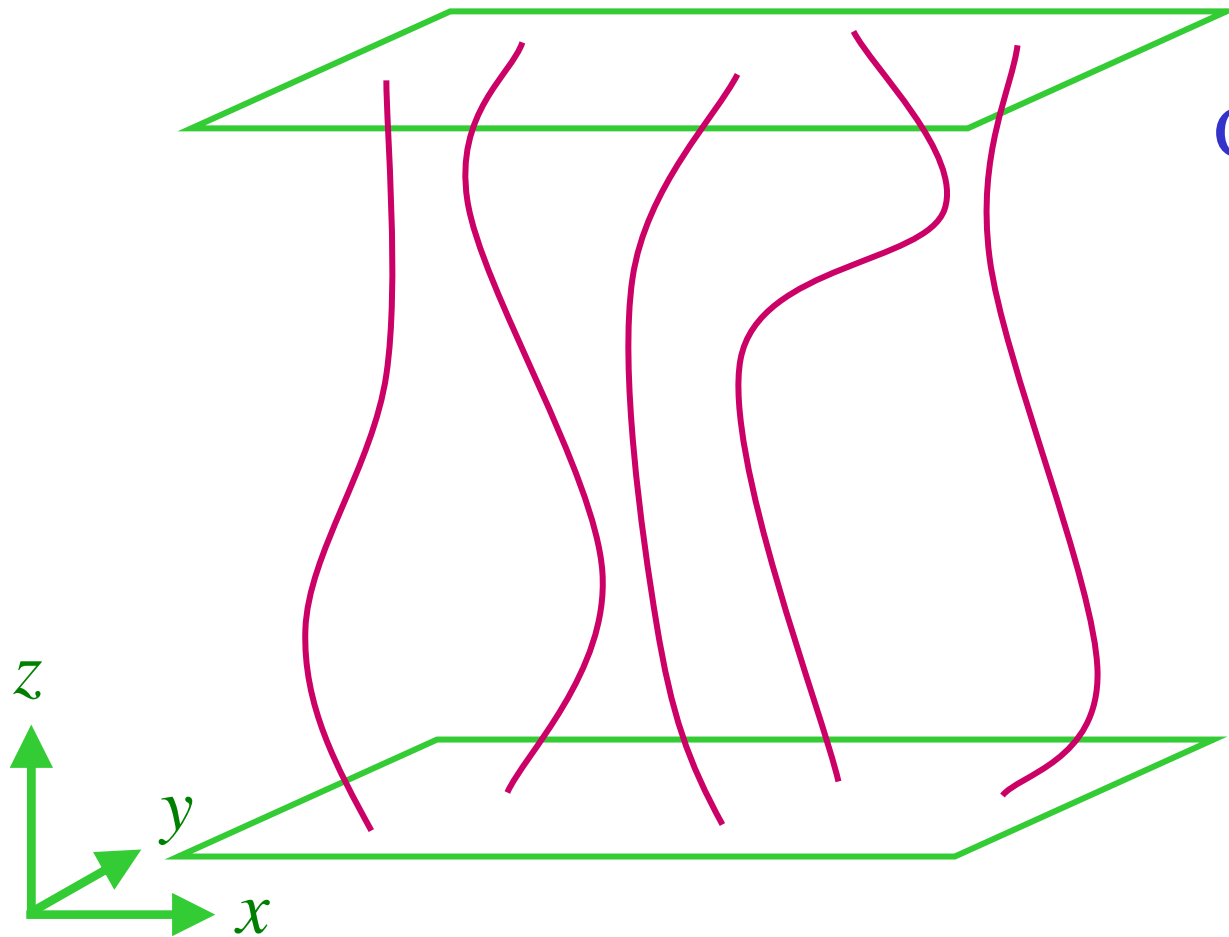
$$H = J \sum_{\langle ij \rangle} (S_i^+ S_j^- + S_i^- S_j^+) - K \sum_{\langle ijkl \rangle \subset \square} (S_i^+ S_j^- S_k^+ S_l^- + S_i^- S_j^+ S_k^- S_l^+)$$

# Boson-vortex duality



Quantum  
mechanics of two-  
dimensional  
bosons: world  
lines of bosons in  
spacetime

# Boson-vortex duality



Classical statistical mechanics of a “dual” three-dimensional superconductor: vortices in a “magnetic” field

Strength of “magnetic” field = density of bosons  
=  $f$  flux quanta per plaquette

## Boson-vortex duality

Statistical mechanics of dual superconductor is invariant under the square lattice space group:

$T_x, T_y$  : Translations by a lattice spacing in the  $x, y$  directions

$R$  : Rotation by 90 degrees.

Magnetic space group:

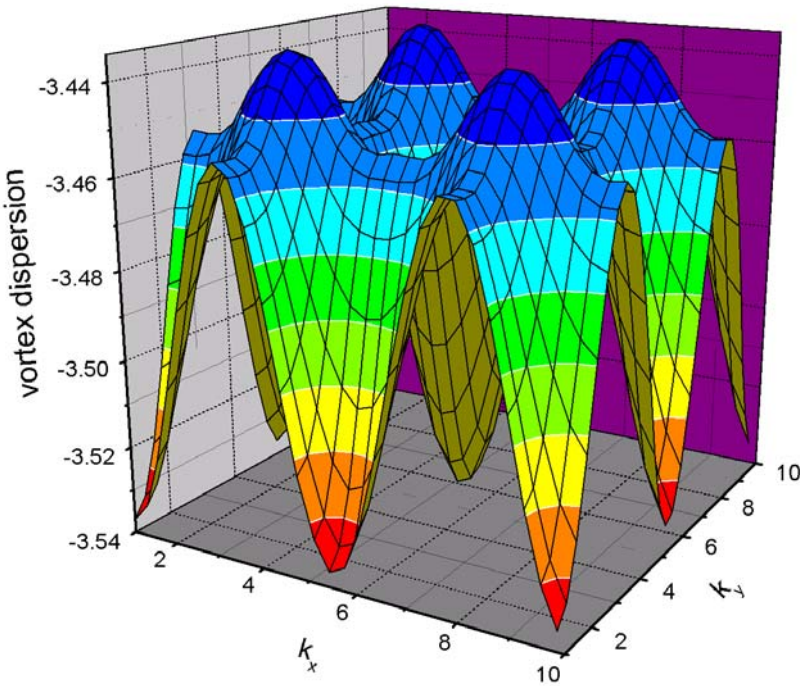
$$T_x T_y = e^{2\pi i f} T_y T_x \ ;$$

$$R^{-1} T_y R = T_x \ ; \ R^{-1} T_x R = T_y^{-1} \ ; \ R^4 = 1$$

Strength of “magnetic” field = density of bosons  
=  $f$  flux quanta per plaquette

# Boson-vortex duality

## Hofstadter spectrum of dual “superconducting” order



At density  $f = p / q$  ( $p, q$  relatively prime integers) there are  $q$  species of vortices,  $\varphi_\ell$  (with  $\ell = 1 \dots q$ ), associated with  $q$  gauge-equivalent regions of the Brillouin zone

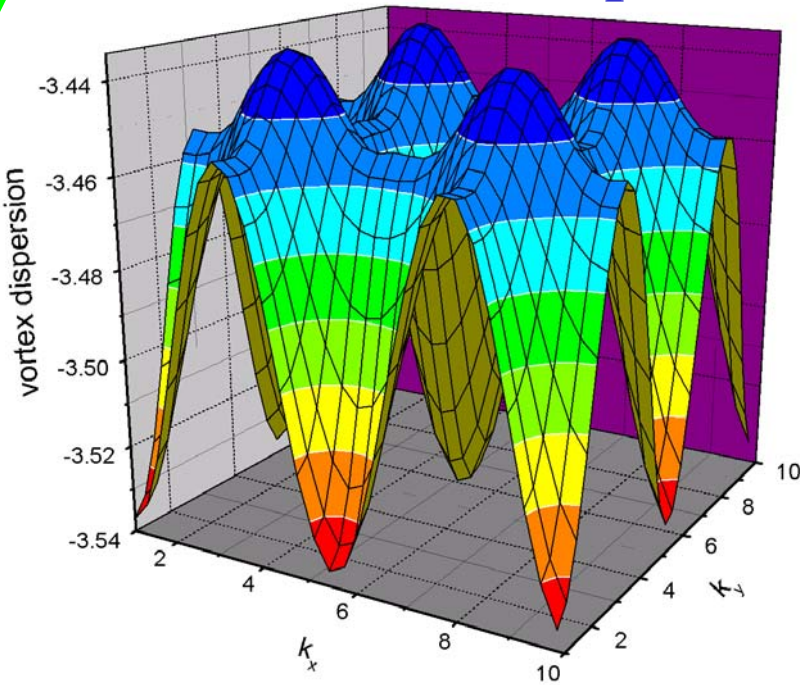
Magnetic space group:

$$T_x T_y = e^{2\pi i f} T_y T_x ;$$

$$R^{-1} T_y R = T_x ; \quad R^{-1} T_x R = T_y^{-1} ; \quad R^4 = 1$$

# Boson-vortex duality

## Hofstadter spectrum of dual “superconducting” order



At density  $f = p / q$  ( $p, q$  relatively prime integers) there are  $q$  species of vortices,  $\varphi_\ell$  (with  $\ell = 1 \dots q$ ), associated with  $q$  gauge-equivalent regions of the Brillouin zone

The  $q$  vortices form a *projective* representation of the space group

$$T_x : \varphi_\ell \rightarrow \varphi_{\ell+1} \quad ; \quad T_y : \varphi_\ell \rightarrow e^{2\pi i \ell f} \varphi_\ell$$

$$R : \varphi_\ell \rightarrow \frac{1}{\sqrt{q}} \sum_{m=1}^q \varphi_m e^{2\pi i \ell m f}$$

# Boson-vortex duality

The  $\varphi_\ell$  fields characterize *both* superconducting and charge order

Superconductor/insulator :  $\langle \varphi_\ell \rangle = 0 / \langle \varphi_\ell \rangle \neq 0$

Charge order:

Status of space group symmetry determined by

density operators  $\rho_Q$  at wavevectors  $Q_{mn} = \frac{2\pi p}{q}(m, n)$

$$\rho_{Q_{mn}} = e^{i\pi mnf} \sum_{\ell=1}^q \varphi_\ell^* \varphi_{\ell+n} e^{2\pi i \ell m f}$$

$$T_x : \rho_Q \rightarrow \rho_Q e^{iQ \cdot \hat{x}} \quad ; \quad T_y : \rho_Q \rightarrow \rho_Q e^{iQ \cdot \hat{y}}$$

$$R : \rho(Q) \rightarrow \rho(RQ)$$

## Boson-vortex duality

The  $\varphi_\ell$  fields characterize *both* superconducting and charge order

Competition between superconducting and charge orders:

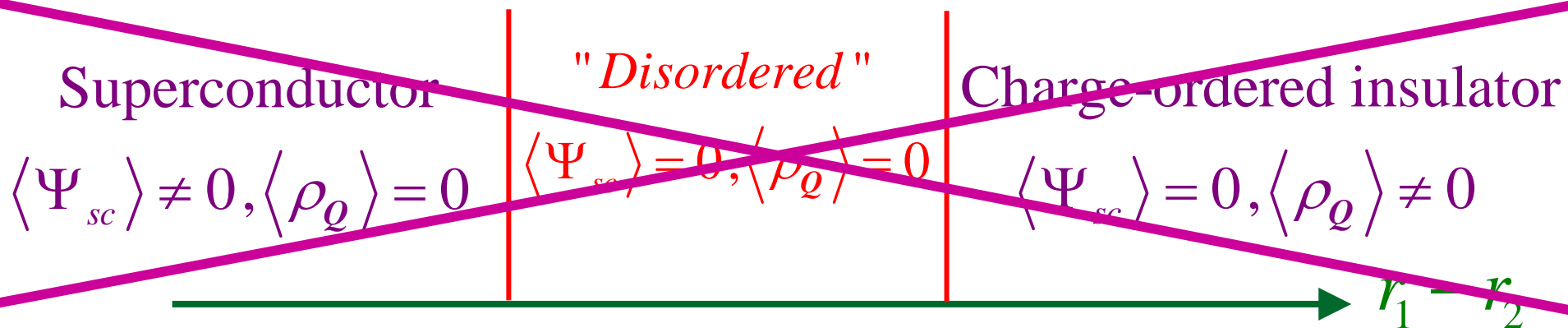
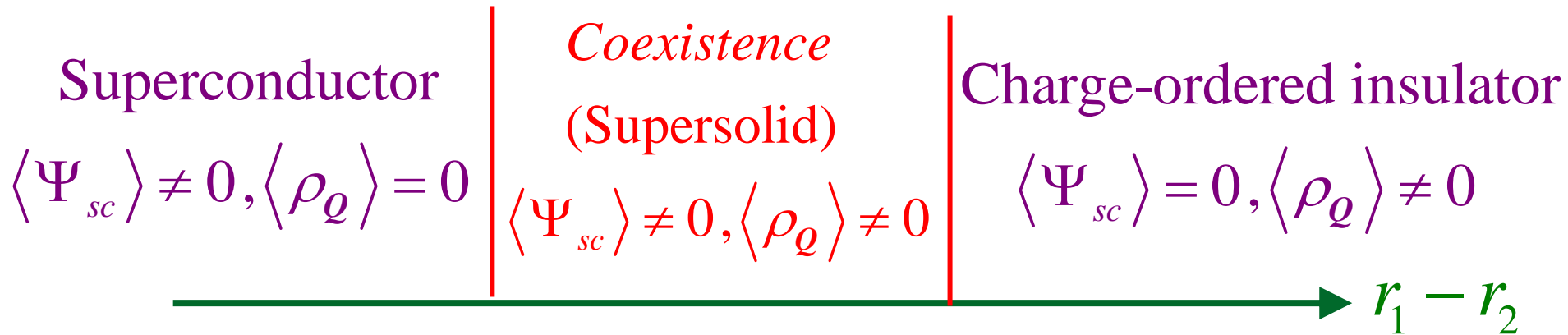
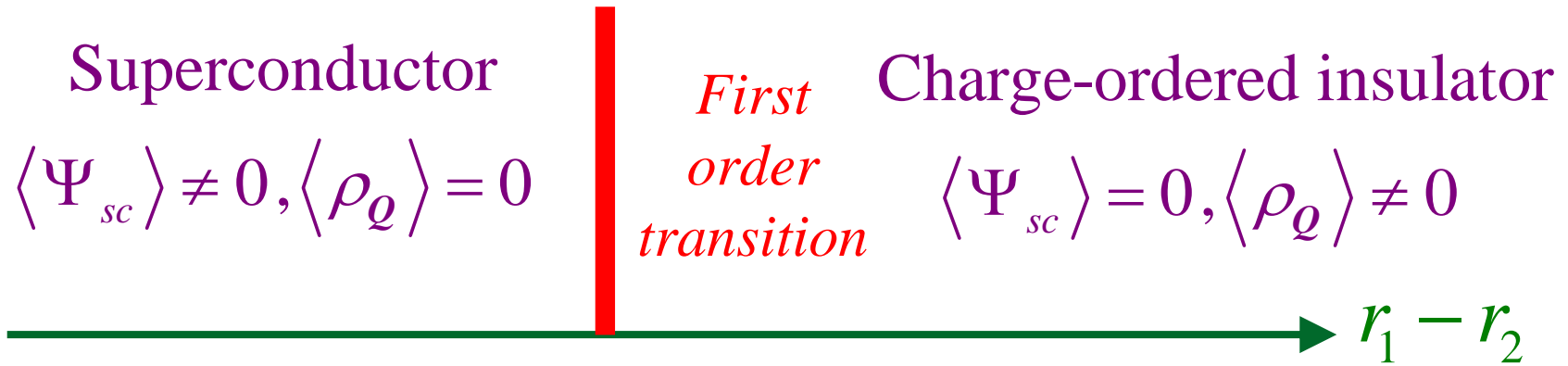
"*Extended LGW*" theory of the  $\varphi_\ell$  fields with the action invariant under the projective transformations:

$$T_x : \varphi_\ell \rightarrow \varphi_{\ell+1} \quad ; \quad T_y : \varphi_\ell \rightarrow e^{2\pi i l f} \varphi_\ell$$

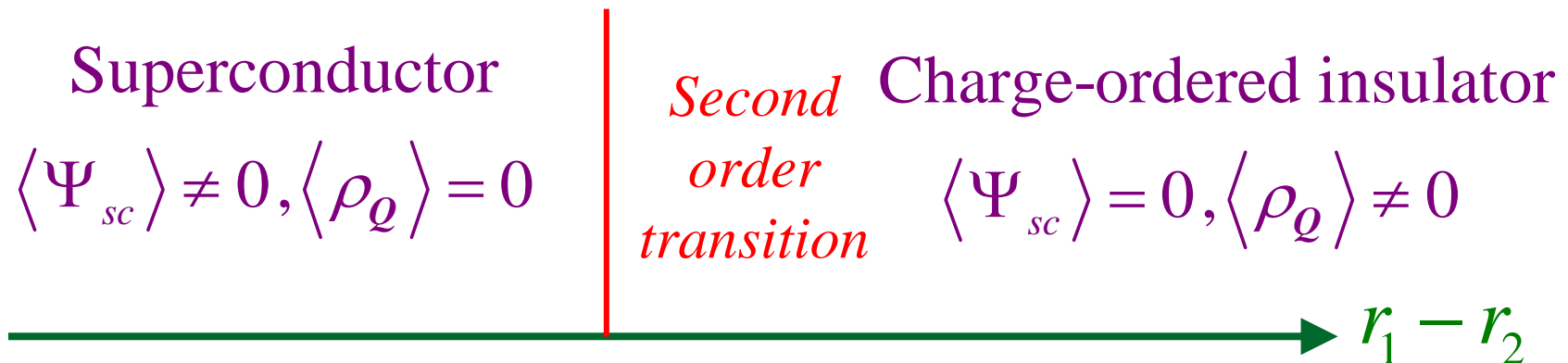
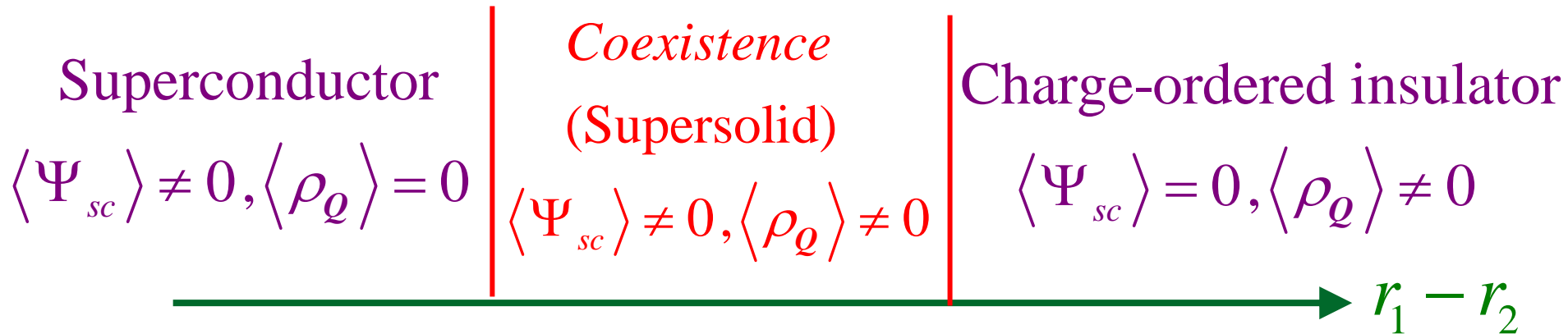
$$R : \varphi_\ell \rightarrow \frac{1}{\sqrt{q}} \sum_{m=1}^q \varphi_m e^{2\pi i l m f}$$

Immediate benefit: There is no intermediate “disordered” phase with neither order (or without “topological” order).

# Analysis of "extended LGW" theory of projective representation

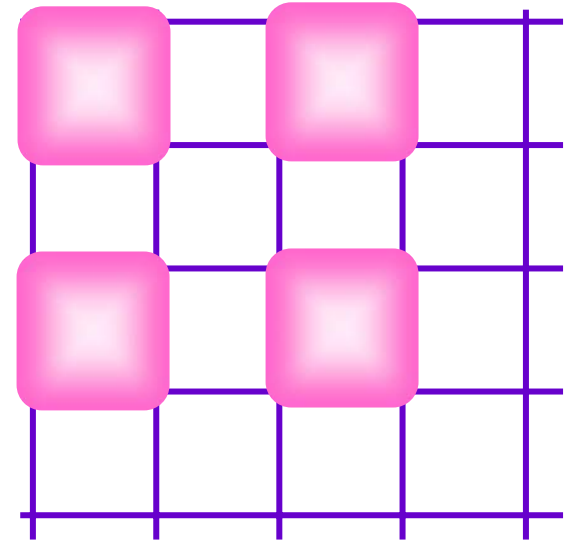
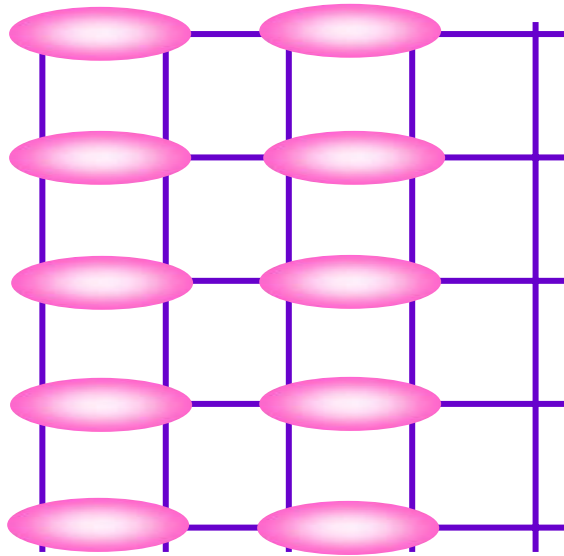
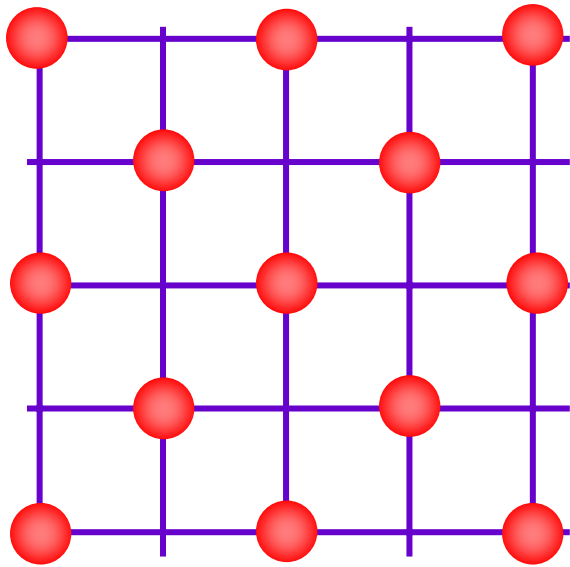


# Analysis of “extended LGW” theory of projective representation



# Analysis of “extended LGW” theory of projective representation

Spatial structure of insulators for  $q=2$  ( $f=1/2$ )

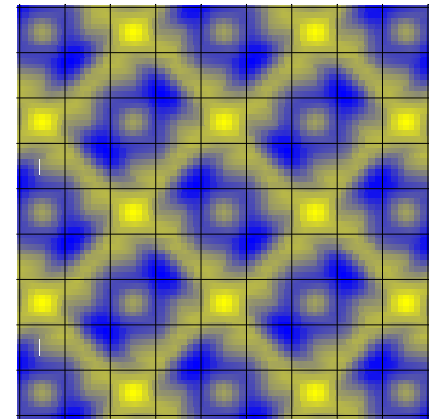
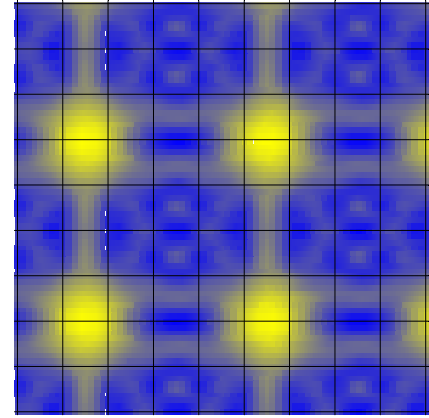
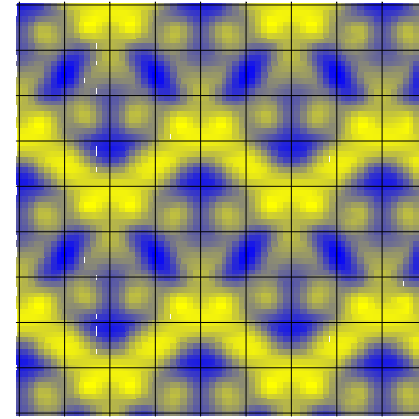
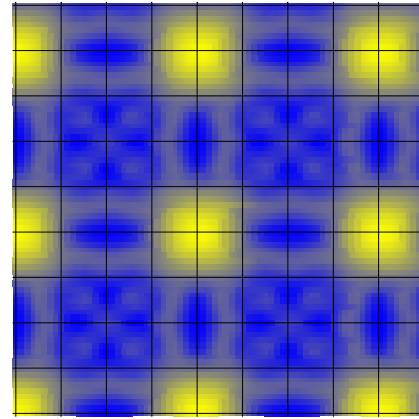
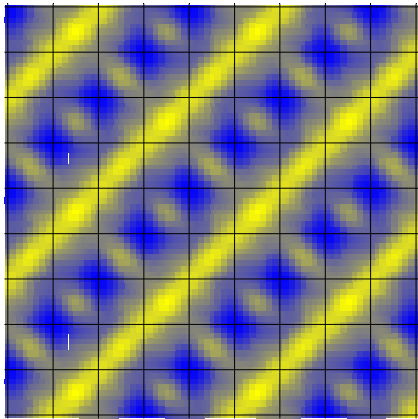
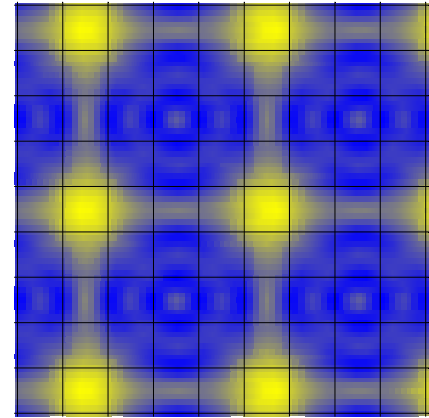
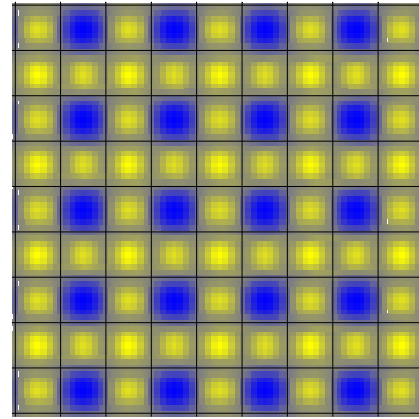
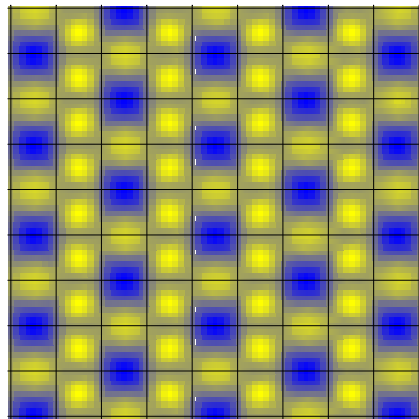
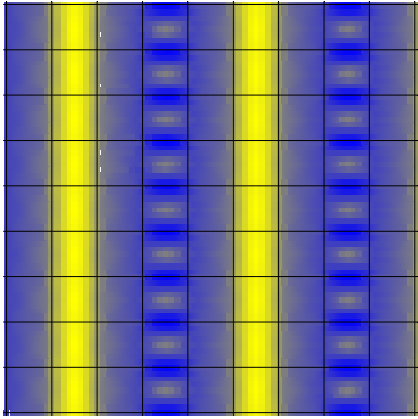


$$\text{pink oval} = \frac{1}{\sqrt{2}} (\text{red circle} + + + \text{red circle})$$

$a \times b$  unit cells;  $\frac{q}{a}$ ,  $\frac{q}{b}$ ,  $\frac{ab}{q}$ , all integers

# Analysis of “extended LGW” theory of projective representation

Spatial structure of insulators for  $q=4$  ( $f=1/4$  or  $3/4$ )



$a \times b$  unit cells;  $\frac{q}{a}$ ,  $\frac{q}{b}$ ,  $\frac{ab}{q}$ , all integers

# Summary

- I. Ferromagnet/paramagnet and superfluid/insulator quantum phase transitions.
- II. Phases flanking the critical point have distinct stable quasiparticle excitations; these quasiparticles disappear at the quantum critical point.
- III. Landau-Ginzburg-Wilson theory provides a theoretical framework for simple quantum phase transitions with a single order parameter.
- IV. Multiple order parameters:
  - \* LGW theory permits a “disordered” phase which is often unphysical.
  - \* Berry phase effects are crucial.
  - \* Quantum fields transform under projective representations of symmetry groups and via exchange of emergent gauge bosons.