Magnetic phases and critical points of insulators and superconductors

Colloquium article:
Reviews of Modern Physics, **75**, 913 (2003).

Reviews:
http://onsager.physics.yale.edu/qafm.pdf
cond-mat/0203363

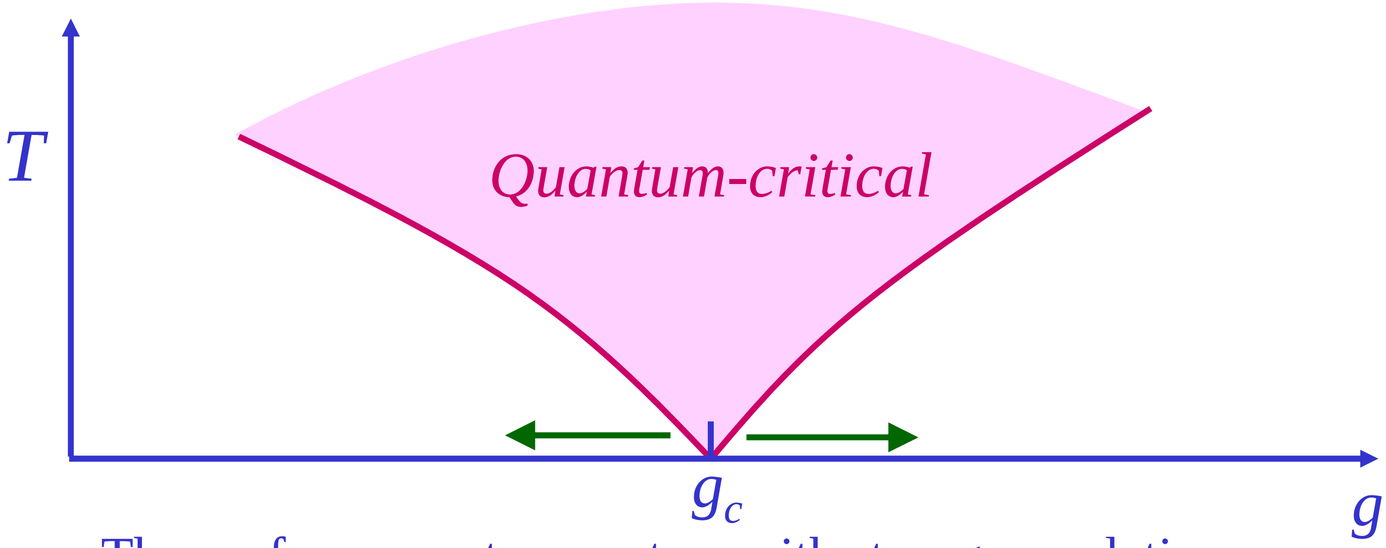
Talks online:
Google Sachdev

YALE UNIVERSITY TERCENTENNIAL
1701 2001

What is a quantum phase transition ?

Non-analyticity in ground state properties as a function of some control parameter g

Why study quantum phase transitions ?



- Theory for a quantum system with strong correlations: describe phases on either side of g_c by expanding in deviation from the quantum critical point.

- Critical point is a novel state of matter without quasiparticle excitations

- Critical excitations control dynamics in the wide *quantum-critical* region at non-zero temperatures.

Outline

A. “Dimerized” Mott insulators with a spin gap

Tuning quantum transitions by applied pressure

B. Spin gap state on the square lattice

Spontaneous bond order

C. Tuning quantum transitions by a magnetic field

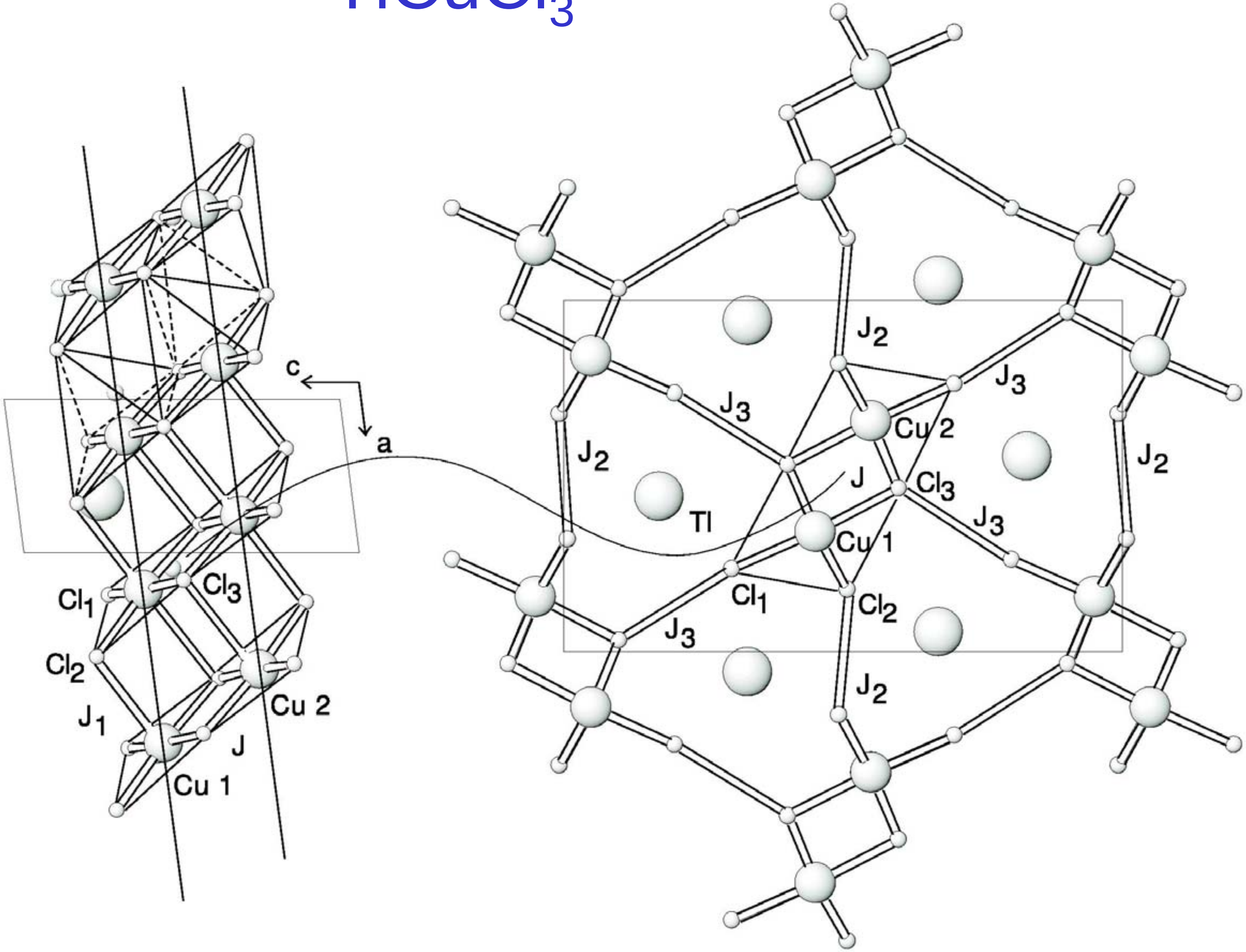
1. Mott insulators

2. Cuprate superconductors

(A) “Dimerized” Mott Insulators with a spin gap

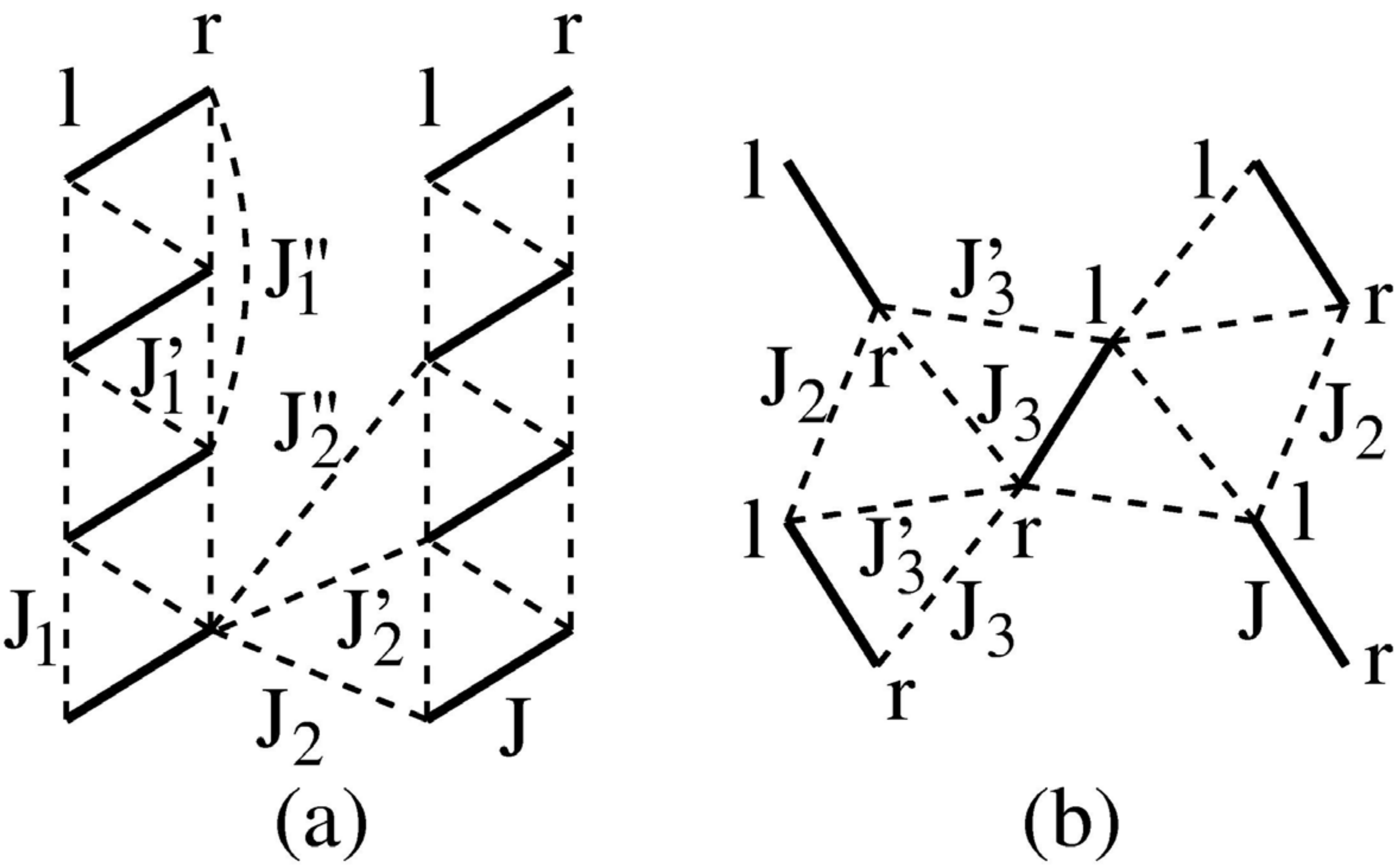
Tuning quantum transitions by applied pressure

$TlCuCl_3$



M. Matsumoto, B. Normand, T.M. Rice, and M. Sigrist, cond-mat/0309440.

$TlCuCl_3$

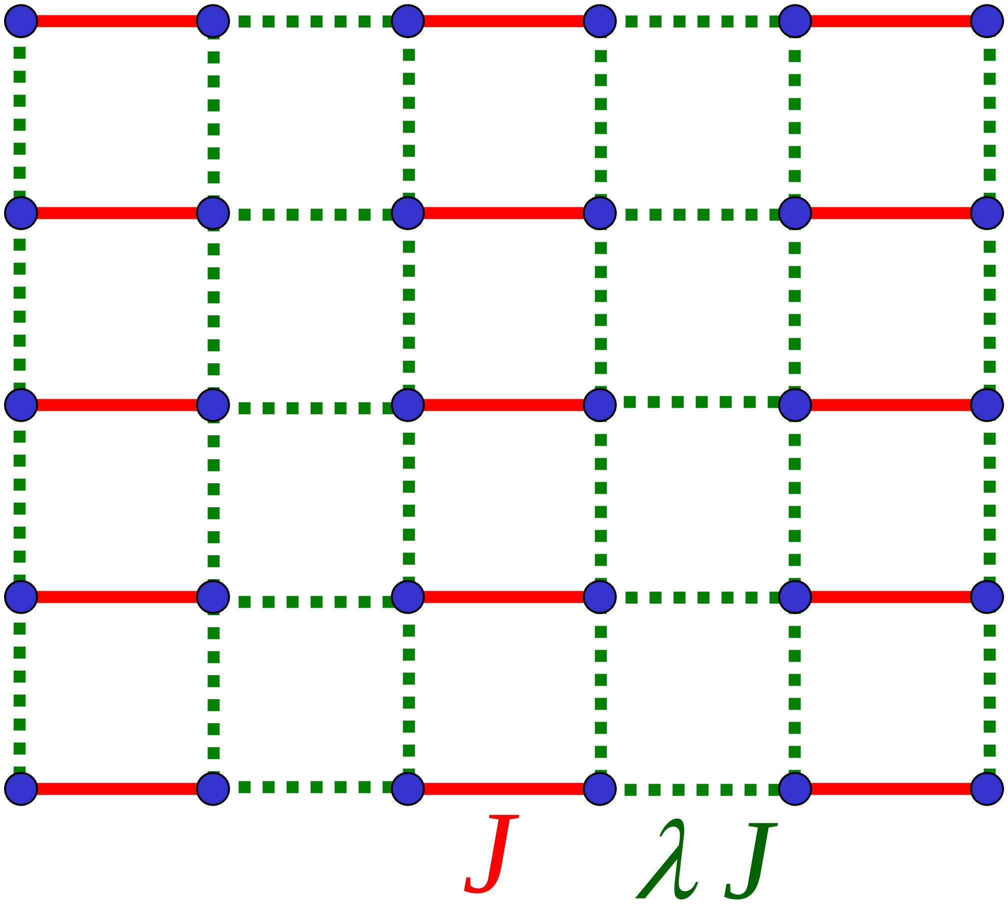


M. Matsumoto, B. Normand, T.M. Rice, and M. Sigrist, cond-mat/0309440.

Coupled Dimer Antiferromagnet

M. P. Gelfand, R. R. P. Singh, and D. A. Huse, *Phys. Rev. B* **40**, 10801-10809 (1989).
N. Katoh and M. Imada, *J. Phys. Soc. Jpn.* **63**, 4529 (1994).
J. Tworzydlo, O. Y. Osman, C. N. A. van Duin, J. Zaanen, *Phys. Rev.* B **59**, 115 (1999).
M. Matsumoto, C. Yasuda, S. Todo, and H. Takayama, *Phys. Rev.* B **65**, 014407 (2002).

S=1/2 spins on coupled dimers

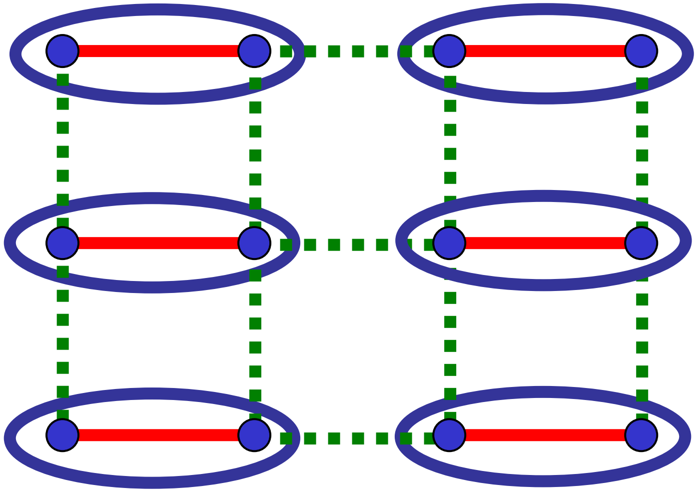


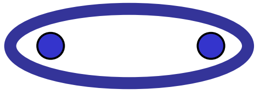
$$H = \sum_{<ij>} J_{ij} \vec{S}_i \cdot \vec{S}_j$$

$$0 \le \lambda \le 1$$

λ close to 0

Weakly coupled dimers





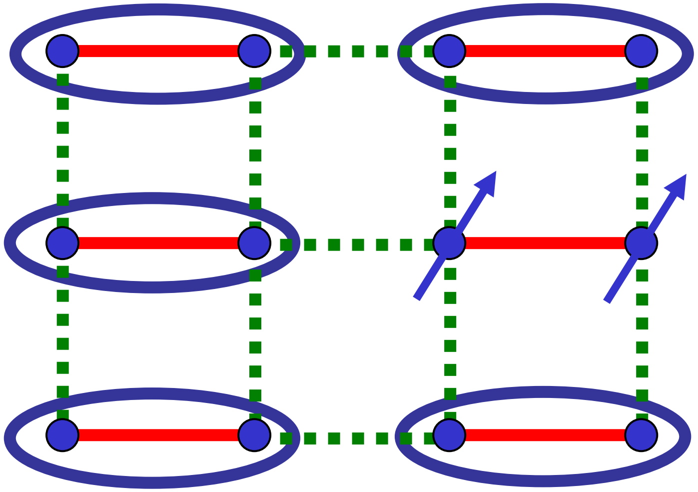
$$= \tfrac{1}{\sqrt{2}}\left(\left|\uparrow\downarrow\right\rangle - \left|\downarrow\uparrow\right\rangle\right)$$

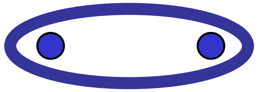
Paramagnetic ground state

$$\left\langle \vec{S}_i \right\rangle = 0$$

$\boxed{\lambda \text{ close to } 0}$

Weakly coupled dimers





$$= \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right)$$

Excitation: S=1 ***triplon*** (***exciton***, spin collective mode)

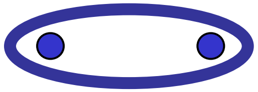
Energy dispersion away from
antiferromagnetic wavevector

$$\varepsilon_p = \Delta + \frac{c_x^2 p_x^2 + c_y^2 p_y^2}{2\Delta}$$

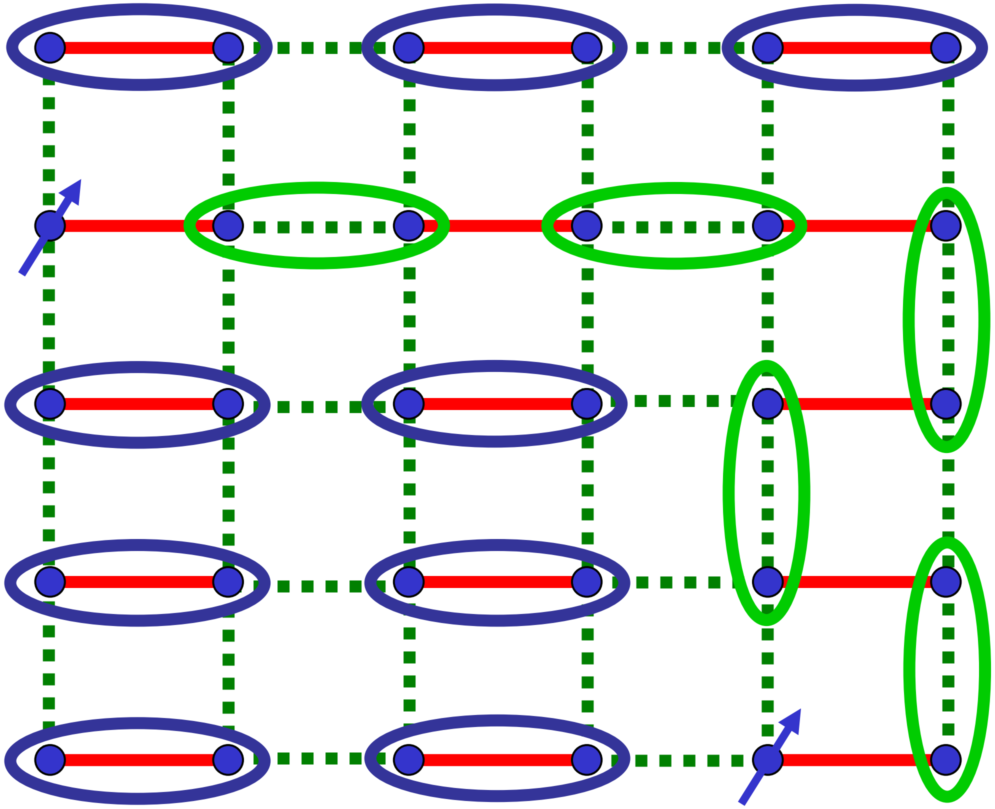
$\Delta \rightarrow$ spin gap

λ close to 0

Weakly coupled dimers

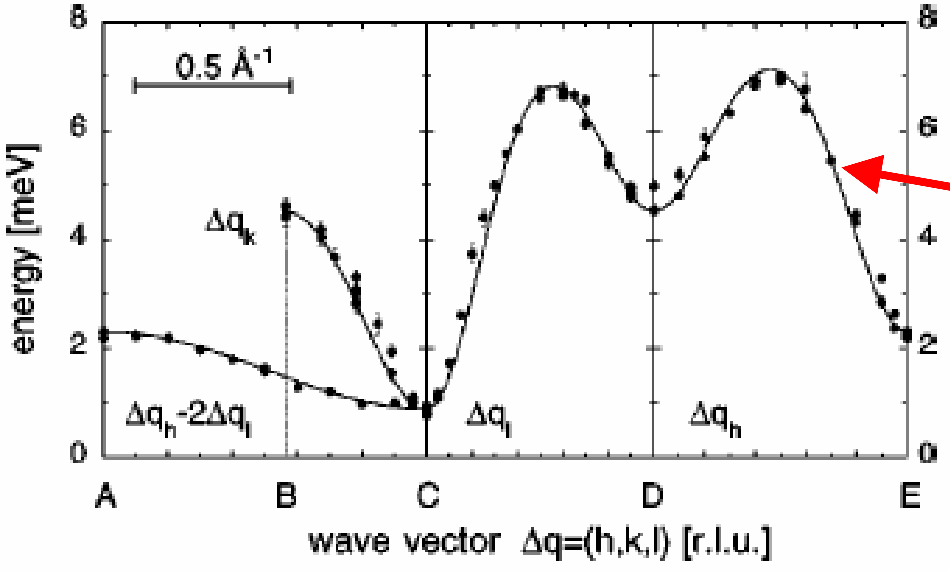


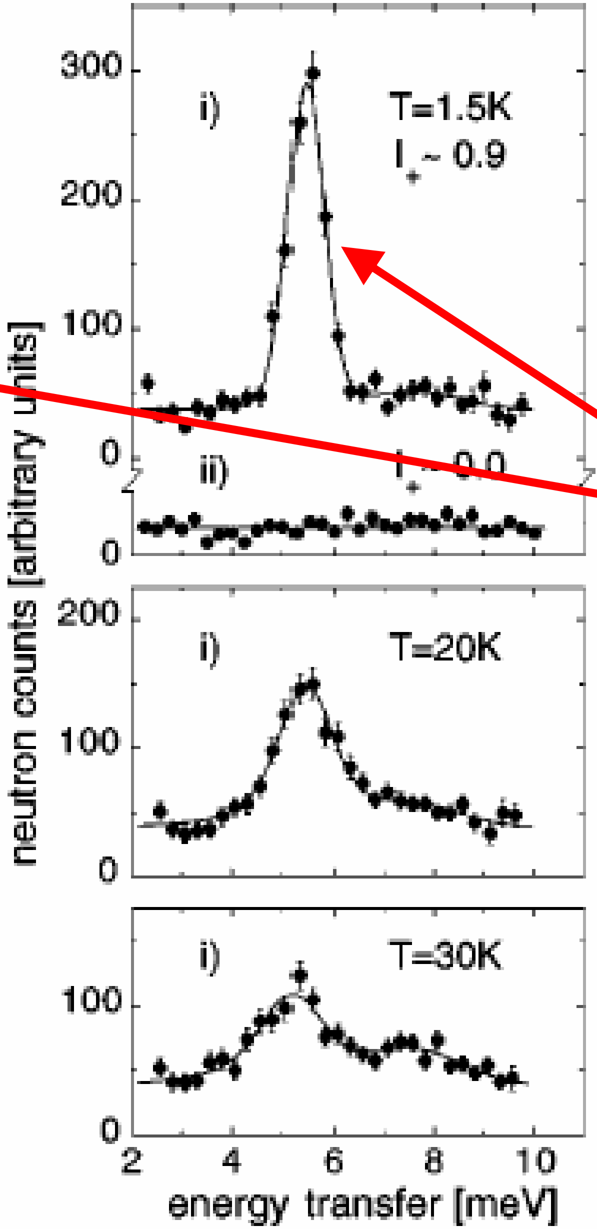
$$= \frac{1}{\sqrt{2}}\left(\left|\uparrow\downarrow\right\rangle - \left|\downarrow\uparrow\right\rangle\right)$$



S=1/2 ***spinons*** are *confined* by a linear potential into a *S*=1 ***triplon***

$TlCuCl_3$





"triplon" or spin exciton

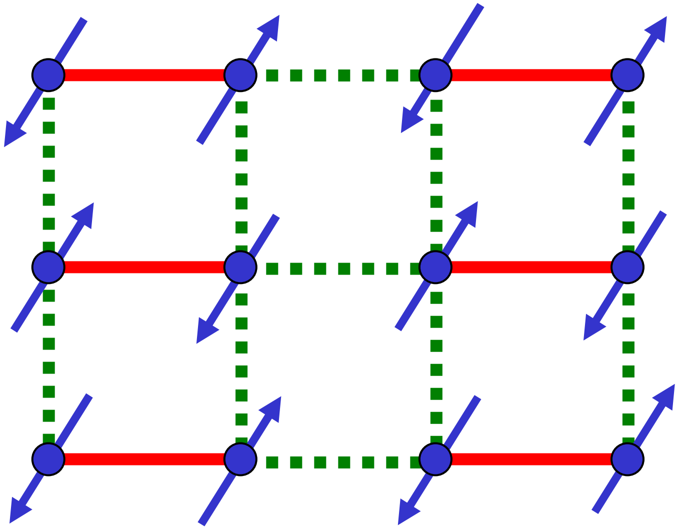
N. Cavadini, G. Heigold, W. Henggeler, A. Furrer, H.-U. Güdel, K. Krämer and H. Mutka, *Phys. Rev.* B 63 172414 (2001).

FIG. 1. Measured neutron profiles in the a^*c^* plane of $TlCuCl_3$ for $i=(1.35,0,0)$, $ii=(0,0,3.15)$ [r.l.u]. The spectrum at $T=1.5$ K

λ close to 1

Square lattice antiferromagnet

Experimental realization: La_2CuO_4

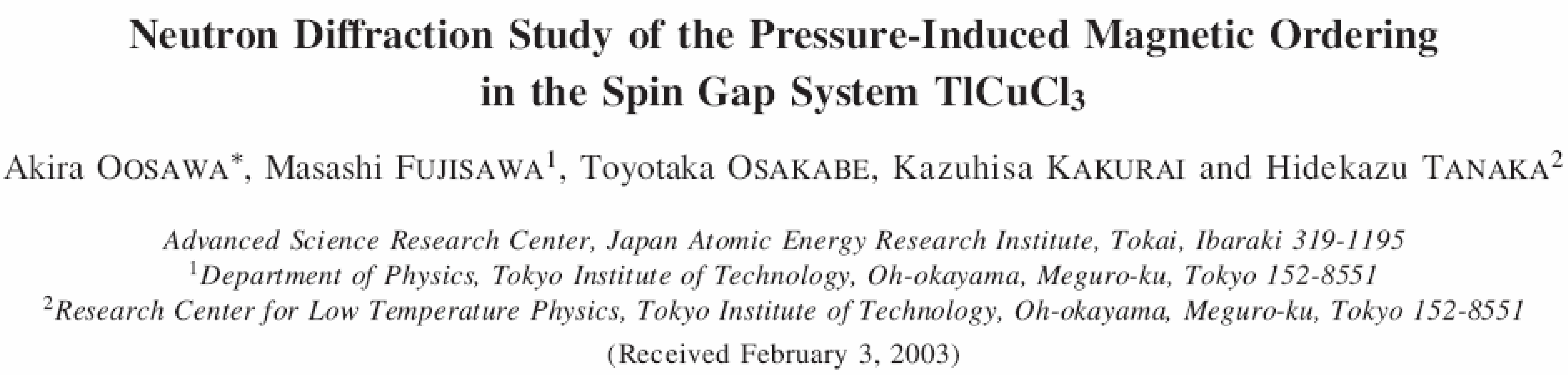


Ground state has long-range magnetic (Neel or spin density wave) order

$$\left\langle \vec{S}_i \right\rangle = (-1)^{i_x+i_y} N_0 \neq 0$$

Excitations: 2 spin waves (***magnons***) $\varepsilon_p = \sqrt{c_x{}^2 p_x{}^2 + c_y{}^2 p_y{}^2}$

TlCuCl_3



Neutron Diffraction Study of the Pressure-Induced Magnetic Ordering in the Spin Gap System TlCuCl_3

Akira OOSAWA*, Masashi FUJISAWA[1], Toyotaka OSAKABE, Kazuhisa KAKURAI and Hidekazu TANAKA[2]

Advanced Science Research Center, Japan Atomic Energy Research Institute, Tokai, Ibaraki 319-1195
[1]*Department of Physics, Tokyo Institute of Technology, Oh-okayama, Meguro-ku, Tokyo 152-8551*
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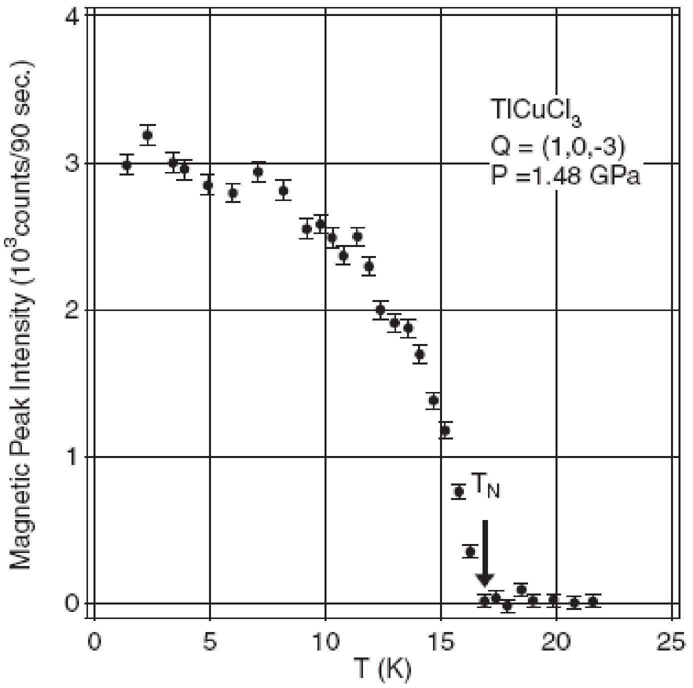
(Received February 3, 2003)



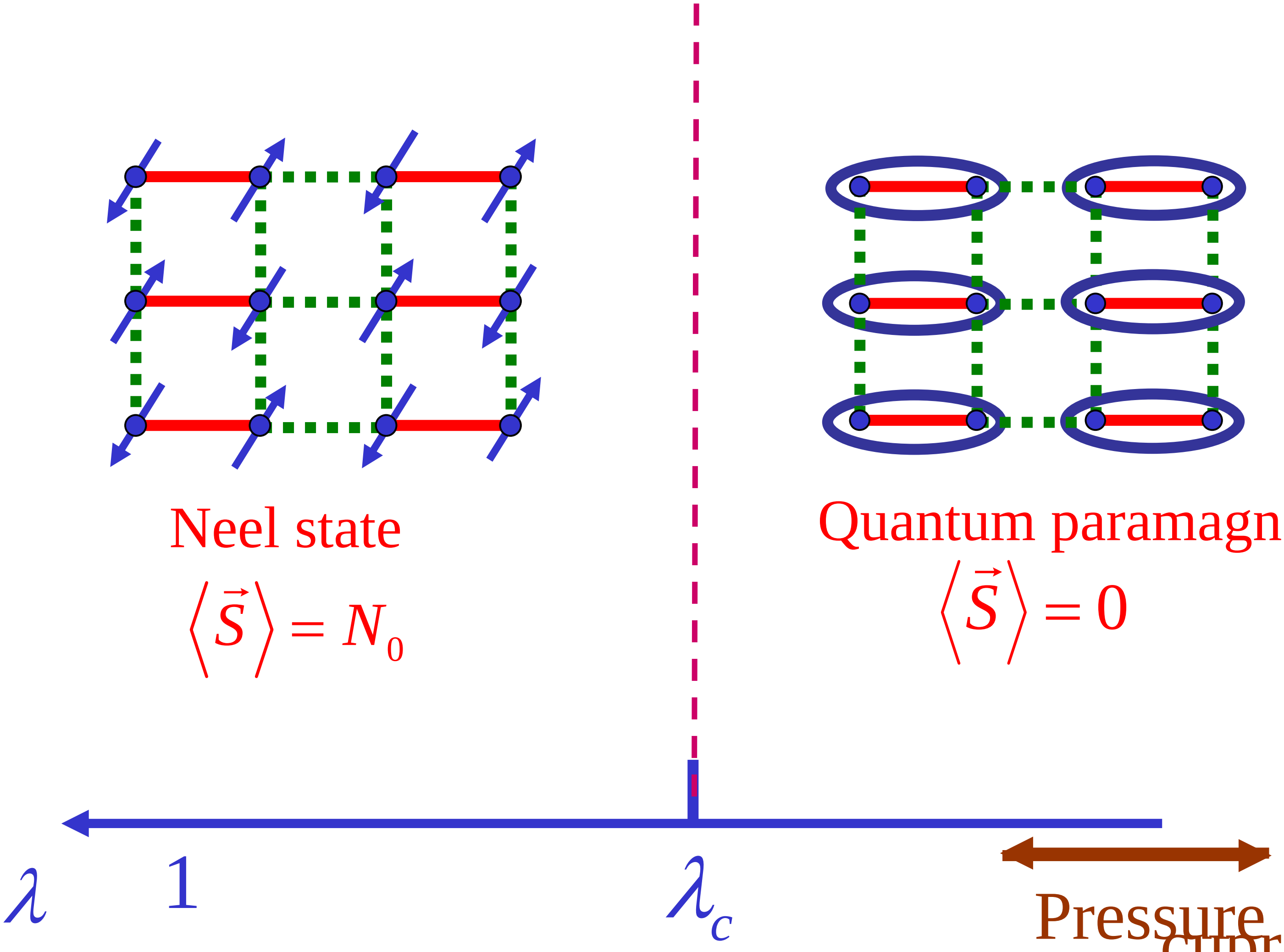
ig. 3. Temperature dependence of the magnetic Bragg peak intensity for $Q = (1, 0, -3)$ reflection measured at $P = 1.48$ GPa in TlCuCl_3.

J. Phys. Soc. Jpn **72**, 1026 (2003)

T=0

$\lambda_c = 0.52337(3)$

M. Matsumoto, C. Yasuda, S. Todo, and H. Takayama, *Phys. Rev.* B **65**, 014407 (2002)



Neel state

$$\langle \vec{S} \rangle = N_0$$

Quantum paramagnet

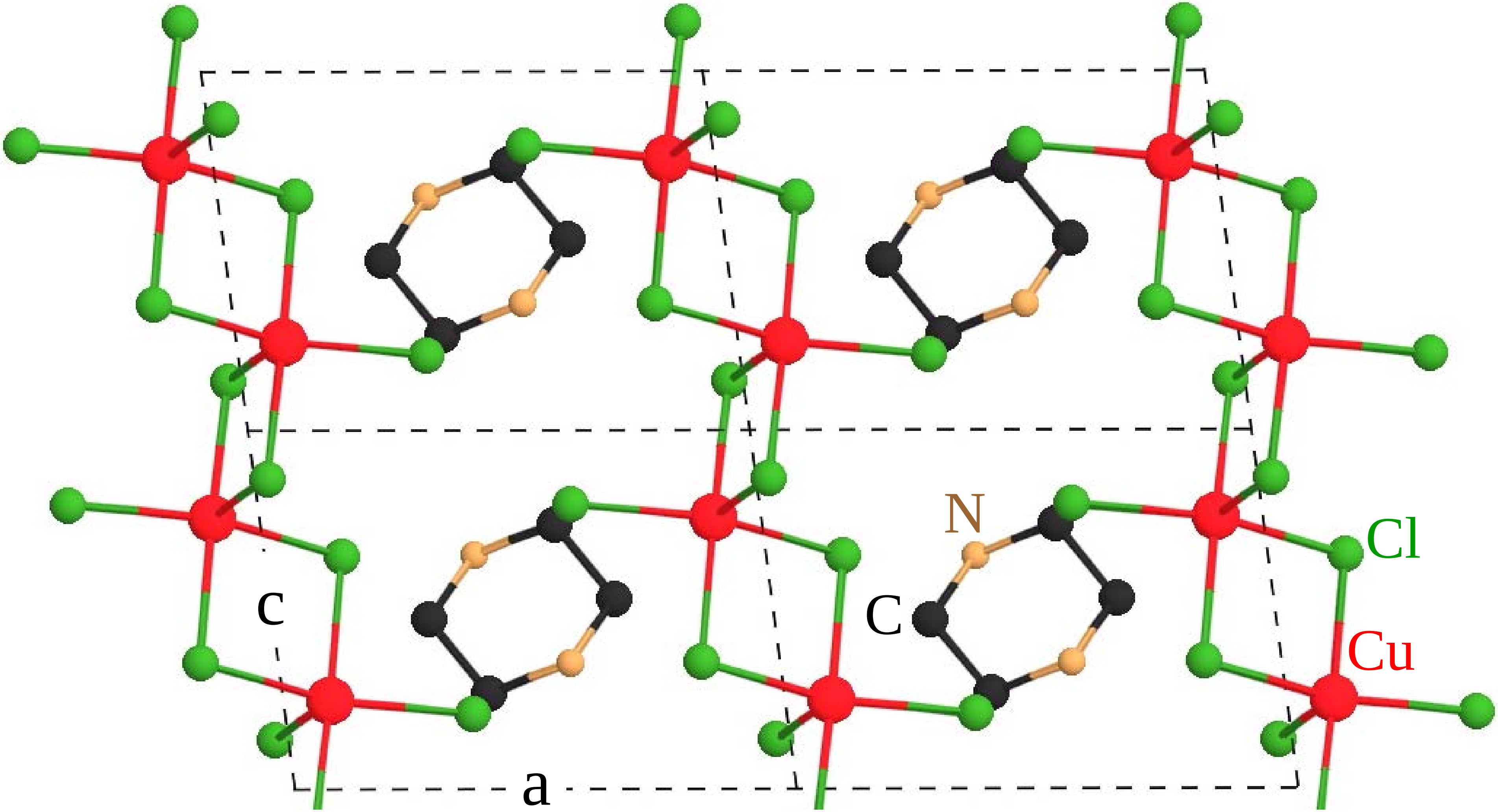
$$\langle \vec{S} \rangle = 0$$

δ in cuprates ?

Pressure in $TlCuCl_3$

PHCC – a two-dimensional antiferromagnet

PHCC = $C_4H_{12}N_2Cu_2Cl_6$



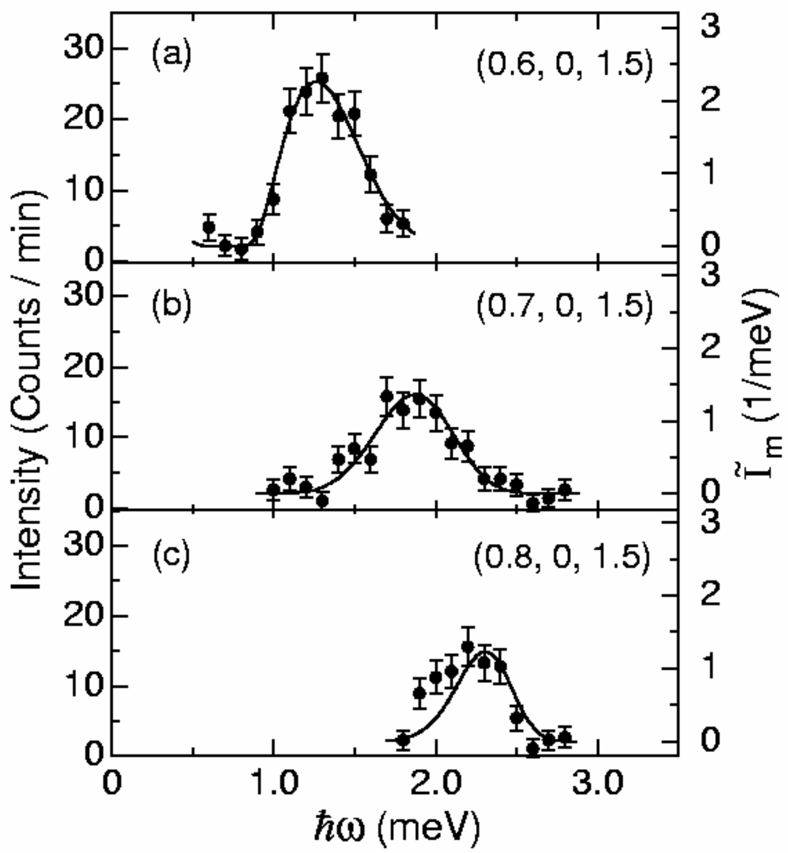
M. B. Stone, I. A. Zaliznyak, D. H. Reich, and C. Broholm, *Phys. Rev*. B **64,** 144405 (2001).

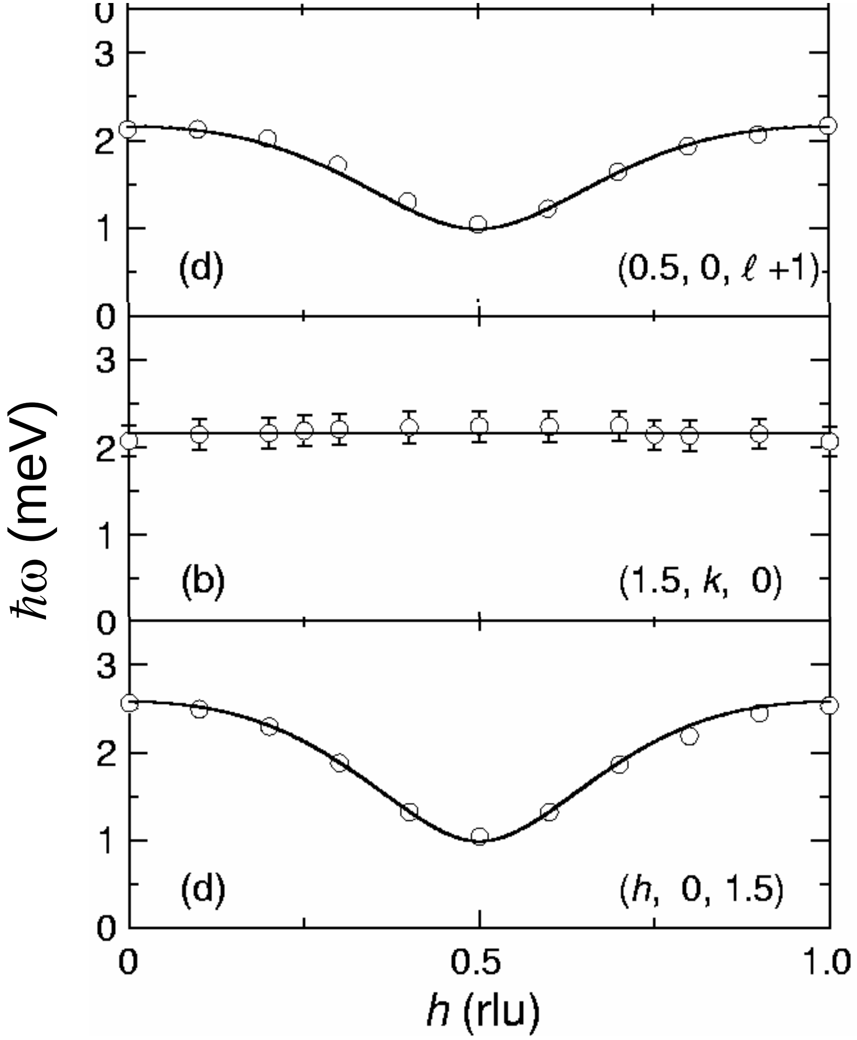
PHCC – a two-dimensional antiferromagnet

Dispersion ⊥ to “chains”

Not chains but planes

M. B. Stone, I. A. Zaliznyak, D. H. Reich, and C. Broholm, *Phys. Rev.* B **64,** 144405 (2001).





PHCC – a two-dimensional antiferromagnet

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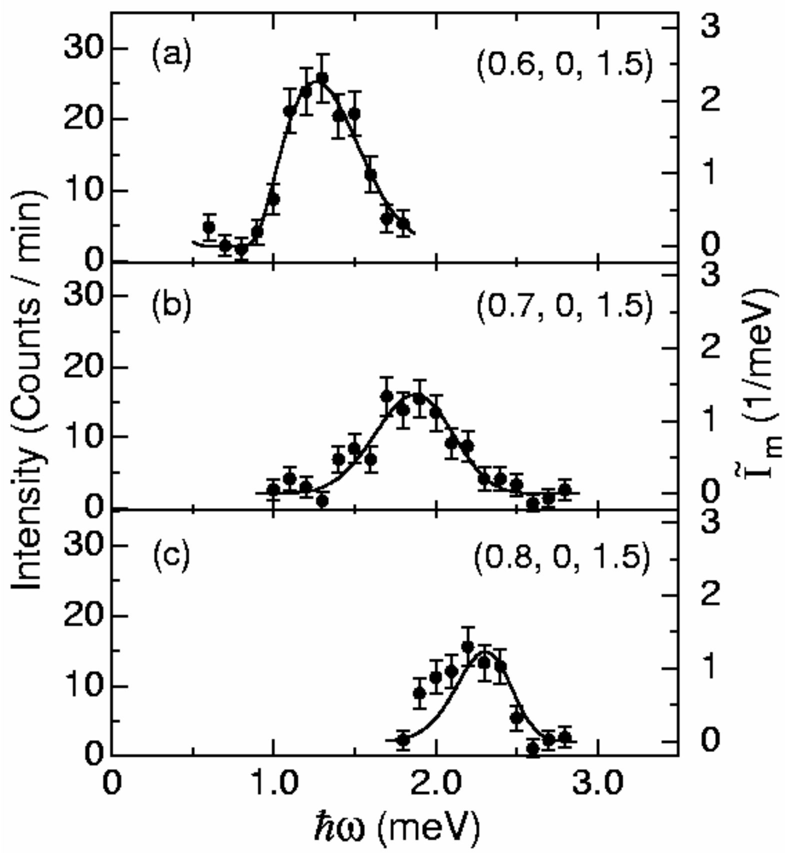
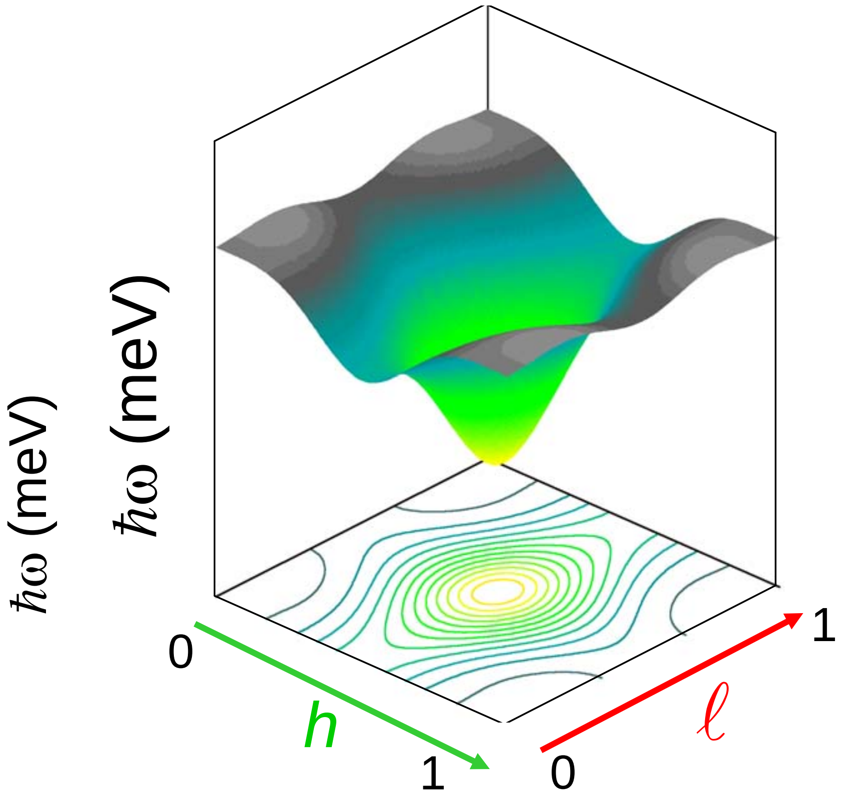




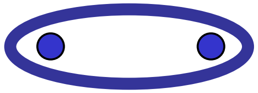
Triplon dispersion

Quantitative theory of experiments and simulations:

method of bond operators

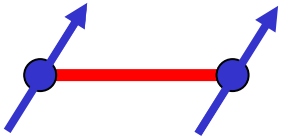
S. Sachdev and R.N. Bhatt, *Phys. Rev.* B **41**, 9323 (1990).

Operators algebra for all states on a single dimer



$$|s\rangle \equiv s^\dagger|0\rangle \equiv \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

$$|t_x\rangle \equiv t_x^\dagger|0\rangle \equiv \frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle\right)$$



$$|t_y\rangle \equiv t_y^\dagger|0\rangle \equiv \frac{i}{\sqrt{2}}\left(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle\right)$$

$$|t_z\rangle \equiv t_z^\dagger|0\rangle \equiv \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$

$$s^\dagger s + t_\alpha^\dagger t_\alpha = 1$$

$$\left[s, s^\dagger\right] = 1$$

$$\left[t_\alpha, t_\beta^\dagger\right] = \delta_{\alpha\beta}$$

Canonical Bose operators with a hard core constraint

$$S_{1\alpha} = \frac{1}{2}\left(s^\dagger t_\alpha + t_\alpha^\dagger s - i\varepsilon_{\alpha\beta\gamma} t_\beta^\dagger t_\gamma\right)$$

$$S_{1\alpha} = \frac{1}{2}\left(-s^\dagger t_\alpha - t_\alpha^\dagger s - i\varepsilon_{\alpha\beta\gamma} t_\beta^\dagger t_\gamma\right)$$

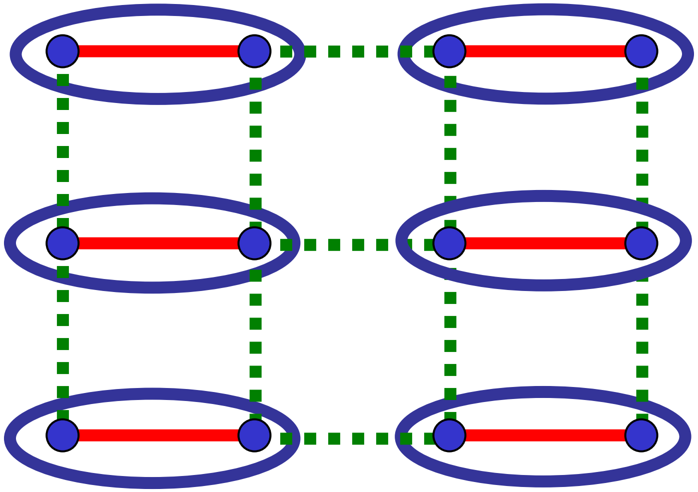
Spin operators on both sites can be expressed in terms of bond operators

Quantitative theory of experiments and simulations:

method of bond operators

S. Sachdev and R.N. Bhatt, *Phys. Rev.* B **41**, 9323 (1990).
A. V. Chubukov and Th. Jolicoeur, *Phys. Rev.* B **44**, 12050 (1991).

Hamiltonian for coupled dimers



Solve constraint by $s = \sqrt{1 - t_\alpha^\dagger t_\alpha}\,,$

$$H_t = \sum_k \left(A(k) t_{k\alpha}^\dagger t_{k\alpha} + \frac{B(k)}{2} \left(t_{k\alpha} t_{-k\alpha} + t_{k\alpha}^\dagger t_{-k\alpha}^\dagger \right) \right) + \cdots$$

Triplon dispersion:

$$\varepsilon(k) = \sqrt{A^2(k) - B^2(k)}$$

$$= \sqrt{\Delta^2 + c_x^2 (k_x - K_x)^2 + c_y^2 (k_y - K_y)^2}$$

Transition to magnetically ordered state occurs when $\Delta \to 0$ and the

t_α bosons condense, leading to $\langle t_\alpha \rangle \neq 0$

Quantitative theory of experiments and simulations:

method of bond operators

S. Sachdev and R.N. Bhatt, *Phys. Rev*. B **41,** 9323 (1990).

Quantum field theory for magnetic ordering transition

Action for t_α bosons near transition:

$$\begin{aligned}\mathcal{S}_t \;=\; \int d^d r d\tau \Big[& t_\alpha^\dagger \frac{\partial t_\alpha}{\partial \tau} + C t_\alpha^\dagger t_\alpha - \frac{D}{2}\left(t_\alpha t_\alpha + \text{H.c.}\right) + K_{1x}|\partial_x t_\alpha|^2 + K_{1y}|\partial_y t_\alpha|^2 \\ & + \frac{1}{2}\left(K_{2x}(\partial_x t_\alpha)^2 + K_{2y}(\partial_y t_\alpha)^2 + \text{H.c.}\right) + \cdots \Big] .\end{aligned}$$

Write

$$t_\alpha = \varphi_\alpha + i\pi_\alpha,$$

and integrate out π_α to obtain

$$\mathcal{S}_\varphi = \int d^d r d\tau \left[\frac{1}{2}\left\{ (\partial_\tau \varphi_\alpha)^2 + c_x^2 \left(\partial_x \varphi_\alpha\right)^2 + c_y^2 \left(\partial_y \varphi_\alpha\right)^2 + (\lambda_c - \lambda)\varphi_\alpha^2 \right\} + \frac{u}{24}\left(\varphi_\alpha^2\right)^2 \right] .$$

This φ^4 field theory also describes the finite temperature Curie transition of a *classical ferromagnet* in $d+1$ spatial dimensions

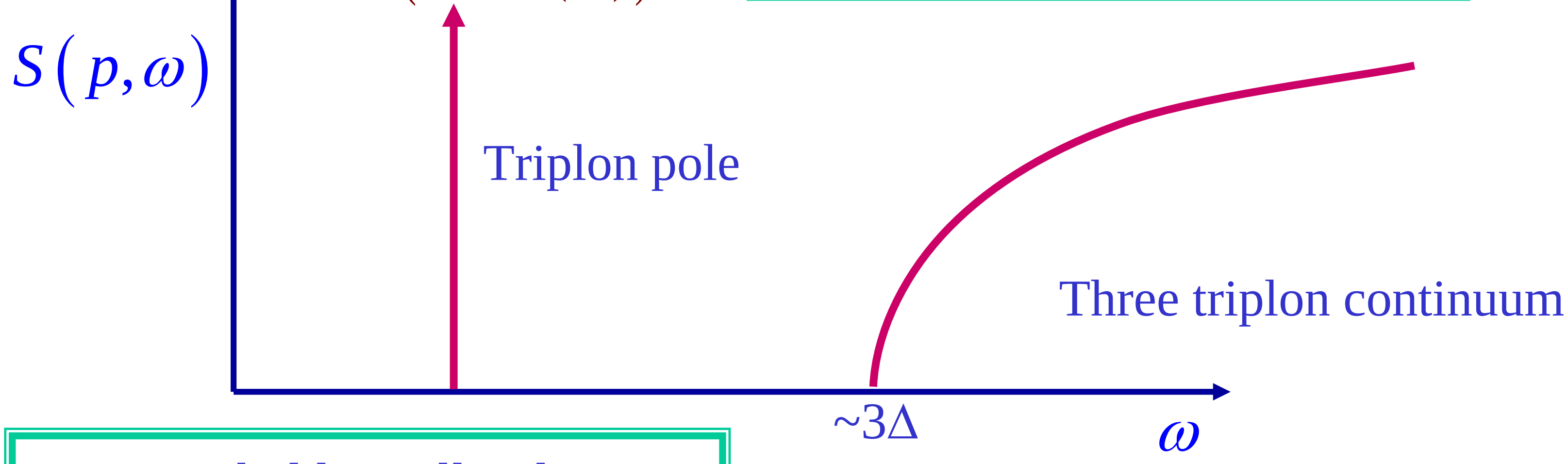
Field theory for quantum criticality

$$S_\varphi = \int d^2x\, d\tau \left[\frac{1}{2}\left(\left(\nabla_x \varphi_\alpha\right)^2 + c^2 \left(\partial_\tau \varphi_\alpha\right)^2 + \left(\lambda_c - \lambda\right)\varphi_\alpha^2 \right) + \frac{u}{4!}\left(\varphi_\alpha^2\right)^2 \right]$$

$\varphi_\alpha \longrightarrow$ 3-component antiferromagnetic order parameter

For $\lambda < \lambda_c$ oscillations of φ_α about $\varphi_\alpha = 0$ lead to the following structure in the dynamic structure factor $S(p,\omega)$

$$\varepsilon(p) = \Delta + \frac{c^2 p^2}{2\Delta} \quad ; \quad \Delta = \sqrt{\lambda_c - \lambda}\,/c$$



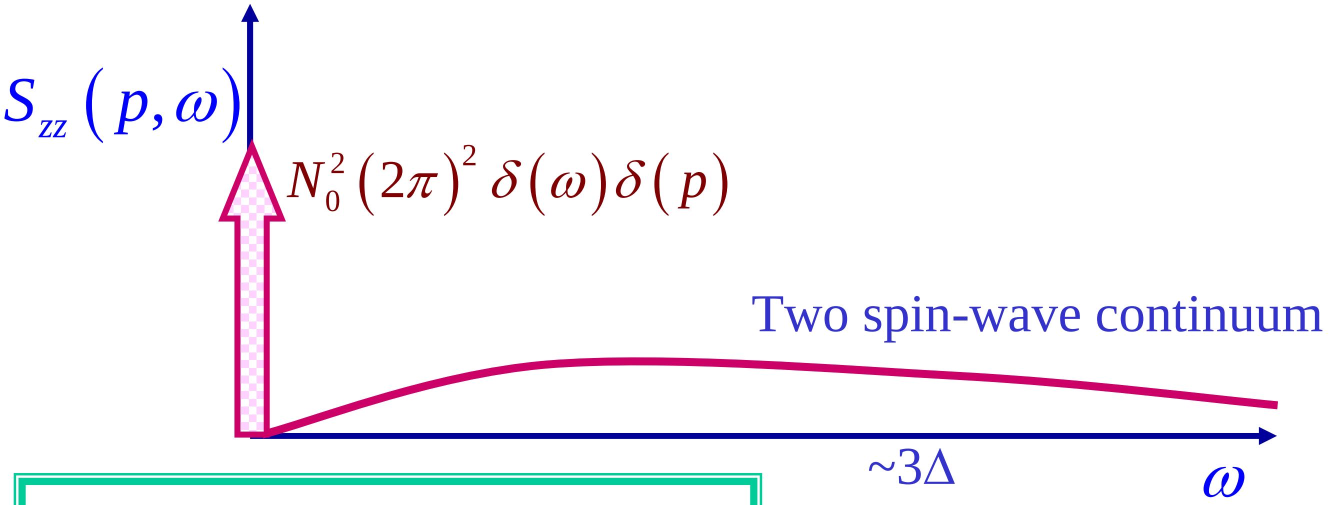
Structure holds to all orders in u

Field theory for quantum criticality

$$S_\varphi = \int d^2x\, d\tau \left[\frac{1}{2}\left(\left(\nabla_x \varphi_\alpha\right)^2 + c^2 \left(\partial_\tau \varphi_\alpha\right)^2 + \left(\lambda_c - \lambda\right)\varphi_\alpha^2 \right) + \frac{u}{4!}\left(\varphi_\alpha^2\right)^2 \right]$$

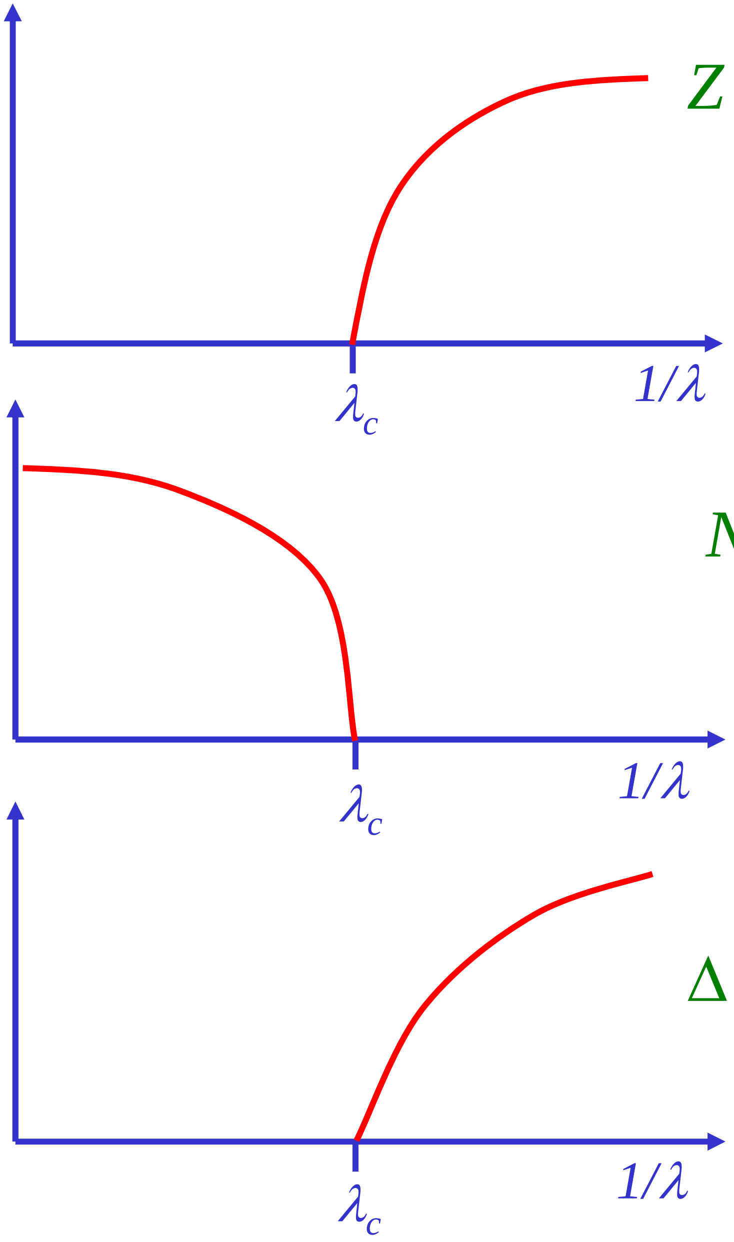
φ_α → 3-component antiferromagnetic order parameter

For $\lambda > \lambda_c$ oscillations of φ_α about $\varphi_z = N_0 \neq 0$ lead to the following *longitudinal* dynamic structure factor $S_{zz}(p,\omega)$



Structure holds to all orders in *u*

Entangled states at λ of order λ_c



Triplon
quasiparticle
weight Z

$$Z \sim (\lambda_c - \lambda)^{\eta\nu}$$

A.V. Chubukov, S. Sachdev, and J.Ye,
Phys. Rev. B **49**, 11919 (1994)

Antiferromagnetic
moment N_0

$$N_0 \sim (\lambda - \lambda_c)^{\beta}$$

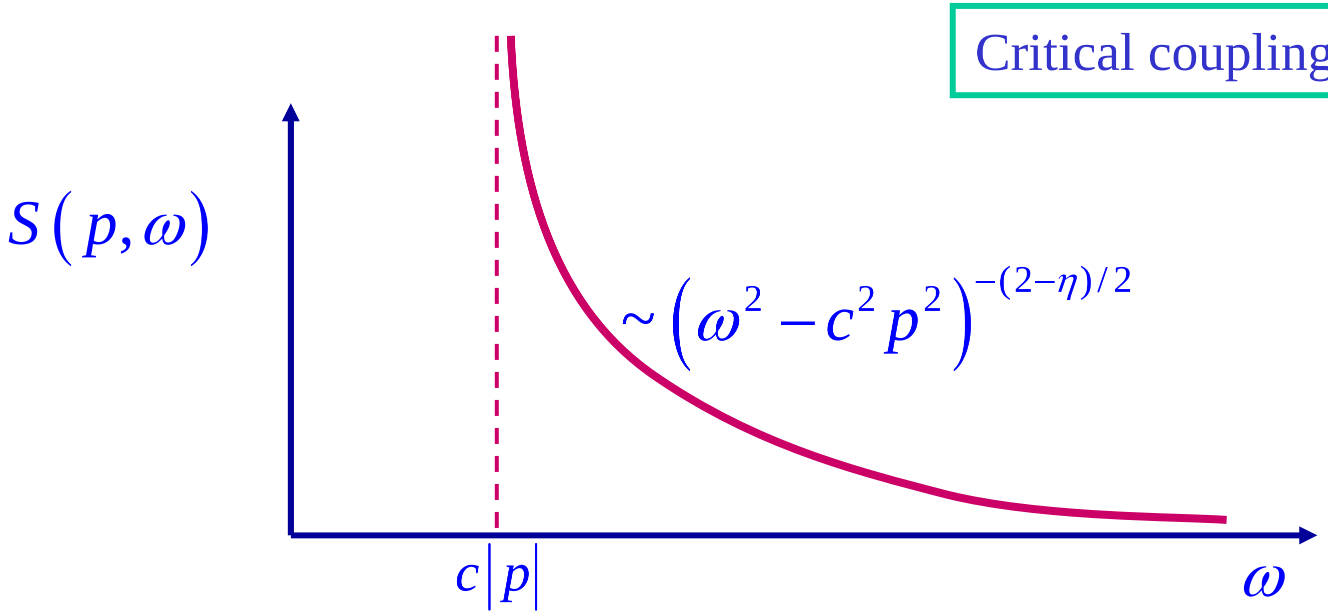
Triplon energy
gap Δ

$$\Delta \sim (\lambda_c - \lambda)^{\nu}$$

Field theory for quantum criticality

Dynamic spectrum at the critical point

Critical coupling $(\lambda = \lambda_c)$

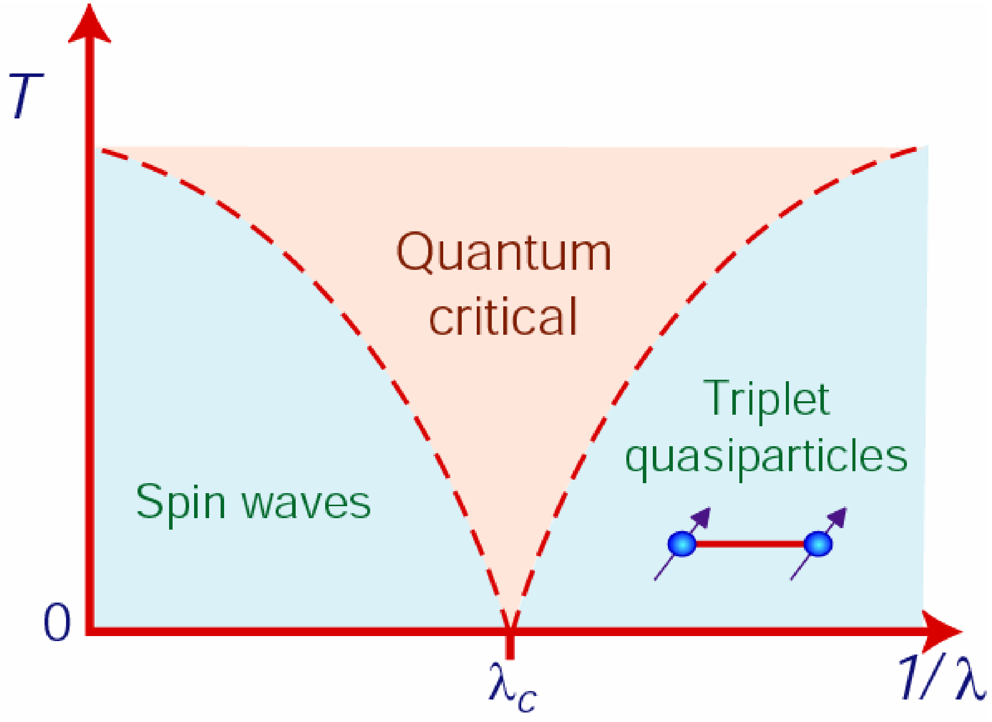


No quasiparticles --- dissipative critical continuum

Field theory for quantum criticality

$$\mathcal{S}_\varphi = \int d^d r \int_0^{1/T} d\tau \left[\frac{1}{2}\left\{(\partial_\tau\varphi_\alpha)^2 + c_x^2\left(\partial_x\varphi_\alpha\right)^2 + c_y^2\left(\partial_y\varphi_\alpha\right)^2 + (\lambda_c-\lambda)\varphi_\alpha^2\right\} + \frac{u}{24}\left(\varphi_\alpha^2\right)^2\right]$$

Here φ_α is the Néel order parameter. Near quantum criticality, the quartic coupling u approaches the Wilson-Fisher fixed point value.



Quantum criticality described by strongly-coupled critical theory with universal dynamic response functions dependent on $\hbar\omega/k_BT$

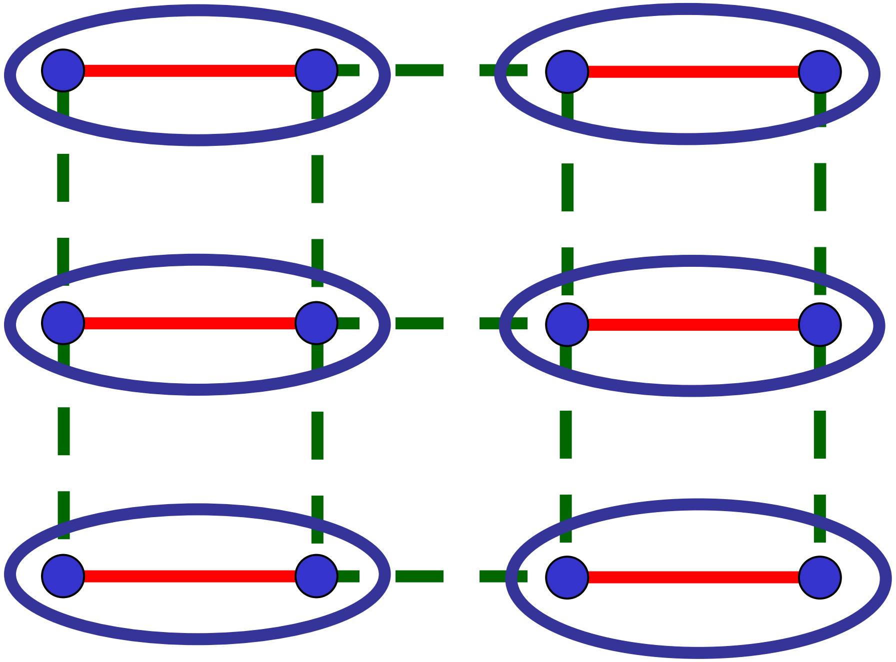
$$\chi(\omega,T) = T^\eta g\left(\hbar\omega/k_BT\right)$$

Triplon scattering amplitude is determined by k_BT alone, and not by the value of microscopic coupling u

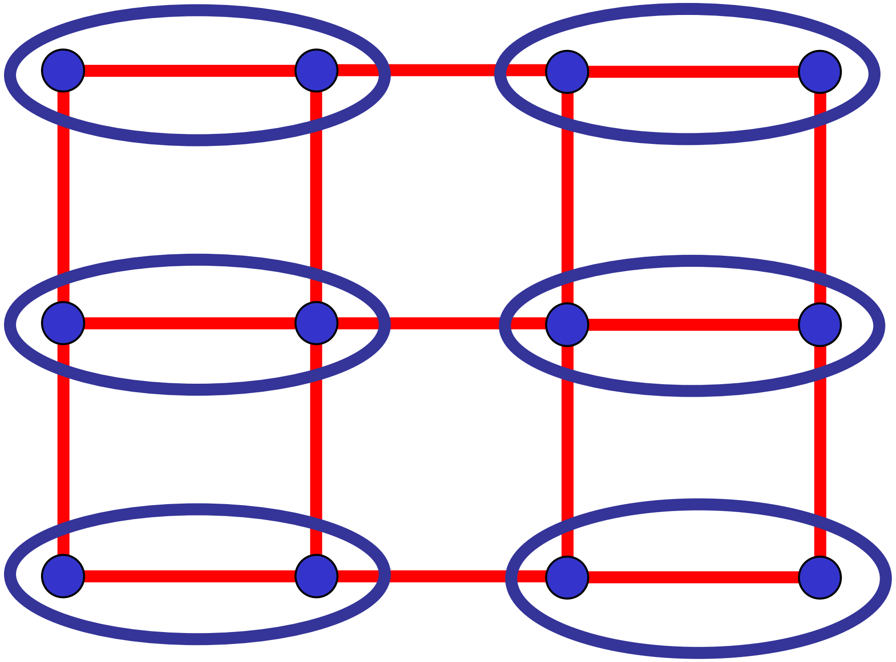
S. Sachdev and J. Ye, *Phys. Rev. Lett.* **69**, 2411 (1992).

(B) Spin gap state on the square lattice:
Spontaneous bond order

Paramagnetic ground state of coupled ladder model



Can such a state with ***bond order*** be the ground state of a system with full square lattice symmetry ?



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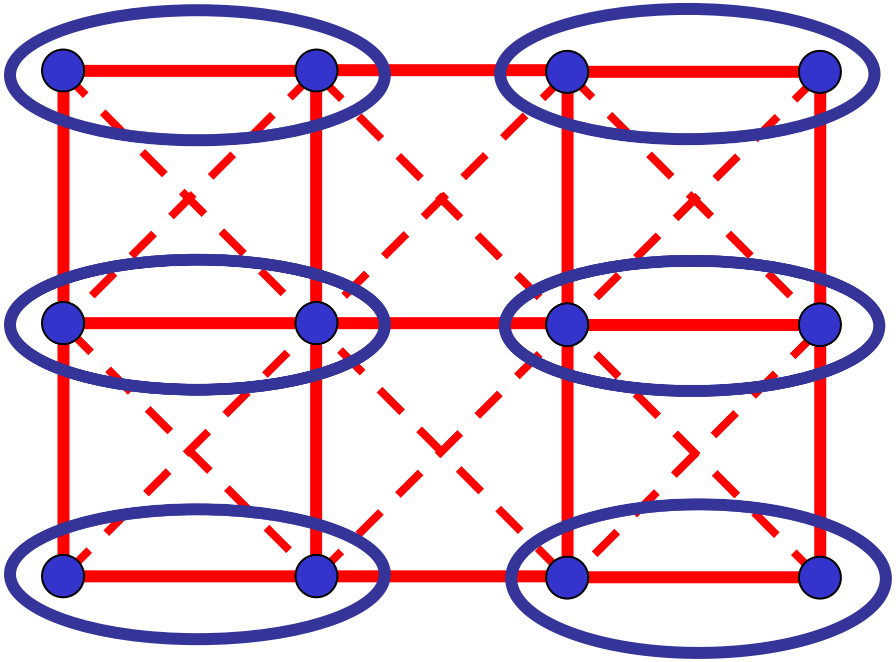
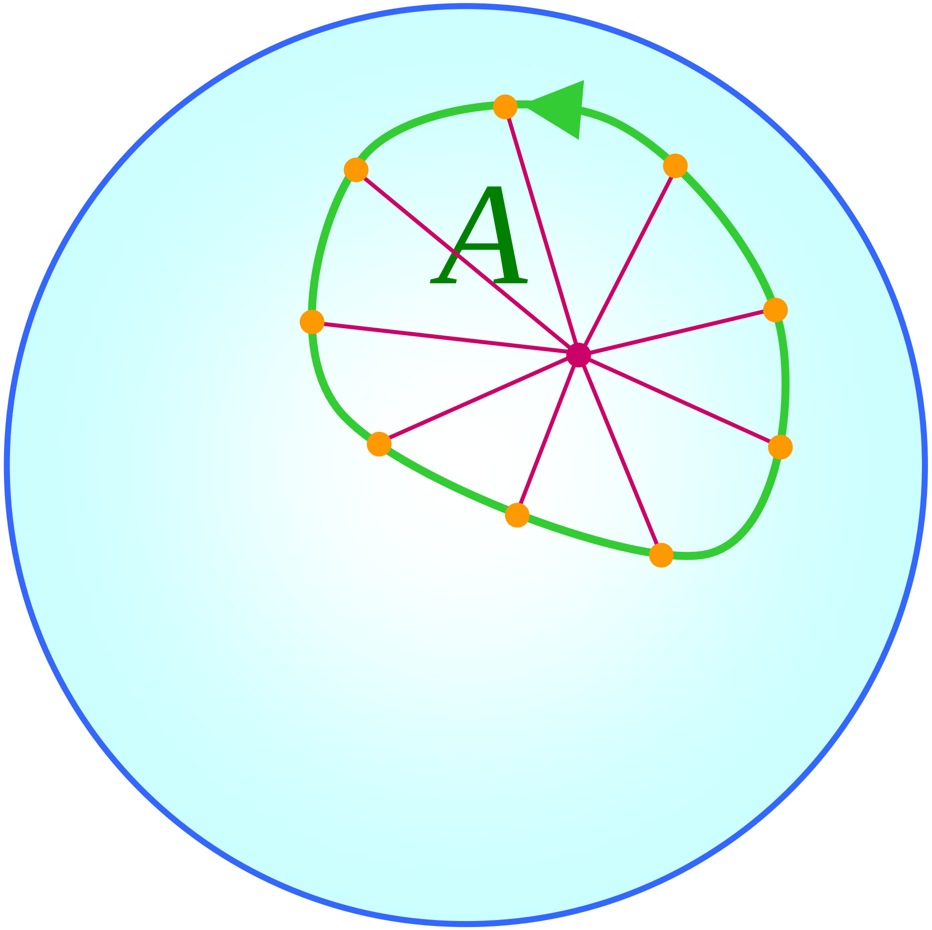


Need additional exchange interactions with full square lattice symmetry to move out of Neel state into paramagnet *e.g.* a second neighbor exchange J_2. This defines a dimensionless coupling $g = J_2 / J$

Collinear spins and compact U(1) gauge theory

Write down path integral for quantum spin fluctuations

Key ingredient: Spin Berry Phases

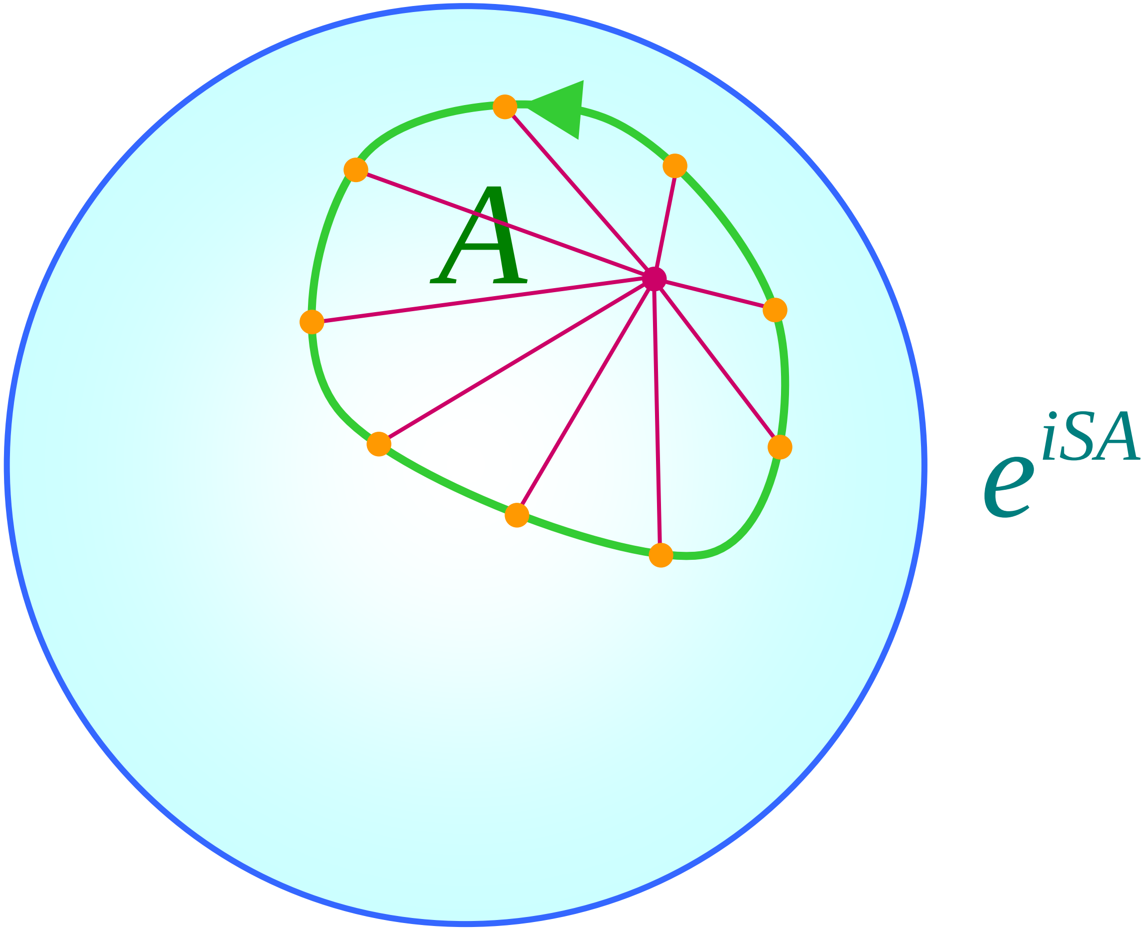


e^{iSA}

Collinear spins and compact U(1) gauge theory

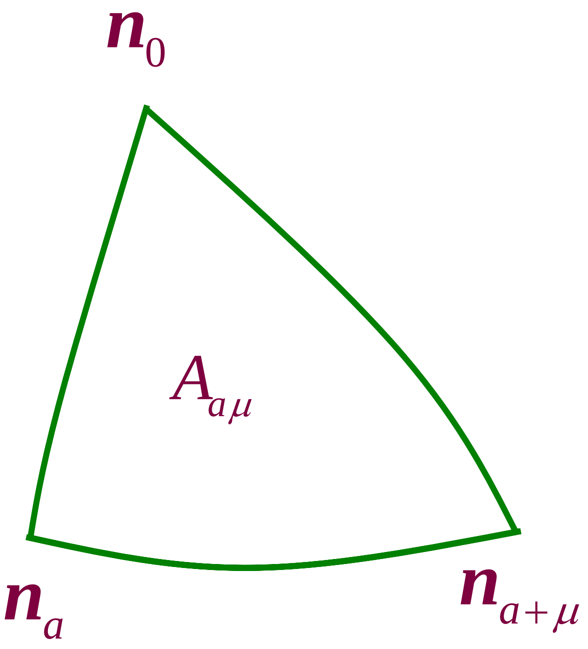
Write down path integral for quantum spin fluctuations

Key ingredient: Spin Berry Phases



e^{iSA}

Discretize imaginary time: path integral is over fields on the sites of a cubic lattice of points *a*



$\boldsymbol{n}_a \sim \eta_a \vec{S}_a$ → Neel order parameter;

η_a → ±1 on two square sublattices ;

$A_{a\mu}$ → oriented area of spherical triangle

formed by $\boldsymbol{n}_a$, $\boldsymbol{n}_{a+\mu}$, and an arbitrary reference point $\boldsymbol{n}_0$

Collinear spins and compact U(1) gauge theory

Partition function on square lattice

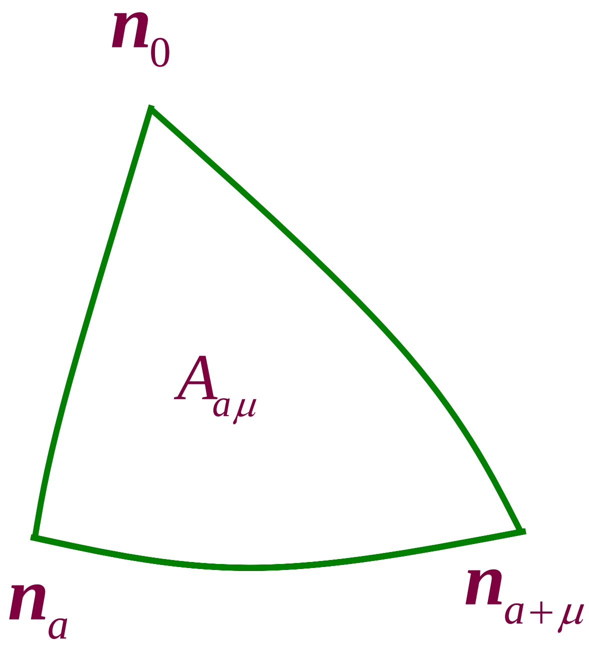
$$Z = \prod_a \int d\boldsymbol{n}_a \delta\left(\boldsymbol{n}_a^2 - 1\right) \exp\left(\frac{1}{g}\sum_{a,\mu} \boldsymbol{n}_a \cdot \boldsymbol{n}_{a+\mu} - \frac{i}{2}\sum_a \eta_a A_{a\tau}\right)$$

Modulus of weights in partition function: those of a classical ferromagnet at "temperature" g

Small $g \Rightarrow$ ground state has Neel order with $\left\langle \boldsymbol{n}_a \right\rangle = N_0 \neq 0$

Large $g \Rightarrow$ paramagnetic ground state with $\left\langle \boldsymbol{n}_a \right\rangle = 0$

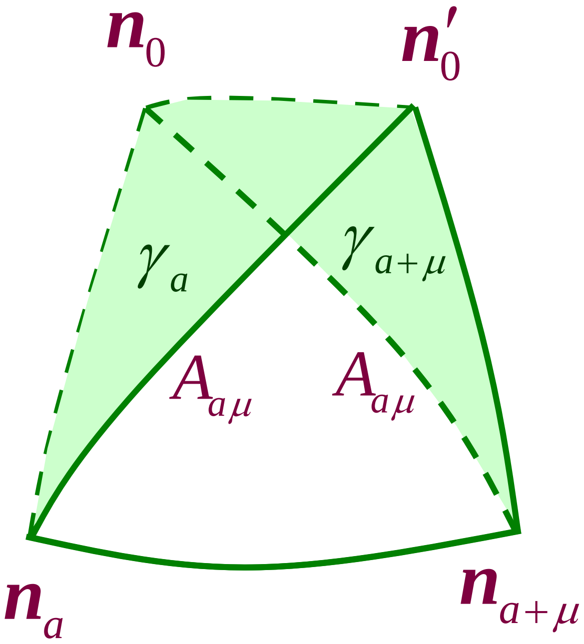
Berry phases lead to large cancellations between different time histories → need an effective action for $A_{a\mu}$ at large g


$\boldsymbol{n}_0$
$A_{a\mu}$
$\boldsymbol{n}_a$
$\boldsymbol{n}_{a+\mu}$

Change in choice of $\boldsymbol{n}_0$ is like a “gauge transformation”

$$A_{a\mu} \to A_{a\mu} - \gamma_{a+\mu} + \gamma_a$$

(γ_a is the oriented area of the spherical triangle formed by $\boldsymbol{n}_a$ and the two choices for $\boldsymbol{n}_0$).



The area of the triangle is uncertain modulo 4π, and the action is invariant under

$$A_{a\mu} \to A_{a\mu} + 4\pi$$

These principles strongly constrain the effective action for $A_{a\mu}$ which provides description of the large g phase

Simplest large g effective action for the $A_{a\mu}$

$$Z = \prod_{a,\mu} \int dA_{a\mu} \exp\left(-\frac{1}{2e^2} \sum_{\square} \cos\left(\frac{1}{2} \left(\Delta_\mu A_{a\nu} - \Delta_\nu A_{a\mu} \right) \right) - \frac{i}{2} \sum_a \eta_a A_{a\tau} \right)$$

with $e^2 \sim g^2$

This is compact QED in $d+1$ dimensions with static charges ± 1 on two sublattices.

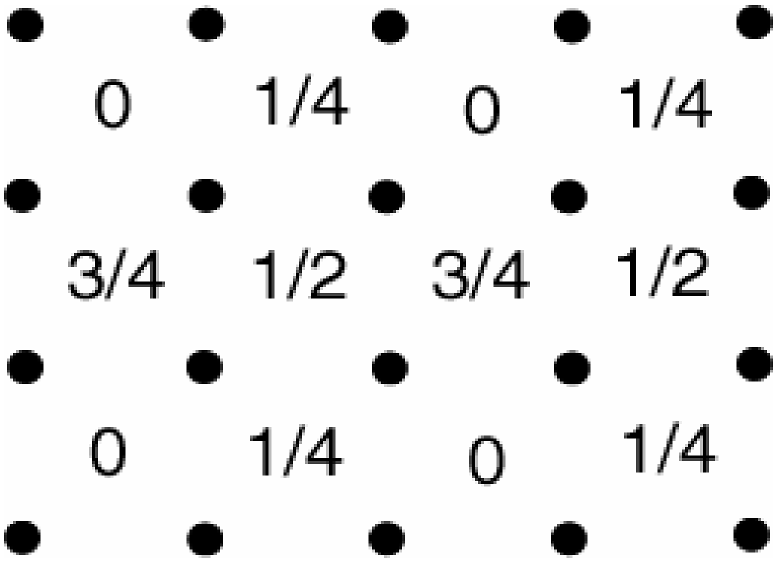
N. Read and S. Sachdev, *Phys. Rev. Lett*. **62**, 1694 (1989).
S. Sachdev and R. Jalabert, *Mod. Phys. Lett*. B **4**, 1043 (1990).
K. Park and S. Sachdev, *Phys. Rev*. B **65**, 220405 (2002).

Exact duality transform on a periodic Gaussian ("Villain") action for compact QED + Berry phases leads to a representation in terms of a "height" model

$$Z_{\text{dual}} = \sum_{\{h_{\bar{\jmath}}\}} \exp\left(-\frac{e^2}{2} \sum_{\bar{\jmath}} \left(\Delta_\mu h_{\bar{\jmath}} - \Delta_\mu \mathcal{X}_{\bar{\jmath}} \right)^2 \right)$$

with the $h_{\bar{\jmath}}$ integer heights.

The Berry phases lead to height 'offsets' $\mathcal{X}_{\bar{\jmath}} = 0, 1/4, 1/2, 3/4$ on the four dual sublattices.

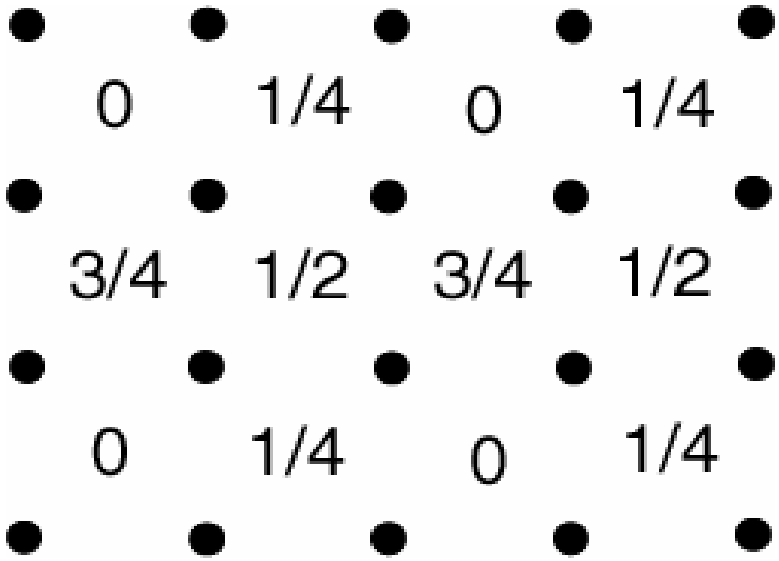


N. Read and S. Sachdev, *Phys. Rev. Lett*. **62**, 1694 (1989).

An alternative representation ia in terms of a Coulomb gas of monopoles

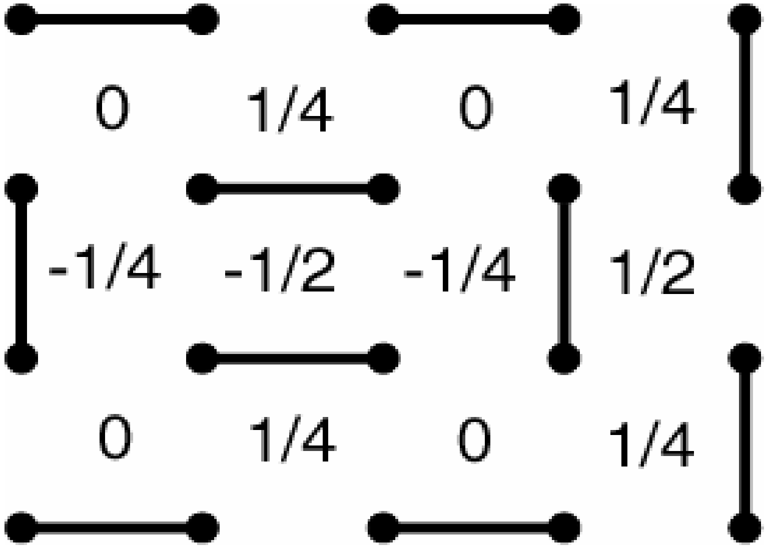
$$Z_{\mathrm{dual}} = \sum_{\{m_{\bar\jmath}\}} \exp\left(-\frac{\pi}{2e^2} \sum_{\bar\jmath,\bar\jmath'} \frac{m_{\bar\jmath} m_{\bar\jmath'}}{|r_{\bar\jmath} - r_{\bar\jmath'}|} + 2\pi i \sum_{\bar\jmath} m_{\bar\jmath} \mathcal{X}_{\bar\jmath} \right)$$

with the $m_{\bar\jmath}$ integer monopole charges. Each monopole carries a Berry phase (F.D.M. Haldane, *Phys. Rev. Lett.* **61**, 1029 (1988)) determined by the fixed $\mathcal{X}_{\bar\jmath} = 0, 1/4, 1/2, 3/4$ on the four dual sublattices.



N. Read and S. Sachdev, *Phys. Rev. Lett*. **62**, 1694 (1989).

For S=1/2 and large e^2 , low energy height configurations are in exact one-to-one correspondence with nearest-neighbor valence bond pairings of the sites square lattice

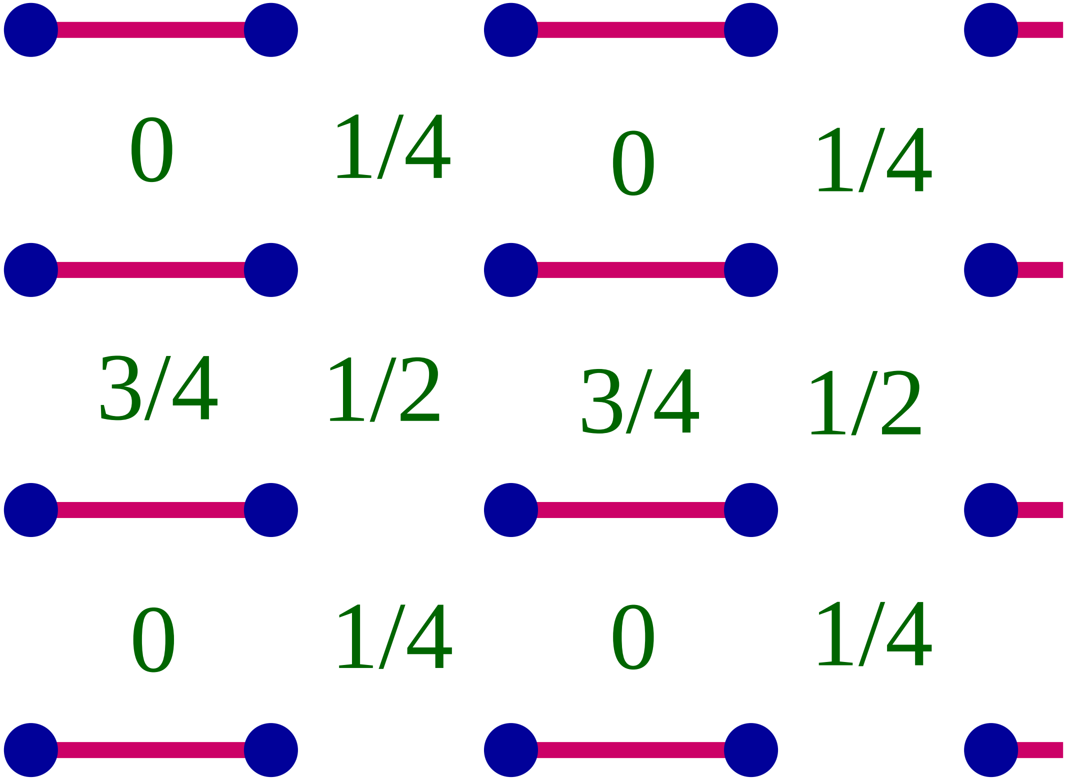


There is no roughening transition for three dimensional interfaces, which are smooth for all couplings

⇒ There is a definite average height of the interface

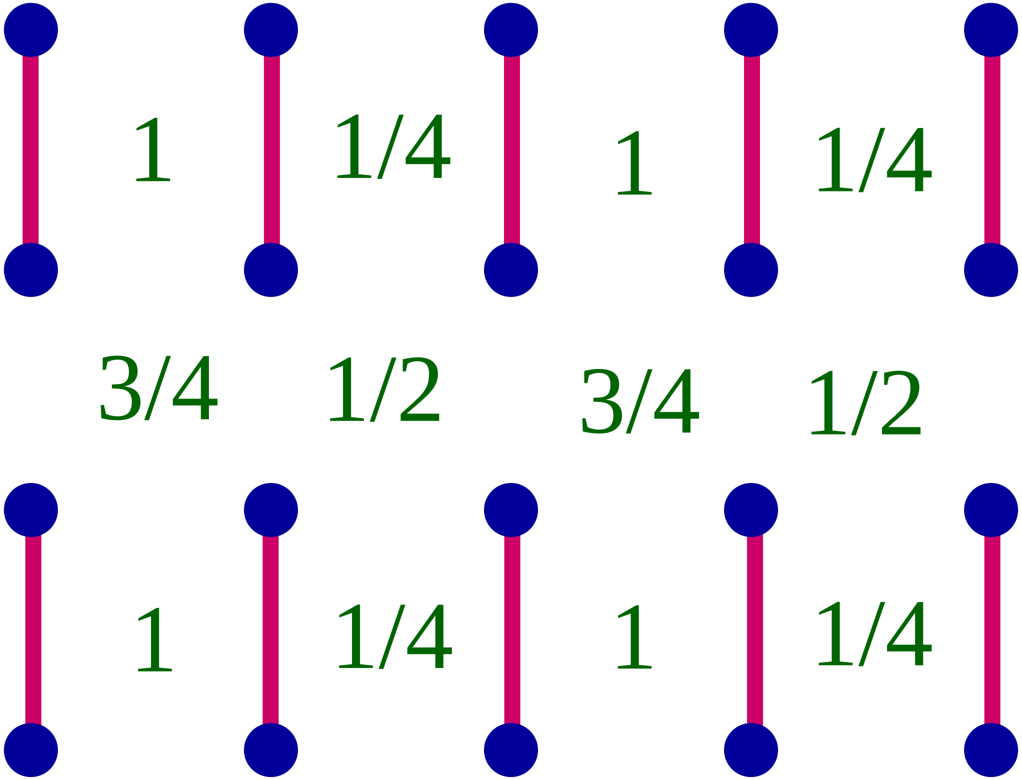
⇒ **Ground state has bond order.**

N. Read and S. Sachdev, *Phys. Rev. Lett*. **62**, 1694 (1989).



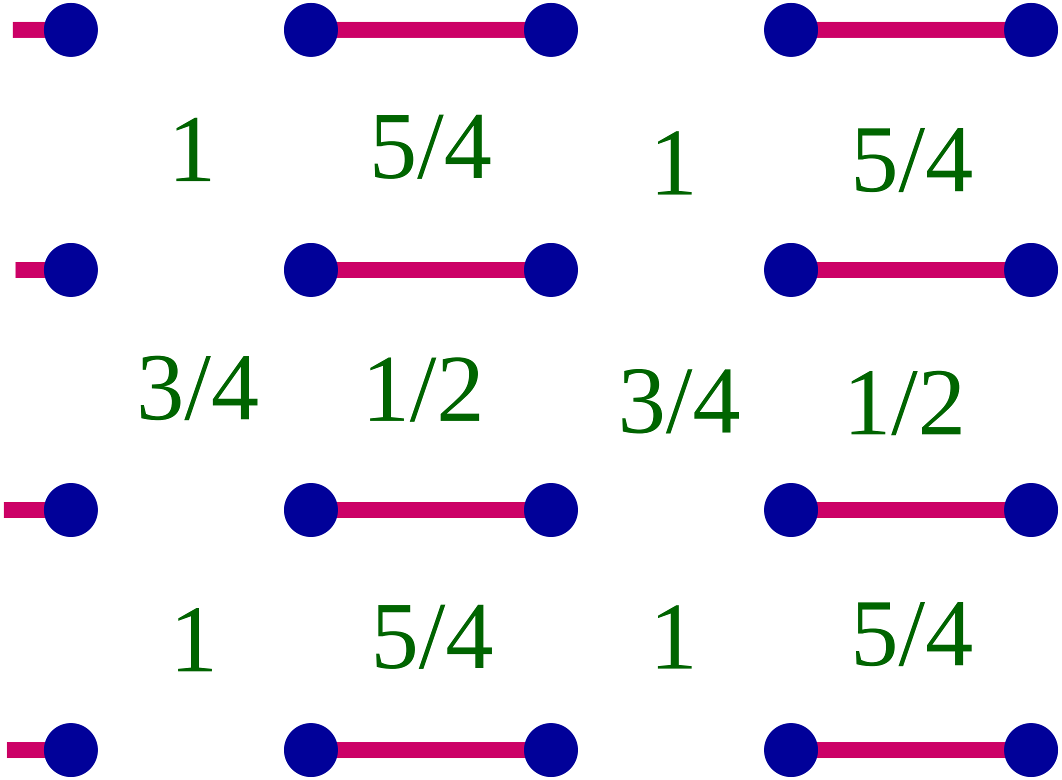
Smooth interface with average height 3/8

W. Zheng and S. Sachdev, *Phys. Rev*. B **40**, 2704 (1989)



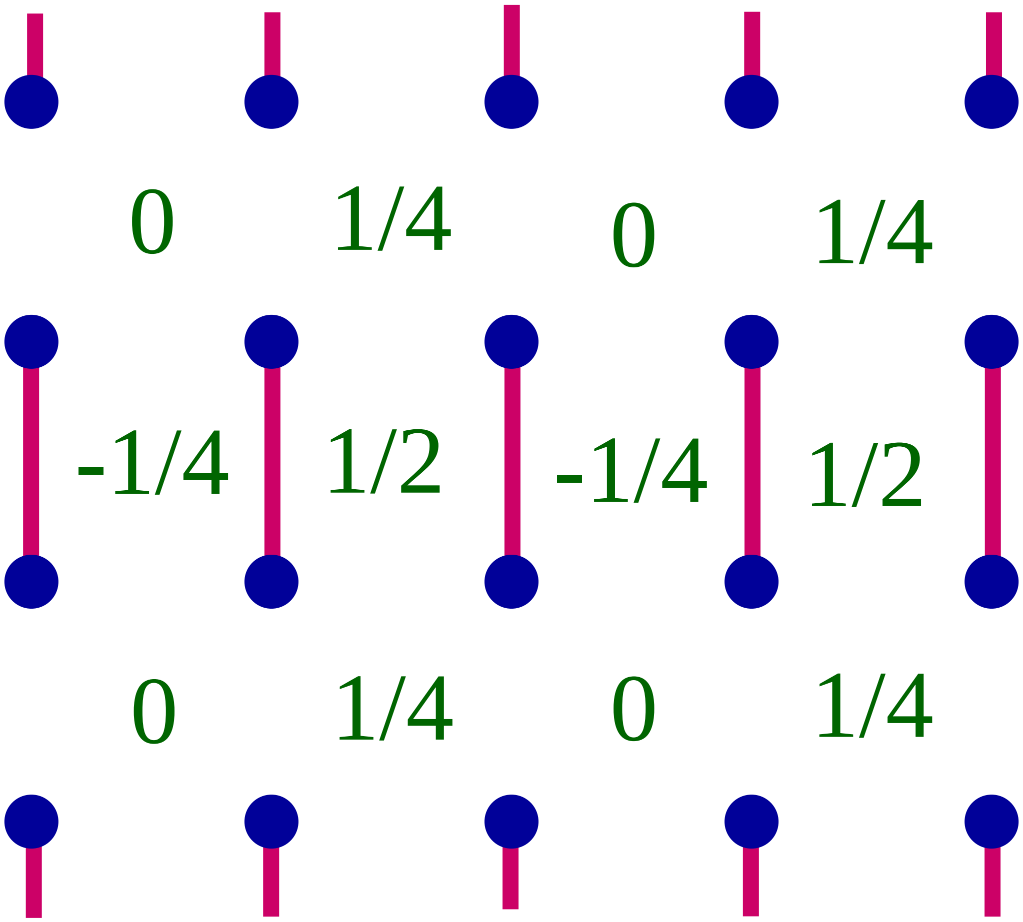
Smooth interface with average height 5/8

W. Zheng and S. Sachdev, *Phys. Rev*. B **40**, 2704 (1989)



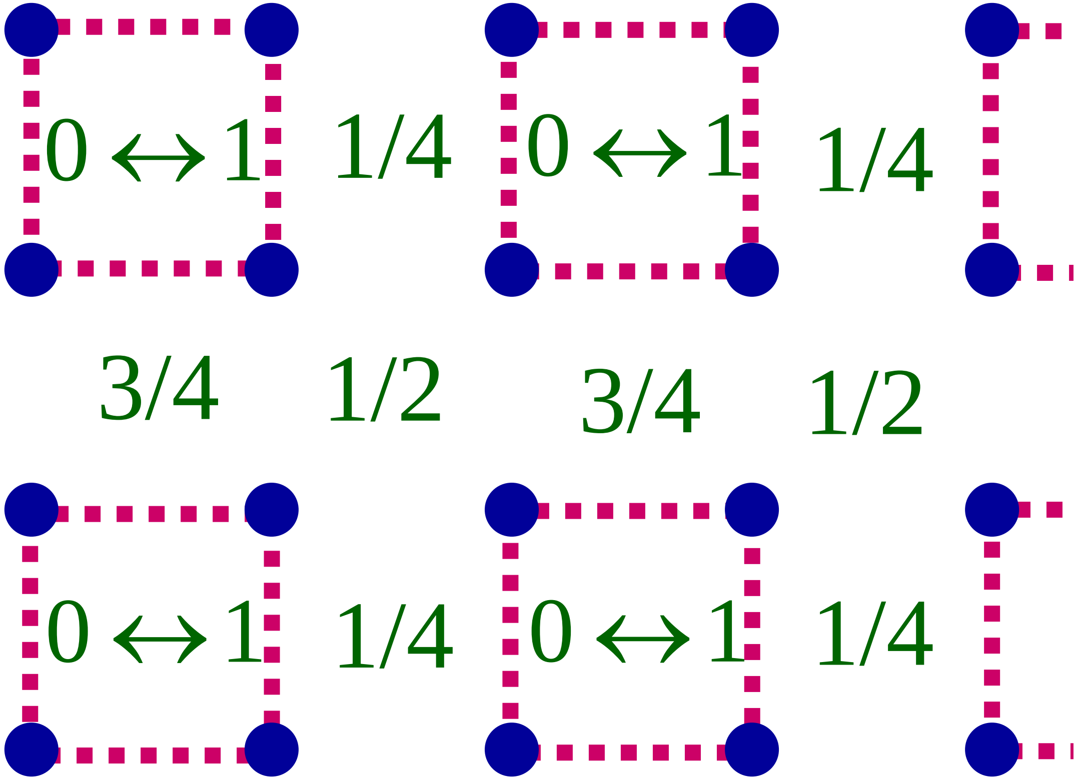
Smooth interface with average height 7/8

W. Zheng and S. Sachdev, *Phys. Rev*. B **40**, 2704 (1989)



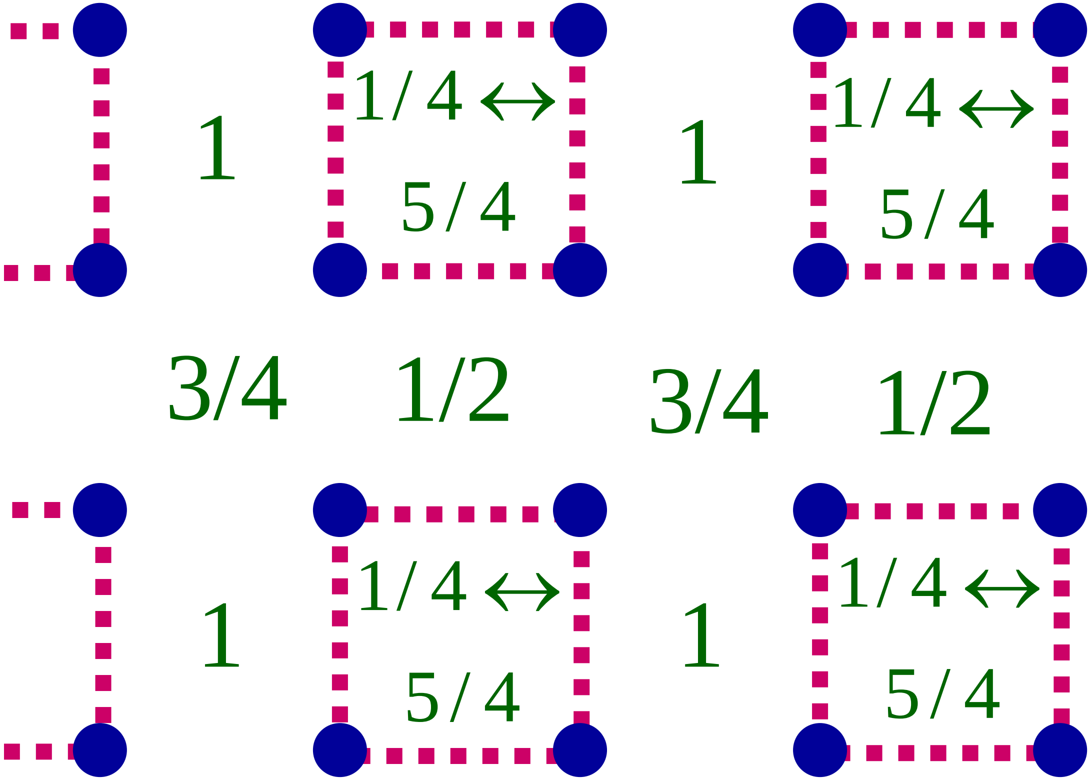
Smooth interface with average height 1/8

W. Zheng and S. Sachdev, *Phys. Rev*. B **40**, 2704 (1989)



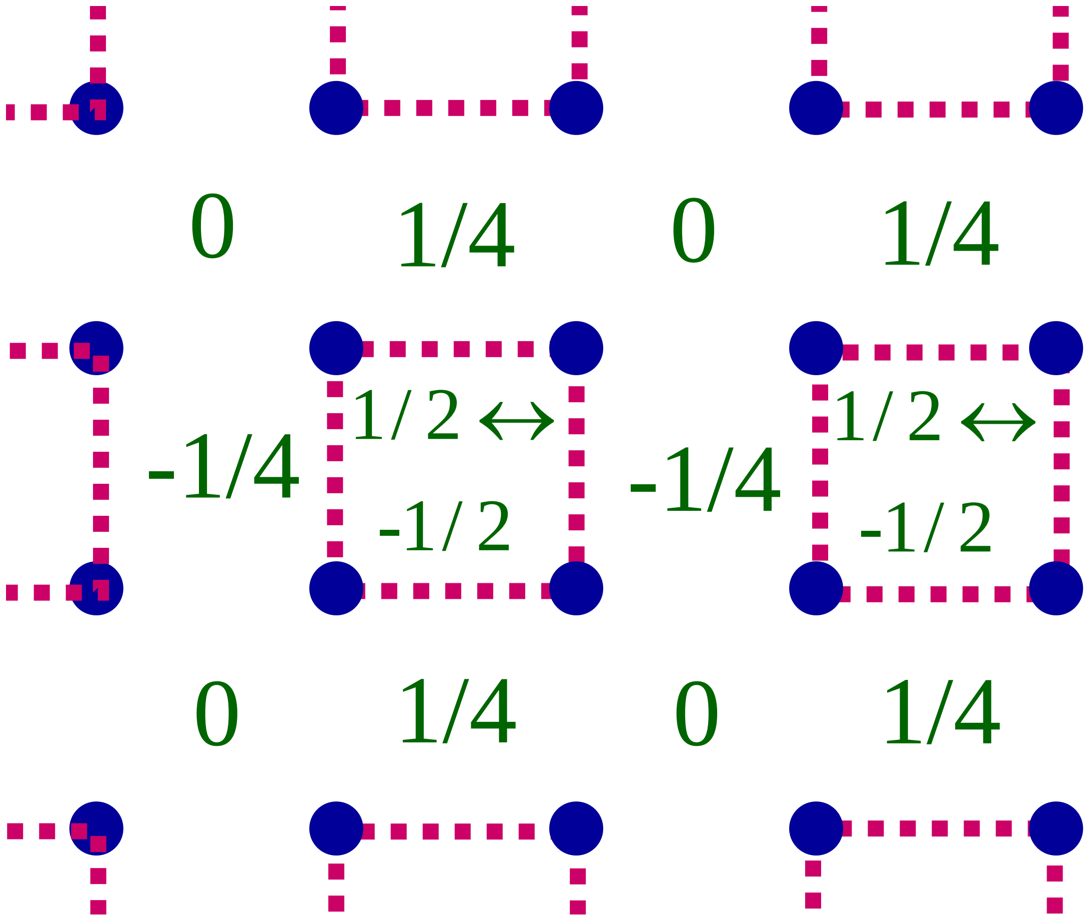
“Disordered-flat” interface with average height 1/2

W. Zheng and S. Sachdev, *Phys. Rev*. B **40**, 2704 (1989)



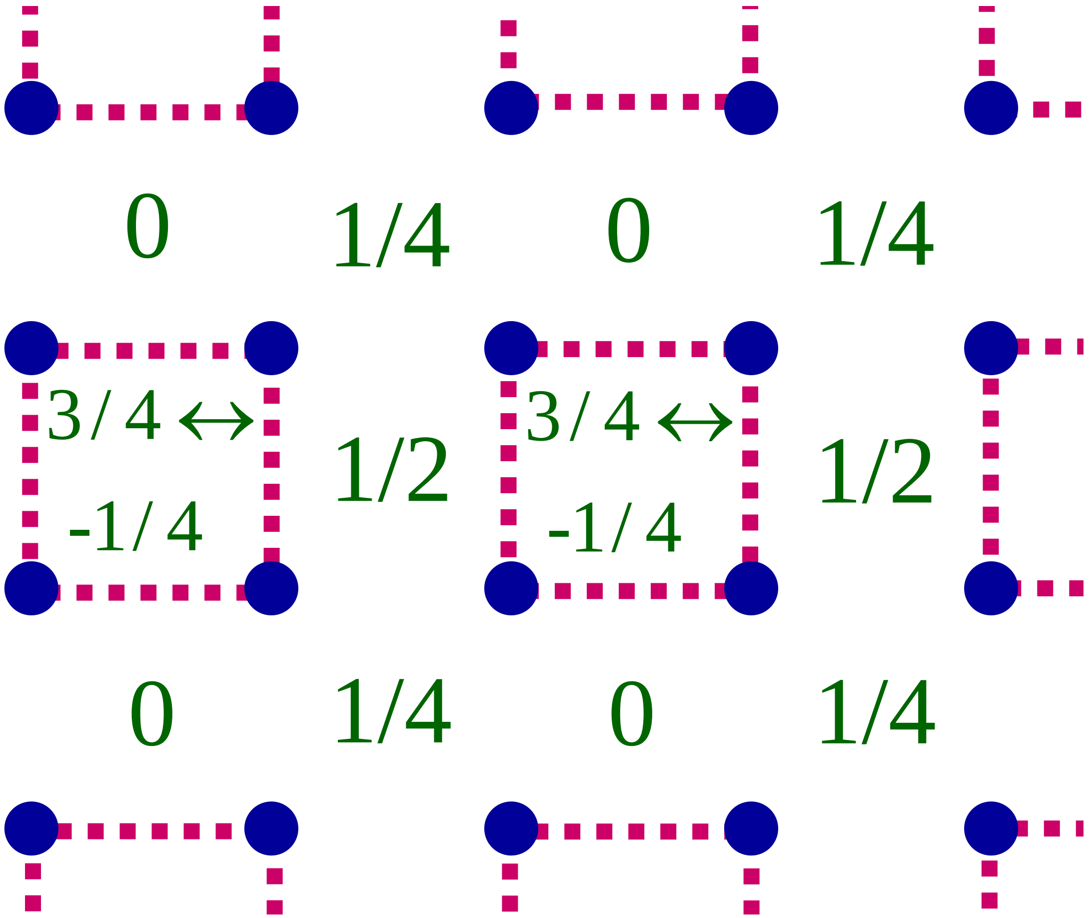
"Disordered-flat" interface with average height 3/4

W. Zheng and S. Sachdev, *Phys. Rev*. B **40**, 2704 (1989)



“Disordered-flat” interface with average height 0

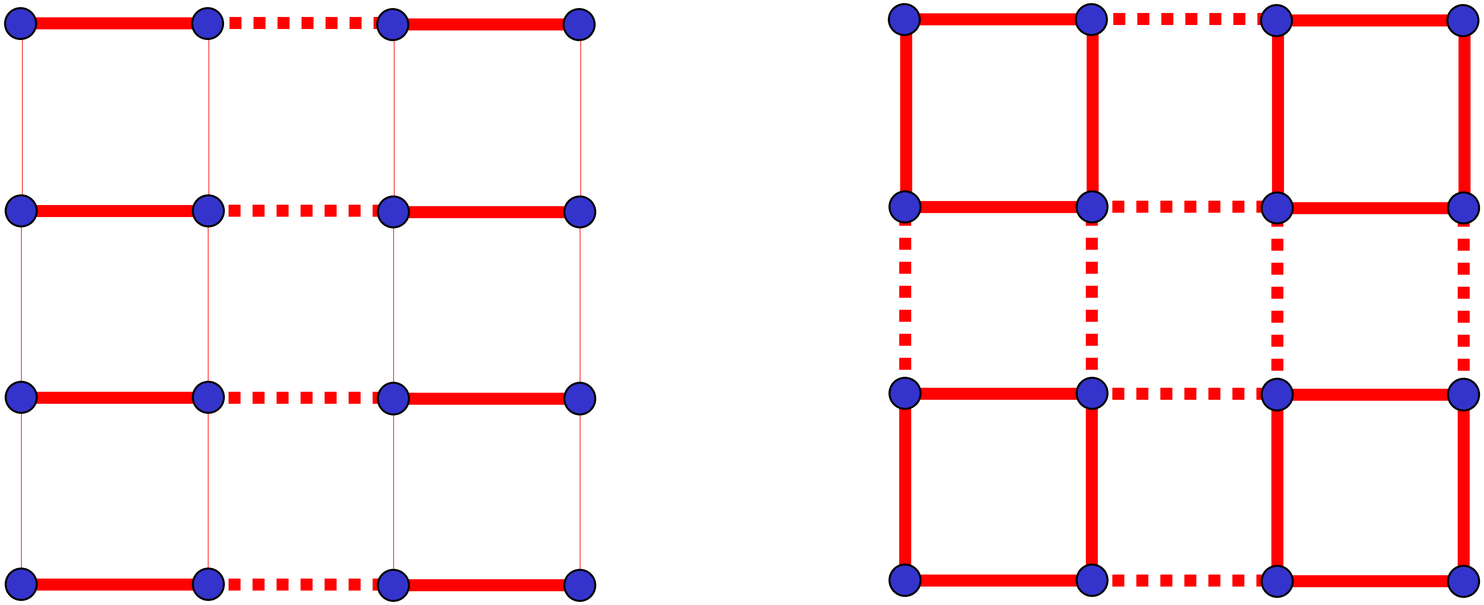
W. Zheng and S. Sachdev, *Phys. Rev*. B **40**, 2704 (1989)



"Disordered-flat" interface with average height 1/4

W. Zheng and S. Sachdev, *Phys. Rev*. B **40**, 2704 (1989)

Two possible bond-ordered paramagnets for *S*=1/2



Distinct lines represent different values of $\left\langle \vec{S}_i \bullet \vec{S}_j \right\rangle$ on links

There is a broken lattice symmetry, and the ground state is at least four-fold degenerate.

N. Read and S. Sachdev, *Phys. Rev. Lett*. **62**, 1694 (1989).

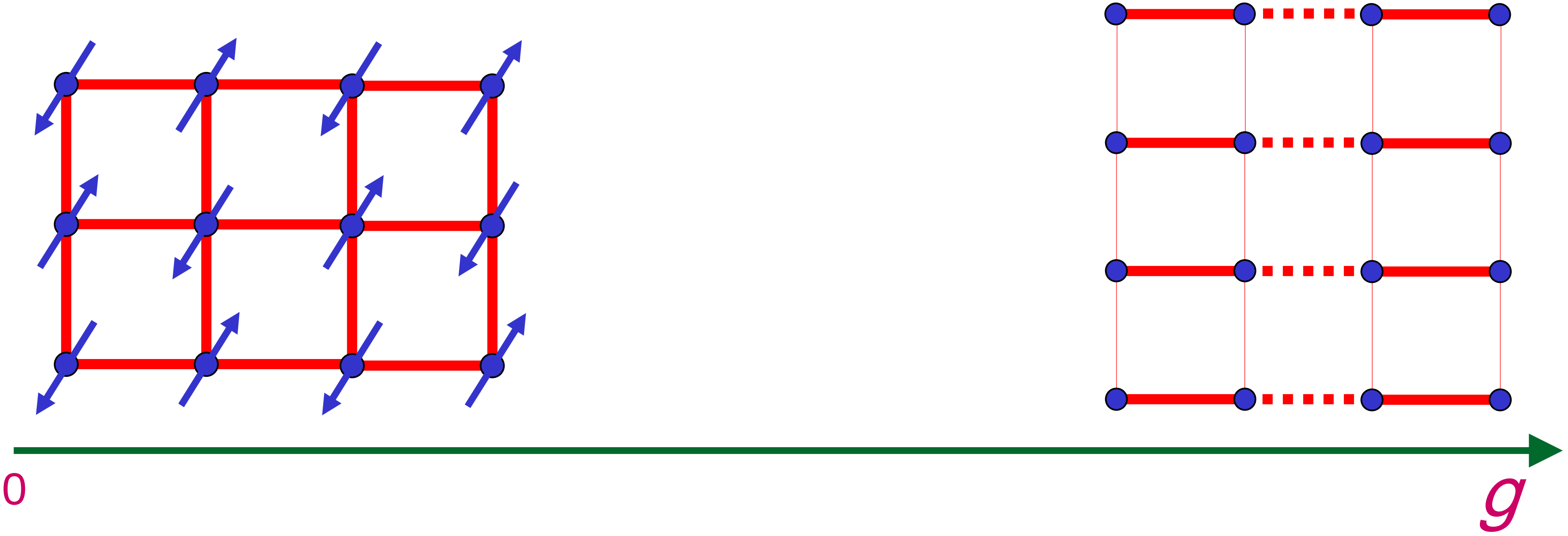
$$Z = \prod_a \int d\boldsymbol{n}_a \, \delta\left(\boldsymbol{n}_a^2 - 1\right) \exp\left(\frac{1}{g} \sum_{a,\mu} \boldsymbol{n}_a \cdot \boldsymbol{n}_{a+\mu} - \frac{i}{2} \sum_a \eta_a A_{a\tau} \right)$$

$\eta_{\mathrm{a}} \rightarrow \pm 1$ on two square sublattices ;

$\boldsymbol{n}_a \sim \eta_a \vec{S}_a \rightarrow$ Neel order parameter;

$A_{a\mu} \rightarrow$ oriented area of spherical triangle

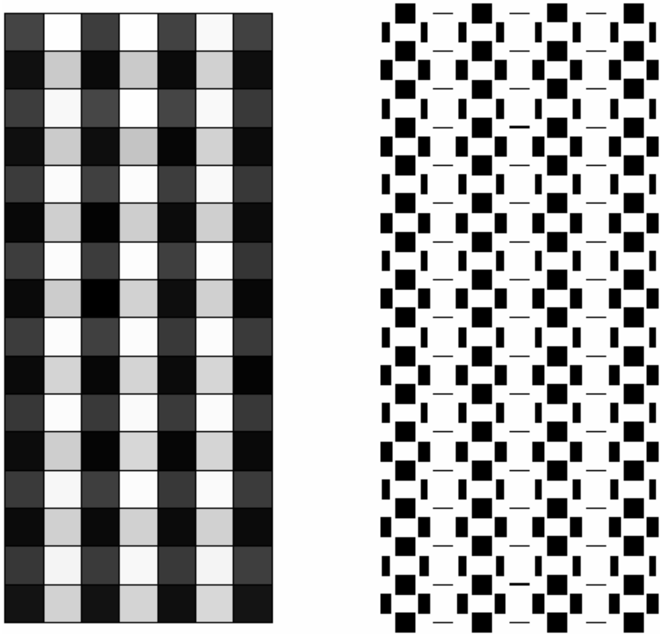
formed by $\boldsymbol{n}_a$, $\boldsymbol{n}_{a+\mu}$, and an arbitrary reference point $\boldsymbol{n}_0$

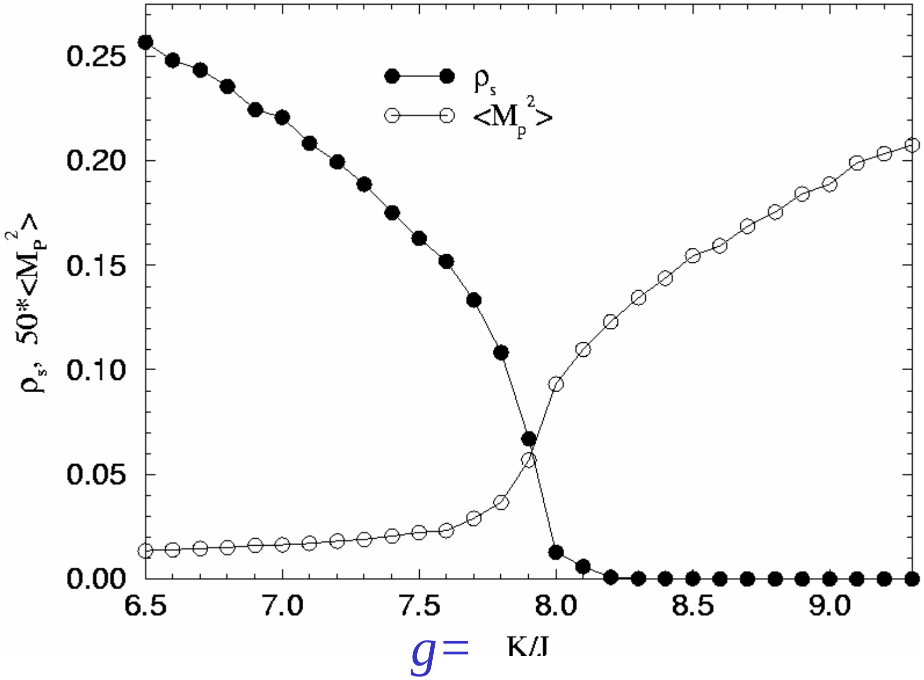


Bond order in a frustrated *S=1/2* XY magnet

A. W. Sandvik, S. Daul, R. R. P. Singh, and D. J. Scalapino, *Phys. Rev. Lett*. **89**, 247201 (2002)

First *<u>large scale</u>* numerical study of the destruction of Neel order in a *S=1/2* antiferromagnet with full square lattice symmetry





$$H = 2J\sum_{\langle ij\rangle}\left(S_i^x S_j^x + S_i^y S_j^y\right) - K\sum_{\langle ijkl\rangle\subset\square}\left(S_i^+ S_j^- S_k^+ S_l^- + S_i^- S_j^+ S_k^- S_l^+\right)$$

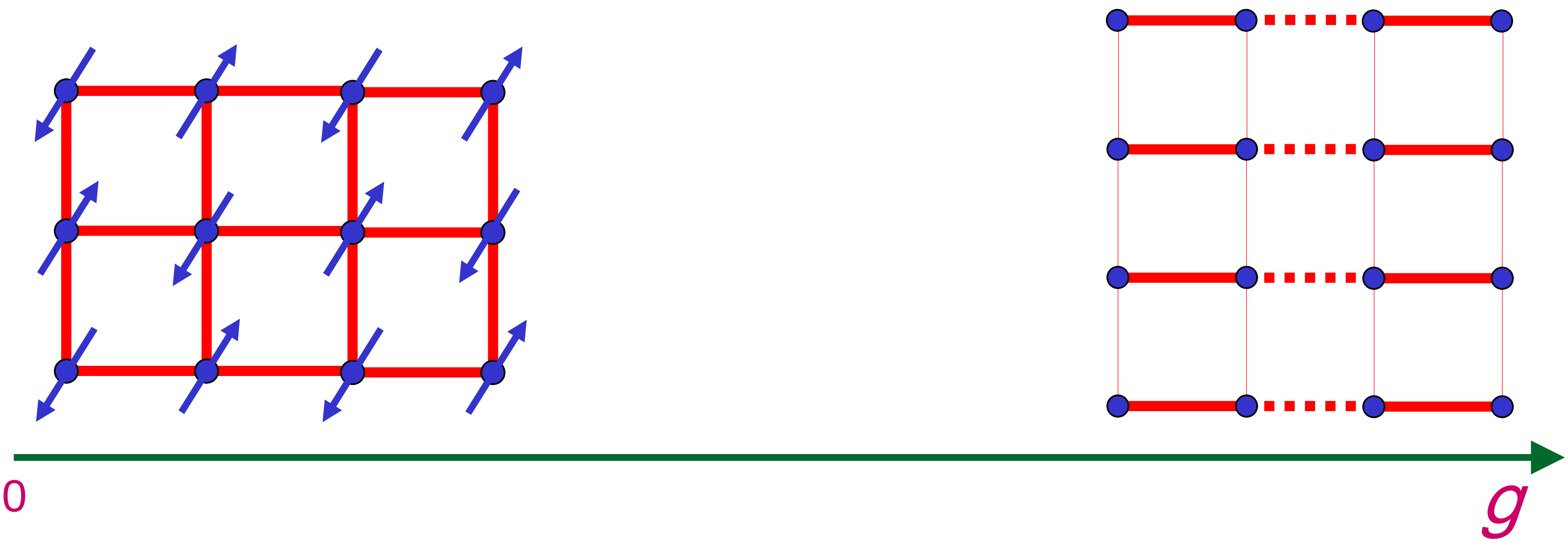
$$Z = \prod_a \int d\boldsymbol{n}_a \delta\left(\boldsymbol{n}_a^2 - 1\right) \exp\left(\frac{1}{g} \sum_{a,\mu} \boldsymbol{n}_a \cdot \boldsymbol{n}_{a+\mu} - \frac{i}{2} \sum_a \eta_a A_{a\tau} \right)$$

$\eta_{\mathrm{a}} \rightarrow \pm 1$ on two square sublattices ;

$\boldsymbol{n}_a \sim \eta_a \vec{S}_a \rightarrow$ Neel order parameter;

$A_{a\mu} \rightarrow$ oriented area of spherical triangle

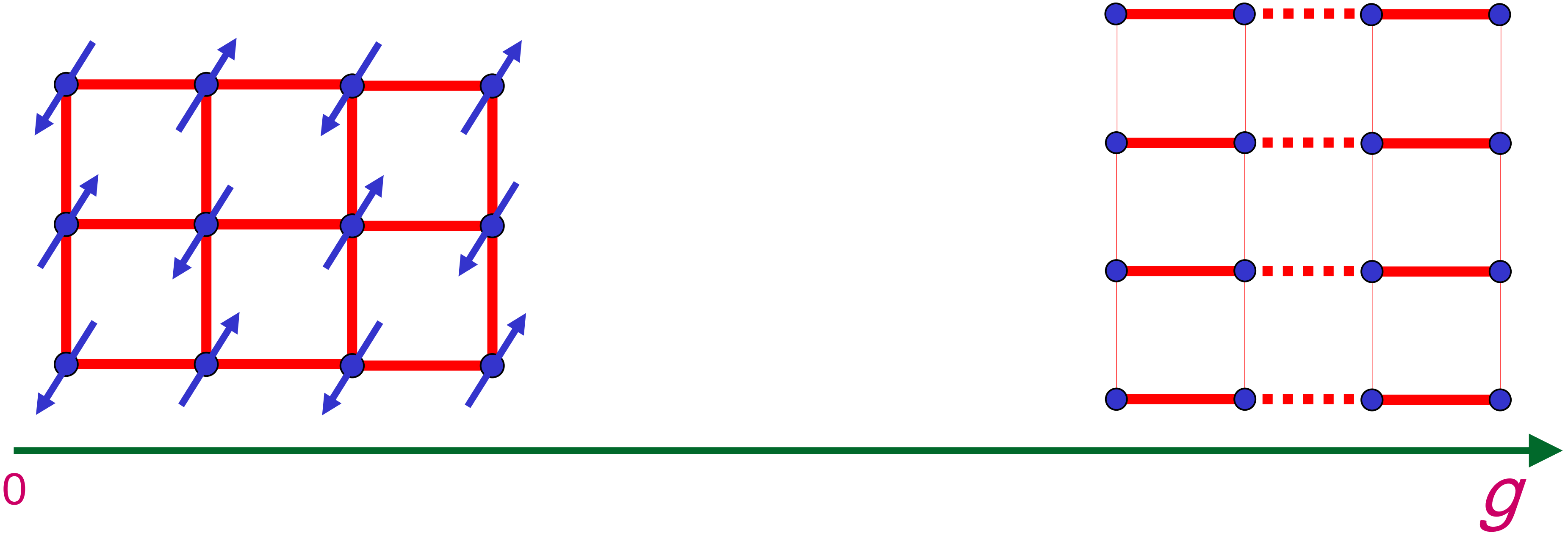
formed by $\boldsymbol{n}_a$, $\boldsymbol{n}_{a+\mu}$, and an arbitrary reference point $\boldsymbol{n}_0$



Alternative formulation to describe transition:

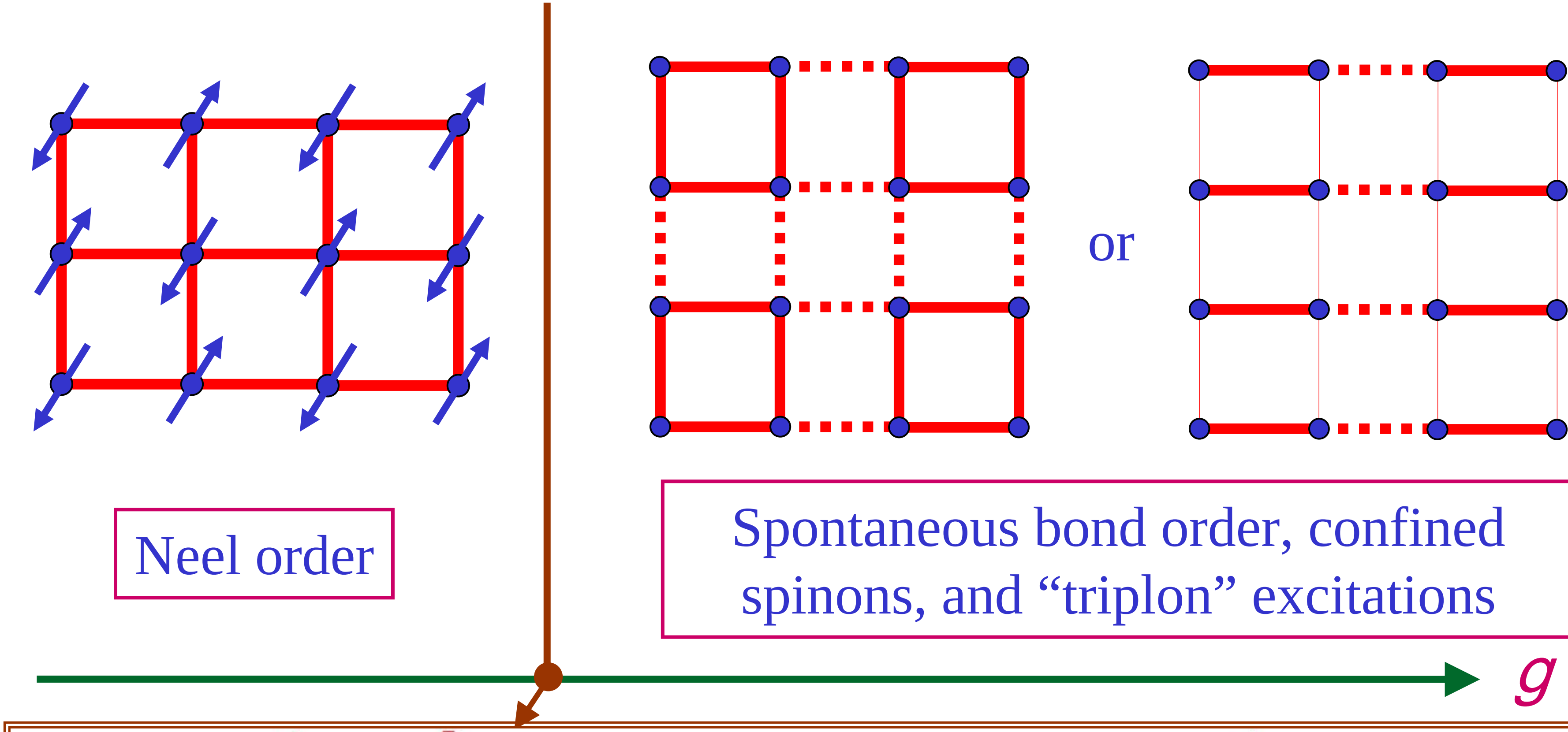
Express theory in terms of a complex spinor $z_{a\alpha}$, $\alpha = \uparrow, \downarrow$, with

$$\begin{aligned}
\mathbf{n}_a &= z^*_{a\alpha} \sigma_{\alpha\beta} z_{a\beta} \\
Z &= \prod_a \int dz_{a\alpha} dA_{a\mu} \delta\left(|z_{a\alpha}|^2 - 1\right) \\
&\qquad \exp\left(\frac{1}{g} \sum_{a,\mu} z^*_{a\alpha} e^{iA_{a\mu}} z_{a+\mu,\alpha} + \text{c.c.} + i \sum_a \eta_a A_{a\tau} \right)
\end{aligned}$$



S. Sachdev and R. Jalabert, *Mod. Phys. Lett.* B **4**, 1043 (1990).
K. Park and S. Sachdev, *Phys. Rev.* B **65**, 220405 (2002).

Phase diagram of S=1/2 square lattice antiferromagnet



Neel order

or

Spontaneous bond order, confined spinons, and “triplon” excitations

$$\mathcal{S}_{\text{critical}} = \int d^2x d\tau \left[\left|(\partial_\mu - iA_\mu) z_\alpha\right|^2 + r\left|z_\alpha\right|^2 + \frac{u}{2}\left(\left|z_\alpha\right|^2\right)^2 + \frac{1}{4e^2}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)^2 \right]$$

where $\mathbf{n} = z_\alpha^* \vec{\sigma}_{\alpha\beta} z_\beta$ and z_α are bosonic spinors

Critical theory is not expressed in terms of order parameter of either phase, but instead contains spinons interacting the a non-compact U(1) gauge force

T. Senthil, A. Vishwanath, L. Balents, S. Sachdev and M.P.A. Fisher, submitted to *Science*

Nature of quantum critical point

$$Z = \prod_a \int dz_{a\alpha} dA_{a\mu} \delta\left(|z_{a\alpha}|^2 - 1\right)$$

$$\exp\left(\frac{1}{g}\sum_{a,\mu} z^*_{a\alpha} e^{iA_{a\mu}} z_{a+\mu,\alpha} + \text{c.c.} + i\sum_a \eta_a A_{a\tau} + \frac{1}{e^2}\sum_{\Box} \cos\left(\Delta_\mu A_{a\nu} - \Delta_\nu A_{a\mu}\right)\right)$$

Use a sequence of simpler models which can be analyzed by duality mappings

A. Non-compact QED with scalar matter
B. Compact QED with scalar matter
C. *N=1*: Compact QED with scalar matter and Berry phases
D. $N \to \infty$ theory
E. Easy plane case for *N=2*

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A. *N=1*, non-compact U(1), no Berry phases

Use $z_a = e^{i\theta_a}$ and then

$$\begin{aligned} Z \;=\; \prod_a \int d\theta_a dA_{a\mu} \exp \Bigg(& -\frac{1}{2e^2} \sum_{\square} \left(\Delta_\mu A_{a\nu} - \Delta_\nu A_{a\mu}\right)^2 \\ & + \frac{1}{g} \sum_{a,\mu} \cos\left(\Delta_\mu \theta_a - A_{a\mu}\right) \Bigg) \end{aligned}$$

Standard duality maps, similar to those discussed earlier, show that this theory is equivalent to an **inverted XY model**, described by the field theory

$$Z_{\text{dual}} = \int \mathcal{D}\psi \exp\left(- \int d^2x d\tau \left(|\partial_\mu \psi|^2 + r|\psi|^2 + \frac{u}{2}|\psi|^4 \right) \right)$$

Here ψ is a *dual* field which orders in the paramagnetic phase *i.e.* $\langle \psi \rangle \neq 0$ where $\langle e^{i\theta} \rangle = 0$, and vice versa. The field ψ is a creation operator for *vortices* in the original theory of a "Ginzburg-Landau superconductor" coupled to "electromagnetism".

C. Dasgupta and B.I. Halperin, *Phys. Rev. Lett.* **47**, 1556 (1981).

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B. *N=1*, compact U(1), no Berry phases

Use $z_a = e^{i\theta_a}$ and then

$$Z = \prod_a \int d\theta_a dA_{a\mu} \exp\left(\frac{1}{e^2} \sum_{\square} \cos\left(\Delta_\mu A_{a\nu} - \Delta_\nu A_{a\mu}\right) + \frac{1}{g} \sum_{a,\mu} \cos\left(\Delta_\mu \theta_a - A_{a\mu}\right) \right)$$

The Dasgupta-Halperin mapping now yields the dual theory

$$Z_{\text{dual}} = \int \mathcal{D}\psi \exp\left(- \int d^2x d\tau \left(|\partial_\mu \psi|^2 + r|\psi|^2 + \frac{u}{2}|\psi|^4 - y_m(\psi + \psi^*) \right) \right)$$

Here y_m is a *monopole fugacity*, and the last term in Z_{dual} accounts for the fact that vortex lines can end in monopoles.

This dual theory is an **inverted XY model in a “magnetic” field** and it has no phase transition. In the direct theory, the monopoles are a relevant perturbation, and they destroy the “superconducting” phase.

Nature of quantum critical point

$$Z = \prod_a \int dz_{a\alpha} dA_{a\mu} \delta\left(|z_{a\alpha}|^2 - 1\right)$$

$$\exp\left(\frac{1}{g}\sum_{a,\mu} z^*_{a\alpha} e^{iA_{a\mu}} z_{a+\mu,\alpha} + \text{c.c.} + i\sum_a \eta_a A_{a\tau} + \frac{1}{e^2}\sum_{\square} \cos\left(\Delta_\mu A_{a\nu} - \Delta_\nu A_{a\mu}\right)\right)$$

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C. *N=1*, compact U(1), Berry phases

Upon including Berry phases, the previous theory becomes

$$Z = \prod_a \int d\theta_a dA_{a\mu} \exp\left(\frac{1}{e^2} \sum_{\square} \cos\left(\Delta_\mu A_{a\nu} - \Delta_\nu A_{a\mu}\right) \right.$$

$$\left. + \frac{1}{g} \sum_{a,\mu} \cos\left(\Delta_\mu \theta_a - A_{a\mu}\right) + i \sum_a \eta_a A_{a\tau} \right)$$

The Dasgupta-Halperin duality can also be extended to this theory, and we obtain

$$Z_{\text{dual}} = \int \mathcal{D}\psi \exp\left(- \int d^2x d\tau \left(|\partial_\mu \psi|^2 + r|\psi|^2 + \frac{u}{2}|\psi|^4 - \tilde{y}_m(\psi^4 + \psi^{*4}) \right) \right)$$

C. *N=1*, compact U(1), Berry phases

$$Z_{\rm dual} = \int \mathcal{D}\psi \exp\left(- \int d^2x d\tau \left(|\partial_\mu \psi|^2 + r|\psi|^2 + \frac{u}{2}|\psi|^4 - \tilde{y}_m(\psi^4 + \psi^{*4}) \right) \right)$$

This is an **inverted XY model with a four-fold anisotropy**, *i.e.* a Z_4 clock model. The four-fold anisotropy is irrelevant at the critical point (J.M. Carmona, A. Pelissetto, E. Vicari, Phys. Rev. B **61**, 15136 (2000)), and hence there is a XY transition to a four-fold degenerate state with $\langle \psi \rangle \neq 0$. In the direct theory, this is the *bond-ordered* paramagnet.

S. Sachdev and R. Jalabert, Mod. Phys. Lett. **4**, 1043 (1990).

C. *N=1*, compact U(1), Berry phases

$$Z_{\text{dual}} = \int \mathcal{D}\psi \exp\left(- \int d^2x d\tau \left(|\partial_\mu \psi|^2 + r|\psi|^2 + \frac{u}{2}|\psi|^4 - \tilde{y}_m(\psi^4 + \psi^{*4}) \right) \right)$$

Reinterpretation by T. Senthil: In the direct theory, the irrelevance of $\tilde{y}_m$ implies that the Berry phases have cancelled out the monopole contributions. So monopoles are 'dangerously irrelevant' at the critical point, and the critical theory is *the same Dasgupta-Halperin inverted XY model describing the non-compact theory without monopoles or Berry phases!*

Nature of quantum critical point

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Identical critical theories!

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$$Z = \prod_a \int dz_{a\alpha} dA_{a\mu} \delta\left(|z_{a\alpha}|^2 - 1\right)$$

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Identical critical theories!

A. Non-compact QED with scalar matter

B. Compact QED with scalar matter

C. *N=1*: Compact QED with scalar matter and Berry phases

D. $N \to \infty$ **theory**

E. Easy plane case for *N*=2

D. $N \to \infty$, compact U(1), Berry phases

Near the critical point of the $N = \infty$ non-compact theory, integrate out z_α quanta (with gap Δ) in the presence of a Dirac monopole with $A_\mu = A_\mu^D$ with magnetic charge q. The functional determinant yields the action of such a monopole, and the scaling dimension of the monopole insertion

$$\mathcal{S}_{\text{monopole}} = N \operatorname{Tr} \ln \left[\frac{-(\partial_\mu - i A_\mu^D)^2 + \Delta^2 + V(r)}{-\partial_\mu^2 + \Delta^2} \right] - \frac{N}{g} \int d^3 r V(r)$$

$$\text{where } \frac{\delta \mathcal{S}_{\text{monopole}}}{\delta V(r)} = 0 \text{ and } V(r \to \infty) = 0.$$

Evaluation of functional determinant for $S = 1/2$ shows

$$\mathcal{S}_{\text{monopole}} = 0.815787 N \ln \left(\frac{\Lambda}{\Delta} \right)$$

This computation shows that the scaling dimension of $q = 4$ monopoles is $3 - 0.815787N$

Monopoles are irrelevant both with and without Berry phases for large N.

G. Murthy and S. Sachdev, *Nucl. Phys.* B **344**, 557 (1990).

E. Easy plane case for *N*=2

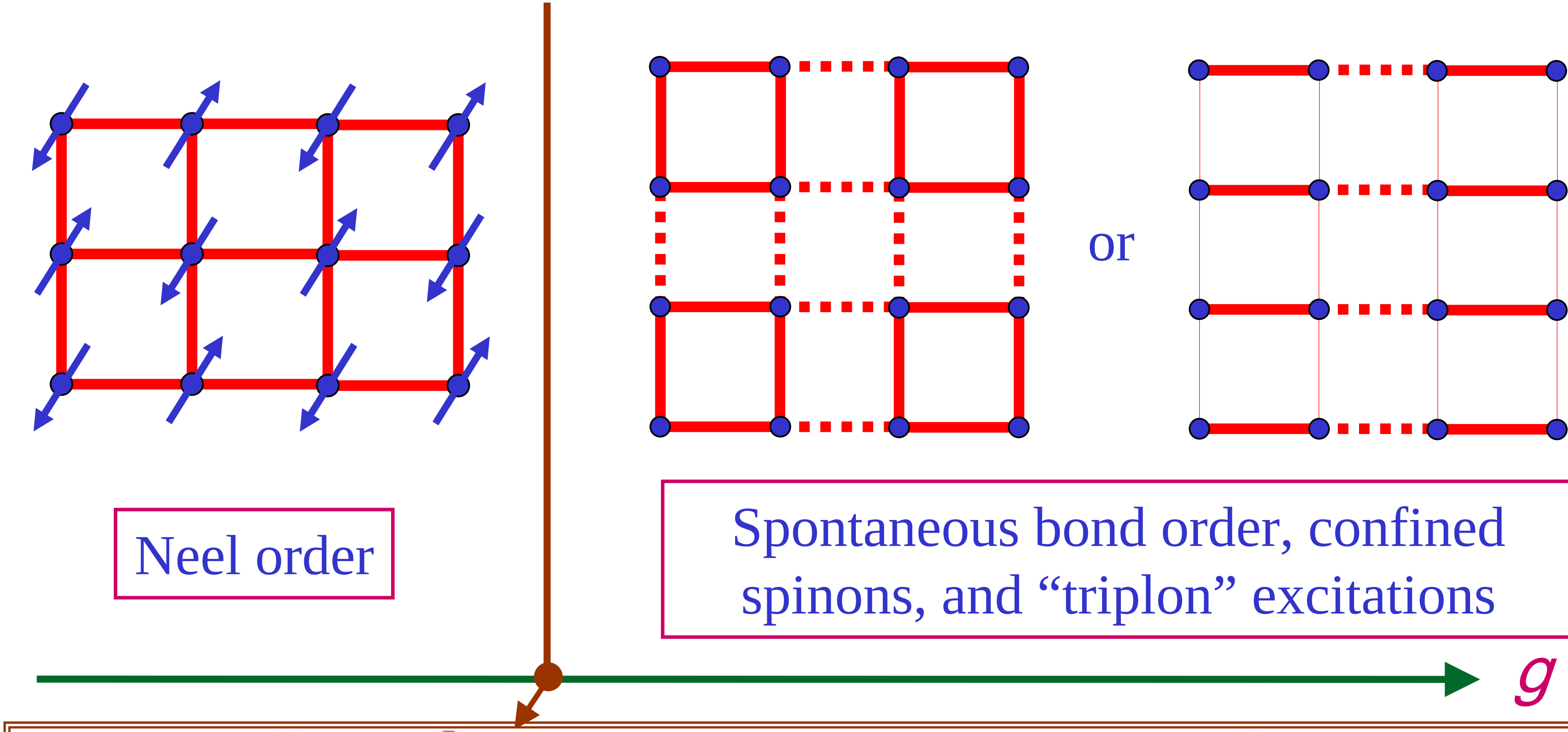
Explicit duality mappings show that the physical situation is as for $N = 1$:

- monopoles are relevant without Berry phases,
- monopoles are irrelevant at the critical point in the presence of Berry phases, and
- monopoles drive the appearance of bond order in the paramagnetic phase.

$$Z_{\text{dual}} = \int \mathcal{D}\psi_1 \mathcal{D}\psi_2 \mathcal{D}a_\mu \exp\Bigg(- \int d^2x d\tau \bigg(|(\partial_\mu - i a_\mu)\psi_1|^2 + |(\partial_\mu - i a_\mu)\psi_2|^2 + r\left(|\psi_1|^2 + |\psi_2|^2\right) + \frac{u}{2}\left(|\psi_1|^4 + |\psi_2|^4\right) - \tilde{y}_m\left((\psi_1^*\psi_2)^4 + (\psi_1\psi_2^*)^4\right)\bigg)\Bigg)$$

C. Lannert, M.P.A. Fisher, and T. Senthil, *Phys. Rev.* B **63**, 134510 (2001).
S. Sachdev and K. Park, *Annals of Physics*, **298**, 58 (2002).
O. Motrunich and A. Vishwanath, to appear.
T. Senthil *et al.*, to appear.

Phase diagram of S=1/2 square lattice antiferromagnet



Neel order

Spontaneous bond order, confined spinons, and “triplon” excitations

$$\mathcal{S}_{\text{critical}} = \int d^2x d\tau \left[\left|(\partial_\mu - iA_\mu)z_\alpha\right|^2 + r\left|z_\alpha\right|^2 + \frac{u}{2}\left(\left|z_\alpha\right|^2\right)^2 + \frac{1}{4e^2}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)^2 \right]$$

where $\mathbf{n} = z_\alpha^* \vec{\sigma}_{\alpha\beta} z_\beta$ and z_α are bosonic spinors

Critical theory is not expressed in terms of order parameter of either phase, but instead contains spinons interacting the a non-compact U(1) gauge force

T. Senthil, A. Vishwanath, L. Balents, S. Sachdev and M.P.A. Fisher, submitted to *Science*

Outline

A. "Dimerized" Mott insulators with a spin gap

Tuning quantum transitions by applied pressure

B. Spin gap state on the square lattice

Spontaneous bond order

C. Tuning quantum transitions by a magnetic field

1. Mott insulators

2. Cuprate superconductors

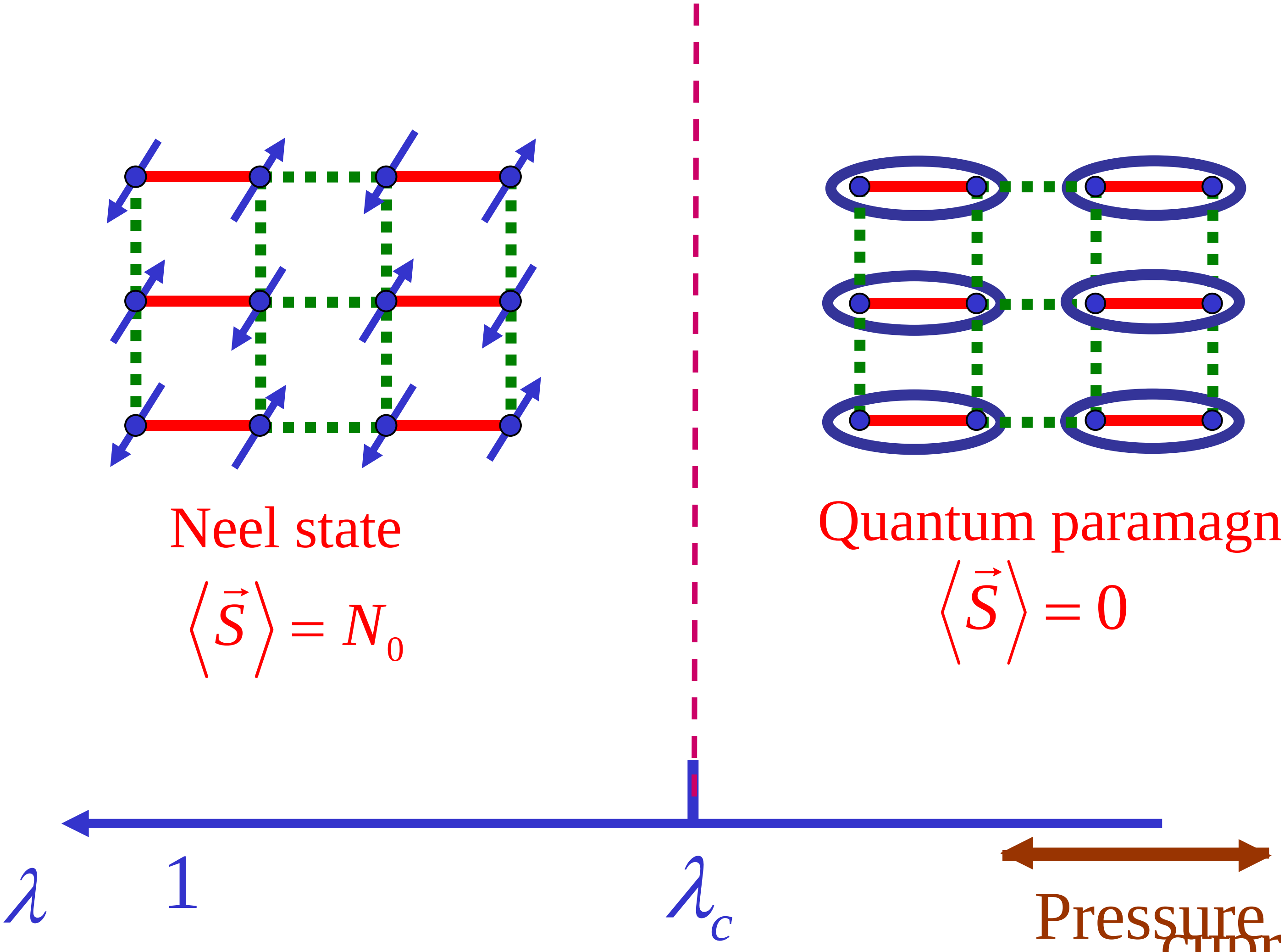
(C) Tuning quantum transitions by a magnetic field

1. Mott insulators

T=0

$\lambda_c = 0.52337(3)$

M. Matsumoto, C. Yasuda, S. Todo, and H. Takayama, *Phys. Rev.* B **65**, 014407 (2002)



Neel state

$\left\langle \vec{S} \right\rangle = N_0$

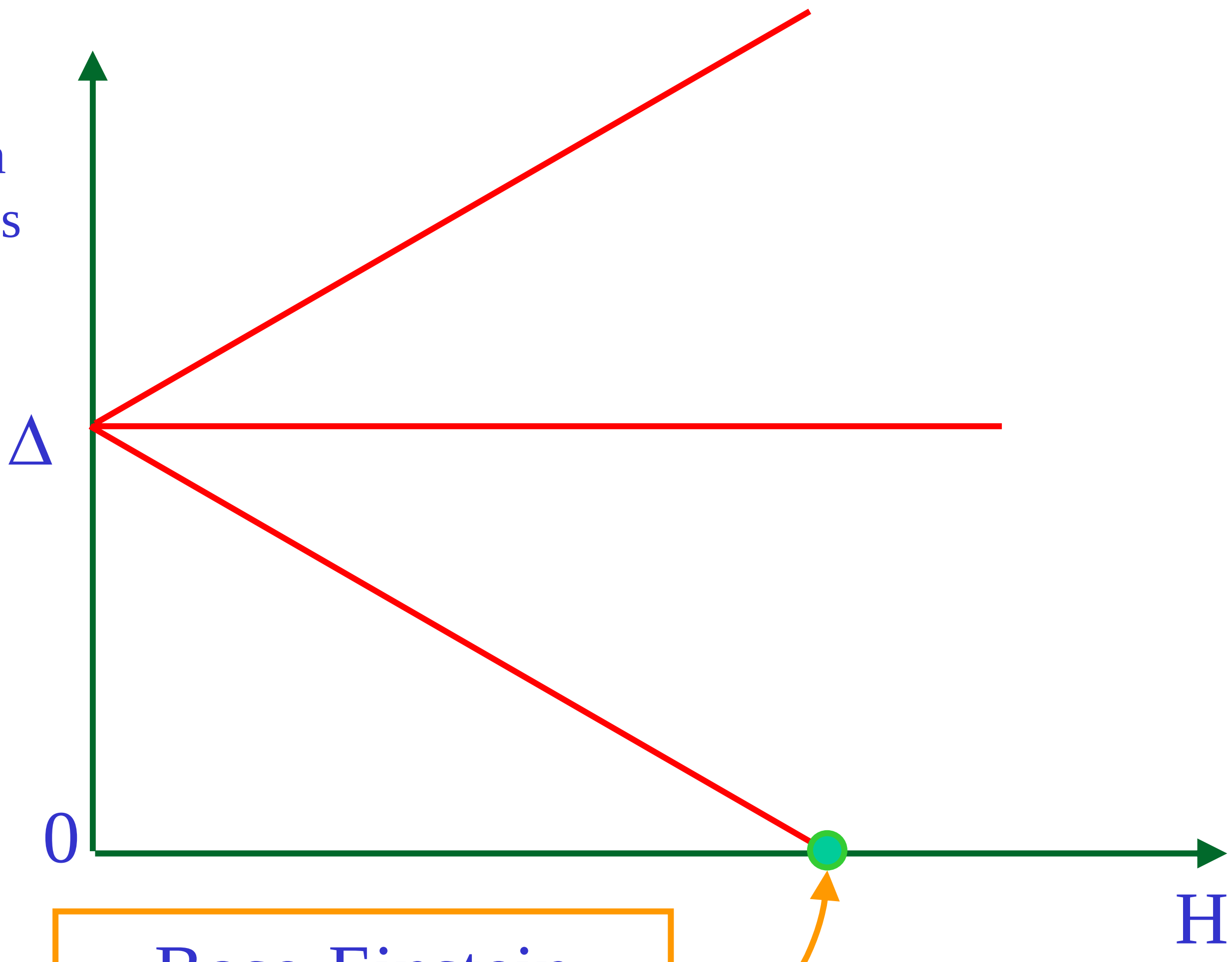
Quantum paramagnet

$\left\langle \vec{S} \right\rangle = 0$

δ in cuprates ?

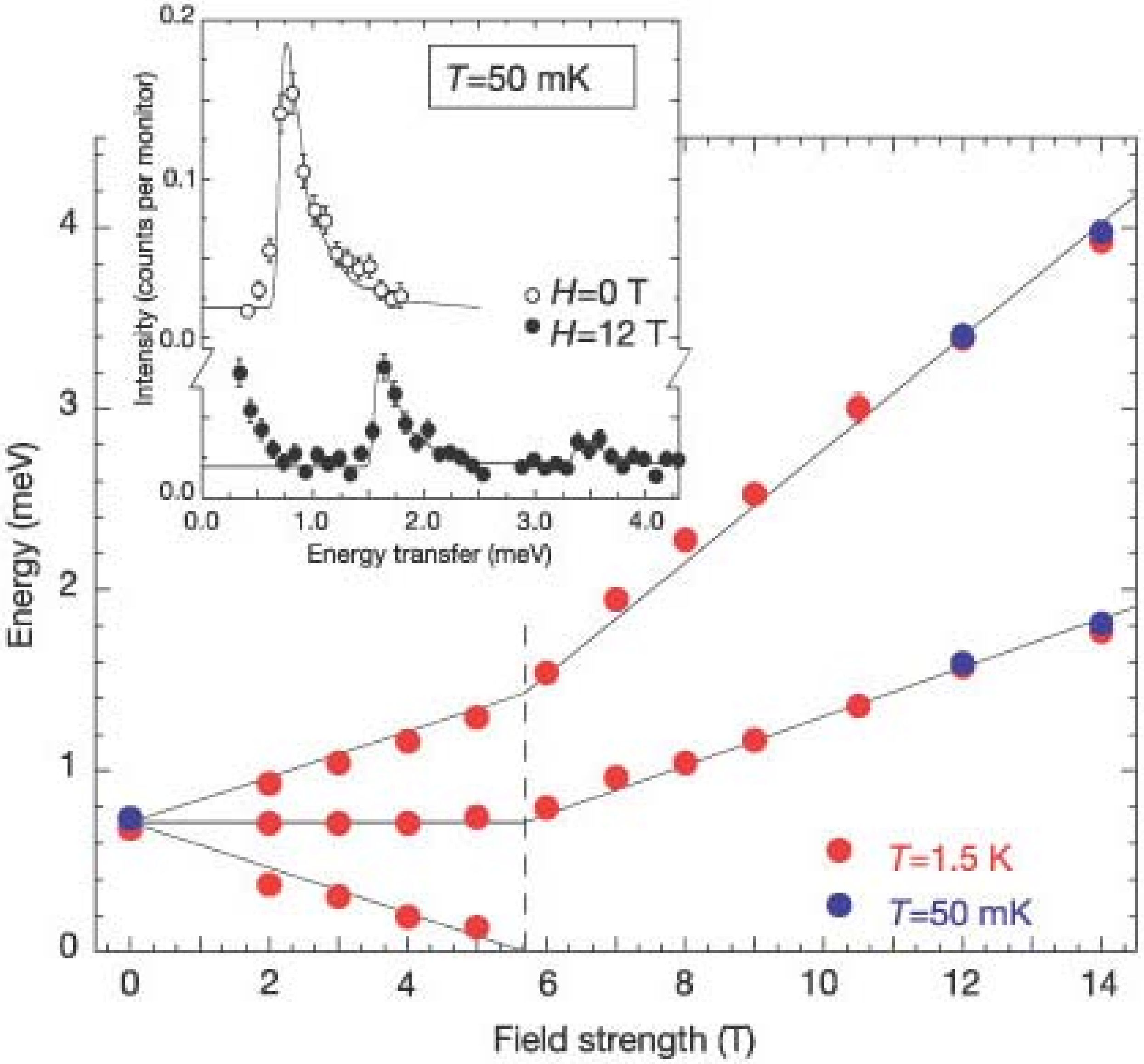
Pressure in $TlCuCl_3$

Effect of a field on paramagnet



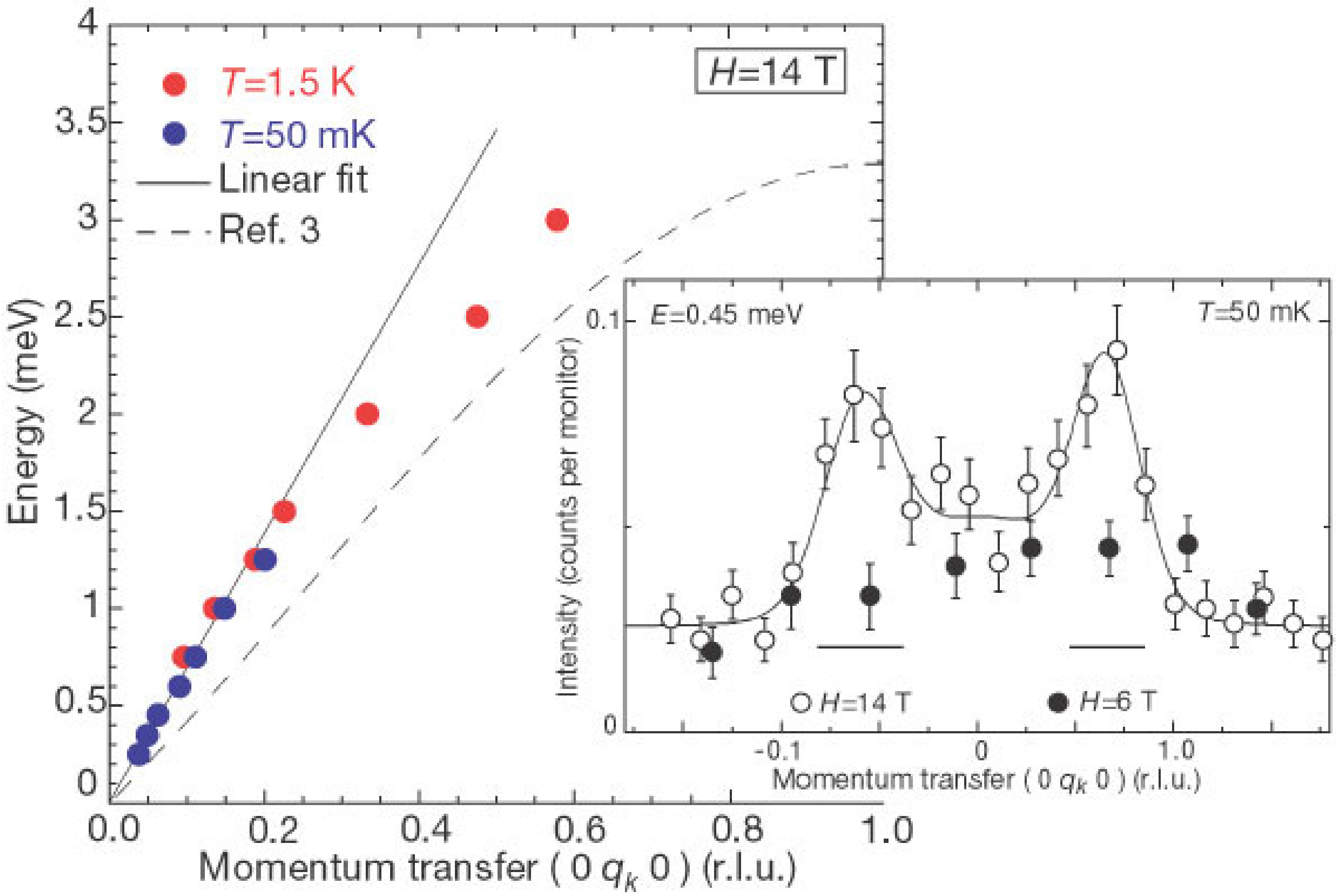
Bose-Einstein condensation of S_z=1 triplon

TlCuCl$_3$



Ch. Rüegg, N. Cavadini, A. Furrer, H.-U. Güdel, K. Krämer, H. Mutka, A. Wildes, K. Habicht, and P. Vorderwisch, *Nature* **423**, 62 (2003).

$TlCuCl_3$



“Spin wave (phonon) above critical field

Ch. Rüegg, N. Cavadini, A. Furrer, H.-U. Güdel, K. Krämer, H. Mutka, A. Wildes, K. Habicht, and P. Vorderwisch, *Nature* **423**, 62 (2003).

Phase diagram in a magnetic field

The Hamiltonian for the triplons is transformed in a magnetic field B_α by

$$H_t \to H_t + iB_\alpha \epsilon_{\alpha\beta\gamma} \sum t^\dagger_\beta t_\gamma$$

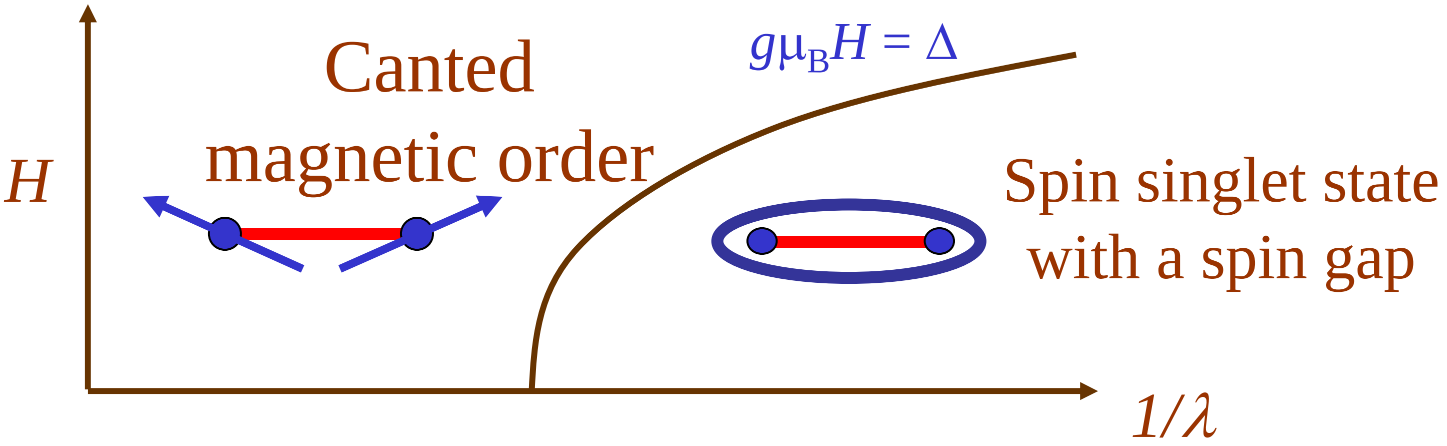
The change in the effective action for the order parameter φ_α is then

$$\partial_\tau \varphi_\alpha \to \partial_\tau \varphi_\alpha + i\epsilon_{\alpha\beta\gamma} H_\beta \varphi_\gamma$$

Take H oriented along z direction. Then the $\varphi_{x,y}$ components are preferred, and have mean-field action

$$\mathcal{S}_{\rm mf} = (\lambda_c - \lambda - H^2)(\varphi_x^2 + \varphi_y^2) + \frac{u}{24}(\varphi_x^2 + \varphi_y^2)^2$$

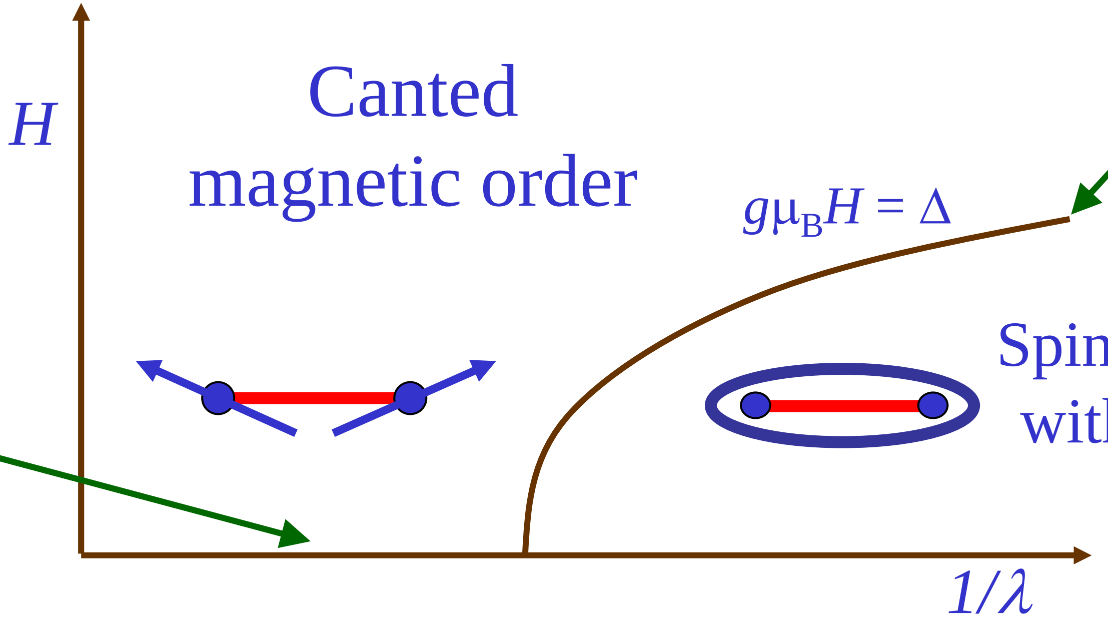
So for $\lambda < \lambda_c$ there is onset of staggered magnetic order in the x, y plane at $H_c = \sqrt{\lambda_c - \lambda}$. For $\lambda > \lambda_c$, the staggered moment $\langle \varphi_x \rangle \sim \sqrt{\lambda - \lambda_c + H^2}$.



Phase diagram in a magnetic field

Elastic scattering intensity

$$I[H] = I[0] + a\left(\frac{H}{J}\right)^2$$



Canted magnetic order

$g\mu_B H = \Delta$

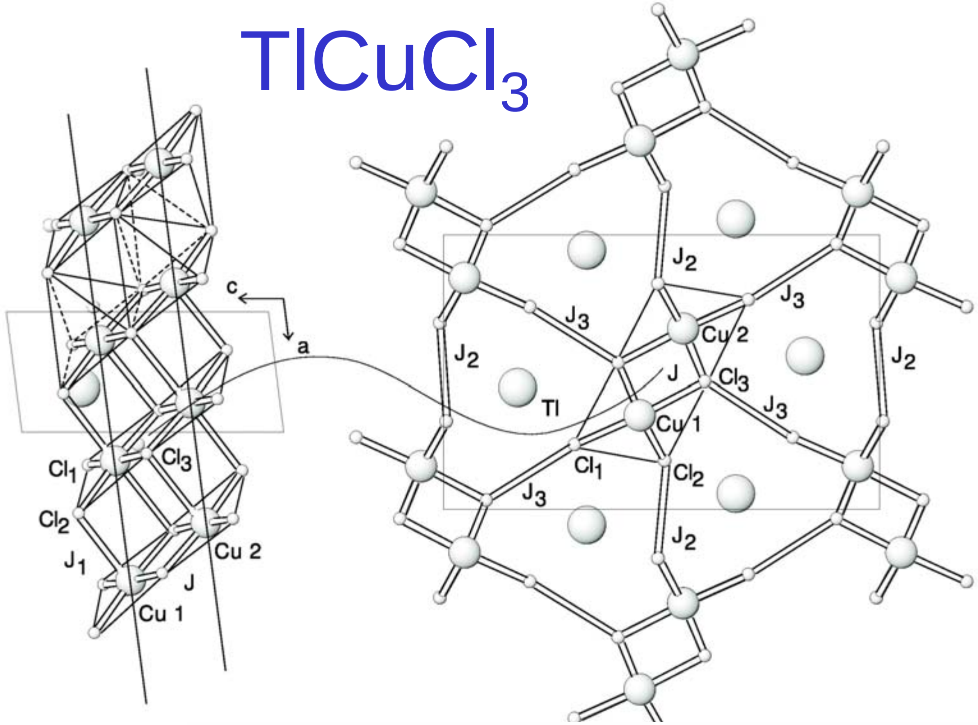
$$H_c \sim \sqrt{\lambda_c - \lambda}$$

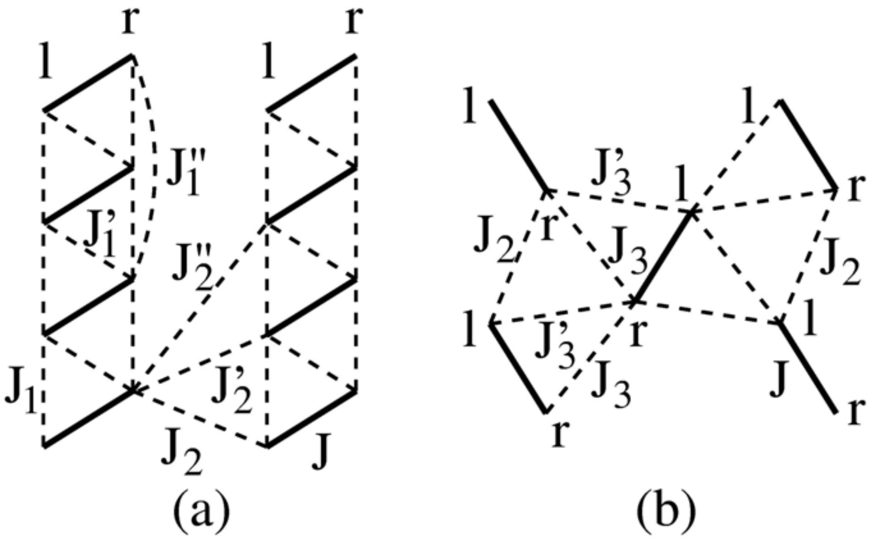
Spin singlet state with a spin gap

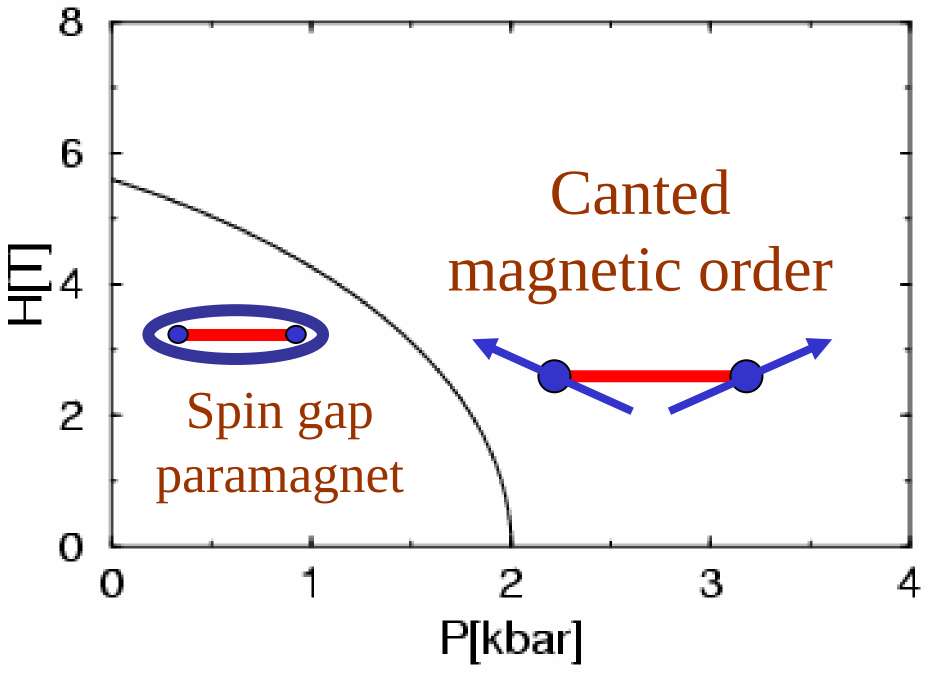
1 Tesla = 0.116 meV

Related theory applies to double layer quantum Hall systems at $\nu=2$

$TlCuCl_3$

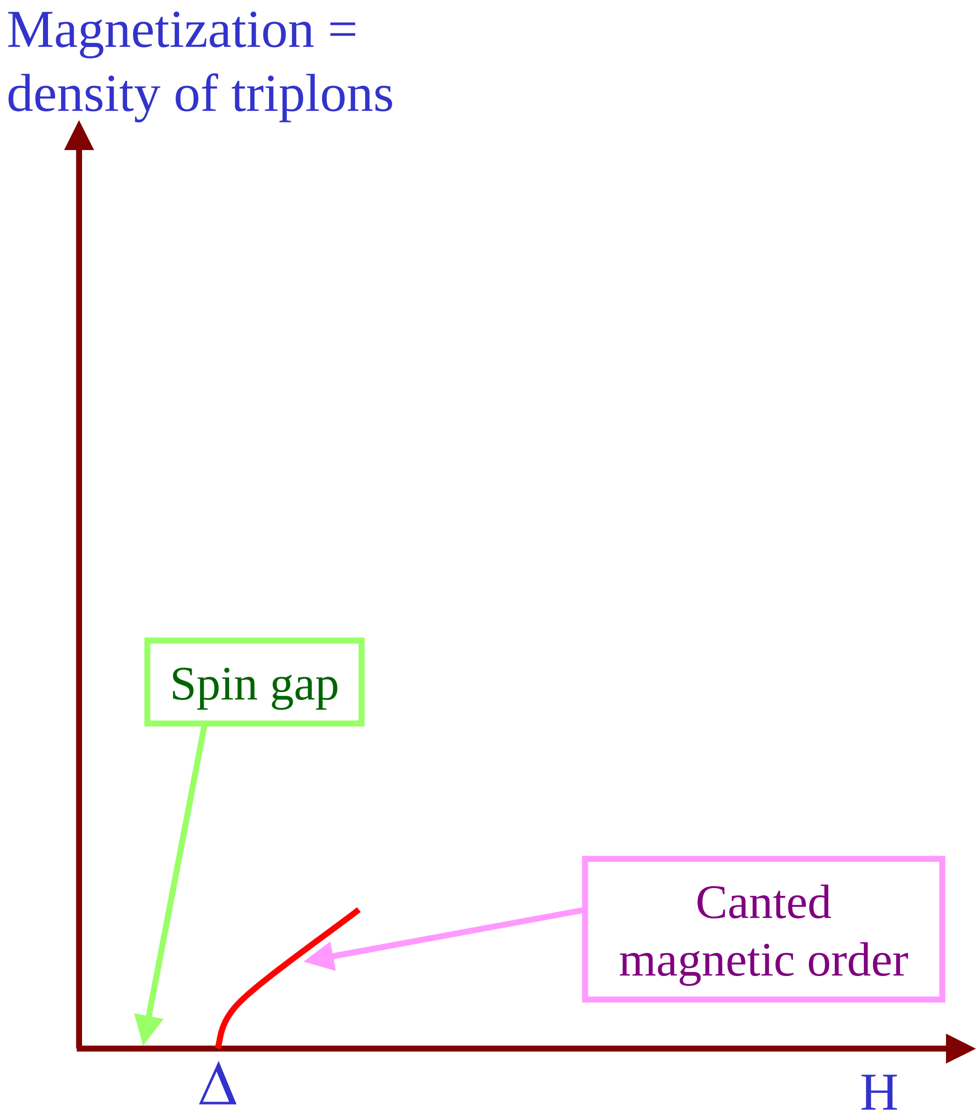






M. Matsumoto,
B. Normand, T.M. Rice,
and M. Sigrist,
cond-mat/0309440.

Phase diagram in a strong magnetic field.

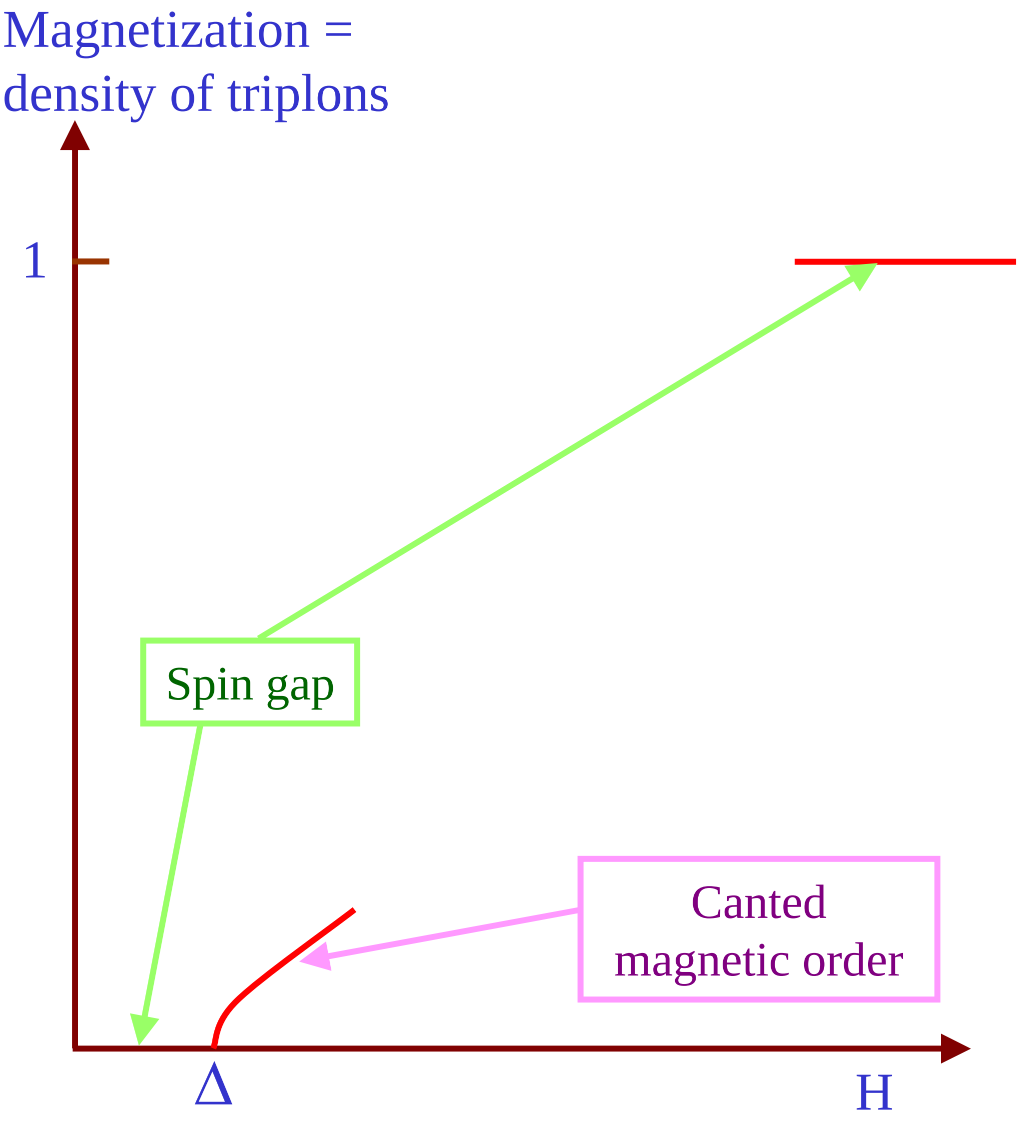


Magnetization = density of triplons

Spin gap

Canted magnetic order

Phase diagram in a strong magnetic field.



Magnetization =
density of triplons

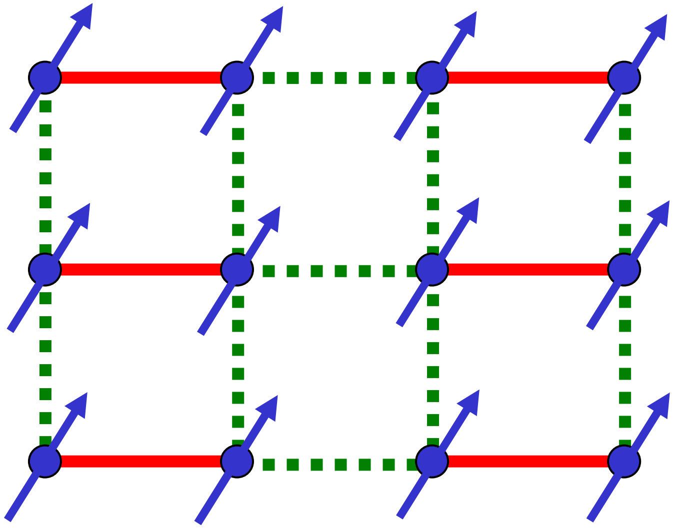
1

Spin gap

Canted
magnetic order

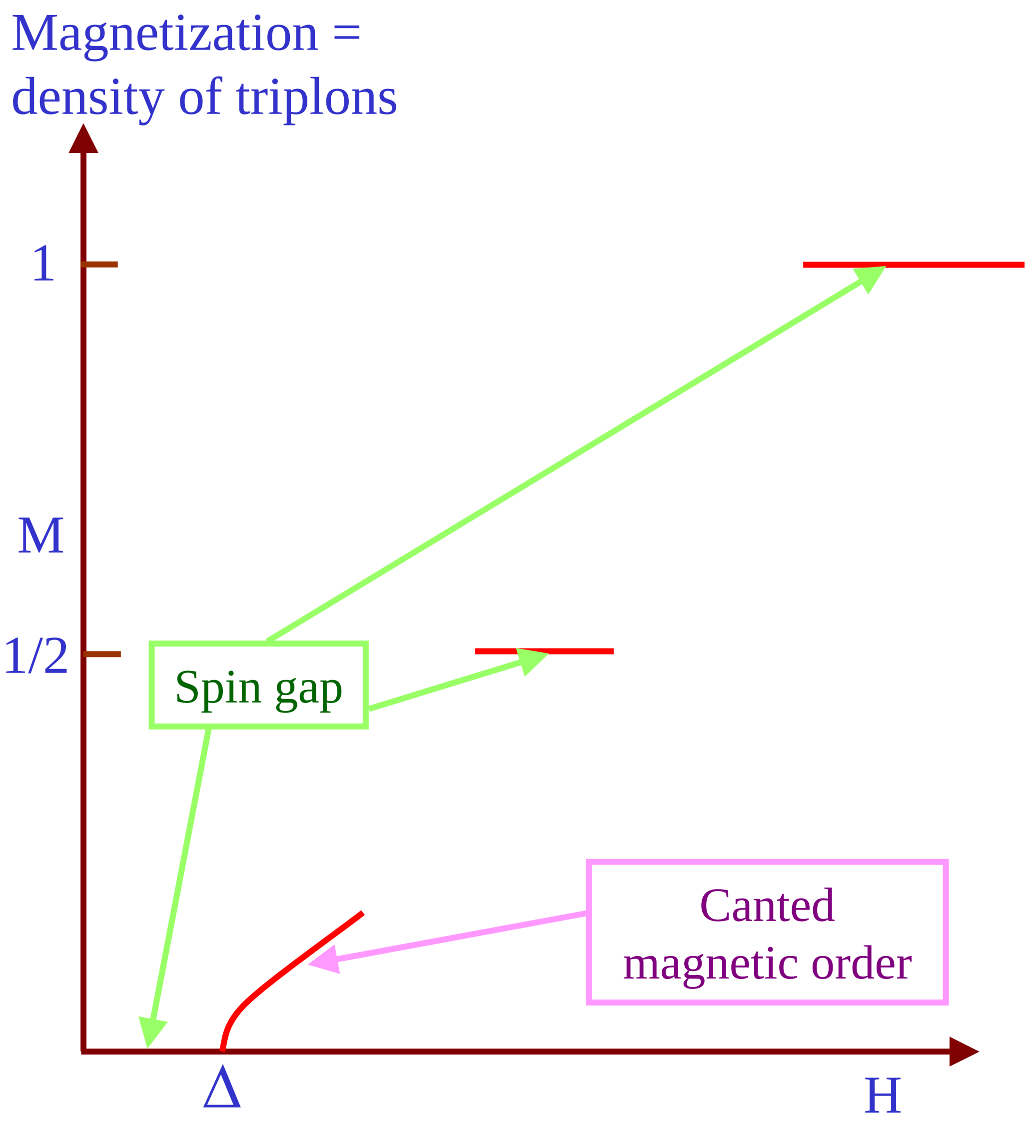
Δ

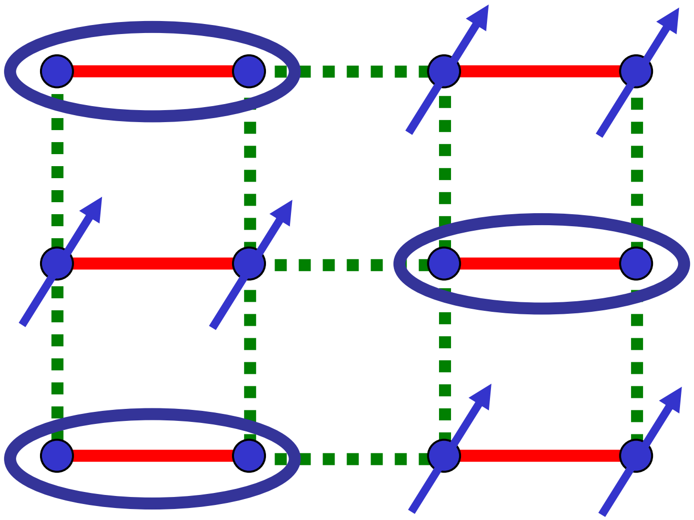
H



At very large H,
magnetization
saturates

Phase diagram in a strong magnetic field.



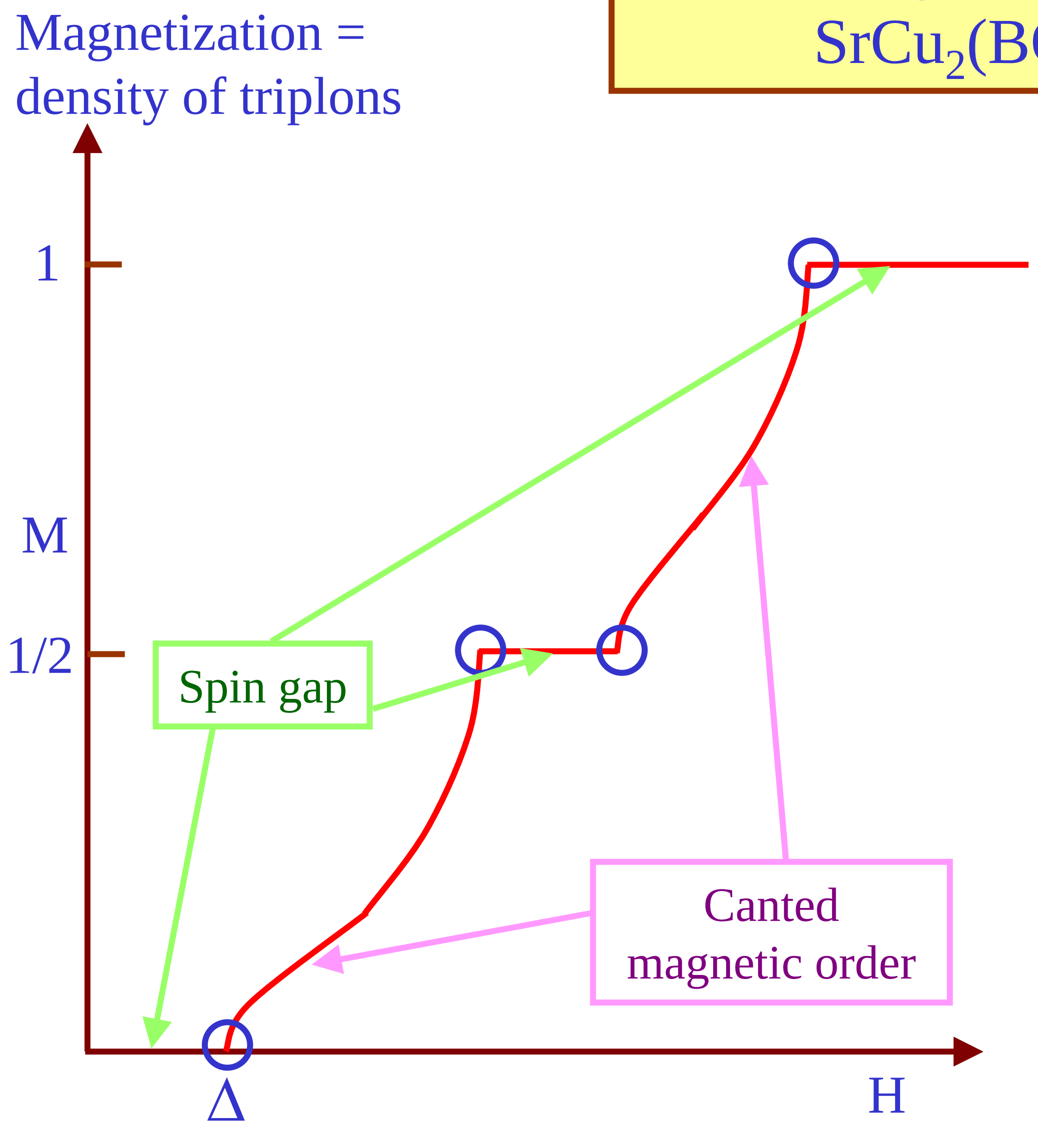


$$\sum_{i<j} J_{ij} S_{zi} S_{zj}$$

Respulsive interactions between triplons can lead to magnetization plateau at any rational fraction

Phase diagram in a strong magnetic field.

Partial magnetization plateau observed in $SrCu_2(BO_3)_2$ and NH_4CuCl_3



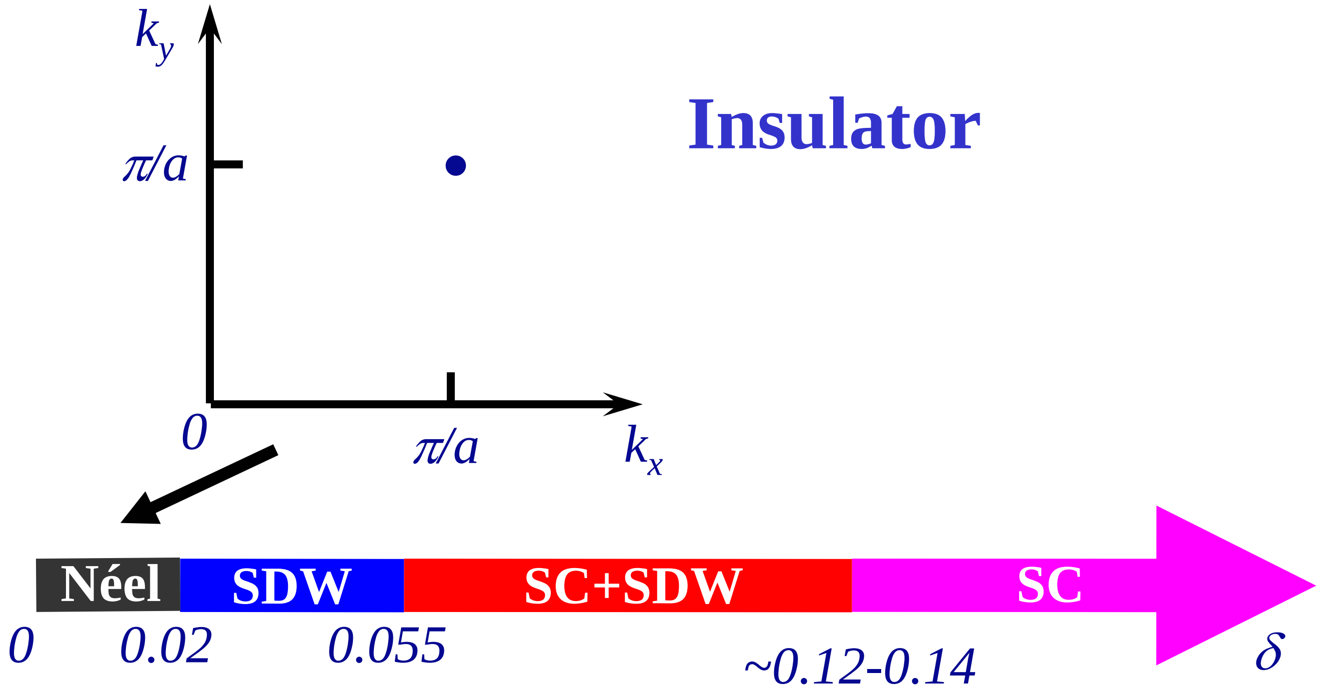
Quantum transitions in and out of plateau are Bose-Einstein condensations of “extra/missing” triplons

(C) Tuning quantum transitions by a magnetic field

2. Cuprate superconductors

Interplay of SDW and SC order in the cuprates

T=0 phases of LSCO

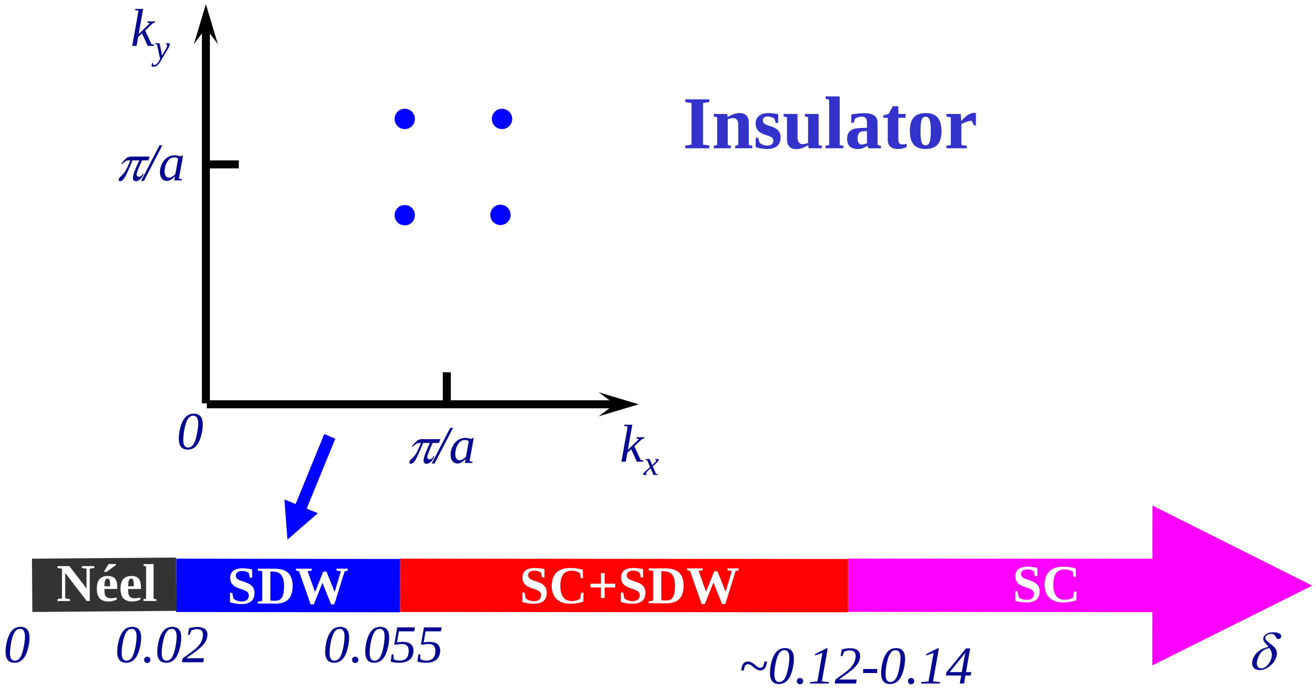


(additional commensurability effects near δ=0.125)

J. M. Tranquada *et al*., *Phys*. *Rev*. B **54**, 7489 (1996).
G. Aeppli, T.E. Mason, S.M. Hayden, H.A. Mook, J. Kulda, *Science* **278**, 1432 (1997).
S. Wakimoto, G. Shirane *et al*., *Phys*. *Rev*. B **60**, R769 (1999).
Y.S. Lee, R. J. Birgeneau, M. A. Kastner *et al*., *Phys*. *Rev*. B **60**, 3643 (1999)
S. Wakimoto, R.J. Birgeneau, Y.S. Lee, and G. Shirane, *Phys*. *Rev*. B **63**, 172501 (2001).

Interplay of SDW and SC order in the cuprates

T=0 phases of LSCO



(additional commensurability effects near δ=0.125)

J. M. Tranquada *et al*., *Phys*. *Rev*. B **54**, 7489 (1996).
G. Aeppli, T.E. Mason, S.M. Hayden, H.A. Mook, J. Kulda, *Science* **278**, 1432 (1997).
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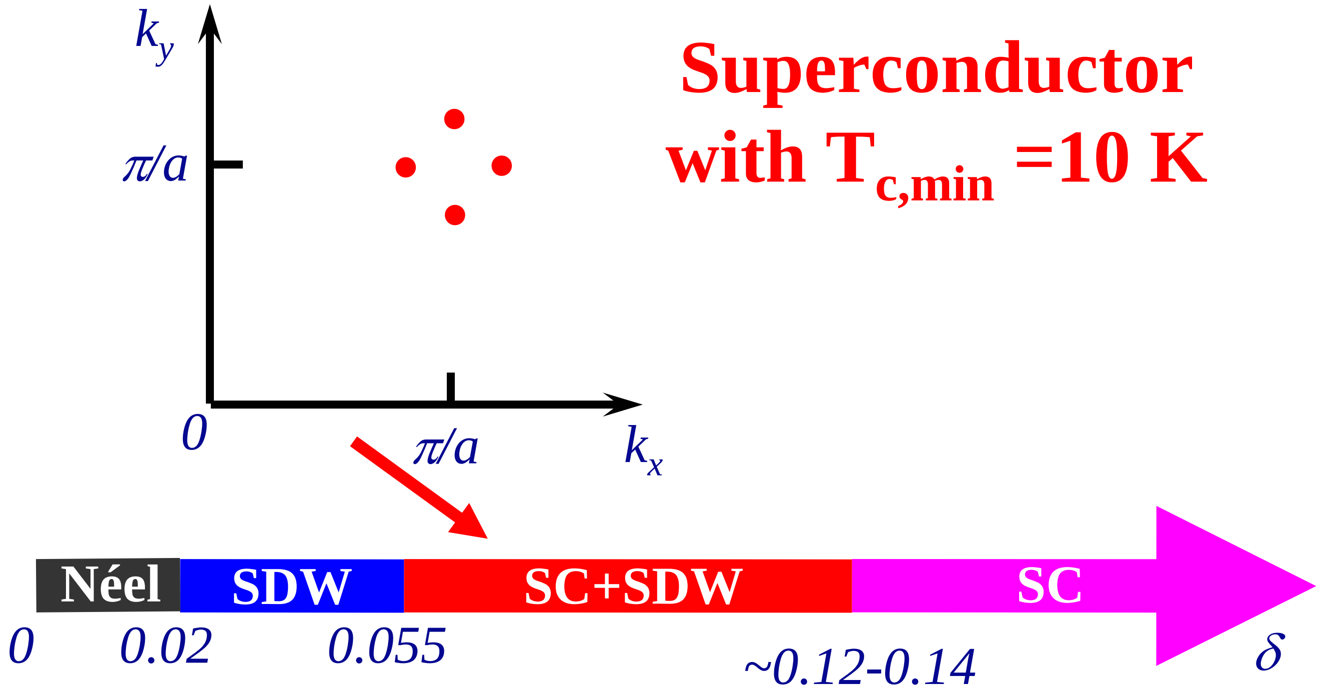
Y.S. Lee, R. J. Birgeneau, M. A. Kastner *et al*., *Phys*. *Rev*. B **60**, 3643 (1999)
S. Wakimoto, R.J. Birgeneau, Y.S. Lee, and G. Shirane, *Phys*. *Rev*. B **63**, 172501 (2001).

Interplay of SDW and SC order in the cuprates

T=0 phases of LSCO



Superconductor with $T_{c,min}$ =10 K

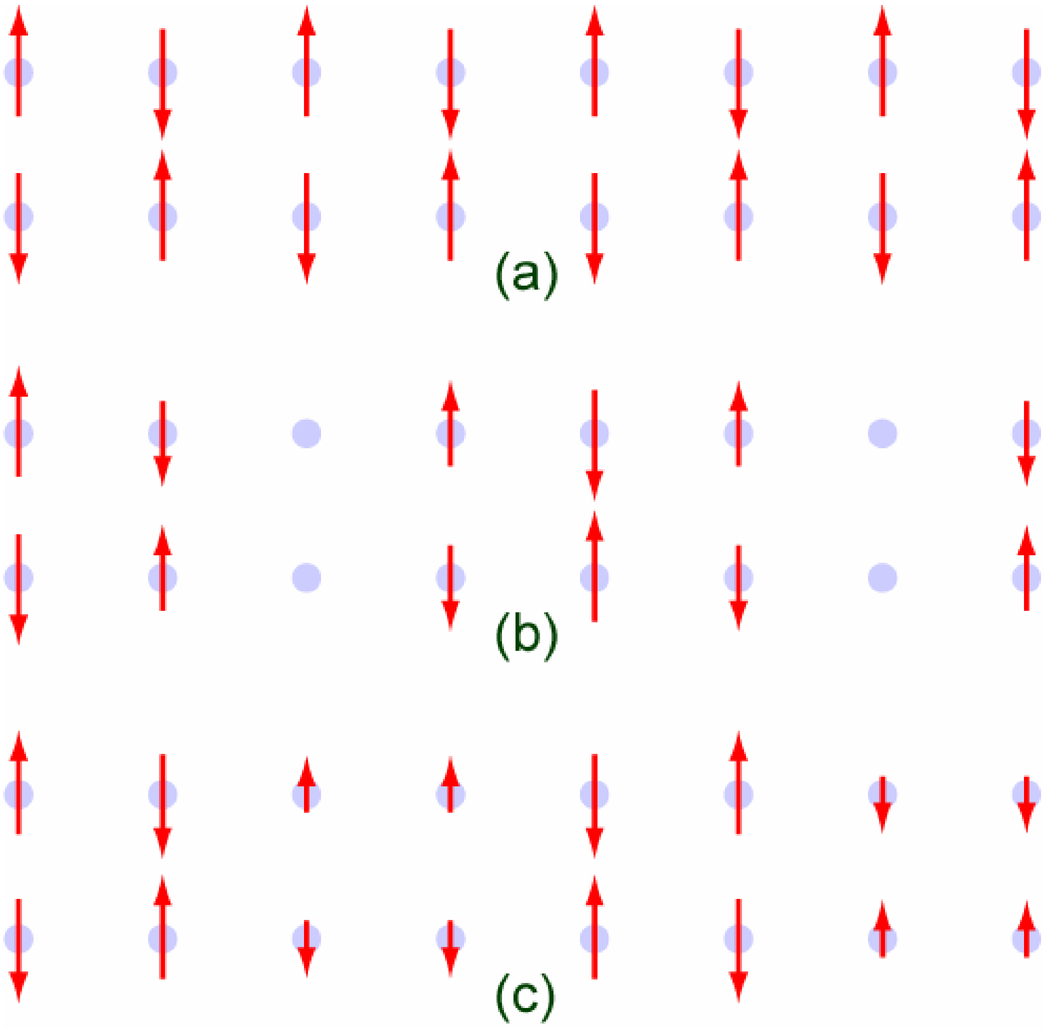
(additional commensurability effects near δ=0.125)

J. M. Tranquada *et al*., *Phys*. *Rev*. B **54**, 7489 (1996).
G. Aeppli, T.E. Mason, S.M. Hayden, H.A. Mook, J. Kulda, *Science* **278**, 1432 (1997).
S. Wakimoto, G. Shirane *et al*., *Phys*. *Rev*. B **60**, R769 (1999).
Y.S. Lee, R. J. Birgeneau, M. A. Kastner *et al*., *Phys*. *Rev*. B **60**, 3643 (1999)
S. Wakimoto, R.J. Birgeneau, Y.S. Lee, and G. Shirane, *Phys*. *Rev*. B **63**, 172501 (2001).

Collinear magnetic (spin density wave) order

$$\langle \boldsymbol{S}_j \rangle = \mathrm{Re}\left[\boldsymbol{\Phi} e^{i\vec{K}\cdot\vec{r}_j} \right] \; ; \; \text{order parameter is complex vector } \boldsymbol{\Phi}$$

$$\text{Collinear spins} \quad \Rightarrow \quad \boldsymbol{\Phi} = \boldsymbol{n} e^{i\theta}$$



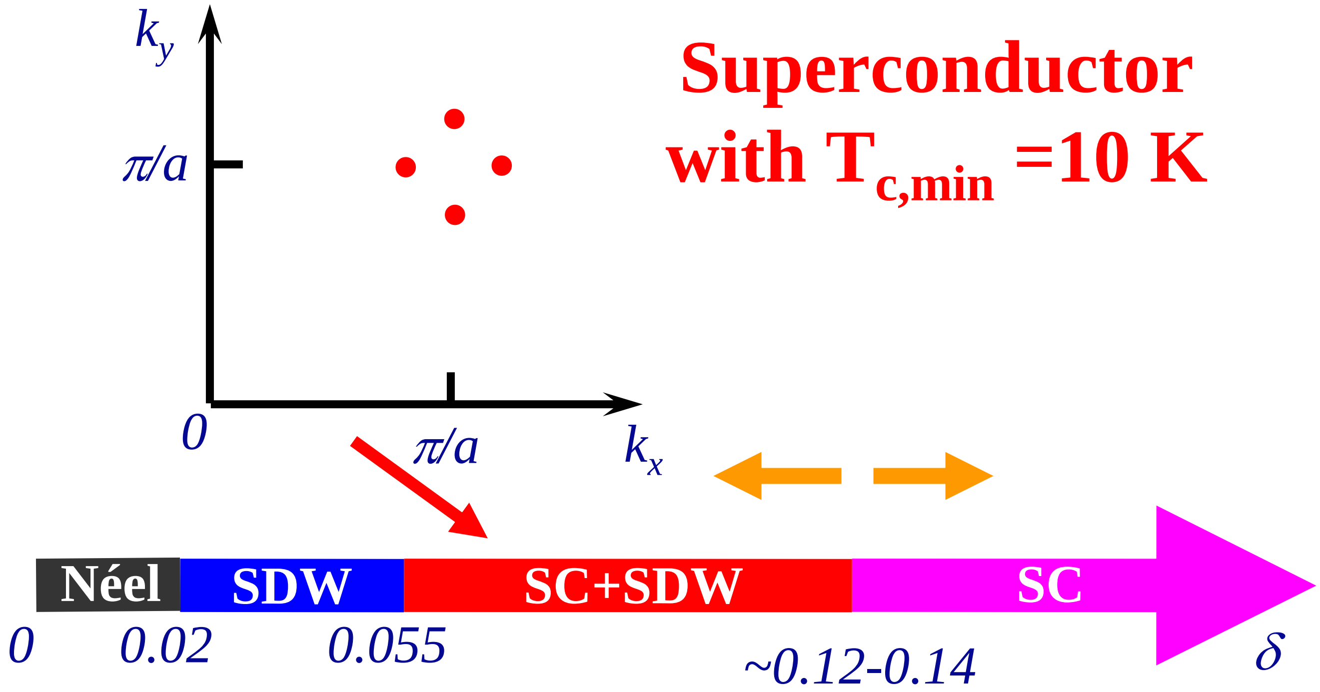
$$\vec{K} = (\pi, \pi) \; ; \; \theta = 0$$

$$\vec{K} = (3\pi/4, \pi) \; ; \; \theta = 0$$

$$\vec{K} = (3\pi/4, \pi) \; ; \; \theta = \pi/8$$

Interplay of SDW and SC order in the cuprates

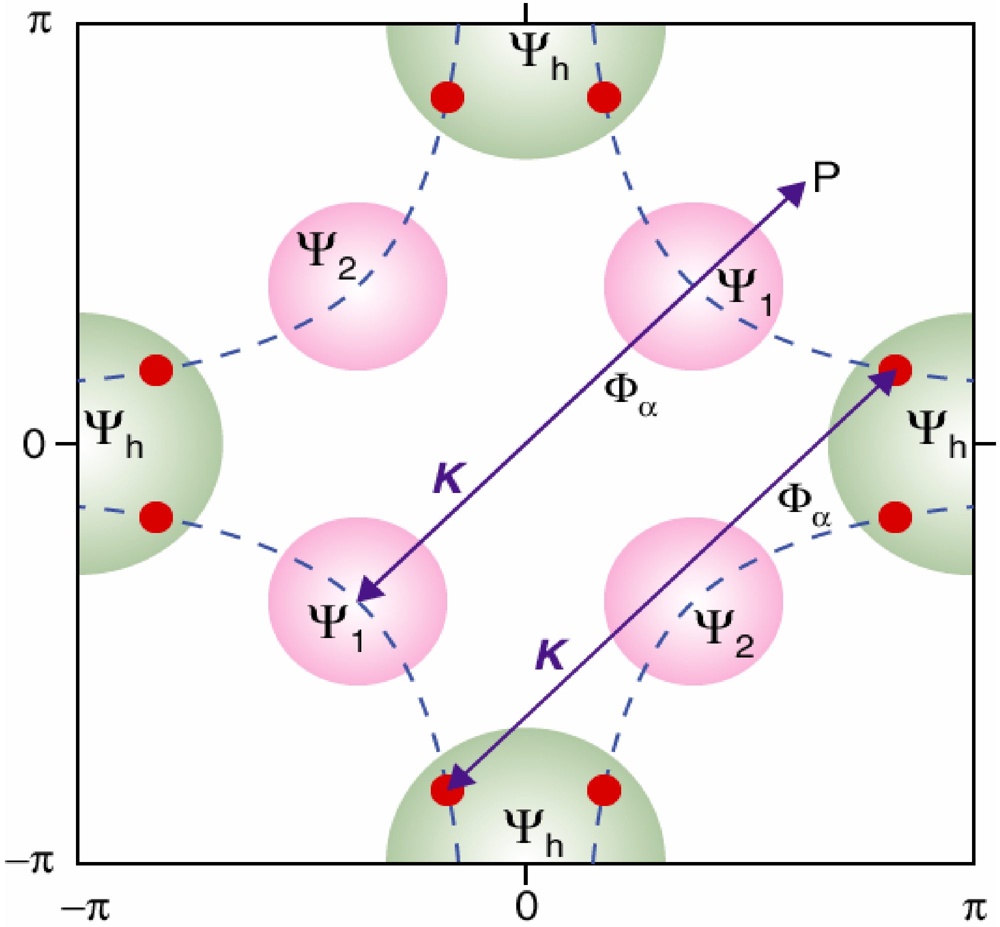
T=0 phases of LSCO



Superconductor with $T_{c,min}$ =10 K

Use simplest assumption of a direct second-order quantum phase transition between SC and SC+SDW phases

Magnetic transition in a *d*-wave superconductor



If $\vec{K}$ does not exactly connect two nodal points, critical theory is as in an insulator

Otherwise, new theory of coupled excitons and nodal quasiparticles

L. Balents, M.P.A. Fisher, C. Nayak, *Int. J. Mod. Phys.* B **12**, 1033 (1998).

Magnetic transition in a *d*-wave superconductor

$$S = \int d^2r\,d\tau \left[\left|\nabla_r \Phi_\alpha\right|^2 + c^2 \left|\partial_\tau \Phi_\alpha\right|^2 + V\left(\Phi_\alpha\right) \right]$$

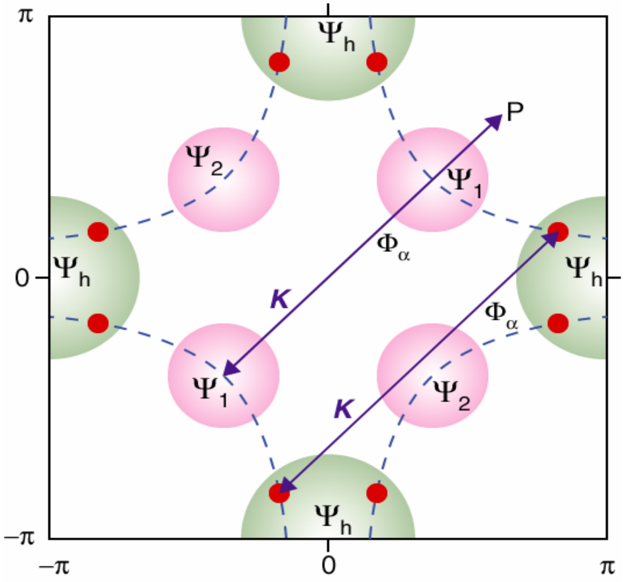
Similar terms present in action for SDW ordering in the insulator

Coupling to the *S*=1/2 Bogoliubov quasiparticles of the *d*-wave superconductor

Trilinear "Yukawa" coupling

$$\int d^2r\,d\tau\,\Phi_\alpha \Psi\Psi$$

is prohibited unless ordering wavevector is fine-tuned.

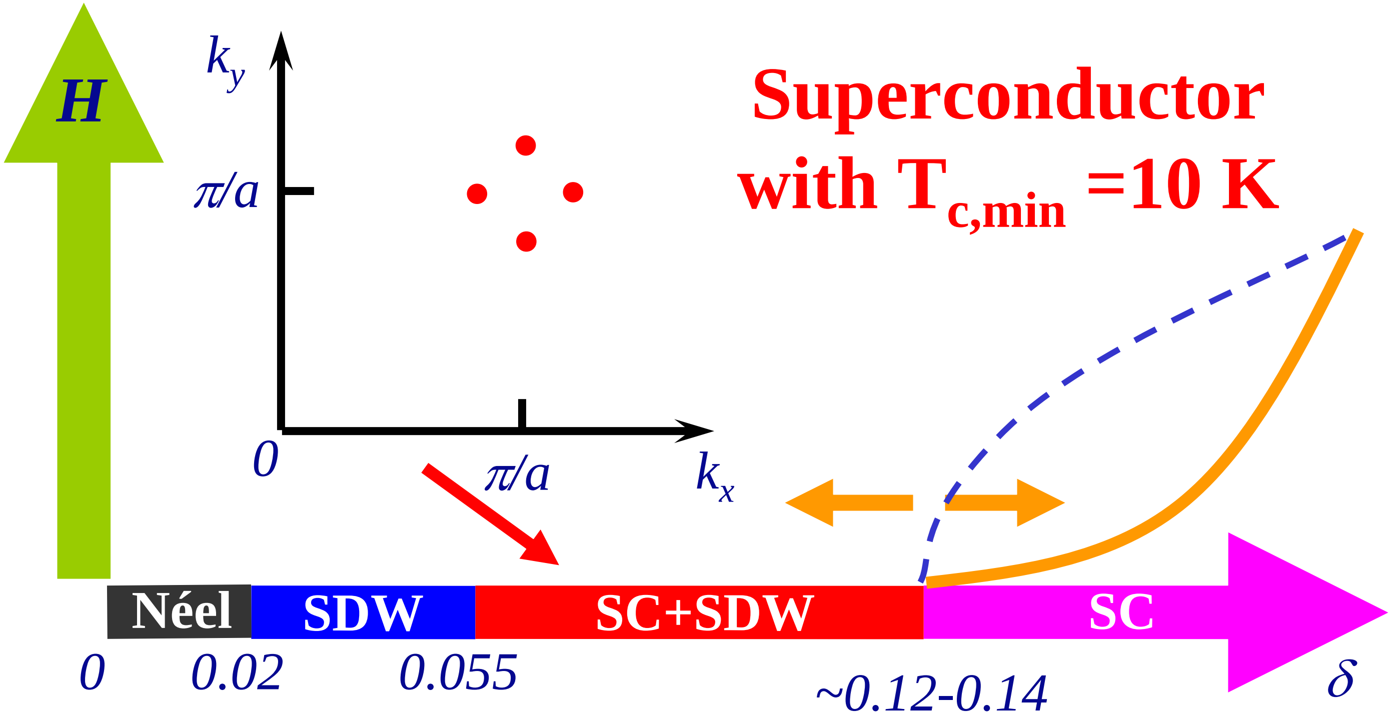


$$\kappa \sum_\alpha \int d^2r\,d\tau \left|\Phi_\alpha\right|^2 \Psi^\dagger\Psi$$ is allowed

Scaling dimension of $\kappa = (1/\nu - 2) < 0 \;\Rightarrow$ irrelevant.

Interplay of SDW and SC order in the cuprates

T=0 phases of LSCO

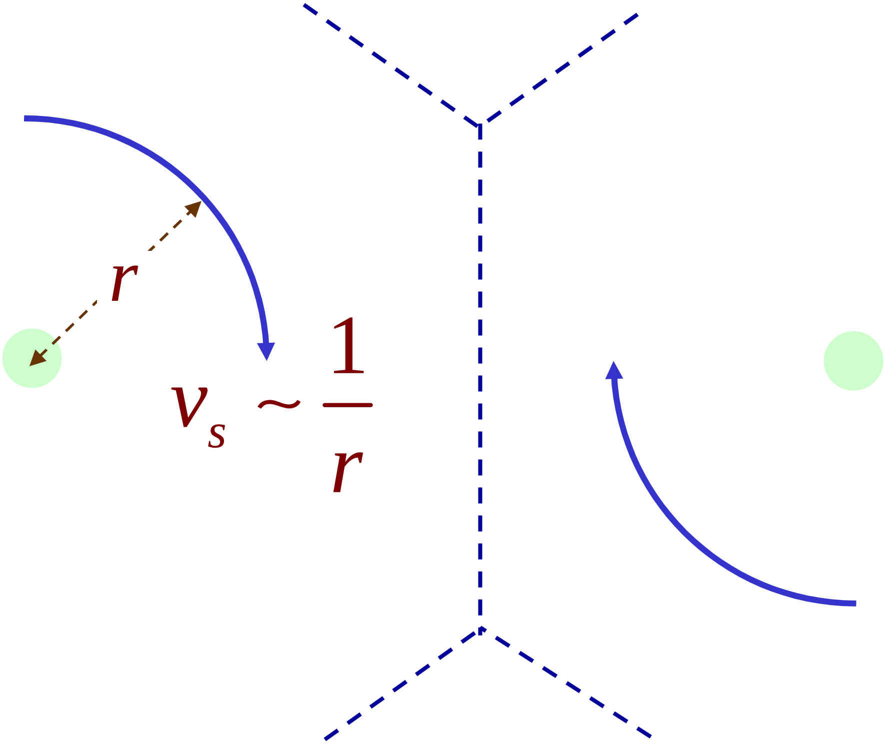


Use simplest assumption of a direct second-order quantum phase transition between SC and SC+SDW phases

Follow intensity of elastic Bragg spots in a magnetic field

Recall, in an insulator intensity would increase ~ H^2

Dominant effect of magnetic field: Abrikosov flux lattice



$$v_s \sim \frac{1}{r}$$

Spatially averaged superflow kinetic energy

$$\sim \left\langle v_s^2 \right\rangle \sim \frac{H}{H_{c2}} \ln \frac{3H_{c2}}{H}$$

Effect of magnetic field on SDW+SC to SC transition

(extreme Type II superconductivity)

Quantum theory for dynamic and critical spin fluctuations

$$S_b = \int d^2 r \int_0^{1/T} d\tau \left[\left| \nabla_r \Phi_\alpha \right|^2 + c^2 \left| \partial_\tau \Phi_\alpha \right|^2 + s \left| \Phi_\alpha \right|^2 + \frac{g_1}{2} \left(\left| \Phi_\alpha \right|^2 \right)^2 + \frac{g_2}{2} \left| \Phi_\alpha^2 \right|^2 \right]$$

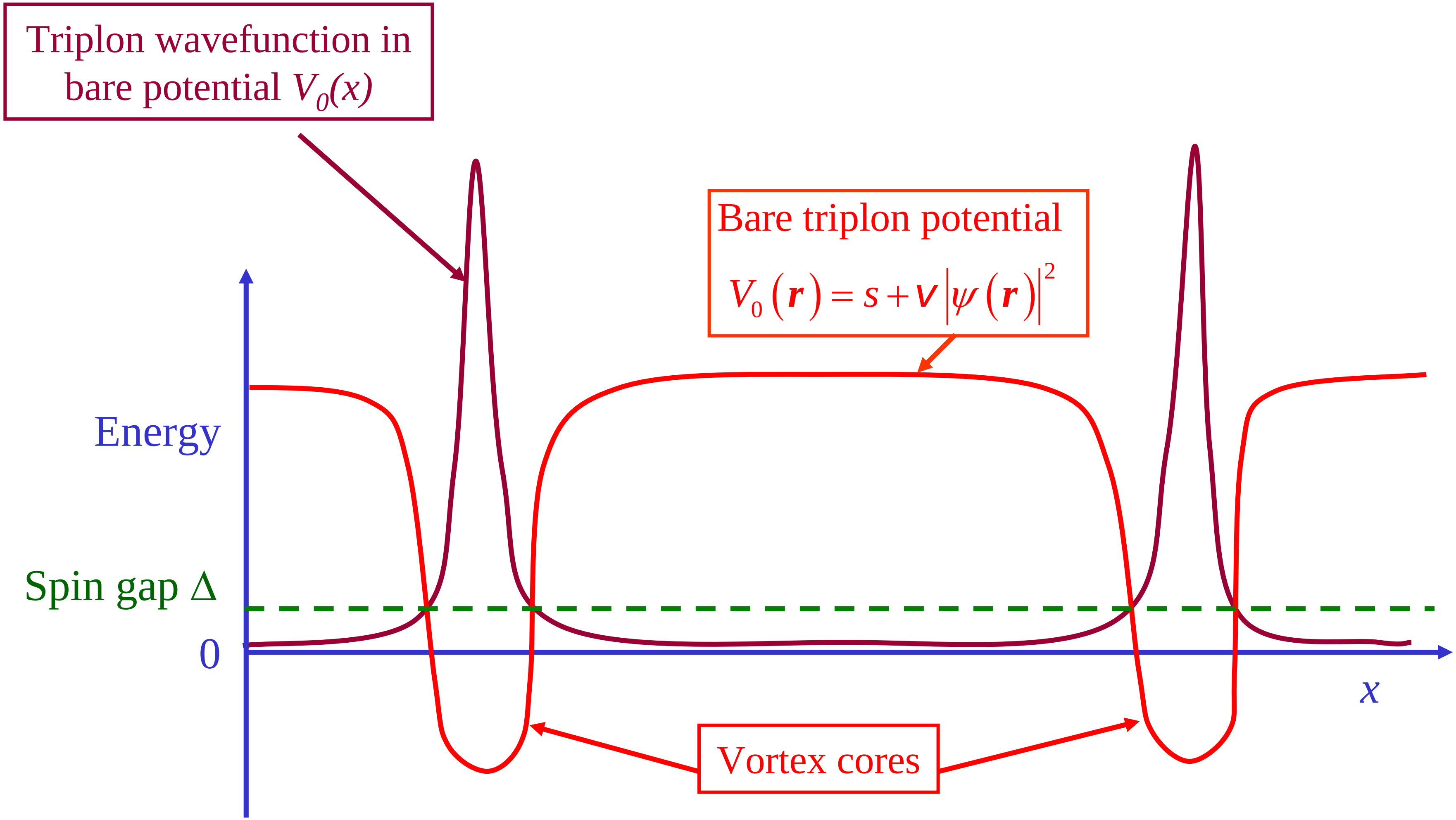
$$S_c = \int d^2 r d\tau \left[\frac{v}{2} \left| \Phi_\alpha \right|^2 \left| \psi \right|^2 \right]$$

$$Z\left[\psi(r)\right] = \int D\Phi(r,\tau) e^{-F_{GL} - S_b - S_c}$$

$$\frac{\delta \ln Z\left[\psi(r)\right]}{\delta \psi(r)} = 0$$

$$F_{GL} = \int d^2 r \left[-\left|\psi\right|^2 + \frac{\left|\psi\right|^4}{2} + \left| \left(\nabla_r - iA \right) \psi \right|^2 \right]$$

Static Ginzburg-Landau theory for non-critical superconductivity



Triplon wavefunction in bare potential $V_0(x)$

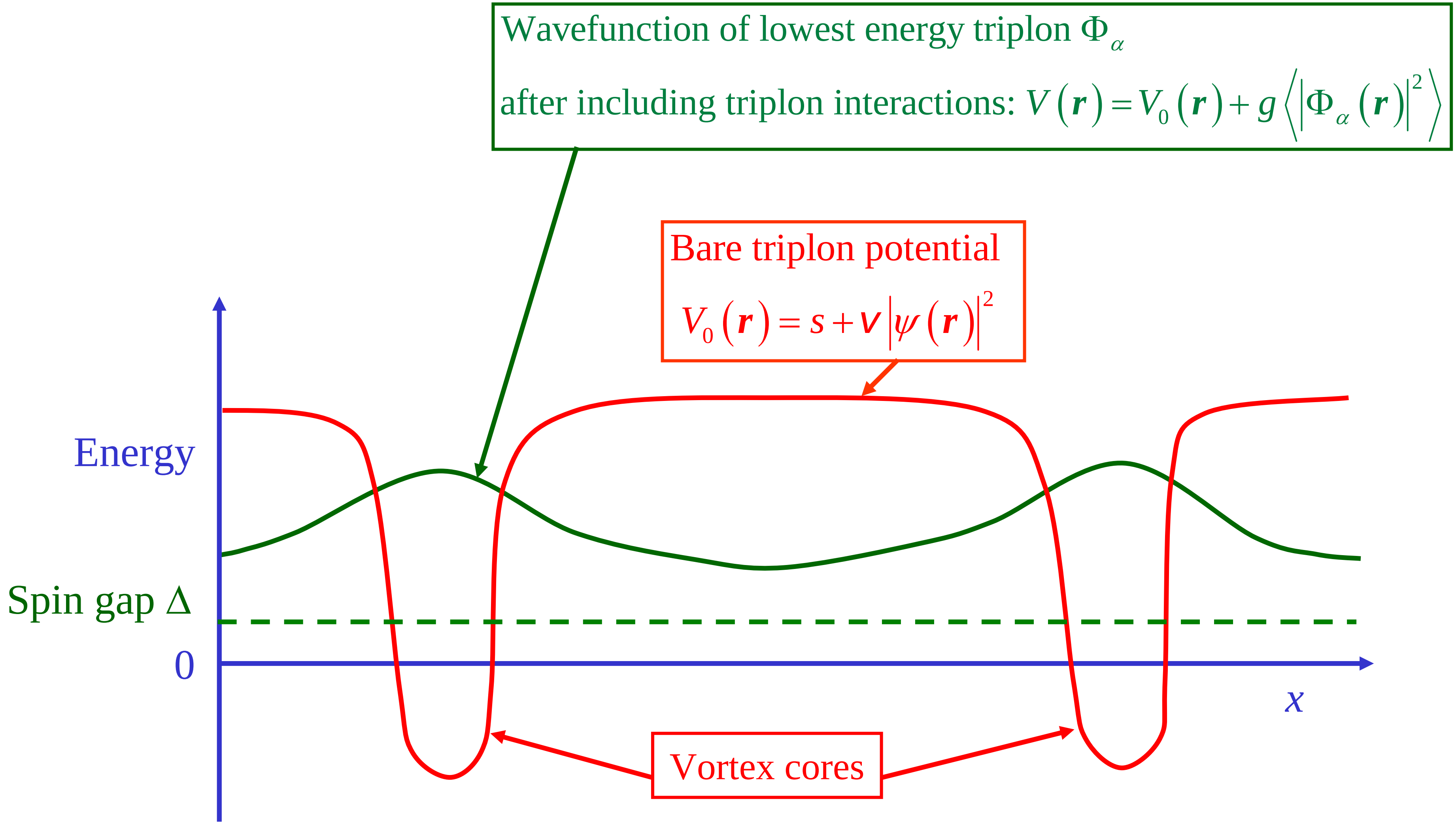
Bare triplon potential

$$V_0(\boldsymbol{r}) = s + \boldsymbol{v}\left|\psi(\boldsymbol{r})\right|^2$$

Vortex cores

D. P. Arovas, A. J. Berlinsky, C. Kallin, and S.-C. Zhang, *Phys. Rev. Lett.* **79**, 2871 (1997) suggested nucleation of static magnetism (with Δ=0) within vortex scores in a first-order transition. However, given the small size of the vortex cores, the magnetism must become dynamic as in a spin gap state.

S. Sachdev, *Phys. Rev.* B **45**, 389 (1992); N. Nagaosa and P. A. Lee, *Phys. Rev.* B **45**, 966 (1992)



Wavefunction of lowest energy triplon Φ_α

after including triplon interactions: $V(\boldsymbol{r}) = V_0(\boldsymbol{r}) + g\left\langle \left|\Phi_\alpha(\boldsymbol{r})\right|^2 \right\rangle$

Bare triplon potential

$V_0(\boldsymbol{r}) = s + v\left|\psi(\boldsymbol{r})\right|^2$

Vortex cores

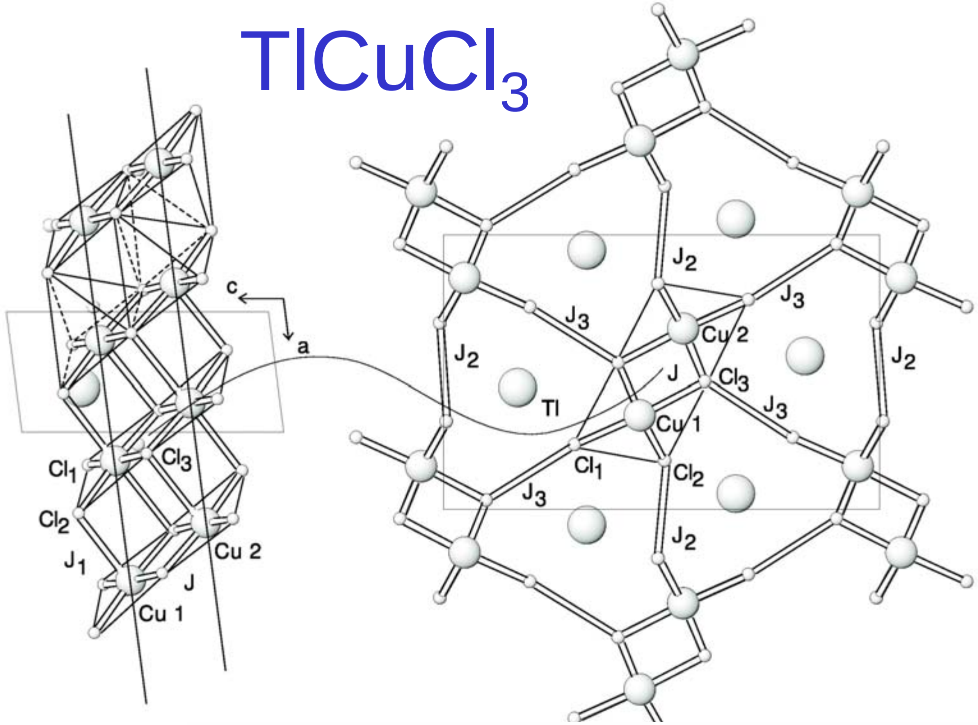
Strongly relevant repulsive interactions between excitons imply that triplons must be extended as $\Delta \to 0$.

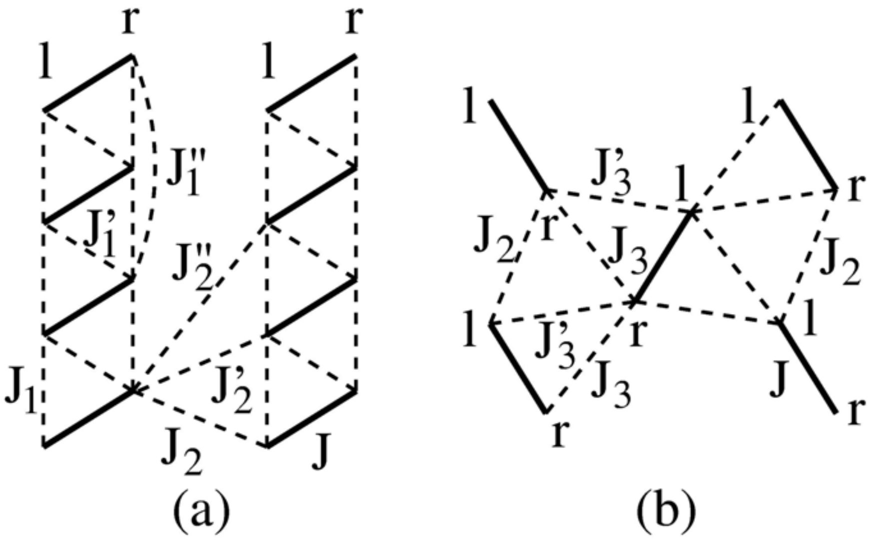
E. Demler, S. Sachdev, and Y. Zhang, *Phys. Rev. Lett.* **87**, 067202 (2001).

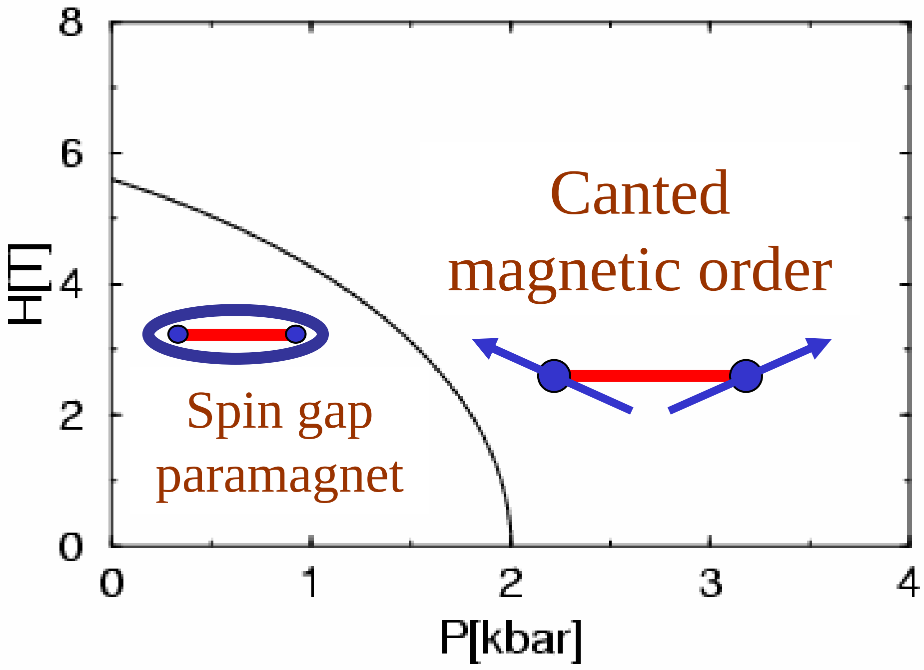
A.J. Bray and M.A. Moore, *J. Phys.* C **15**, L7 65 (1982).

J.A. Hertz, A. Fleishman, and P.W. Anderson, *Phys. Rev. Lett.* **43**, 942 (1979).

TlCuCl$_3$

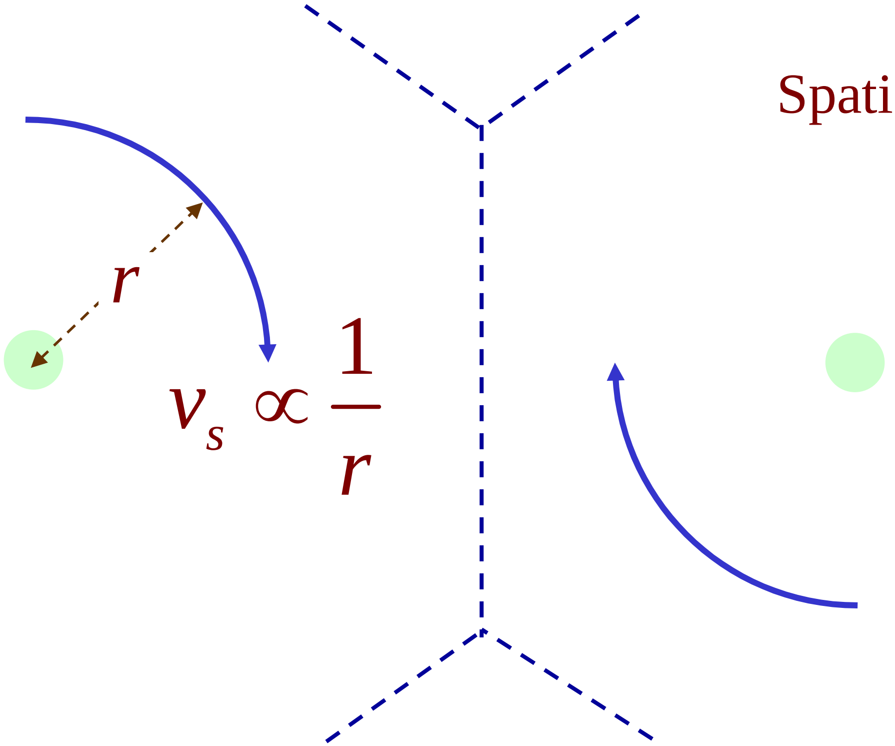






M. Matsumoto,
B. Normand, T.M. Rice,
and M. Sigrist,
cond-mat/0309440.

Phase diagram of SC and SDW order in a magnetic field



$$v_s \propto \frac{1}{r}$$

Spatially averaged superflow kinetic energy

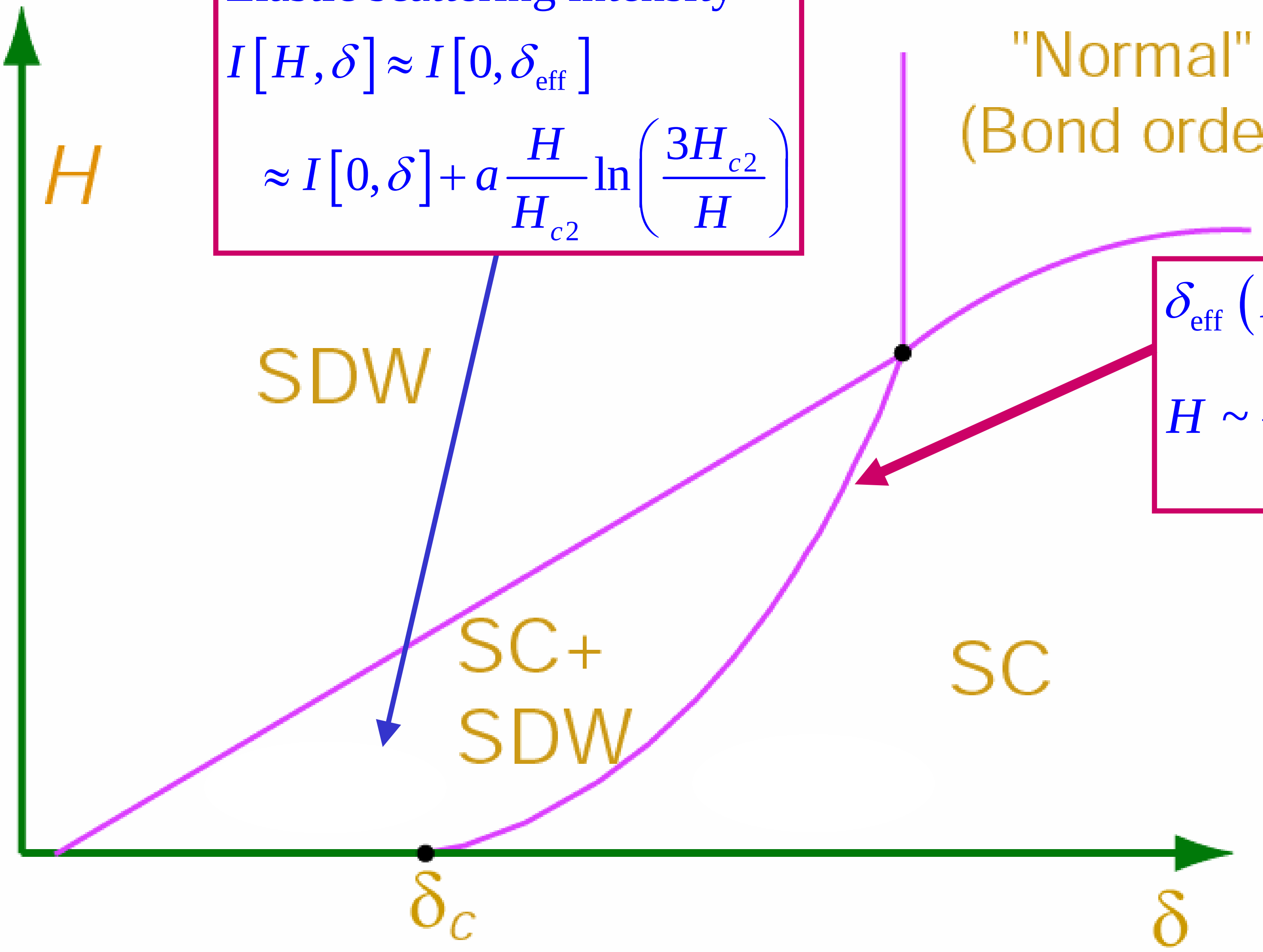
$$\left\langle v_s^2 \right\rangle \propto \frac{H}{H_{c2}} \ln \frac{3H_{c2}}{H}$$

The suppression of SC order appears to the SDW order as a ***uniform*** effective "doping" δ :

$$\delta_{\text{eff}}\left(H\right) = \delta - C\frac{H}{H_{c2}}\ln\left(\frac{3H_{c2}}{H}\right)$$

E. Demler, S. Sachdev, and Ying Zhang, *Phys. Rev. Lett.* **87**, 067202 (2001).

Phase diagram of SC and SDW order in a magnetic field



Elastic scattering intensity

$$I[H,\delta] \approx I[0,\delta_{\text{eff}}]$$

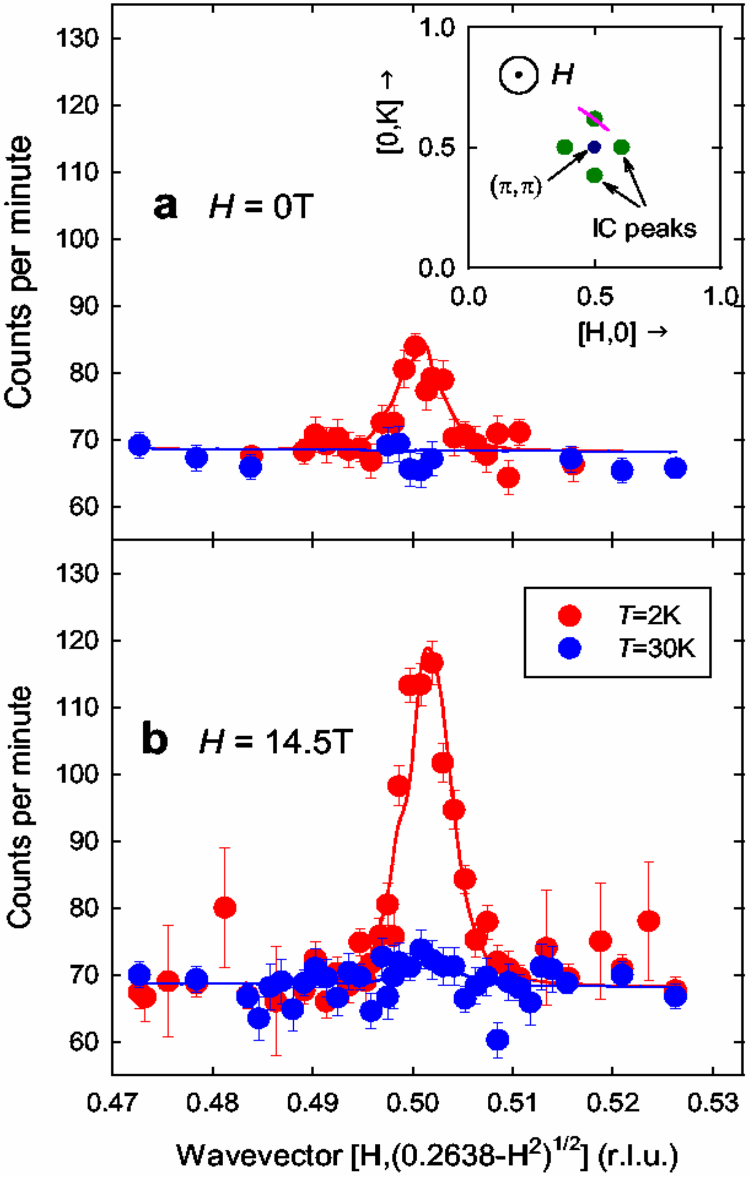
$$\approx I[0,\delta] + a\frac{H}{H_{c2}}\ln\left(\frac{3H_{c2}}{H}\right)$$

$$\delta_{\text{eff}}(H) = \delta_c \quad \Rightarrow$$

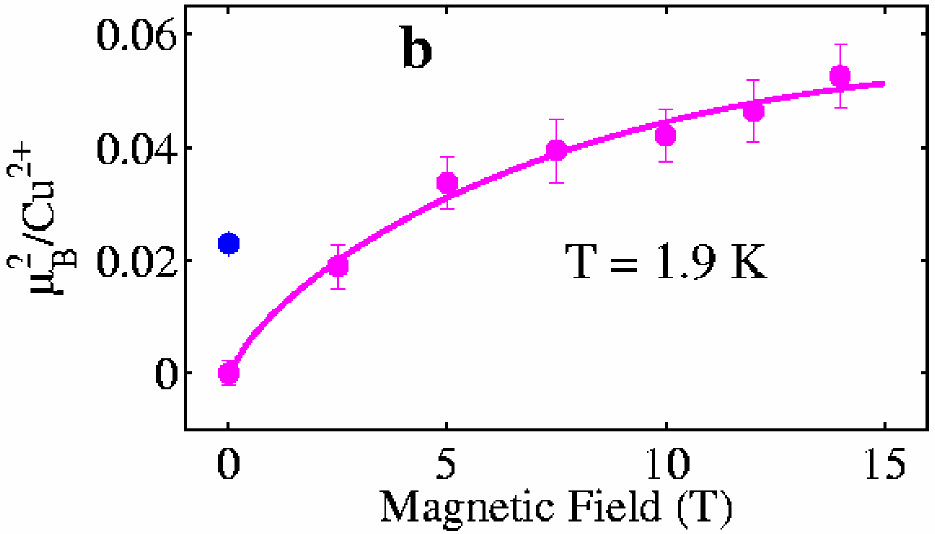
$$H \sim \frac{(\delta-\delta_c)}{\ln\left(1/(\delta-\delta_c)\right)}$$

E. Demler, S. Sachdev, and Ying Zhang, *Phys. Rev. Lett.* **87**, 067202 (2001).

Neutron scattering of $La_{2-x}Sr_xCuO_4$ at x=0.1



B. Lake, H. M. Rønnow, N. B. Christensen, G. Aeppli, K. Lefmann, D. F. McMorrow, P. Vorderwisch, P. Smeibidl, N. Mangkorntong, T. Sasagawa, M. Nohara, H. Takagi, T. E. Mason, *Nature*, **415**, 299 (2002).



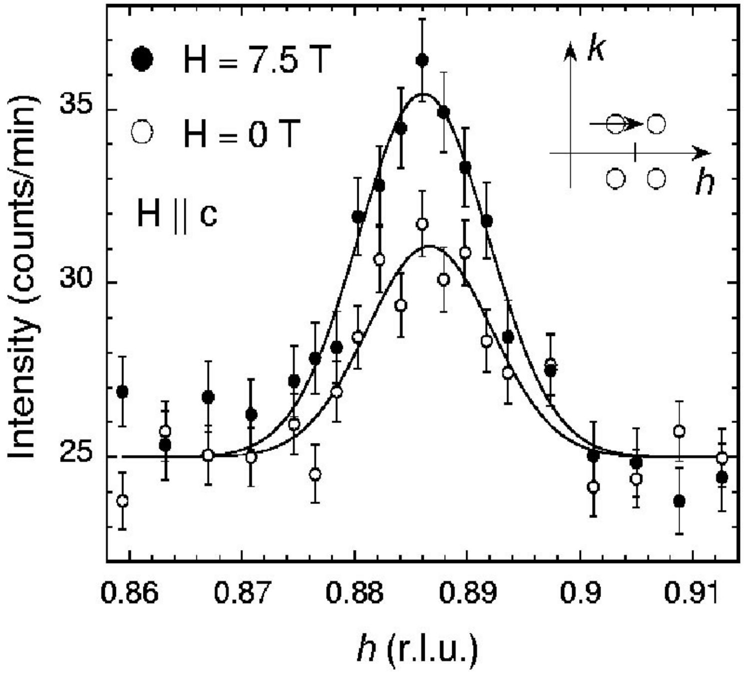
Solid line - fit to : $I(H) = a \frac{H}{H_{c2}} \ln\left(\frac{H_{c2}}{H}\right)$

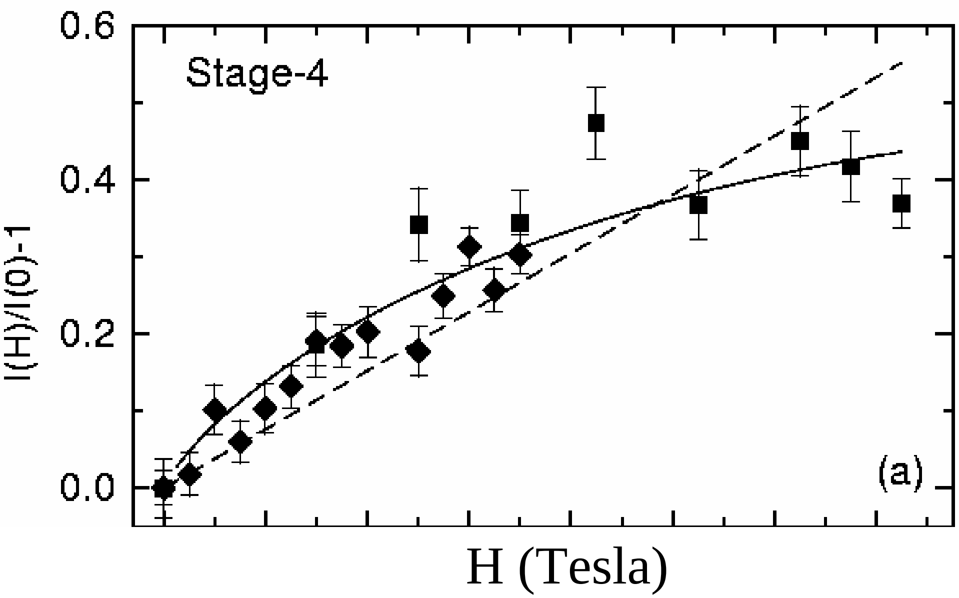
See also S. Katano, M. Sato, K. Yamada, T. Suzuki, and T. Fukase, *Phys. Rev.* B **62**, R14677 (2000).

Neutron scattering measurements of static spin correlations of the superconductor+spin-density-wave (SC+CM) in a magnetic field

Elastic neutron scattering off La_2CuO_{4+y}

B. Khaykovich, Y. S. Lee, S. Wakimoto,
K. J. Thomas, M. A. Kastner,
and R.J. Birgeneau, *Phys. Rev.* B **66**, 014528 (2002).



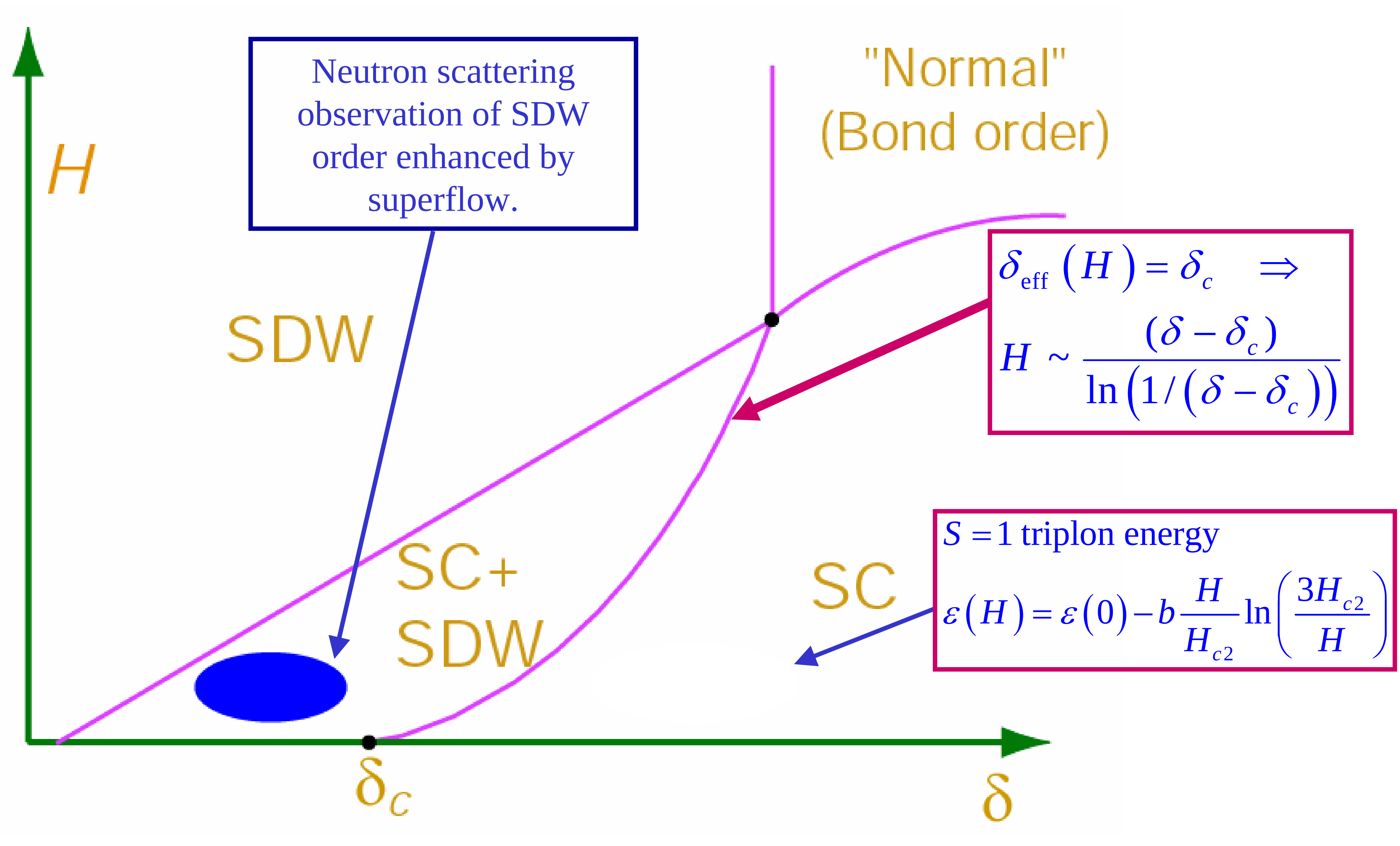


Solid line --- fit to : $$\frac{I(H)}{I(0)} = 1 + a\frac{H}{H_{c2}}\ln\left(\frac{3.0H_{c2}}{H}\right)$$

a is the only fitting parameter

Best fit value - $a = 2.4$ with $H_{c2} = 60$ T

Phase diagram of a superconductor in a magnetic field



Neutron scattering observation of SDW order enhanced by superflow.

$$\delta_{\text{eff}}(H)=\delta_c \quad \Rightarrow$$

$$H \sim \frac{(\delta-\delta_c)}{\ln\left(1/(\delta-\delta_c)\right)}$$

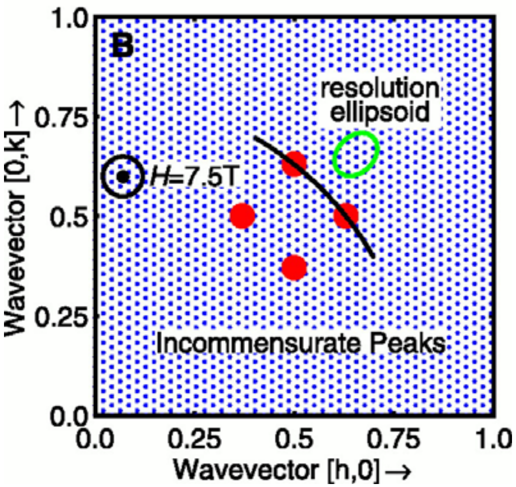
$S=1$ triplon energy

$$\varepsilon(H)=\varepsilon(0)-b\frac{H}{H_{c2}}\ln\left(\frac{3H_{c2}}{H}\right)$$

E. Demler, S. Sachdev, and Ying Zhang, *Phys. Rev. Lett.* **87**, 067202 (2001).

Neutron scattering measurements of dynamic spin correlations of the superconductor (SC) in a magnetic field

B. Lake, G. Aeppli, K. N. Clausen, D. F. McMorrow, K. Lefmann,
N. E. Hussey, N. Mangkorntong, M. Nohara, H. Takagi, T. E. Mason,
and A. Schröder, *Science* **291**, 1759 (2001).



Peaks at $(0.5, 0.5) \pm (0.125, 0)$

and $(0.5,0.5) \pm (0,0.125)$

$\Rightarrow$ dynamic SDW of period 8

Neutron scattering off $\mathrm{La}_{2\text{-}\delta}\mathrm{Sr}_{\delta}\mathrm{CuO}_4$ ($\delta = 0.163$, ***SC phase***)
at low temperatures in H=0 (red dots) and H=7.5T (blue dots)

Neutron scattering measurements of dynamic spin correlations of the superconductor (SC) in a magnetic field

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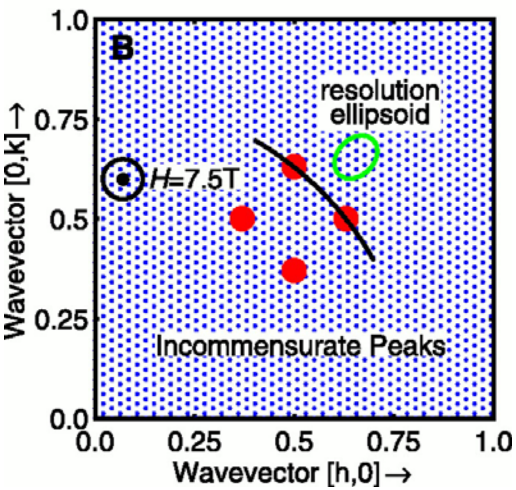
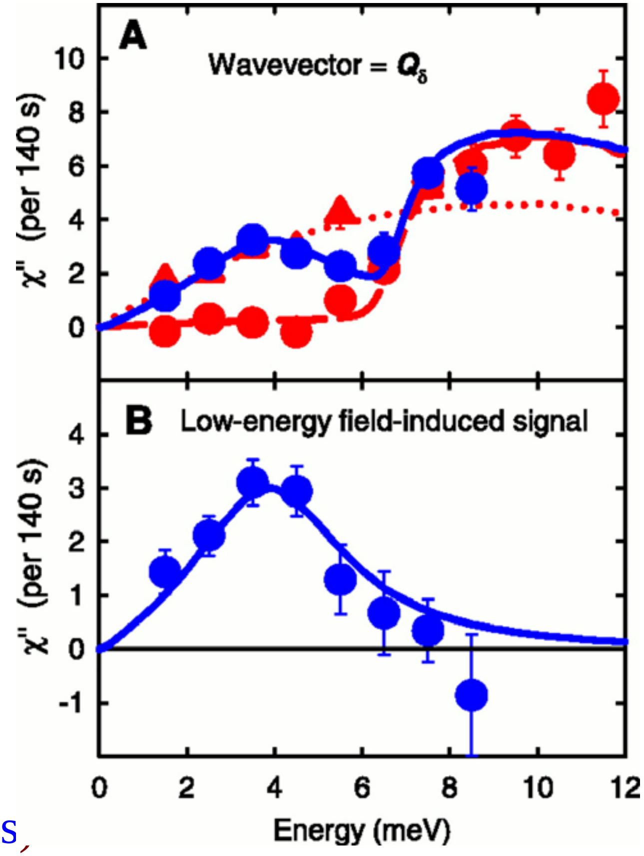
and A. Schröder, *Science* **291**, 1759 (2001).



Peaks at $(0.5, 0.5) \pm (0.125, 0)$

and $(0.5,0.5) \pm (0,0.125)$

$\Rightarrow$ dynamic SDW of period 8

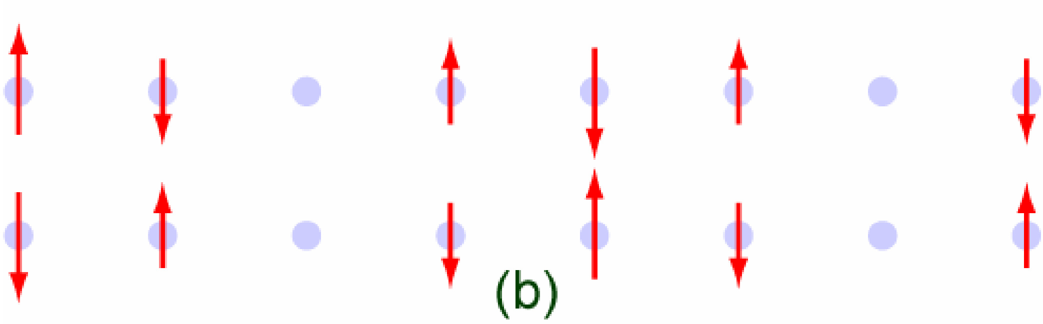


Neutron scattering off $La_{2-\delta}Sr_\delta CuO_4$ ($\delta = 0.163$, ***SC phase***)
at low temperatures in H=0 (red dots) and H=7.5T (blue dots

Collinear magnetic (spin density wave) order

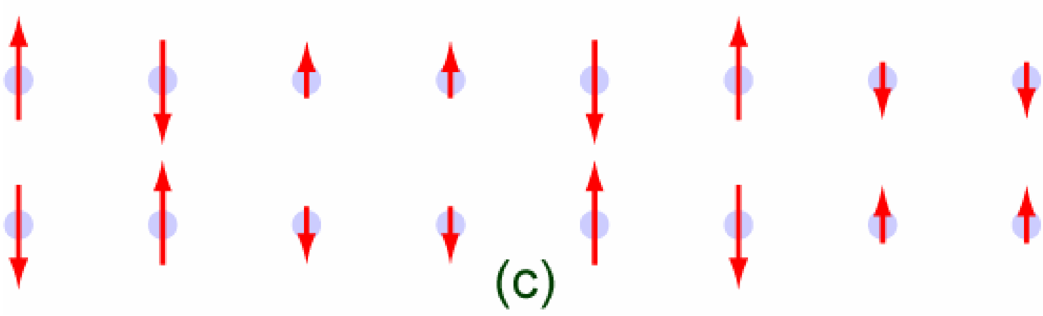
$\langle \boldsymbol{S}_j \rangle = \mathrm{Re}\left[\Phi e^{i\vec{K}\cdot\vec{r}_j} \right]$; order parameter is complex vector Φ

Collinear spins $\Rightarrow$ $\Phi = \boldsymbol{n} e^{i\theta}$, and there is modulation in the *bond order* parameter $Q(\vec{r}_j) \equiv \langle \boldsymbol{S}_j \bullet \boldsymbol{S}_{j+a_x} \rangle$ at wavevector $2\vec{K}$



(b)

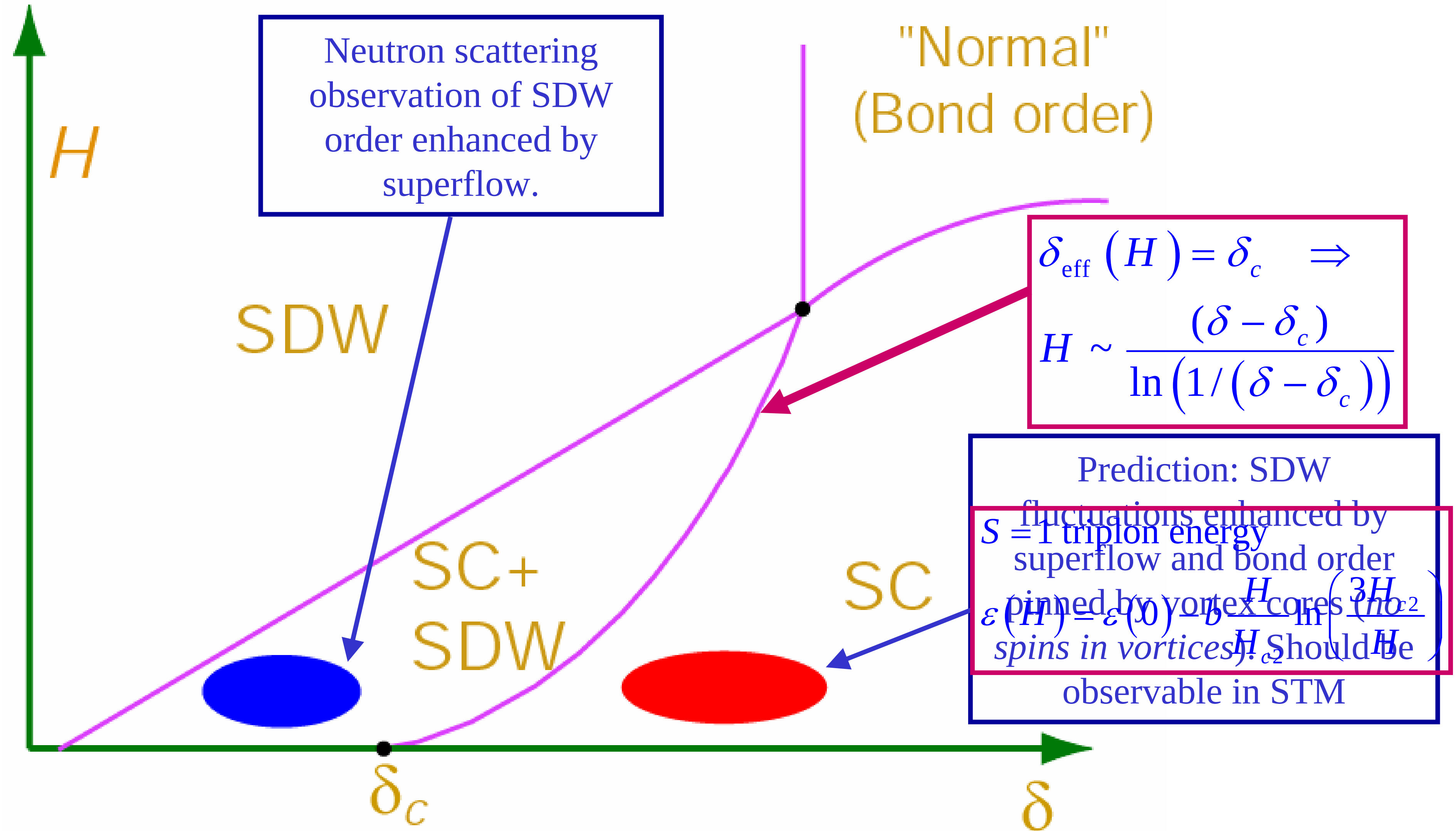
$$\vec{K} = (3\pi/4, \pi) \;\; ; \;\; \theta = 0$$



(c)

$$\vec{K} = (3\pi/4, \pi) \;\; ; \;\; \theta = \pi/8$$

Phase diagram of a superconductor in a magnetic field



Neutron scattering observation of SDW order enhanced by superflow.

$$\delta_{\text{eff}}(H) = \delta_c \quad \Rightarrow$$

$$H \sim \frac{(\delta - \delta_c)}{\ln\left(1/(\delta - \delta_c)\right)}$$

Prediction: SDW fluctuations enhanced by superflow and bond order pinned by vortex cores (no *spins in vortices*). Should be observable in STM

$S = 1$ triplon energy

$$\varepsilon(H) = \varepsilon(0) - b\frac{H}{H_{c2}}\ln\left(\frac{3H_{c2}}{H}\right)$$

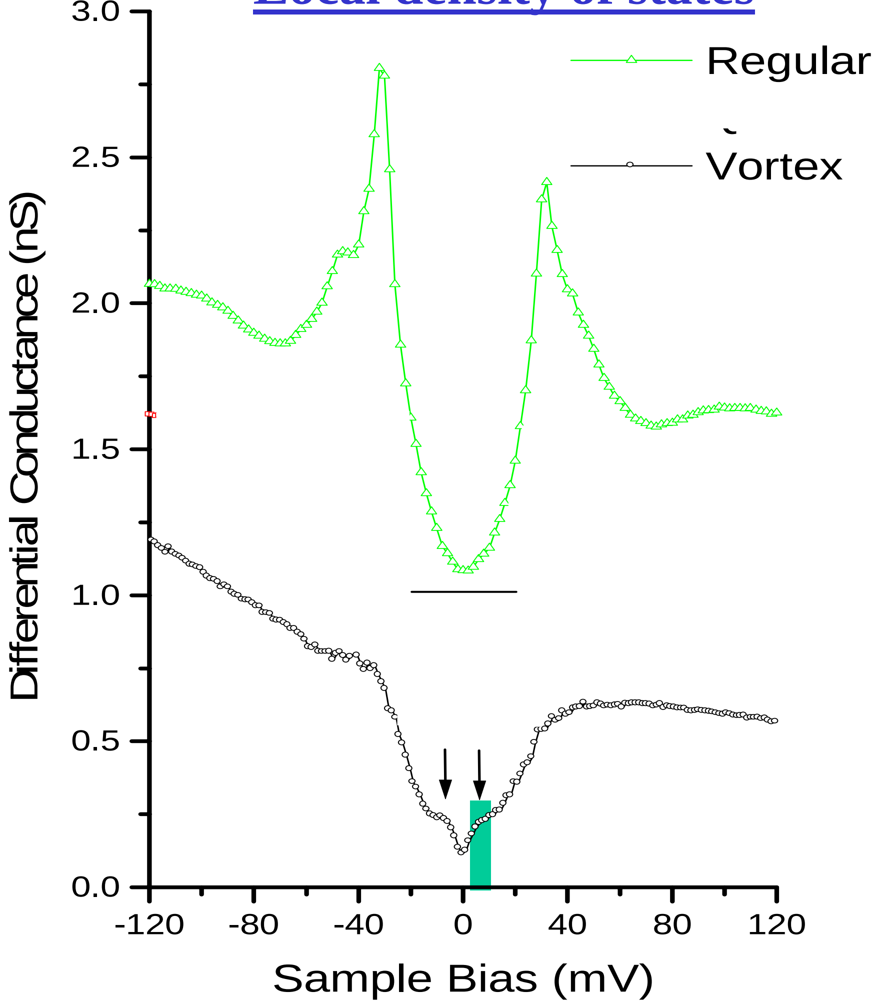
K. Park and S. Sachdev *Physical Review* B **64**, 184510 (2001);
Y. Zhang, E. Demler and S. Sachdev, *Physical Review* B **66**, 094501 (2002).

E. Demler, S. Sachdev, and Ying Zhang, *Phys. Rev. Lett.* **87**, 067202 (2001).

STM around vortices induced by a magnetic field in the superconducting state

J. E. Hoffman, E. W. Hudson, K. M. Lang, V. Madhavan, S. H. Pan,
H. Eisaki, S. Uchida, and J. C. Davis, *Science* **295**, 466 (2002).

Local density of states

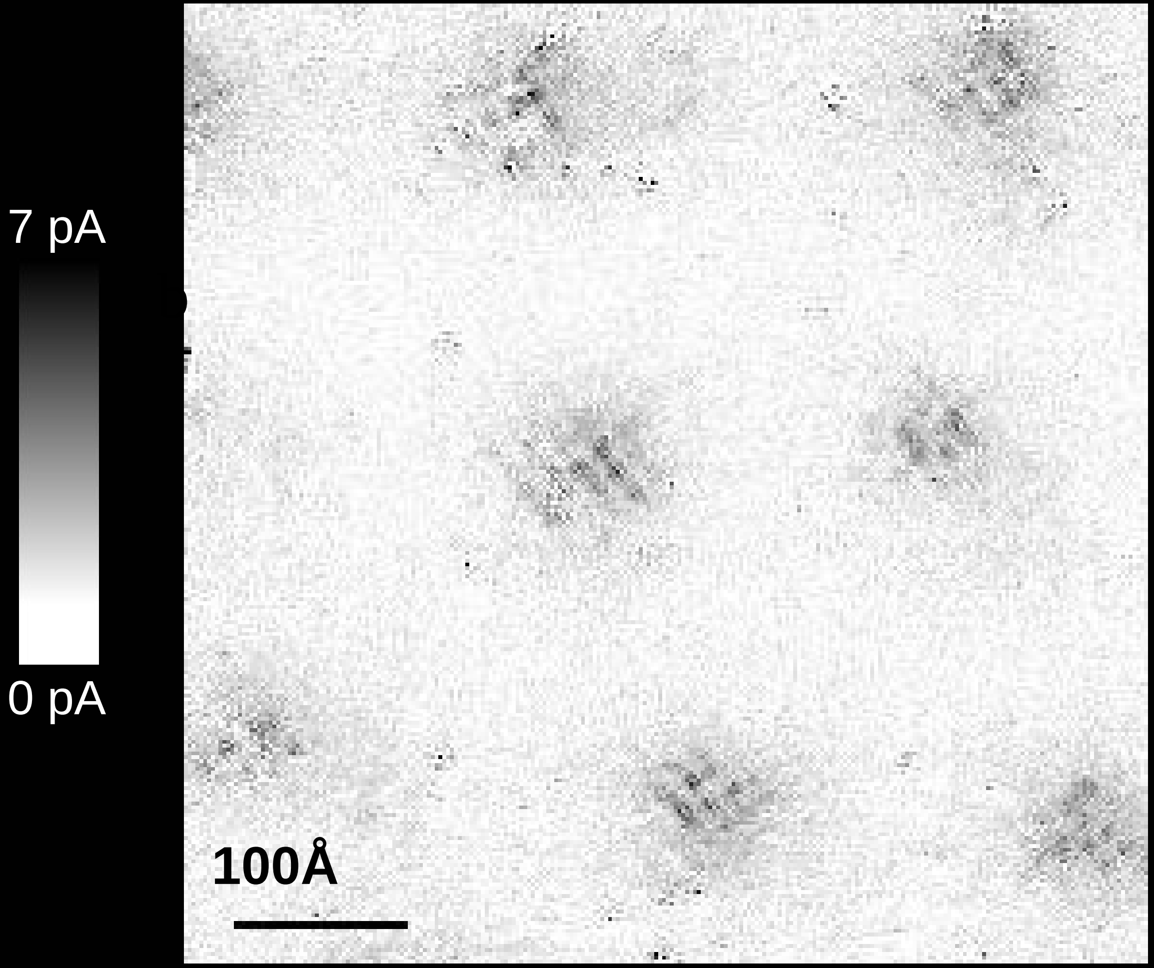


S.H. Pan *et al. Phys. Rev. Lett.* **85**, 1536 (2000).

1Å spatial resolution image of integrated LDOS of $Bi_2Sr_2CaCu_2O_{8+\delta}$

(1meV to 12 meV) at B=5 Tesla.

Vortex-induced LDOS of $Bi_2Sr_2CaCu_2O_{8+\delta}$ integrated from 1meV to 12meV



Our interpretation:
LDOS modulations are signals of bond order of period 4 revealed in vortex halo

See also:
S. A. Kivelson, E. Fradkin,
V. Oganesyan, I. P. Bindloss,
J. M. Tranquada,
A. Kapitulnik, and
C. Howald,
cond-mat/0210683.

J. Hoffman E. W. Hudson, K. M. Lang,
V. Madhavan, S. H. Pan, H. Eisaki, S. Uchida,
and J. C. Davis, *Science* 295, 466 (2002).

Conclusions

I. Introduction to magnetic quantum criticality in coupled dimer antiferromagnet.

II. Berry phases and bond order in square lattice antiferromagnets.

III. Theory of quantum phase transitions provides semi-quantitative predictions for neutron scattering measurements of spin-density-wave order in superconductors; theory also proposes a connection to STM experiments.

IV. Spontaneous bond order in spin gap state on the square lattice: possible connection to modulations observed in vortex halo.