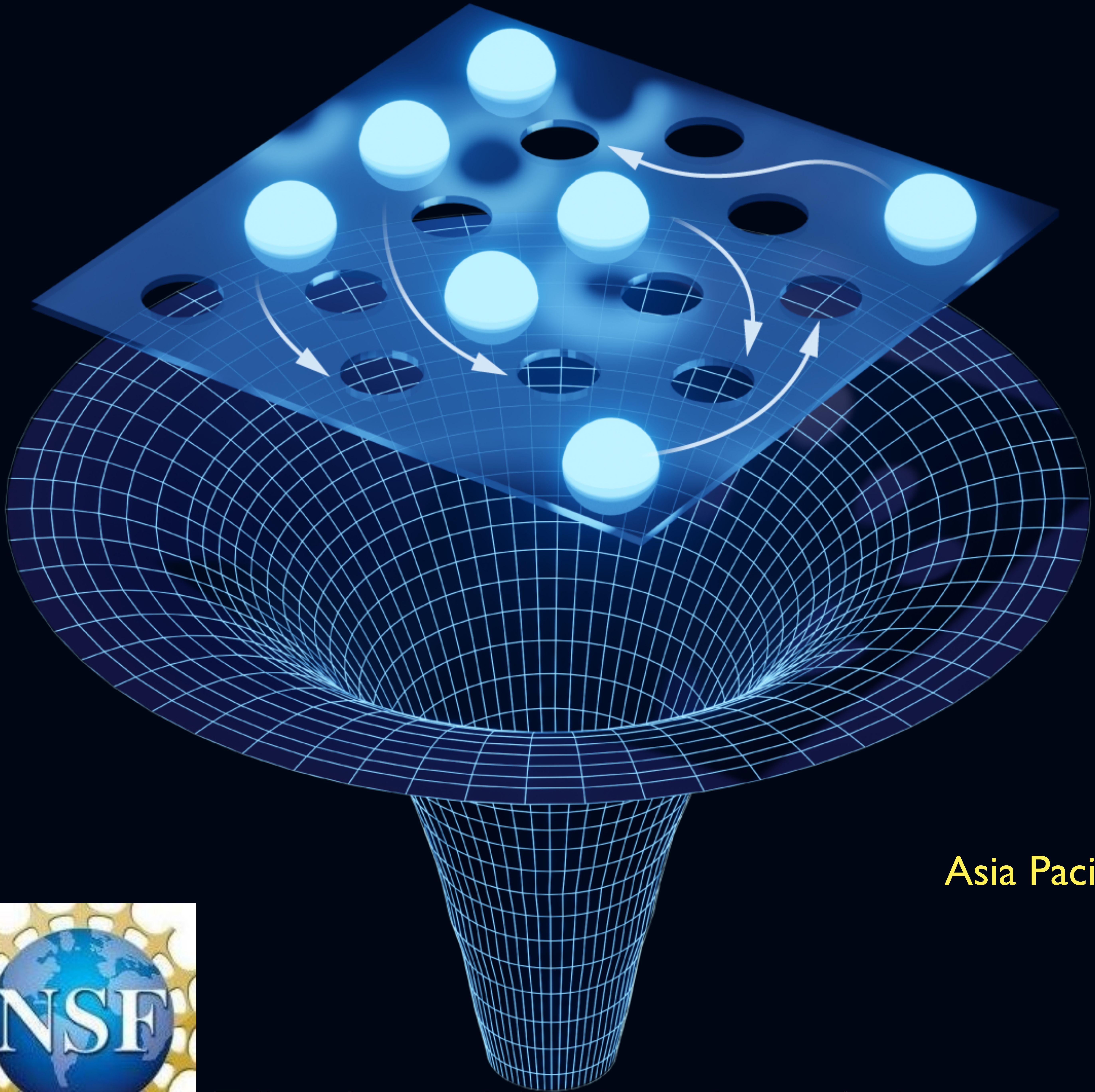


Quantum statistical mechanics of black holes and strange metals

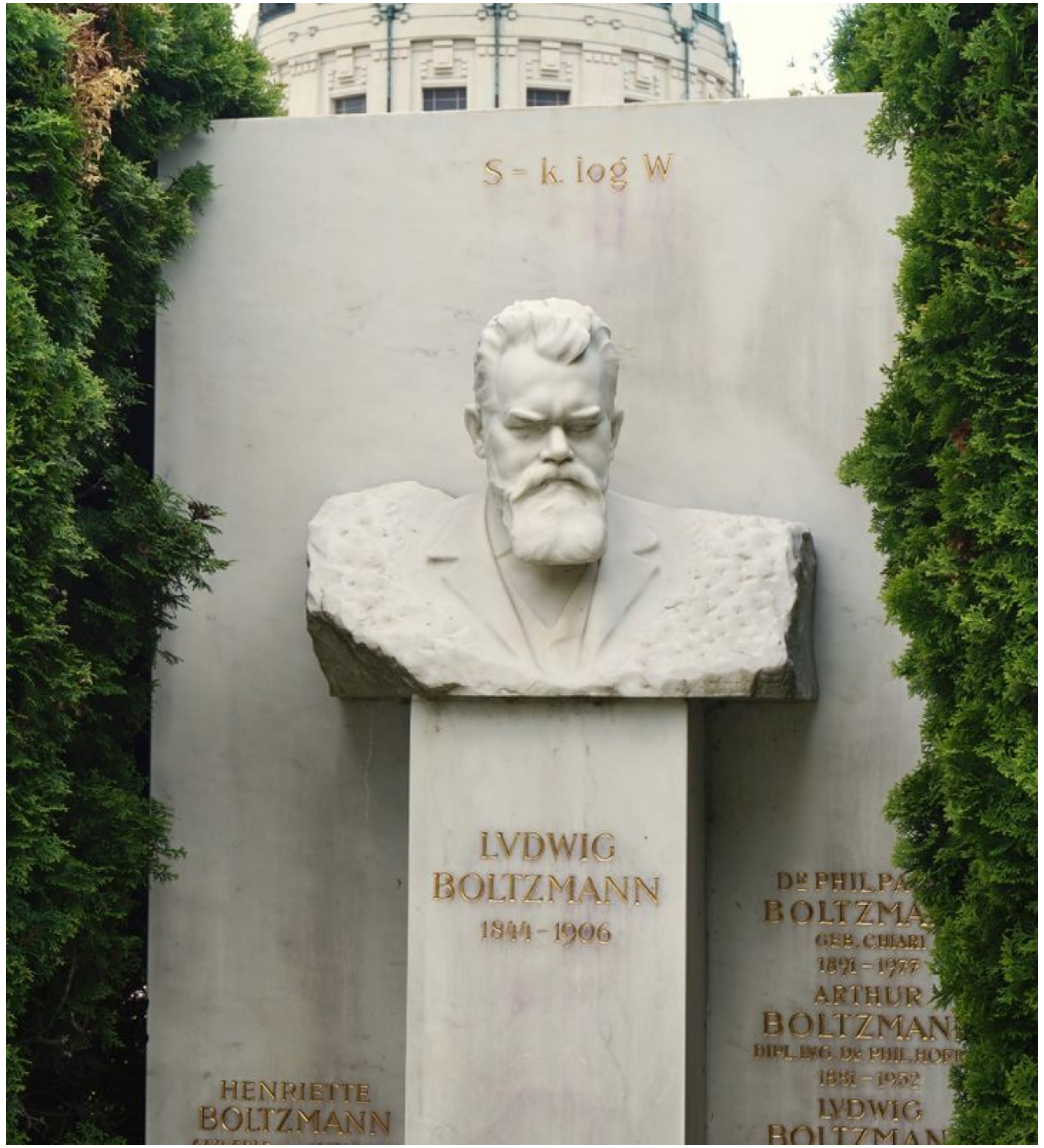
Colloquium
Asia Pacific Center for Theoretical Physics
Pohang, South Korea
November 4, 2022

Subir Sachdev

Talk online: sachdev.physics.harvard.edu

Foundations
by
Boltzmann

Statistical interpretation of entropy (1870)

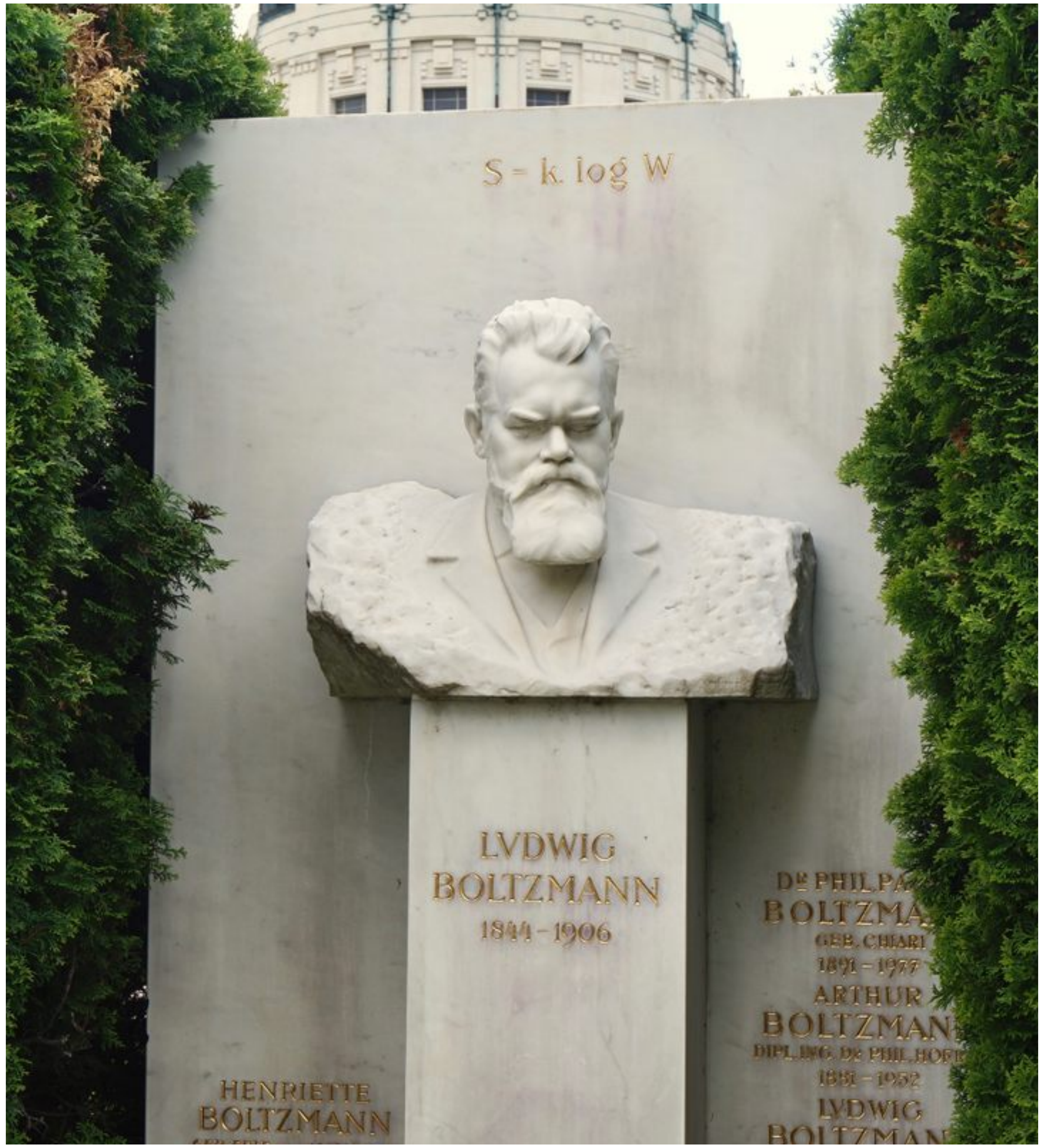


Ludwig Boltzmann
20 February 1844 - September 5, 1906
Vienna, Austria

$$S = k_B \log W$$

Density of quantum states $D(E) = \exp(S(E)/k_B)$

Boltzmann equation (1872)
Dilute classical gas



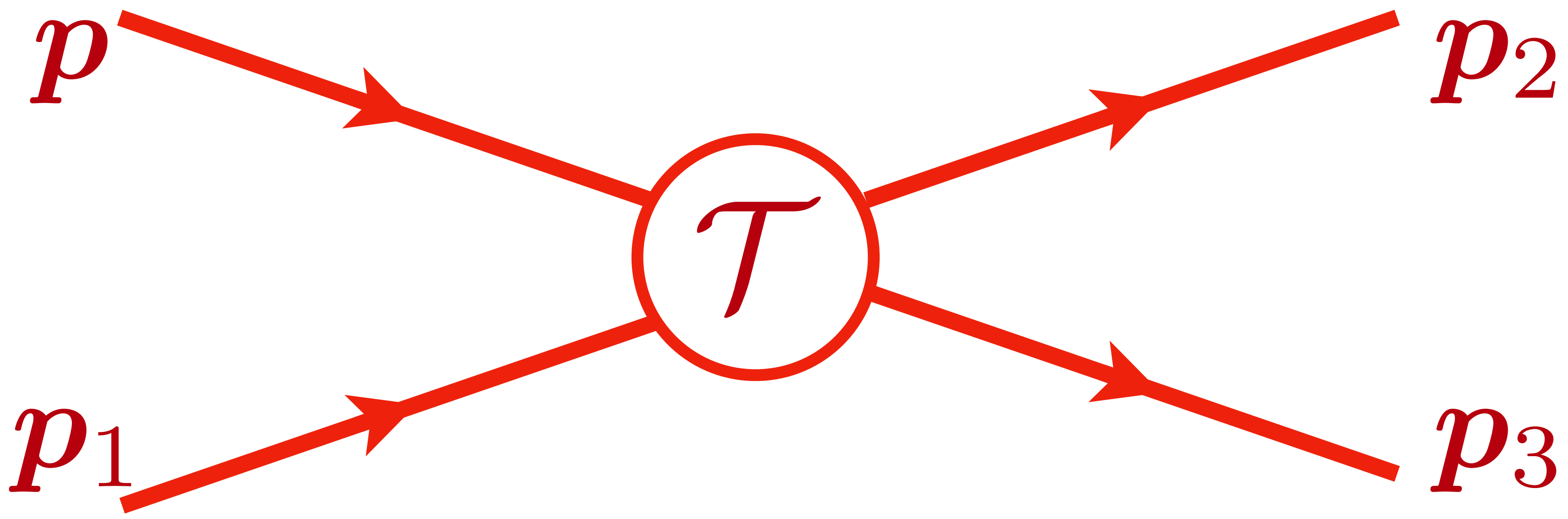
Ludwig Boltzmann
20 February 1844 - September 5, 1906
Vienna, Austria

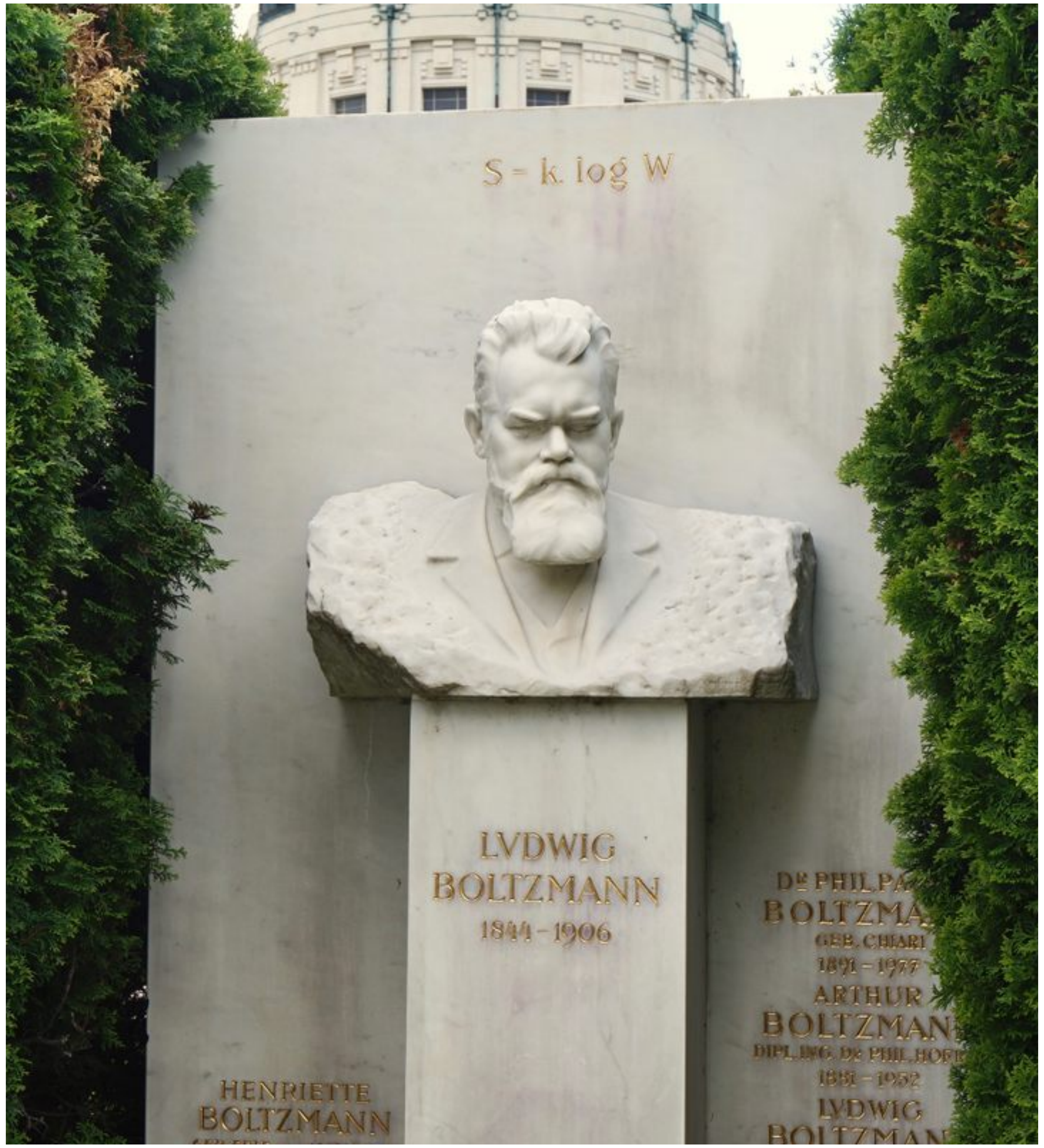
Molecular chaos: successive collisions are statistically independent

$$\frac{\partial f_{\boldsymbol{p}}}{\partial t} + \frac{\partial \varepsilon_{\boldsymbol{p}}}{\partial \boldsymbol{p}} \cdot \nabla_{\boldsymbol{r}} f_{\boldsymbol{p}} + \boldsymbol{F} \cdot \nabla_{\boldsymbol{p}} f_{\boldsymbol{p}} =$$

$$-2\pi \int_{\boldsymbol{p}_{1,2,3}} |\mathcal{T}|^2 \delta(\varepsilon_{\boldsymbol{p}} + \varepsilon_{\boldsymbol{p}_1} - \varepsilon_{\boldsymbol{p}_2} - \varepsilon_{\boldsymbol{p}_3}) \delta(\boldsymbol{p} + \boldsymbol{p}_1 - \boldsymbol{p}_2 - \boldsymbol{p}_3)$$

$$\times \left[f_{\boldsymbol{p}} f_{\boldsymbol{p}_1} - f_{\boldsymbol{p}_2} f_{\boldsymbol{p}_3} \right]$$





Ludwig Boltzmann
20 February 1844 - September 5, 1906
Vienna, Austria

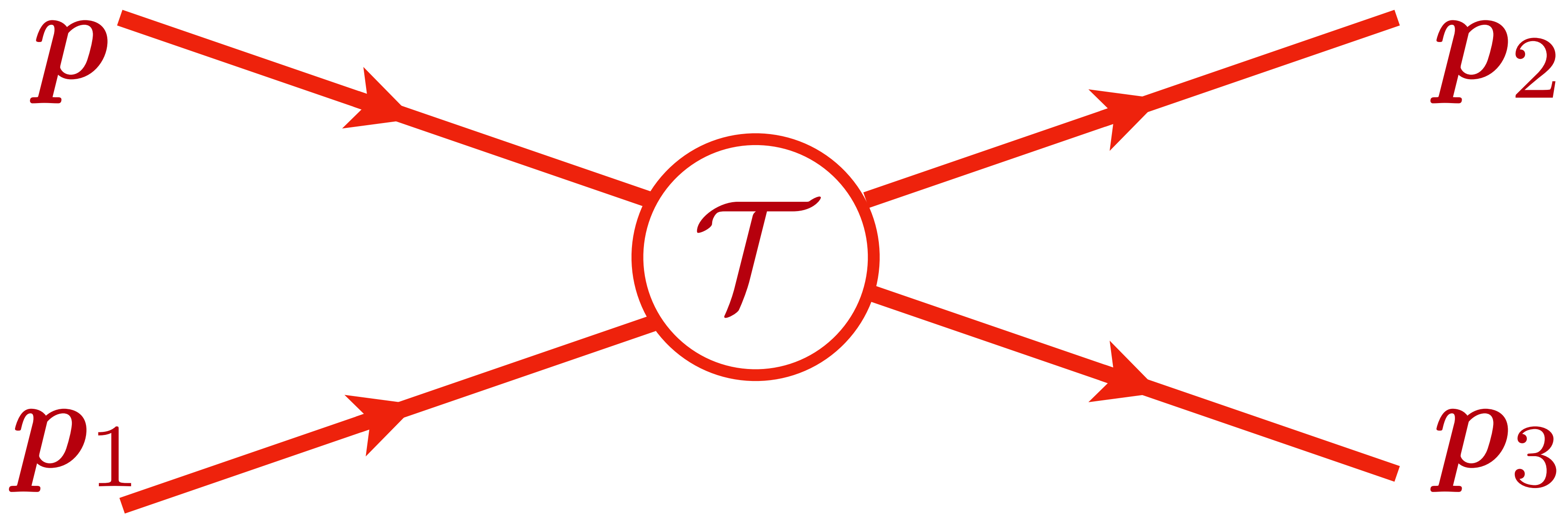
Quantum Boltzmann equation (Landau)
Dense gas of electrons

Neglects quantum interference (entanglement)
between successive collisions

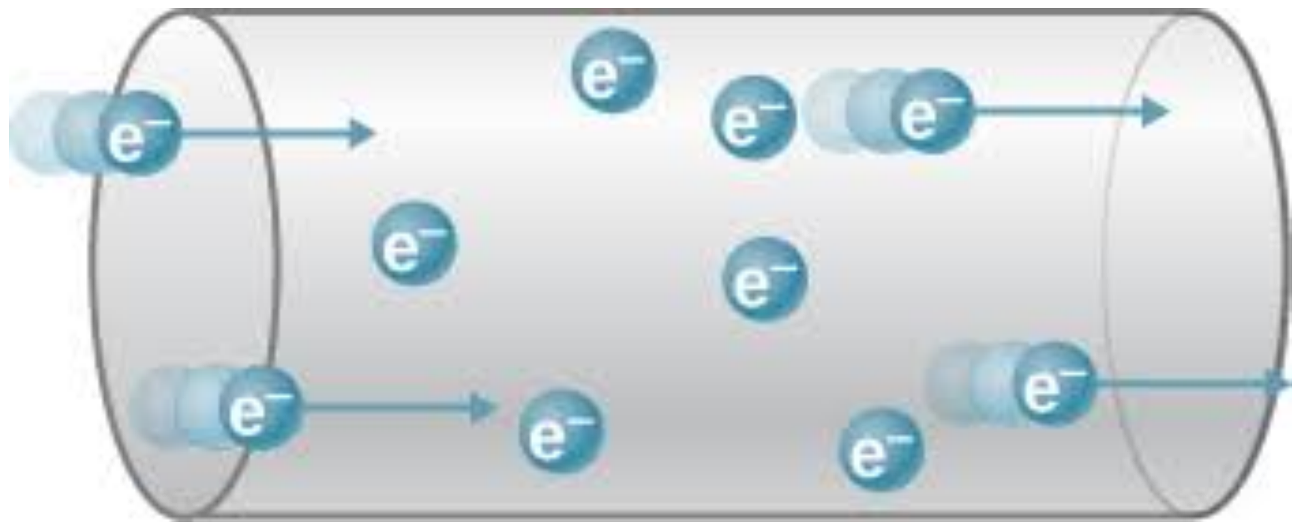
$$\frac{\partial f_{\boldsymbol{p}}}{\partial t} + \frac{\partial \varepsilon_{\boldsymbol{p}}}{\partial \boldsymbol{p}} \cdot \nabla_{\boldsymbol{r}} f_{\boldsymbol{p}} + \boldsymbol{F} \cdot \nabla_{\boldsymbol{p}} f_{\boldsymbol{p}} =$$

$$- 2\pi \int_{\boldsymbol{p}_{1,2,3}} |\mathcal{T}|^2 \delta(\varepsilon_{\boldsymbol{p}} + \varepsilon_{\boldsymbol{p}_1} - \varepsilon_{\boldsymbol{p}_2} - \varepsilon_{\boldsymbol{p}_3}) \delta(\boldsymbol{p} + \boldsymbol{p}_1 - \boldsymbol{p}_2 - \boldsymbol{p}_3)$$

$$\times \left[f_{\boldsymbol{p}} f_{\boldsymbol{p}_1} (1 - f_{\boldsymbol{p}_2})(1 - f_{\boldsymbol{p}_3}) - f_{\boldsymbol{p}_2} f_{\boldsymbol{p}_3} (1 - f_{\boldsymbol{p}})(1 - f_{\boldsymbol{p}_1}) \right]$$



Current flow with electrons in Copper

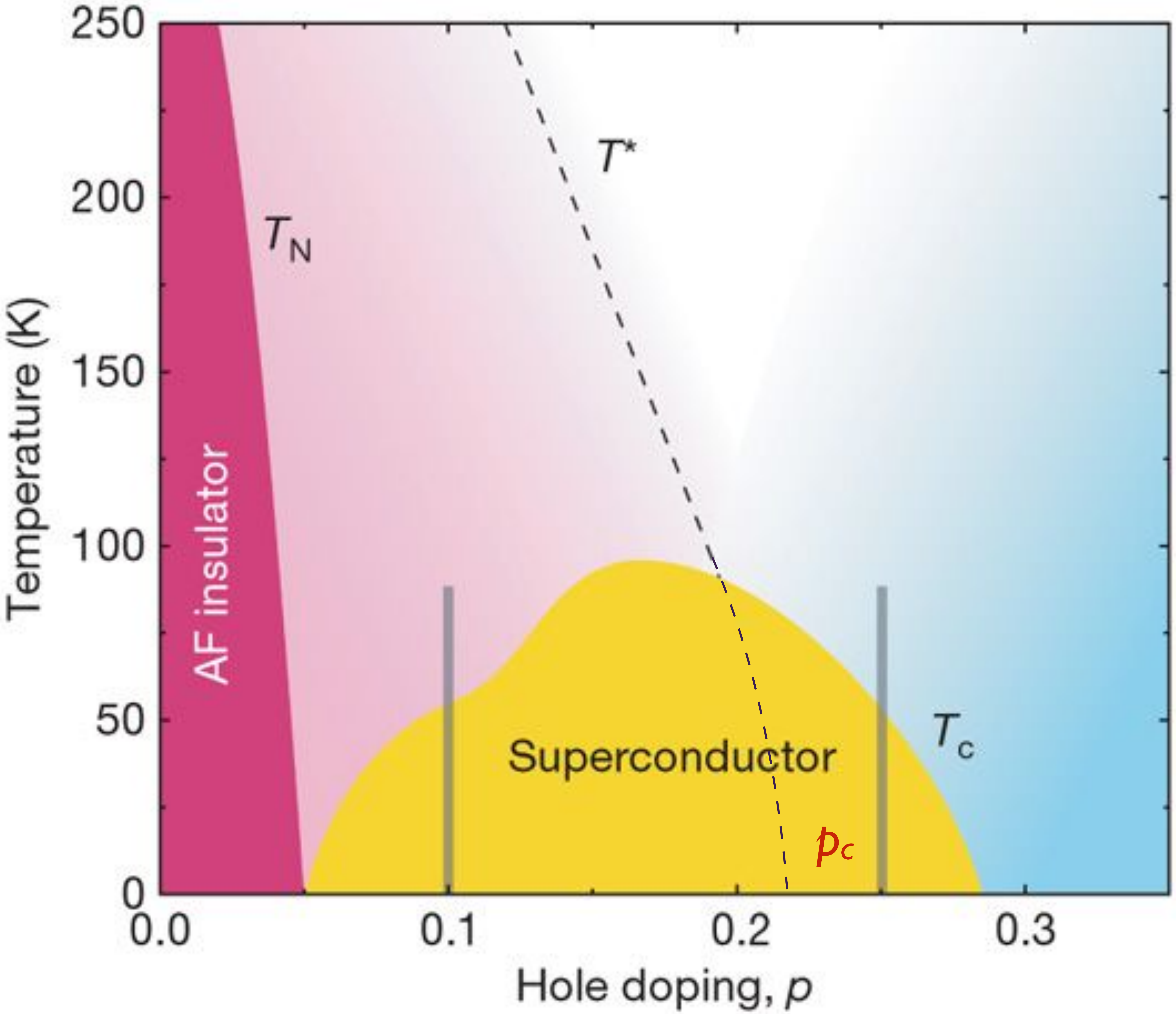


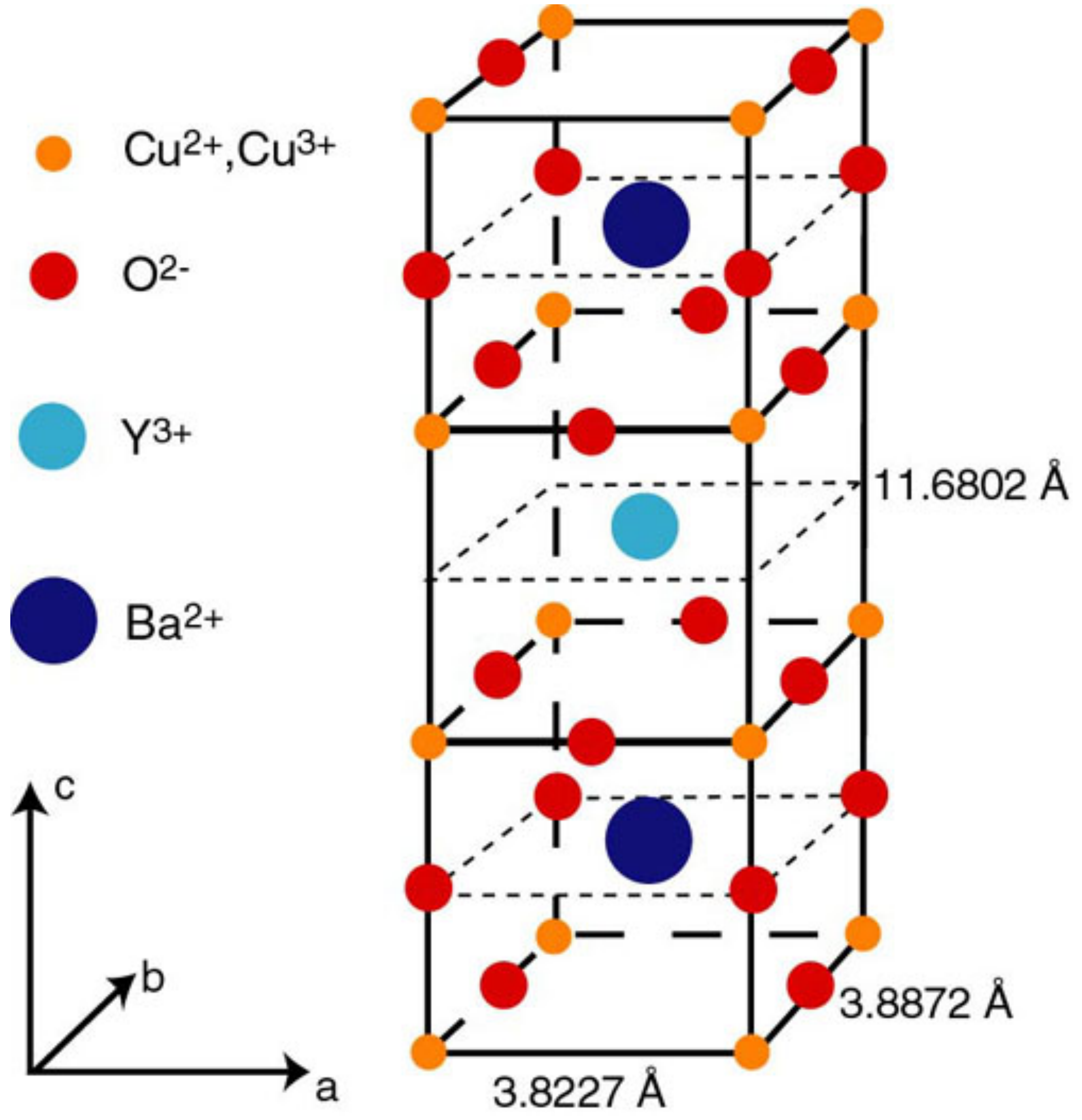
Flow of electrons described by Boltzmann equation $\Rightarrow$
typical scattering time $\tau \sim 1/T^2$, resistivity $\rho(T) = \rho(0) + AT^2$

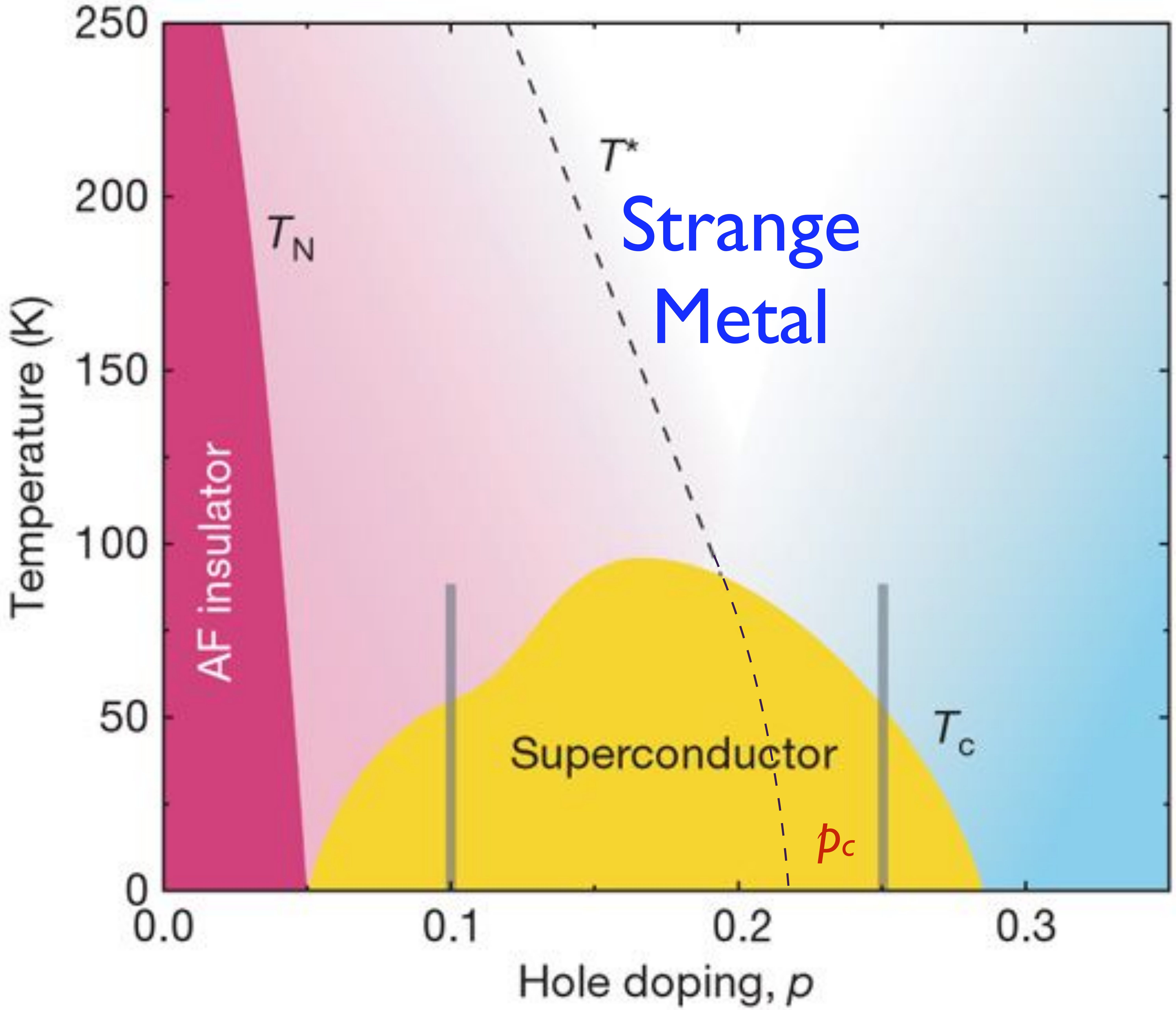
The time τ is much longer than a limiting 'Planckian time' $\dfrac{\hbar}{k_B T}$.

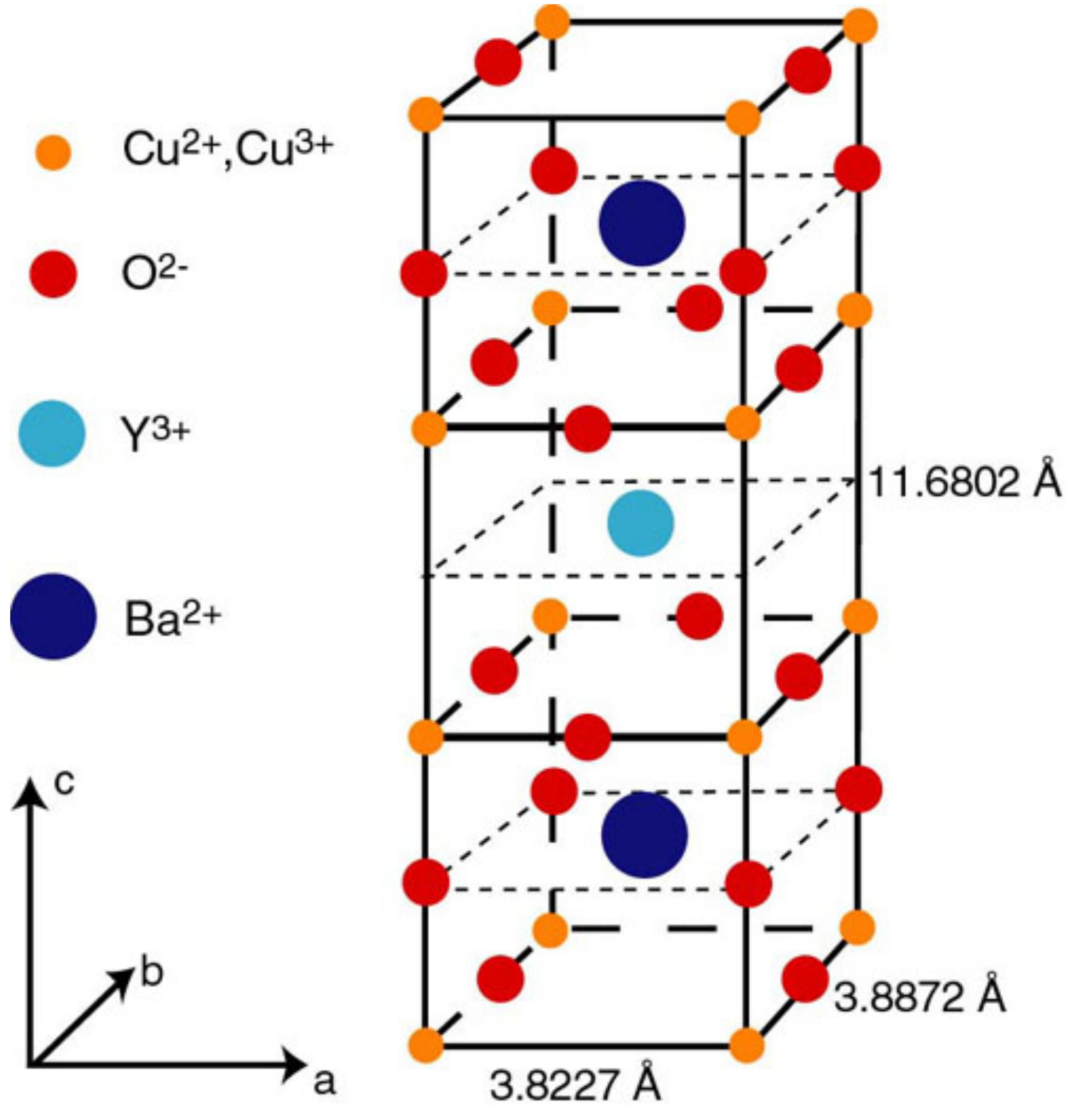
The long scattering time implies that individual electrons are well-defined.

The motion of electrons is 'ballistic' or 'integrable'
up to the long time τ, after which it is chaotic.


250
200
150
100
50
0
Temperature (K)
0.0
0.1
0.2
0.3
Hole doping, p
AF insulator
T_N
T^*
Superconductor
T_c
p_c


Cu^{2+},Cu^{3+}
O^{2-}
Y^{3+}
Ba^{2+}
c
b
a
11.6802 Å
3.8872 Å
3.8227 Å


250
200
150
100
50
0
Temperature (K)
0.0
0.1
0.2
0.3
Hole doping, p
AF insulator
T_N
T^*
Strange
Metal
Superconductor
T_c
p_c


Cu^{2+}, Cu^{3+}
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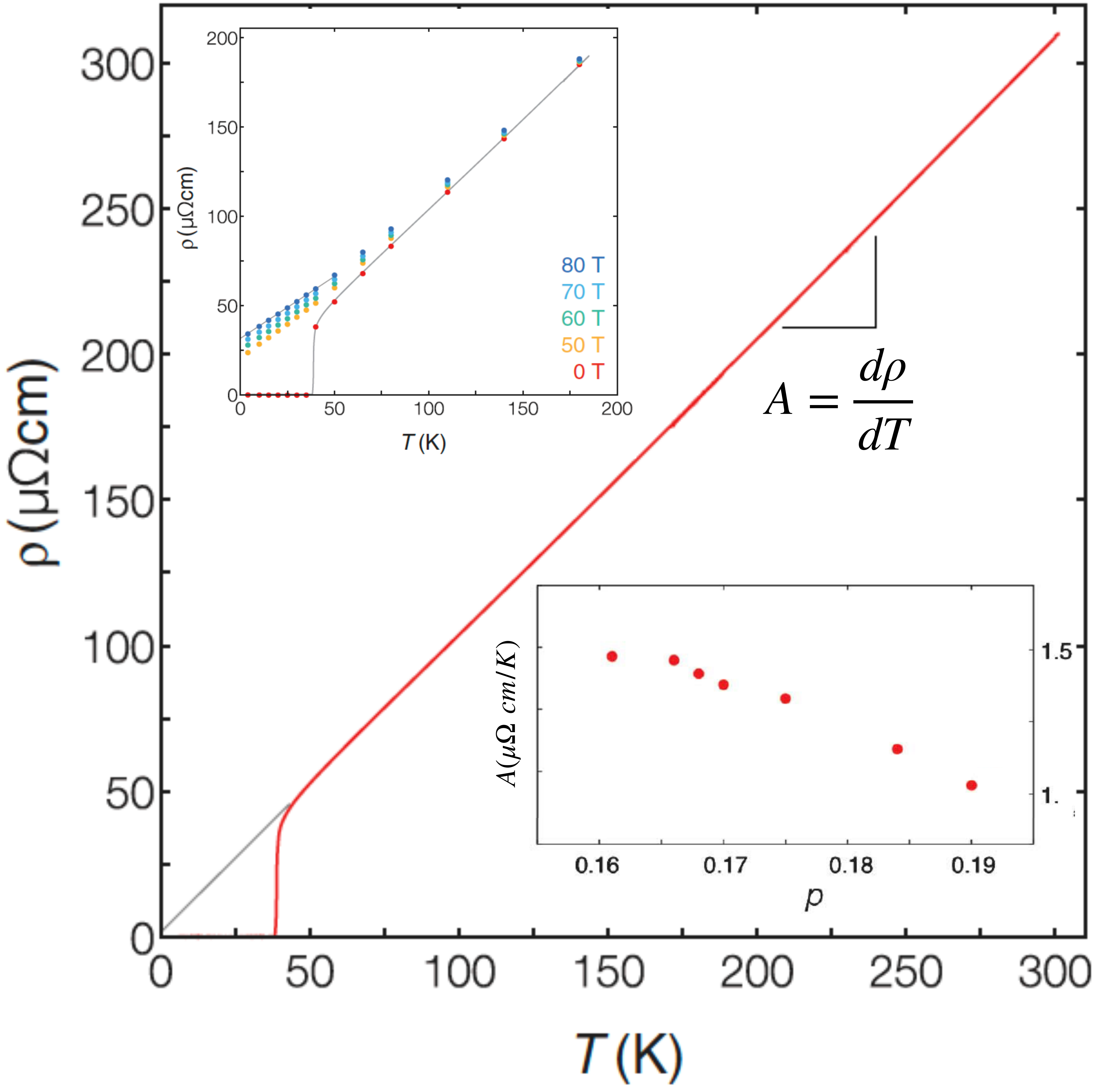
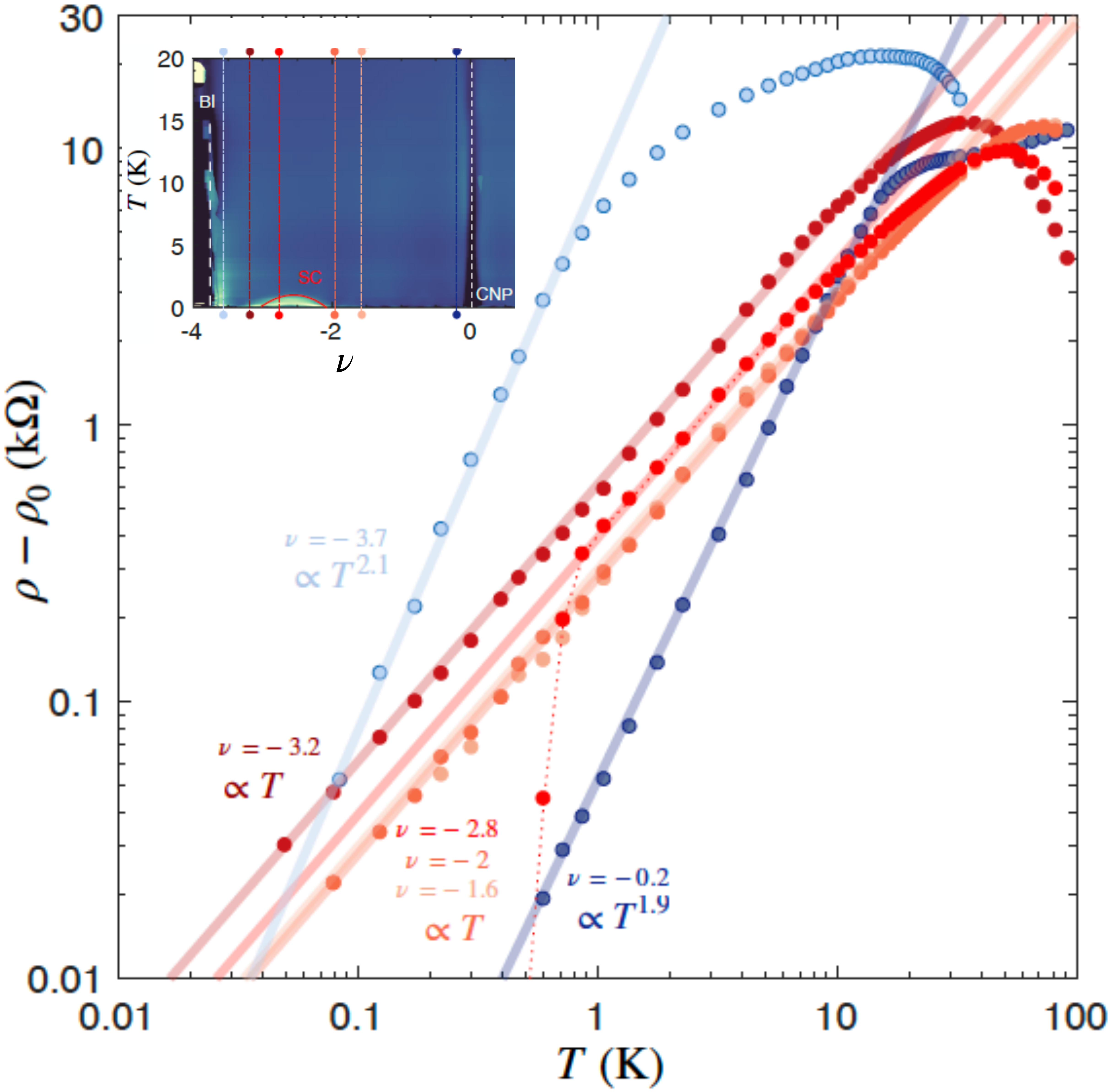


LSCO: Giraldo-Gallo et al. 2018

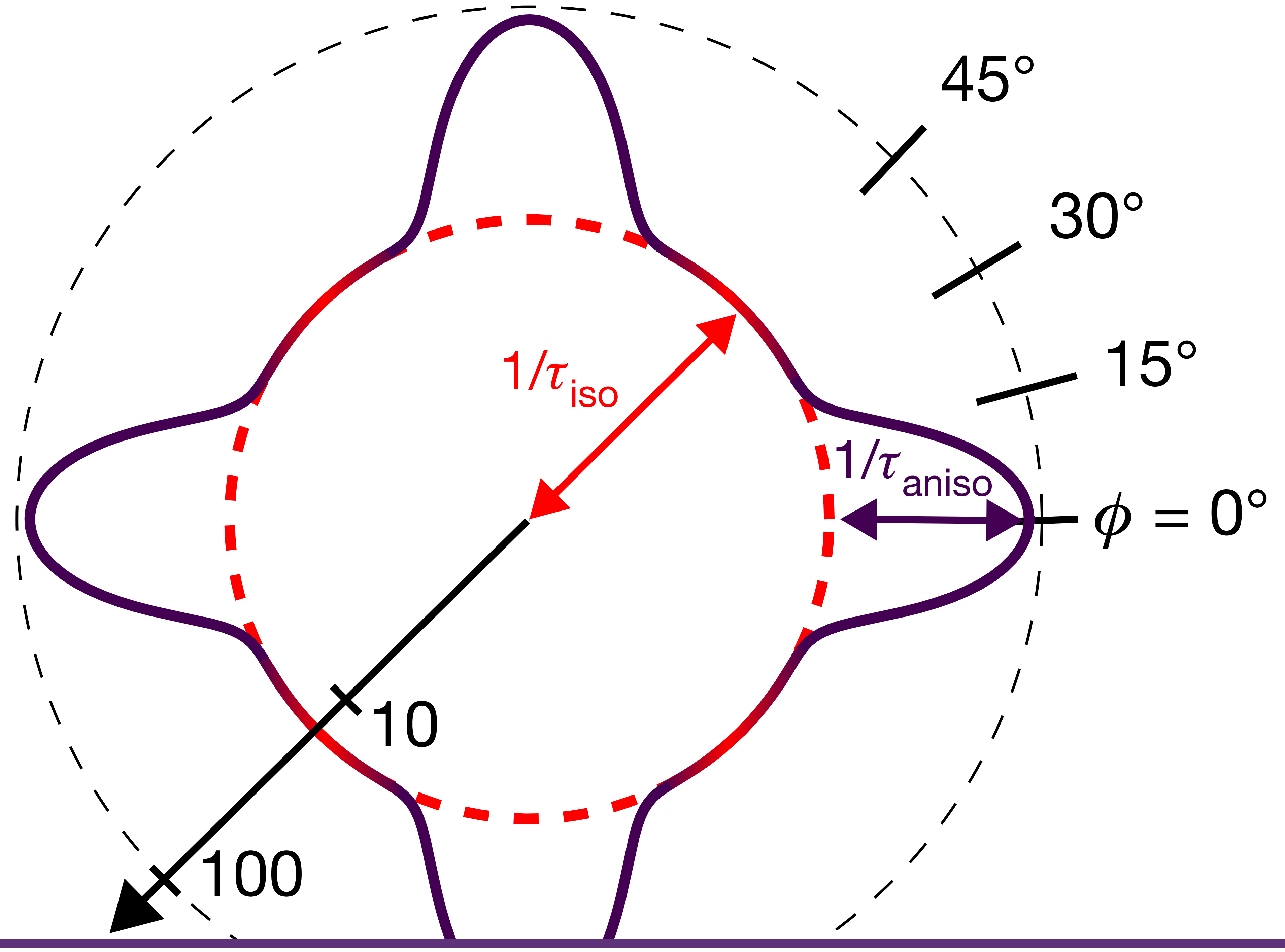


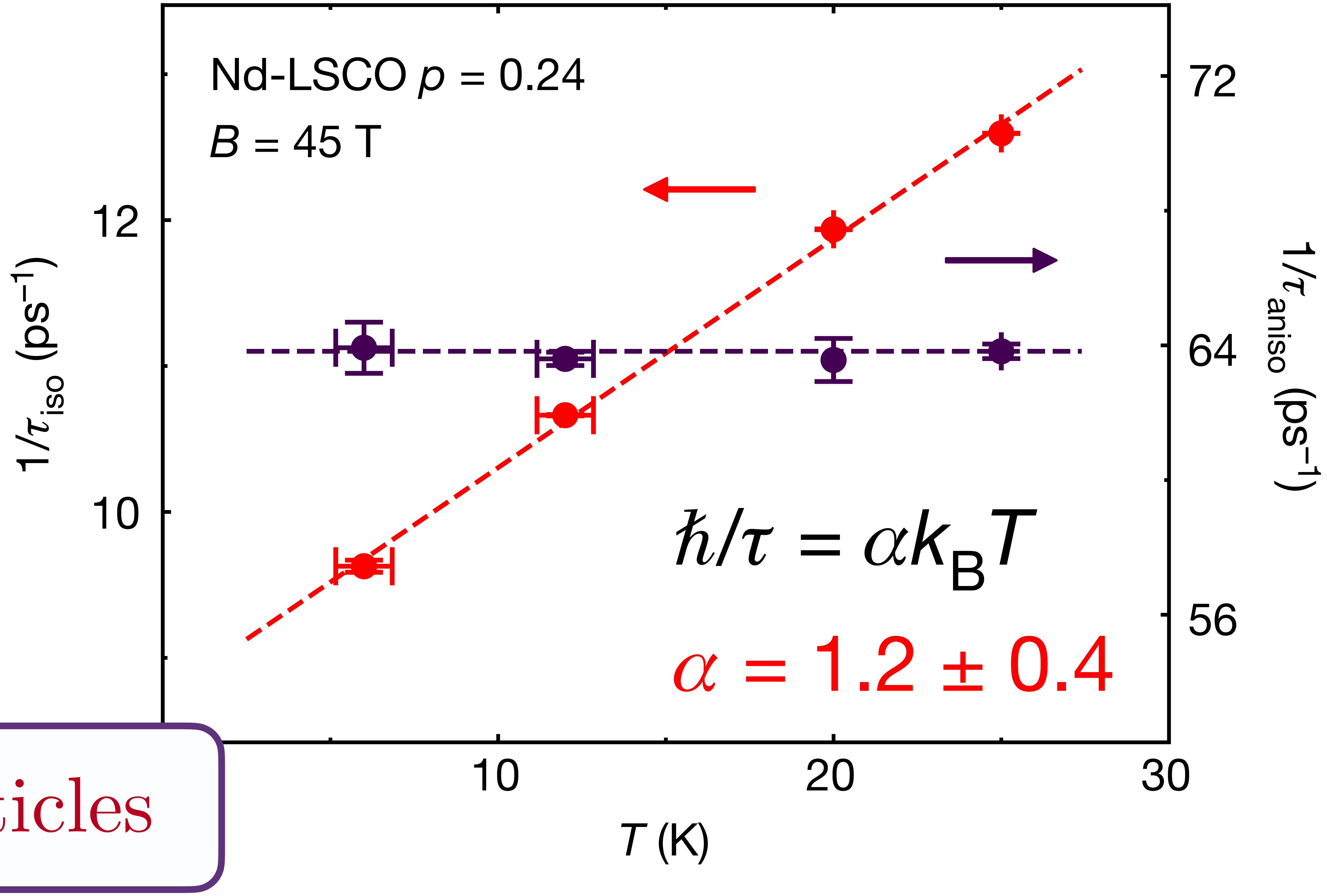
MATBG: Jaoui et al. 2021

Linear-in temperature resistivity from an isotropic Planckian scattering rate

Nature **595**, 667-672 (2021)

G. Grissonnanche, Y. Fang, A. Legros, S. Verret, F. Laliberté, C. Collignon, J. Zhou, D. Graf, P. Goddard, L. Taillefer, B. J. Ramshaw





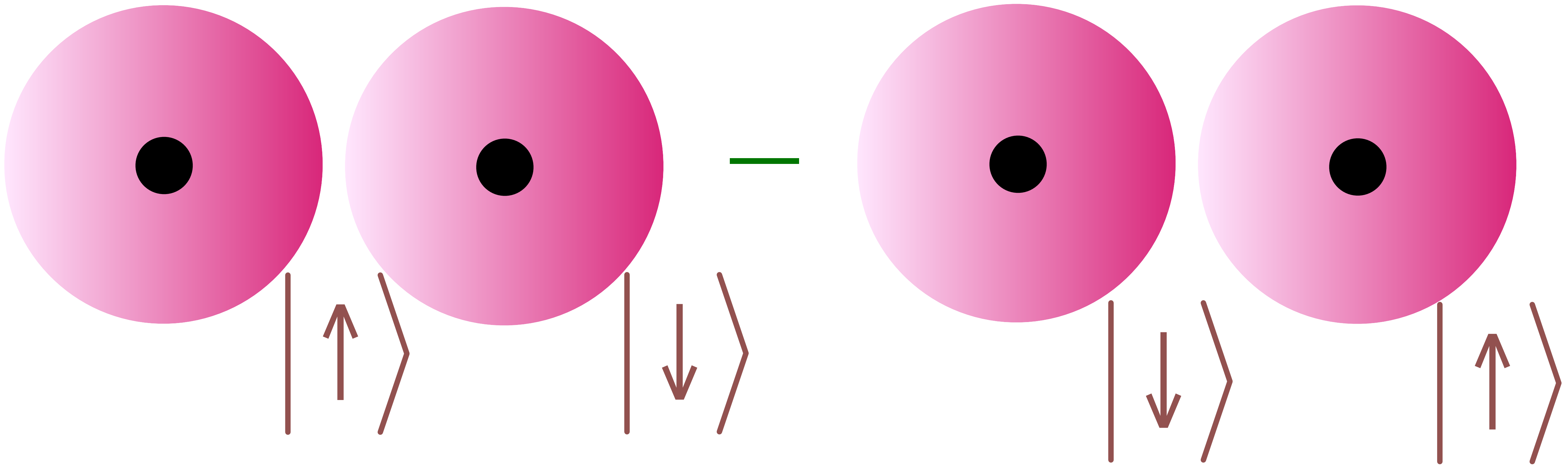
Current flow without quasiparticles

No Boltzmann-Landau quasiparticle description $\Rightarrow$
Many particle quantum entanglement
from quantum interference between “collisions”

Sachdev-Ye-Kitaev Model

Quantum Entanglement

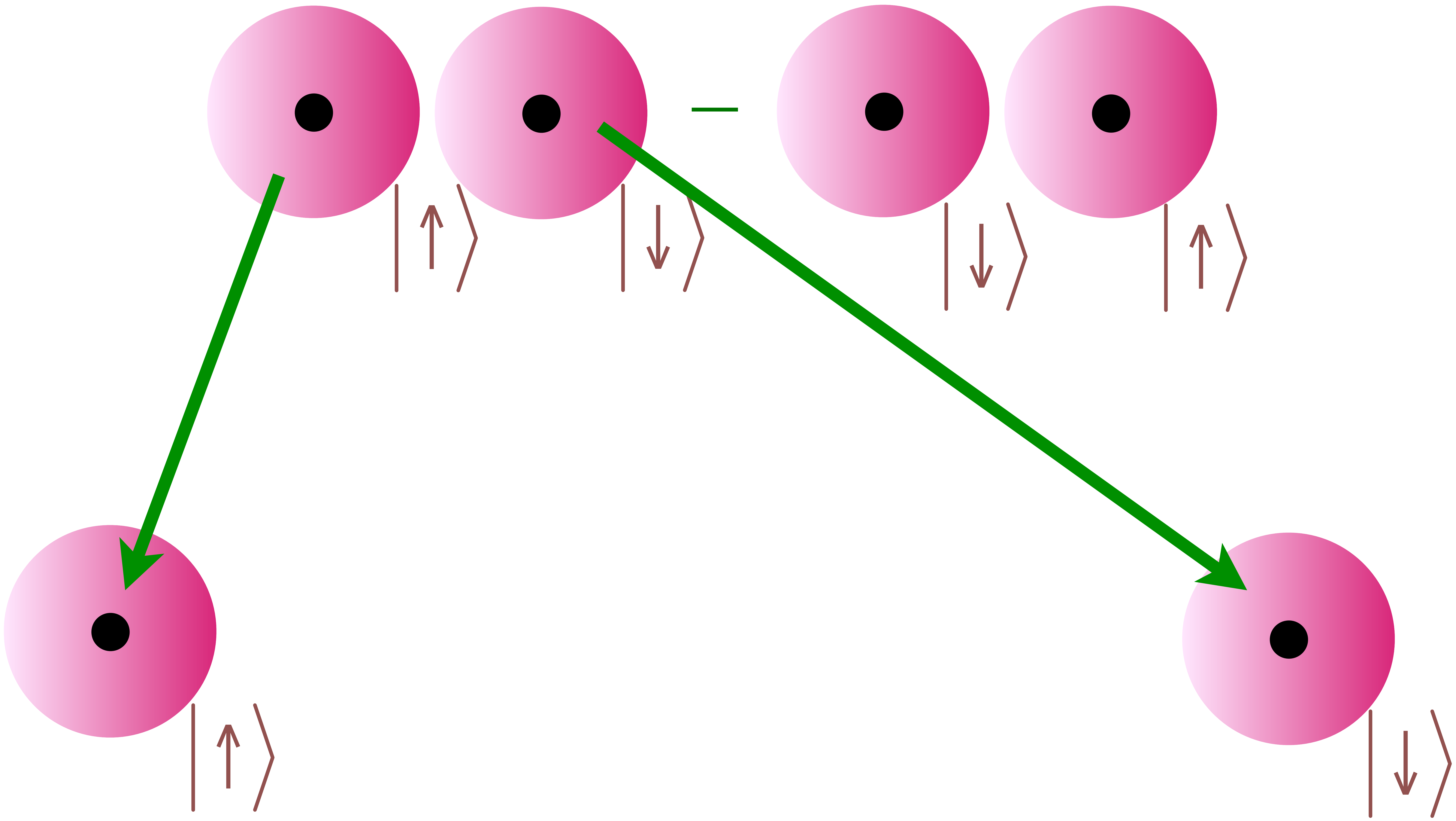
Einstein, Podolsky, Rosen (1935)



$$|\uparrow\rangle \, |\downarrow\rangle - |\downarrow\rangle \, |\uparrow\rangle$$

Quantum Entanglement

Einstein, Podolsky, Rosen (1935)



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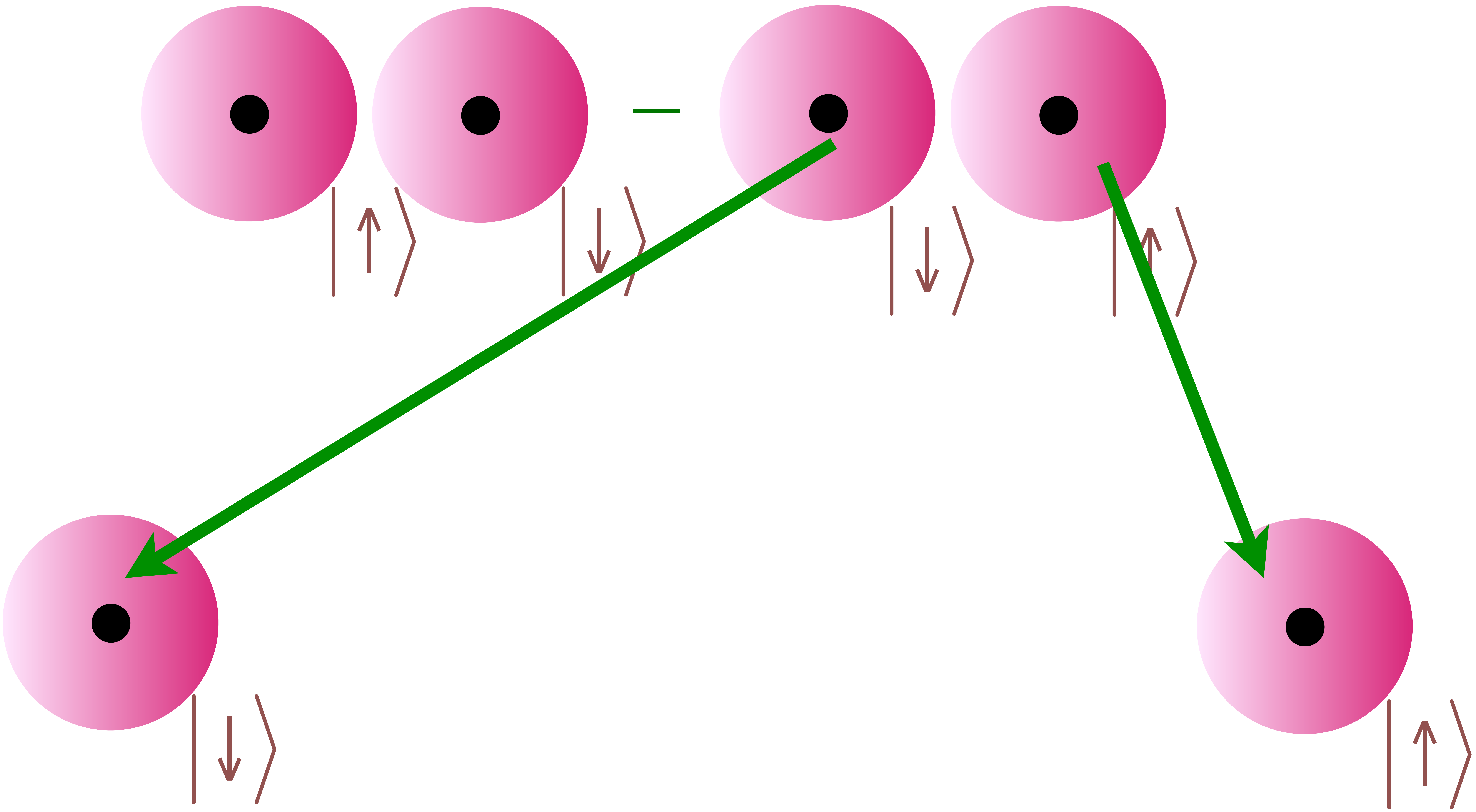
Einstein, Podolsky, Rosen (1935)



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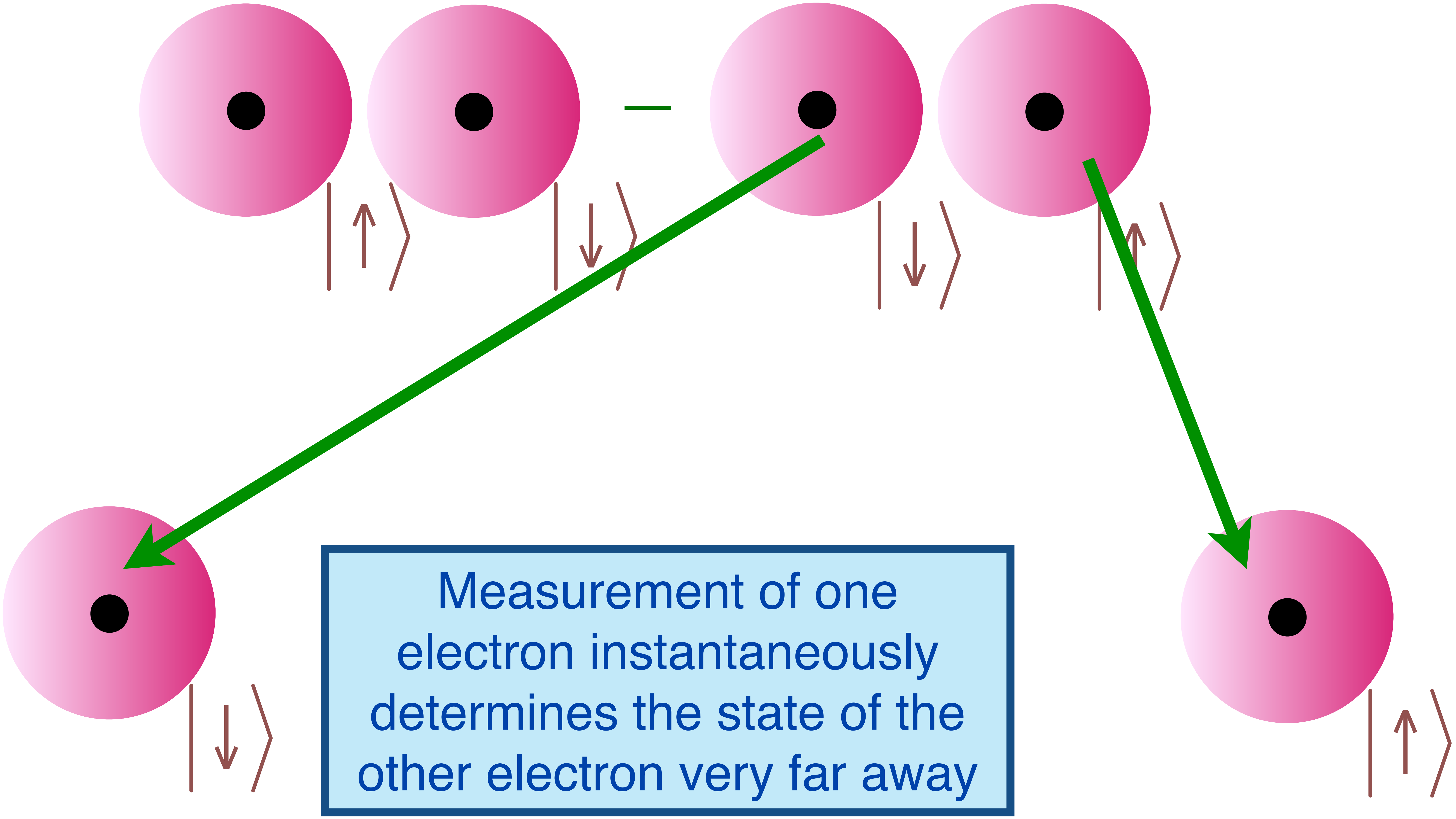
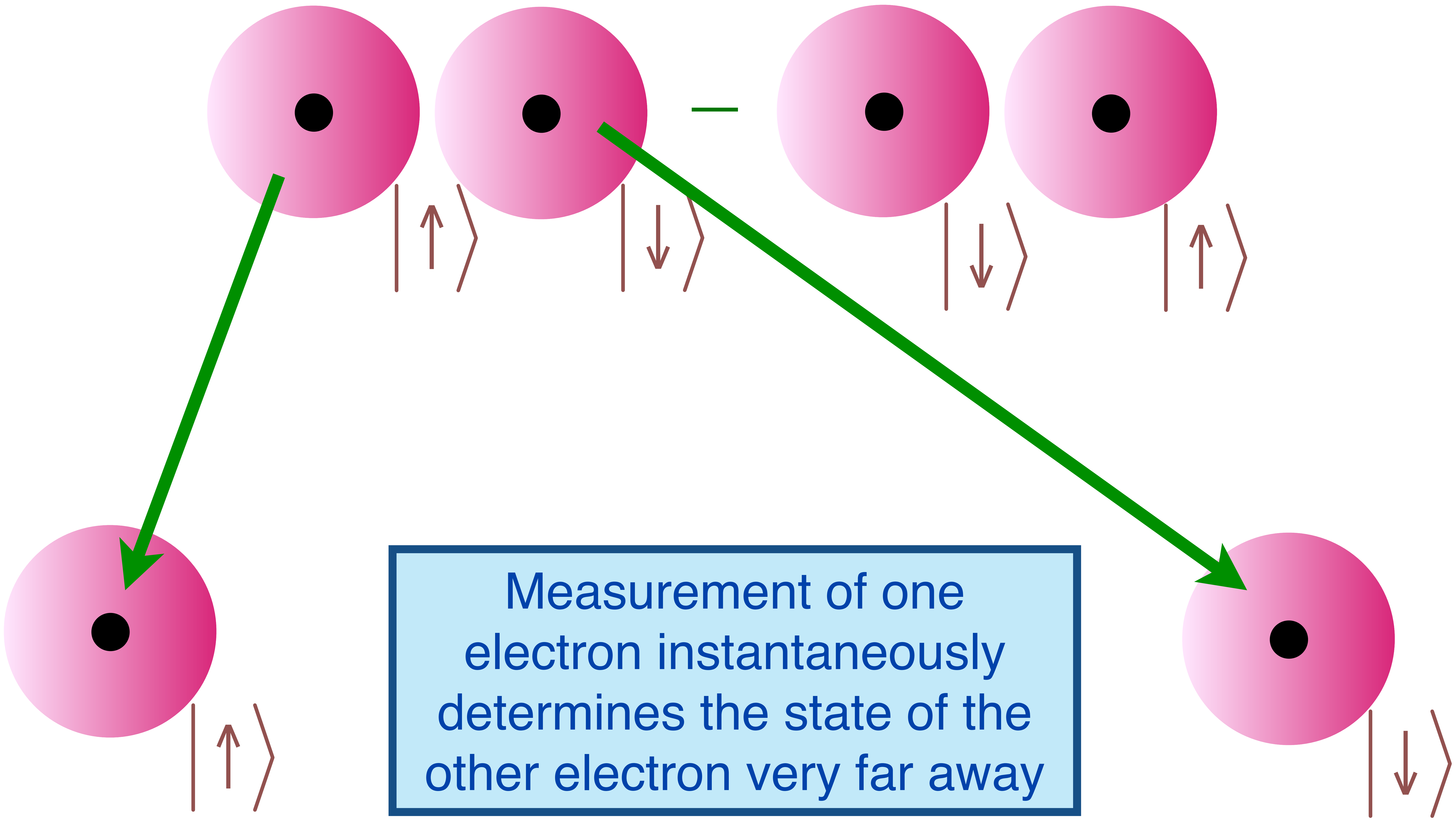


Measurement of one electron instantaneously determines the state of the other electron very far away

Quantum Entanglement

Einstein, Podolsky, Rosen (1935)



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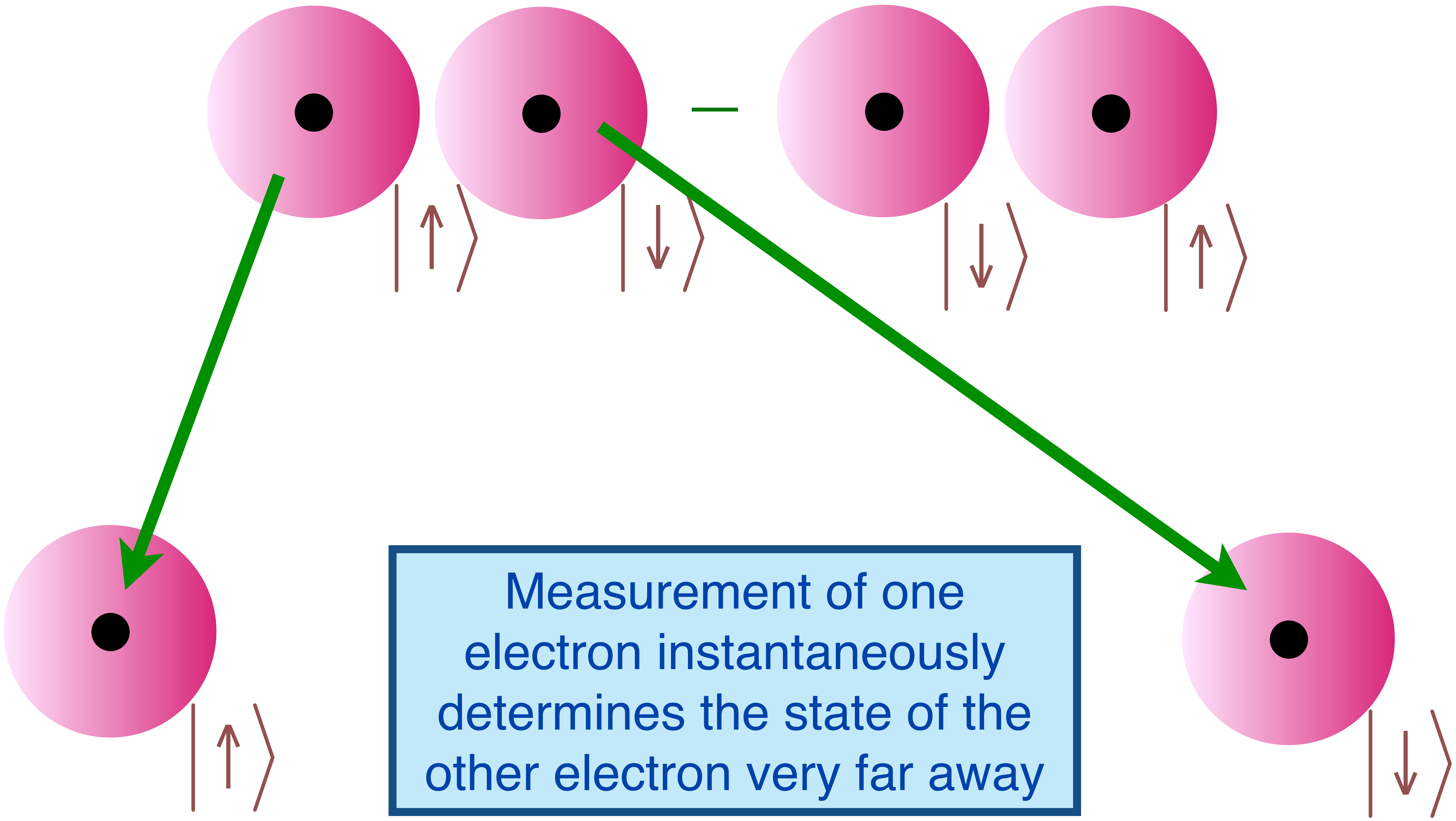
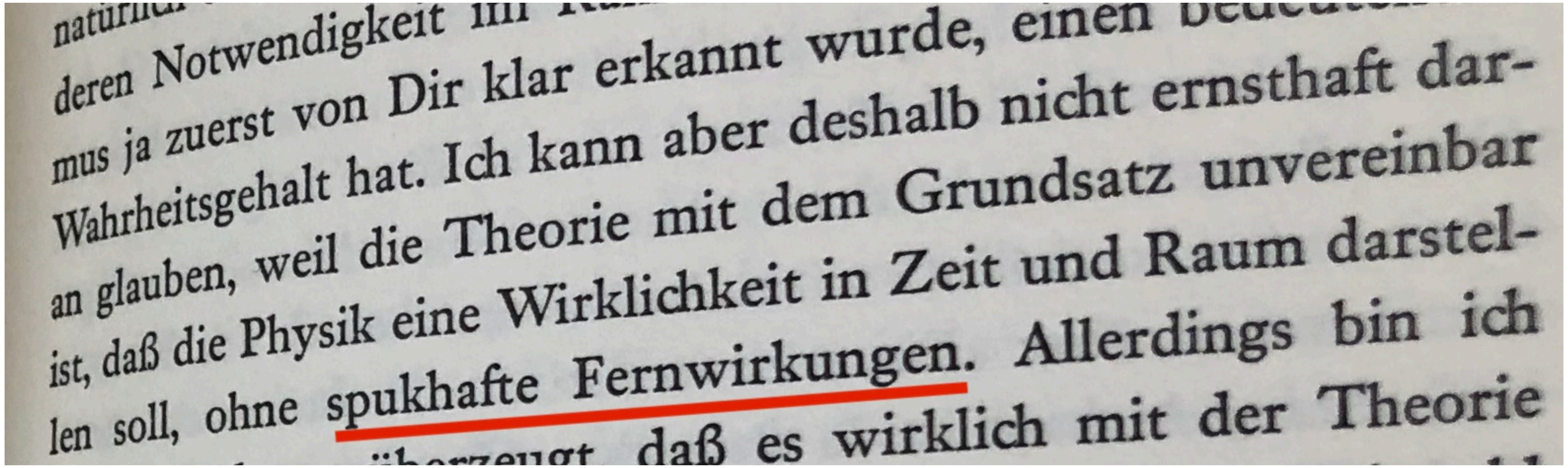
Quantum Entanglement

Einstein, Podolsky, Rosen (1935)

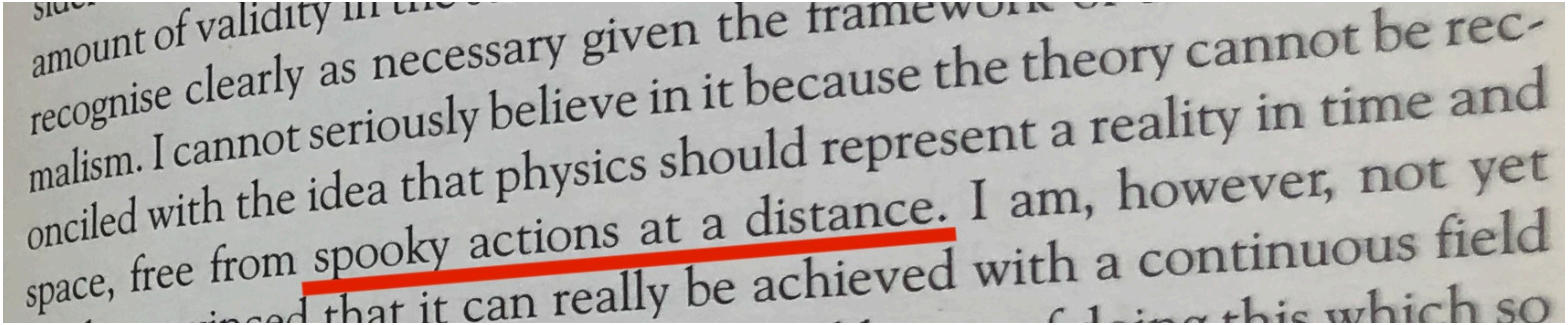


Measurement of one electron instantaneously determines the state of the other electron very far away

Spooky action at a distance !



deren Notwendigkeit im
mus ja zuerst von Dir klar erkannt wurde, einen
Wahrheitsgehalt hat. Ich kann aber deshalb nicht ernsthaft daran glauben, weil die Theorie mit dem Grundsatz unvereinbar ist, daß die Physik eine Wirklichkeit in Zeit und Raum darstellen soll, ohne spukhafte Fernwirkungen. Allerdings bin ich
überzeugt, daß es wirklich mit der Theorie



amount of validity in the
recognise clearly as necessary given the framework
malism. I cannot seriously believe in it because the theory cannot be reconciled with the idea that physics should represent a reality in time and space, free from spooky actions at a distance. I am, however, not yet
that it can really be achieved with a continuous field
of doing this which so

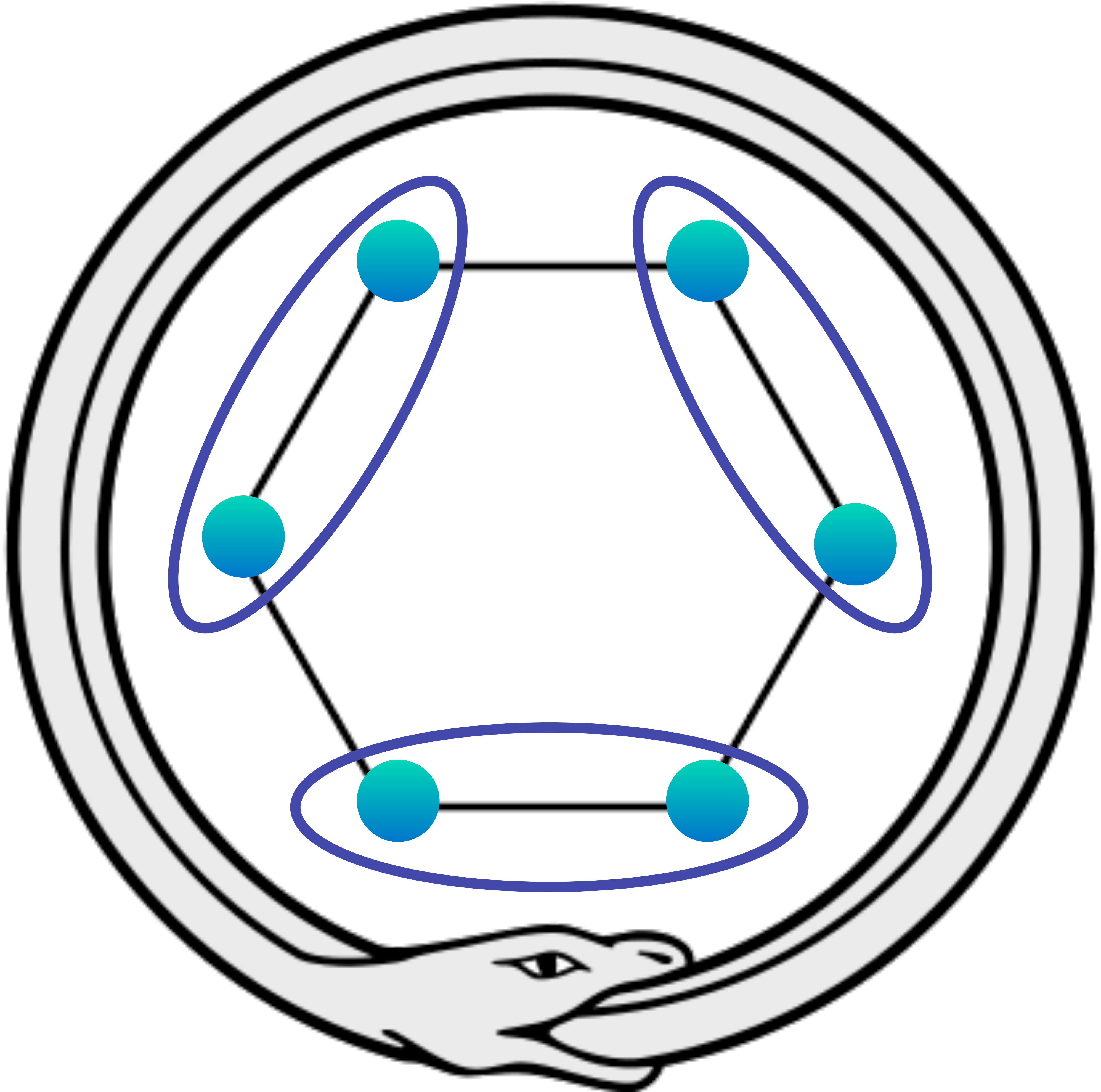
I cannot seriously believe in it because the theory cannot be reconciled with the idea that physics should represent a reality in time and space, free from spooky actions at distance

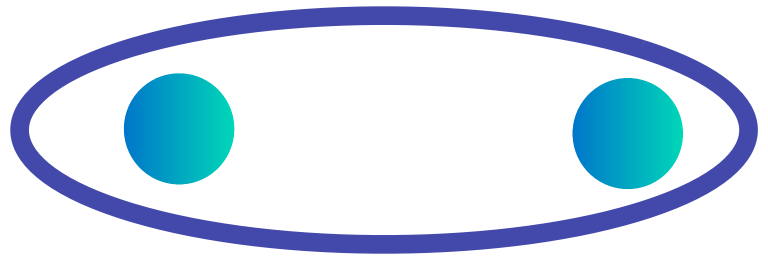
Albert Einstein to Max Born, 3 March 1947

Acknowledgement: Ahmed Almheiri

Kekule's spooky dream

Here Kekulé spoke of the creation of the theory. He said that he had discovered the ring shape of the benzene molecule after having a reverie or day-dream of a snake seizing its own tail*





$= |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$

*Wikipedia

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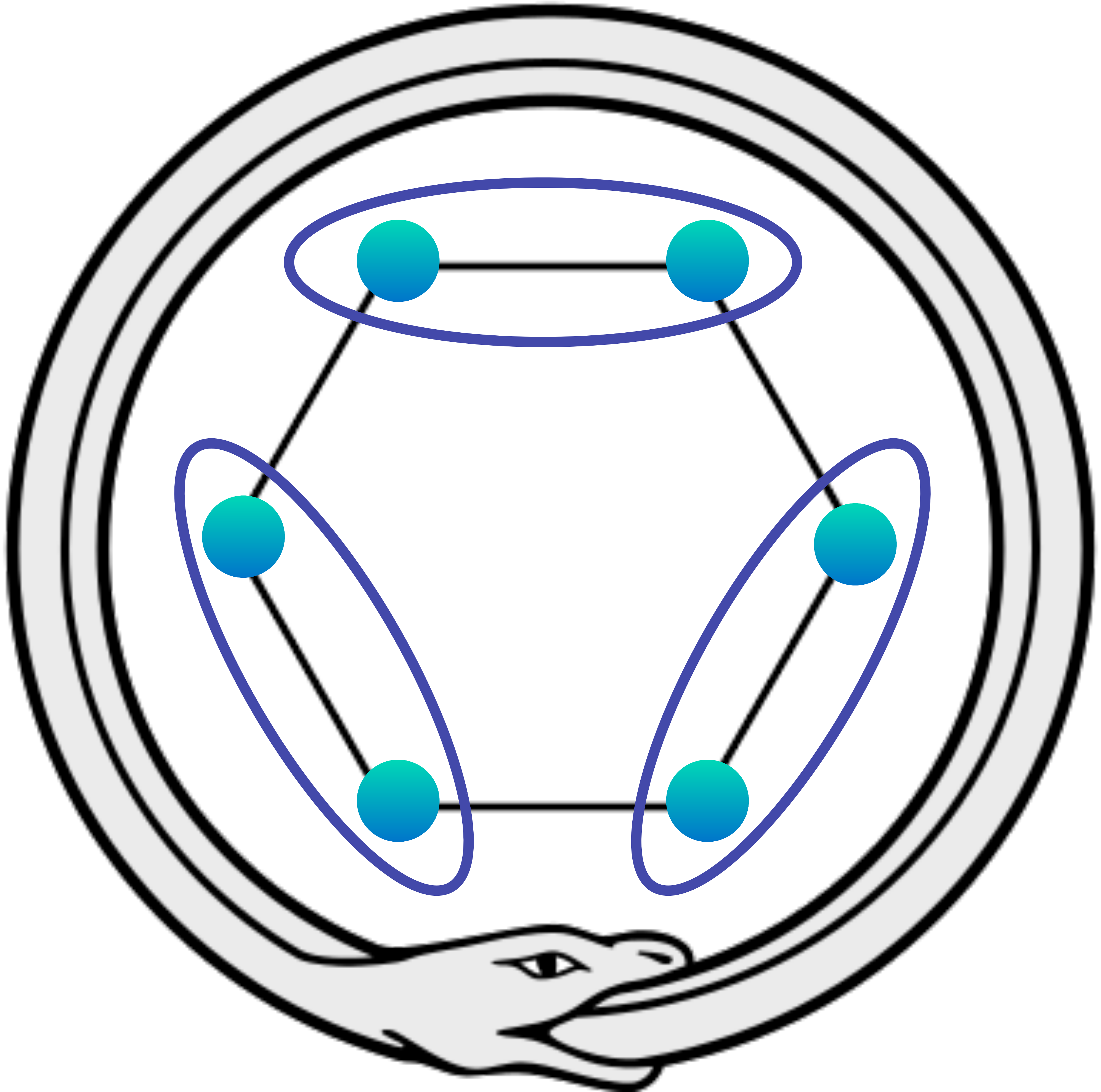

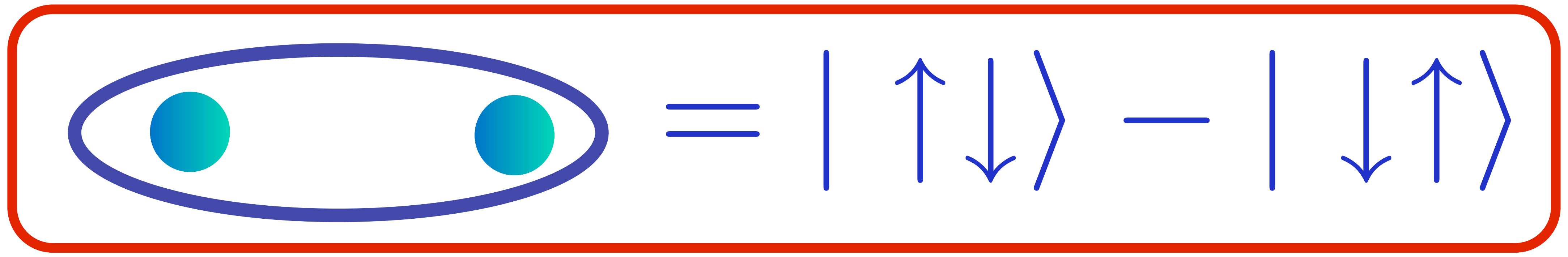

$$= |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$$

*Wikipedia

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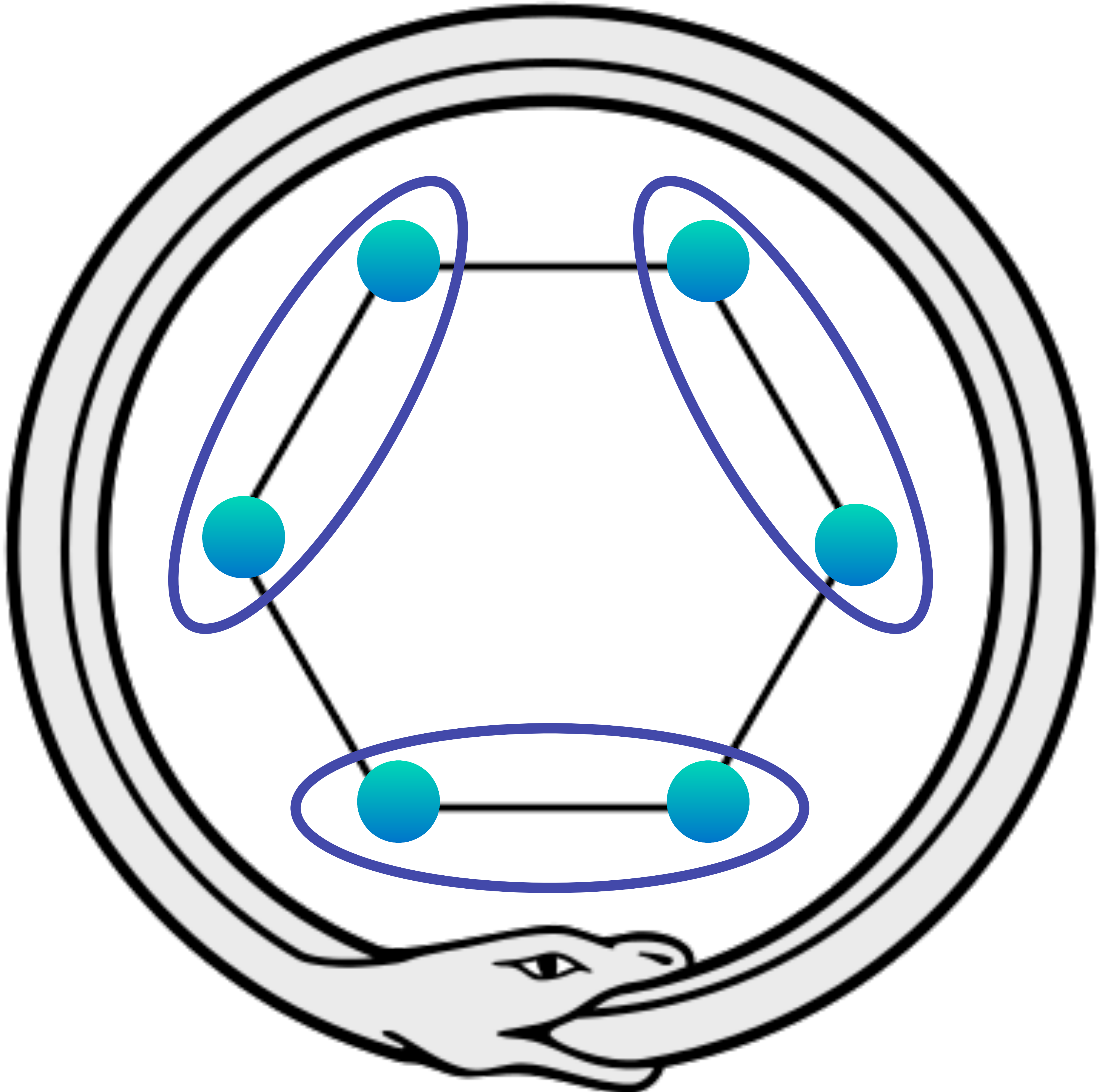

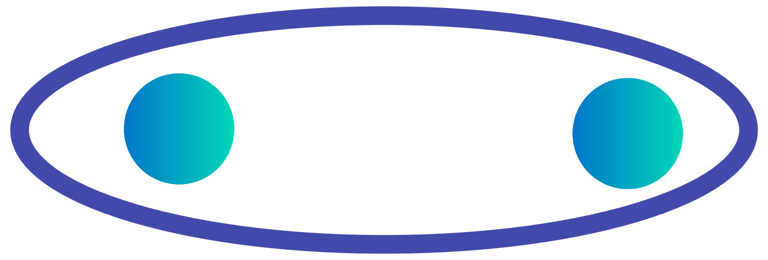

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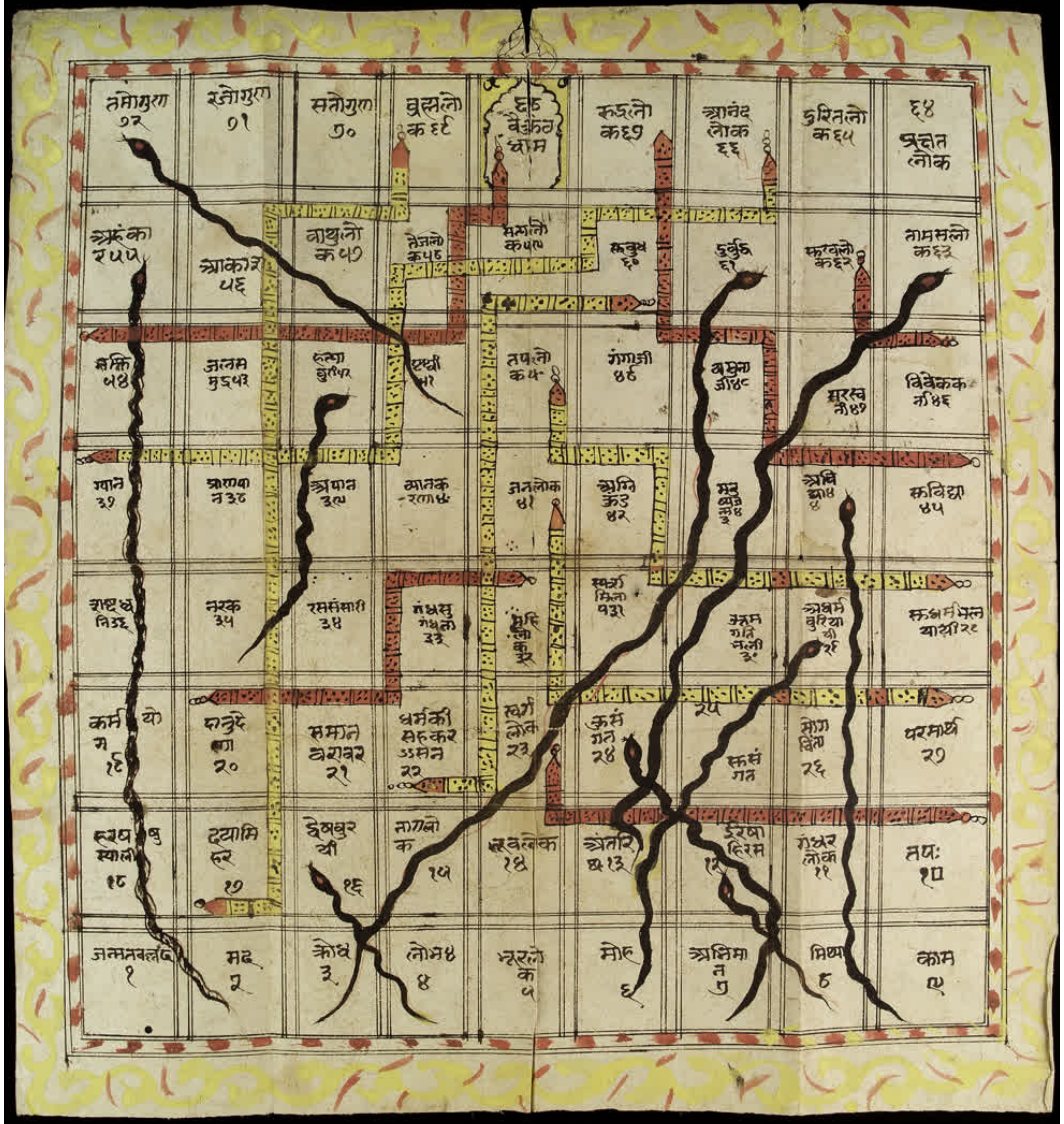


My spooky dream*

Ancient Indian game of Snakes and Ladders

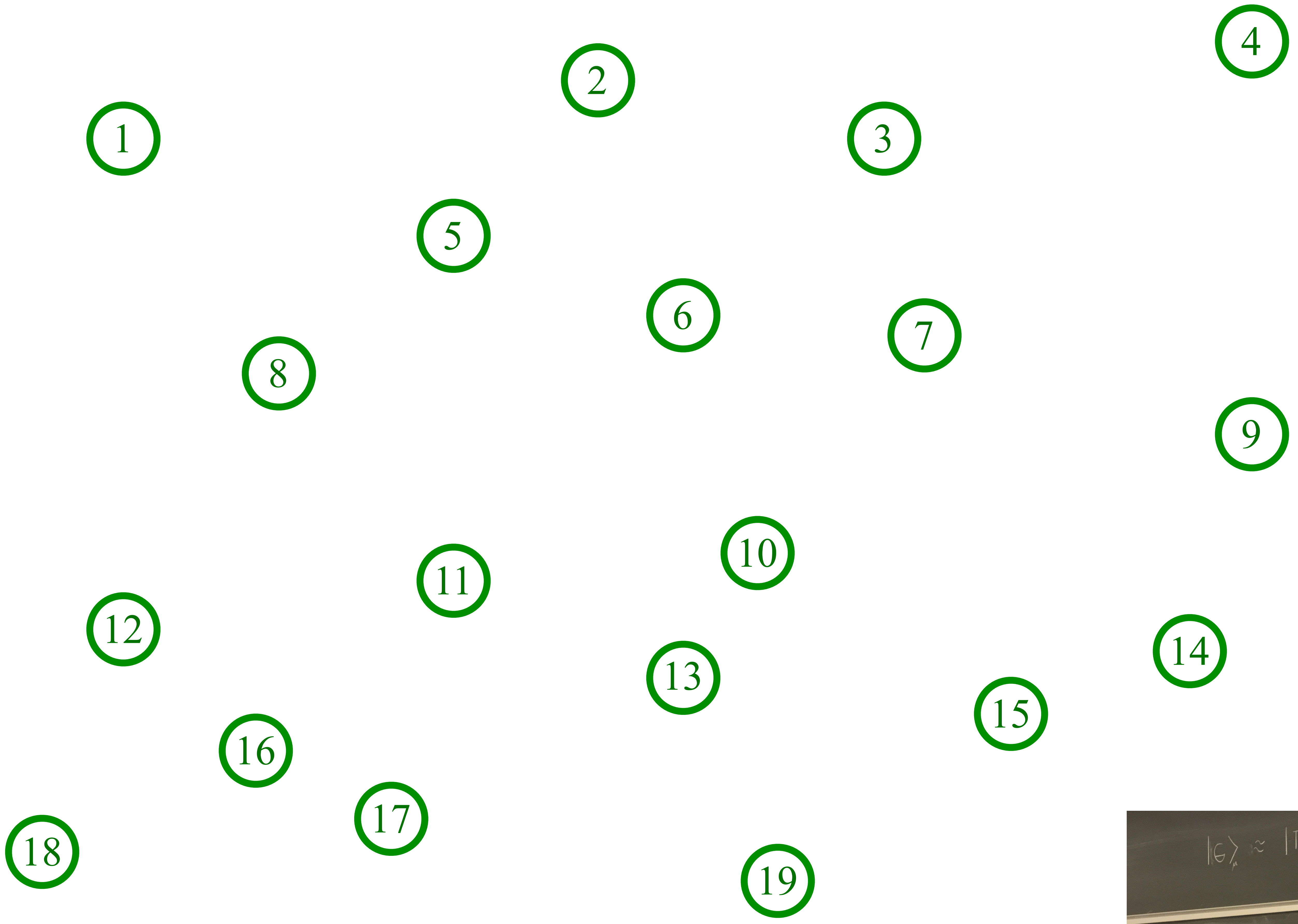
*Not true

A solvable model of multi-particle entanglement which accounts for quantum interference between successive collisions:

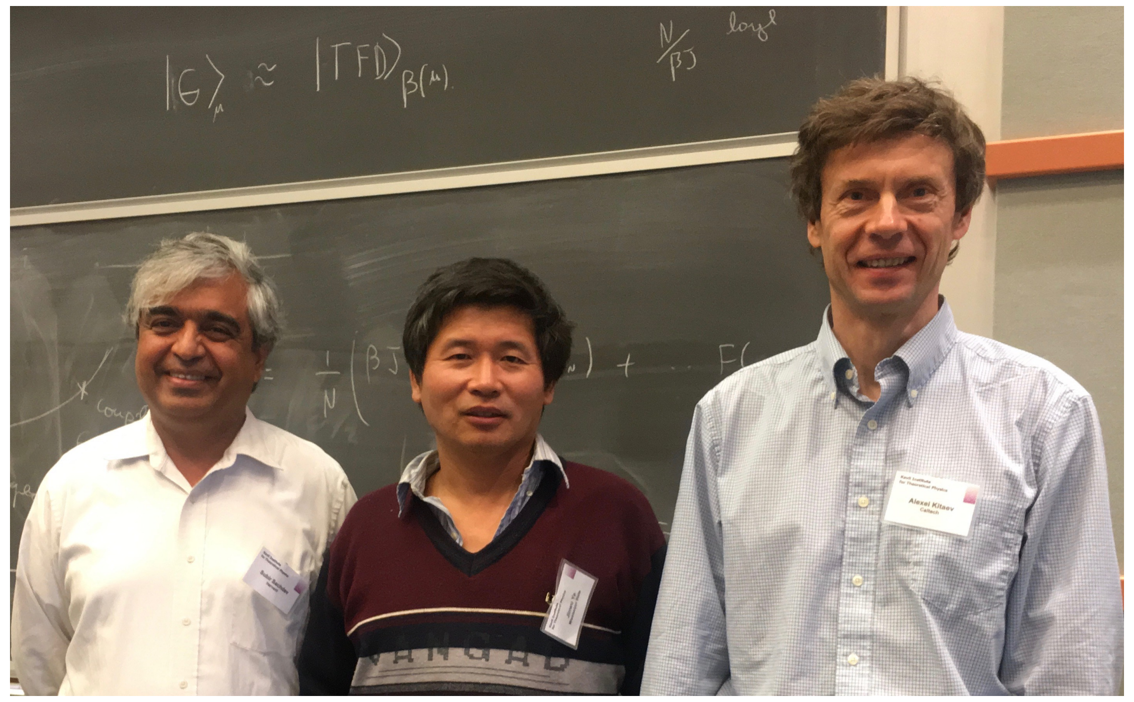
leading to a metal with no particle-like excitations

The SYK model

Sachdev, Ye (1993); Kitaev (2015)

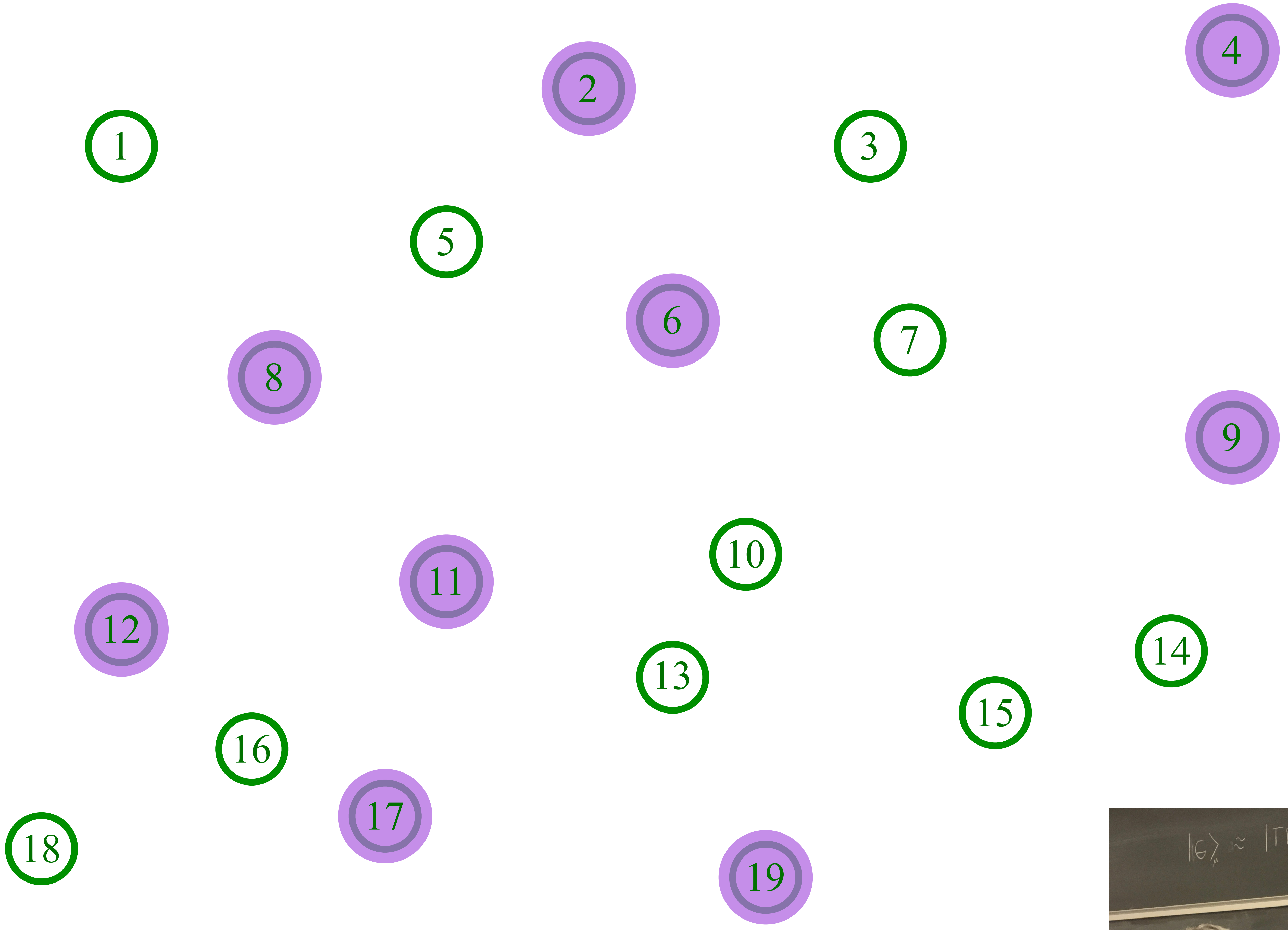


Pick a set of random positions

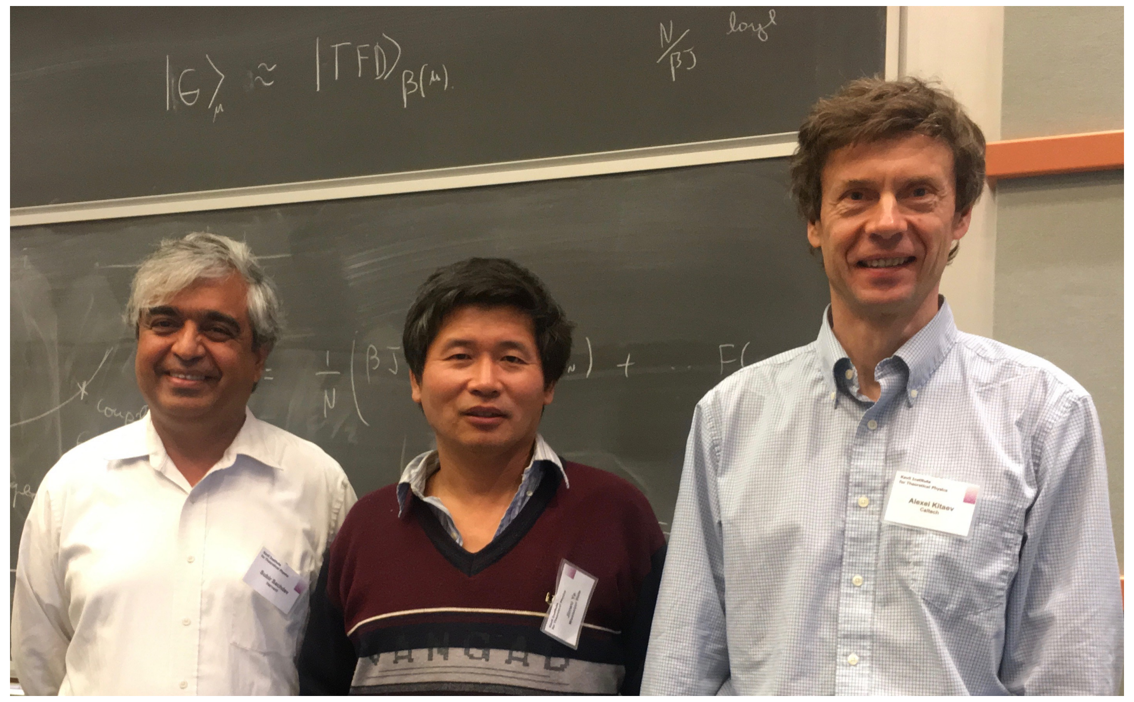


The SYK model

Sachdev, Ye (1993); Kitaev (2015)



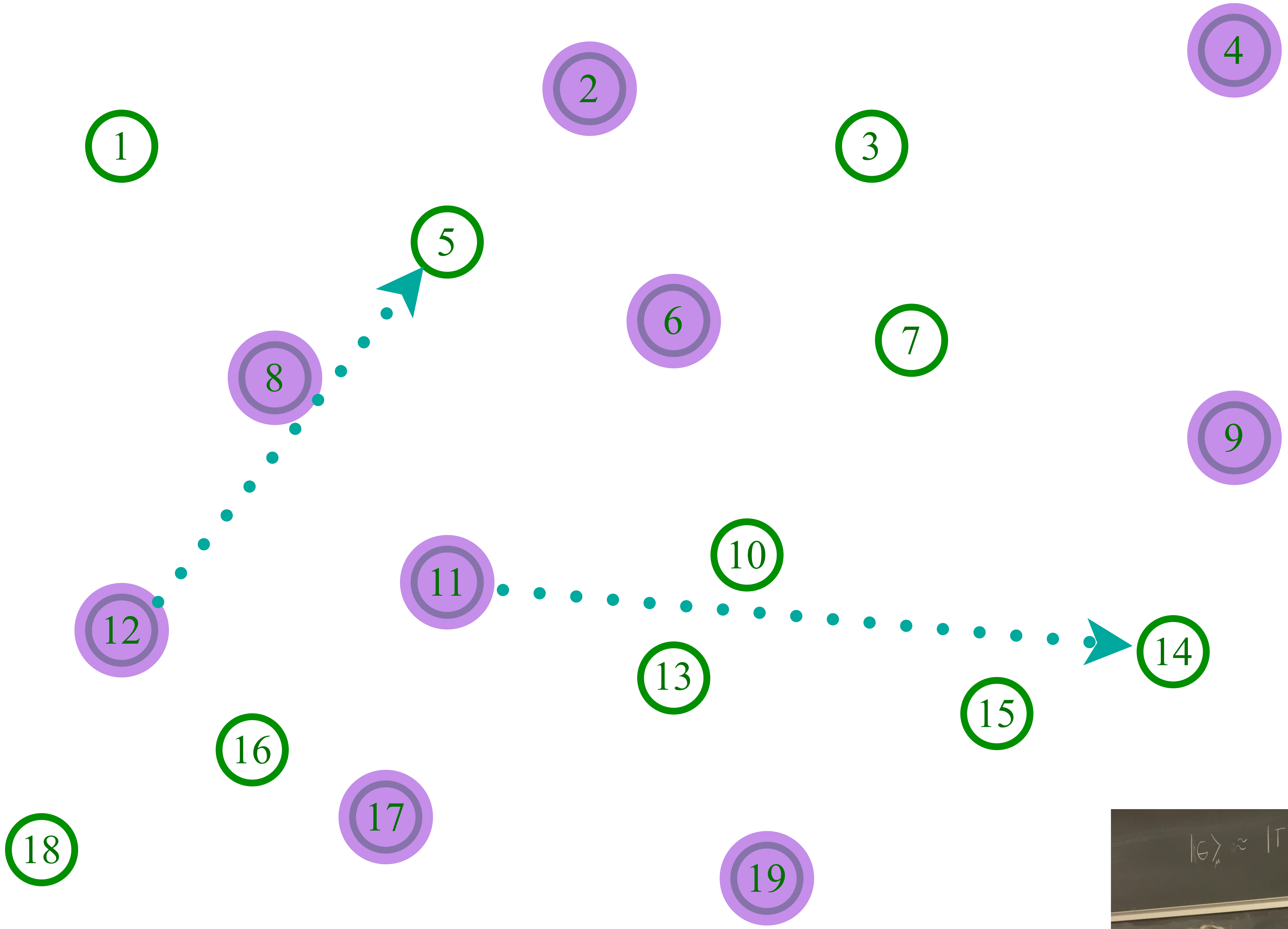
Place electrons randomly on some sites

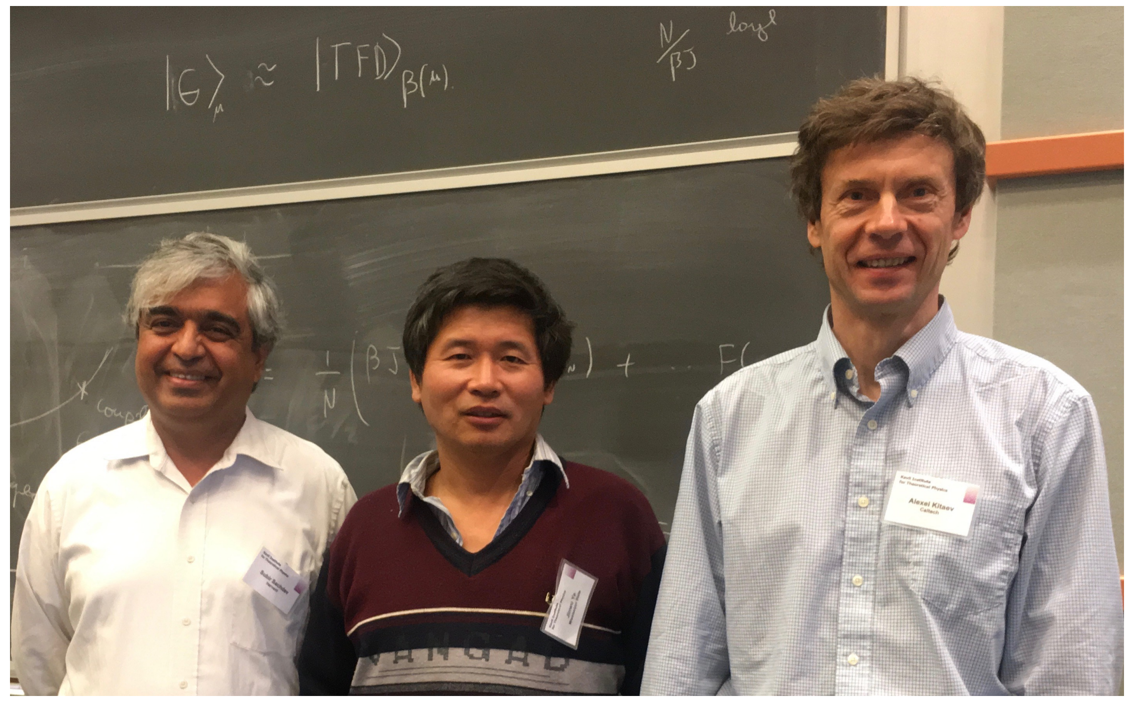


The SYK model

Sachdev, Ye (1993); Kitaev (2015)

$$U_{11,12;5,14}$$





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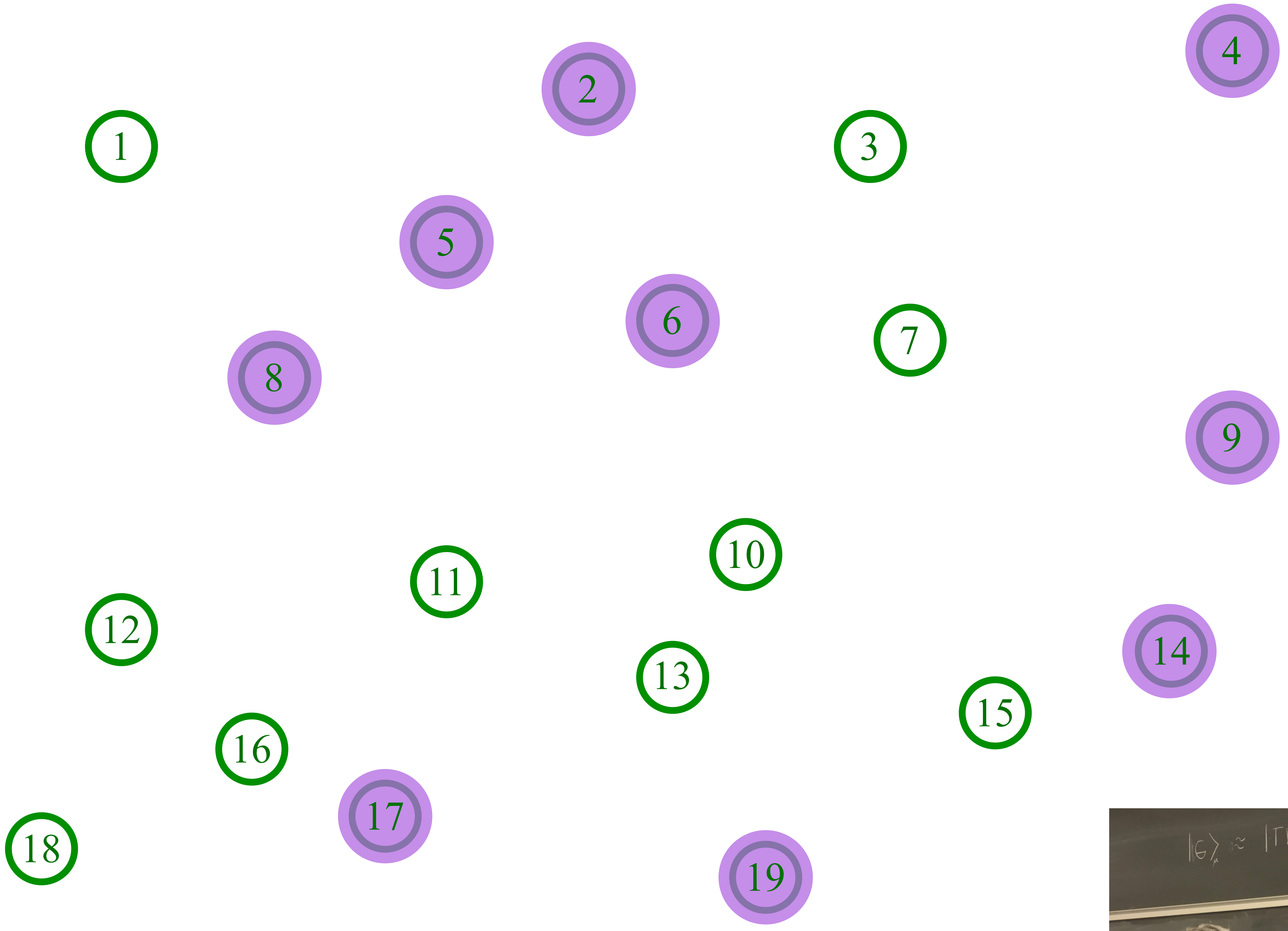
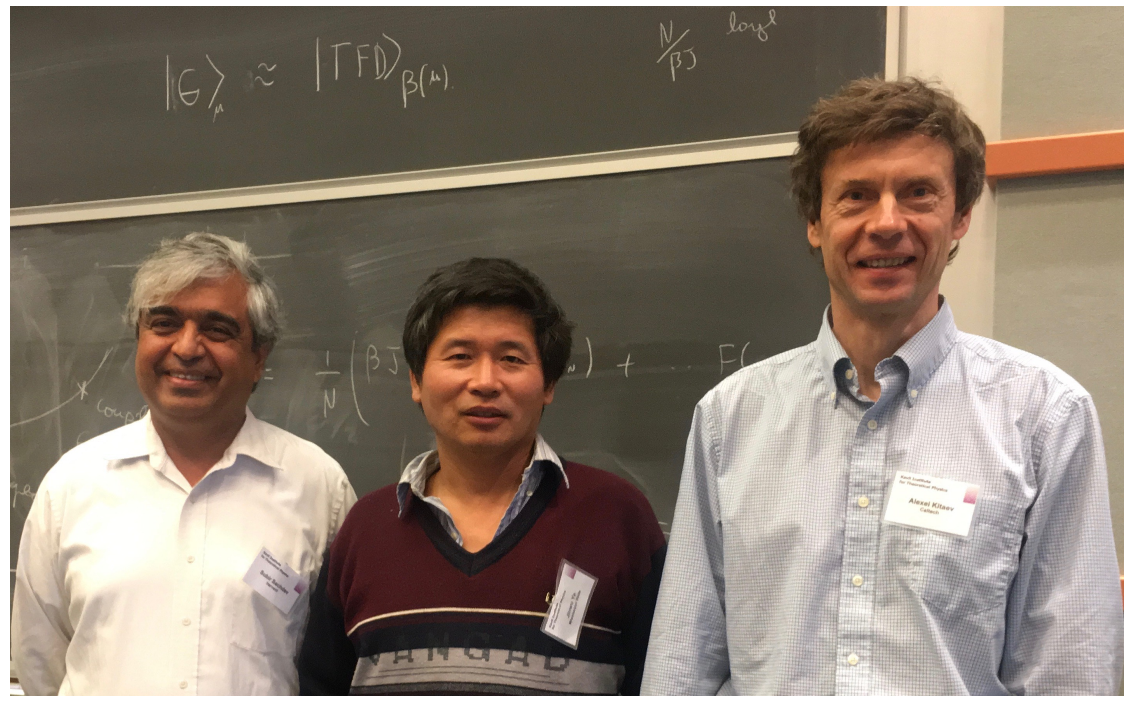
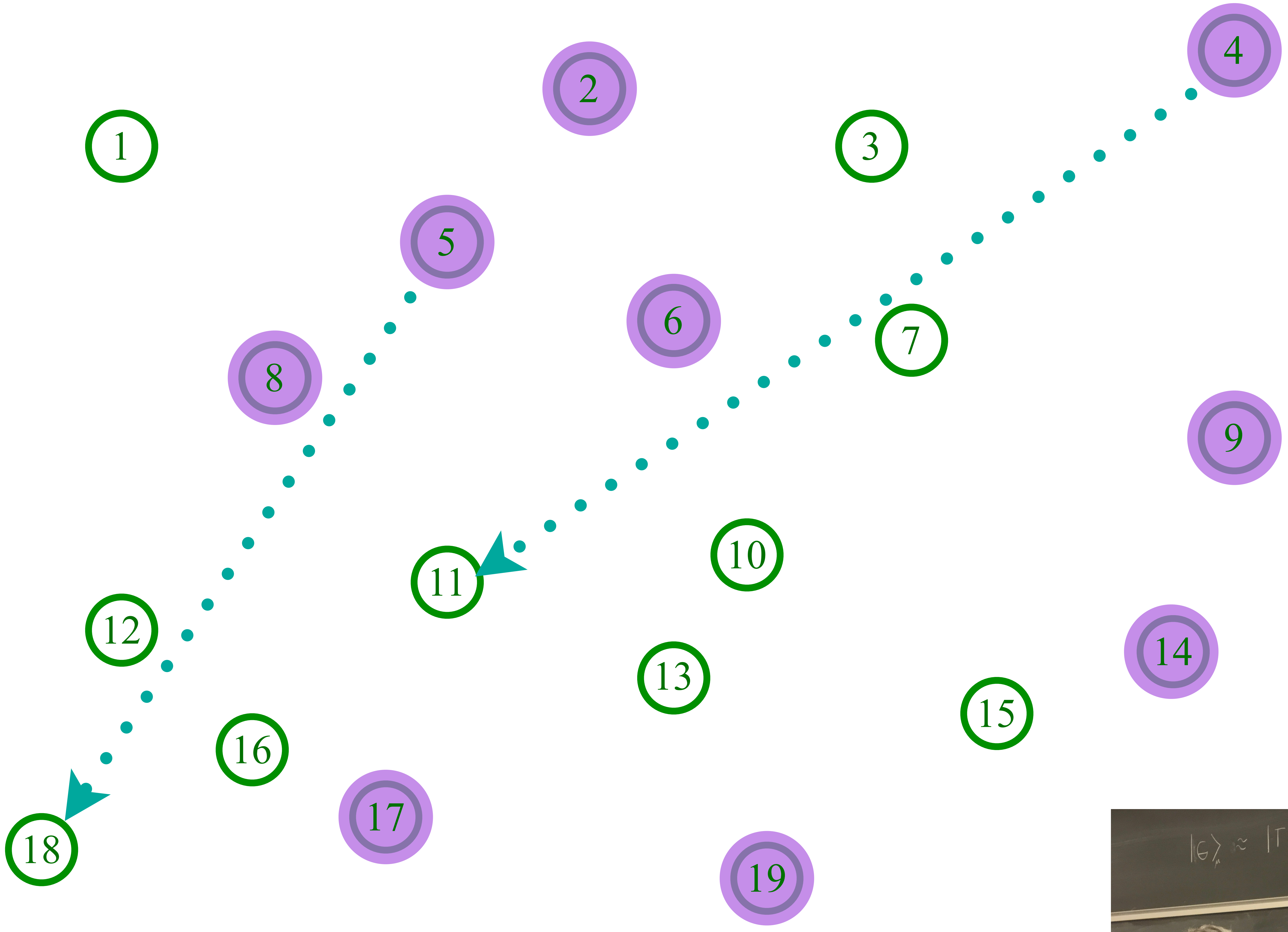


Entangle electrons pairwise randomly



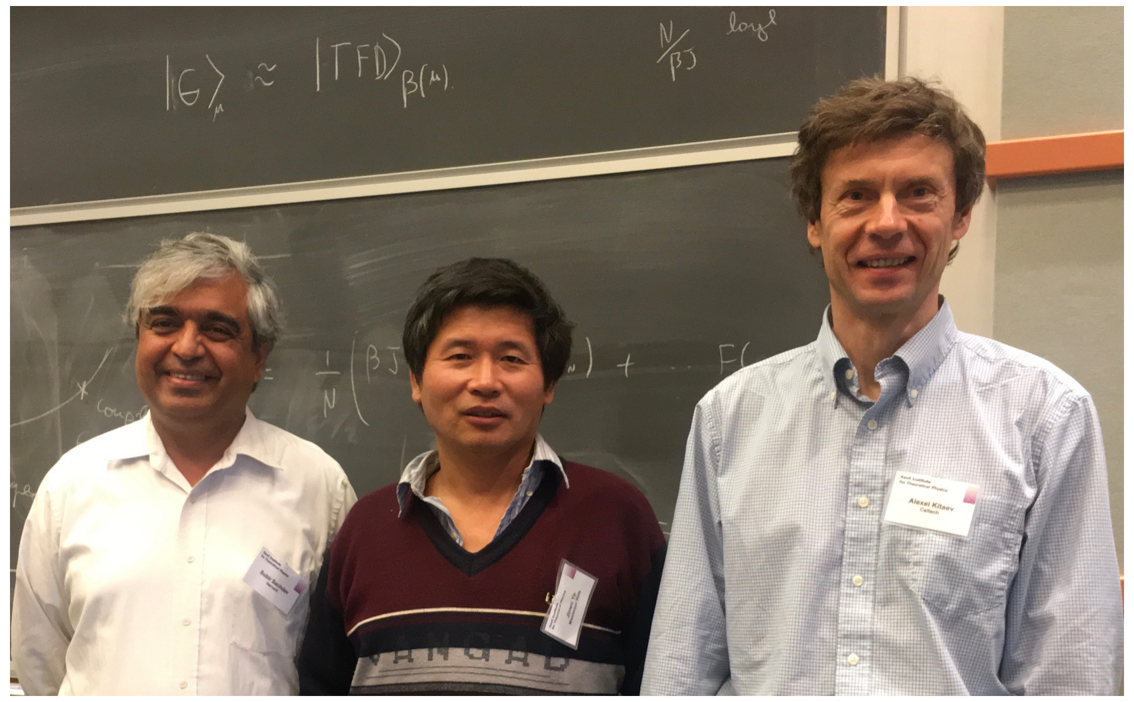
The SYK model

Sachdev, Ye (1993); Kitaev (2015)



$$U_{4,5;11,18}$$

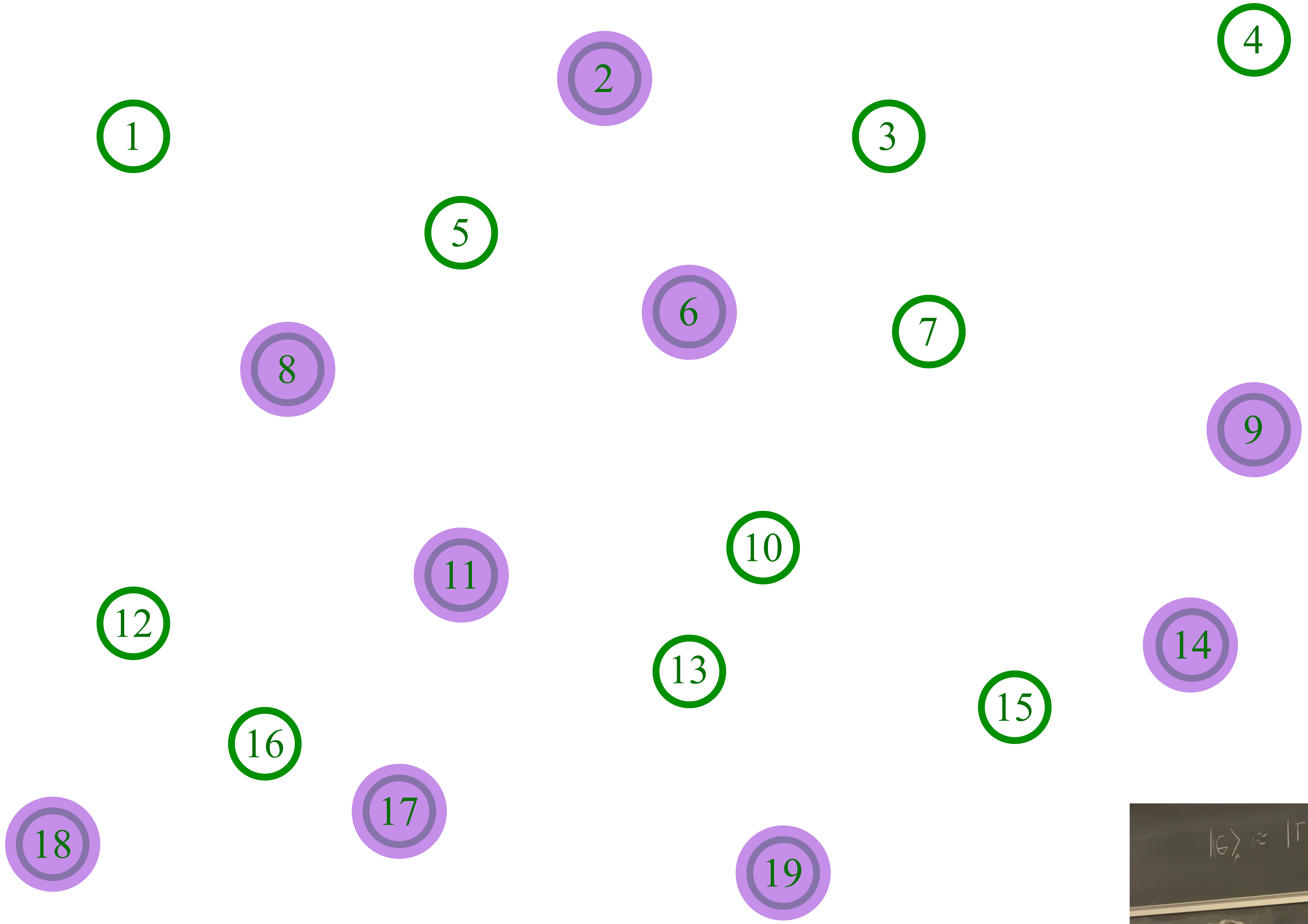
Entangle electrons pairwise randomly



The SYK model

Sachdev, Ye (1993); Kitaev (2015)

$$U_{4,5;11,18}$$



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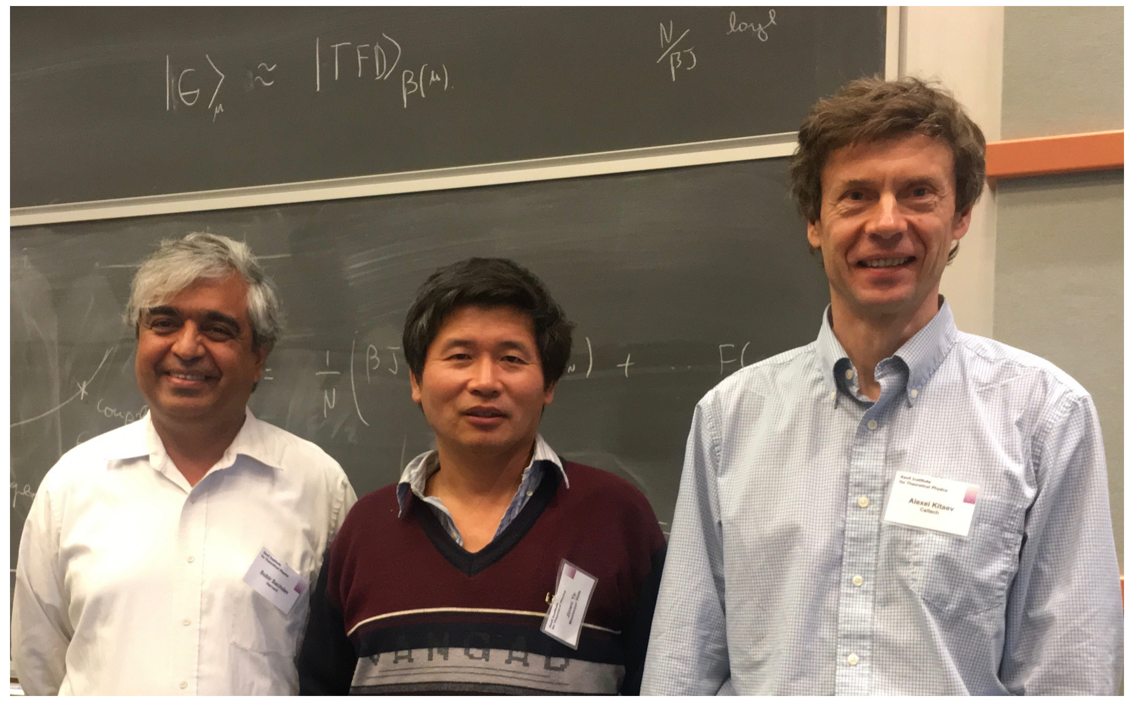
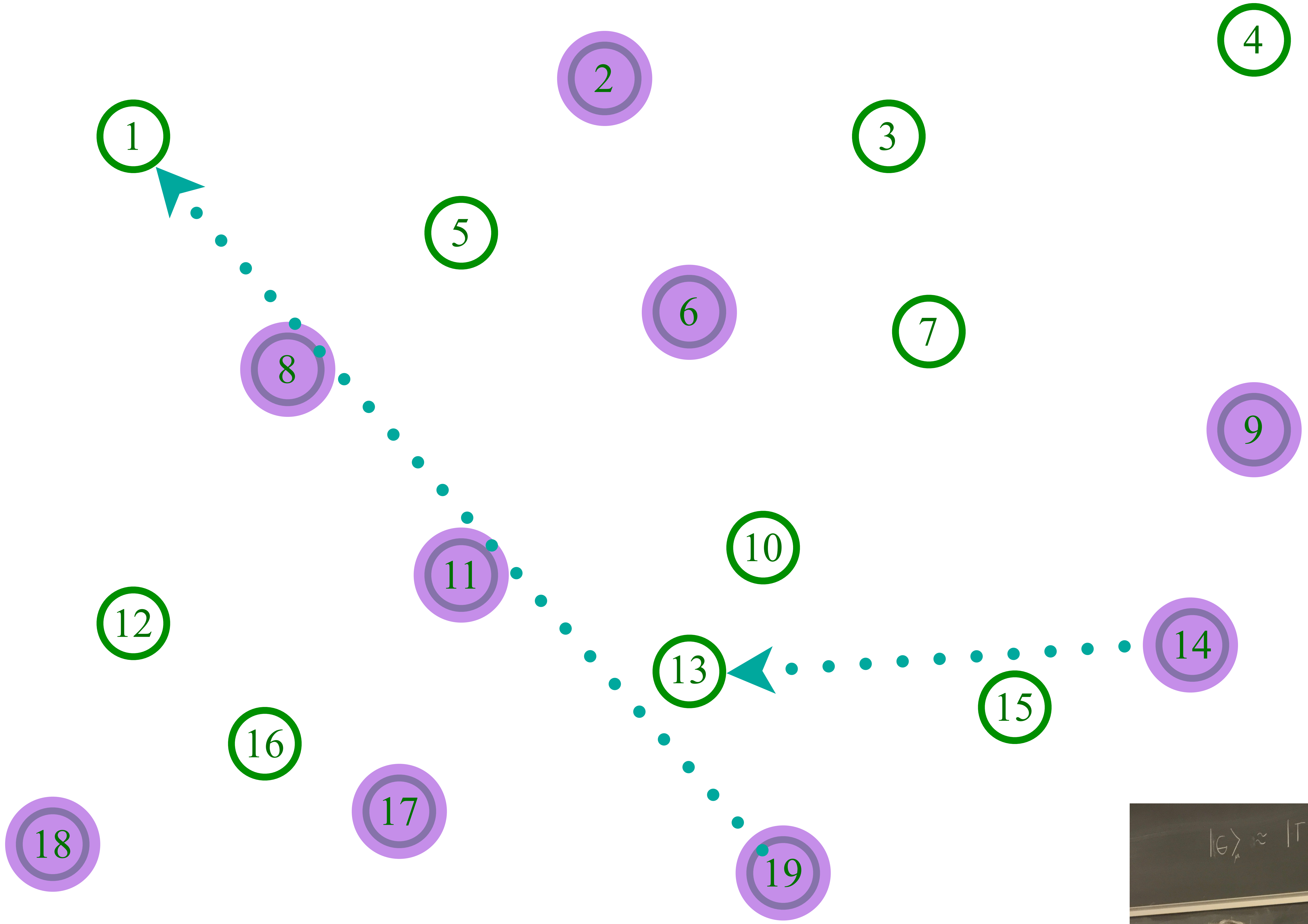


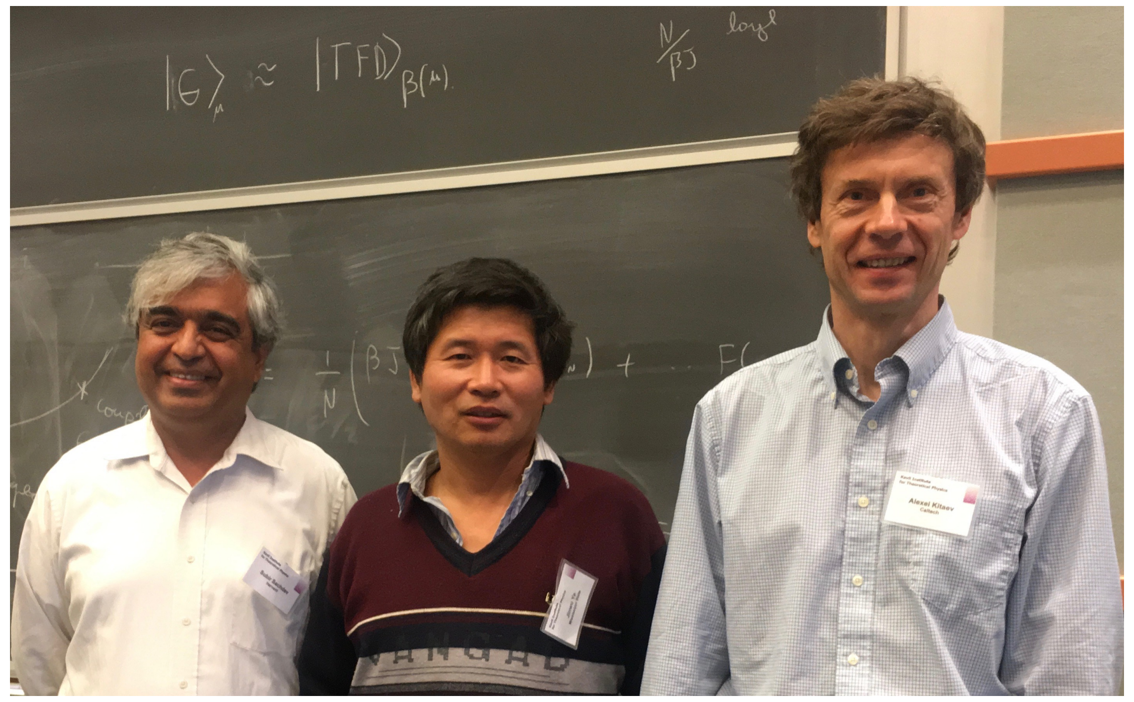
The SYK model

Sachdev, Ye (1993); Kitaev (2015)



$$U_{14,19;1,13}$$

Entangle electrons pairwise randomly



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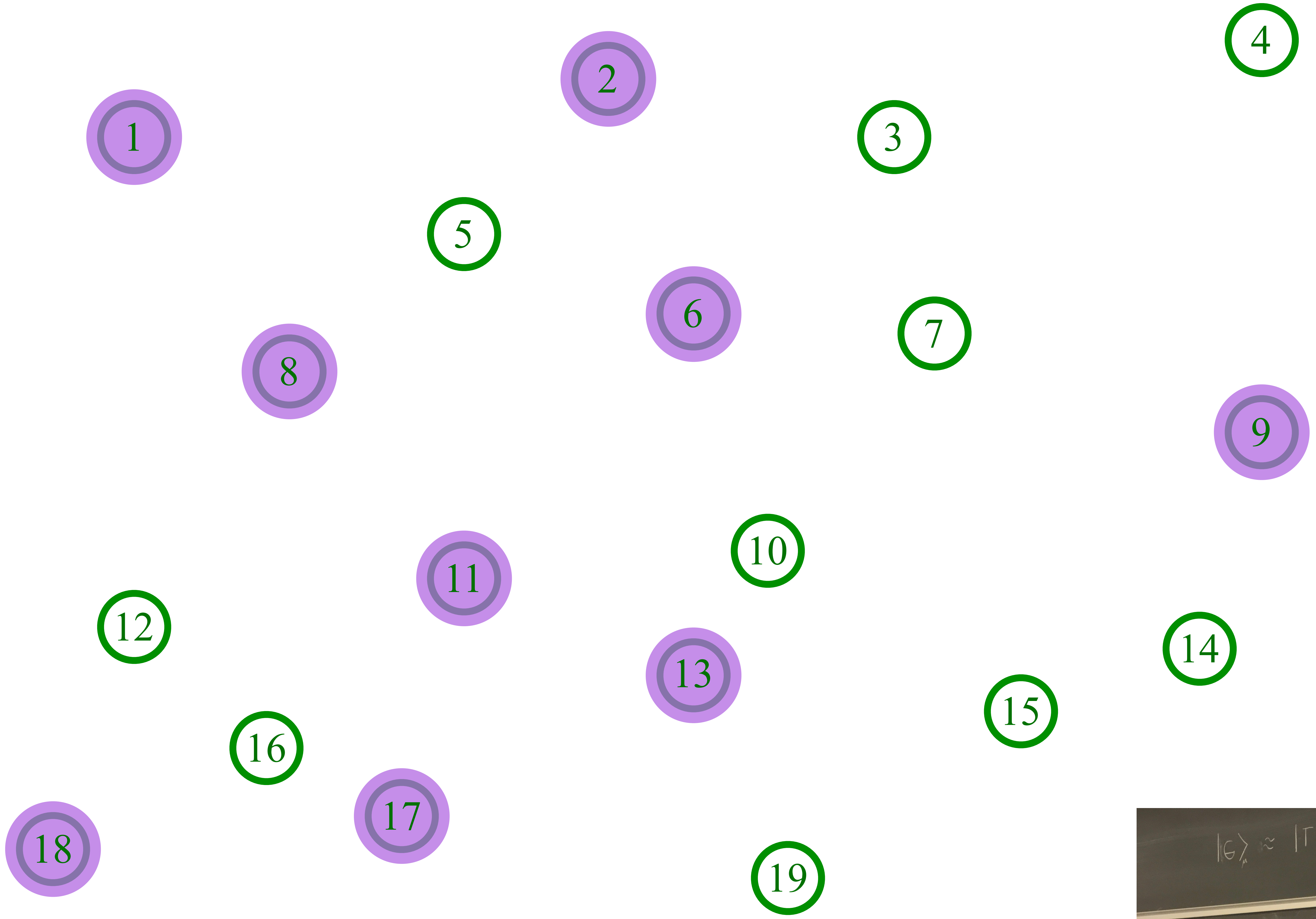
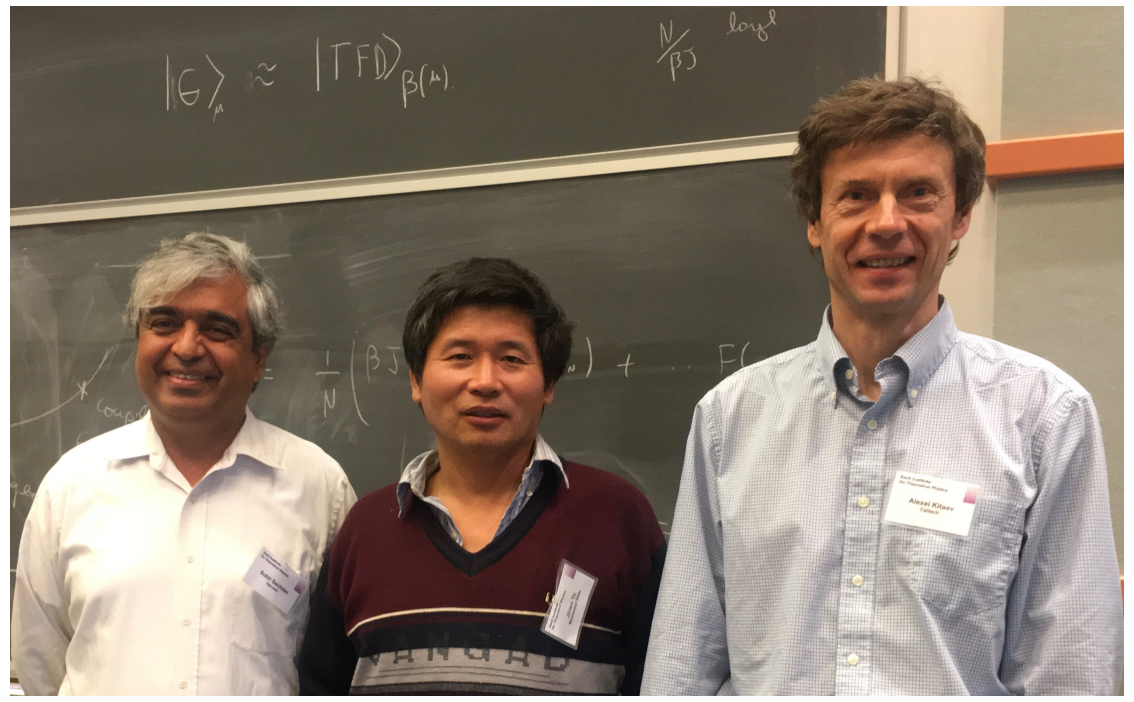


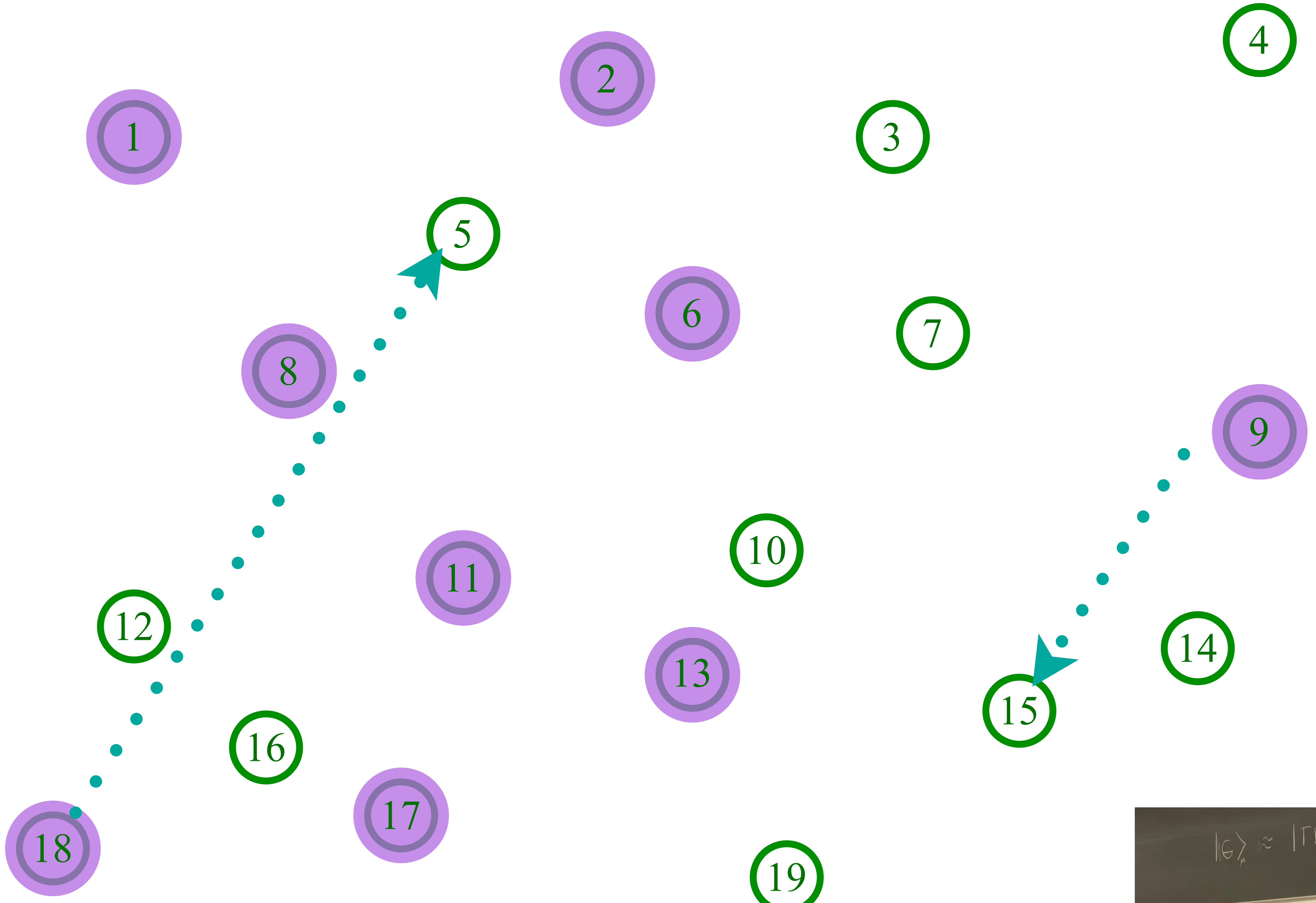
Entangle electrons pairwise randomly



The SYK model

Sachdev, Ye (1993); Kitaev (2015)

$$U_{9,18;5,15}$$



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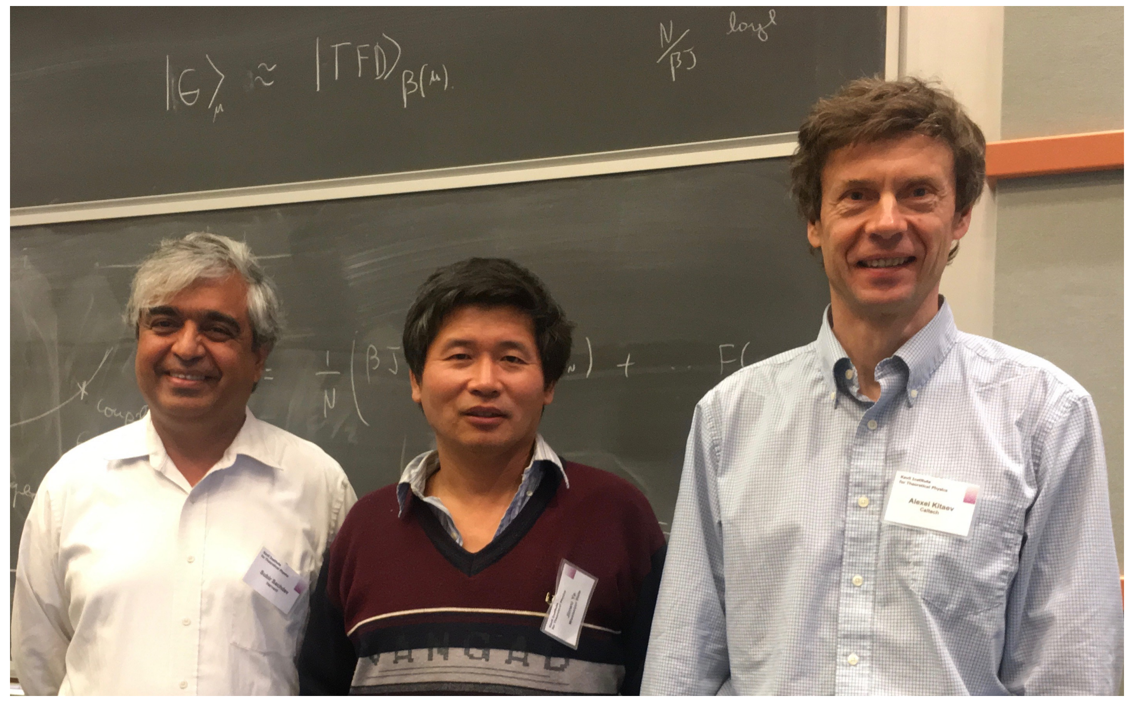

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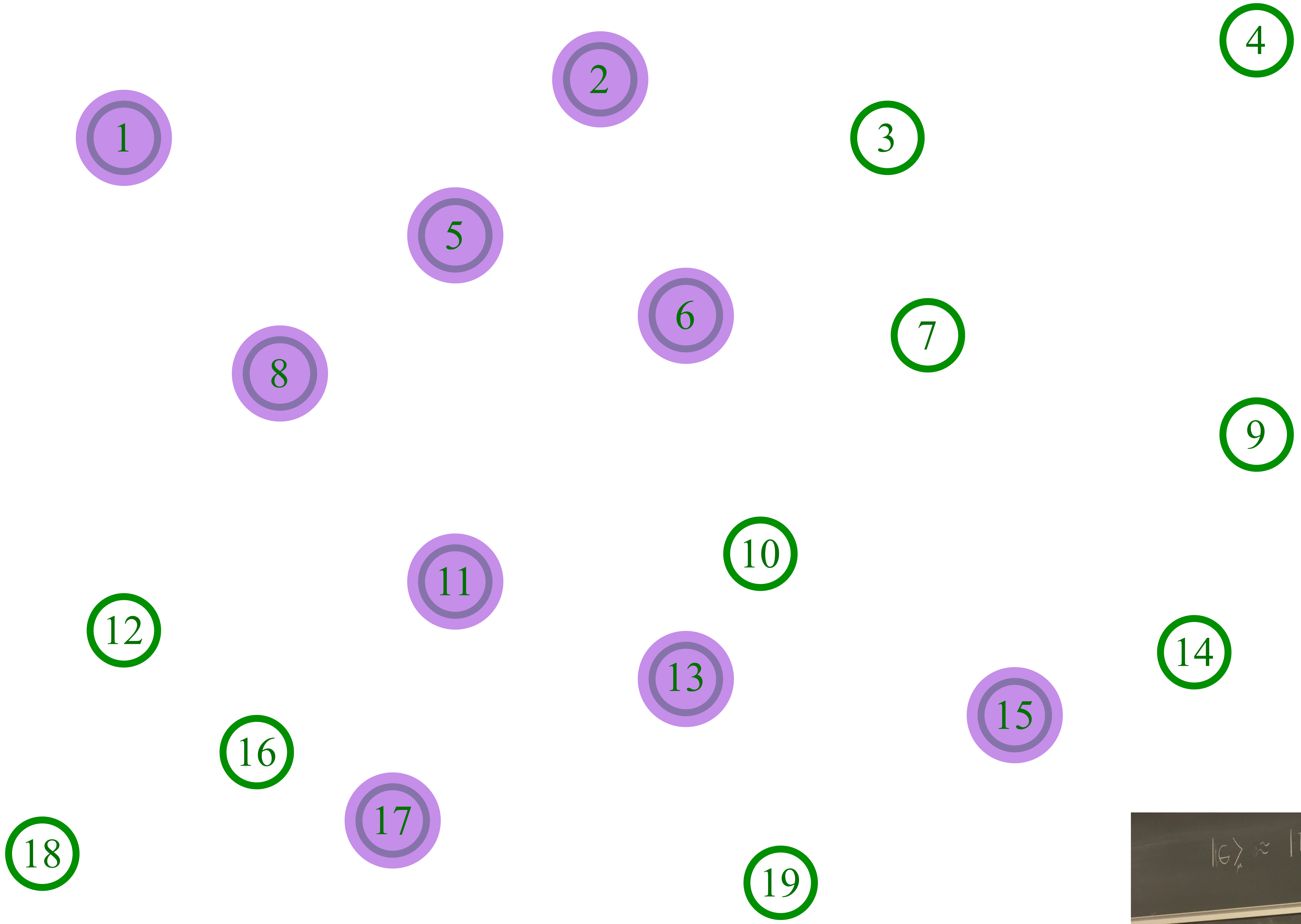

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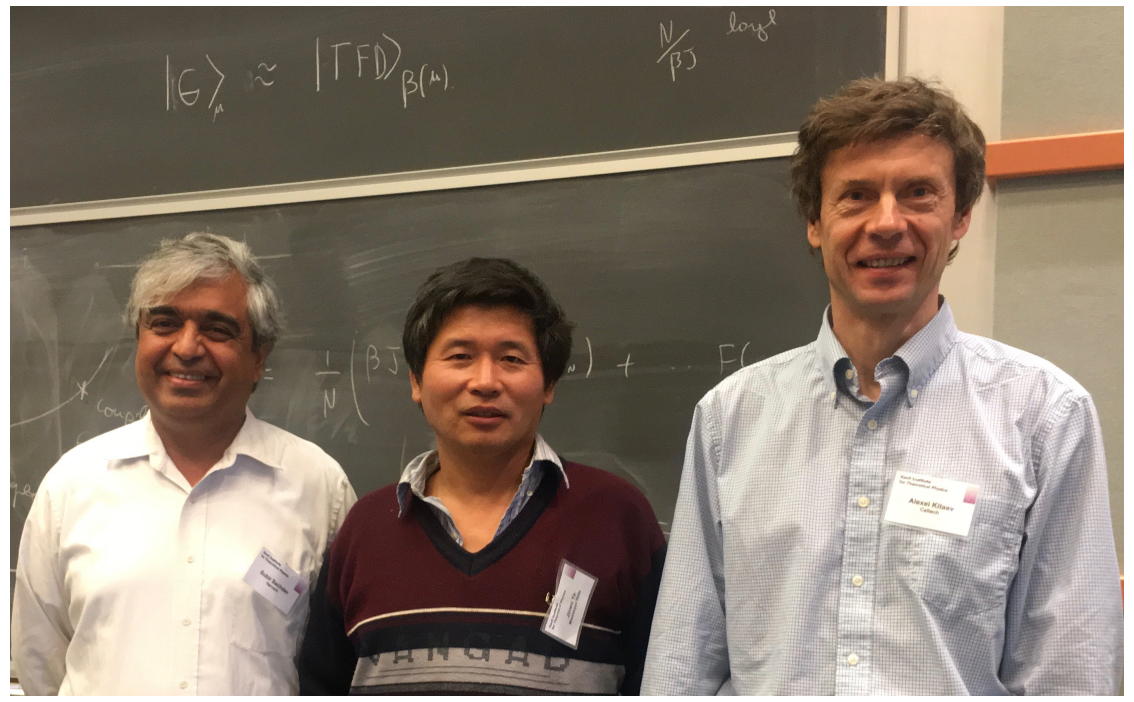
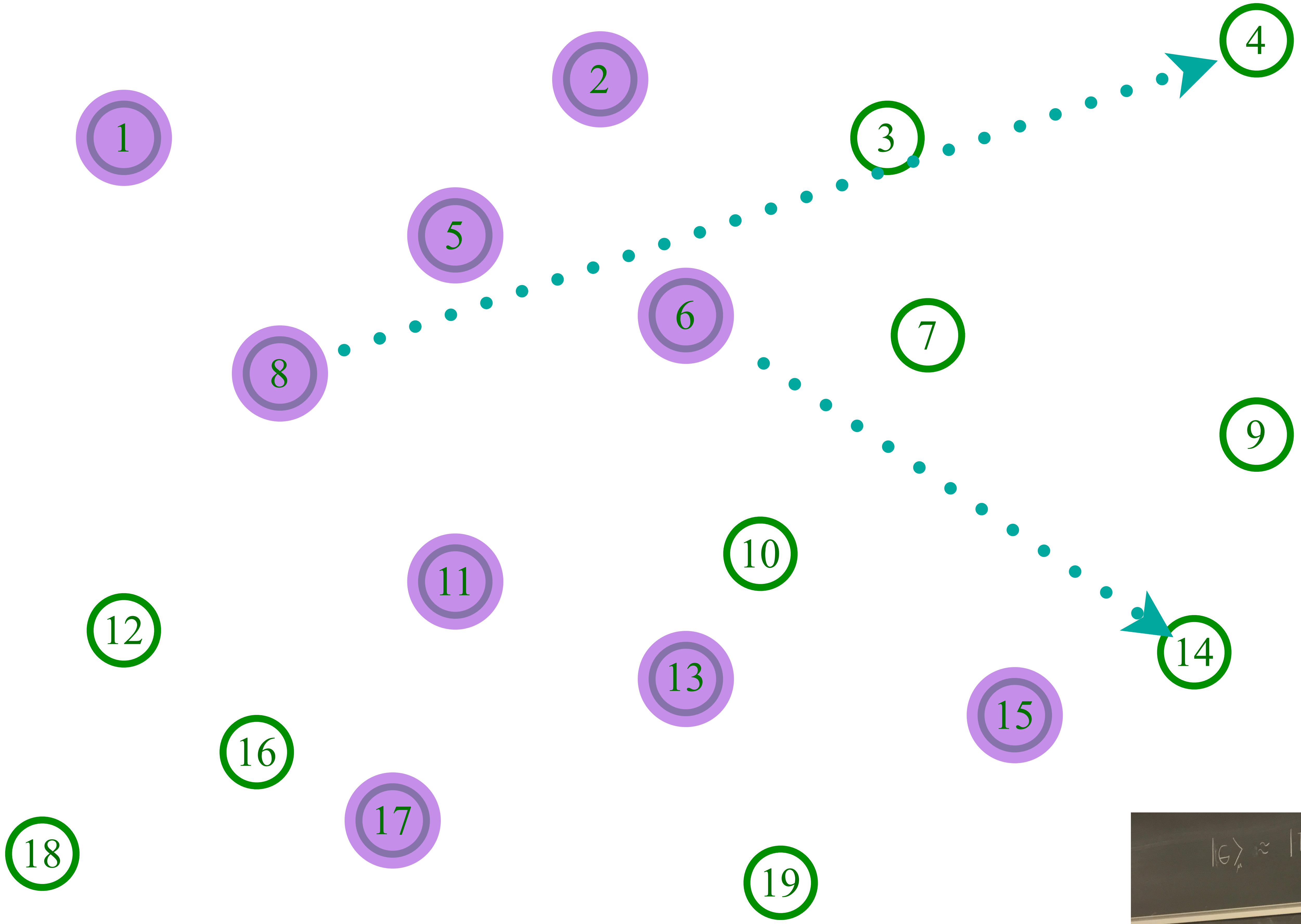


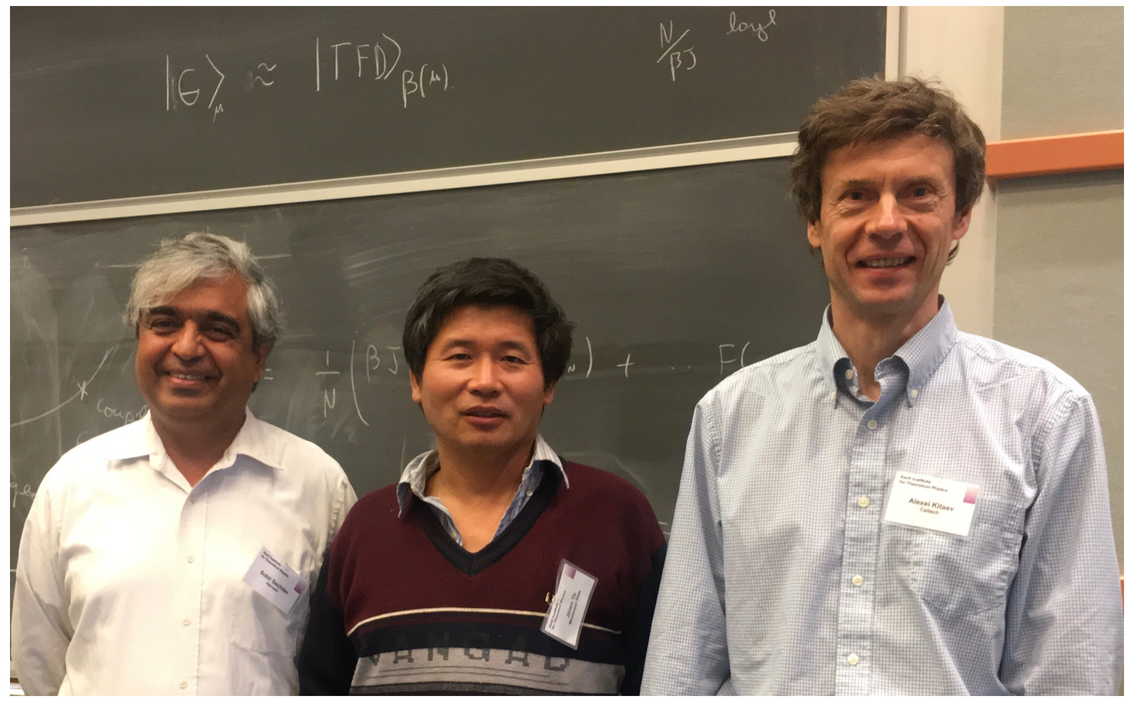
The SYK model

Sachdev, Ye (1993); Kitaev (2015)



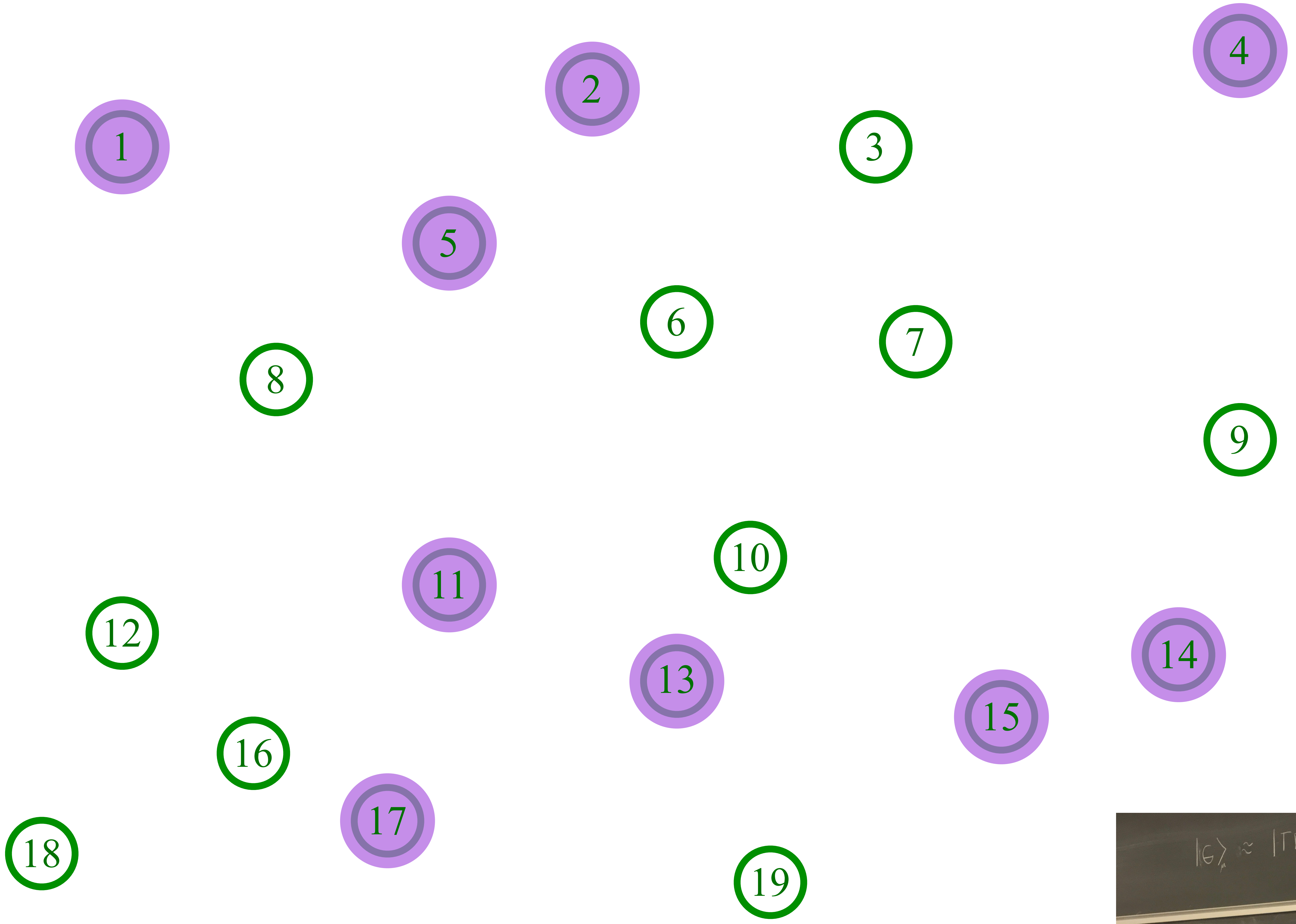
$$U_{6,8;4,14}$$

Entangle electrons pairwise randomly



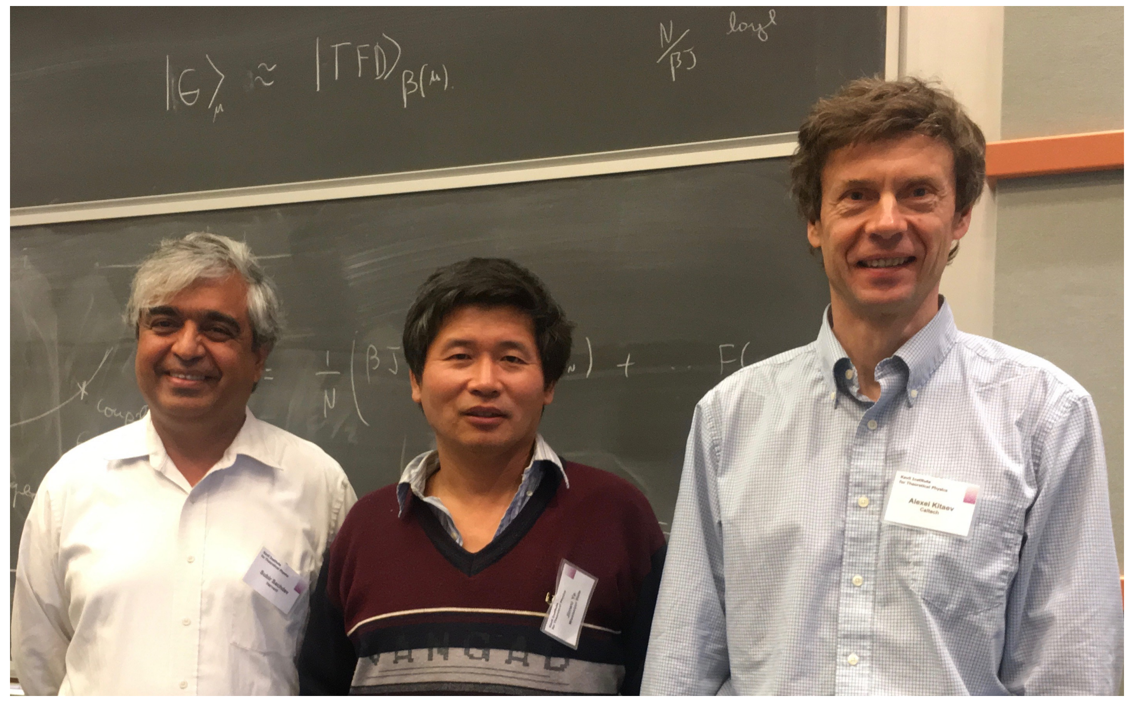
The SYK model

Sachdev, Ye (1993); Kitaev (2015)



$$U_{6,8;4,14}$$

Entangle electrons pairwise randomly



The SYK model

(See also: the "2-Body Random Ensemble" in nuclear physics; did not obtain the large *N* limit;
T.A. Brody, J. Flores, J.B. French, P.A. Mello, A. Pandey, and S.S.M. Wong, Rev. Mod. Phys. **53**, 385 (1981))

$$\mathcal{H} = \frac{1}{(2N)^{3/2}} \sum_{\alpha,\beta,\gamma,\delta=1}^{N} U_{\alpha\beta;\gamma\delta}\, c_\alpha^\dagger c_\beta^\dagger c_\gamma c_\delta - \mu \sum_\alpha c_\alpha^\dagger c_\alpha$$

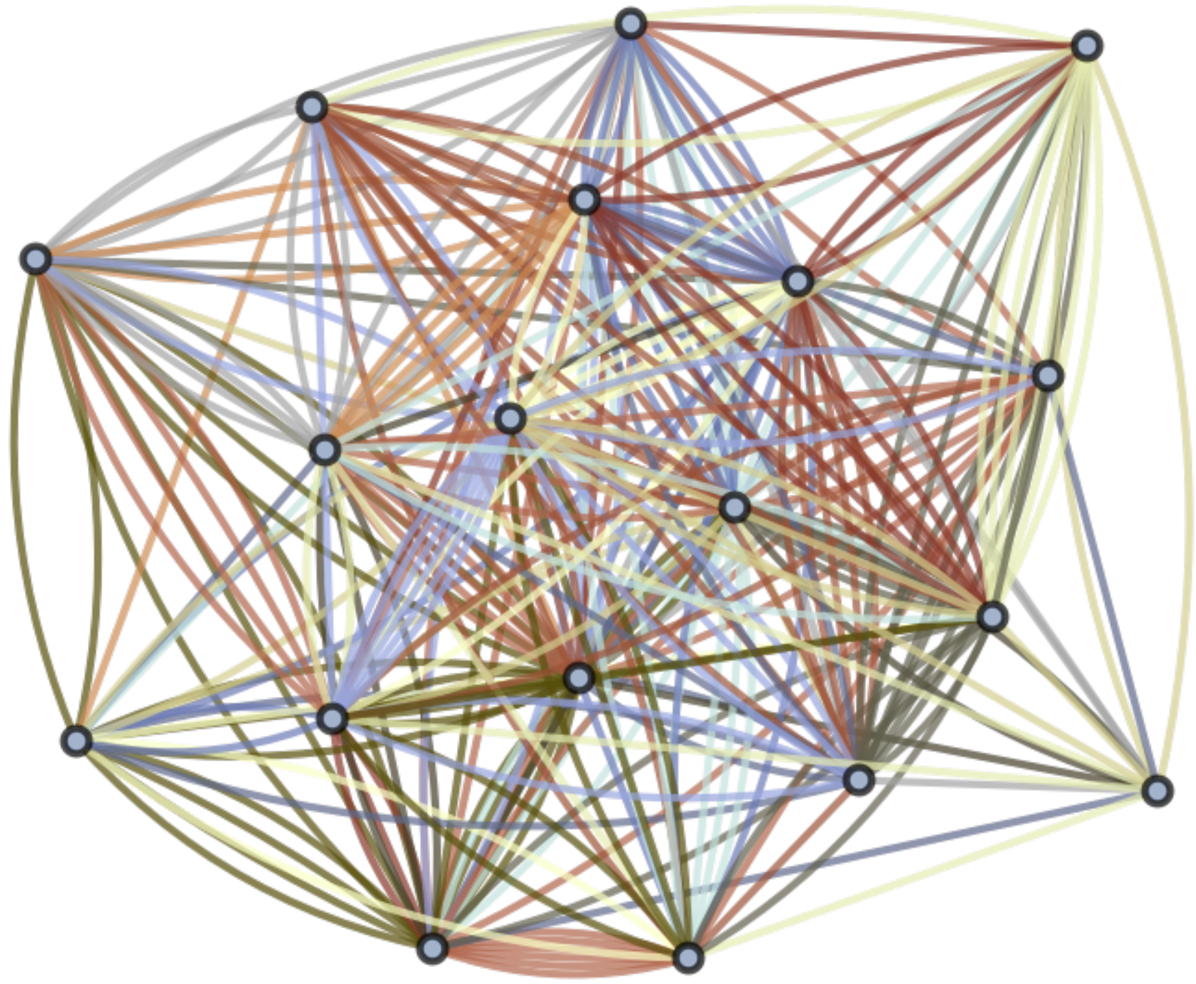
$$c_\alpha c_\beta + c_\beta c_\alpha = 0 \quad , \quad c_\alpha c_\beta^\dagger + c_\beta^\dagger c_\alpha = \delta_{\alpha\beta}$$

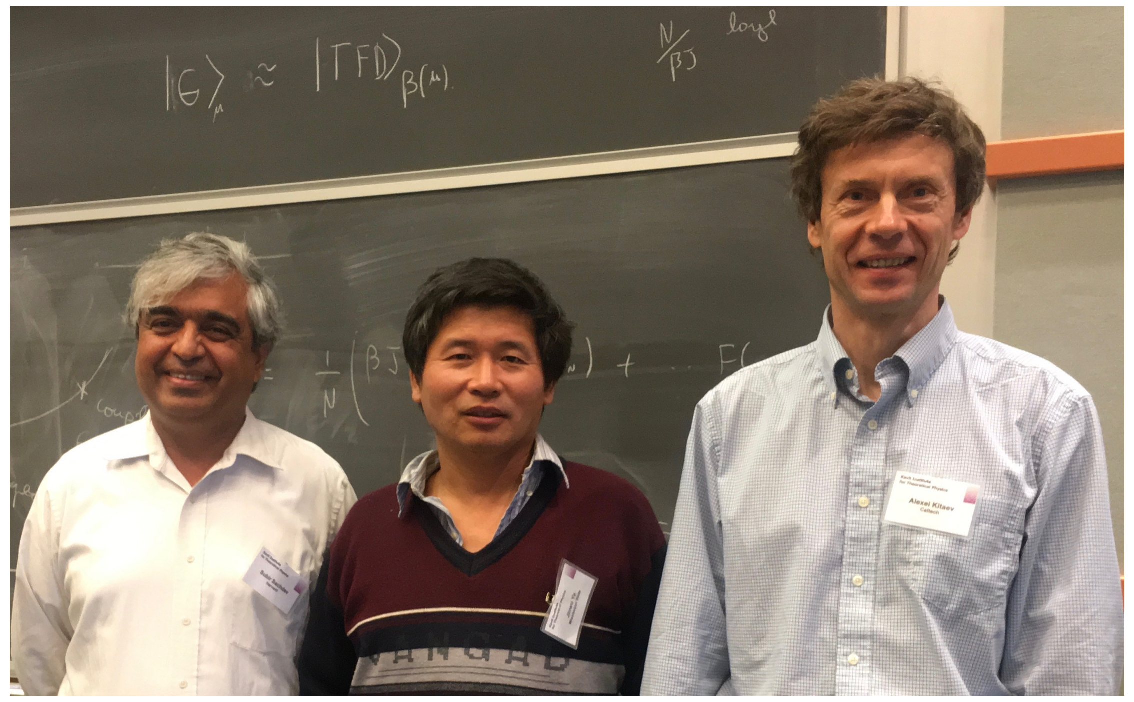
$$\mathcal{Q} = \frac{1}{N} \sum_\alpha c_\alpha^\dagger c_\alpha \,; \quad [\mathcal{H}, \mathcal{Q}] = 0\,; \quad 0 \leq \mathcal{Q} \leq 1$$

$U_{\alpha\beta;\gamma\delta}$ are independent random variables with $\overline{U_{\alpha\beta;\gamma\delta}} = 0$ and $\overline{|U_{\alpha\beta;\gamma\delta}|^2} = U^2$
$N \to \infty$ yields critical strange metal.

S. Sachdev and J. Ye, PRL **70**, 3339 (1993)

A. Kitaev, unpublished; S. Sachdev, PRX **5**, 041025 (2015)



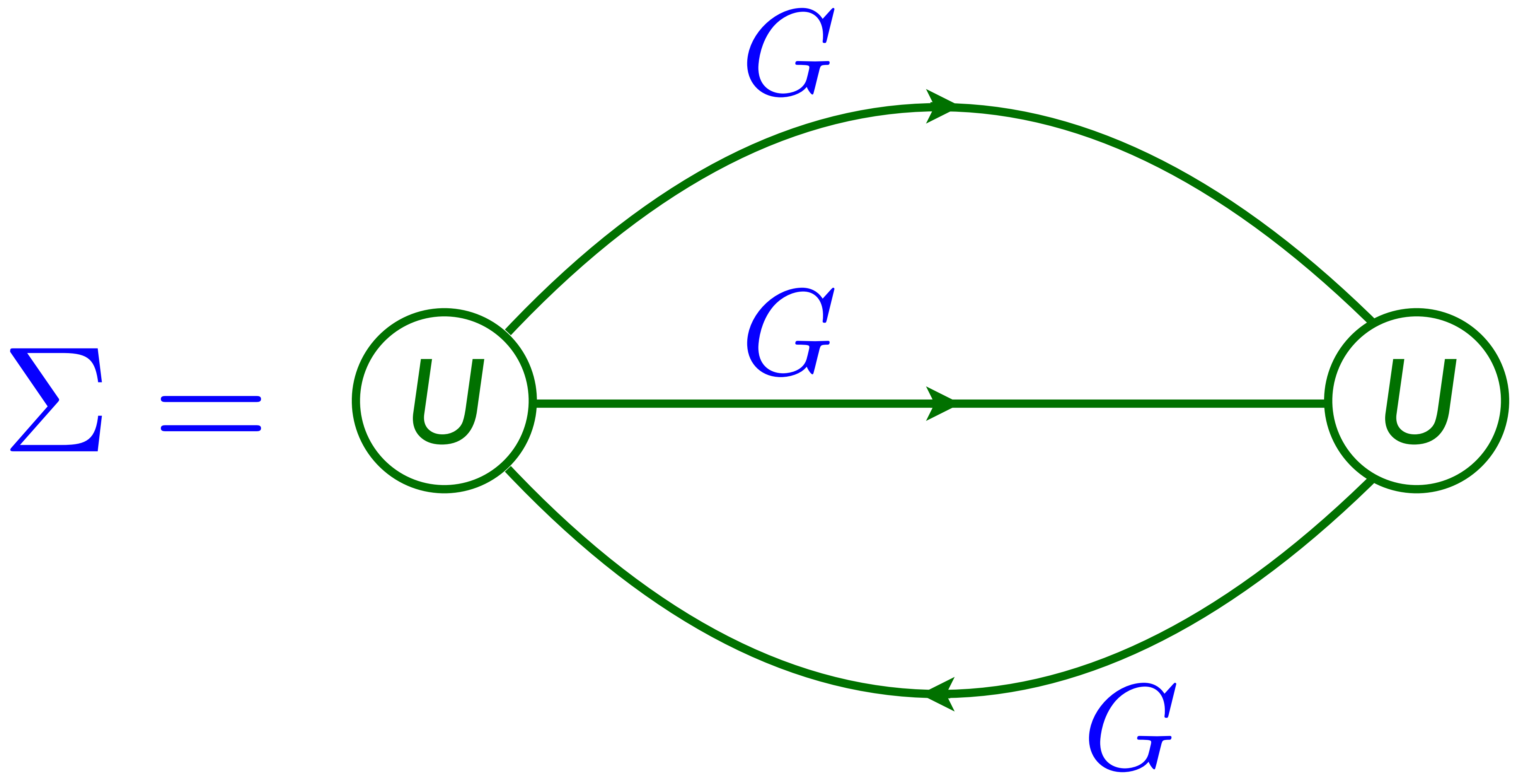


The SYK model

Feynman graph expansion in $U_{\alpha\beta;\gamma\delta}$, and graph-by-graph average, yields exact equations in the large N limit:

$$G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = -U^2 G^2(\tau) G(-\tau)$$

$$G(\tau = 0^-) = \mathcal{Q}.$$



S. Sachdev and J. Ye,
PRL **70**, 3339 (1993)

The SYK model

$$G_*(\tau) = -C\,\frac{e^{-2\pi\mathcal{E}T\tau}}{\sqrt{1+e^{-4\pi\mathcal{E}}}}\left(\frac{T}{\sin(\pi T\tau)}\right)^{1/2}.$$

$$G^R_*(\omega) =$$

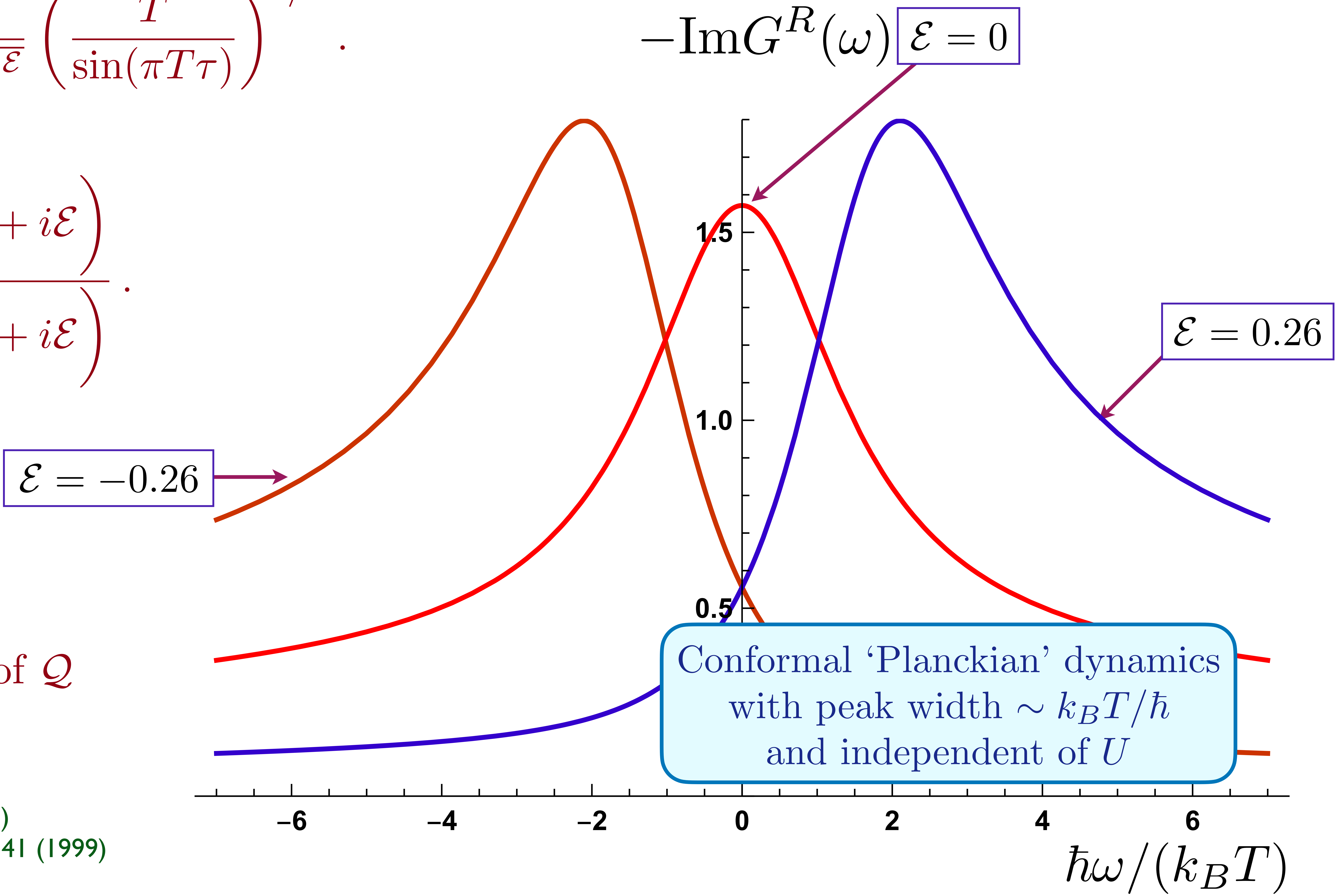
$$\frac{-iCe^{-i\theta}}{(2\pi T)^{1/2}}\,\frac{\Gamma\left(\frac{1}{4}-\frac{i\omega}{2\pi T}+i\mathcal{E}\right)}{\Gamma\left(\frac{3}{4}-\frac{i\omega}{2\pi T}+i\mathcal{E}\right)}.$$

$$e^{2\pi\mathcal{E}} = \frac{\sin(\pi/4+\theta)}{\sin(\pi/4-\theta)}$$

$$C = \left(\frac{\pi}{U^2\cos(2\theta)}\right)^{1/4}$$

$\mathcal{E}$ is a known function of $\mathcal{Q}$

(Luttinger relation)



Conformal ‘Planckian’ dynamics with peak width $\sim k_BT/\hbar$ and independent of U

S. Sachdev and J. Ye, PRL **70**, 3339 (1993)
A. Georges and O. Parcollet PRB **59**, 5341 (1999)
S. Sachdev, PRX **5**, 041025 (2015)

The SYK model

$$
\begin{aligned}
\mathcal{Z}(\mathcal{Q},T) &= \mathrm{Tr}_{\mathcal{Q}} \exp\left(-\frac{\mathcal{H}}{T}\right) = \exp(-F/T)\,;\ \text{Entropy } S = -\frac{\partial F}{\partial T}. \\
&\equiv \int_{E_0^-}^{\infty} D(E) e^{-E/T}\,; \quad D(E) = \sum_i \delta(E - E_i)\,; \quad \mathcal{H}\left|\Psi_i\right\rangle = E_i \left|\Psi_i\right\rangle
\end{aligned}
$$

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A CFT in d spatial dimensions has an entropy density $S \sim T^d$. The SYK model is a 0+1 dimensional CFT, and we obtain a T-independent entropy:

$$\lim_{T\to 0}\lim_{N\to\infty} \frac{S}{N} = s_0$$

A. Georges, O. Parcollet, and S. Sachdev,
PRB **63**, 134406 (2001)

The constant s_0 is a known universal number. At $\mathcal{Q} = 1/2$, we have

$$s_0 = \frac{\text{Catalan}}{\pi} + \frac{\ln 2}{4} = 0.464847699170805107492692486833\ldots\,.$$

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$$
\lim_{T\to 0} \lim_{N \to \infty} \frac{S}{N} = s_0 \quad \boxed{\Rightarrow \quad D(E) \overset{?}{=} e^{N s_0} \delta(E - E_0)}
$$

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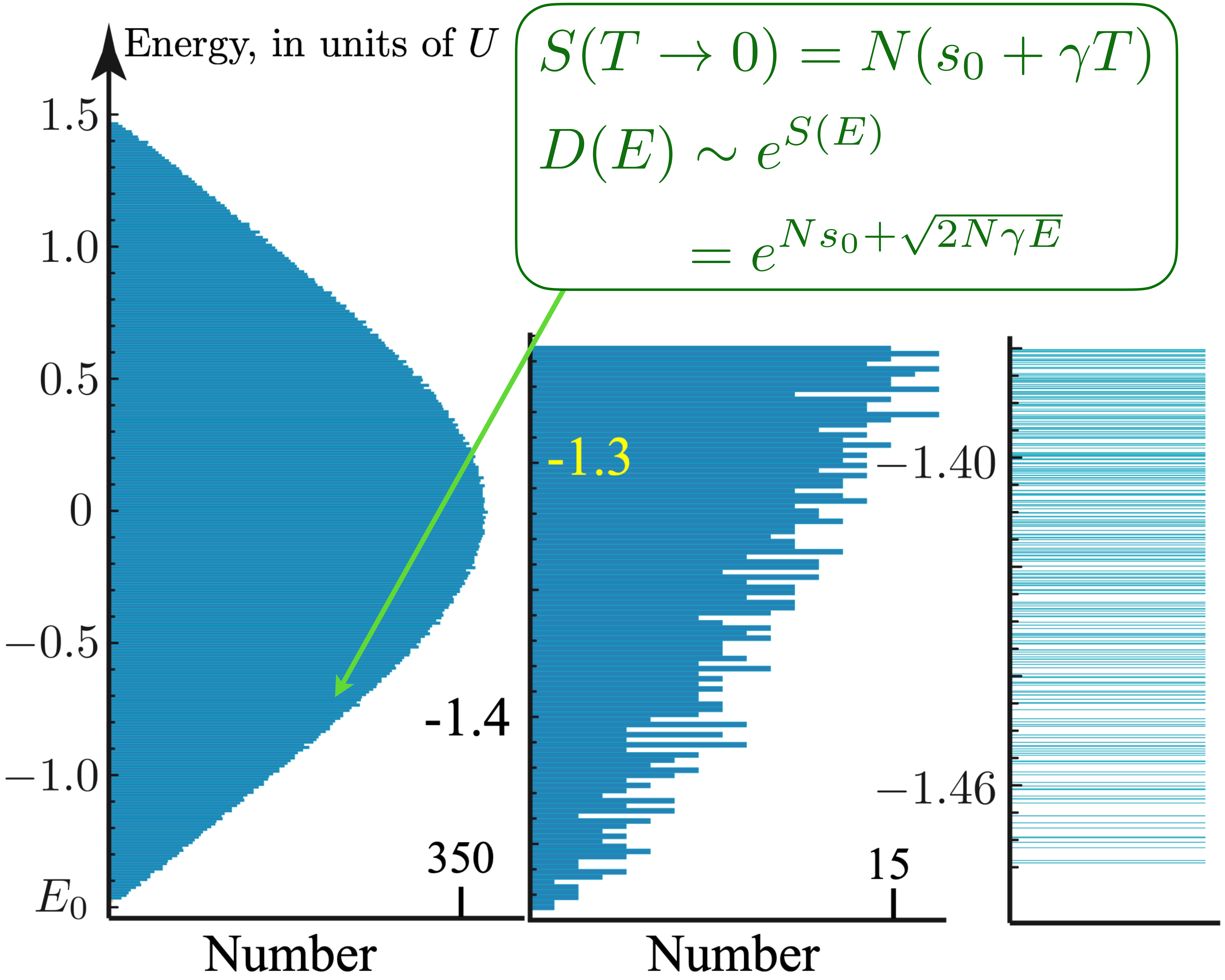
Conformal perturbations of saddle-point theory:

$$\boxed{S = N(s_0+\gamma T) \quad \Rightarrow \quad D(E) \sim \exp\left(Ns_0 + \sqrt{2N\gamma E}\right) \ \text{for } E \sim NU}$$

$\gamma = \#/U$ is non-universal.

Many-body density of states

$$D(E) = \sum_i \delta(E - E_i); \quad E_0 + E_i \Rightarrow \text{Many body eigenvalue}$$



$$S(T \to 0) = N(s_0 + \gamma T)$$

$$D(E) \sim e^{S(E)}$$

$$= e^{N s_0 + \sqrt{2N\gamma E}}$$

Complex SYK model

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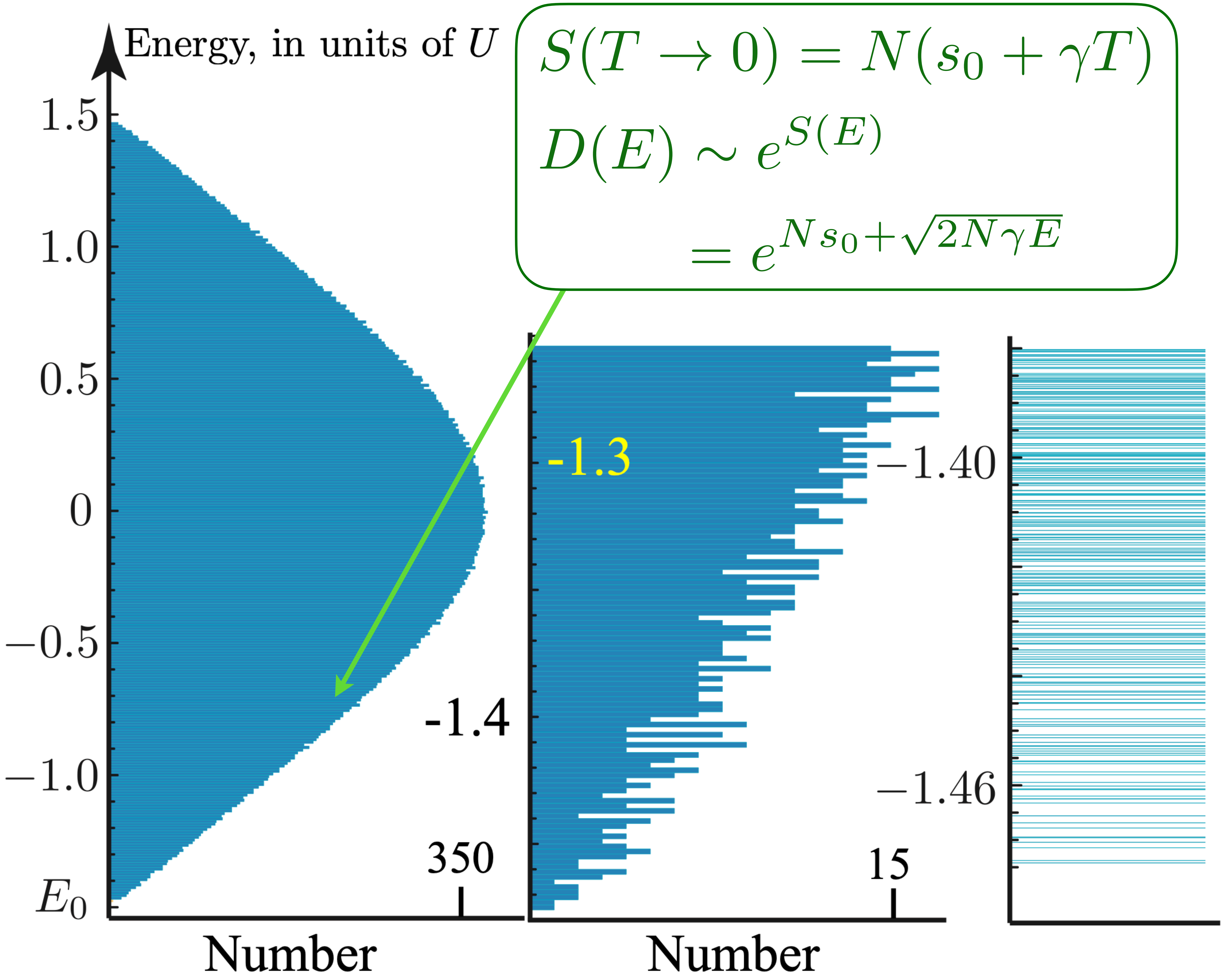

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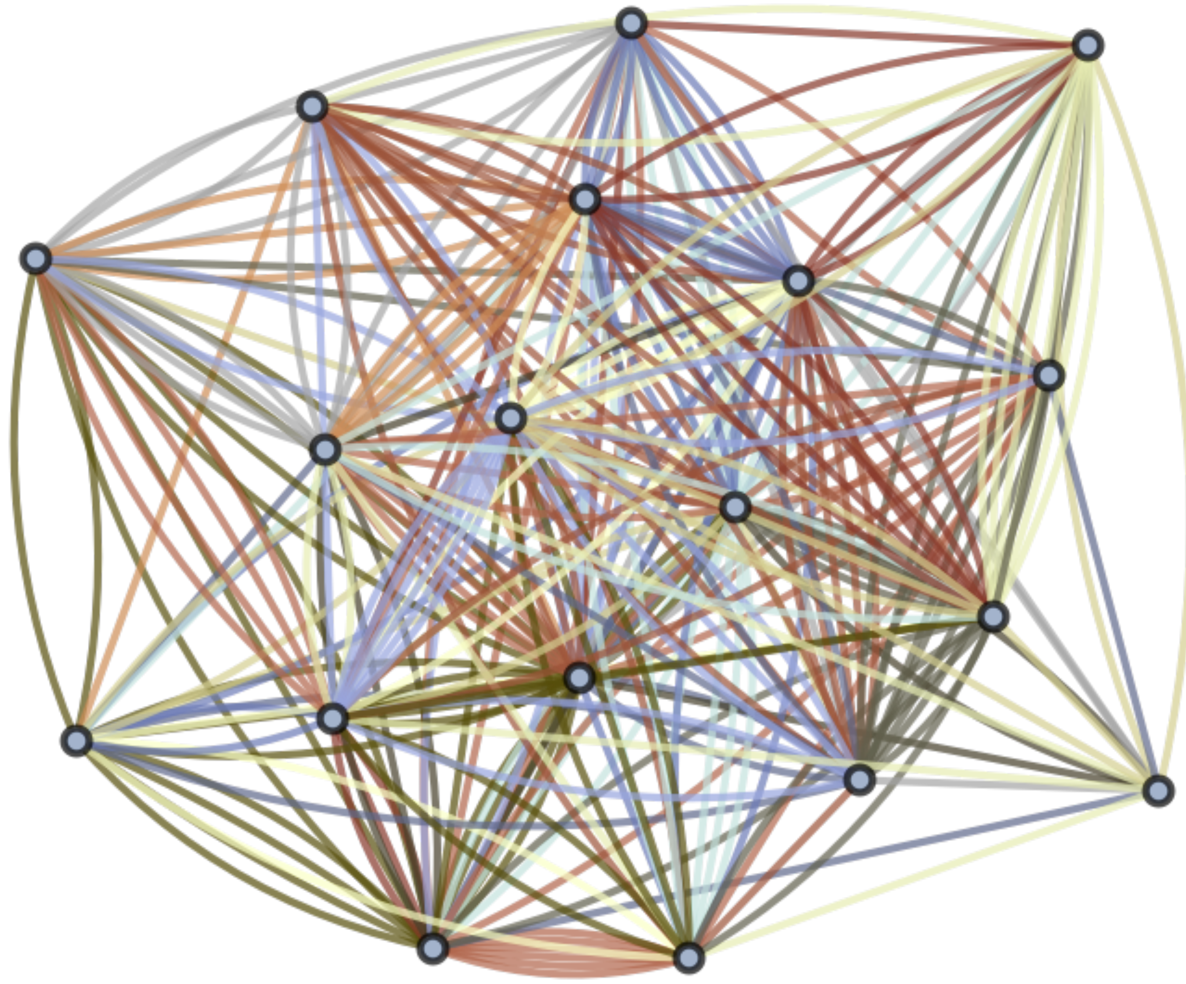
$$D(E) \sim e^{S(E)}$$

$$= e^{N s_0 + \sqrt{2N\gamma E}}$$

Energy level spacing $\sim e^{-N s_0}$!

Complex SYK model

The Sachdev-Ye-Kitaev (SYK) model



The disorder-averaged partition is exactly this G-Σ theory:

$$\mathcal{Z} = \int \mathcal{D}G(\tau_1, \tau_2)\mathcal{D}\Sigma(\tau_1, \tau_2) \exp(-NI)$$

$$I = \ln\det\left[\delta(\tau_1 - \tau_2)(\partial_{\tau_1} + \mu) - \Sigma(\tau_1, \tau_2)\right] + \int d\tau_1 d\tau_2 \left[\Sigma(\tau_1, \tau_2)G(\tau_2, \tau_1) + (U^2/2)G^2(\tau_2, \tau_1)G^2(\tau_1, \tau_2)\right]$$

Saddle-point equations for $G(\tau_1 - \tau_2)$ and $\Sigma(\tau_1 - \tau_2)$:

$$G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = -J^2 G^2(\tau) G(-\tau)$$

$$G(\tau = 0^-) = \mathcal{Q}.$$

G-Σ path integral

At frequencies $\ll U$, the time derivative in the determinant is less important, and without it the path integral is invariant under the reparametrization and gauge transformations

$$\tau = f(\sigma)$$

$$G(\tau_1,\tau_2) = [f'(\sigma_1)f'(\sigma_2)]^{-1/4}\, e^{-i\phi(\sigma_1)+i\phi(\sigma_2)}\, \widetilde{G}(\sigma_1,\sigma_2)$$

$$\Sigma(\tau_1,\tau_2) = [f'(\sigma_1)f'(\sigma_2)]^{-3/4}\, e^{-i\phi(\sigma_1)+i\phi(\sigma_2)}\, \widetilde{\Sigma}(\sigma_1,\sigma_2)$$

where $f(\sigma)$ and $\phi(\sigma)$ are arbitrary functions.

A. Georges and O. Parcollet
PRB **59**, 5341 (1999)
A. Kitaev, 2015
S. Sachdev, PRX **5,** 041025 (2015)

Time reparametrization and phase soft modes

The dominant fluctuations of the bilocal Green's function $G(\tau_1, \tau_2)$ involve a deformation of the conformal Green's function by a time reparameterization and a gauge transformation.

$$G(\tau_1, \tau_2) = [f'(\tau_1) f'(\tau_2)]^{1/4} G_*(f(\tau_1) - f(\tau_2)) e^{i\phi(\tau_1) - i\phi(\tau_2)}$$

Then the path integral is approximated by

$$\mathcal{Z} = \int \mathcal{D}f(\tau) \mathcal{D}\phi(\tau) e^{-E_0/T + N s_0 - N I_{\text{eff}}[f, \phi]} \,,$$

where $E_0 \propto N$ is the ground state energy.

A. Kitaev (2015); J. Maldacena and D. Stanford, arXiv:1604.07818;
R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, arXiv.1612.00849;
S. Sachdev, PRX **5,** 041025 (2015)

$$
\begin{aligned}
\mathcal{Z} &= \mathrm{Tr}\exp\left(-\frac{\mathcal{H}}{k_B T}\right) \\
&\approx \exp\left(N\frac{s_0}{k_B}\right)\int\frac{\mathcal{D}f(\tau)\mathcal{D}\phi(\tau)}{||\mathrm{SL(2,R)}||}\exp\left(-\frac{1}{\hbar}I_{\text{eff}}\left[f(\tau),\phi(\tau)\right]\right) \\
I_{\text{eff}}[f,\phi] &= \frac{NK}{2}\int_0^{1/T} d\tau(\partial_\tau\phi + i(2\pi\mathcal{E}T)\partial_\tau f)^2 - \frac{N\gamma}{4\pi^2}\int_0^{1/T} d\tau\,\{\tan(\pi T f(\tau)),\tau\},
\end{aligned}
$$

where $f(\tau)$ is a monotonic map from $[0,1/T]$ to $[0,1/T]$. The conformal group is d spatial dimensions is SO(d+2, 1), and PSL(2,R)≅SO(2,1), and the Schwarzian

$$
\{g,\tau\} \equiv \frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2 ,
$$

vanishes for $g(\tau) = (a\tau + b)/(c\tau + d)$ a SL(2,R) transformation $(ad - bc = 1)$. The couplings K, γ, and $\mathcal{E}$ can be related to thermodynamic derivatives.

A. Kitaev (2015); J. Maldacena and D. Stanford, arXiv:1604.07818;
R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, arXiv.1612.00849;
S. Sachdev, PRX **5,** 041025 (2015)

The Sachdev-Ye-Kitaev (SYK) model

$$\mathcal{Z}(\mathcal{Q},T) = \mathrm{Tr}_{\mathcal{Q}} \exp\left(-\frac{\mathcal{H}}{T}\right) = \exp(-F/T)\,; \text{ Entropy } S = -\frac{\partial F}{\partial T}.$$

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Exact path integral over time reparameterizations:

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A. Georges, O. Parcollet, and S. Sachdev,
PRB **63**, 134406 (2001)

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J. S. Cotler et al., JHEP 05 (2017) 118

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Yingfei Gu, A. Kitaev, S. Sachdev, and G. Tarnopolsky, JHEP 02 (2020) 157

A. Georges, O. Parcollet, and S. Sachdev, PRB **63**, 134406 (2001)

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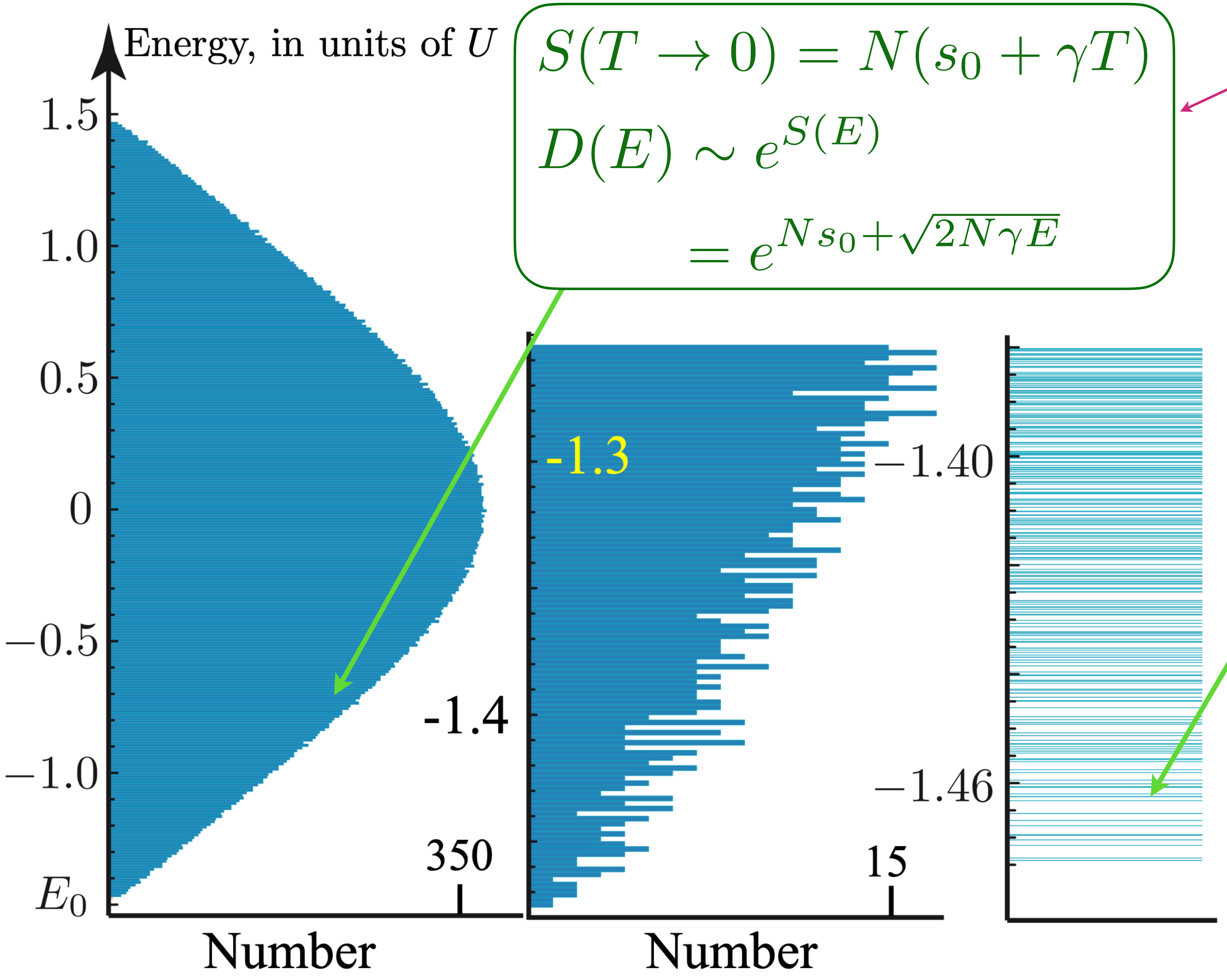

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$$D(E) \sim N^{-1}\, e^{N s_0} \sinh(\sqrt{2N\gamma E})$$

$$D(E) \sim e^{N s_0} \sqrt{2\gamma E/N}$$

Complex SYK model

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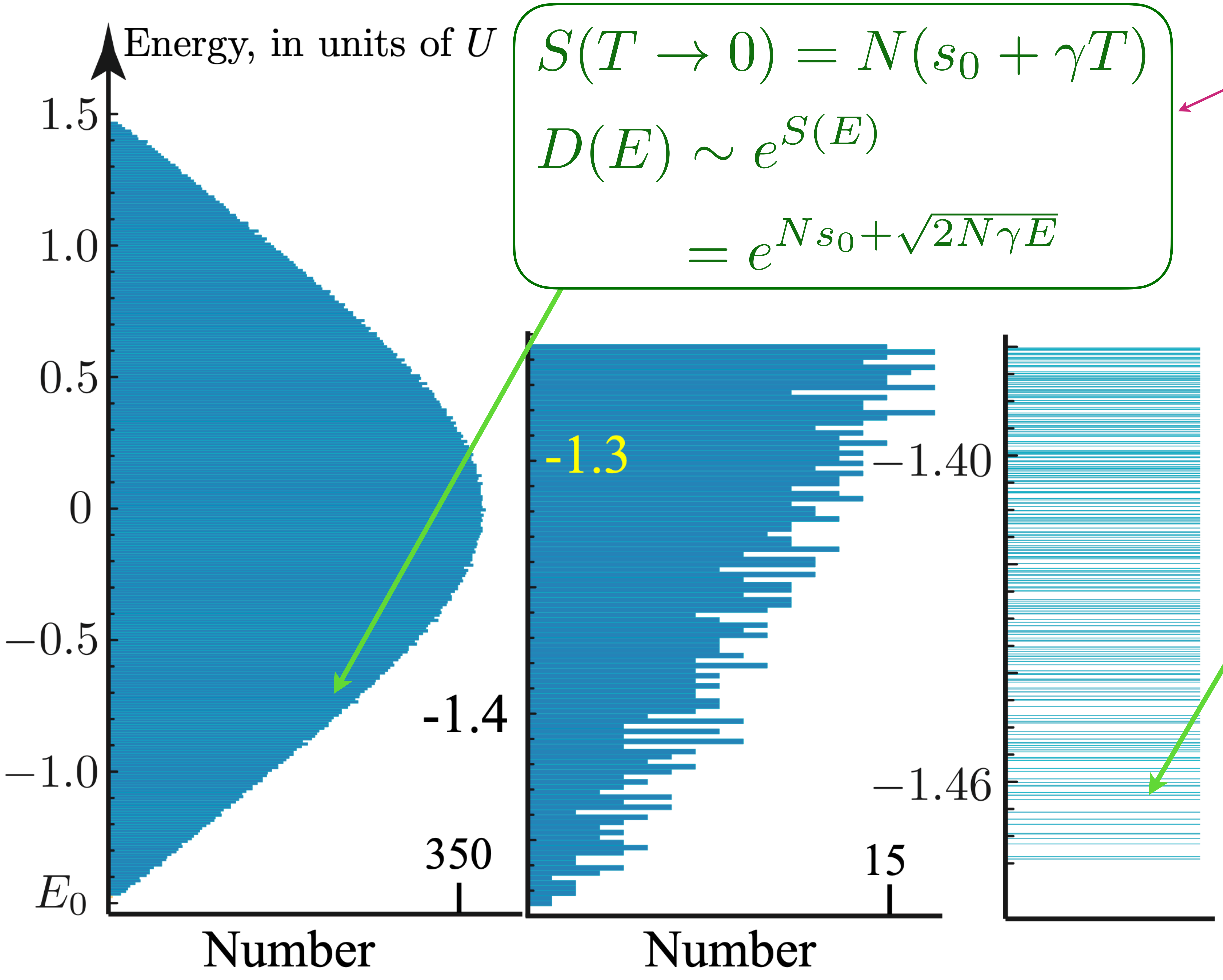


$$D(E) \sim e^{N s_0} \sqrt{2 \gamma E / N}$$

No exponentially large degeneracy, but exponentially small level spacing!

No quasiparticle decomposition: wavefunctions change chaotically from one state to the next.

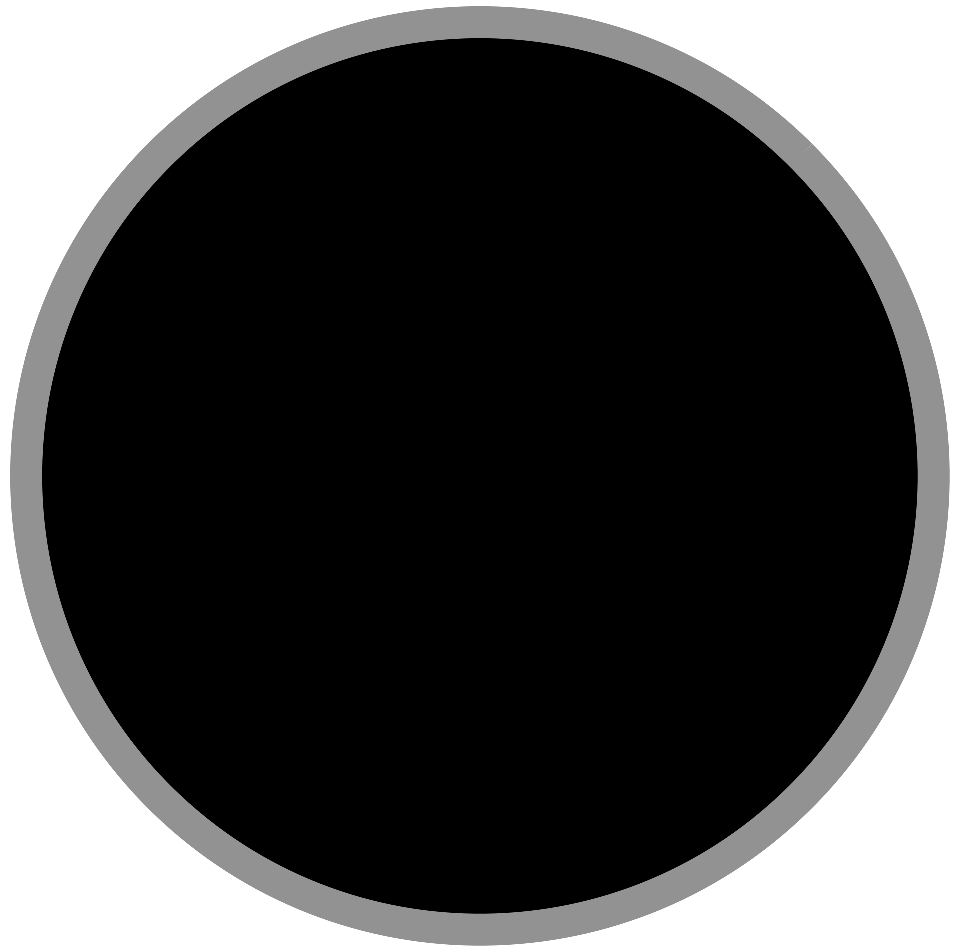
Complex SYK model

Quantum black holes

Black Holes

Objects so dense that light is gravitationally bound to them.

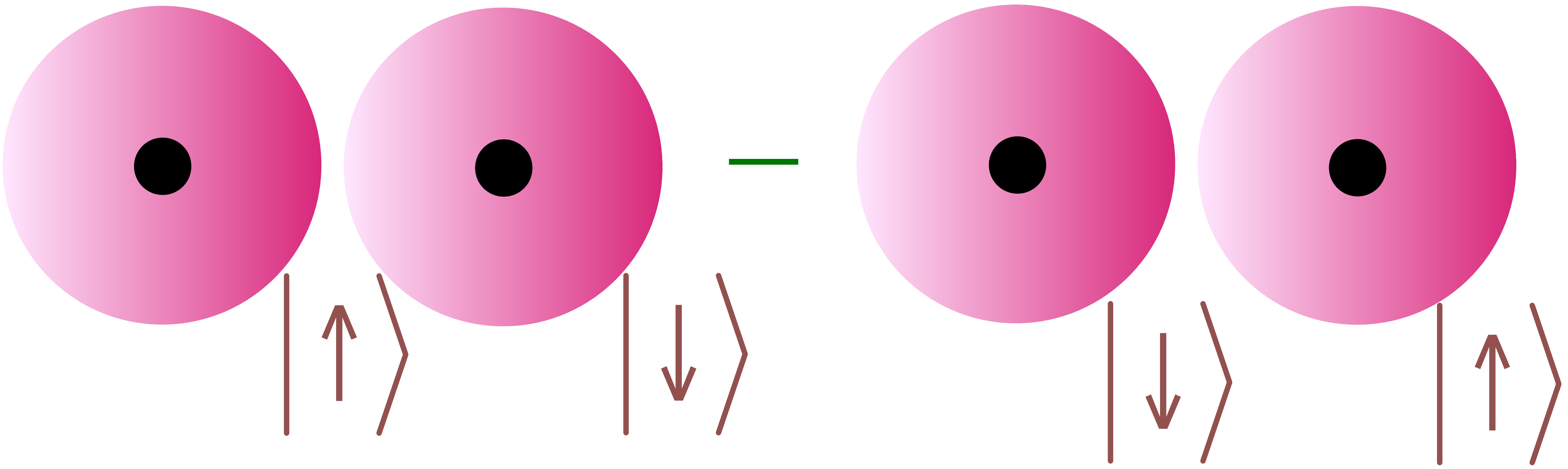
Horizon radius $R = \dfrac{2GM}{c^2}$



G Newton's constant, c velocity of light, M mass of black hole

For $M =$ earth's mass, $R \approx 9\,mm$!

Quantum Entanglement across a black hole horizon



$$|\uparrow\rangle \; |\downarrow\rangle \; - \; |\downarrow\rangle \; |\uparrow\rangle$$

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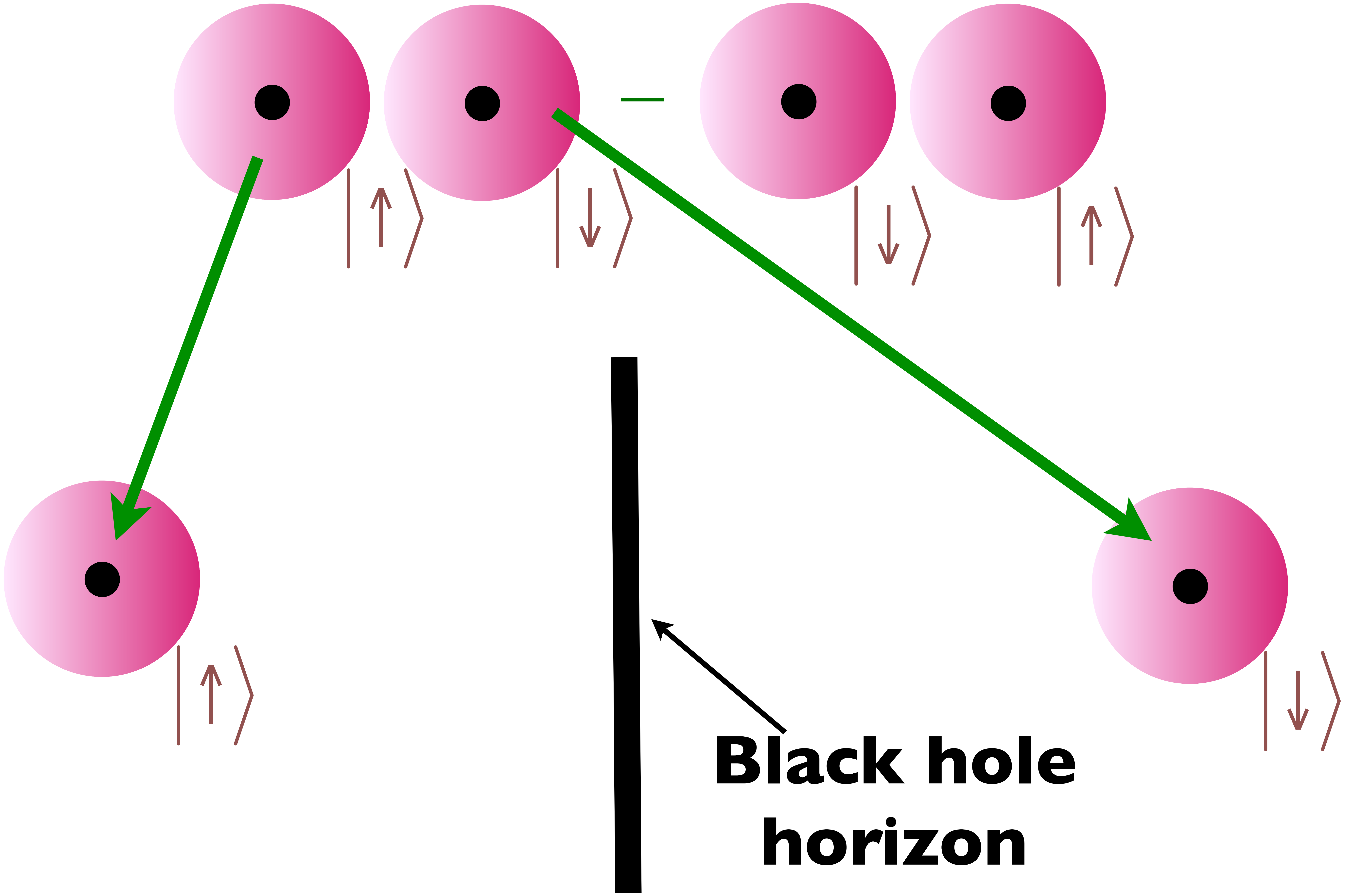

Black hole horizon

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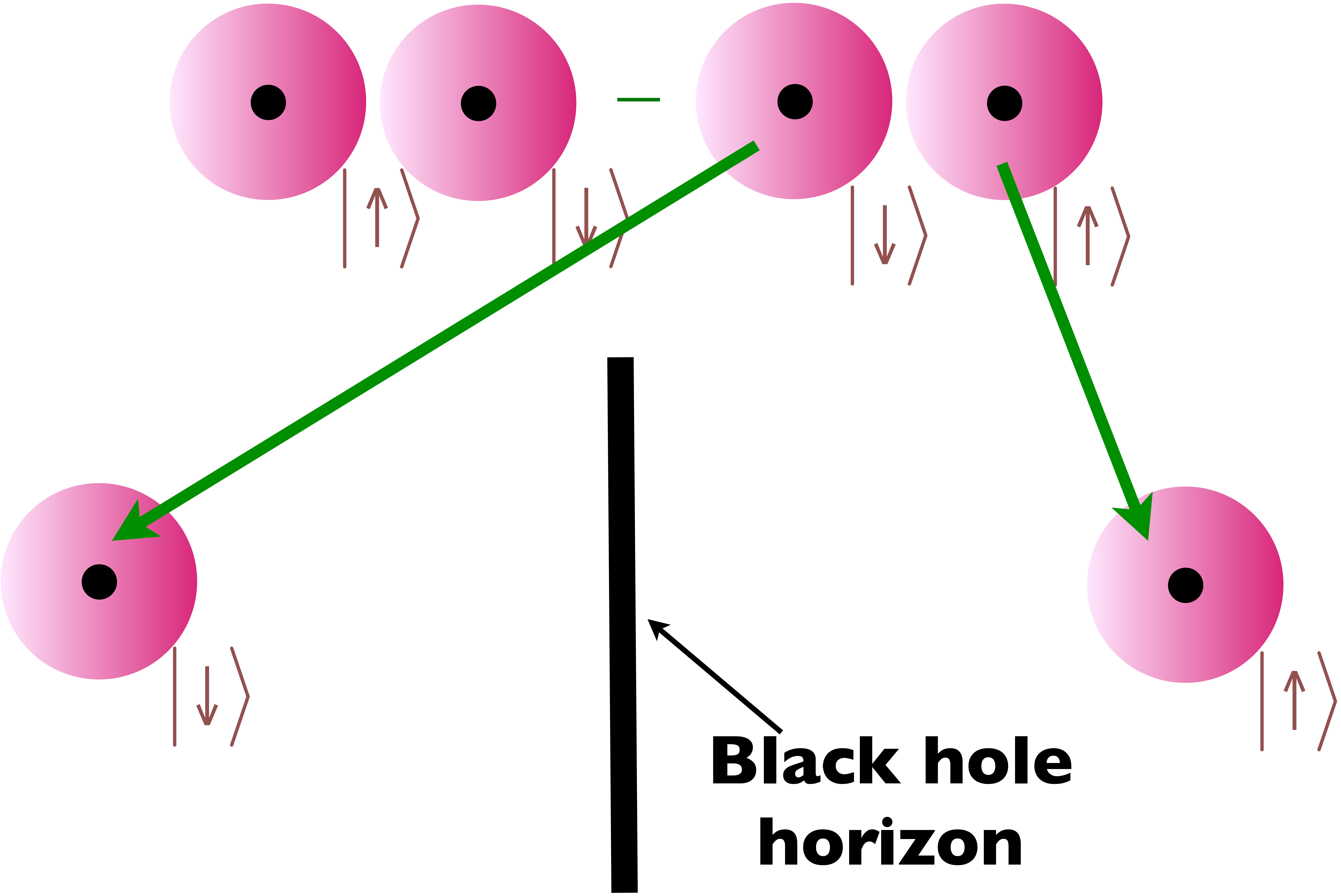
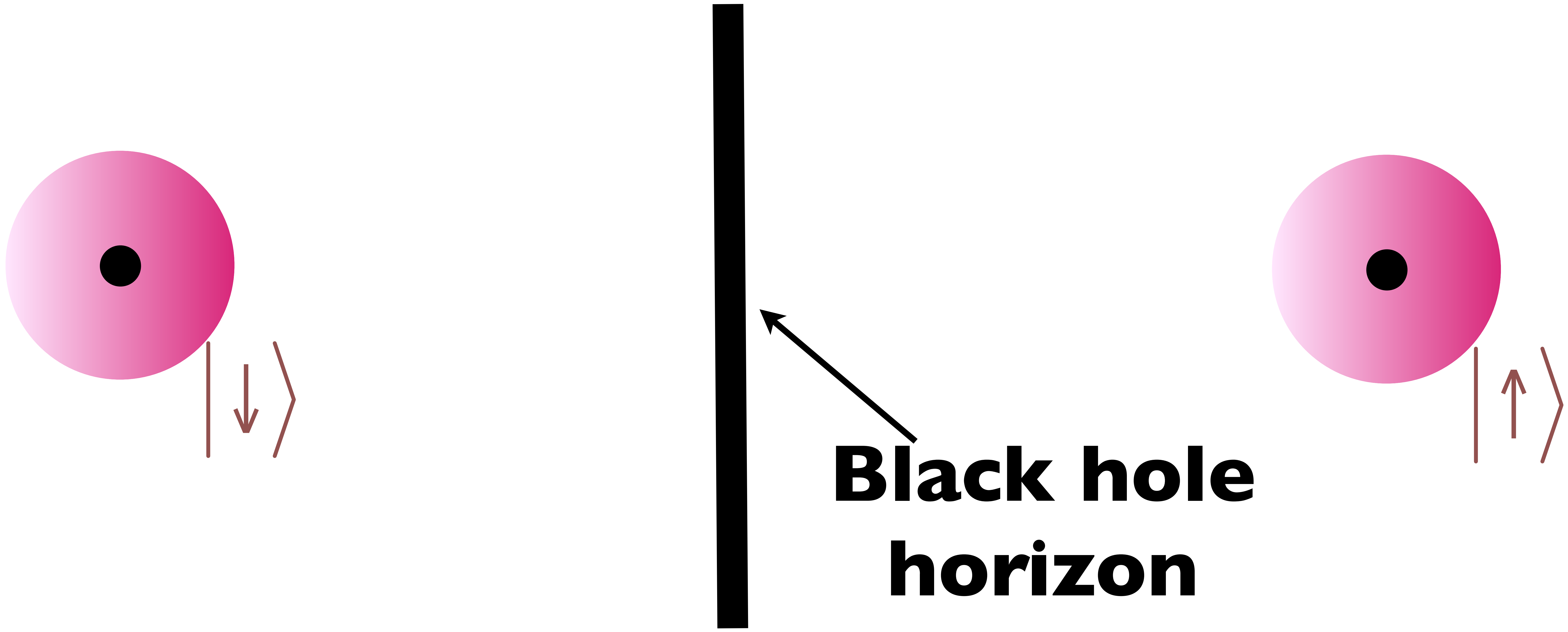


Black hole horizon

Quantum Entanglement across a black hole horizon

Hawking (1975) used other arguments to show that black hole horizons have a temperature

(The entanglement reasoning: to an outside observer, the state of the electron inside the black hole cannot be known, and so the outside electron is in a random state.)



Quantum black holes

- Black holes have an entropy and a temperature, $T_H = \hbar c^3/(8\pi G M k_B)$.
- The entropy is proportional to their surface area. $S = A k_B c^3/(4G\hbar)$.

J. D. Bekenstein, PRD **7**, 2333 (1973)
S. W. Hawking, Nature **248**, 30 (1974)

Quantum black holes

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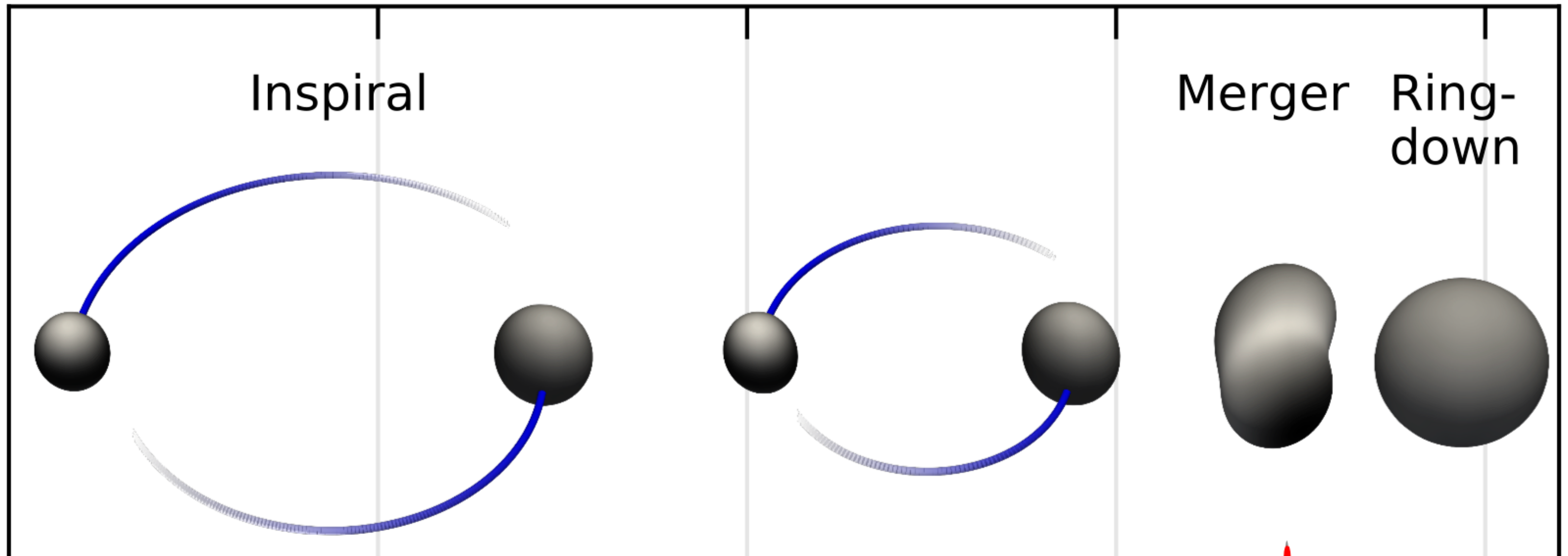
- The entropy is proportional to their surface area. $S = A k_B c^3/(4G\hbar)$.
- They relax to thermal equilibrium in a time $\sim 8\pi GM/c^3 = \hbar/(k_B T_H)$ which is Planckian!



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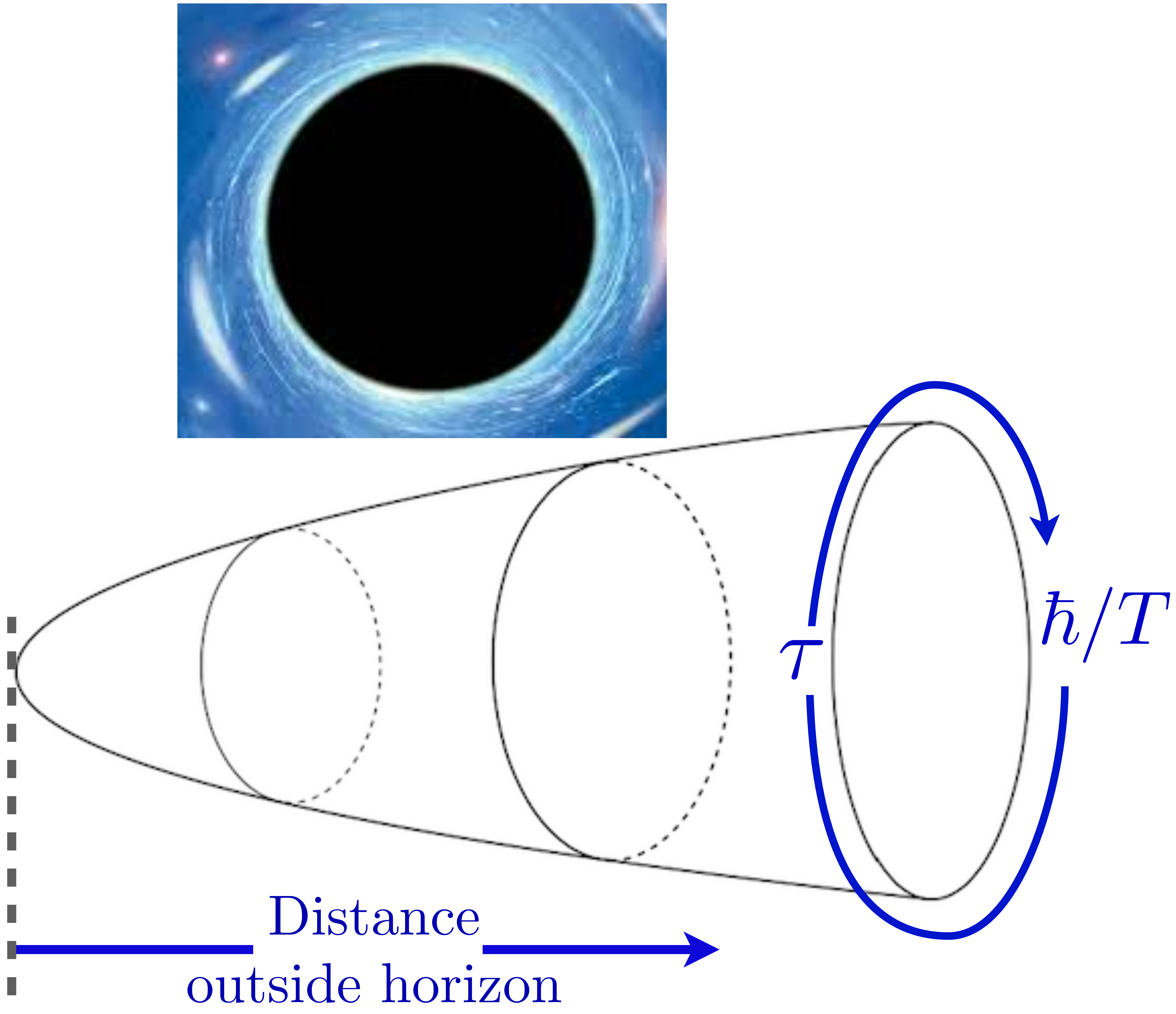
Thermodynamics of quantum black holes with charge $\mathcal{Q}$:

$$\mathcal{Z}(\mathcal{Q},T) = \int \mathcal{D}g_{\mu\nu}\mathcal{D}A_\mu \exp\left(-\frac{1}{\hbar}\, I^{(3+1)}_{\text{Einstein gravity+Maxwell EM}}[g_{\mu\nu}, A_\mu]\right)$$



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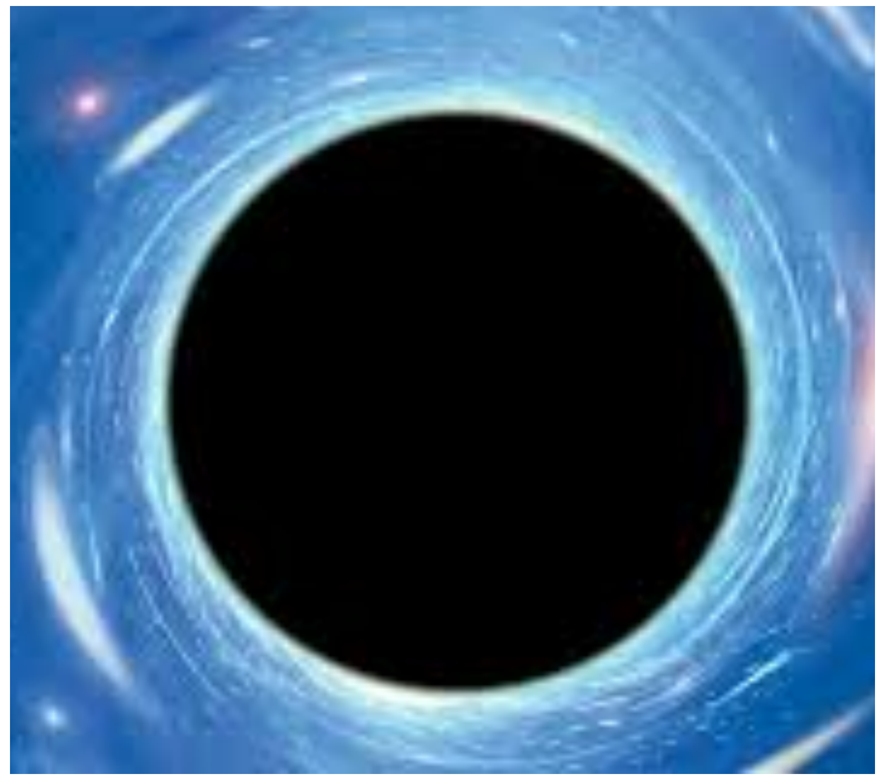
$$= \exp\Big(S_{BH}\Big) \times \left(\quad ....????... \quad \right)$$

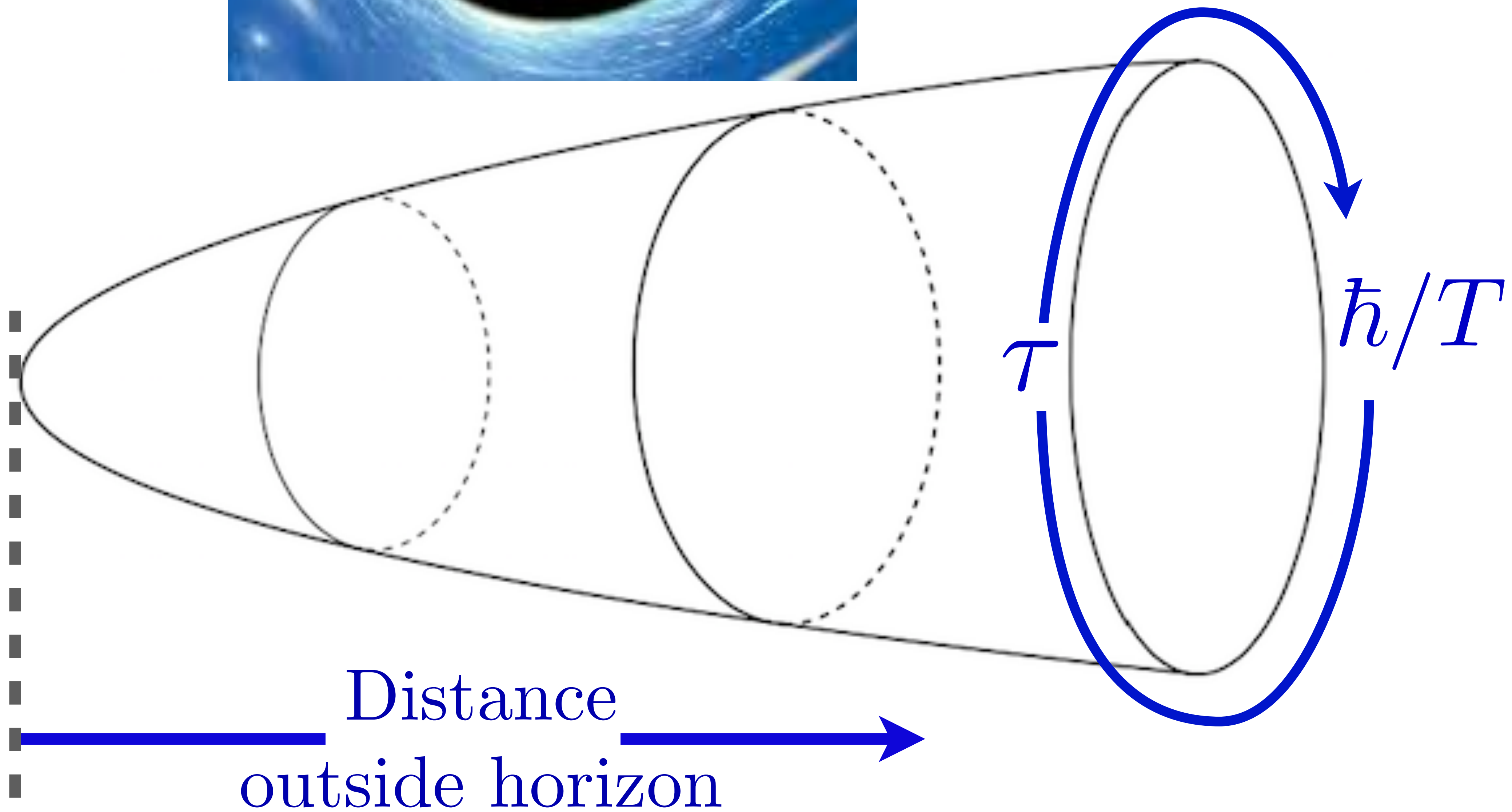
Gibbons, Hawking (1977)
Chambin, Emparan, Johnson, Myers (1999)



$$S_{BH}(T \to 0, \mathcal{Q}) = \frac{A(T)c^3}{4G\hbar} = \frac{A_0 c^3}{4G\hbar}\left(1 + \frac{2(\pi A_0)^{1/2}T}{\hbar c}\right)$$

$A_0 = 2G\mathcal{Q}^2/c^4$ is the area of the charged black hole horizon at $T = 0$.

Obtained from the saddle-point of the gravity path integral in the imaginary time spacetime outside the black hole.



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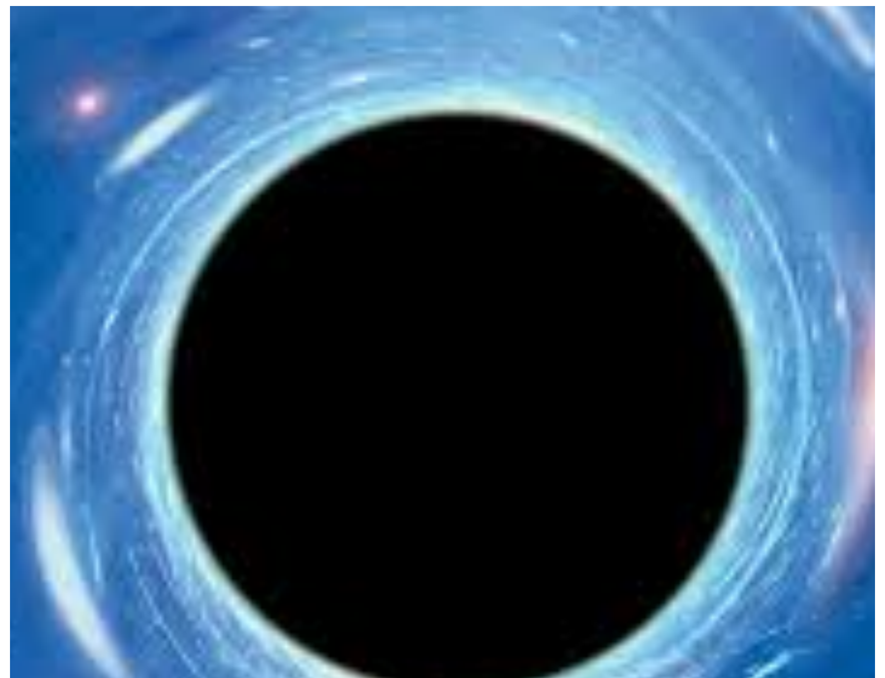

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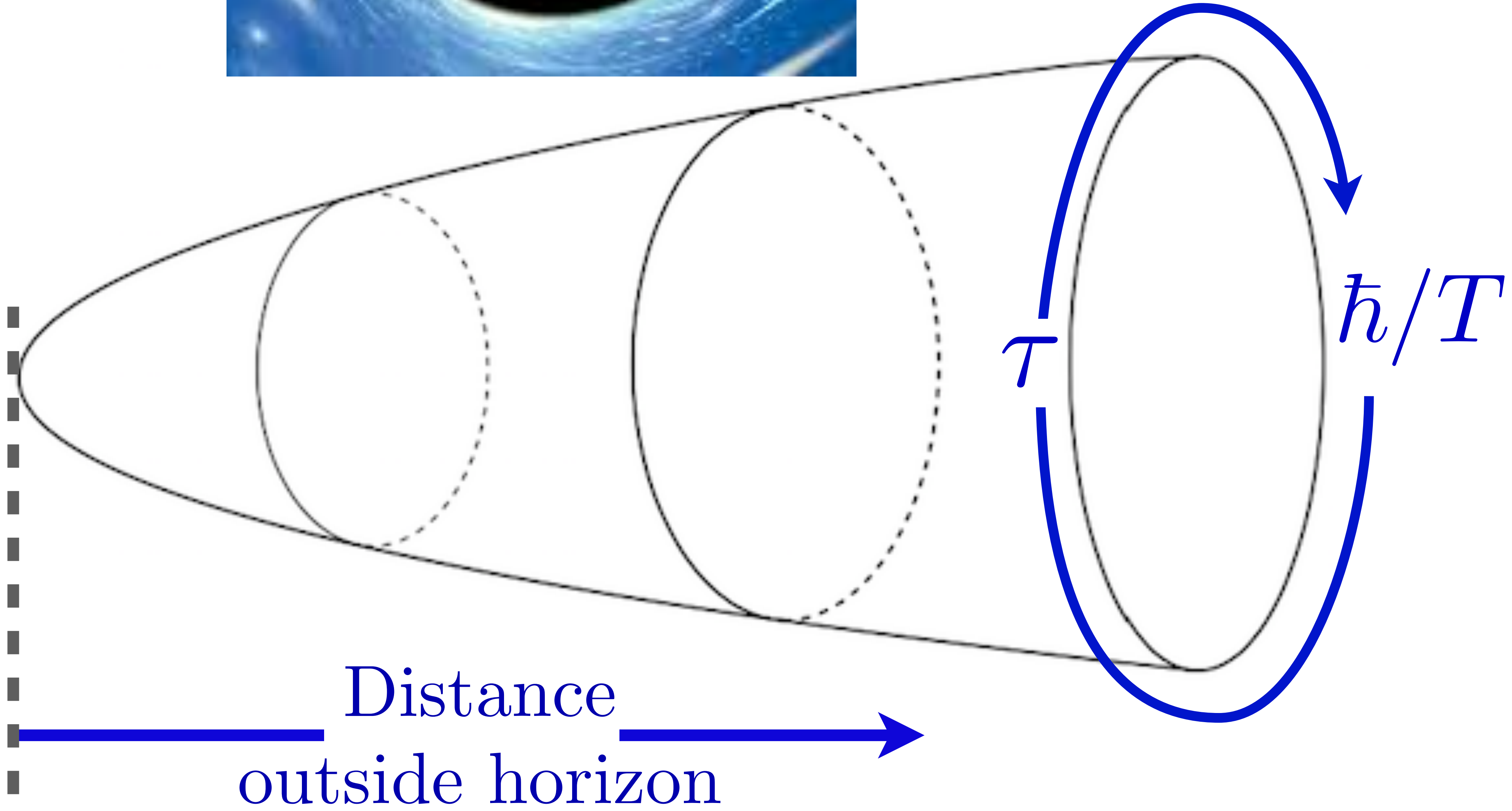
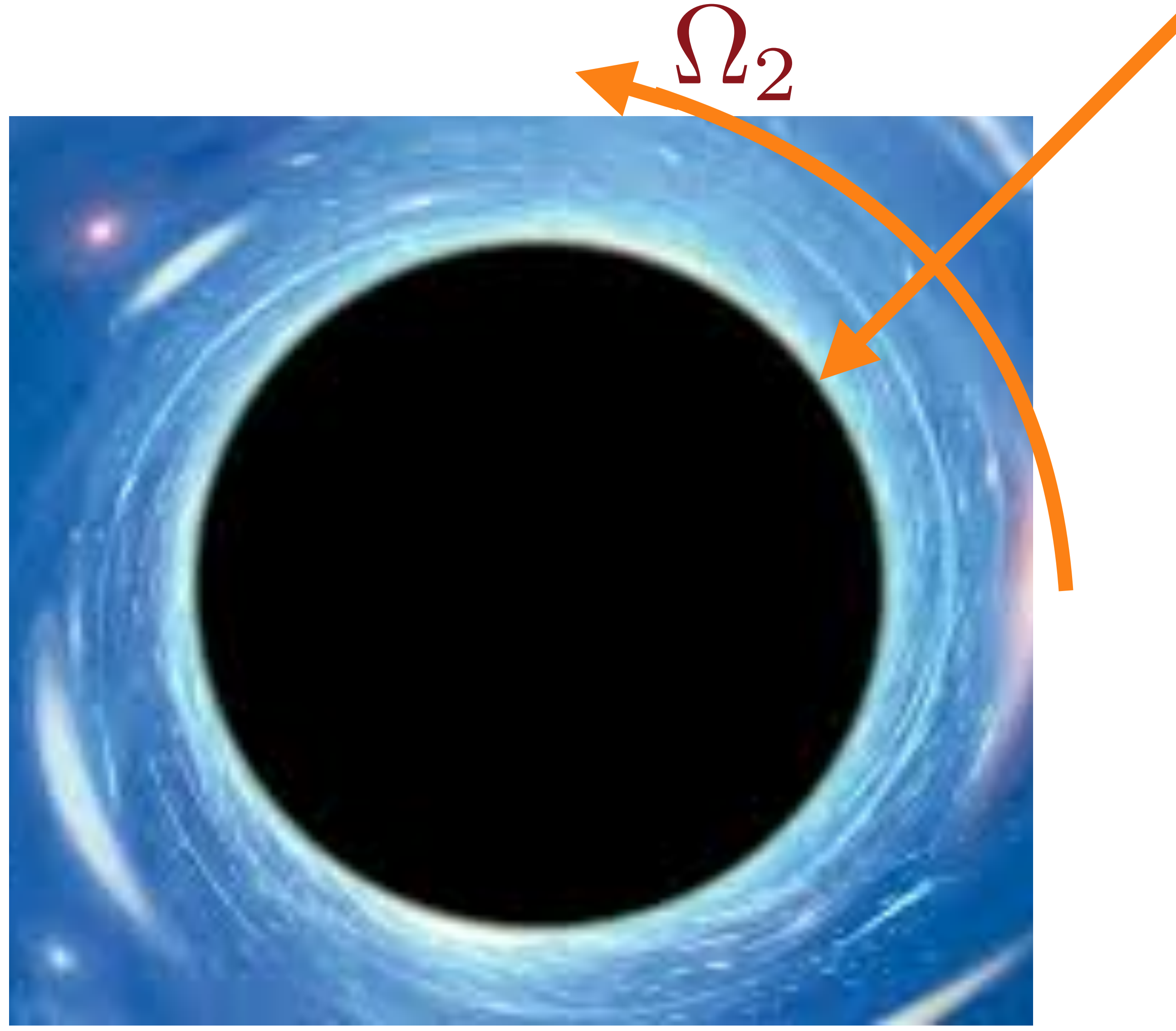
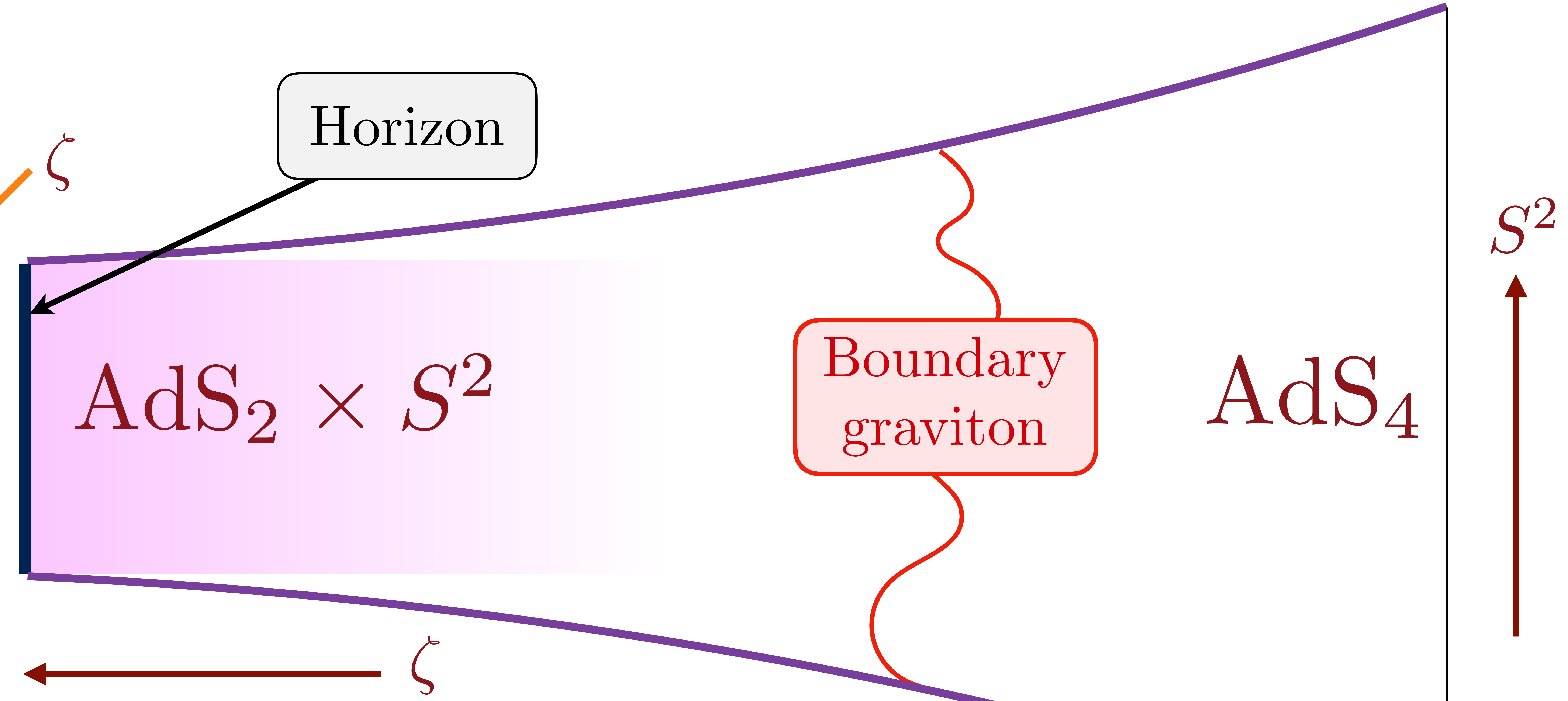


Note the similarity to the large *N* entropy of the SYK model !
(along with other similarities)

Sachdev PRL 2010

Reissner-Nordstrom black hole of Einstein-Maxwell theory





Dimensional reduction from 3+1 dimensions to 1+1 dimensions (AdS_2) at low energies!

The isometry group of AdS_2 is the 0+1 dimensional conformal group SL(2,R).

Faulkner, Liu, McGreevy, Vegh 2009; Sachdev 2010

Thermodynamics of quantum black holes with charge $\mathcal{Q}$:

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Maldacena, Stanford, Yang (2016)

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$$= \int \mathcal{D}f(\tau)\mathcal{D}\phi(\tau)\exp\left(-\frac{1}{\hbar}I_{\text{SYK}}[\text{time reparameterizations } f(\tau), \text{ phase rotations } \phi(\tau)]\right)$$

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Sachdev (2010); Kitaev (2015); Sachdev (2015); Bagrets, Altland, Kamenev (2016); Maldacena, Stanford, Yang (2016); Moitra, Trivedi, Vishal (2018) ; Gaikwad, Joshi, Mandal, Wadia (2018); Sachdev (2019); Iliesiu, Turaci (2020)

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$$S(T\to 0,\mathcal{Q}) = \frac{A(T)c^3}{4G\hbar} - \frac{3}{2}\ln\left(\frac{(\hbar c^5/G)^{1/2}}{T}\right)$$

The $\ln T$ term is the SYK/boundary-graviton correction to Bekenstein-Hawking.

There is also a

$$-\frac{559}{180}\ln\left(\frac{A_0 c^3}{\hbar G}\right)$$

term from other massless modes; Sen (2011)
Iliesiu, Murthy, Turiaci (2022)

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Black hole questions and answers

Can we find a quantum simulation of the inside of a black hole whose $D(E)$ matches the Bekenstein-Hawking entropy computed outside the black hole?

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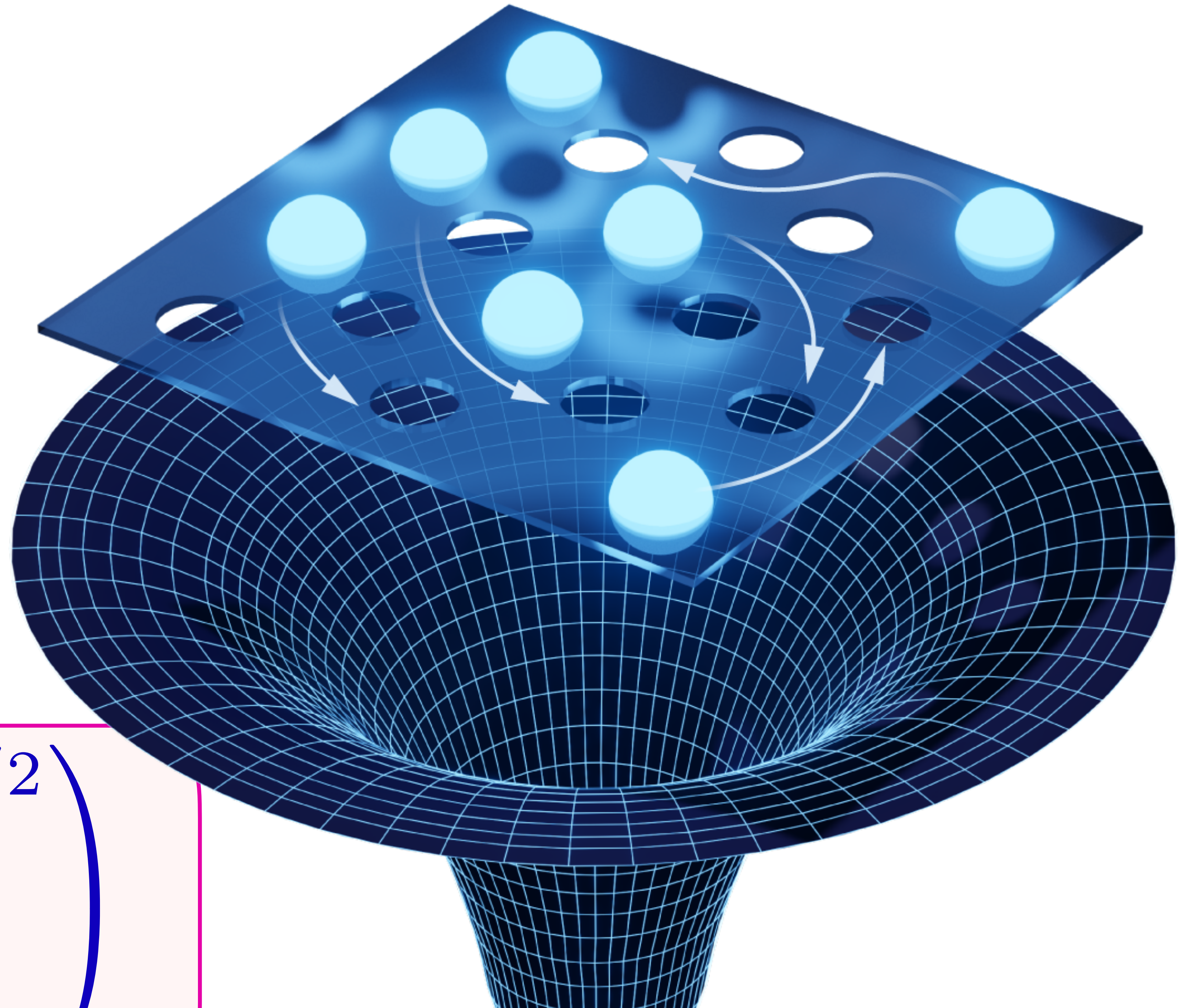
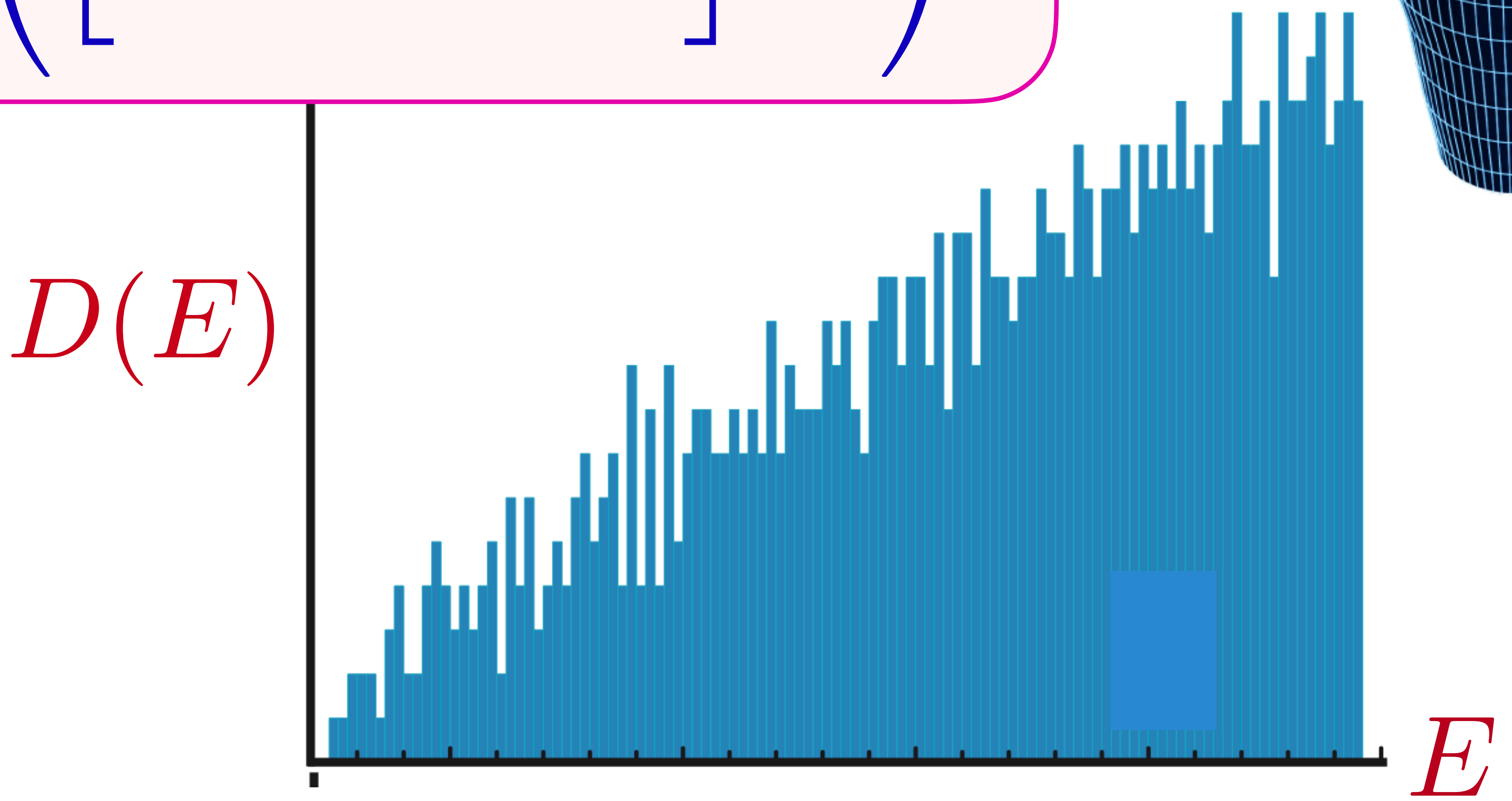


- For generic charged black holes in 3+1 dimensions, the SYK model yields, in terms of $A_0 = 2G\mathcal{Q}^2/c^4$ the horizon area at $T=0$:

$$D(E) \sim \left(\frac{A_0 c^3}{\hbar G}\right)^{-347/90} \exp\left(\frac{A_0 c^3}{4\hbar G}\right) \sinh\left(\left[\frac{\sqrt{\pi} A_0^{3/2} c^2}{\hbar^2 G} E\right]^{1/2}\right)$$

There is no degeneracy, but an exponentially small level spacing down to the ground state.



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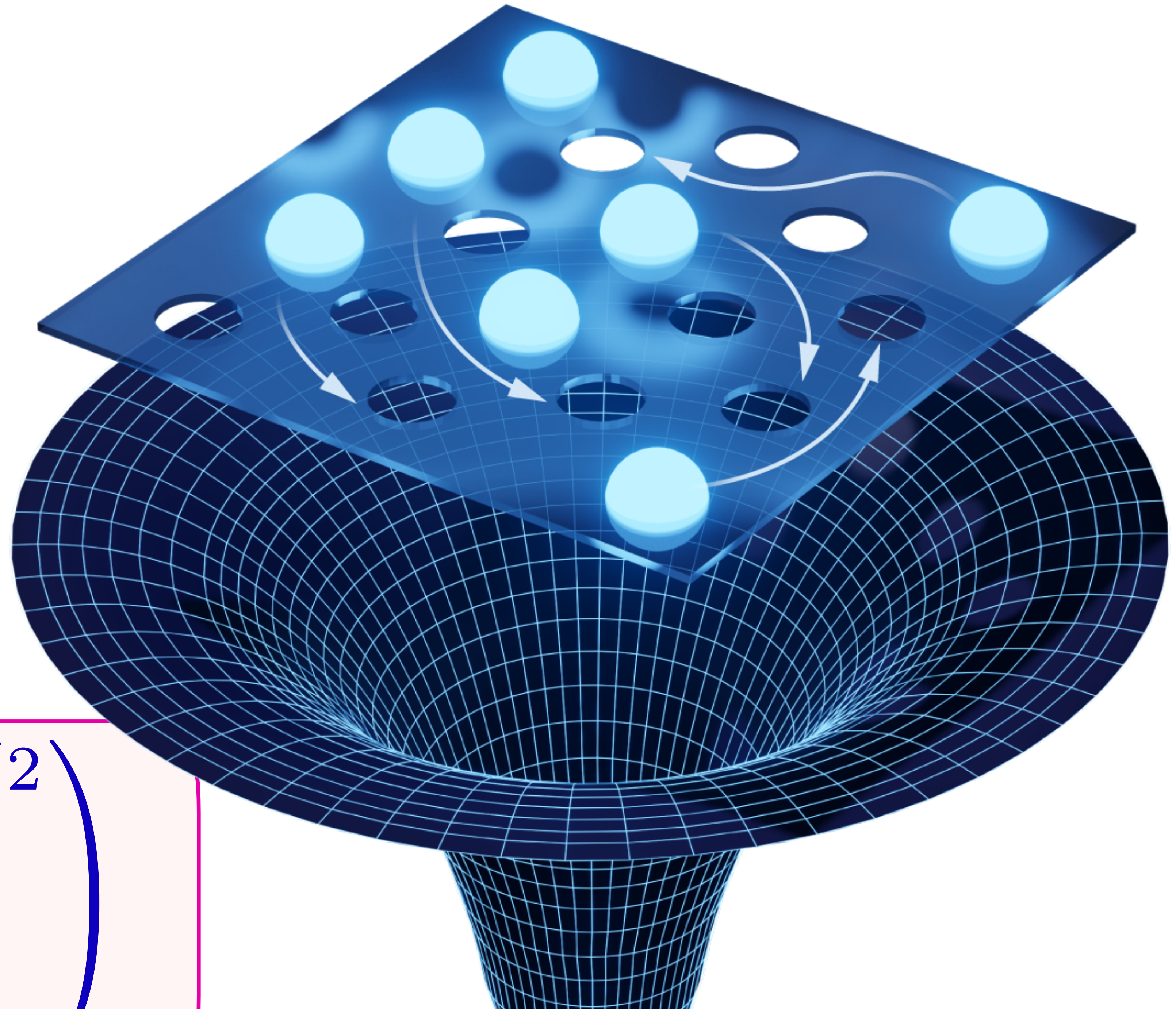

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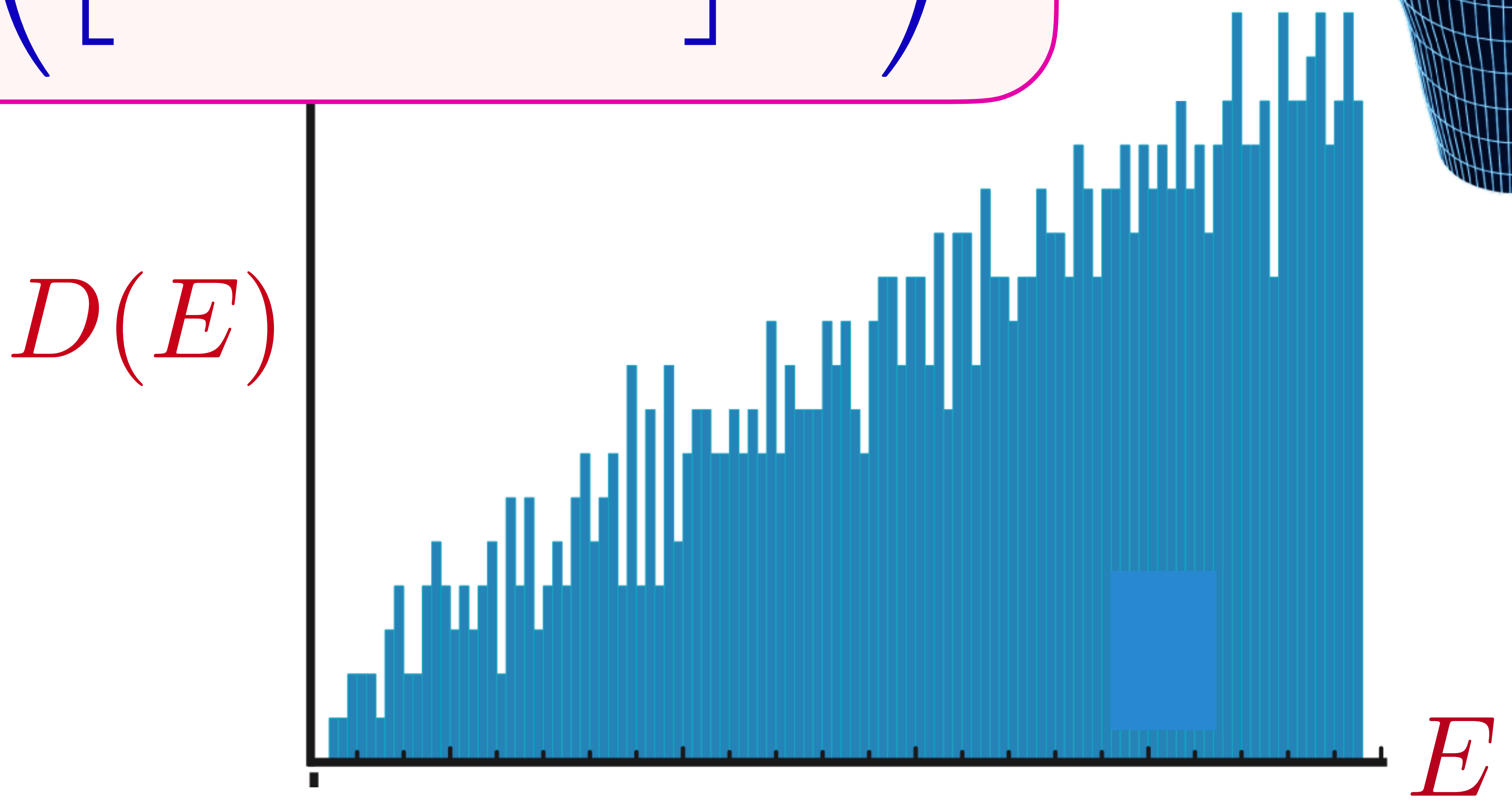

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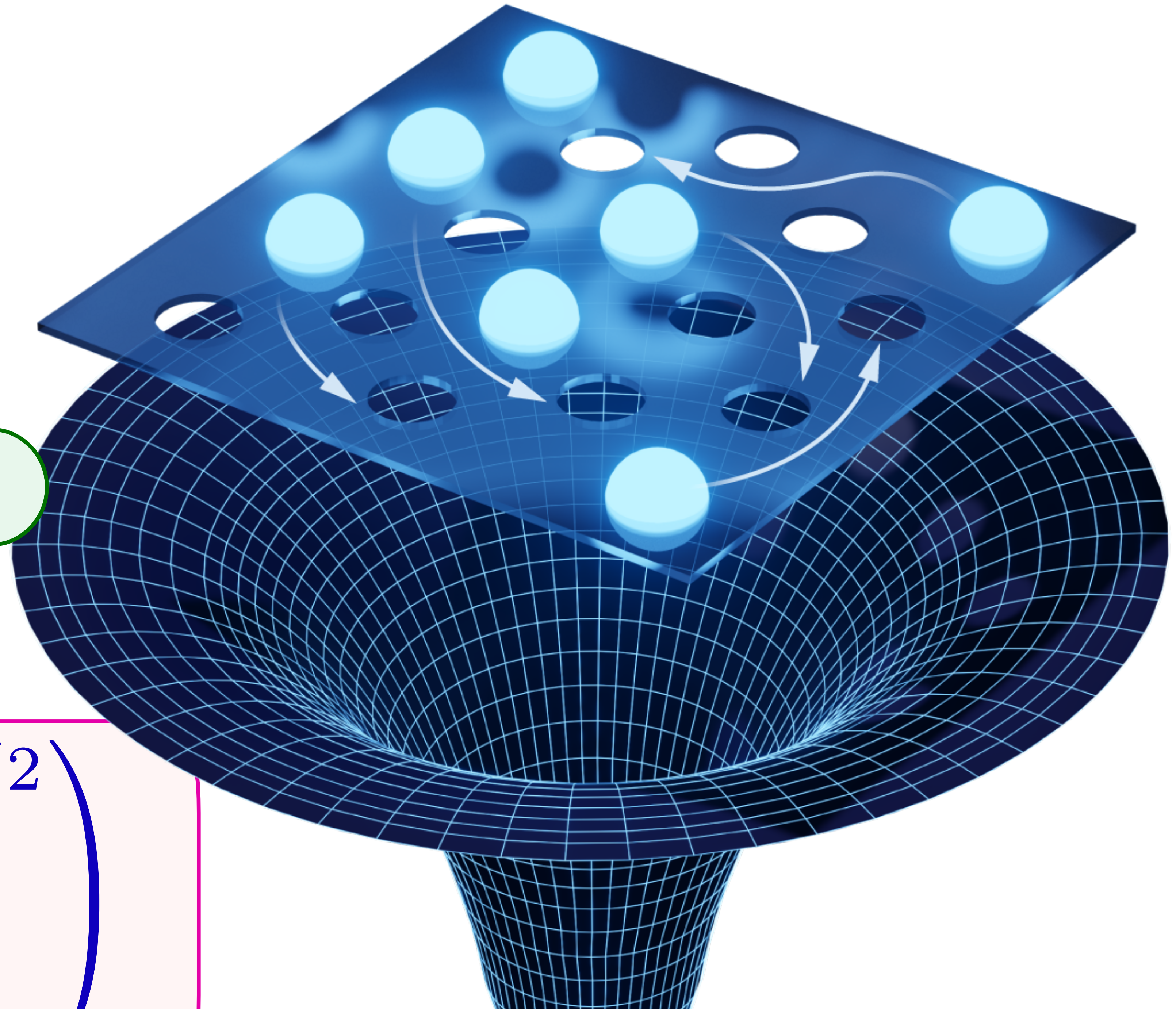

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Developments from the SYK model

Bekenstein-Hawking

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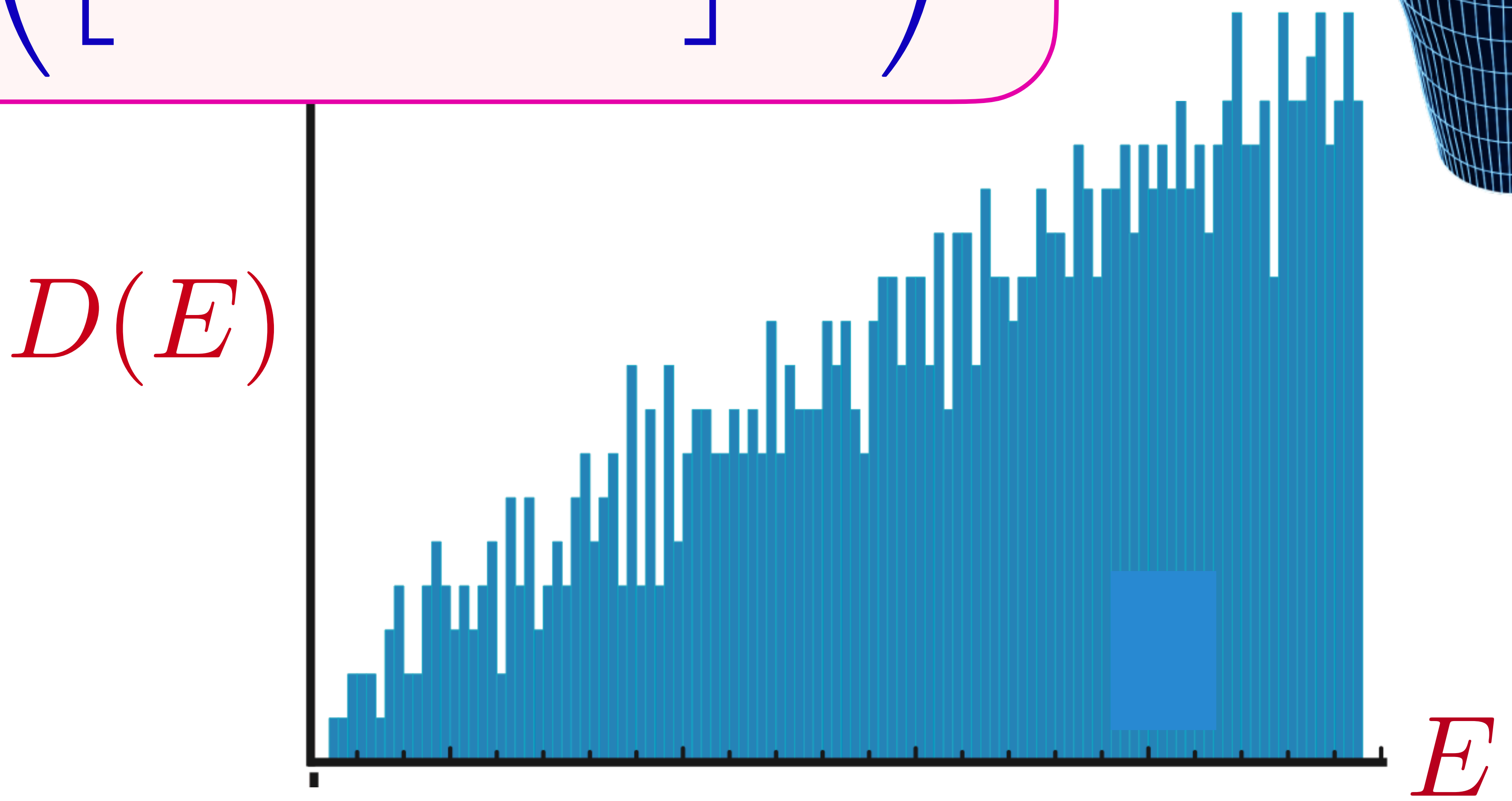

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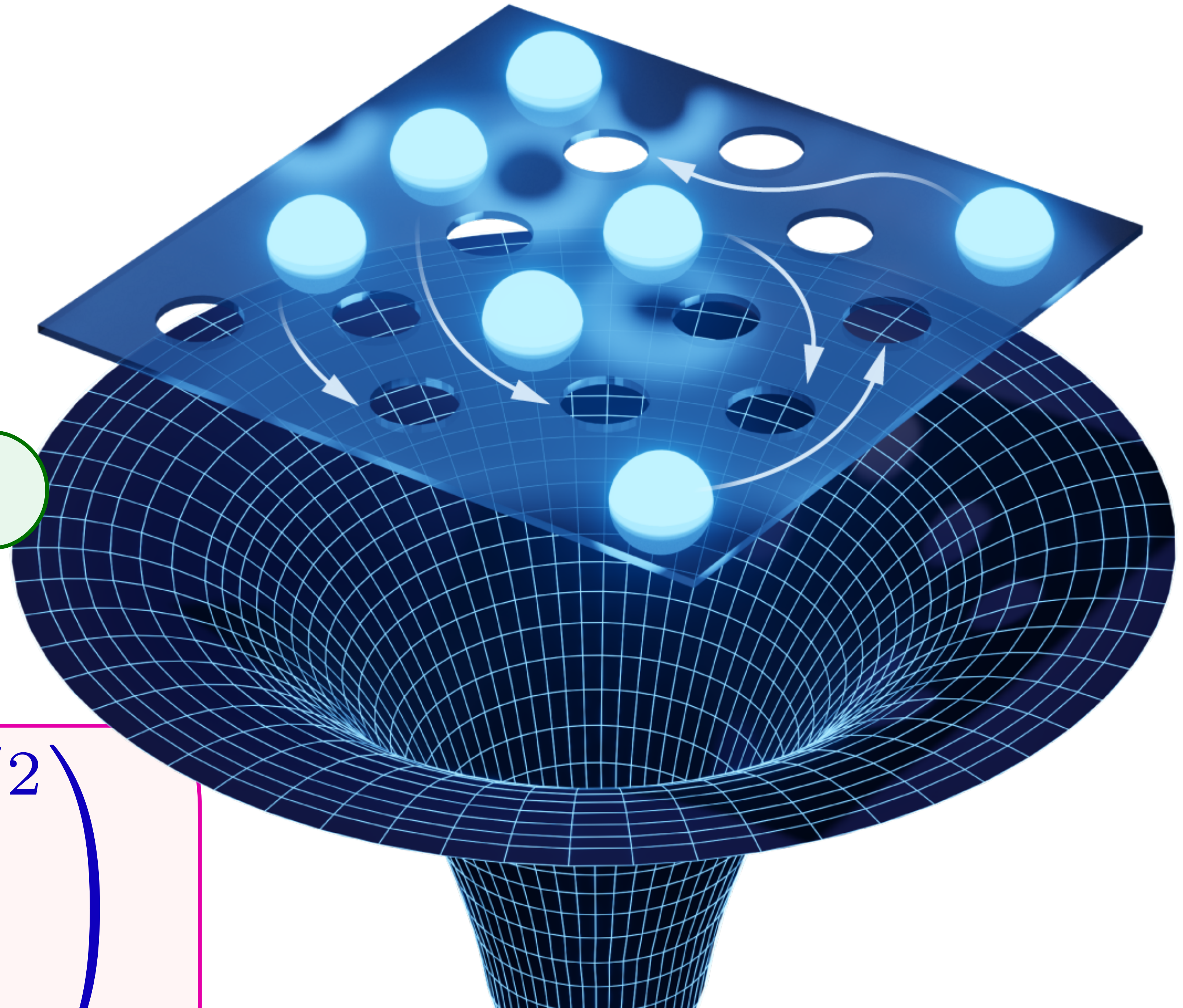

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Iliesiu, Murthy, Turiaci (2022)

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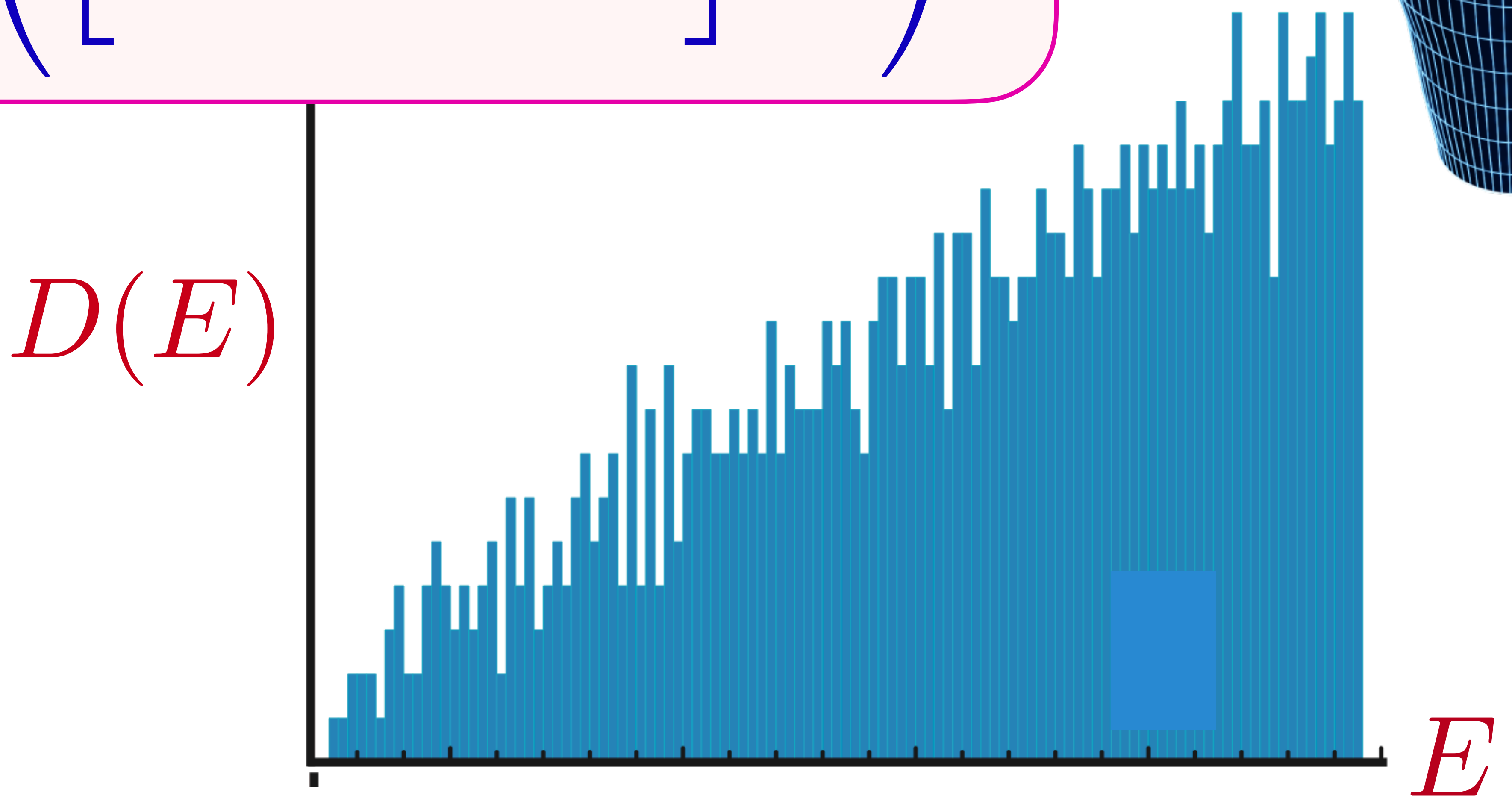

Black hole questions and answers

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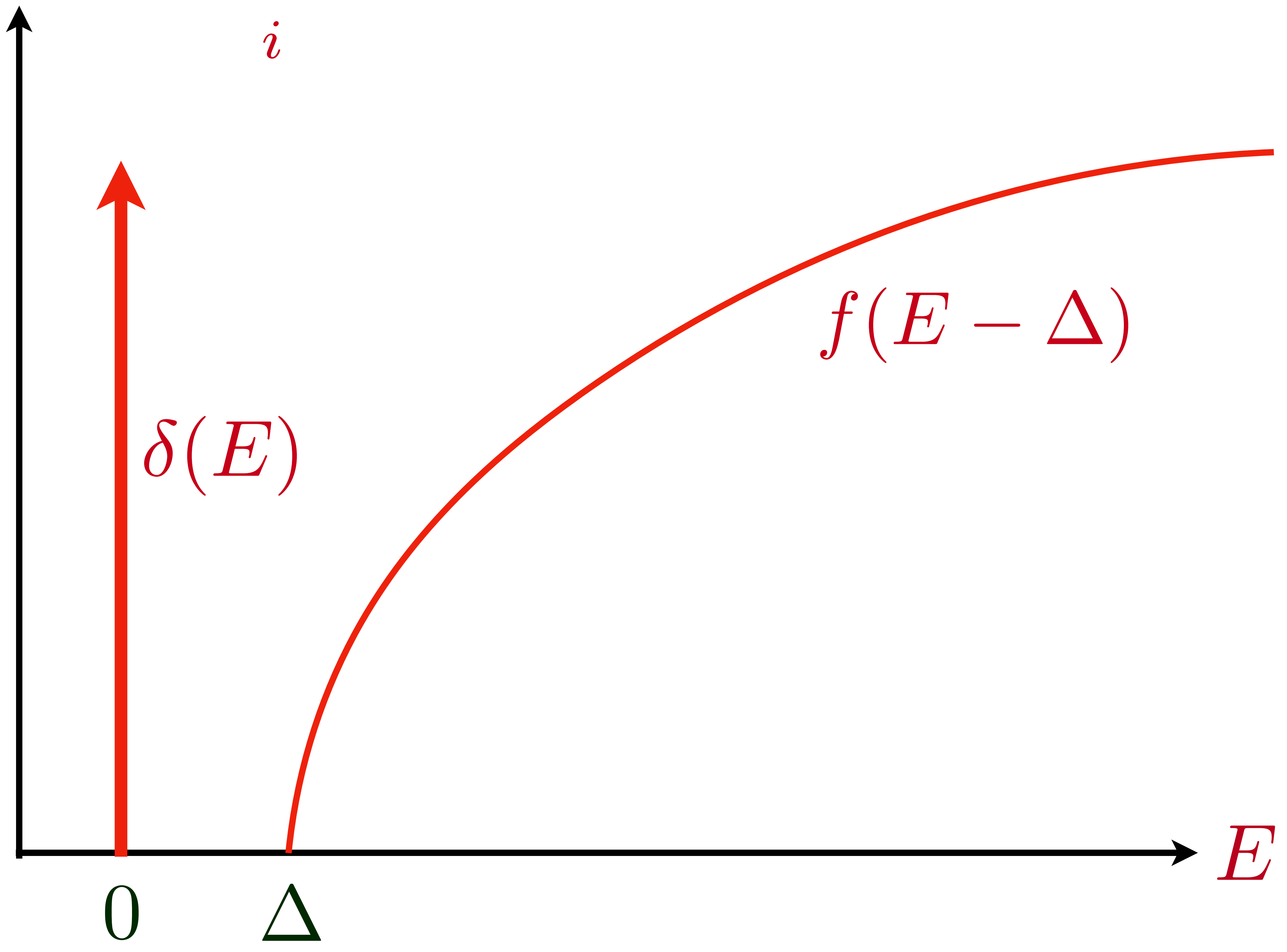
Yes, for charged black holes:

- With sufficient low energy supersymmetry, string theory yields:

$$D(E) = \exp\left(\frac{A_0 c^3}{4\hbar G}\right)\delta(E) + \theta(E-\Delta) f(E-\Delta) + \ldots$$

There are exponentially many degenerate BPS ground states, and an energy gap Δ above the ground state.

$$D(E) = \sum_i \delta(E - E_i)$$



M. Heydeman, L.V. Iliesiu, G. J. Turiaci, and W. Zhao, 2020
L.V. Iliesiu, S. Murthy, G. J. Turiaci, 2022

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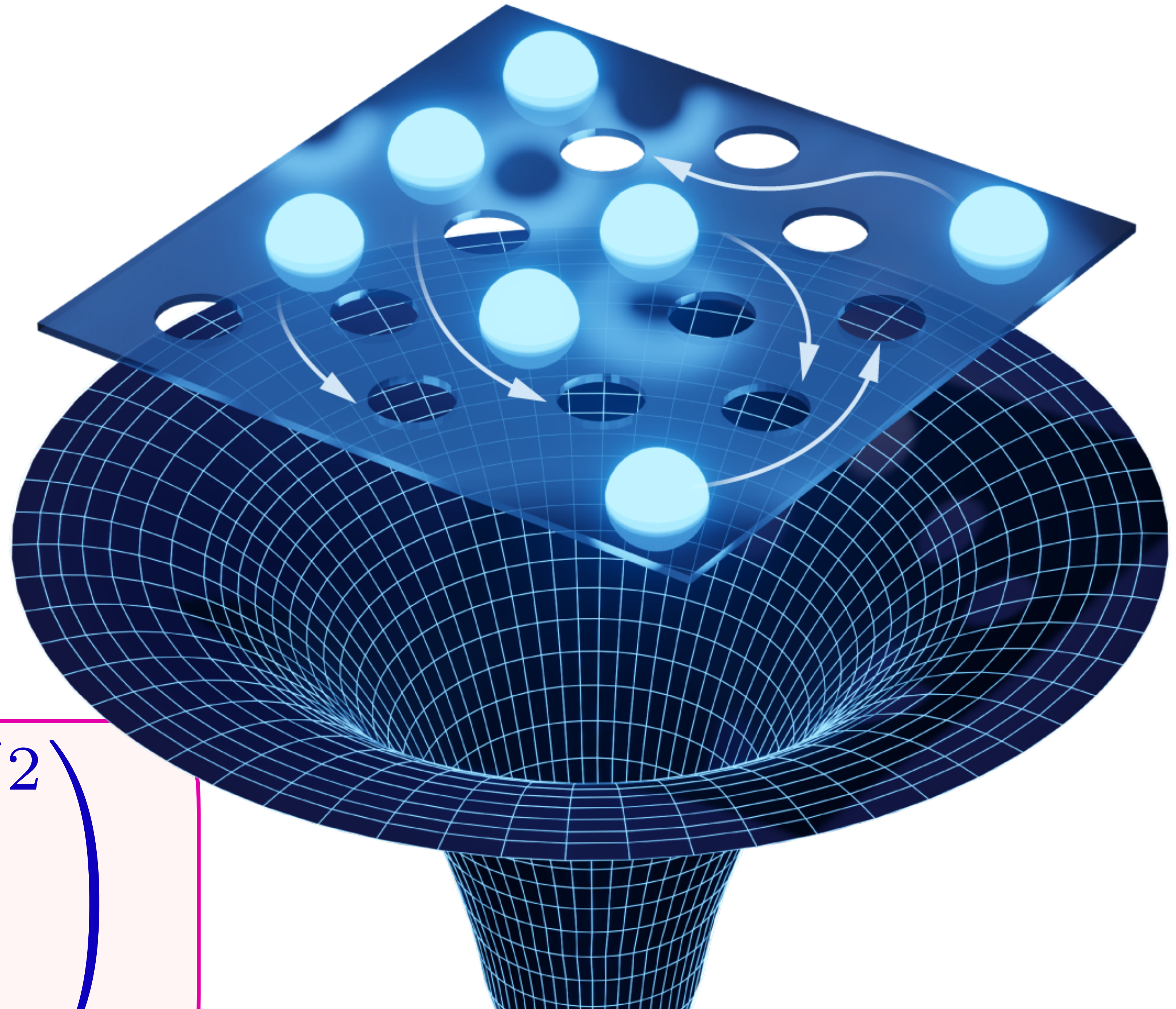

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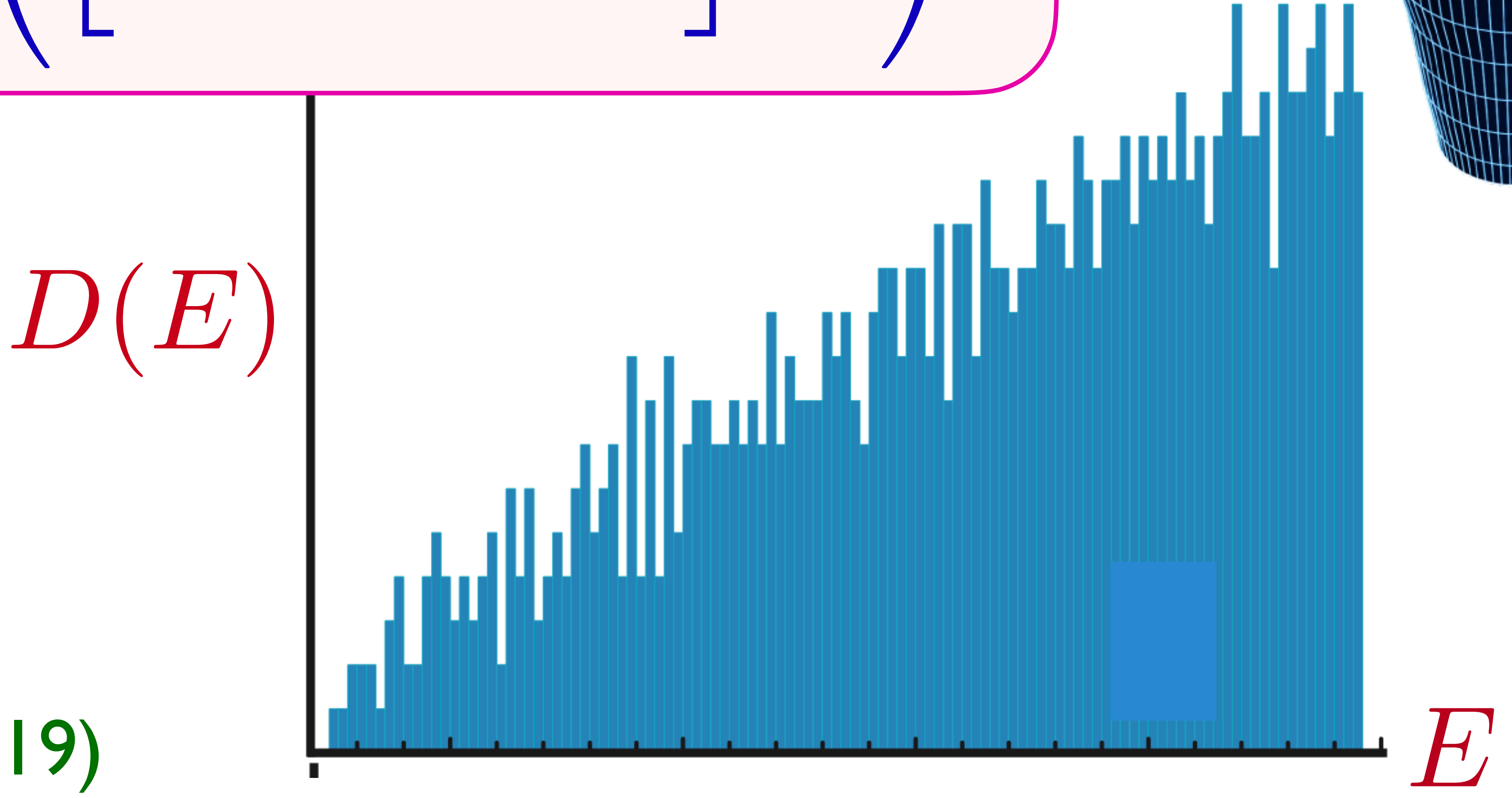
- 'Wormhole' contributions to this quantum simulation have led to an understanding of the Page curve of entanglement entropy of evaporating black holes.

Saad, Shenker, Stanford (2019)



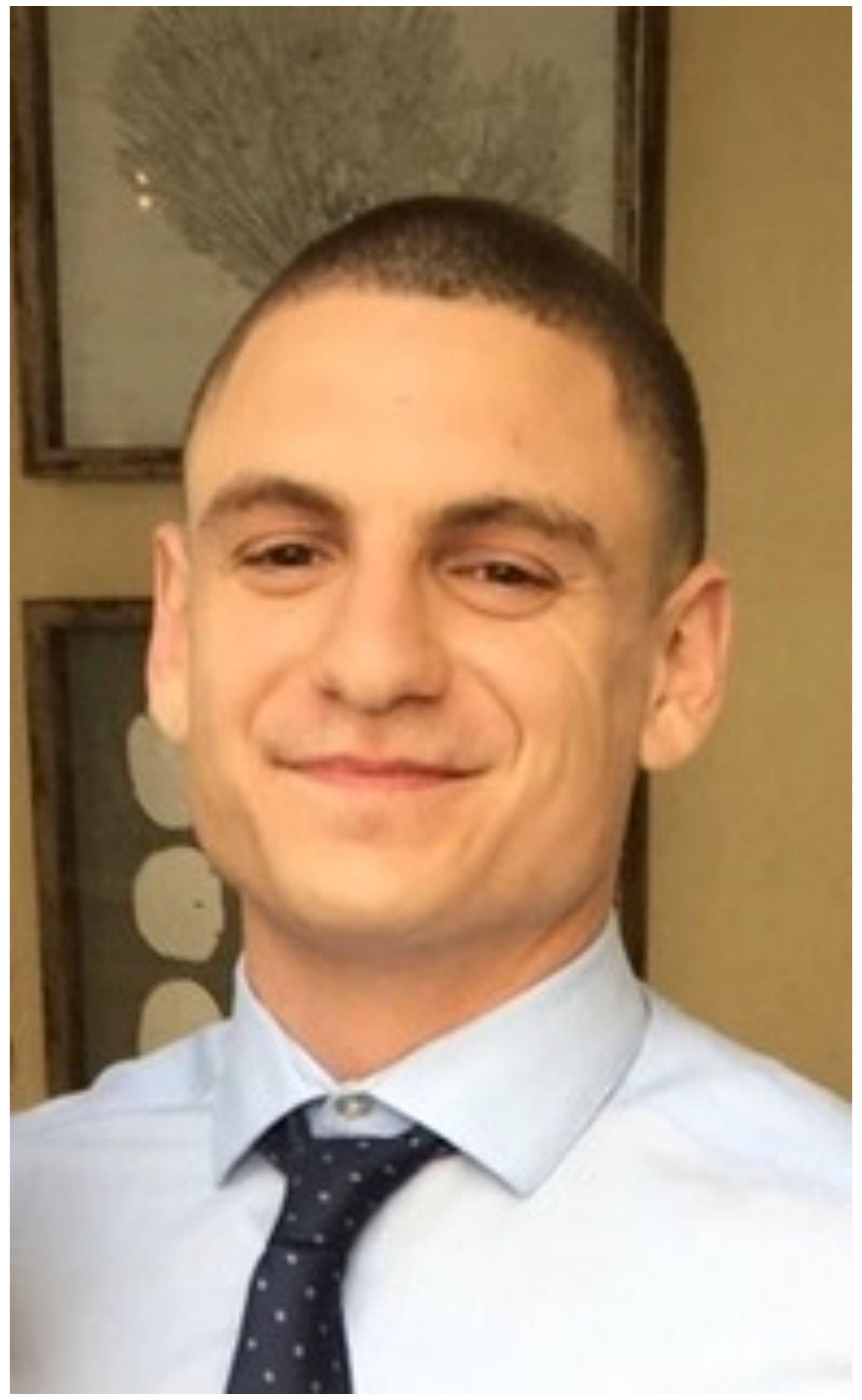
Strange metals

Aavishkar Patel

Flatiron Institute, NYC

Haoyu Guo

Harvard



Ilya Esterlis

Harvard → Wisconsin

arXiv: 2103.08615, 2203.04990, 2207.08841

E. E. Aldape, T. Cookmeyer, Aavishkar A. Patel, and Ehud Altman, arXiv:2012.00763

Properties of a strange metal:

1. Resistivity $\rho(T) = \rho_0 + AT + \ldots$ as $T \to 0$
   <u>and</u> $\rho(T) < h/e^2$ (in $d = 2$).
   Metals with $\rho(T) > h/e^2$ are <u>bad metals</u>.

2. Specific heat $\sim T \ln(1/T)$ as $T \to 0$.

S.A. Hartnoll and A.P. MacKenzie, arXiv:2107.07802

3. Optical conductivity

$$\sigma(\omega) = \frac{K}{\dfrac{1}{\tau_{\text{trans}}(\omega)} - i\omega \dfrac{m^*_{\text{trans}}(\omega)}{m}} \quad ; \quad \frac{1}{\tau_{\text{trans}}(\omega)} \sim |\omega| \Phi_\sigma \left(\frac{\hbar\omega}{k_B T} \right)$$

B. Michon……A. Georges, arXiv:2205.04030

4. Photoemission: nearly "marginal Fermi liquid" electron spectral density:

$$\mathrm{Im}\Sigma(\omega) \sim |\omega|^{2\alpha} \Phi_\Sigma \left(\frac{\hbar\omega}{k_B T} \right) \quad \text{with } \alpha \approx 1/2 \quad ; \quad \frac{1}{\tau(\omega)} \sim |\omega| \Phi_\Sigma \left(\frac{\hbar\omega}{k_B T} \right)$$

T.J. Reber….D. Dessau, Nature Communications **10**, 5737 (2019)

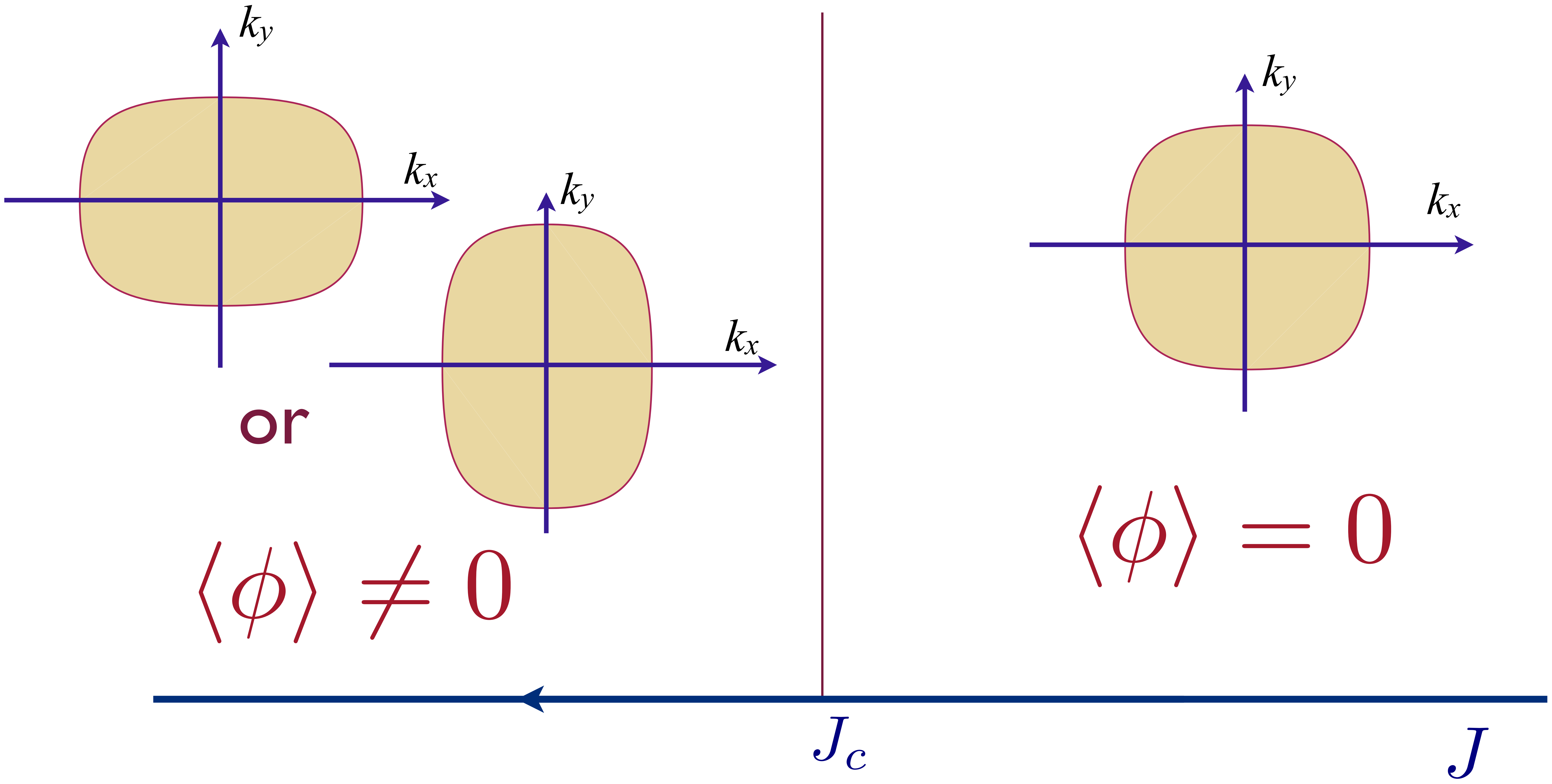
Yukawa-SYK models

$$\mathcal{H} = -\mu \sum_i \psi_i^\dagger \psi_i + \sum_\ell \frac{1}{2}\left(\pi_\ell^2 + \omega_0^2 \phi_\ell^2\right) + \frac{1}{N}\sum_{ij\ell} g_{ij\ell}\psi_i^\dagger \psi_j \phi_\ell \,,$$

$g_{ij\ell}$ independent random numbers with zero mean. Large N limit leads to Migdal-Eliashberg equations $\Sigma_\psi \sim g^2 G_\psi G_\phi$, $\Sigma_\phi \sim g^2 G_\psi G_\psi$.

W. Fu, D. Gaiotto, J. Maldacena, and S. Sachdev, PRD **95**, 026009 (2017)
J. Murugan, D. Stanford, and E. Witten, JHEP 08, 146 (2017)
A. A. Patel and S. Sachdev, PRB **98**, 125134 (2018)
E. Marcus and S. Vandoren, JHEP 01, 166 (2018)
Yuxuan Wang, PRL **124**, 017002 (2020)
I. Esterlis and J. Schmalian, PRB **100**, 115132 (2019)
Yuxuan Wang and A. V. Chubukov, PRR **2**, 033084 (2020)
E. E. Aldape, T. Cookmeyer, A. A. Patel, and E. Altman, arXiv:2012.00763
Jaewon Kim, E. Altman, and Xiangyu Cao, PRB **103**, 081113 (2021)
W. Wang, A. Davis, G. Pan, Yuxuan Wang, and Zi Yang Meng, PRB **103**, 195108 (2021)
I. Esterlis, H. Guo, A. A. Patel, and S. Sachdev, PRB **103**, 235129 (2021).

Quantum criticality of Ising-nematic ordering in a metal



Pomeranchuk instability as a function of coupling J

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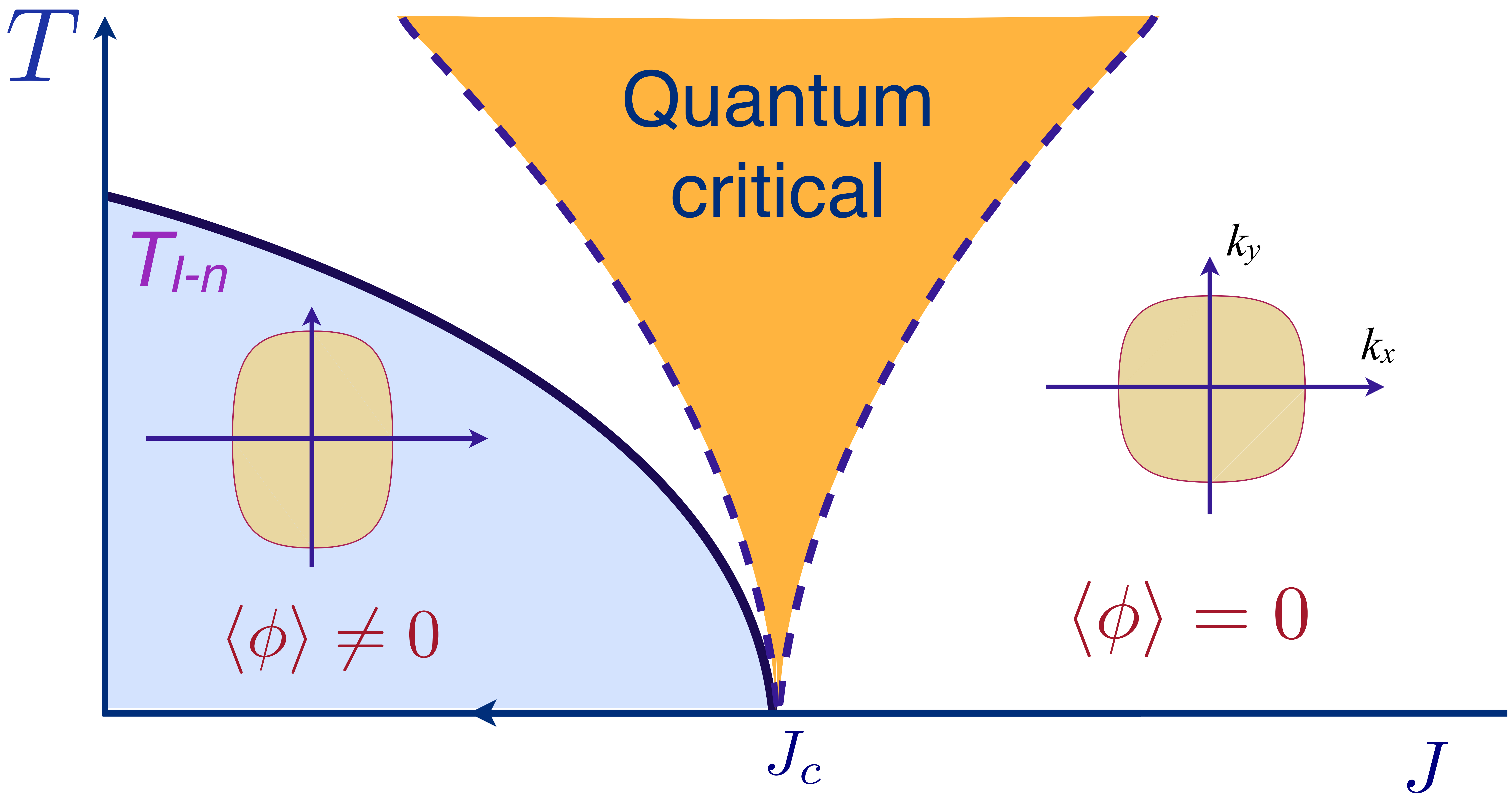
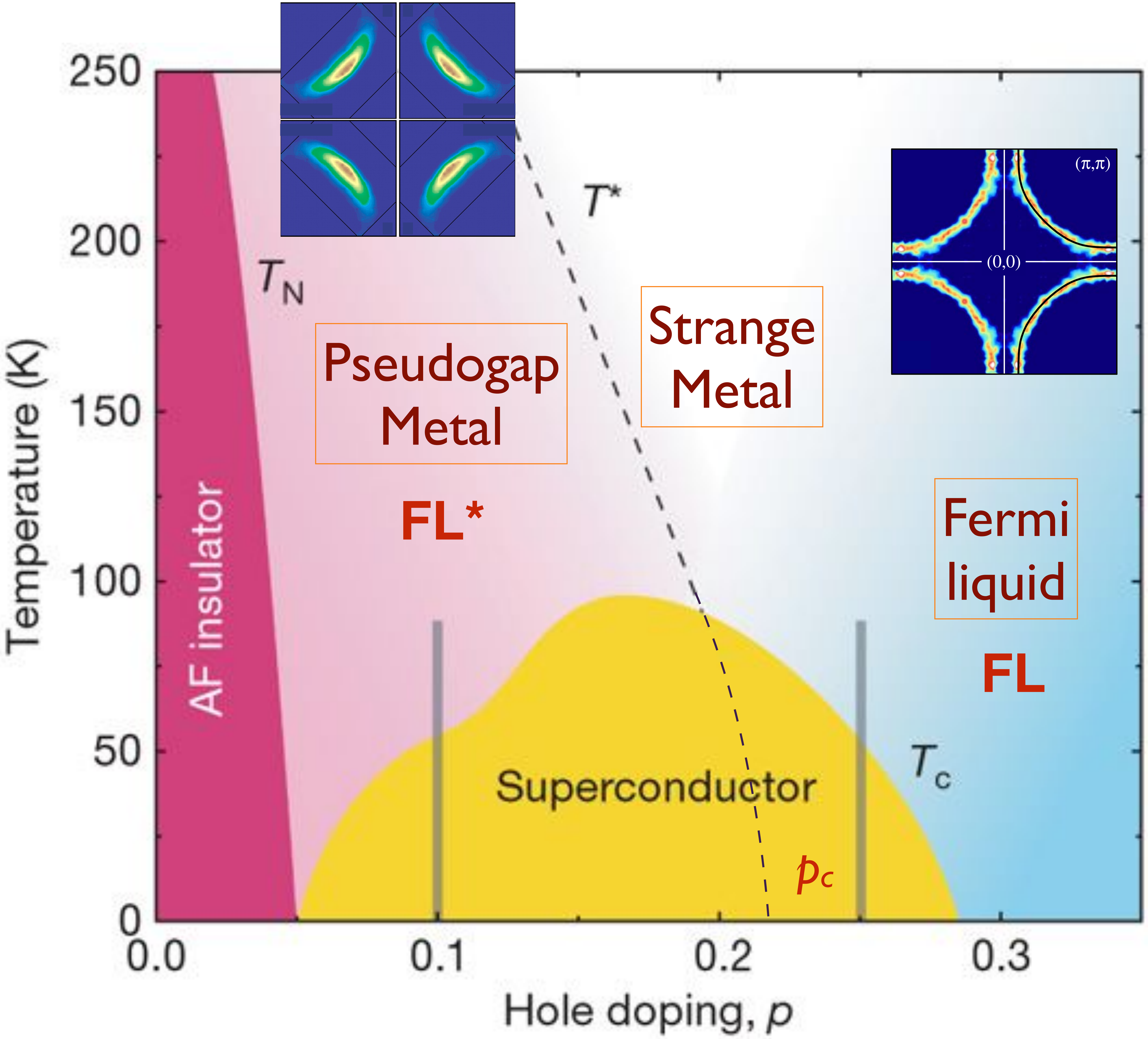


Phase diagram as a function of T and J



Our results also apply
to the transition from a

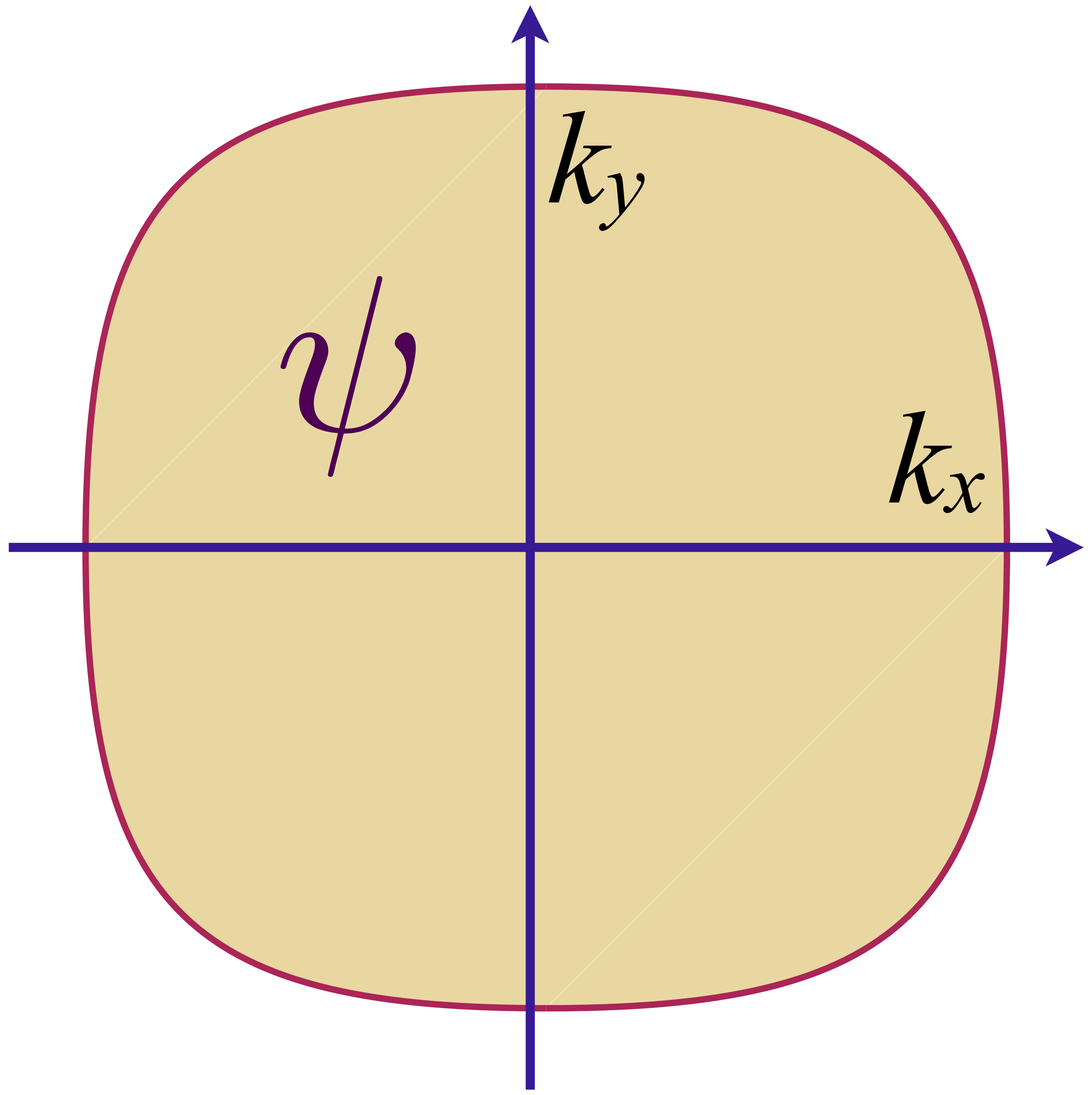
fractionalized Fermi liquid (FL*)
to a Fermi liquid (FL)

in a single-band Hubbard model,
or in a Kondo lattice model;

a Higgs field ("slave boson")
takes the place of ϕ.

Fermi surface

$$\mathcal{L}_\psi = \psi_{\boldsymbol{k}}^\dagger \left(\frac{\partial}{\partial \tau} + \varepsilon(\boldsymbol{k}) \right) \psi_{\boldsymbol{k}}$$



$$-J\, \psi^\dagger(\boldsymbol{r})\psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})\psi(\boldsymbol{r})$$

Fermi surface coupled to a critical boson

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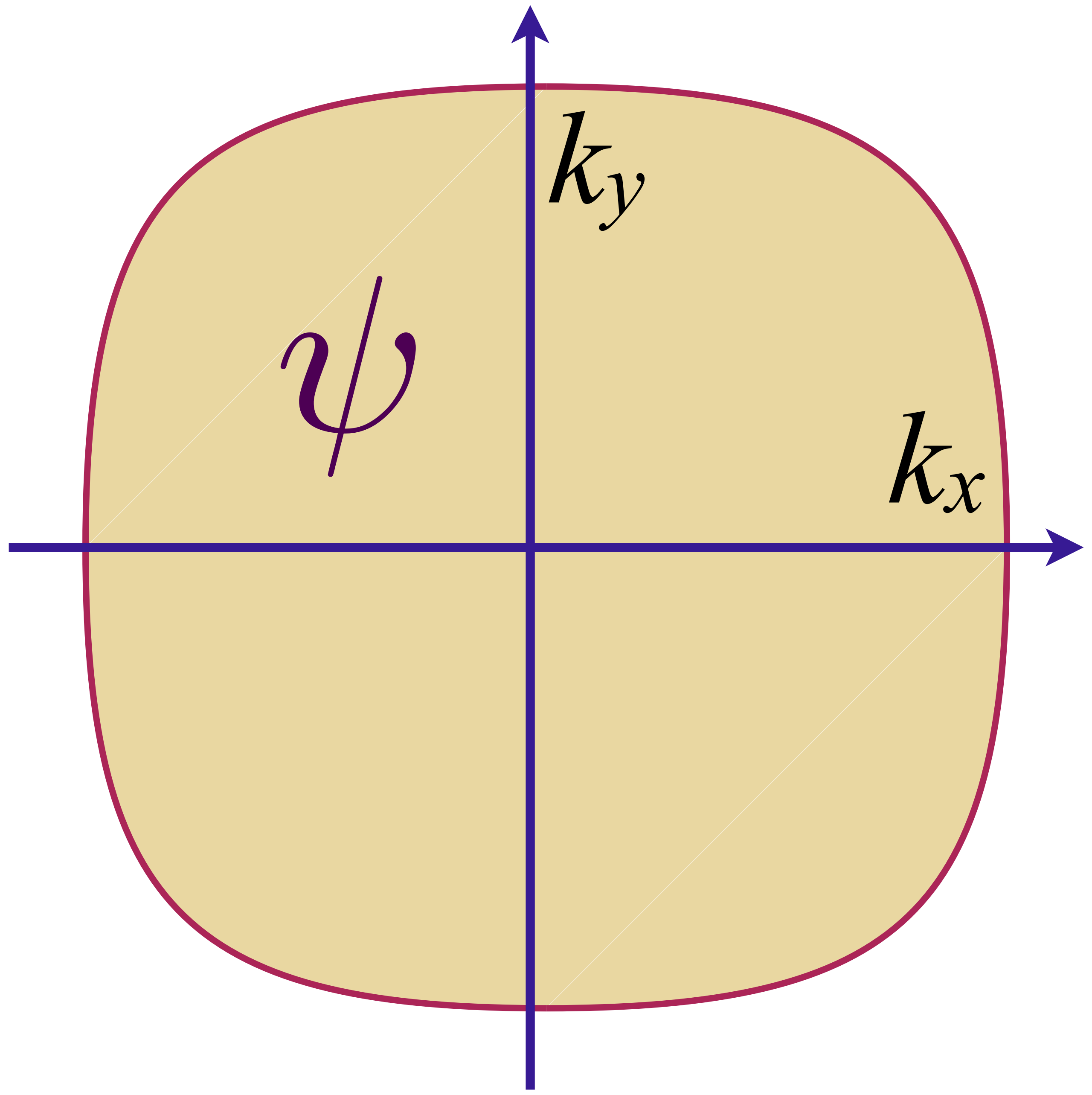

a critical boson ϕ
e.g. Ising-nematic order

$$\frac{[\phi(\boldsymbol{r})]^2}{J} + \psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})\,\phi(\boldsymbol{r})$$

Fermi surface coupled to a critical boson

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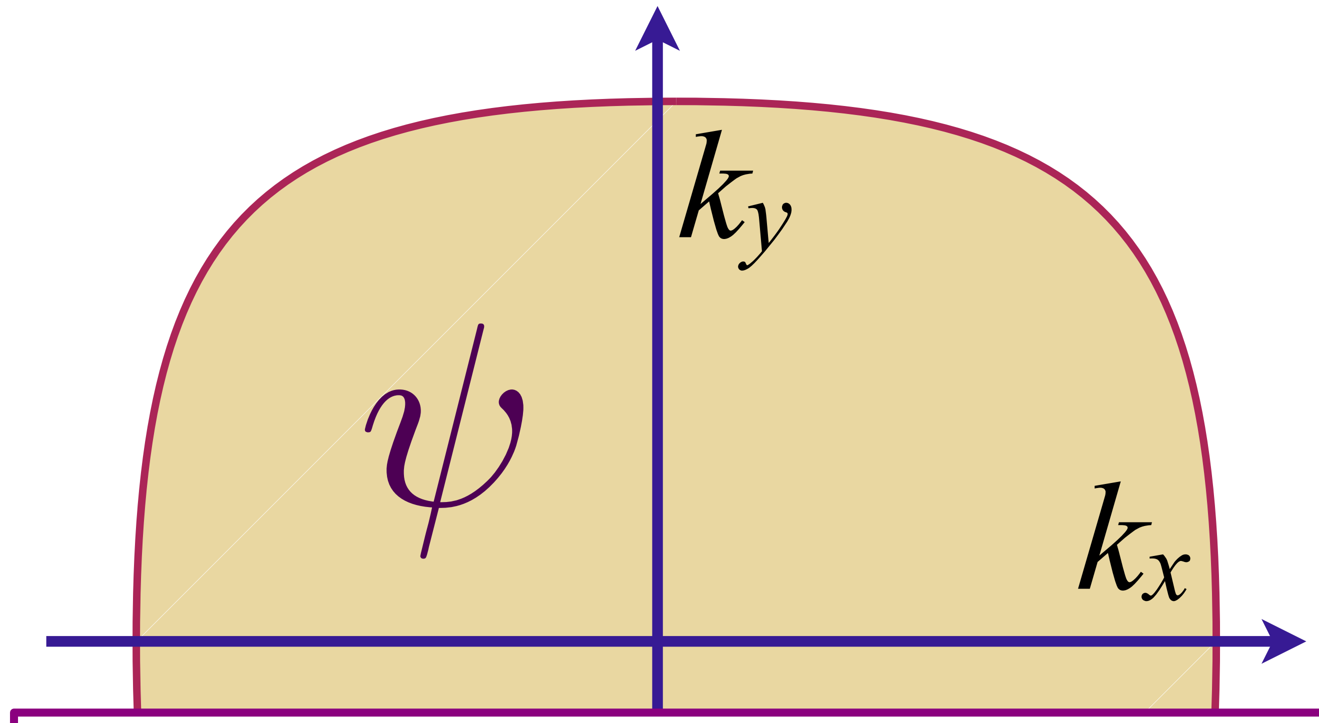


$$\frac{[\phi(\boldsymbol{r})]^2}{J} + \psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})\,\phi(\boldsymbol{r})$$

Solve in a large N limit with Yukawa coupling

$$\frac{g_{ij\ell}}{N} \int d^2r d\tau\, \psi_i^\dagger(r,\tau)\psi_j(r,\tau)\phi_l(r,\tau) \quad , \quad \overline{g_{ijl}} = 0 \quad , \quad \overline{|g_{ijl}|^2} = g^2$$

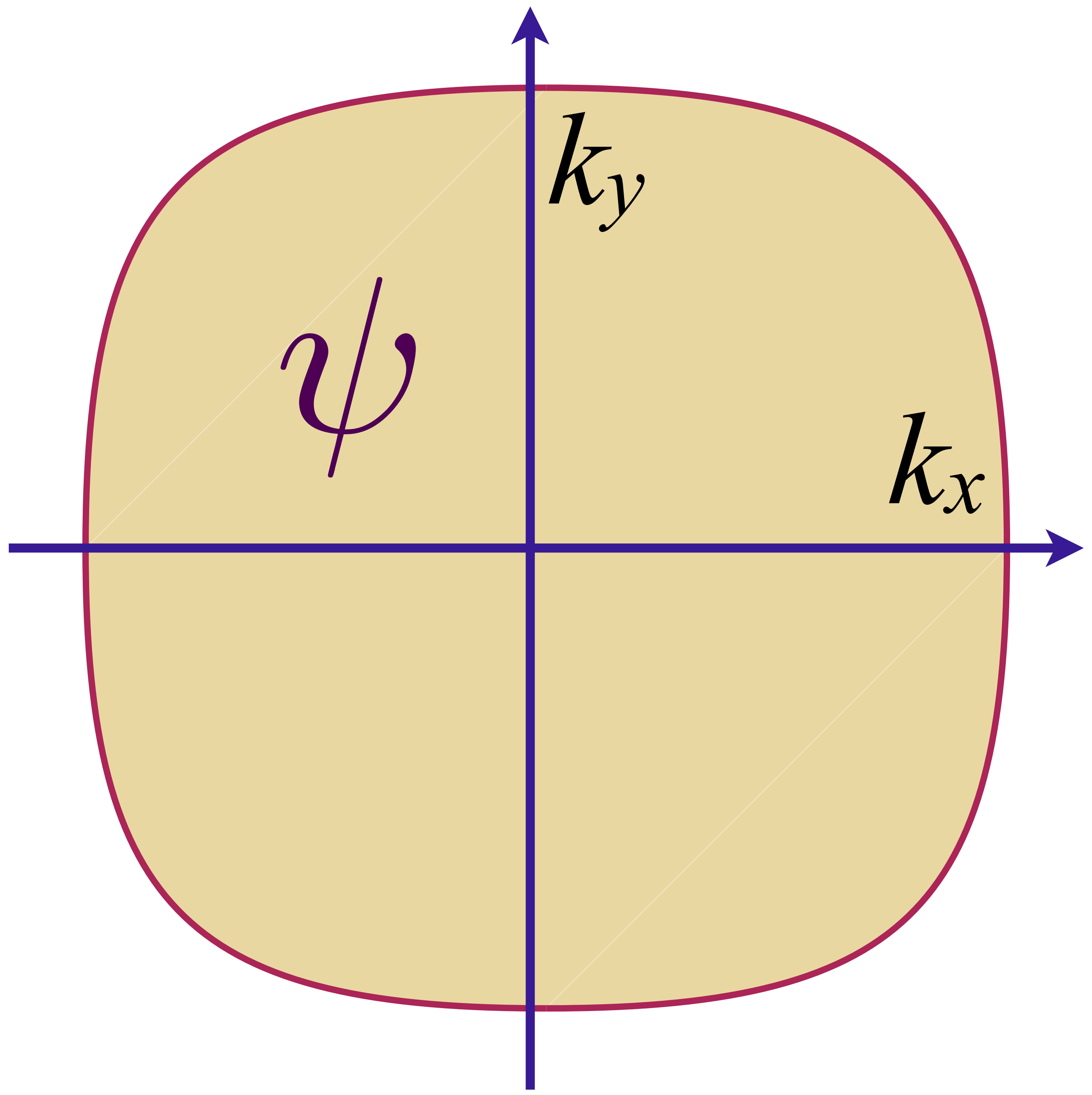
to obtain Eliashberg solution for electron (G) and boson (D) Green's functions at small ω:

$$\Sigma(\hat{\boldsymbol{k}}, i\omega) \sim -i\text{sgn}(\omega)|\omega|^{2/3}, \quad G(\boldsymbol{k}, i\omega) = \frac{1}{i\omega - \varepsilon(\boldsymbol{k}) - \Sigma(\hat{\boldsymbol{k}}, i\omega)}, \quad D(\boldsymbol{q}, i\Omega) = \frac{1}{\Omega^2 + q^2 + \gamma|\Omega|/q}$$

P.A. Lee (1989)

Fermi surface coupled to a critical boson

$$\mathcal{L}_\psi = \psi^\dagger_{\boldsymbol{k}} \left(\frac{\partial}{\partial \tau} + \varepsilon(\boldsymbol{k}) \right) \psi_{\boldsymbol{k}}$$



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S. A. Hartnoll, P. K. Kovtun, M. Muller, and S.S. PRB **76**, 144502 (2007)
D. L. Maslov, V. I. Yudson, and A. V. Chubukov PRL **106**, 106403 (2011)
S. A. Hartnoll, R. Mahajan, M. Punk, and S.S. PRB **89**, 155130 (2014)
A. Eberlein, I. Mandal, and S.S. PRB **94**, 045133 (2016)

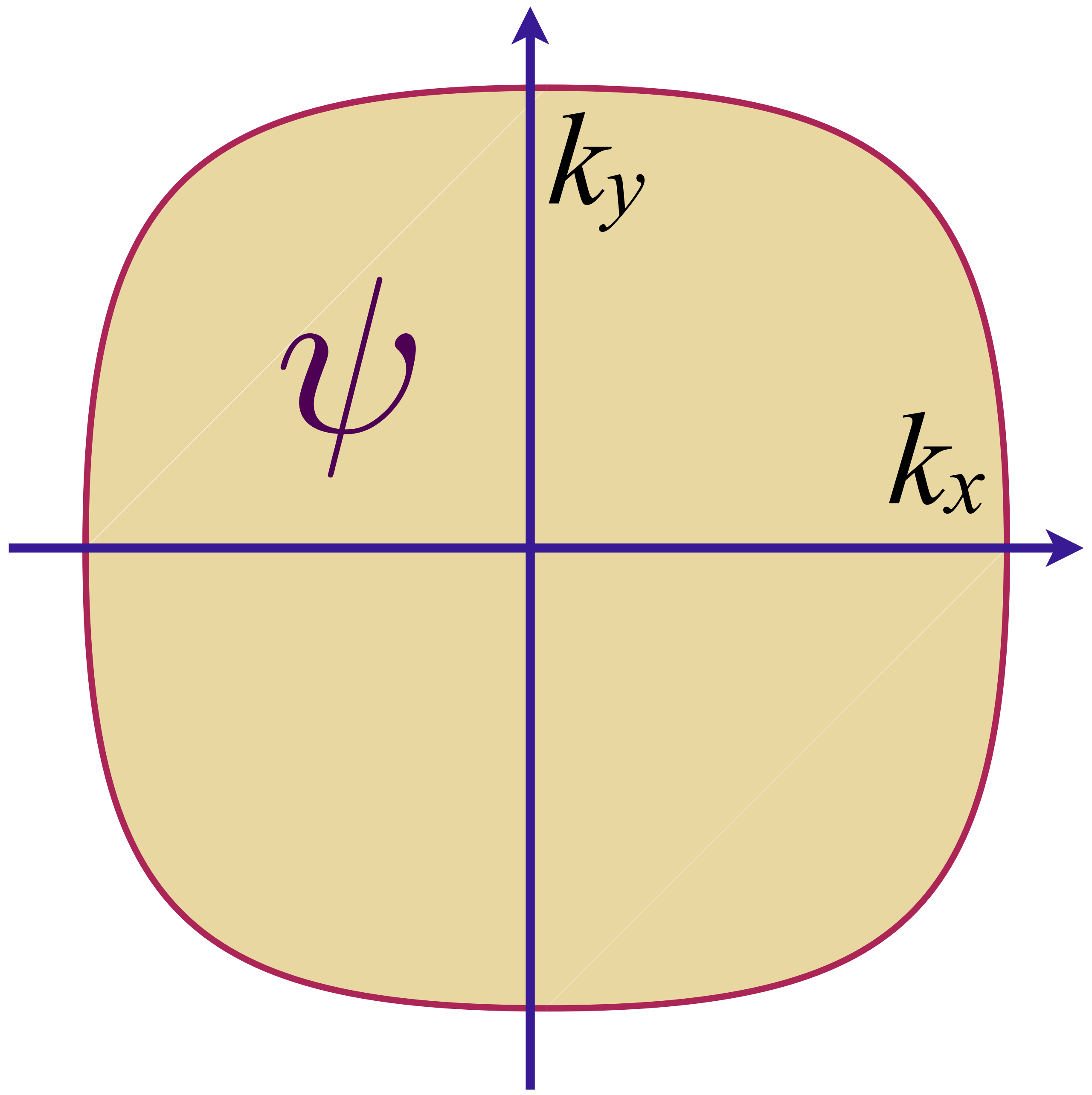
Transport—a perfect metal!

Conservation of momentum and fermion-boson drag imply:

$$\mathrm{Re}\left[\sigma(\omega)\right] = D\delta(\omega) + \ldots$$

Fermi surface coupled to a critical boson

$$\mathcal{L}_\psi = \psi_{\boldsymbol{k}}^\dagger \left(\frac{\partial}{\partial \tau} + \varepsilon(\boldsymbol{k}) \right) \psi_{\boldsymbol{k}}$$



a critical boson ϕ
e.g. Ising-nematic order

$$\frac{[\phi(\boldsymbol{r})]^2}{J} + \psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})\,\phi(\boldsymbol{r})$$

Transport—a perfect metal!

Conservation of momentum and fermion-boson drag imply:

$$\sigma(\omega) \sim \frac{1}{-i\omega} + |\omega|^0 + \cdots \quad (\omega^{-2/3} \text{ term has vanishing co-efficient})$$

Haoyu Guo, Aavishkar Patel, Ilya Esterlis, S.S. PRB **106**, 115151 (2022)

Fermi surface coupled to a critical boson:

No spatial disorder

A non-Fermi liquid but NOT a strange metal

Fermi surface coupled to a critical boson with disorder

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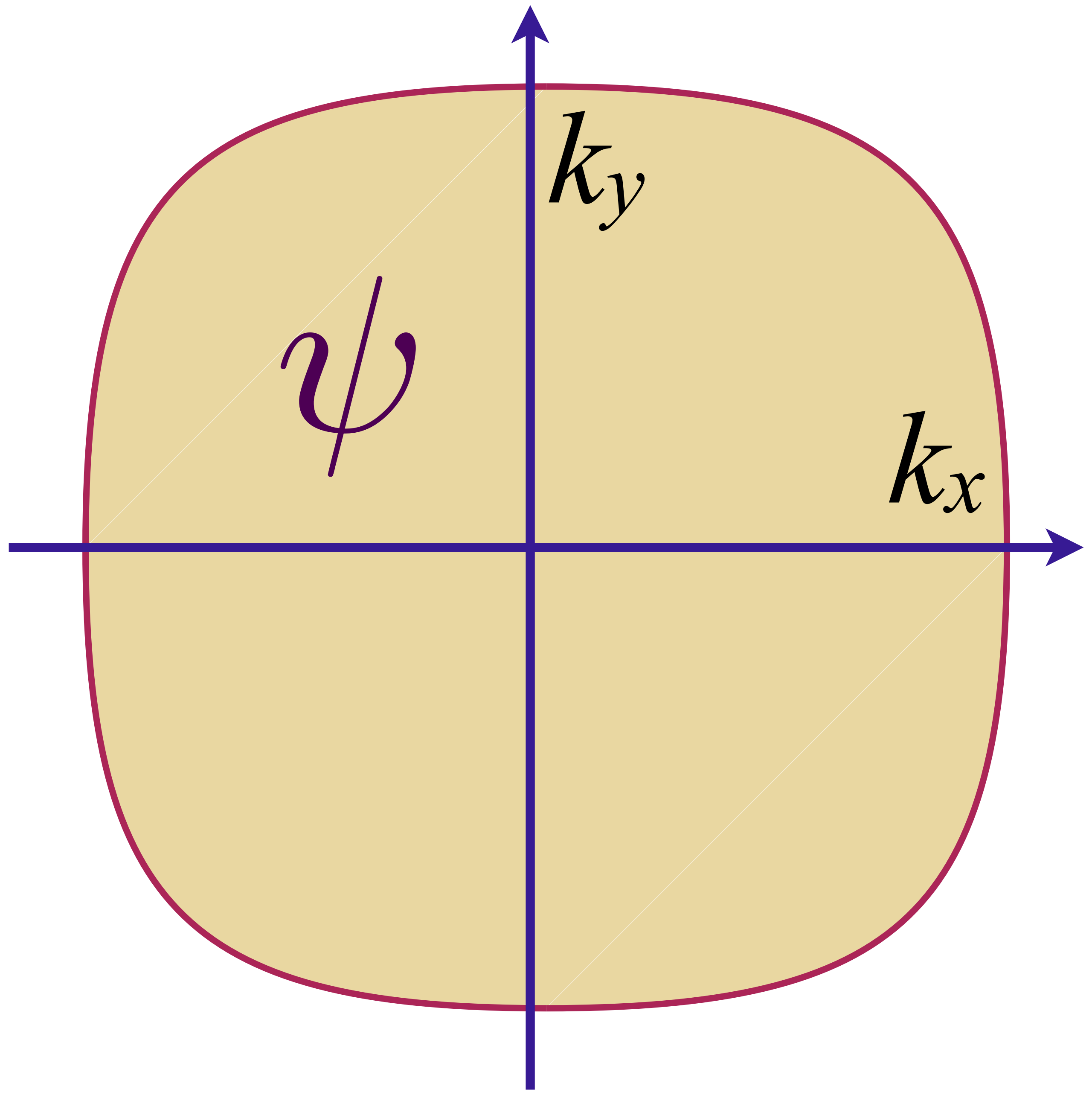

a critical boson ϕ
e.g. Ising-nematic order

$$\frac{[\phi(\boldsymbol{r})]^2}{J} + \psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})\,\phi(\boldsymbol{r})$$

$$+v(\boldsymbol{r})\psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})$$

Spatially random potential $v(\boldsymbol{r})$ with $\overline{v(\boldsymbol{r})} = 0$, $\overline{v(\boldsymbol{r})v(\boldsymbol{r}')} = v^2\delta(\boldsymbol{r} - \boldsymbol{r}')$

Fermi surface coupled to a critical boson with disorder

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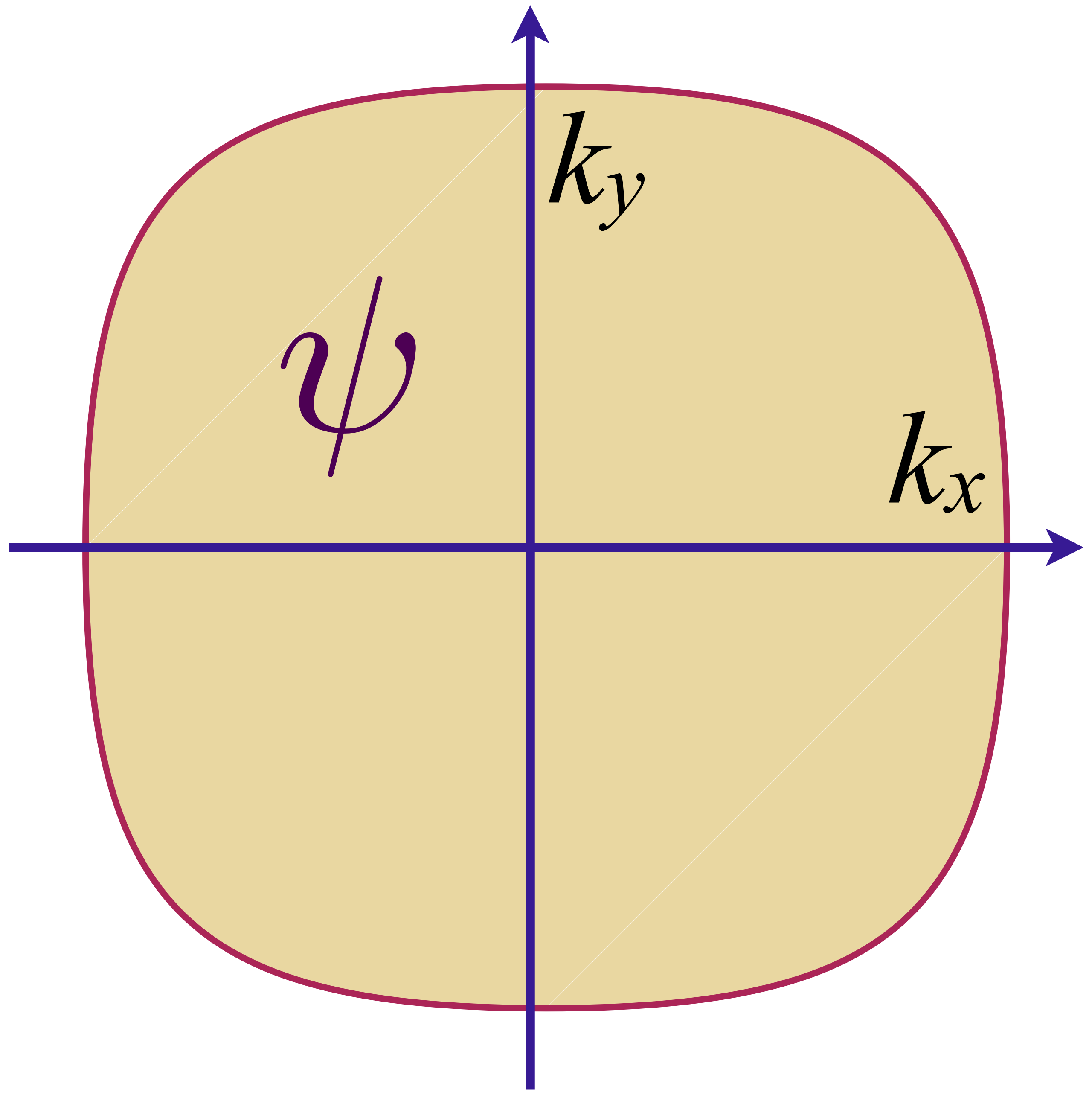


$$\frac{[\phi(\boldsymbol{r})]^2}{J + J'(\boldsymbol{r})} + \psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})\,\phi(\boldsymbol{r})$$

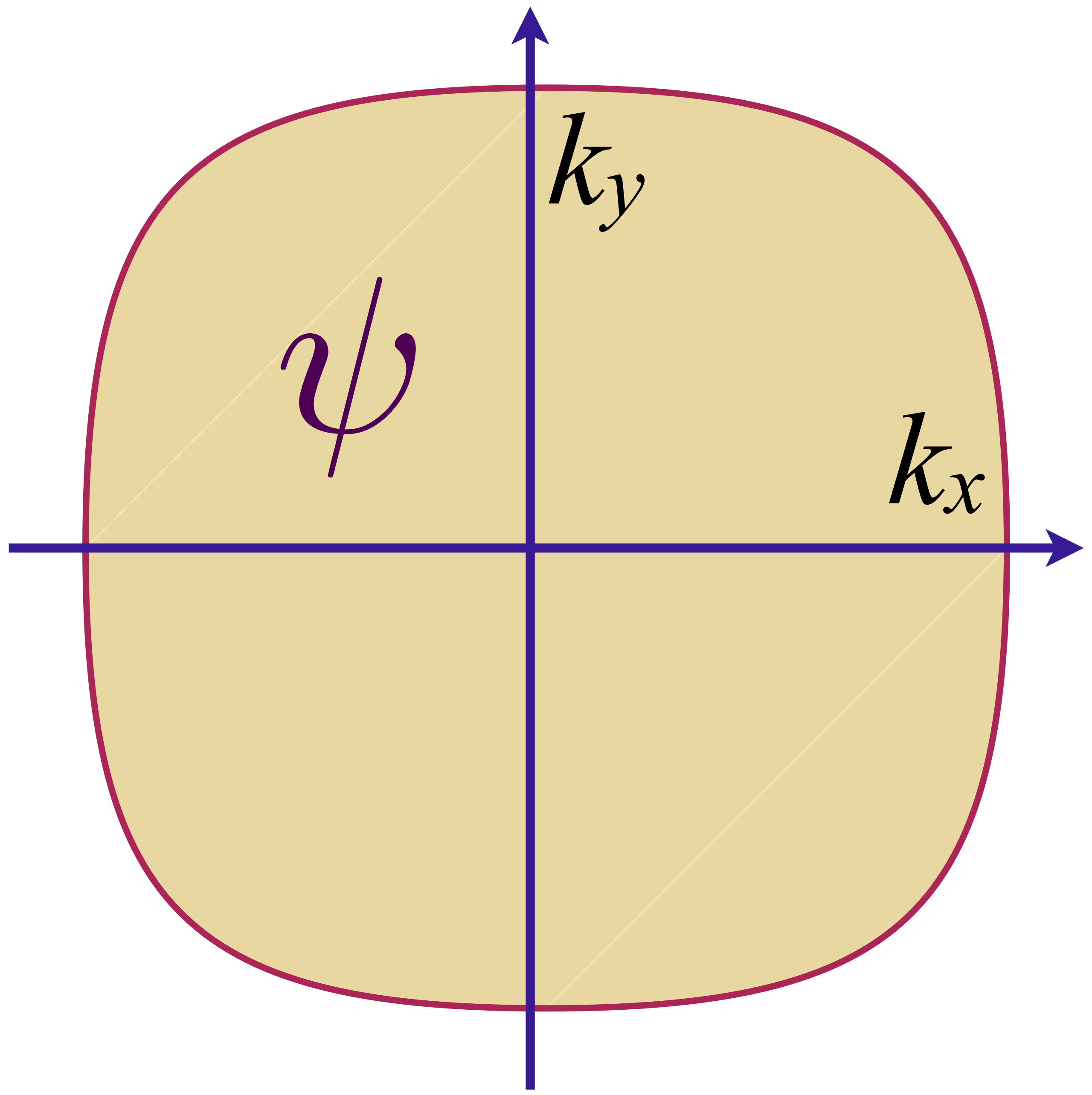
$$+v(\boldsymbol{r})\psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})$$

Fermi surface coupled to a critical boson with disorder

$$\mathcal{L}_\psi = \psi^\dagger_{\boldsymbol{k}} \left(\frac{\partial}{\partial \tau} + \varepsilon(\boldsymbol{k}) \right) \psi_{\boldsymbol{k}}$$

a critical boson ϕ

e.g. Ising-nematic order



$$[\phi(\boldsymbol{r})]^2 + [g + g'(\boldsymbol{r})] \, \psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r}) \, \phi(\boldsymbol{r})$$

$$+ v(\boldsymbol{r})\psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})$$

ϕ^2 "mass" disorder $J'(\boldsymbol{r})$ is strongly relevant;
rescale ϕ to move disorder to the Yukawa coupling;

Spatially random Yukawa coupling $g'(\boldsymbol{r})$ with $\overline{g'(\boldsymbol{r})} = 0$, $\overline{g'(\boldsymbol{r})g'(\boldsymbol{r}')} = g'^2\delta(\boldsymbol{r} - \boldsymbol{r}')$

Spatially random potential $v(\boldsymbol{r})$ with $\overline{v(\boldsymbol{r})} = 0$, $\overline{v(\boldsymbol{r})v(\boldsymbol{r}')} = v^2\delta(\boldsymbol{r} - \boldsymbol{r}')$

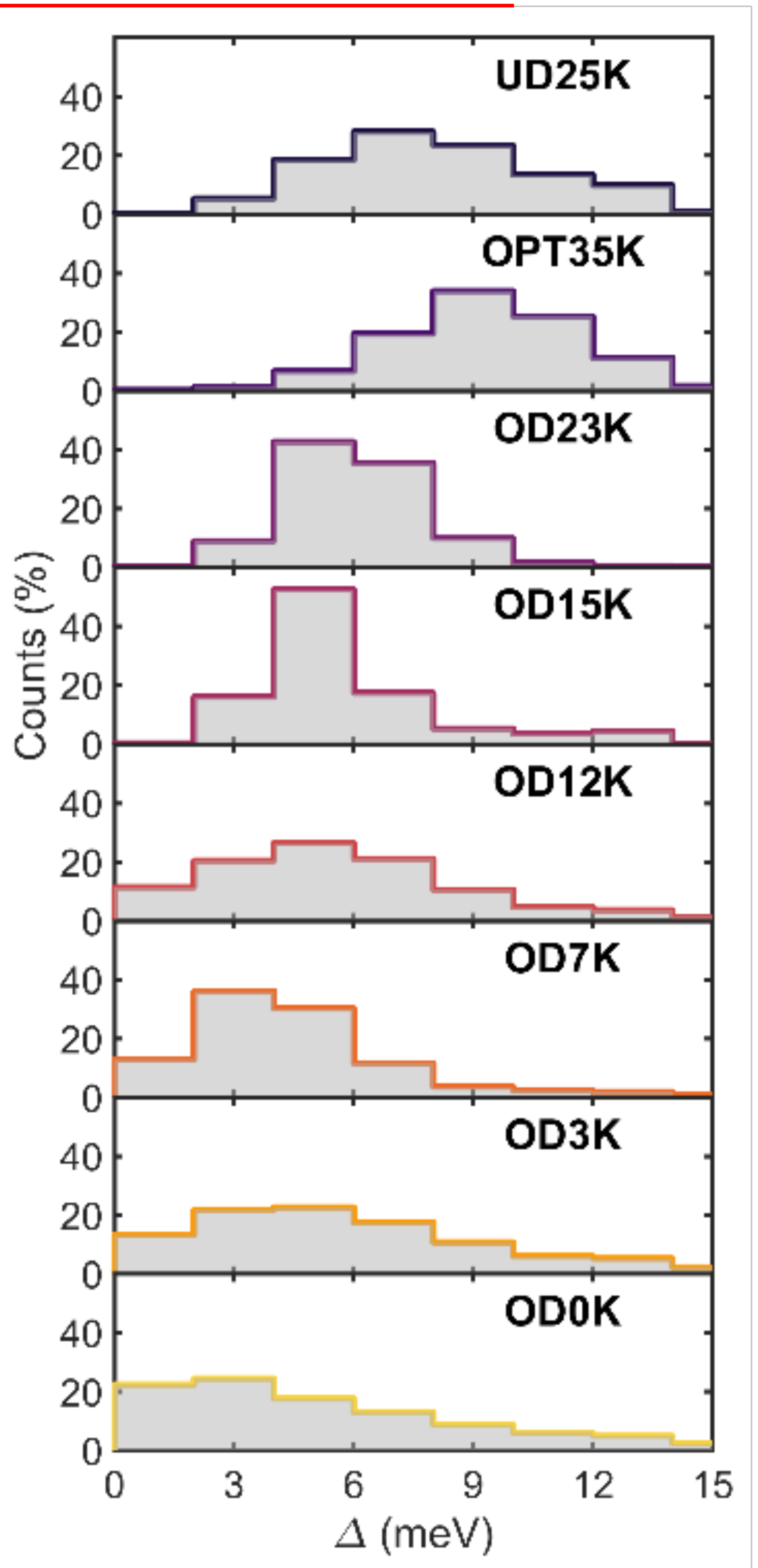
Spatially random interactions!

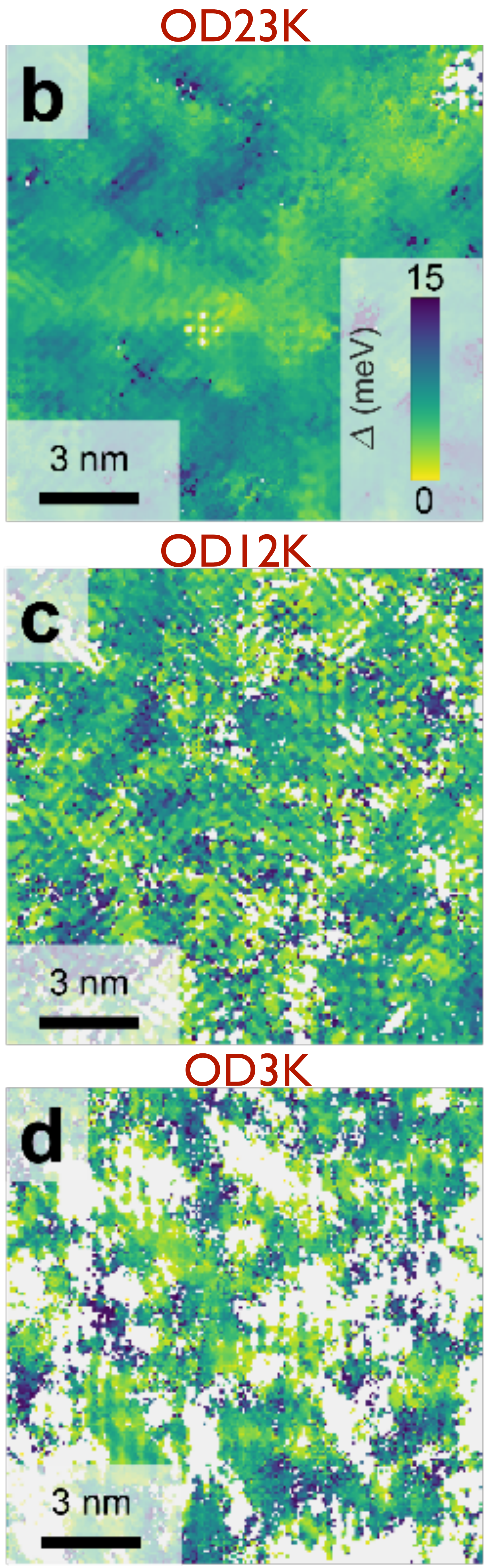
Puddle formation, persistent gaps, and non-mean-field breakdown of superconductivity in overdoped $(Pb,Bi)_2Sr_2CuO_{6+\delta}$

Willem O. Tromp, Tjerk Benschop, Jian-Feng Ge,
Irene Battisti, Koen M. Bastiaans, Damianos Chatzopoulos,
Amber Vervloet, Steef Smit, Erik van Heumen,
Mark S. Golden, Yinkai Huang, Takeshi Kondo, Yi Yin,
Jennifer E. Hoffman, Miguel Antonio Sulangi, Jan Zaanen,
Milan P. Allan

Our scanning tunneling spectroscopy measurements in the overdoped regime of the $(Pb,Bi)_2Sr_2CuO_{6+\delta}$ high-temperature superconductor show the emergence of puddled superconductivity, featuring nanoscale superconducting islands in a metallic matrix

arXiv:2205.09740





Fermi surface coupled to a critical boson with disorder

All results are obtained from the large N saddle-point and response functions of this G-Σ-D-Π theory:

$$\mathcal{Z} = \int \mathcal{D}G\,\mathcal{D}\Sigma\,\mathcal{D}D\,\mathcal{D}\Pi \exp(-NS_{\rm all})$$

$$\begin{aligned}
S_{\rm all} = & -\ln\det(\partial_\tau + \varepsilon(\mathbf{k}) - \mu + \Sigma) + \frac{1}{2}\ln\det(-\partial_\tau^2 + \mathbf{q}^2 + m_b^2 - \Pi) \\
& + \int d\tau d^2r \int d\tau' d^2r' \Big[-\Sigma(\tau',\mathbf{r}';\tau,\mathbf{r})G(\tau,\mathbf{r};\tau',\mathbf{r}') + \frac{1}{2}\Pi(\tau',\mathbf{r}';\tau,\mathbf{r})D(\tau,\mathbf{r};\tau',\mathbf{r}') \\
& + \frac{g^2}{2}G(\tau,\mathbf{r};\tau',\mathbf{r}')G(\tau',\mathbf{r}';\tau,\mathbf{r})D(\tau,\mathbf{r};\tau',\mathbf{r}') + \frac{v^2}{2}G(\tau,\mathbf{r};\tau',\mathbf{r}')G(\tau',\mathbf{r}';\tau,\mathbf{r})\delta(\mathbf{r}-\mathbf{r}' \\
& + \frac{g'^2}{2}G(\tau,\mathbf{r};\tau',\mathbf{r}')G(\tau',\mathbf{r}';\tau,\mathbf{r})D(\tau,\mathbf{r};\tau',\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')\Big]\,.
\end{aligned}$$

Fermi surface coupled to a critical boson with disorder

All results are obtained from the large N saddle-point and response functions of this G-Σ-D-Π theory:

$$\mathcal{Z} = \int \mathcal{D}G\,\mathcal{D}\Sigma\,\mathcal{D}D\,\mathcal{D}\Pi \exp(-NS_{\rm all})$$

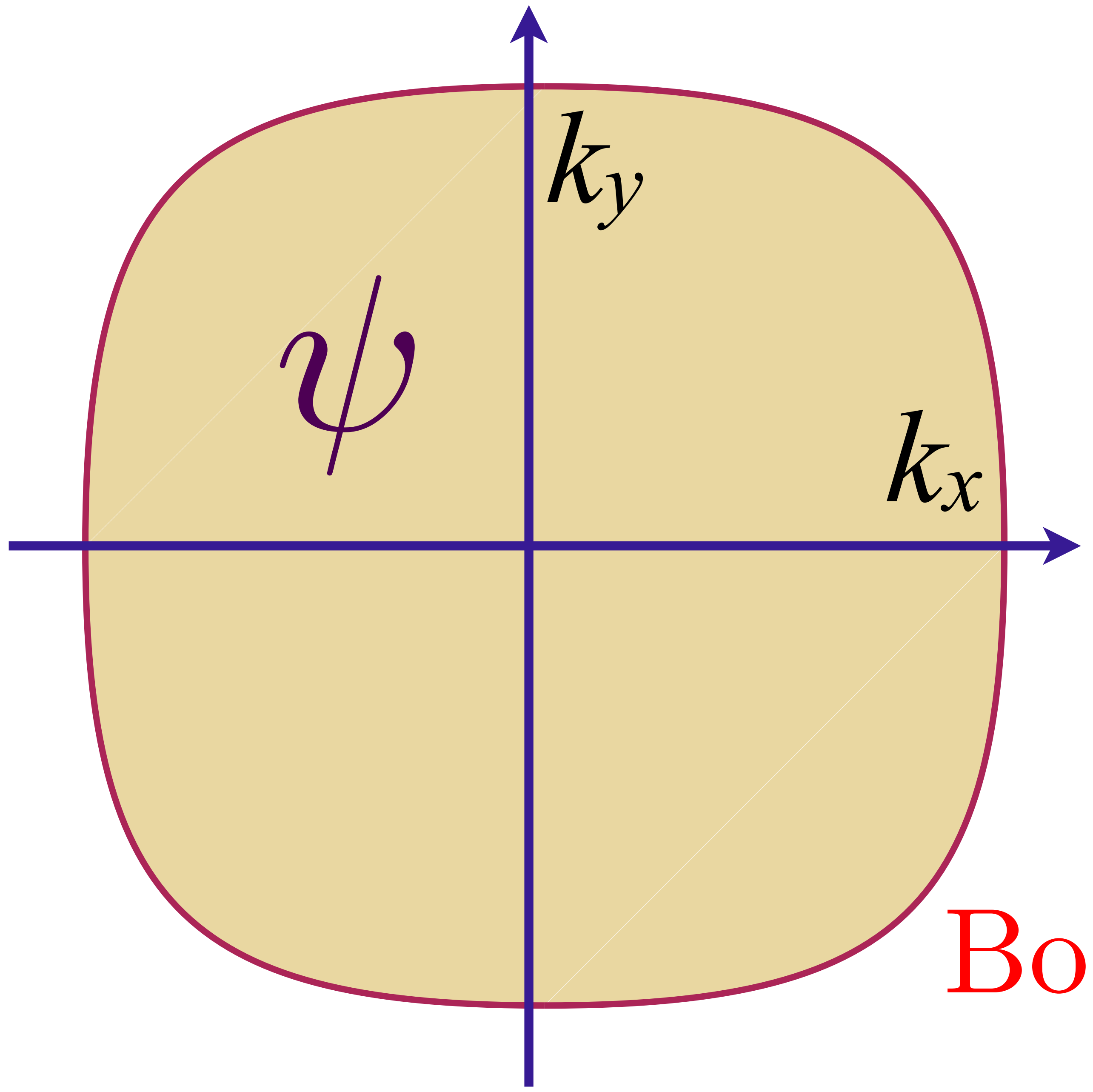
Saddle-point equations

$$\begin{aligned}
\Sigma(\tau,\mathbf{r}) &= g^2 D(\tau,\mathbf{r})G(\tau,\mathbf{r}) + v^2 G(\tau,\mathbf{r})\delta^2(\mathbf{r}) + {g'}^2 G(\tau,\mathbf{r})D(\tau,\mathbf{r})\delta^2(\mathbf{r}),\\
\Pi(\tau,\mathbf{r}) &= -g^2 G(-\tau,-\mathbf{r})G(\tau,\mathbf{r}) - {g'}^2 G(-\tau,\mathbf{r})G(\tau,\mathbf{r})\delta^2(\mathbf{r}),\\
G(i\omega,\mathbf{k}) &= \frac{1}{i\omega - \varepsilon(\mathbf{k}) + \mu - \Sigma(i\omega,\mathbf{k})},\\
D(i\Omega,\mathbf{q}) &= \frac{1}{\Omega^2 + \mathbf{q}^2 + m_b^2 - \Pi(i\Omega,\mathbf{q})}.
\end{aligned}$$

Fermi surface coupled to a critical boson with disorder

$$\mathcal{L}_\psi = \psi_{\boldsymbol{k}}^\dagger \left(\frac{\partial}{\partial \tau} + \varepsilon(\boldsymbol{k}) \right) \psi_{\boldsymbol{k}}$$

a critical boson ϕ
e.g. Ising-nematic order



$$[\phi(\boldsymbol{r})]^2 + [g + g'(\boldsymbol{r})]\, \psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})\, \phi(\boldsymbol{r})$$

$$+ v(\boldsymbol{r})\psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})$$

Boson Green's function: $D(q, i\Omega) \sim 1/(q^2 + \gamma|\Omega|)$

Fermion self energy:

$$\Sigma(i\omega) \sim -iv^2 \text{sgn}(\omega) - i \left(\frac{g^2}{v^2} + g'^2 \right) \omega \ln(1/|\omega|); \quad \frac{1}{\tau_{\text{in}}(\omega)} \sim \left(\frac{g^2}{v^2} + g'^2 \right) |\omega|$$

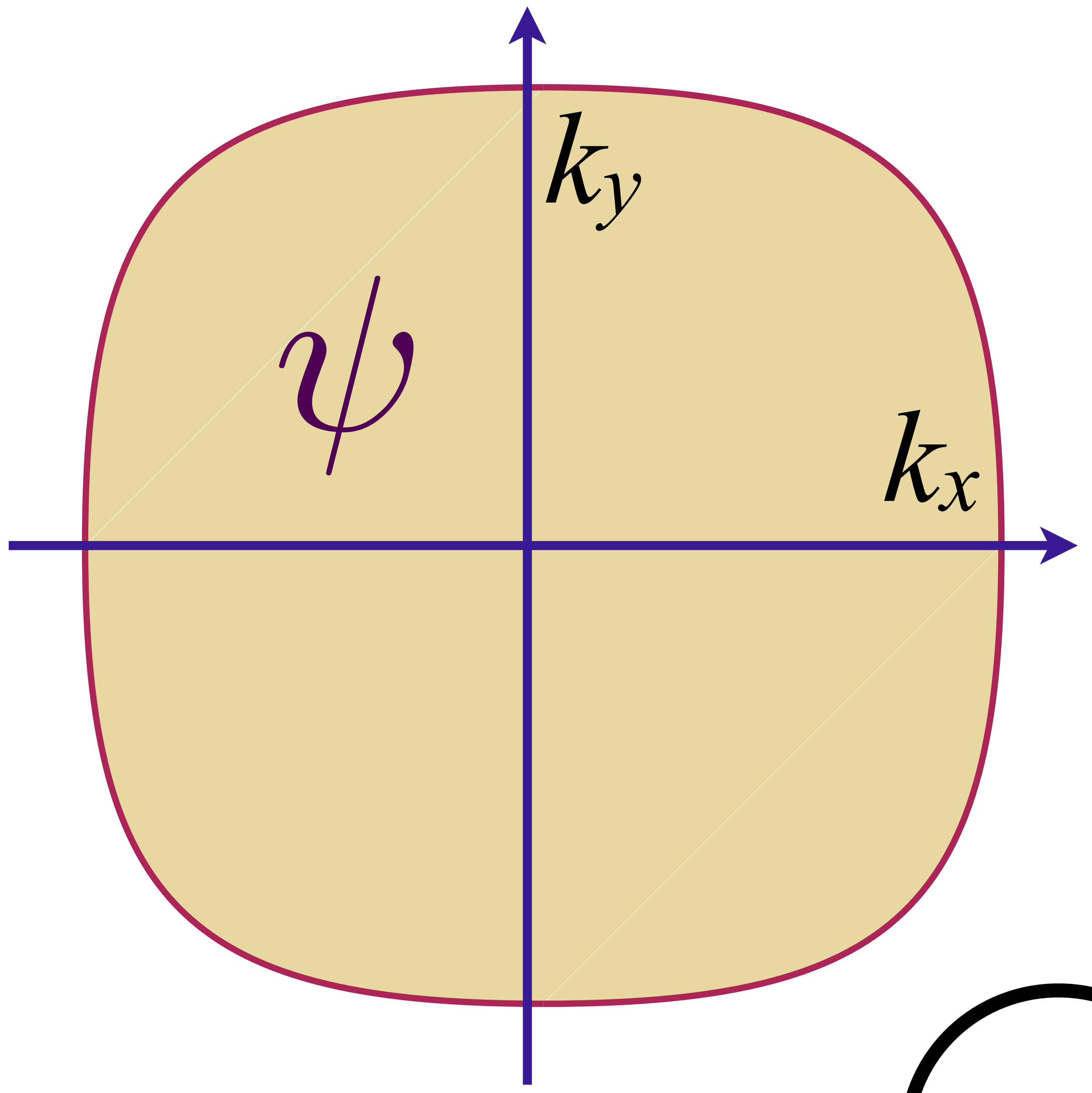
Marginal Fermi liquid self energy and $T\ln(1/T)$ specific heat

Fermi surface coupled to a critical boson with disorder

$$\mathcal{L}_\psi = \psi^\dagger_{\boldsymbol{k}}\left(\frac{\partial}{\partial\tau}+\varepsilon(\boldsymbol{k})\right)\psi_{\boldsymbol{k}}$$

a critical boson ϕ

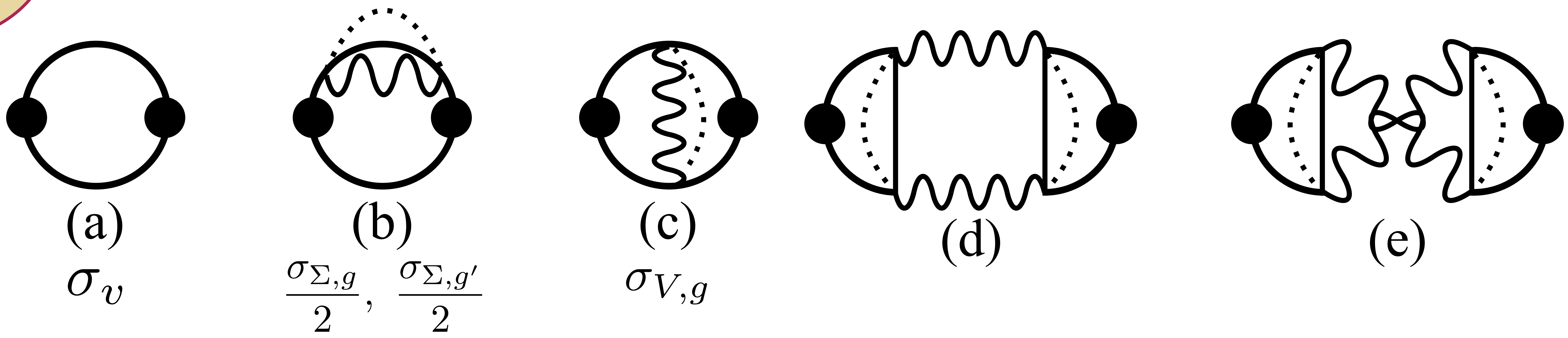
e.g. Ising-nematic order



$$[\phi(\boldsymbol{r})]^2 + [g+g'(\boldsymbol{r})]\,\psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})\,\phi(\boldsymbol{r})$$

$$+v(\boldsymbol{r})\psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})$$

Conductivity:



+ all ladders and bubbles…..

Fermi surface coupled to a critical boson with disorder

$$\text{Conductivity: } \sigma(\omega) \sim \frac{1}{\dfrac{1}{\tau_{\rm trans}(\omega)} - i\omega \, \dfrac{m^*_{\rm trans}(\omega)}{m}}$$

$$\frac{1}{\tau_{\rm trans}(\omega)} \sim v^2 + g'^2 |\omega| \quad ; \quad \frac{m^*_{\rm trans}(\omega)}{m} \sim \frac{2g'^2}{\pi} \ln(\Lambda/\omega)$$

Residual resistivity is determined by v^2; Linear-in-T resistivity determined by g'^2; Transport insensitive to g

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$$\text{Electron Green's function: } G(\omega) \sim \frac{1}{\omega \, \dfrac{m^*(\omega)}{m} - \varepsilon(\boldsymbol{k}) + i \left(\dfrac{1}{\tau_e} + \dfrac{1}{\tau_{\rm in}(\omega)} \right) \mathrm{sgn}(\omega)}$$

$$\frac{1}{\tau_e} \sim v^2 \quad ; \quad \frac{1}{\tau_{\rm in}(\omega)} \sim \left(\frac{g^2}{v^2} + g'^2 \right) |\omega| \quad ; \quad \frac{m^*(\omega)}{m} \sim \frac{2}{\pi} \left(\frac{g^2}{v^2} + g'^2 \right) \ln(\Lambda/\omega)$$

Residual resistivity is determined by v^2; Linear-in-T resistivity determined by g'^2; Transport insensitive to g; Marginal Fermi liquid self energy and $T \ln(1/T)$ specific heat.

Fermi surface coupled to a critical boson:

No spatial disorder

A non-Fermi liquid but NOT a strange metal

Fermi surface coupled to a critical boson:

Potential disorder v

A marginal Fermi liquid but NOT a strange metal

Fermi surface coupled to a critical boson:

Interaction disorder g'

A marginal Fermi liquid AND a strange metal

Summary

- SYK: a solvable toy model without particle-like excitations, exhibiting thermalization and many-body chaos in a time of order $\hbar/(k_B T)$, independent of microscopic energy scales.

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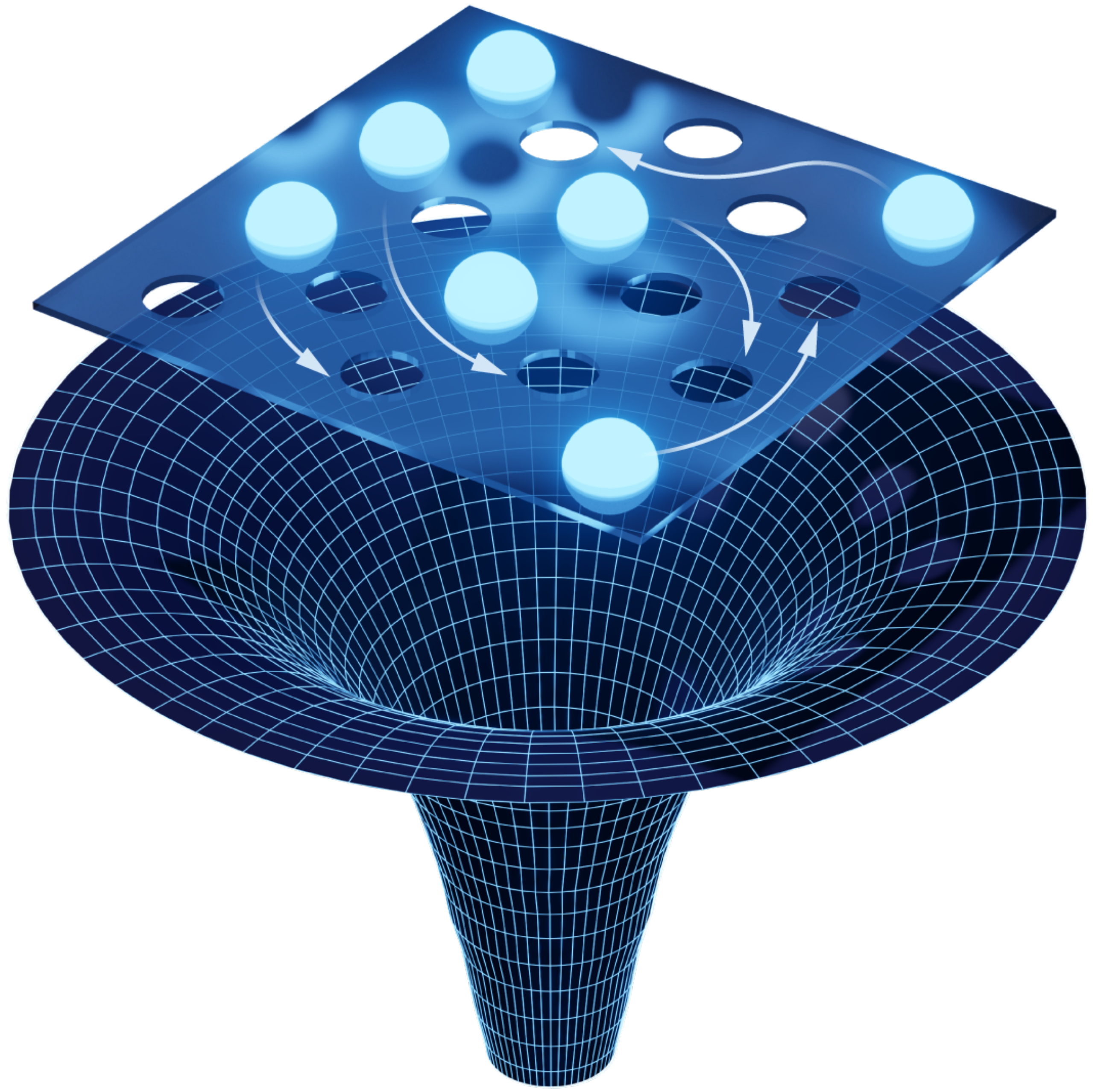
- Toy SYK model captures the correct universal low energy quantum theory of charged black holes, and provides a Hamiltonian realization of black hole microstates.



Summary

- SYK: a solvable toy model without particle-like excitations, exhibiting thermalization and many-body chaos in a time of order $\hbar/(k_B T)$, independent of microscopic energy scales.

- Toy SYK model captures the correct universal low energy quantum theory of charged black holes, and provides a Hamiltonian realization of black hole microstates.

- Linear-T resistivity, $T \ln(1/T)$ specific heat, $\sim 1/\omega$ optical conductivity, and marginal Fermi liquid electron spectrum *all* arise from a SYK-like model with spatially random interactions in a two-dimensional quantum-critical metal.

Aavishkar Patel, Haoyu Guo, Ilya Esterlis, S.S. arXiv: 2203.04990