Strange metals
and
black holes

Subir Sachdev

Dynamics and Disorder in
Quantum Many Body Systems Far from Equilibrium
Les Houches Summer School, August 19-21, 2019

PHYSICS
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Remarkable recent observation of 'Planckian' strange metal transport in cuprates, pnictides, magic-angle graphene, and ultracold atoms: the resistivity, ρ, is

$$\rho = \frac{m^*}{ne^2}\frac{1}{\tau}$$

with a universal scattering rate

$$\frac{1}{\tau} \approx \frac{k_B T}{\hbar}\,,$$

independent of the strength of interactions!

Remarkable recent observation of 'Planckian' strange metal transport in cuprates, pnictides, magic-angle graphene, and ultracold atoms: the resistivity is associated with a universal scattering time $\approx \hbar/(k_B T)$.

Universal *T*-linear resistivity and Planckian dissipation in overdoped cuprates

NATURE PHYSICS | VOL 15 | FEBRUARY 2019 | 142–147

A. Legros[1,2], S. Benhabib[3], W. Tabis[3,4], F. Laliberté[1], M. Dion[1], M. Lizaire[1], B. Vignolle[3], D. Vignolles[3], H. Raffy[5], Z. Z. Li[5], P. Auban-Senzier[5], N. Doiron-Leyraud[1], P. Fournier[1,6], D. Colson[2], L. Taillefer[1,6*] and C. Proust[3,6*]

arXiv:1902.01034

Planckian dissipation and scale invariance in a quantum-critical disordered pnictide

Yasuyuki Nakajima,[1,2] Tristin Metz,[2] Christopher Eckberg,[2] Kevin Kirshenbaum,[2] Alex Hughes,[2] Renxiong Wang,[2] Limin Wang,[2] Shanta R. Saha,[2] I-Lin Liu,[2,3,4] Nicholas P. Butch,[2,4] Zhonghao Liu,[5,6] Sergey V. Borisenko,[5] Peter Y. Zavalij,[7] and Johnpierre Paglione[2,8]

Strange metal in magic-angle graphene with near Planckian dissipation

Yuan Cao,[1,*] Debanjan Chowdhury,[1,*] Daniel Rodan-Legrain,[1] Oriol Rubies-Bigordà,[1] Kenji Watanabe,[2] Takashi Taniguchi,[2] T. Senthil,[1,†] and Pablo Jarillo-Herrero[1,†]

arXiv:1901.03710

Bad metallic transport in a cold atom Fermi-Hubbard system

Science **363**, 379–382 (2019)

Peter T. Brown[1], Debayan Mitra[1], Elmer Guardado-Sanchez[1], Reza Nourafkan[2], Alexis Reymbaut[2], Charles-David Hébert[2], Simon Bergeron[2], A.-M. S. Tremblay[2,3], Jure Kokalj[4,5], David A. Huse[1], Peter Schauß[1*], Waseem S. Bakr[1]†

Material		n (10^{27} m^{-3})	m^* (m_0)	A_1 / d (Ω / K)	$h / (2e^2 T_F)$ (Ω / K)	α
Bi2212	$p = 0.23$	6.8	8.4 ± 1.6	8.0 ± 0.9	7.4 ± 1.4	1.1 ± 0.3
Bi2201	$p \sim 0.4$	3.5	7 ± 1.5	8 ± 2	8 ± 2	1.0 ± 0.4
LSCO	$p = 0.26$	7.8	9.8 ± 1.7	8.2 ± 1.0	8.9 ± 1.8	0.9 ± 0.3
Nd-LSCO	$p = 0.24$	7.9	12 ± 4	7.4 ± 0.8	10.6 ± 3.7	0.7 ± 0.4
PCCO	$x = 0.17$	8.8	2.4 ± 0.1	1.7 ± 0.3	2.1 ± 0.1	0.8 ± 0.2
LCCO	$x = 0.15$	9.0	3.0 ± 0.3	3.0 ± 0.45	2.6 ± 0.3	1.2 ± 0.3
TMTSF	$P = 11$ kbar	1.4	1.15 ± 0.2	2.8 ± 0.3	2.8 ± 0.4	1.0 ± 0.3

Slope of *T*-linear resistivity vs Planckian limit in seven materials.

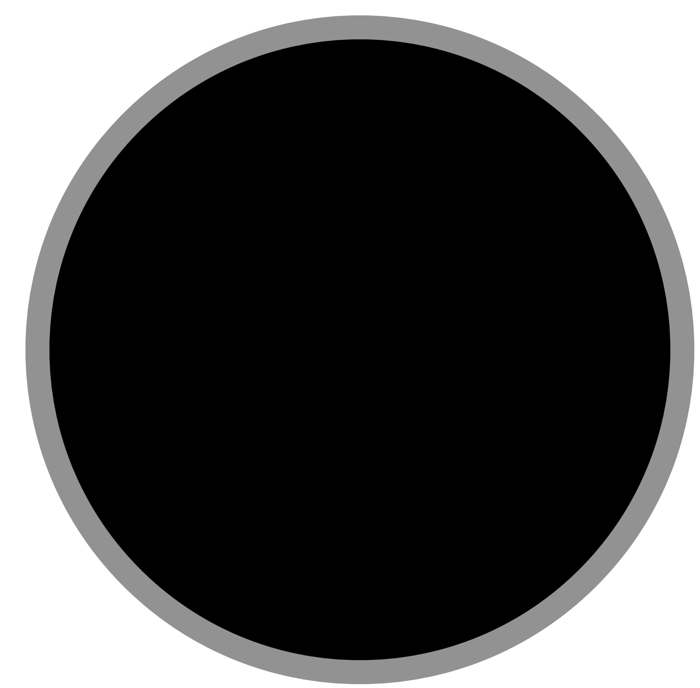
$$\frac{1}{\tau} = \alpha \frac{k_B T}{\hbar}$$

A. Legros, S. Benhabib, W. Tabis, F. Laliberté, M. Dion, M. Lizaire, B. Vignolle, D. Vignolles, H. Raffy, Z. Z. Li, P. Auban-Senzier, N. Doiron-Leyraud, P. Fournier, D. Colson, L. Taillefer, and C. Proust, Nature Physics **15**, 142 (2019)

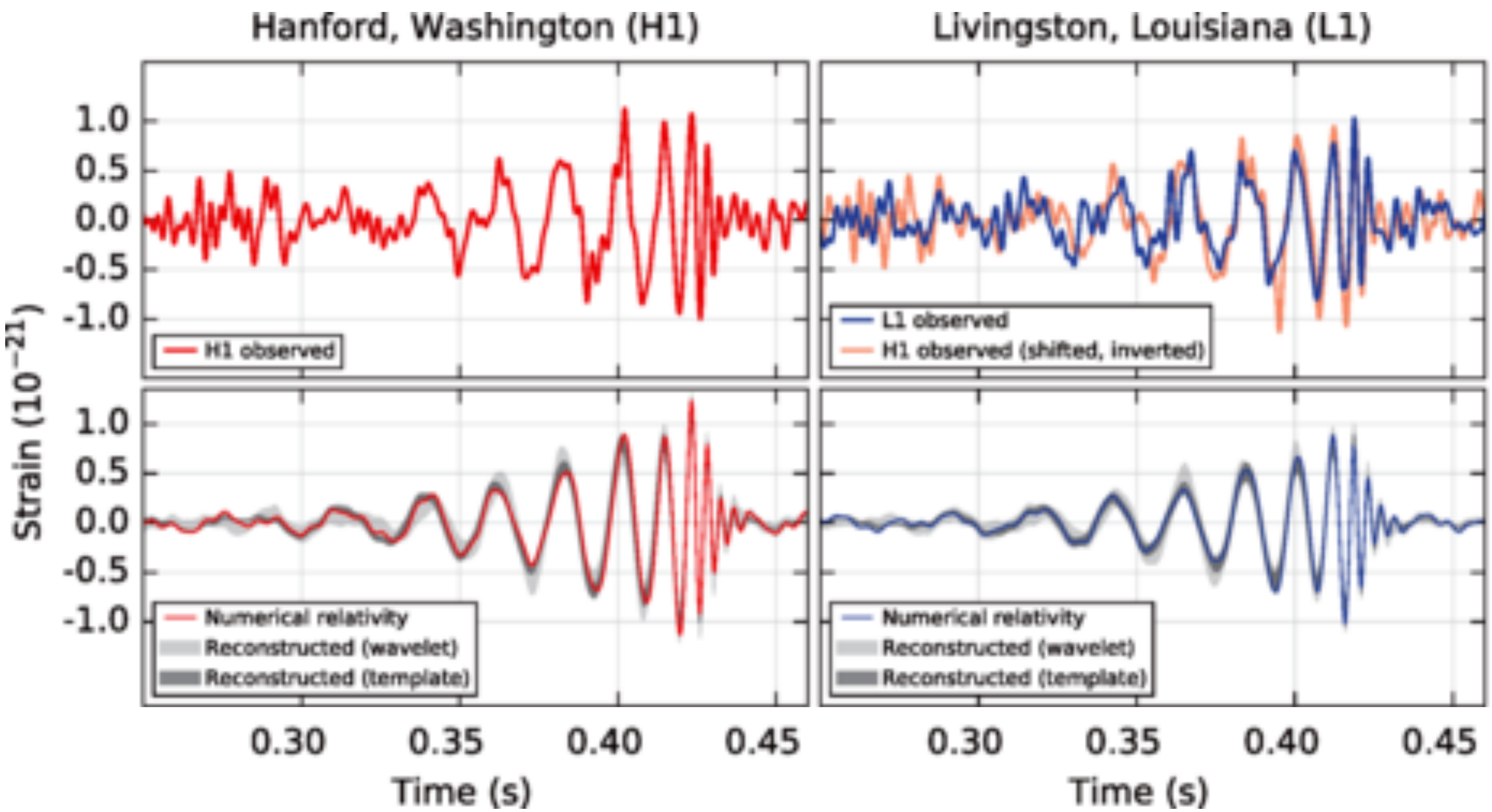
Black Holes

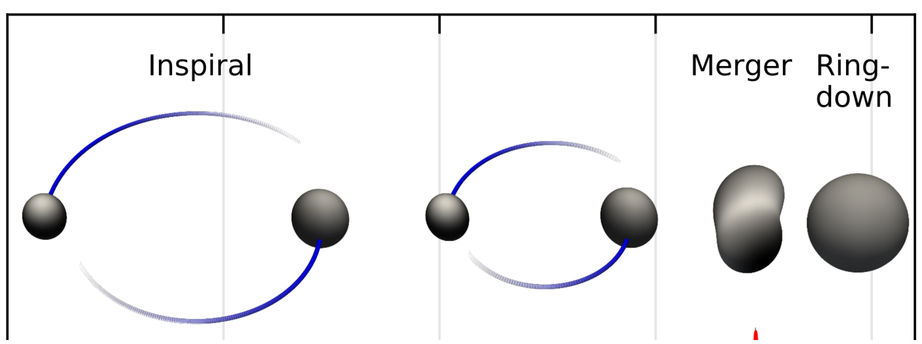
Objects so dense that light is gravitationally bound to them.

In Einstein's theory, the region inside the black hole **horizon** is disconnected from the rest of the universe.



$$\text{Horizon radius } R = \frac{2GM}{c^2}$$





LIGO
September 14, 2015

- The ring-down is predicted by General Relativity to happen in a time $\frac{8\pi GM}{c^3} \sim 8$ milliseconds. Curiously this happens to equal $\boxed{\frac{\hbar}{k_B T_H}}$: so the ring down can also be viewed as the approach of a quantum system to thermal equilibrium at the fastest possible rate!

Black holes

- Black holes have an entropy and a temperature, $T_H = \hbar c^3/(8\pi G M k_B)$.
- The entropy is proportional to their surface area.
- They relax to thermal equilibrium in a Planckian time $\sim \hbar/(k_B T_H)$.

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Holography:
Quantum black holes "look like" quantum many-particle systems without quasiparticle excitations, residing "on" the surface of the black hole

J. Maldacena, IJTP **38**, 1113 (1999); S.S. Gubser, I.R. Klebanov, and A.M. Polyakov Phys. Lett. B **428**, 105 (1998);
E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998)

1. Quantum matter with quasiparticles:
random matrix model

2. Quantum matter without quasiparticles:
the complex SYK model

3. Fluctuations, and the Schwarzian

4. Models of strange metals

5. Einstein-Maxwell theory of charged
black holes in AdS space

1. Quantum matter with quasiparticles: random matrix model

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What are quasiparticles ?

- **Quasiparticles are additive excitations:** The low-lying excitations of the many-body system can be identified as a set $\{n_\alpha\}$ of quasiparticles with energy ε_α

$$E = \sum_\alpha n_\alpha \varepsilon_\alpha + \sum_{\alpha,\beta} F_{\alpha\beta} n_\alpha n_\beta + \ldots$$

In a lattice system of N sites, this parameterizes the energy of $\sim e^{\alpha N}$ states in terms of poly(N) numbers.

Ordinary metals and quasiparticles

- Quasiparticles eventually collide with each other. Such collisions eventually leads to thermal equilibration in a chaotic quantum state, but the equilibration takes a long time. In a Fermi liquid, this time diverges as

$$\tau_{\rm eq} \sim \frac{\hbar E_F^3}{U^2 (k_B T)^2} \quad , \quad \text{as } T \to 0,$$

  where U is the strength of interactions, and E_F is the Fermi energy.

- Similarly, a quasiparticle model implies a resistivity

$$\rho = \frac{m^*}{ne^2}\frac{1}{\tau} \sim U^2 T^2 \quad \text{with} \;\; \tau \sim \tau_{\rm eq}$$

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$$\rho = \frac{m^*}{ne^2}\frac{1}{\tau} \sim U^2 T^2 \quad \text{with} \;\; \tau \sim \tau_{\rm eq}$$

- These times are much longer than the 'Planckian time' $\hbar/(k_B T)$, which we will find in systems without quasiparticle excitations.

$$\tau \sim \tau_{\rm eq} \gg \frac{\hbar}{k_B T} \quad , \quad \text{as } T \to 0.$$

A simple model of a metal with quasiparticles

$$H = \frac{1}{(N)^{1/2}} \sum_{i,j=1}^{N} t_{ij} c_i^\dagger c_j - \mu \sum_i c_i^\dagger c_i$$

$$c_i c_j + c_j c_i = 0 \quad , \quad c_i c_j^\dagger + c_j^\dagger c_i = \delta_{ij}$$

$$\frac{1}{N} \sum_i c_i^\dagger c_i = \mathcal{Q}$$

t_{ij} are independent random variables with $\overline{t_{ij}} = 0$ and $\overline{|t_{ij}|^2} = t^2$

Fermions occupying the eigenstates of a
N x *N* random matrix

A simple model of a metal with quasiparticles

Feynman graph expansion in $t_{ij..}$, and graph-by-graph average, yields exact equations in the large N limit:

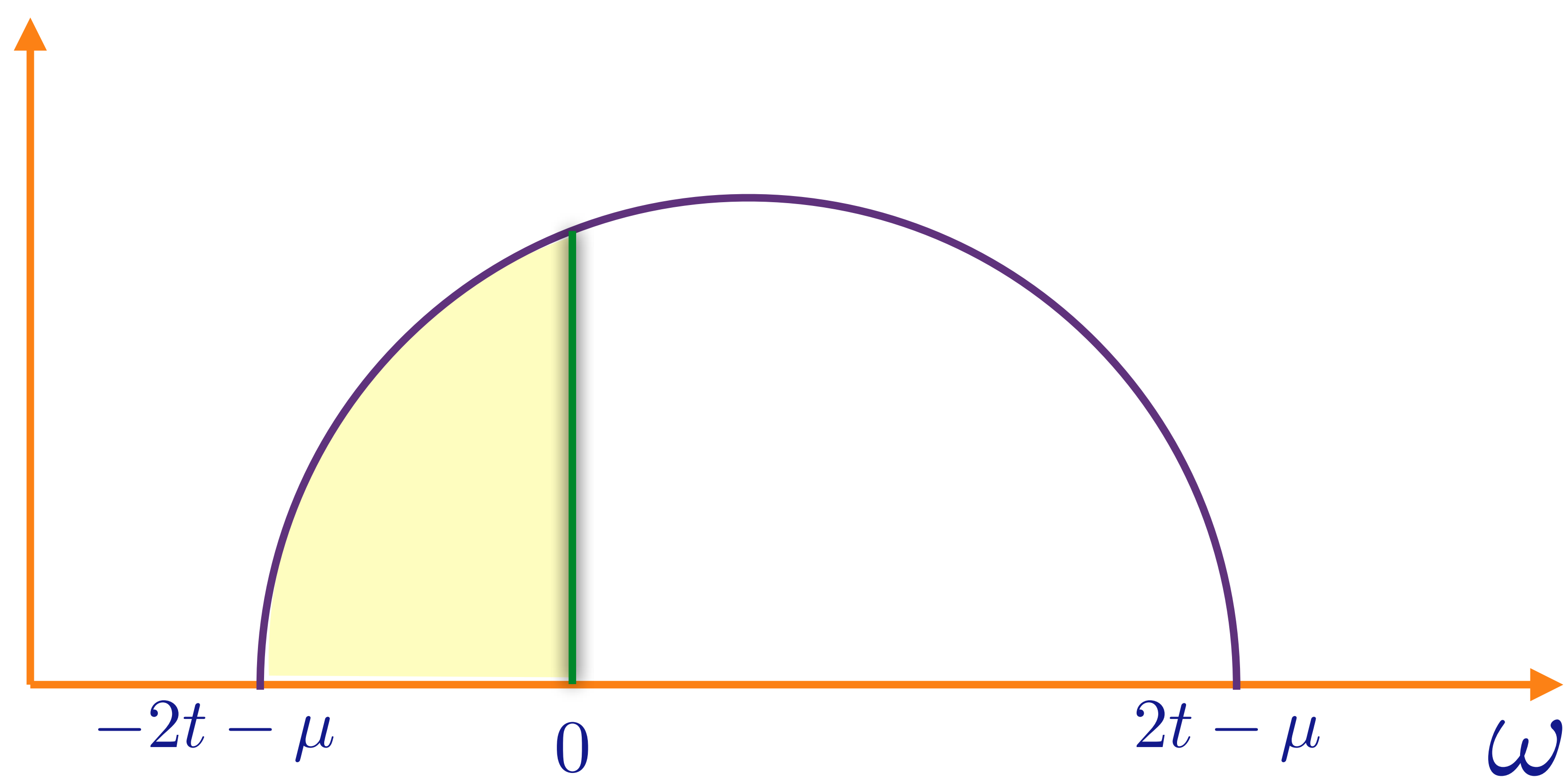
$$G(\tau) \equiv -T_\tau \left\langle c_i(\tau) c_i^\dagger(0) \right\rangle$$

$$G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = t^2 G(\tau)$$

$$G(\tau = 0^-) = \mathcal{Q}.$$

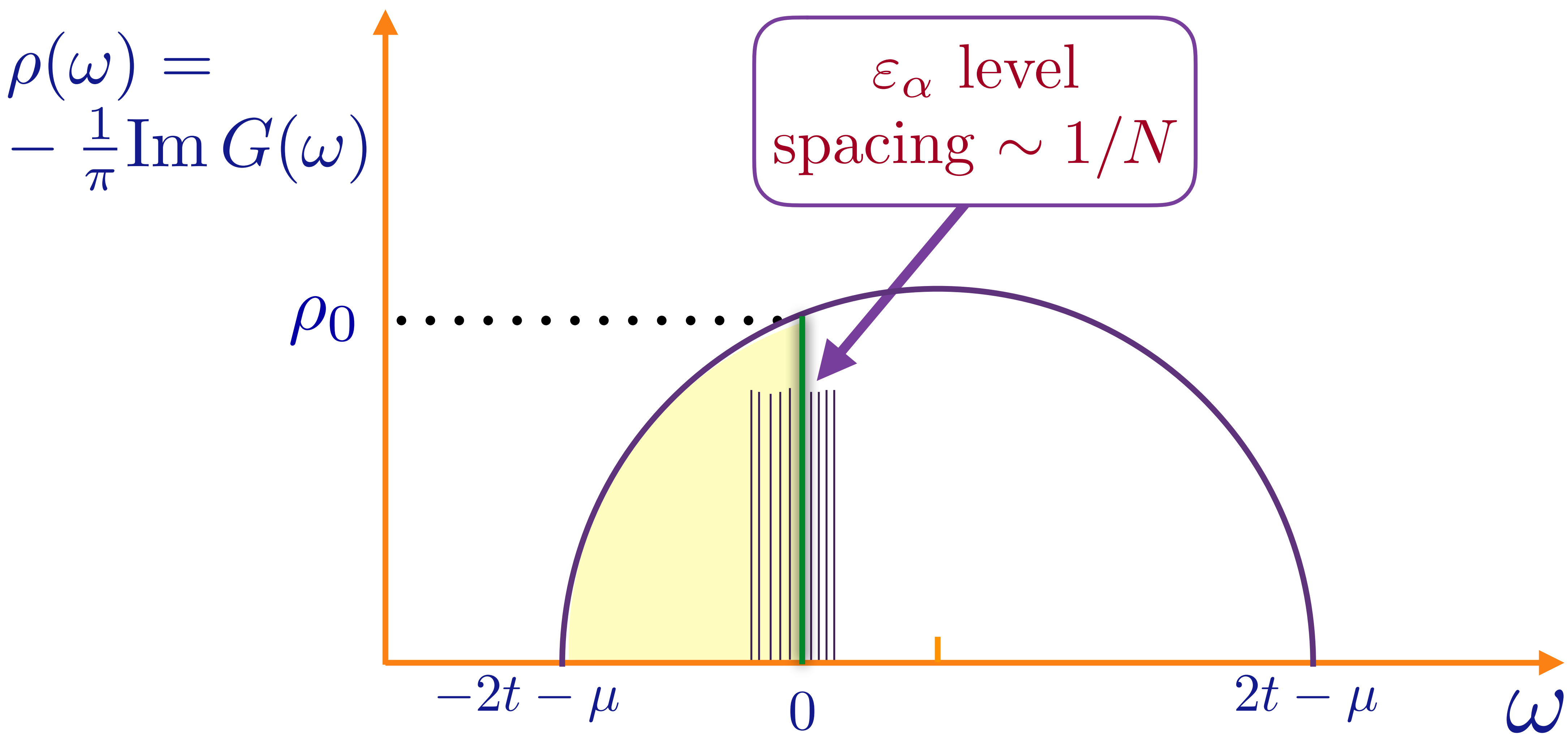
$G(\omega)$ can be determined by solving a quadratic equation.

$\rho(\omega) = -\frac{1}{\pi} \mathrm{Im}\, G(\omega)$



A simple model of a metal with quasiparticles

Let ε_α be the eigenvalues of the matrix $t_{ij}/\sqrt{N}$. The fermions will occupy the lowest NQ eigenvalues, upto the Fermi energy E_F. The single-particle density of states is
$\rho(\omega) = (1/N)\sum_\alpha \delta(\omega - \varepsilon_\alpha)$, and $\rho_0 \equiv \rho(\omega = 0)$.



A simple model of a metal with quasiparticles

The grand potential $\Omega(T)$ at low T is (from the Sommerfeld expansion)

$$\Omega(T) - E_0 = N\left(-\frac{\pi^2}{6}\rho_0 T^2 + \mathcal{O}(T^4)\right) + \ldots$$

where $\rho_0 \equiv \rho(0)$ is the *single* particle density of states at the Fermi level.

We can also define the *many* body density of states, $D(E)$, via

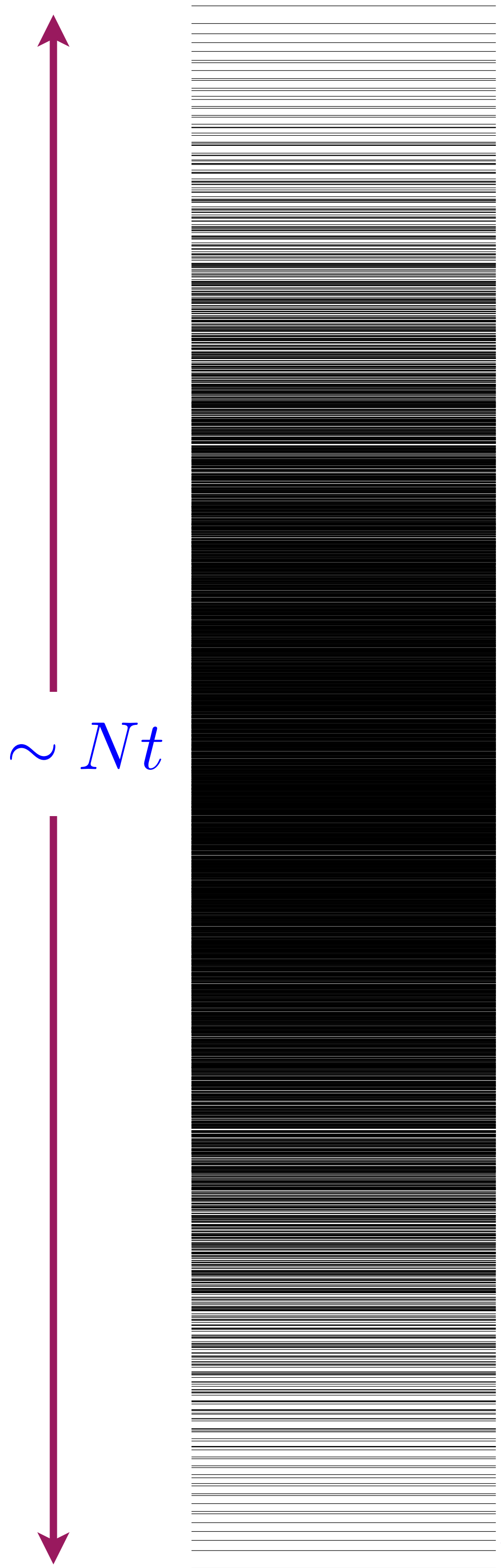
$$Z = e^{-\Omega(T)/T} = \int_{-\infty}^{\infty} dE D(E) e^{-E/T}$$

The inversion from $\Omega(T)$ to $D(E)$ has to performed with care (it need not commute with the $1/N$ expansion), and we obtain

$$\boxed{D(E) \sim \exp\left(\pi\sqrt{\frac{2N\rho_0(E-E_0)}{3}}\right) \quad , \quad E > E_0 \; , \; \frac{1}{N} \ll \rho_0(E-E_0) \ll N}$$

and $D(E) = 0$ for $E < E_0$. This is related to the asymptotic growth of the partitions of an integer, $p(n) \sim \exp(\pi\sqrt{2n/3})$. Near the lower bound, there are large sample-to-sample fluctuations due to variations in the lowest quasiparticle energies.

A simple model of a metal with quasiparticles



Many-body level spacing $\sim 2^{-N}$

Quasiparticle excitations with spacing $\sim 1/N$

There are 2^N many body levels with energy

$$E = \sum_{\alpha=1}^{N} n_\alpha \varepsilon_\alpha,$$

where $n_\alpha = 0, 1$. Shown are all values of E for a single cluster of size $N = 12$. The ε_α have a level spacing $\sim 1/N$.

A simple model of a metal with quasiparticles

Now add weak interactions

$$H = \frac{1}{(N)^{1/2}} \sum_{i,j=1}^{N} t_{ij} c_i^\dagger c_j - \mu \sum_i c_i^\dagger c_i + \frac{1}{(2N)^{3/2}} \sum_{i,j,k,\ell=1}^{N} U_{ij;k\ell}\, c_i^\dagger c_j^\dagger c_k c_\ell$$

$U_{ij;k\ell}$ are independent random variables with $\overline{U_{ij;k\ell}} = 0$ and $\overline{|U_{ij;k\ell}|^2} = U^2$. We compute the lifetime of a quasiparticle, τ_α, in an exact eigenstate $\psi_\alpha(i)$ of the free particle Hamitonian with energy ε_α. By Fermi's Golden rule, for ε_α at the Fermi energy

$$\begin{aligned}\frac{1}{\tau_\alpha} &= \pi U^2 \rho_0^3 \int d\varepsilon_\beta d\varepsilon_\gamma d\varepsilon_\delta f(\varepsilon_\beta)(1 - f(\varepsilon_\gamma))(1 - f(\varepsilon_\delta))\delta(\varepsilon_\alpha + \varepsilon_\beta - \varepsilon_\gamma - \varepsilon_\delta) \\ &= \frac{\pi^3 U^2 \rho_0^3}{4} T^2\end{aligned}$$

where ρ_0 is the density of states at the Fermi energy, and $f(\epsilon) = 1/(e^{\epsilon/T} + 1)$ is the Fermi function.

Fermi liquid state: Two-body interactions lead to a scattering time of quasiparticle excitations from in (random) single-particle eigenstates which diverges as $\sim T^{-2}$ at the Fermi level.

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The complex SYK model

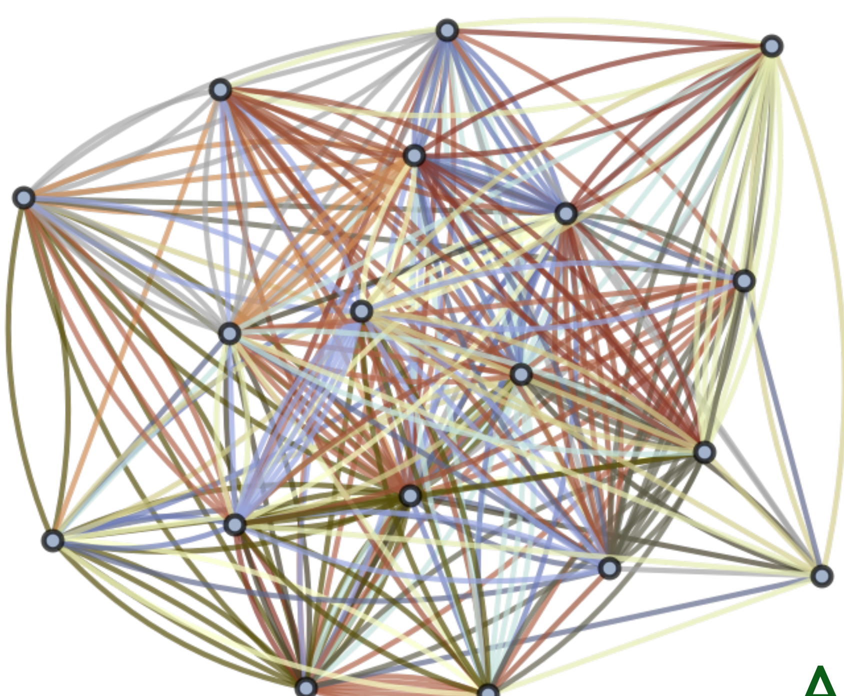
(See also: the "2-Body Random Ensemble" in nuclear physics; did not obtain the large N limit;
T.A. Brody, J. Flores, J.B. French, P.A. Mello, A. Pandey, and S.S.M. Wong, Rev. Mod. Phys. **53**, 385 (1981))

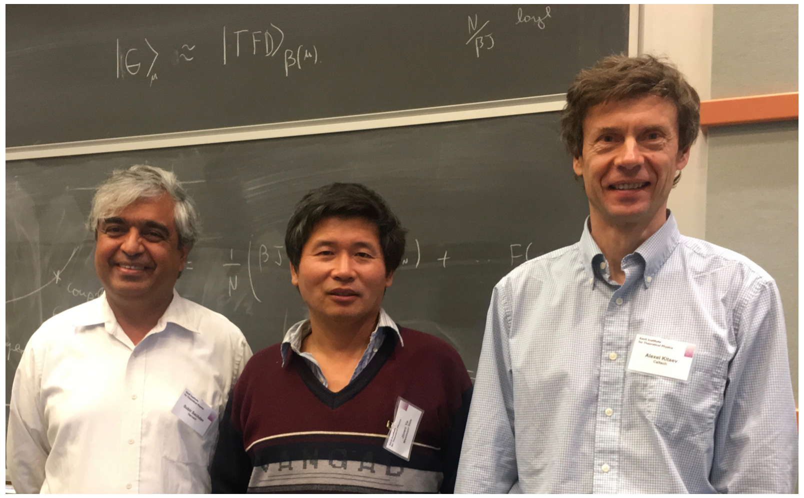
$$H = \frac{1}{(2N)^{3/2}} \sum_{\alpha,\beta,\gamma,\delta=1}^{N} U_{\alpha\beta;\gamma\delta}\, c^\dagger_\alpha c^\dagger_\beta c_\gamma c_\delta + \epsilon \sum_\alpha c^\dagger_\alpha c_\alpha$$

$$c_\alpha c_\beta + c_\beta c_\alpha = 0 \quad , \quad c_\alpha c^\dagger_\beta + c^\dagger_\beta c_\alpha = \delta_{\alpha\beta}$$

$$\mathcal{Q} = \frac{1}{N} \sum_\alpha c^\dagger_\alpha c_\alpha$$

$U_{\alpha\beta;\gamma\delta}$ are independent random variables with $\overline{U_{\alpha\beta;\gamma\delta}} = 0$ and $\overline{|U_{\alpha\beta;\gamma\delta}|^2} = U^2$
$N \to \infty$ yields critical strange metal.





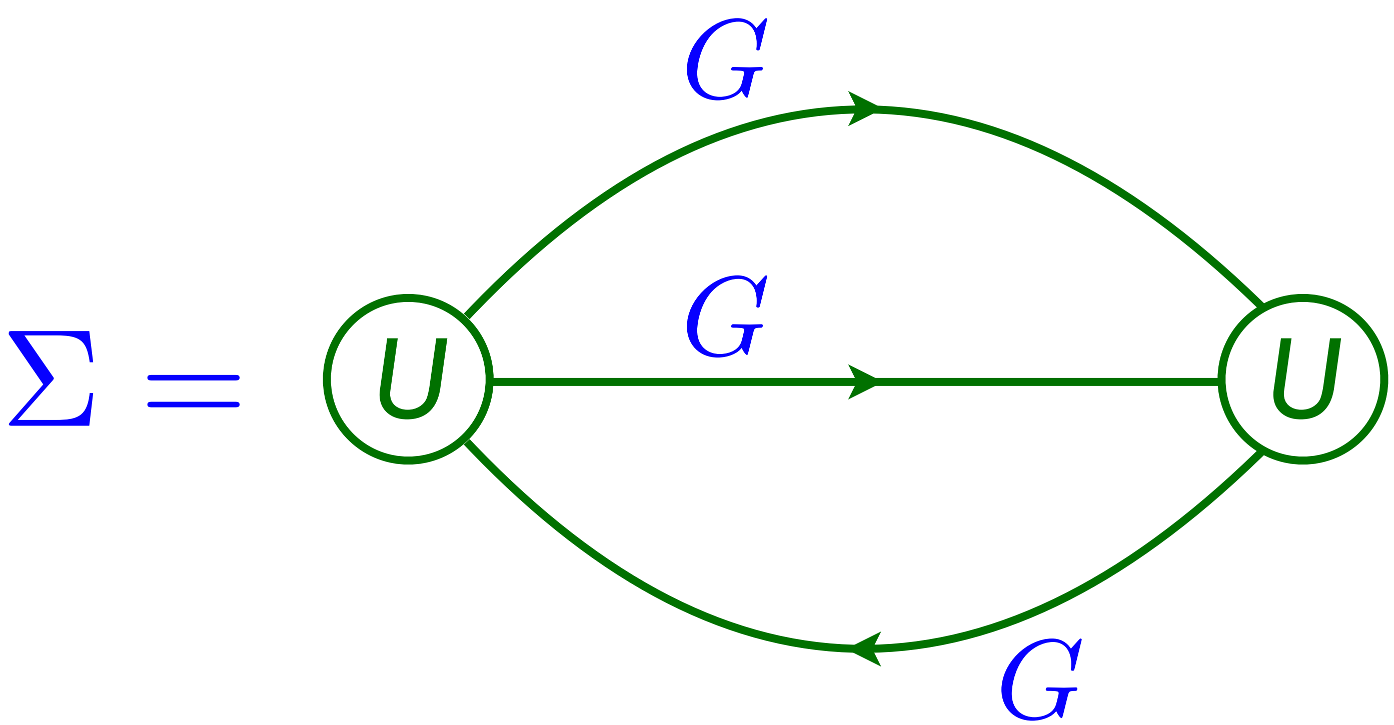
S. Sachdev and J. Ye, PRL **70**, 3339 (1993)

A. Kitaev, unpublished; S. Sachdev, PRX **5**, 041025 (2015)

The complex SYK model

Feynman graph expansion in $U_{\alpha\beta;\gamma\delta}$, and graph-by-graph average, yields exact equations in the large N limit:

$$\boxed{\begin{gathered} G(i\omega) = \frac{1}{i\omega - \epsilon - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = -U^2 G^2(\tau) G(-\tau) \\ G(\tau = 0^-) = \mathcal{Q}. \end{gathered}}$$



S. Sachdev and J. Ye,
PRL **70**, 3339 (1993)

The complex SYK model

The large N limit is given by the sum of "melon" Feynman graphs

For long times $\tau > 0$

$$\left\langle c_\alpha(\tau) c_\alpha^\dagger(0) \right\rangle = \frac{A}{\sqrt{\tau}}$$

$$\left\langle c_\alpha^\dagger(\tau) c_\alpha(0) \right\rangle = e^{-2\pi\mathcal{E}} \frac{A}{\sqrt{\tau}}$$

The parameter $\mathcal{E} = \mathbb{C}\left(\epsilon/U\right)$ determines the particle-hole asymmetry, and has a universal "Luttinger" relation to $\mathcal{Q}$.

In a Fermi liquid,

$$\left\langle c_\alpha(\tau) c_\alpha^\dagger(0) \right\rangle = \left\langle c_\alpha^\dagger(\tau) c_\alpha(0) \right\rangle = \widetilde{A}/\tau$$

The complex SYK model

Solution of these equations, and of the free energy, yields universal results for the SYK model with q fermion terms. These results are *quantitatively* unchanged by adding additional higher q fermion terms:

- At long times, and at $T = 0$, $G(\tau) \sim |\tau|^{-2\Delta}$ with $\Delta = 1/q$ ($\Rightarrow$ indication there are no quasiparticles)

- At general charge $\mathcal{Q}$, there is a spectral symmetry determined by a parameter $\mathcal{E}$:

$$G(\tau) \sim \begin{cases} -\tau^{-2\Delta} & \tau > 0 \\ e^{-2\pi\mathcal{E}}(-\tau)^{-2\Delta} & \tau < 0 \end{cases} \quad , \quad T = 0$$

- There is a universal 'Luttinger relation' between $-\infty < \mathcal{E} < \infty$ and the total charge $0 < \mathcal{Q} < 1$

$$\begin{aligned} e^{2\pi\mathcal{E}} &= \frac{\sin(\pi\Delta + \theta)}{\sin(\pi\Delta - \theta)} \\ \mathcal{Q} &= \frac{1}{2} - \frac{\theta}{\pi} + \left(\Delta - \frac{1}{2}\right)\frac{\sin(2\theta)}{\sin(2\pi\Delta)} \end{aligned}$$

A. Georges, O. Parcollet, and S. Sachdev, PRB **63**, 134406 (2001)
R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, PRB **95**, 155131 (2017)

The complex SYK model

Solution of these equations, and of the free energy, yields universal results for the SYK model with q fermion terms. These results are *quantitatively* unchanged by adding additional higher q fermion terms:

- At $T > 0$, we obtain a solution with a conformal structure

$$G(\tau) = -A \frac{e^{-2\pi \mathcal{E} T \tau}}{\sqrt{1 + e^{-4\pi \mathcal{E}}}} \left(\frac{T}{\sin(\pi T \tau)} \right)^{1/2} \quad , \quad 0 < \tau < 1/T \, ,$$

where the 'particle-hole asymmetry' is determined by $\mathcal{E}$

A. Georges and O. Parcollet PRB **59**, 5341 (1999)
S. Sachdev, PRX **5**, 041025 (2015)

The complex SYK model

The equations for the Green's function can also be solved at non-zero T. At $\epsilon = \mathcal{E} = 0$ we "guess" the solution

$$G(\tau) = B\, \mathrm{sgn}(\tau) \left| \frac{\pi T}{\sin(\pi T \tau)} \right|^{\rho}$$

Then the self-energy is

$$\Sigma(\tau) = U^2 B^3 \mathrm{sgn}(\tau) \left| \frac{\pi T}{\sin(\pi T \tau)} \right|^{3\rho}$$

A. Georges and O. Parcollet
PRB **59**, 5341 (1999)

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Taking Fourier transforms, we have as a function of the Matsubara frequency ω_n

$$\begin{aligned} G(i\omega_n) &= [iB\Pi(\rho)]\,\frac{T^{\rho-1}\,\Gamma\left(\dfrac{\rho}{2}+\dfrac{\omega_n}{2\pi T}\right)}{\Gamma\left(1-\dfrac{\rho}{2}+\dfrac{\omega_n}{2\pi T}\right)} \\ \Sigma_{\text{sing}}(i\omega_n) &= \left[iU^2B^3\Pi(3\rho)\right]\,\frac{T^{3\rho-1}\,\Gamma\left(\dfrac{3\rho}{2}+\dfrac{\omega_n}{2\pi T}\right)}{\Gamma\left(1-\dfrac{3\rho}{2}+\dfrac{\omega_n}{2\pi T}\right)}\,, \end{aligned}$$

A. Georges and O. Parcollet
PRB **59**, 5341 (1999)

The complex SYK model

$$G(i\omega_n) = [iB\Pi(\rho)] \frac{T^{\rho-1}\,\Gamma\left(\frac{\rho}{2} + \frac{\omega_n}{2\pi T}\right)}{\Gamma\left(1 - \frac{\rho}{2} + \frac{\omega_n}{2\pi T}\right)}$$

$$\Sigma_{\rm sing}(i\omega_n) = \left[iU^2 B^3 \Pi(3\rho)\right] \frac{T^{3\rho-1}\,\Gamma\left(\frac{3\rho}{2} + \frac{\omega_n}{2\pi T}\right)}{\Gamma\left(1 - \frac{3\rho}{2} + \frac{\omega_n}{2\pi T}\right)},$$

where we have dropped a less-singular term in Σ, and

$$\Pi(s) \equiv \pi^{s-1} 2^s \cos\left(\frac{\pi s}{2}\right) \Gamma(1-s).$$

Now the singular part of Dyson's equation is

A. Georges and O. Parcollet
PRB **59**, 5341 (1999)

$$G(i\omega_n)\Sigma_{\rm sing}(i\omega_n) = -1$$

Remarkably, the Γ functions appear with just the right arguments, so that there is a solution of the Dyson equation at $\rho = 1/2$!

So the Green's functions display thermal 'damping' at a scale set by T alone, which is independent of U.

The complex SYK model

$$G^R_{\mathrm{SYK}}(\epsilon, \hbar\omega/(k_B T)) =$$

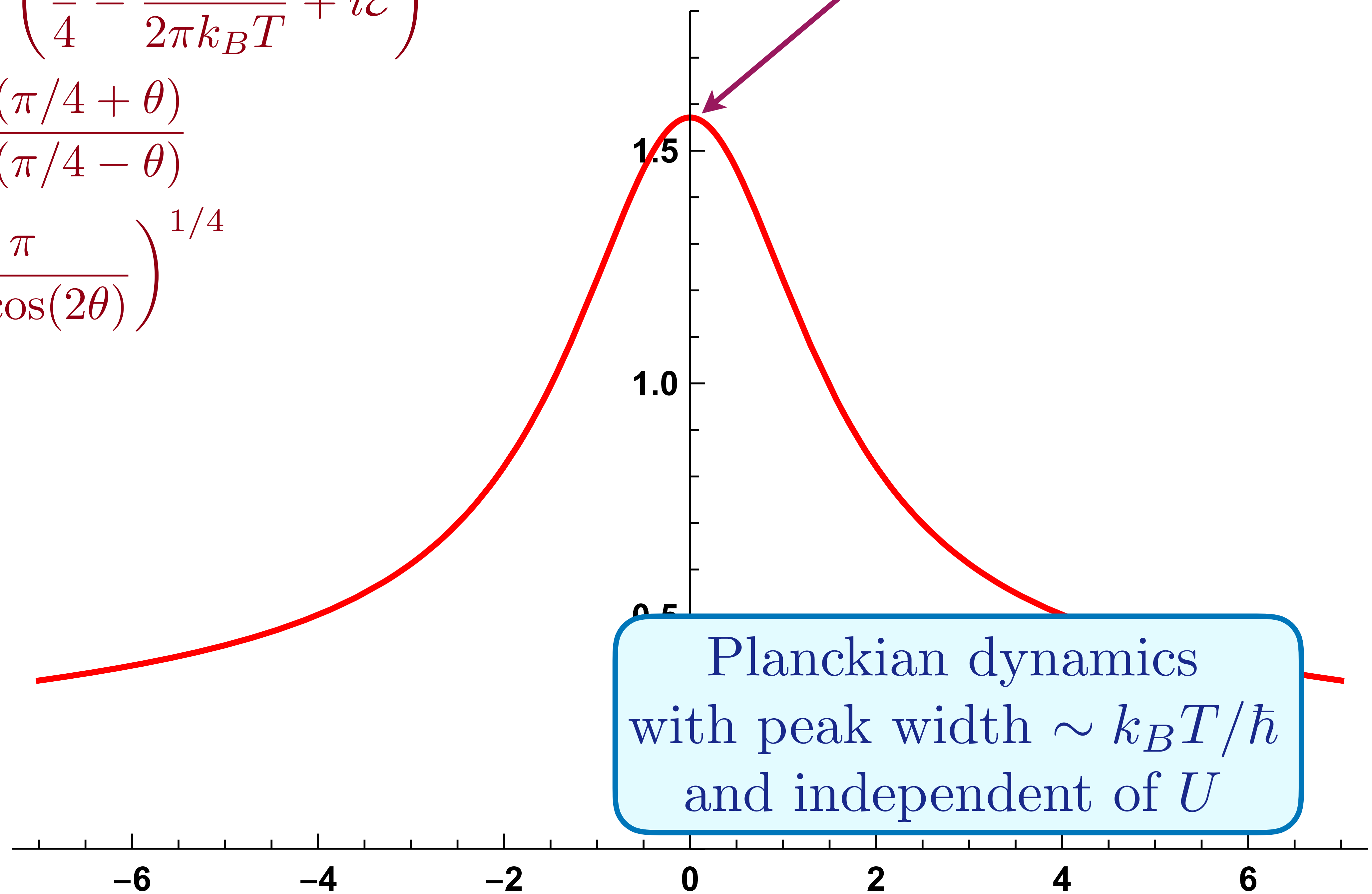
$$\frac{-iCe^{-i\theta}}{(2\pi T)^{1/2}} \frac{\Gamma\left(\frac{1}{4} - \frac{i\hbar\omega}{2\pi k_B T} + i\mathcal{E}\right)}{\Gamma\left(\frac{3}{4} - \frac{i\hbar\omega}{2\pi k_B T} + i\mathcal{E}\right)}.$$

$$e^{2\pi\mathcal{E}} = \frac{\sin(\pi/4 + \theta)}{\sin(\pi/4 - \theta)}$$

$$C = \left(\frac{\pi}{U^2 \cos(2\theta)}\right)^{1/4}$$

$$\boxed{\mathcal{E} = \mathbb{C}\frac{\epsilon}{U}}$$

$-\mathrm{Im}G^R(\omega)$ $\boxed{\mathcal{E} = 0}$



Planckian dynamics with peak width $\sim k_B T/\hbar$ and independent of U

$\hbar\omega/(k_B T)$

A. Georges and O. Parcollet PRB **59**, 5341 (1999)
S. Sachdev, PRX **5**, 041025 (2015)

The complex SYK model

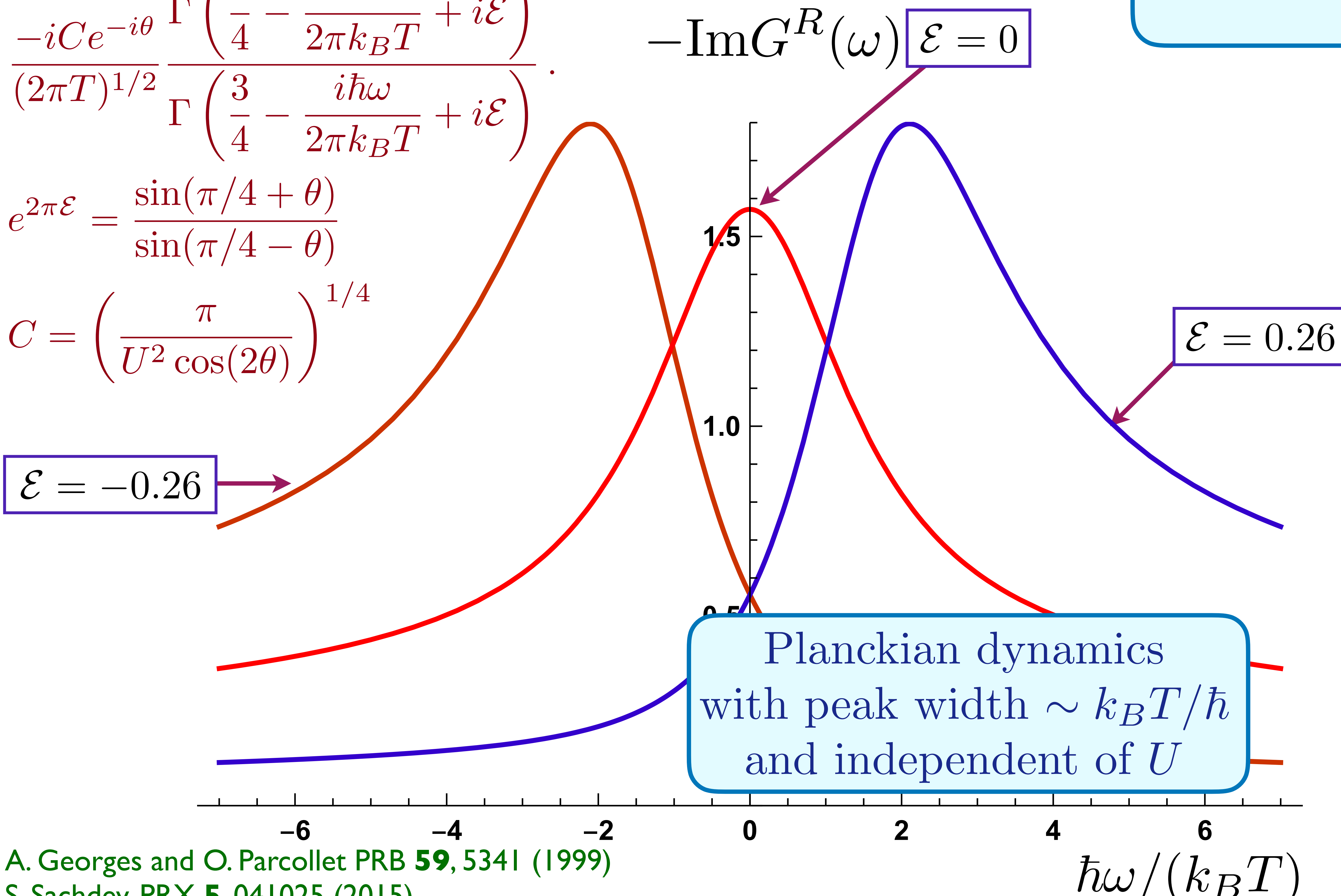
$$G^R_{\rm SYK}(\epsilon, \hbar\omega/(k_B T)) =$$

$$\frac{-iCe^{-i\theta}}{(2\pi T)^{1/2}} \frac{\Gamma\left(\frac{1}{4} - \frac{i\hbar\omega}{2\pi k_B T} + i\mathcal{E}\right)}{\Gamma\left(\frac{3}{4} - \frac{i\hbar\omega}{2\pi k_B T} + i\mathcal{E}\right)} .$$

$$e^{2\pi\mathcal{E}} = \frac{\sin(\pi/4 + \theta)}{\sin(\pi/4 - \theta)}$$

$$C = \left(\frac{\pi}{U^2 \cos(2\theta)}\right)^{1/4}$$

$$\boxed{\mathcal{E} = \mathbb{C}\frac{\epsilon}{U}}$$



A. Georges and O. Parcollet PRB **59**, 5341 (1999)
S. Sachdev, PRX **5**, 041025 (2015)

The complex SYK model

We now examine the behavior of the chemical potential, μ, as $T \to 0$ at fixed $\mathcal{Q}$. For this we relate the long-time 'conformal' Greens function, (valid for $\tau \gg 1/U$) to its short-time behavior. In particular at $|\omega_n| \gg U$ we have

$$G(i\omega_n) = \frac{1}{i\omega_n} - \frac{\mu}{(i\omega_n)^2} + \ldots$$

which implies for the spectral density of the Green's function, $\rho(\Omega)$

$$\mu = -\int_{-\infty}^{\infty} \frac{d\Omega}{\pi}\, \Omega \rho(\Omega),$$

which makes it evident that μ depends only upon the particle-hole asymmetric part of the spectral density. Next, we can relate the Ω integrals to the derivative of the imaginary time correlator

$$\mu = -\partial_\tau G(\tau = 0^+) - \partial_\tau G(\tau = (1/T)^-).$$

The complex SYK model

We pull out an explicitly particle-hole asymmetric part of $G(\tau)$ by defining

$$G(\tau) \equiv e^{-2\pi\mathcal{E}T\tau} G_c(\tau) \quad , \quad 0 < \sigma < \frac{1}{T}.$$

where G_c will be given by a particle-hole symmetric conformal form at low T and low ω. Then we obtain

$$\begin{aligned} \mu &= 2\pi\mathcal{E}T \left[G(\tau = 0^+) + G(\tau = (1/T)^-) \right] \\ &\qquad + \text{terms dependent on } G_c \\ &= -2\pi\mathcal{E}T + \text{terms dependent on } G_c \end{aligned}$$

It can be shown that all the terms dependent upon G_c have a T dependence that is weaker than linear in T provided $\mathcal{Q}$ is held fixed. Hence we have

$$\mu = \mu_0 - 2\pi\mathcal{E}T + \text{terms vanishing as } T^p \text{ with } p > 1$$

with μ_0 a non-universal constant. From this relation we obtain

The complex SYK model

with μ_0 a non-universal constant. From this relation we obtain

$$\left(\frac{\partial \mu}{\partial T}\right)_{\mathcal{Q}} = -2\pi\mathcal{E} \quad , \quad T \to 0,$$

Using a Maxwell relation we then have

$$\frac{1}{N}\left(\frac{\partial S}{\partial \mathcal{Q}}\right)_T = 2\pi\mathcal{E} \neq 0 \quad \text{as } T \to 0.$$

The complex SYK model

Solution of these equations and corresponding evaluation of the free energy yields the following universal results (*i.e.* all results are *quantitatively* unchanged by adding additional higher q fermion terms):

- There is a non-vanishing entropy in the zero temperature limit

$$S(T \to 0) = N s_0 + \ldots$$

A. Georges, O. Parcollet, and S. Sachdev, PRB **63**, 134406 (2001)

The complex SYK model

Solution of these equations and corresponding evaluation of the free energy yields the following universal results (*i.e.* all results are *quantitatively* unchanged by adding additional higher q fermion terms):

- There is a non-vanishing entropy in the zero temperature limit

$$S(T \to 0) = N s_0 + \ldots$$

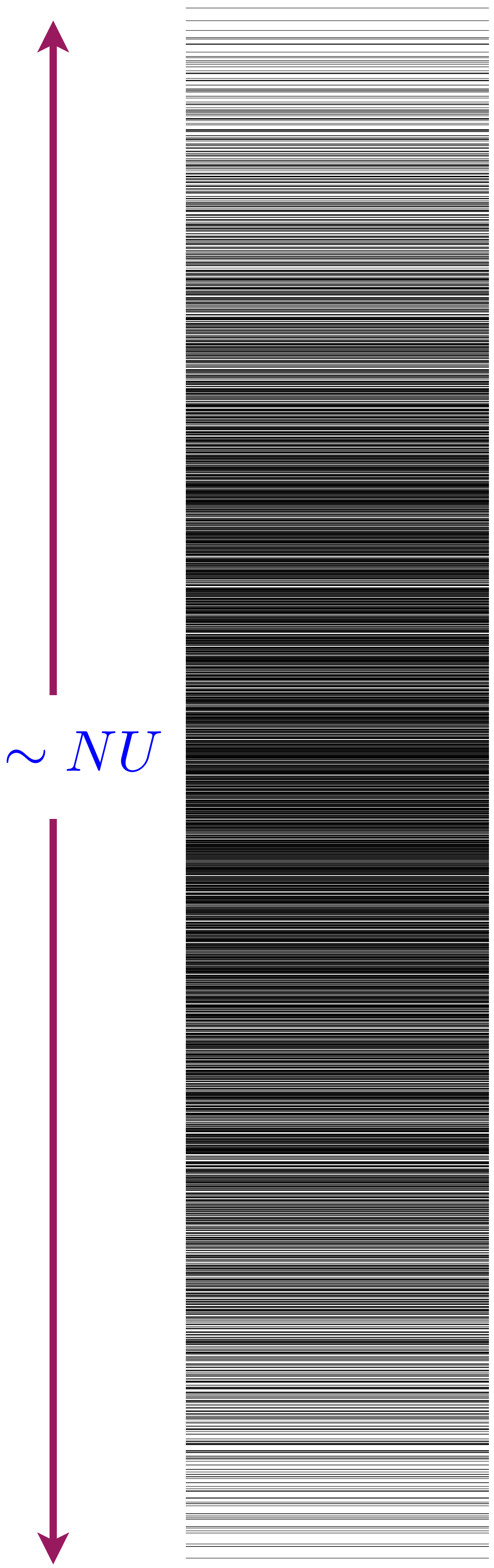
- The saddle point equations imply the relation

$$\boxed{\frac{ds_0}{d\mathcal{Q}} = 2\pi\mathcal{E}}$$

Integrating this relation from $s_0 = 0$, $\mathcal{Q} = 0$, allows us to compute s_0 as a function of $\mathcal{Q}$.

A. Georges, O. Parcollet, and S. Sachdev, PRB **63**, 134406 (2001)

The complex SYK model



Many-body level spacing $\sim 2^{-N} = e^{-N \ln 2}$

Non-quasiparticle excitations with spacing $\sim e^{-N s_0}$

There are 2^N many body levels with energy E. Shown are all values of E for a single cluster of size $N = 12$. The $T \to 0$ state has an entropy $S_{GPS} = N s_0$, where $s_0 < \ln 2$ is determined by integrating

$$\boxed{\frac{d s_0}{d\mathcal{Q}} = 2\pi \mathcal{E} \,.}$$

At $\mathcal{Q} = 1/2$,

$$s_0 \quad = \quad \frac{G}{\pi} + \frac{\ln(2)}{4} = 0.464848\ldots$$

where G is Catalan's constant.

GPS: A. Georges, O. Parcollet, and S. Sachdev, PRB **63**, 134406 (2001)

W. Fu and S. Sachdev, PRB **94**, 035135 (2016)

1. Quantum matter with quasiparticles:
   random matrix model

2. Quantum matter without quasiparticles:
   the complex SYK model

3. Fluctuations, and the Schwarzian

4. Models of strange metals

5. Einstein-Maxwell theory of charged
   black holes in AdS space

The SYK model

$$G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = -U^2 G^2(\tau) G(-\tau)$$

$$\Sigma(z) = \mu - \frac{1}{A}\sqrt{z} + \ldots \quad , \quad G(z) = \frac{A}{\sqrt{z}}$$

S. Sachdev and J. Ye, Phys. Rev. Lett. **70**, 3339 (1993)

The SYK model

$$G(i\omega) = \frac{1}{\cancel{i\omega} + \cancel{\mu} - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = -U^2 G^2(\tau) G(-\tau)$$

$$\Sigma(z) = \cancel{\mu} - \frac{1}{A}\sqrt{z} + \ldots \quad , \quad G(z) = \frac{A}{\sqrt{z}}$$

At frequencies $\ll U$, the $i\omega + \mu$ can be dropped, and without it equations are invariant under the reparametrization and gauge transformations. The singular part of the self-energy and the Green's function obey

$$\int_0^\beta d\tau_2\, \Sigma_{\rm sing}(\tau_1, \tau_2) G(\tau_2, \tau_3) = -\delta(\tau_1 - \tau_3)$$

$$\Sigma_{\rm sing}(\tau_1, \tau_2) = -U^2 G^2(\tau_1, \tau_2) G(\tau_2, \tau_1)$$

A. Kitaev, 2015

S. Sachdev, PRX **5,** 041025 (2015)

The complex SYK model

$$\int_0^\beta d\tau_2\, \Sigma_(\tau_1, \tau_2) G(\tau_2, \tau_3) = -\delta(\tau_1 - \tau_3)$$

$$\Sigma(\tau_1, \tau_2) = -U^2 G^2(\tau_1, \tau_2) G(\tau_2, \tau_1)$$

These equations are invariant under

$$\tau = f(\sigma)$$

$$G(\tau_1, \tau_2) = [f'(\sigma_1) f'(\sigma_2)]^{-1/4} \frac{g(\sigma_1)}{g(\sigma_2)} \widetilde{G}(\sigma_1, \sigma_2)$$

$$\Sigma(\tau_1, \tau_2) = [f'(\sigma_1) f'(\sigma_2)]^{-3/4} \frac{g(\sigma_1)}{g(\sigma_2)} \widetilde{\Sigma}(\sigma_1, \sigma_2)$$

where $f(\sigma)$ and $g(\sigma)$ are arbitrary functions.
By using $f(\sigma) = \tan(\pi T \sigma)/(\pi T)$ and
$g(\sigma) = e^{-2\pi \mathcal{E} T \sigma}$, we can now obtain
the $T > 0$ solution from the $T = 0$ solution.

A. Kitaev, 2015
S. Sachdev, PRX **5,** 041025 (2015)

The SYK model

Let us write the large N saddle point solutions of S as

$$\begin{aligned} G_s(\tau_1-\tau_2) &\sim (\tau_1-\tau_2)^{-1/2} \\ \Sigma_s(\tau_1-\tau_2) &\sim (\tau_1-\tau_2)^{-3/2}. \end{aligned}$$

The saddle point will be invariant under a reperamaterization $f(\tau)$ when choosing $G(\tau_1,\tau_2)=G_s(\tau_1-\tau_2)$ leads to a transformed $\widetilde{G}(\sigma_1,\sigma_2)=G_s(\sigma_1-\sigma_2)$ (and similarly for Σ). It turns out this is true only for the SL(2, R) transformations under which

$$f(\tau)=\frac{a\tau+b}{c\tau+d} \quad , \quad ad-bc=1.$$

So the (approximate) reparametrization symmetry is spontaneously broken down to SL(2, R) by the saddle point.

A. Kitaev

Fluctuations

- The saddle-point

$$G(\tau_1-\tau_2) = -A\frac{e^{-2\pi\mathcal{E}T(\tau_1-\tau_2)}}{\sqrt{1+e^{-4\pi\mathcal{E}}}}\left(\frac{T}{\sin(\pi T(\tau_1-\tau_2))}\right)^{2\Delta}$$

is invariant only under PSL(2, R) transformations which map the thermal circle onto itself, and an associated gauge transformation

$$\frac{\tan(\pi T f(\tau))}{\pi T} = \frac{a\dfrac{\tan(\pi T\tau)}{\pi T}+b}{c\dfrac{\tan(\pi T\tau)}{\pi T}+d} \quad , \quad ad-bc=1,$$

$$-i\phi(\tau) = -i\phi_0 + 2\pi\mathcal{E}T(\tau - f(\tau))$$

A. Kitaev, 2015

R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, PRB **95**, 155131 (2017)

Infinite-range (SYK) model without quasiparticles

After introducing replicas $a = 1 \dots n$, and integrating out the disorder, the partition function can be written as

$$Z = \int \mathcal{D}c_{ia}(\tau) \exp\left[-\sum_{ia} \int_0^\beta d\tau \, c_{ia}^\dagger \left(\frac{\partial}{\partial \tau} - \mu \right) c_{ia} \right.$$
$$\left. - \frac{U^2}{4N^3} \sum_{ab} \int_0^\beta d\tau d\tau' \left| \sum_i c_{ia}^\dagger(\tau) c_{ib}(\tau') \right|^4 \right] .$$

For simplicity, we neglect the replica indices, and introduce the identity

$$1 = \int \mathcal{D}G(\tau_1, \tau_2) \mathcal{D}\Sigma(\tau_1, \tau_2) \exp\left[-N \int_0^\beta d\tau_1 d\tau_2 \Sigma(\tau_1, \tau_2) \Big(G(\tau_2, \tau_1) \right.$$
$$\left. + \frac{1}{N} \sum_i c_i(\tau_2) c_i^\dagger(\tau_1) \Big) \right] .$$

Infinite-range (SYK) model without quasiparticles

Then the partition function can be written as a path integral with an action S analogous to a Luttinger-Ward functional

$$Z = \int \mathcal{D}G(\tau_1,\tau_2)\mathcal{D}\Sigma(\tau_1,\tau_2)\exp(-NS)$$

$$S = \ln\det\left[\delta(\tau_1-\tau_2)(\partial_{\tau_1}+\mu) - \Sigma(\tau_1,\tau_2)\right] + \int d\tau_1 d\tau_2 \left[\Sigma(\tau_1,\tau_2)G(\tau_2,\tau_1) + (U^2/2)G^2(\tau_2,\tau_1)G^2(\tau_1,\tau_2)\right]$$

At frequencies $\ll U$, the time derivative in the determinant is less important, and without it the path integral is invariant under the reparametrization and gauge transformations

A. Georges and O. Parcollet
PRB **59**, 5341 (1999)
A. Kitaev, 2015
S. Sachdev, PRX **5,** 041025 (2015)

$$\tau = f(\sigma)$$

$$G(\tau_1,\tau_2) = [f'(\sigma_1)f'(\sigma_2)]^{-1/4}\frac{g(\sigma_1)}{g(\sigma_2)}G(\sigma_1,\sigma_2)$$

$$\Sigma(\tau_1,\tau_2) = [f'(\sigma_1)f'(\sigma_2)]^{-3/4}\frac{g(\sigma_1)}{g(\sigma_2)}\Sigma(\sigma_1,\sigma_2)$$

where $f(\sigma)$ and $g(\sigma)$ are arbitrary functions.

The SYK model

Reparametrization and phase zero modes

We can write the path integral for the SYK model as

$$\mathcal{Z} = \int \mathcal{D}G(\tau_1, \tau_1)\mathcal{D}\Sigma(\tau_1, \tau_2)e^{-NS[G,\Sigma]}$$

for a known action $S[G, \Sigma]$. We find the saddle point, G_s, Σ_s, and only focus on the "Nambu-Goldstone" modes associated with breaking reparameterization and U(1) gauge symmetries by writing

$$G(\tau_1, \tau_2) = [f'(\tau_1)f'(\tau_2)]^{1/4} G_s(f(\tau_1) - f(\tau_2))e^{i\phi(\tau_1) - i\phi(\tau_2)}$$

(and similarly for Σ). Then the path integral is approximated by

$$\mathcal{Z} = \int \mathcal{D}f(\tau)\mathcal{D}\phi(\tau)e^{-E_0/T + Ns_0 - NS_{\text{eff}}[f,\phi]},$$

where $E_0 \propto N$ is the ground state energy.

J. Maldacena and D. Stanford, arXiv:1604.07818;
R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, arXiv.1612.00849;
S. Sachdev, PRX **5,** 041025 (2015); J. Maldacena, D. Stanford, and Zhenbin Yang, arXiv:1606.01857;
K. Jensen, arXiv:1605.06098; J. Engelsoy, T.G. Mertens, and H. Verlinde, arXiv:1606.03438

Fluctuations

Symmetry arguments, and explicit computations, show that the effective action is

$$\boxed{S_{\rm eff}[f,\phi] = \frac{NK}{2}\int_0^{1/T} d\tau (\partial_\tau \phi + i(2\pi \mathcal{E} T)\partial_\tau f)^2 - \frac{N\gamma}{4\pi^2}\int_0^{1/T} d\tau \left\{\tan(\pi T f(\tau)), \tau\right\},}$$

where $f(\tau)$ is a monotonic map from $[0,1/T]$ to $[0,1/T]$, the couplings K, γ, and $\mathcal{E}$ can be related to thermodynamic derivatives and we have used the Schwarzian:

$$\{g,\tau\} \equiv \frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2 .$$

Specifically, an argument constraining the effective at $T = 0$ is

$$S_{\rm eff}\left[f(\tau) = \frac{a\tau + b}{c\tau + d}, \phi(\tau) = 0\right] = 0,$$

and this is origin of the Schwarzian.

J. Maldacena and D. Stanford, arXiv:1604.07818;
R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, PRB **95**, 155131 (2017);
A. Gaikwad, L.K. Joshi, G. Mandal, and S.R. Wadia, arXiv:1802.07746

Fluctuations

We use the parameterization

$$f(\tau) \equiv \tau + \epsilon(\tau), \tag{1}$$

and express the action in terms $\phi(\tau)$ and $\epsilon(\tau)$. The energy and density operators are defined by

$$\delta E(\tau) - \mu \delta \mathcal{Q}(\tau) = \frac{1}{N} \frac{\delta S_{\phi,\epsilon}}{\delta \epsilon'(\tau)} \quad , \quad \delta \mathcal{Q}(\tau) = \frac{i}{N} \frac{\delta S_{\phi,\epsilon}}{\delta \phi'(\tau)}. \tag{2}$$

Introducing,

$$\widetilde{\phi}(\tau) = \phi(\tau) + i 2\pi \mathcal{E} T \epsilon(\tau) \tag{3}$$

and expanding $S_{\rm eff}$ to quadratic order in ϕ and ϵ, we obtain the Gaussian action

$$\frac{S_{\phi,\epsilon}}{N} = \frac{KT}{2} \sum_{\omega_n \neq 0} \omega_n^2 \left| \widetilde{\phi}(\omega_n) \right|^2 + \frac{T\gamma}{8\pi^2} \sum_{|\omega_n| \neq 0, 2\pi T} \omega_n^2 (\omega_n^2 - 4\pi^2 T^2) |\epsilon(\omega_n)|^2 + \ldots \tag{4}$$

where ω_n is a Matsubara frequency. Note the restrictions on $n = 0, \pm 1$ frequencies, which are needed to eliminate the zero modes associated with $PSL(2, \mathbb{R})$ and U(1) invariances.

R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, PRB **95**, 155131 (2017)

Fluctuations

In terms of $\widetilde{\phi}(\tau)$ and $\epsilon(\tau)$, the fluctuations in the thermodynamic observables are

$$\begin{aligned} \delta\mathcal{Q}(\tau) &= iK\widetilde{\phi}'(\tau) \\ \delta E(\tau) - \mu_0 \delta\mathcal{Q}(\tau) &= -\frac{\gamma}{4\pi^2}\left[\epsilon'''(\tau) + 4\pi^2 T^2 \epsilon'(\tau)\right] + i2\pi K\mathcal{E}T\widetilde{\phi}'(\tau). \end{aligned} \quad (5)$$

Now we compute the correlators of these observables in the Gaussian action. We have for the two-point correlator of $\widetilde{\phi}(\tau)$

$$\begin{aligned} \left\langle \widetilde{\phi}(\tau)\widetilde{\phi}(0) \right\rangle &= \frac{T}{NK} \sum_{\omega_n \neq 0} \frac{e^{i\omega_n \tau}}{\omega_n^2} \\ &= \frac{1}{NKT}\left[\frac{1}{2}\left(T\tau - \frac{1}{2}\right)^2 - \frac{1}{24}\right] \quad \text{for } 0 < T\tau < 1, \end{aligned} \quad (6)$$

and extended periodically for all τ with period $1/T$. Similar for $\epsilon(\tau)$

$$\begin{aligned} \langle \epsilon(\tau)\epsilon(0)\rangle &= \frac{4\pi^2 T}{N\gamma} \sum_{|\omega_n| \neq 0, 2\pi T} \frac{e^{i\omega_n \tau}}{\omega_n^2(\omega_n^2 - 4\pi^2 T^2)} \\ &= \frac{1}{N\gamma T^3}\left[\frac{1}{24} + \frac{1}{4\pi^2} - \frac{1}{2}\left(T\tau - \frac{1}{2}\right)^2 + \frac{5}{8\pi^2}\cos(2\pi T\tau) + \frac{1}{2\pi}\left(T\tau - \frac{1}{2}\right)\sin(2\pi T\tau)\right] \\ &\quad \text{for } 0 < T\tau < 1. \end{aligned} \quad (7)$$

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Fluctuations

We confirm that the correlators of the conserved densities are τ-independent; their second moment correlators, which define the matrix of static susceptibility correlators, are given by

$$\begin{aligned}\chi_s &= \frac{1}{N}\begin{pmatrix} -(\partial^2\Omega/\partial\mu^2)_T & -(\partial^2\Omega/\partial\mu\partial T)_\mu \\ -T(\partial^2\Omega/\partial\mu\partial T)_\mu & -T(\partial^2\Omega/\partial T^2)_\mu \end{pmatrix} \\ &= \frac{1}{T}\begin{pmatrix} \left\langle(\delta\mathcal{Q})^2\right\rangle & \left\langle(\delta E-\mu\delta\mathcal{Q})\delta\mathcal{Q}\right\rangle/T \\ \left\langle(\delta E-\mu\delta\mathcal{Q})\delta\mathcal{Q}\right\rangle & \left\langle(\delta E-\mu\delta\mathcal{Q})^2\right\rangle/T \end{pmatrix} \\ &= \frac{1}{N}\begin{pmatrix} K & 2\pi K\mathcal{E} \\ 2\pi K\mathcal{E}T & (\gamma+4\pi^2\mathcal{E}^2K)T \end{pmatrix}\end{aligned} \tag{8}$$

From this we obtain the relationship between the couplings K and γ in the effective action. After application of some thermodynamic identities, we can write these as

$$K=\left(\frac{\partial\mathcal{Q}}{\partial\mu}\right)_T \quad , \quad \gamma=-\left(\frac{\partial^2F}{\partial T^2}\right)_{\mathcal{Q}} \quad , \quad 2\pi\mathcal{E}=-\lim_{T\to0}\left(\frac{\partial\mu}{\partial T}\right)_{\mathcal{Q}} . \tag{9}$$

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Fluctuations

We can also evaluate the path integral over the Gaussian action in (4). Here we consider only the integral over the Schwarzian modes, and consider the phase modes later. From such a Gaussian integral we find

$$\ln Z(T) = N s_0 + \frac{N\gamma T}{2} - \frac{1}{2} \sum_{|\omega_n| \neq 0, 2\pi T} \ln \left[\frac{T\gamma}{4\pi^2} \omega_n^2 (\omega_n^2 - 4\pi^2 T^2) \right] \tag{10}$$

Evaluating the summation using ζ function regularization we find

$$\ln Z(T) = -\frac{E_0}{T} + N s_0 + \frac{N\gamma T}{2} - \frac{3}{2} \ln \left(\frac{c_1 J}{T} \right) \tag{11}$$

where c_1 is a non-universal constant. We can now invert the following equation

$$Z(T) = \int dE D_s(E) e^{-E/T} \tag{12}$$

to obtain the density of states of the Schwarzian

$$D_s(E) \propto e^{N s_0} \sinh \left(\sqrt{2N\gamma(E - E_0)} \right) \tag{13}$$

Note that the $e^{\sqrt{N\gamma(E-E_0)}}$ component of this is similar to the quasiparticle case, but the complete expression is very different.

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Fluctuations

2.5.1 Partition function

Going beyond the large N limit, and to higher order in T, the grand partition function is given by a path integral of the effective action in (1.8)

$$Z(\beta) = \exp\left(-\beta\Omega_0\right) \int \frac{\mathcal{D}\varphi}{\mathrm{SL}(2,\mathbb{R})} \frac{\mathcal{D}\lambda}{\mathrm{U}(1)} \exp\left(-I_{\text{eff}}[\varphi,\lambda]\right) \,, \tag{2.87}$$

where we have divided the integral by the volume of $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{U}(1)$ since we should view $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{U}(1)$ as a gauge symmetry. Related partition functions have also been evaluated recently in Ref. [36].

The Schwarzian path integral over φ was evaluated exactly in Ref. [35]. Given the boundary conditions of $\varphi(\tau)$ and $\lambda(\tau)$ above (1.8), it is useful to parameterize these fields by

$$\begin{aligned} \varphi(\tau) &= \tau + \overline{\varphi}(\tau) \\ \lambda(\tau) &= \frac{2\pi p}{\beta}\tau + \overline{\lambda}(\tau)\,, \end{aligned} \tag{2.88}$$

where the 'winding number' p is an integer, and $\overline{\varphi}$ and $\overline{\lambda}$ are then periodic functions of τ with period β. In the first term in the action (1.8), we can absorb $\overline{\varphi}$ by a shift in $\overline{\lambda}$; then the remaining dependence on $\overline{\varphi}$ is only in Schwarzian, and the path integral over $\overline{\varphi}$ reduces to precisely that in Ref. [35]. So it remains to only evaluate the path integral over λ defined by

$$Z_Q(\beta) = \int \frac{\mathcal{D}\lambda}{\mathrm{U}(1)} \exp\left[-\frac{NK}{2}\int_0^\beta d\tau \left(\lambda'(\tau) + i\frac{2\pi\mathcal{E}}{\beta}\right)^2\right] \,. \tag{2.89}$$

Yingfei Gu, A. Kitaev, S. Sachdev, and G. Tarnopolsky, to appear

Fluctuations

The path integral $Z_Q(\beta)$ can also be evaluated exactly: it represents a single quantum rotor in the presence of a field coupling linearly to its angular momentum. Employing (2.88), we have

$$Z_Q(\beta) = \left(\sum_{p=-\infty}^{\infty} \exp\left[-\frac{2\pi^2 NK}{\beta}(p + i\mathcal{E})^2 \right] \right) \int \frac{\mathcal{D}\overline{\lambda}}{\mathrm{U}(1)} \exp\left[-\frac{NK}{2} \int_0^\beta d\tau \left(\overline{\lambda}'(\tau) \right)^2 \right]. \tag{2.90}$$

The first term in (2.90) is more easily evaluated at very low temperatures, $\beta J \gg N$, by the Poisson summation formula. The second term is just the imaginary time amplitude for a 'free particle' of mass $1/(NK)$ to return to its starting point in a time β [37]. In this manner, we obtain

$$\begin{aligned} Z_Q(\beta) &= \left(\sum_{n=-\infty}^{\infty} \sqrt{\frac{\beta}{2\pi NK}} \exp\left[-\frac{\beta n^2}{2NK} + 2\pi \mathcal{E} n \right] \right) \sqrt{\frac{2\pi NK}{\beta}}, \\ &= \sum_{n=-\infty}^{\infty} \exp\left[-\frac{\beta n^2}{2NK} + 2\pi \mathcal{E} n \right]. \end{aligned} \tag{2.91}$$

The integer n clearly has the interpretation of the charge shift in $Q + n$ in (2.84).

Now we combine the result for the path integral over the Schwarzian in Ref. [35] with (2.91) to obtain the complete result for $Z(\beta)$

$$\begin{aligned} Z(\beta) &\propto \exp\left(-\beta \Omega_0\right) \left(\sum_{n=-\infty}^{\infty} \exp\left[-\frac{\beta n^2}{2NK} + 2\pi \mathcal{E} n \right] \right) \\ &\quad \times \left(\frac{N\gamma}{2\pi\beta} \right)^{3/2} \exp\left(\frac{N\gamma}{2\beta} \right). \end{aligned} \tag{2.92}$$

Yingfei Gu, A. Kitaev, S. Sachdev, and G. Tarnopolsky, to appear

Fluctuations

At very low temperatures $\beta J \gg N$, only the charge $n = 0$ contributes significantly to the sum, and then the logarithm of (2.92) yields the leading $1/N$ correction to the Helmhotz free energy [7], defined from the canonical partition function

$$
\begin{aligned}
F &= \Omega + \mu Q \\
&= E_0(Q) - NT\mathcal{S}(Q) - \frac{N\gamma T^2}{2} + \frac{3T}{2}\ln(J/T) + \ldots . \qquad (2.93)
\end{aligned}
$$

Note that γ is the coefficient of the linear-T specific heat at fixed Q.

At higher temperatures $1 \ll \beta J \ll N$, we should use the winding number p summation in (2.90). Then the corresponding expression for $Z(\beta)$ is

$$
\begin{aligned}
Z(\beta) &\propto \exp\left(-\beta\Omega_0\right)\left(\sum_{p=-\infty}^{\infty}\exp\left[-\frac{2\pi^2 NK}{\beta}(p+i\mathcal{E})^2\right]\right) \\
&\quad \times \sqrt{\frac{2\pi NK}{\beta}}\left(\frac{N\gamma}{2\pi\beta}\right)^{3/2}\exp\left(\frac{N\gamma}{2\beta}\right) . \qquad (2.94)
\end{aligned}
$$

Note that the prefactor of the exponentials is now $\sim \beta^{-2}$, in contrast to the $\beta^{-3/2}$ in (2.92). At $1 \ll \beta J \ll N$, only the winding number $p = 0$ term in (2.94) is important, and the logarithm yields the $1/N$ correction to the grand potential

$$
\Omega = \Omega_0 - \frac{N(\gamma + 4\pi^2\mathcal{E}^2 K)T^2}{2} + 2T\ln(J/T) + \ldots . \qquad (2.95)
$$

Now $\gamma + 4\pi^2\mathcal{E}^2 K$ is co-efficient of the linear-T specific heat at fixed μ [8]. Note also the change in the co-efficient of the $\ln(J/T)$ term from $3/2$ in (2.93) to 2 in (2.95).

Yingfei Gu, A. Kitaev, S. Sachdev, and G. Tarnopolsky, to appear

2.5.2 Inverse Laplace transform

Finally, we turn to the evaluation of the density of states, $D(\mathbb{E})$. We will perform the inverse Laplace transform of (2.85) using the expression in (2.92) for $Z(\beta)$. In this transform, it is important that we regard μ as independent of β, as should be clear from (2.81) and (2.85).

The inverse Laplace transform of (2.92) proceeds just as in Ref. [35], and we obtain

$$D(\mathbb{E}) \propto \exp\left(N\mathcal{S}(Q)\right) {\sum_n}' \exp\left(2\pi\mathcal{E}n\right) \sinh\left(\sqrt{2N\gamma(\mathbb{E} - \mathbb{E}_0(Q) - n^2/(2NK))}\right), \qquad (2.96)$$

where the prime on the summation indicates that it only extends over values of n for which the argument of the sinh is real, *i.e.* for

$$n^2 < 2NK(\mathbb{E} - \mathbb{E}_0(Q)). \qquad (2.97)$$

The result (2.96) is plotted in Fig. 2. We can obtain a clearer physical interpretation of (2.96) by simplifying its dependence on n. We note from (1.7) that

$$N\mathcal{S}(Q) + 2\pi\mathcal{E}n \approx N\mathcal{S}(Q+n), \qquad (2.98)$$

and use (2.84) to obtain one of our main results

$$D(\mathbb{E}) \propto {\sum_n}' \exp\left(N\mathcal{S}(Q+n)\right) \sinh\left(\sqrt{2N\gamma(\mathbb{E} - \mathbb{E}_0(Q) - n^2/(2NK))}\right). \qquad (2.99)$$

This expression has a clear meaning: there is a square-root threshold for each charge sector $Q+n$ at the energy $\mathbb{E}_0(Q) + n^2/(2NK)$, which equals $\mathbb{E}_0(Q+n)$ at $T=0$ by (2.84). The amplitude of the density of states in each sector is, as expected, the exponential of its entropy $N\mathcal{S}(Q+n)$. Now we can appreciate the role of the 'shift' proportional to $\mathcal{E}$ in the effective action in (1.8): it is needed to correct the entropy of the charge sectors from $N\mathcal{S}(Q)$ to $N\mathcal{S}(Q+n)$. The sinh form of the density of states in each charge sector is the same as that in the Majorana SYK model [32, 35, 7].

Yingfei Gu, A. Kitaev, S. Sachdev, and G. Tarnopolsky, to appear

Fluctuations

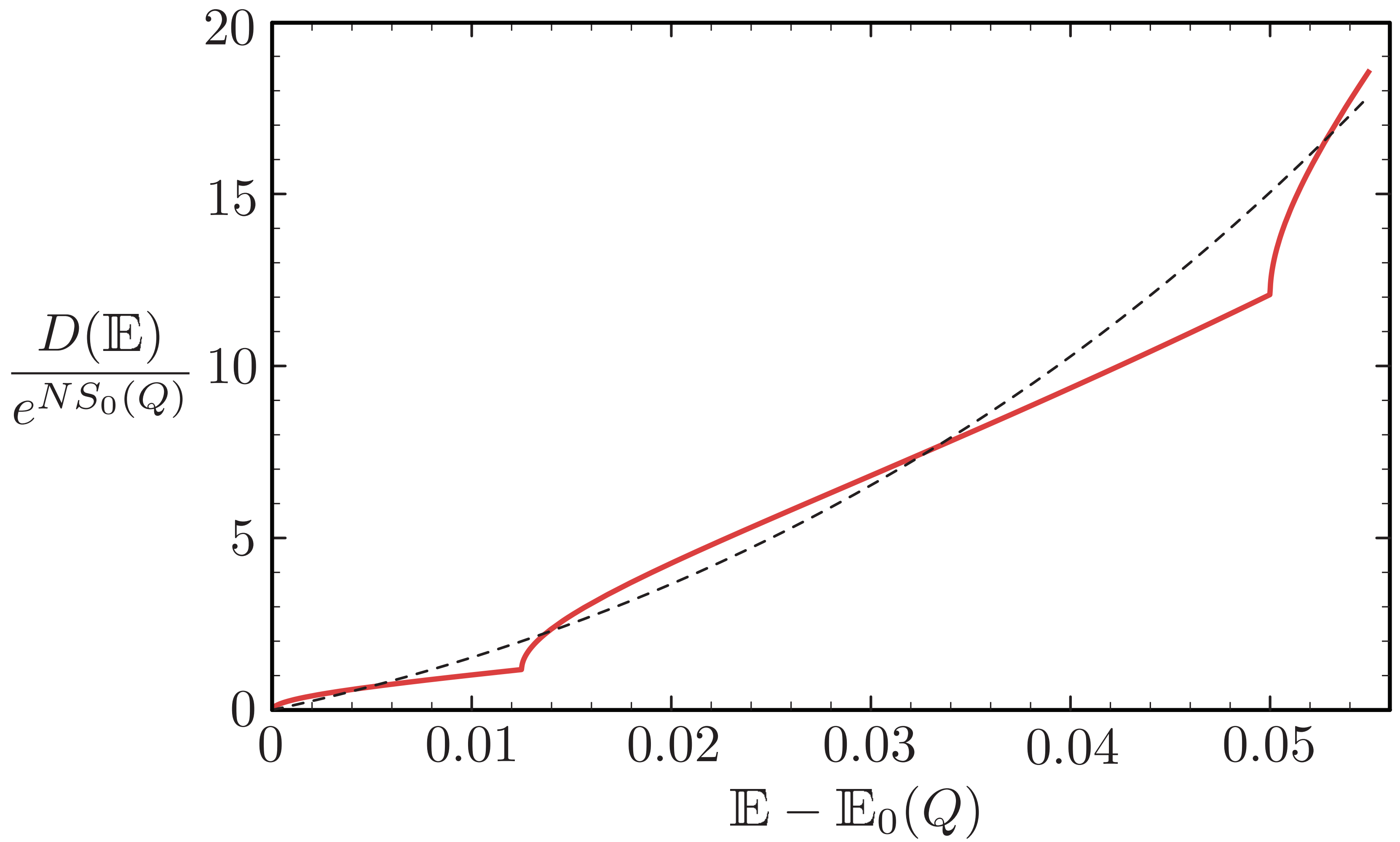


Figure 2: The full line is the density of states in (2.96) for $N = 40$ with $\gamma = K = 2\pi\mathcal{E} = 1$. It has square-root thresholds at $\mathbb{E} - \mathbb{E}_0(Q) = n^2/(2NK)$ with n integer. The dashed line is the approximate form valid when (2.100) holds, and is obtained from (2.101); it corresponds to ignoring the winding modes in the λ path integral in (2.87). Note there is *no* delta function at $\mathbb{E} - \mathbb{E}_0(Q) = 0$. A delta function is present for the SYK model with unbroken $\mathcal{N} = 2$ supersymmetry [21, 35], and in supersymmetric black holes with AdS_2 horizons [19, 20], and it accounts completely for the $T = 0$ entropy in these cases.

Yingfei Gu, A. Kitaev, S. Sachdev, and G. Tarnopolsky, to appear

Fluctuations

Note that the thresholds in (2.99) are separated by energies of order J/N, and so the expression (2.99) is best used at $\mathbb{E} - \mathbb{E}_0(Q)$ of order J/N. At larger energies

$$J/N \ll (\mathbb{E} - \mathbb{E}_0(Q)) \ll NJ\,, \tag{2.100}$$

we should take the inverse Laplace transform of only the $p = 0$ term in (2.94). Actually, it is easier to use the equivalent approximation of converting the n summation in (2.96) to an integration, to obtain in the regime (2.100)

$$\begin{aligned} D(\mathbb{E}) \quad \propto \quad & \exp\left(N\mathcal{S}(Q)\right) \int_{-\sqrt{2NK(\mathbb{E}-\mathbb{E}_0(Q))}}^{\sqrt{2NK(\mathbb{E}-\mathbb{E}_0(Q))}} dn \exp\left(2\pi\mathcal{E}n\right) \\ & \times \sinh\left(\sqrt{2N\gamma(\mathbb{E} - \mathbb{E}_0(Q) - n^2/(2NK))}\right)\,, \end{aligned} \tag{2.101}$$

for $\mathbb{E} > \mathbb{E}_0(Q)$, and $D(\mathbb{E}) = 0$ otherwise. This result is shown as the dashed line in Fig. 2. It vanishes linearly in $\mathbb{E} - \mathbb{E}_0(Q)$ at threshold; but (2.100) does not hold near threshold and the square-root threshold in (2.96) is the correct result. To leading exponential accuracy, we can evaluate the integral in (2.101) in the saddle-point approximation to obtain

$$D(\mathbb{E}) \propto \exp\left(N\mathcal{S}(Q) + \sqrt{2N(\gamma + 4\pi^2\mathcal{E}^2K)(\mathbb{E} - \mathbb{E}_0(Q))}\right)\,. \tag{2.102}$$

This is the result expected from the inverse Laplace transform of the grand potential in (2.95).

Yingfei Gu, A. Kitaev, S. Sachdev, and G. Tarnopolsky, to appear

Fluctuations

An *exact* path integral over the effective action leads to the following physical consequences

- The ground state energy with fermion number $N\mathcal{Q} + p$ (p integer) varies as

$$E_p = E_0 + \frac{p^2}{2NK}$$

This identifies K with the compressibility $K = d\mathcal{Q}/d\mu$ at $T = 0$.

- The low temperature corrections to the entropy are

$$S(T \to 0, \mathcal{Q}) = N\Big[s_0 + \gamma T + \ldots\Big] + 2\ln(U/T)\ldots$$

This defines γ as the co-efficient of the linear-in-T specific heat (at fixed $\mathcal{Q}$)

D. Stanford and E. Witten, arXiv:1703.04612; A. Kitaev and S. Josephine Suh, arXiv:1711.08467

Fluctuations

An *exact* path integral over the effective action leads to the following physical consequences

- The *many*-body density of states, $D(E)$, is related to the grand potential, $\Omega(T)$ by

$$Z = e^{-\Omega(T)/T} = \int_{-\infty}^{\infty} dE D(E) e^{-E/T}$$

We obtain

$$D(E) = \sum_{p=-\infty}^{\infty} e^{2\pi p \mathcal{E}} \, d\left(E - E_p\right)$$

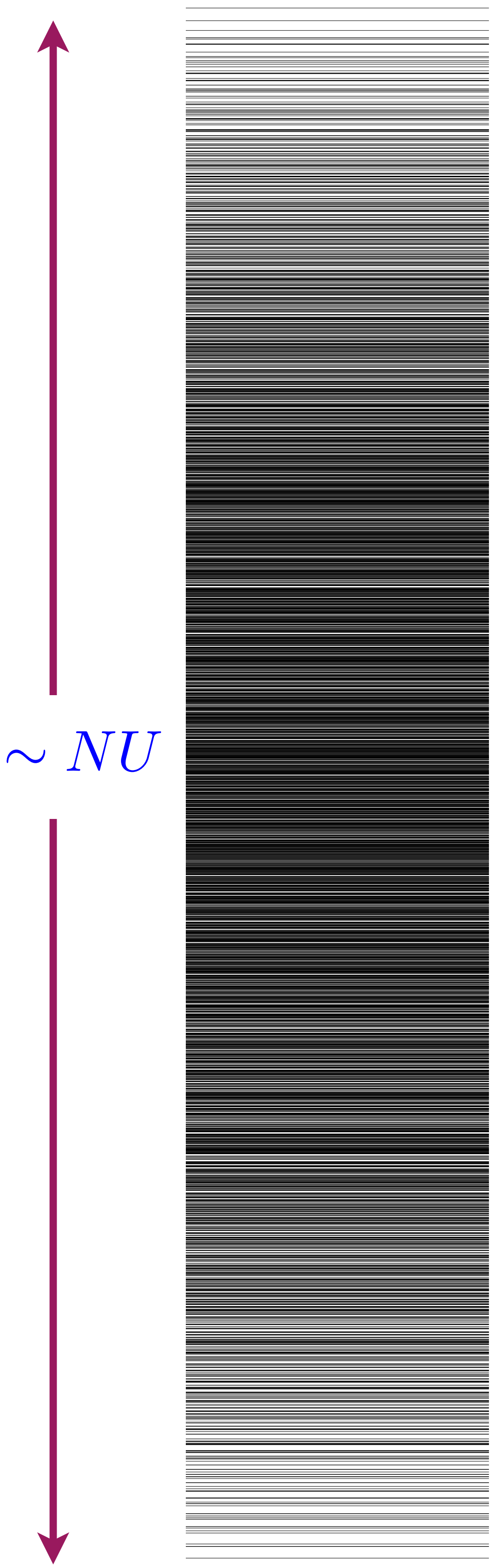
where $N\mathcal{Q} + p$ is the integer fermion number,

$$d(E) \sim \exp\left(N s_0\right) \sinh\left(\sqrt{2N\gamma E}\right) \quad , \quad E > 0 \; , \; e^{-cN} \ll \gamma E \ll N$$

There are exponentially more low energy states than for the quasiparticle case, and $D(E)$ self-averages down to energies exponentially small in N.

J. S. Cotler, G. Gur-Ari, M. Hanada, J. Polchinski, P. Saad, S. H. Shenker, D. Stanford, A. Streicher, and M. Tezuka, arXiv:1611.04650;
A.M. Garcia-Garcia and J.J.M. Verbaarschot, arXiv:1701.06593; D. Bagrets, A. Altland, and A. Kamenev, arXiv:1702.08902;
D. Stanford and E. Witten, arXiv:1703.04612; A. Kitaev and S.J. Suh, arXiv:1711.08467

The complex SYK model



Many-body level spacing $\sim 2^{-N} = e^{-N \ln 2}$

Non-quasiparticle excitations with spacing $\sim e^{-N s_0}$

There are 2^N many body levels with energy E. Shown are all values of E for a single cluster of size $N = 12$. The $T \to 0$ state has an entropy $S_{GPS} = N s_0$, where $s_0 < \ln 2$ is determined by integrating

$$\boxed{\frac{ds_0}{d\mathcal{Q}} = 2\pi\mathcal{E}\,.}$$

At $\mathcal{Q} = 1/2$,

$$s_0 \quad = \quad \frac{G}{\pi} + \frac{\ln(2)}{4} = 0.464848\ldots$$

where G is Catalan's constant.

GPS: A. Georges, O. Parcollet, and S. Sachdev, PRB **63**, 134406 (2001)

W. Fu and S. Sachdev, PRB **94**, 035135 (2016)

Fluctuations

An *exact* path integral over the effective action leads to the following physical consequences

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$$D(E) = \sum_{p=-\infty}^{\infty} e^{2\pi p \mathcal{E}} \, d\left(E - E_p\right)$$

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Fluctuations

An *exact* path integral over the effective action leads to the following physical consequences

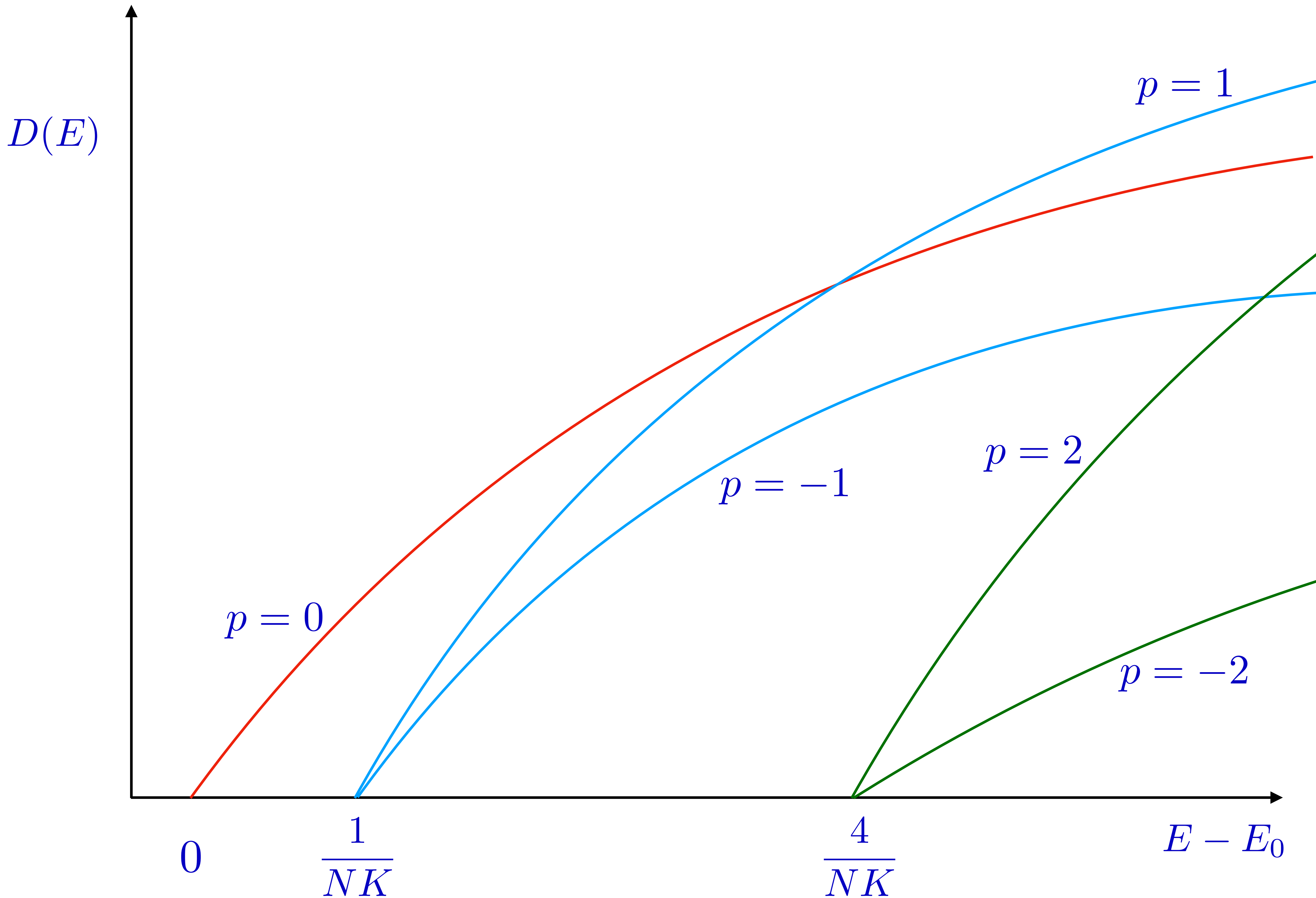
- At charge $N\mathcal{Q} + p$, the prefactor of the $\sinh(\sqrt{2N\gamma(E - E_p)})$ term is

$$\exp\left[N s_0(\mathcal{Q}) + 2\pi p \mathcal{E}\right] \approx \exp\left[N s_0(\mathcal{Q} + p/N)\right]$$

using

$$\boxed{\frac{ds_0}{d\mathcal{Q}} = 2\pi\mathcal{E}}$$

Yingfei Gu, A. Kitaev, S. Sachdev, and G. Tarnopolsky, to appear



Yingfei Gu, A. Kitaev, S. Sachdev, and G. Tarnopolsky, to appear

Fluctuations

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using

$$\boxed{\frac{ds_0}{d\mathcal{Q}} = 2\pi\mathcal{E}}$$

Yingfei Gu, A. Kitaev, S. Sachdev, and G. Tarnopolsky, to appear

4.2. *Gravitational contributions to the four-point function*

Suppose that we have operators V, W, which are dual to two different fields that are free in AdS_2 before coupling to gravity. The gravitational contribution to the four-point function can be computed as follows. (Some four-point functions were also considered in [3]. These steps are identical to the ones discussed in [16], since the effective action is the same.) We start from the factorized expression for the four-point function, $\langle V(t_1)V(t_2)W(t_3)W(t_4)\rangle = \frac{1}{t_{12}^{2\Delta}}\frac{1}{t_{34}^{2\Delta}}$. We then insert the reparametrizations (3.11) and (4.5) into (4.4) and expand to linear order in ε to obtain

$$\frac{1}{t_{12}^{2\Delta}} \longrightarrow \mathcal{B}(u_1,u_2)\frac{\Delta}{\left[2\sin\frac{u_{12}}{2}\right]^{2\Delta}}, \qquad \mathcal{B}(u_1,u_2) \equiv \left[\varepsilon'(u_1)+\varepsilon'(u_2)-\frac{\varepsilon(u_1)-\varepsilon(u_2)}{\tan\frac{u_{12}}{2}}\right]. \tag{4.8}$$

We make a similar replacement for $t_{34}^{-2\Delta}$, and then contract the factors of ε using the propagator (4.7). This gives the $O(1/C) = O(G)$ contribution to the four-point function. Note that the bilocal operator $\mathcal{B}$ is SL(2) invariant.[12] The final expression depends on the relative ordering of the four points. When $u_4 < u_3 < u_2 < u_1$ we obtain the factorized expression

$$\frac{\langle V_1V_2W_3W_4\rangle_{\text{grav}}}{\langle V_1V_2\rangle\langle W_3W_4\rangle} = \Delta^2\langle\mathcal{B}(u_1,u_2)\mathcal{B}(u_3,u_4)\rangle = \frac{\Delta^2}{2\pi C}\left(-2+\frac{u_{12}}{\tan\frac{u_{12}}{2}}\right)\left(-2+\frac{u_{34}}{\tan\frac{u_{34}}{2}}\right). \tag{4.9}$$

As discussed in [16], this expression can be viewed as arising from energy fluctuations. Each two-point function generates an energy fluctuation which then affects the other. Since energy is conserved, the result does not depend on the relative distance between the pair of points. In other words, we can think of it as

$$\langle V_1 V_2 W_3 W_4 \rangle_{\text{grav}} = \partial_M \langle V_1 V_2 \rangle \partial_M \langle W_3 W_4 \rangle \frac{1}{-\partial_M^2 S(M)} = \partial_\beta \langle V_1 V_2 \rangle \partial_\beta \langle W_3 W_4 \rangle \frac{1}{\partial_\beta^2 \log Z(\beta)}, \quad (4.10)$$

where M is the mass of the black hole background, or β its temperature, and $S(M)$ or $\log Z$ are its entropy or partition function.[13] Both expressions give the same answer, thanks to thermodynamic identities between entropy and mass.[14] If one expands as $u_{12} \to 0$ we get a leading term going like u_{12}^2, which one would identify with an operator of dimension two. In this case this is the Schwarzian itself, which is also the energy, and it is conserved (3.15). Its two-point functions are constant.[15]

It is also interesting to evaluate the correlator in the other ordering $u_4 < u_2 < u_3 < u_1$. We get

$$\frac{\langle V_1 W_3 V_2 W_4 \rangle_{\text{grav}}}{\langle V_1 V_2 \rangle \langle W_3 W_4 \rangle} = \frac{\Delta^2}{2\pi C} \left[\left(-2 + \frac{u_{12}}{\tan \frac{u_{12}}{2}} \right) \left(-2 + \frac{u_{34}}{\tan \frac{u_{34}}{2}} \right) + \right.$$
$$\left. + \frac{2\pi [\sin(\frac{u_1 - u_2 + u_3 - u_4}{2}) - \sin(\frac{u_1 + u_2 - u_3 - u_4}{2})]}{\sin \frac{u_{12}}{2} \sin \frac{u_{34}}{2}} + \frac{2\pi u_{23}}{\tan \frac{u_{12}}{2} \tan \frac{u_{34}}{2}} \right]. \quad (4.11)$$

This expression interpolates between (4.9) when $u_3 = u_2$ and an expression like (4.9), but with $u_{34} \to -2\pi + u_{34}$, when $u_3 = u_1$. Note that now the answer depends on the overall separation of the two pairs. This dependence, which involves the second sine term in the numerator as well as the u_{23} factor, looks like we are exciting the various zero modes of the Schwarzian action, including the exponential ones. It is interesting to continue (4.11) to Lorentzian time and into the chaos region, which involves the correlator in the out-of-time-order form

$$\langle V(a) W_3(b+\hat{u}) V(0) W(\hat{u}) \rangle \sim \frac{\beta \Delta^2}{C} e^{\frac{2\pi \hat{u}}{\beta}}, \qquad \frac{\beta}{2\pi} \ll \hat{u} \ll \frac{\beta}{2\pi} \log \frac{C}{\beta}, \tag{4.12}$$

where $a, b \sim \beta$. Here we restored the temperature dependence in (4.11) by multiplying by an overall factor of $\frac{\beta}{2\pi}$ and sending $u_i \to \frac{2\pi}{\beta} u_i$.

We can also connect (4.12) to a scattering process. It is peculiar that in this setup the two particles do not scatter since they behave like free fields on a fixed AdS_2 background. On the other hand, they create a dilaton profile which gives rise to a nontrivial interaction once we relate the AdS_2 time to the boundary time. The net result is the same as what is usually produced by the scattering of shock waves; see Appendix B. Here we see that the gravitational effects are very delocalized; we can remove them from the bulk and take them into account in terms of the boundary degree of freedom $t(u)$.

1. Quantum matter with quasiparticles:
   random matrix model

2. Quantum matter without quasiparticles:
   the complex SYK model

3. Fluctuations, and the Schwarzian

$\boxed{\text{4. Models of strange metals}}$

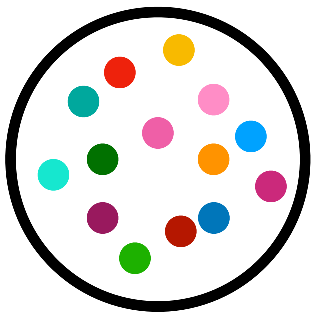
5. Einstein-Maxwell theory of charged
   black holes in AdS space

The complex SYK model

$$H = \frac{1}{(2N)^{3/2}} \sum_{\alpha,\beta,\gamma,\delta=1}^{N} U_{\alpha\beta;\gamma\delta}\, c_\alpha^\dagger c_\beta^\dagger c_\gamma c_\delta + \epsilon \sum_\alpha c_\alpha^\dagger c_\alpha$$

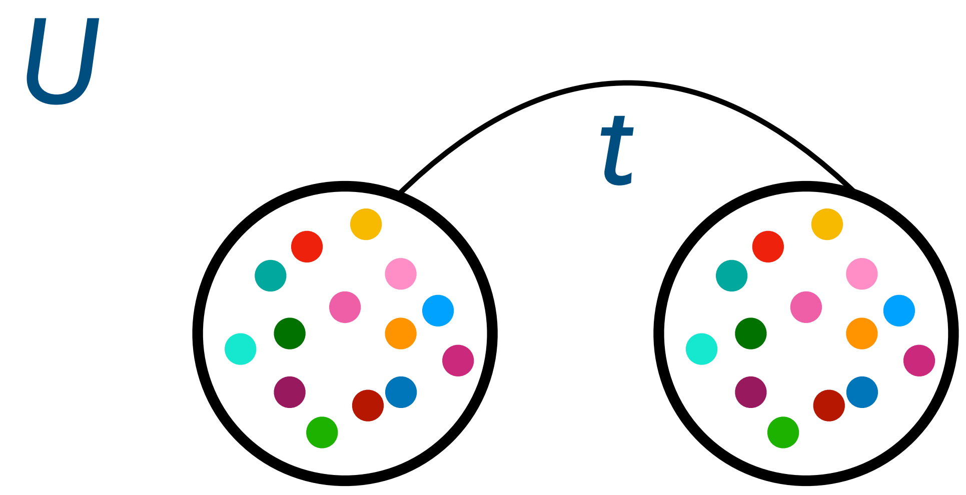
$U_{\alpha\beta;\gamma\delta}$ are independent random variables

with $\overline{U_{\alpha\beta;\gamma\delta}} = 0$ and $\overline{|U_{\alpha\beta;\gamma\delta}|^2} = U^2$



$$H = \frac{1}{(2N)^{3/2}} \sum_{i} \sum_{\alpha,\beta,\gamma,\delta=1}^{N} U_{\alpha\beta;\gamma\delta}\, c_{i\alpha}^{\dagger} c_{i\beta}^{\dagger} c_{i\gamma} c_{i\delta} - t \sum_{\langle ij \rangle} \sum_{\alpha} c_{i\alpha}^{\dagger} c_{j\alpha}$$

Equivalent to an
"eternal traversable wormhole"
between two black holes with
AdS_2 horizons



J. Maldacena and Xiao-Liang Qi, arXiv:1804.00491

Generalized SYK models

$$H = \frac{1}{(2N)^{3/2}} \sum_{k_a} \sum_{\alpha,\beta,\gamma,\delta=1}^{N} U_{\alpha\beta;\gamma\delta}(k_a)\, c^\dagger_{k_1\alpha} c^\dagger_{k_2\beta} c_{k_3\gamma} c_{k_4\delta}$$

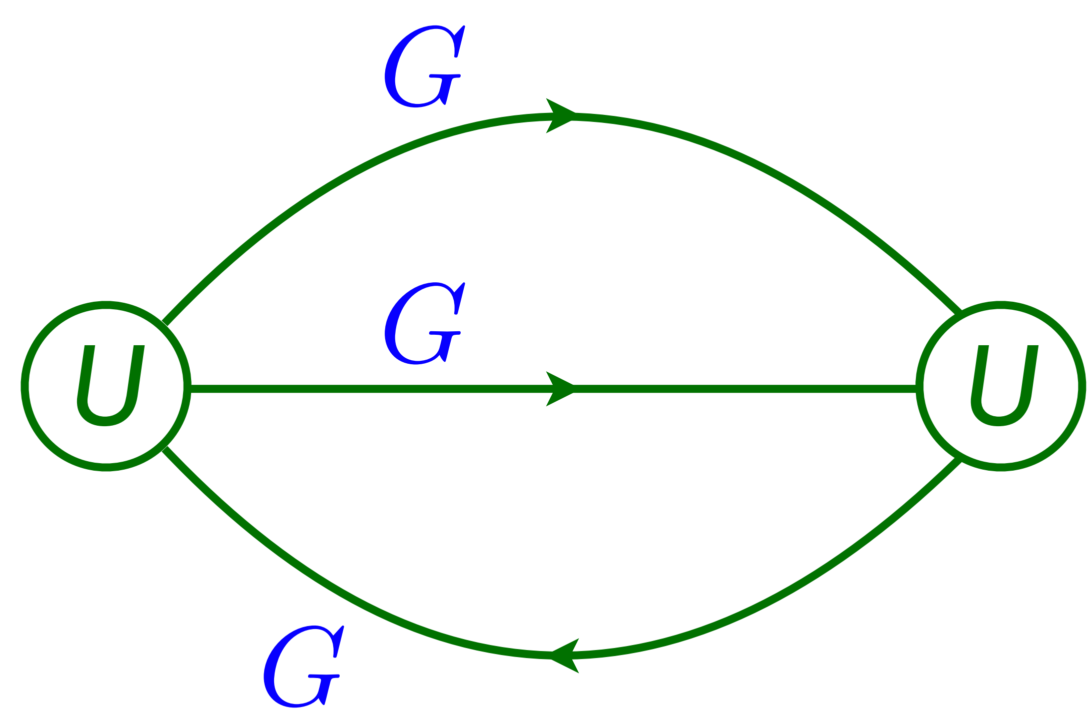
$$+ \sum_{k\alpha} \epsilon_k c^\dagger_{k\alpha} c_{k\alpha}$$

$U_{\alpha\beta;\gamma\delta}(k_a)$ is a random function of $\alpha\beta\gamma\delta$ (as before)
ϵ_k has a range of values of width W.

The large N limit is still given by the sum of "melon" diagrams.

$$G(k, i\omega) = \frac{1}{i\omega - \epsilon_k - \Sigma(k, i\omega)}$$

$\Sigma =$



A lattice SYK model

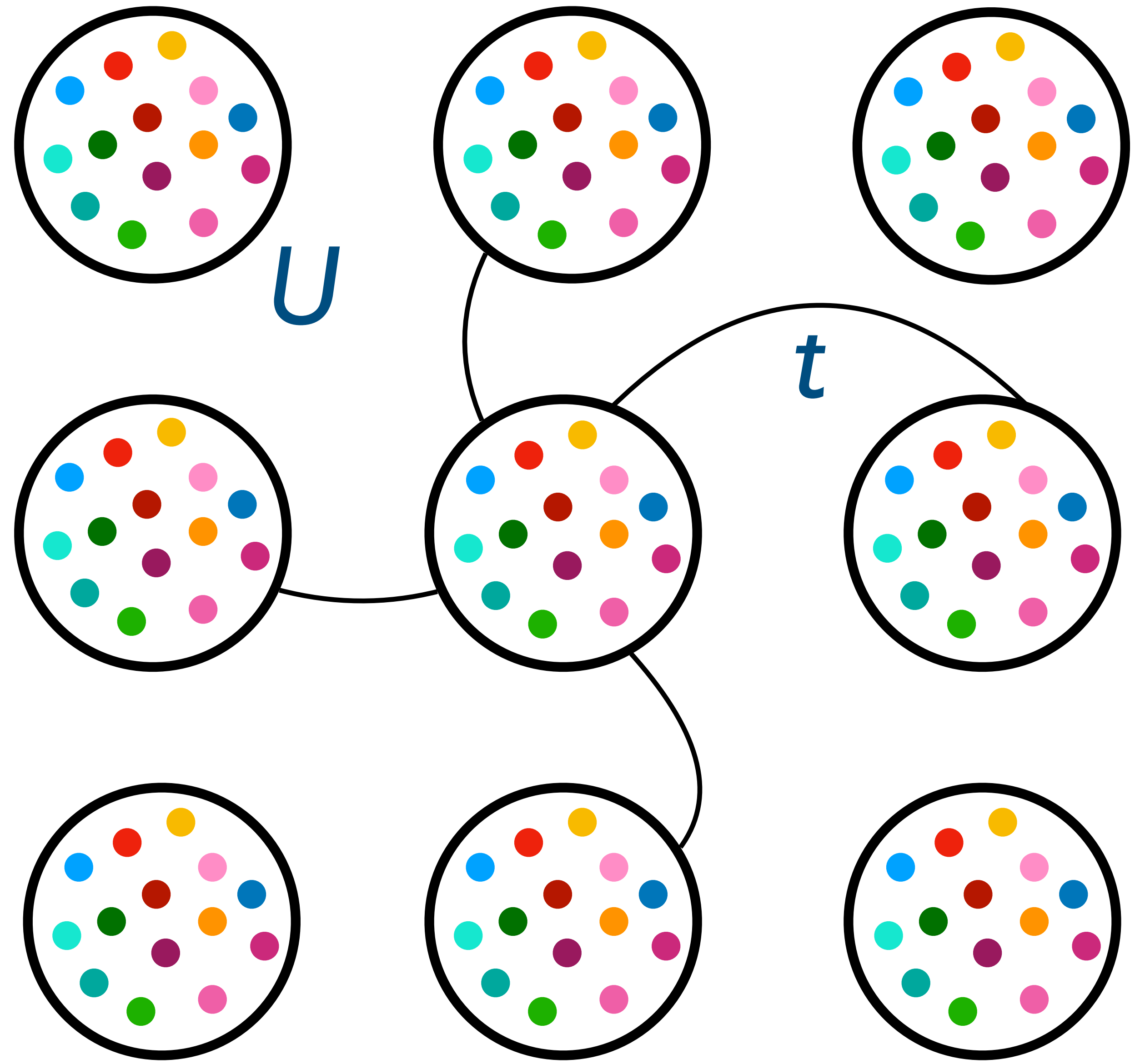
$$H = \frac{1}{(2N)^{3/2}} \sum_i \sum_{\alpha,\beta,\gamma,\delta=1}^{N} U_{\alpha\beta;\gamma\delta}\, c_{i\alpha}^\dagger c_{i\beta}^\dagger c_{i\gamma} c_{i\delta} - t \sum_{\langle ij \rangle} \sum_{\alpha} c_{i\alpha}^\dagger c_{j\alpha}$$

Choose U on-site,
and the same on all sites;
yields 'incoherent metal'
with no Fermi surface
for $t^2/U \ll k_B T \ll U$ with

$$G(\mathbf{k}, \omega) = G_{\rm SYK}(\epsilon, \hbar\omega/(k_B T))$$

independent of $\mathbf{k}$.
There is linear-in-T resistivity
but only with bad metal
behavior with $\rho > h/e^2$, and
co-efficient dependent upon U:

$$\rho \sim \frac{h}{e^2} \frac{k_B T}{t^2/U}$$



Xue-Yang Song, Chao-Ming Jian, and L. Balents, PRL **119**, 216601 (2017);
Pengfei Zhang, PRB **96**, 205138 (2017); Debanjan Chowdhury, Yochai Werman,
Erez Berg, T. Senthil, PRX **8**, 031024 (2018); Aavishkar A. Patel, John McGreevy,
Daniel P. Arovas, Subir Sachdev, PRX **8**, 021049 (2018)

See also Antoine Georges and Olivier Parcollet PRB **59**, 5341 (1999)

A lattice SYK model

$$H = \frac{1}{(2N)^{3/2}} \sum_i \sum_{\alpha,\beta,\gamma,\delta=1}^{N} U_{i,\alpha\beta;\gamma\delta}\, c^\dagger_{i\alpha} c^\dagger_{i\beta} c_{i\gamma} c_{i\delta} - t \sum_{\langle ij \rangle} \sum_\alpha c^\dagger_{i\alpha} c_{j\alpha}$$

Choose U on-site,
and independent between sites;
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$$G(\mathbf{k},\omega) = G_{\rm SYK}(\epsilon, \hbar\omega/(k_B T))$$

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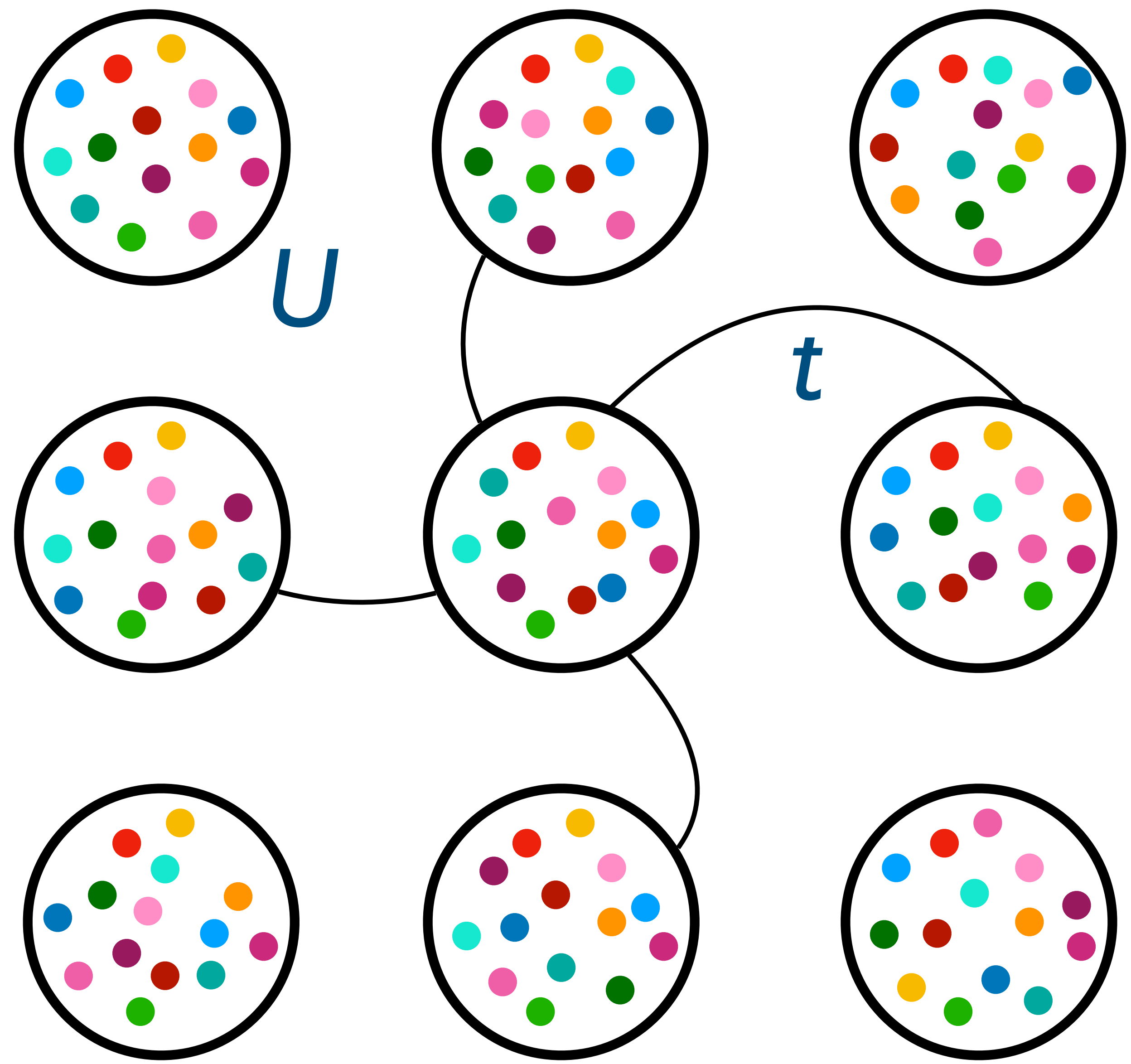

Xue-Yang Song, Chao-Ming Jian, and L. Balents, PRL **119**, 216601 (2017);
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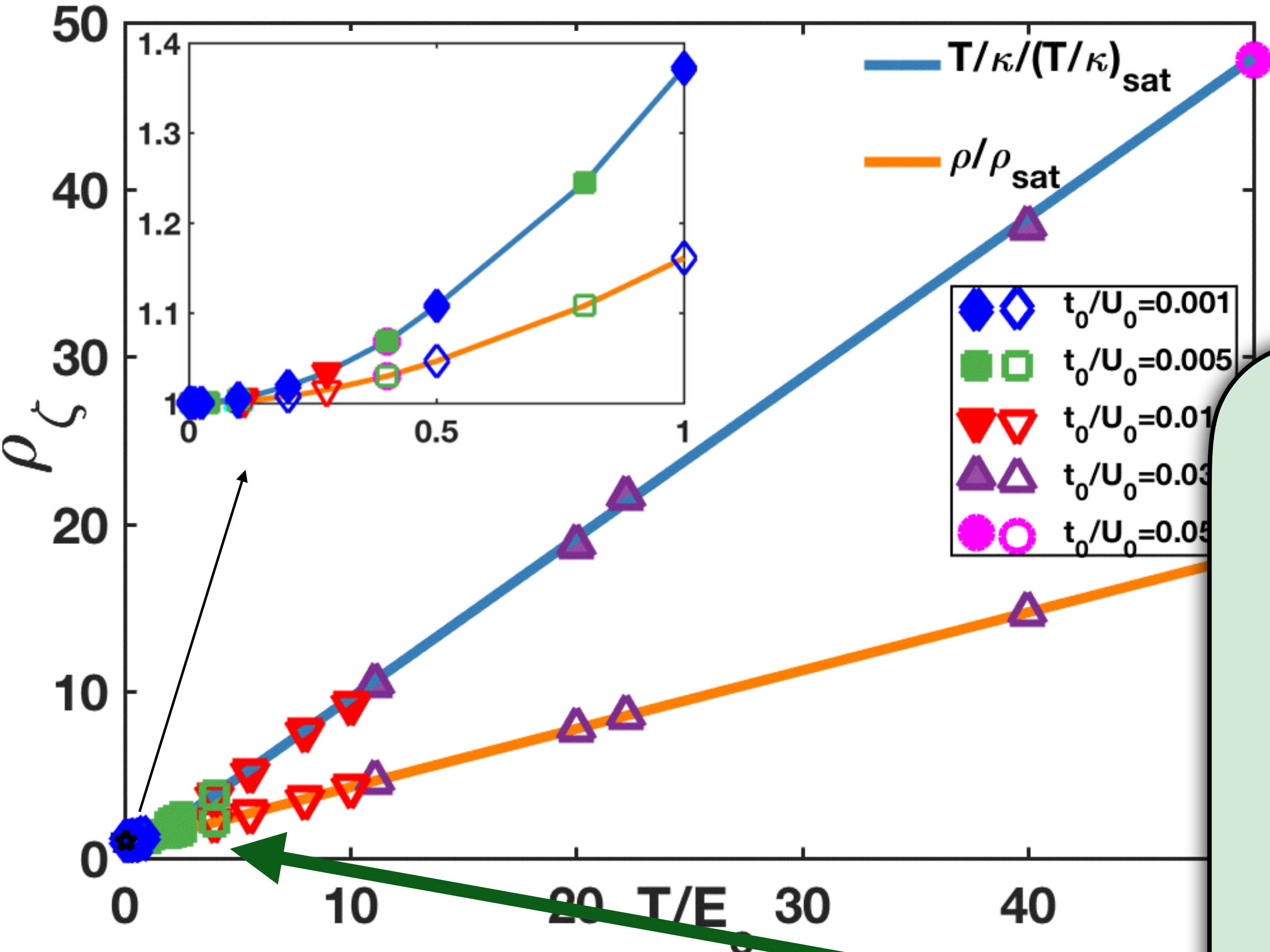
See also Antoine Georges and Olivier Parcollet PRB **59**, 5341 (1999)

Coupled SYK Islands

Low 'coherence' scale

$$E_c \sim \frac{t_0^2}{U}$$



For $T < E_c$, the resistivity, ρ, and entropy density, s, are

$$\rho = \frac{h}{e^2}\left[c_1 + c_2\left(\frac{T}{E_c}\right)^2\right]$$

$$s \sim s_0\left(\frac{T}{E_c}\right)$$

Xue-Yang Song, Chao-Ming Jian, and L. Balents, PRL **119**, 216601 (2017)

See also A. Georges and O. Parcollet PRB **59**, 5341 (1999)

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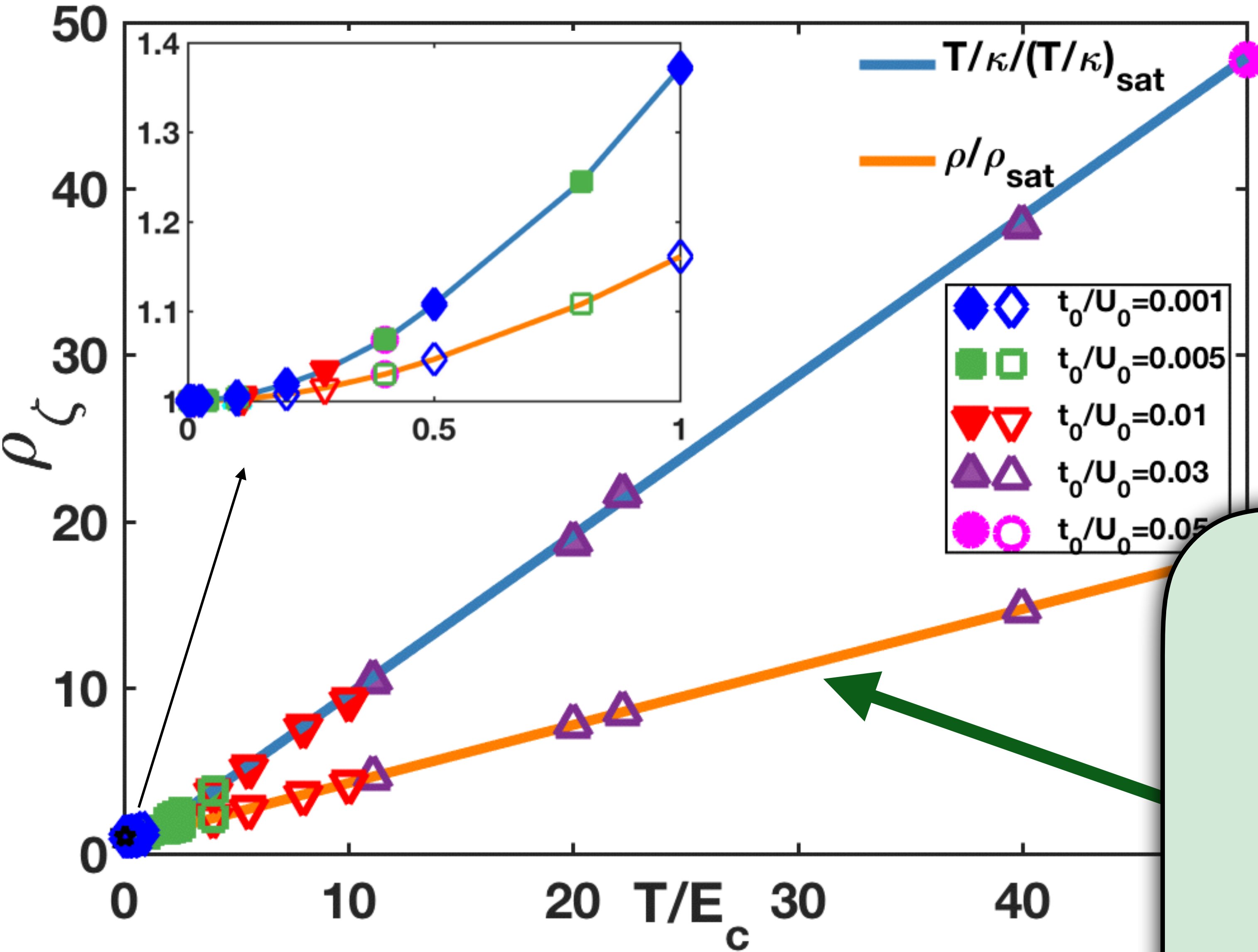


For $E_c < T < U$, the resistivity, ρ, and entropy density, s, are

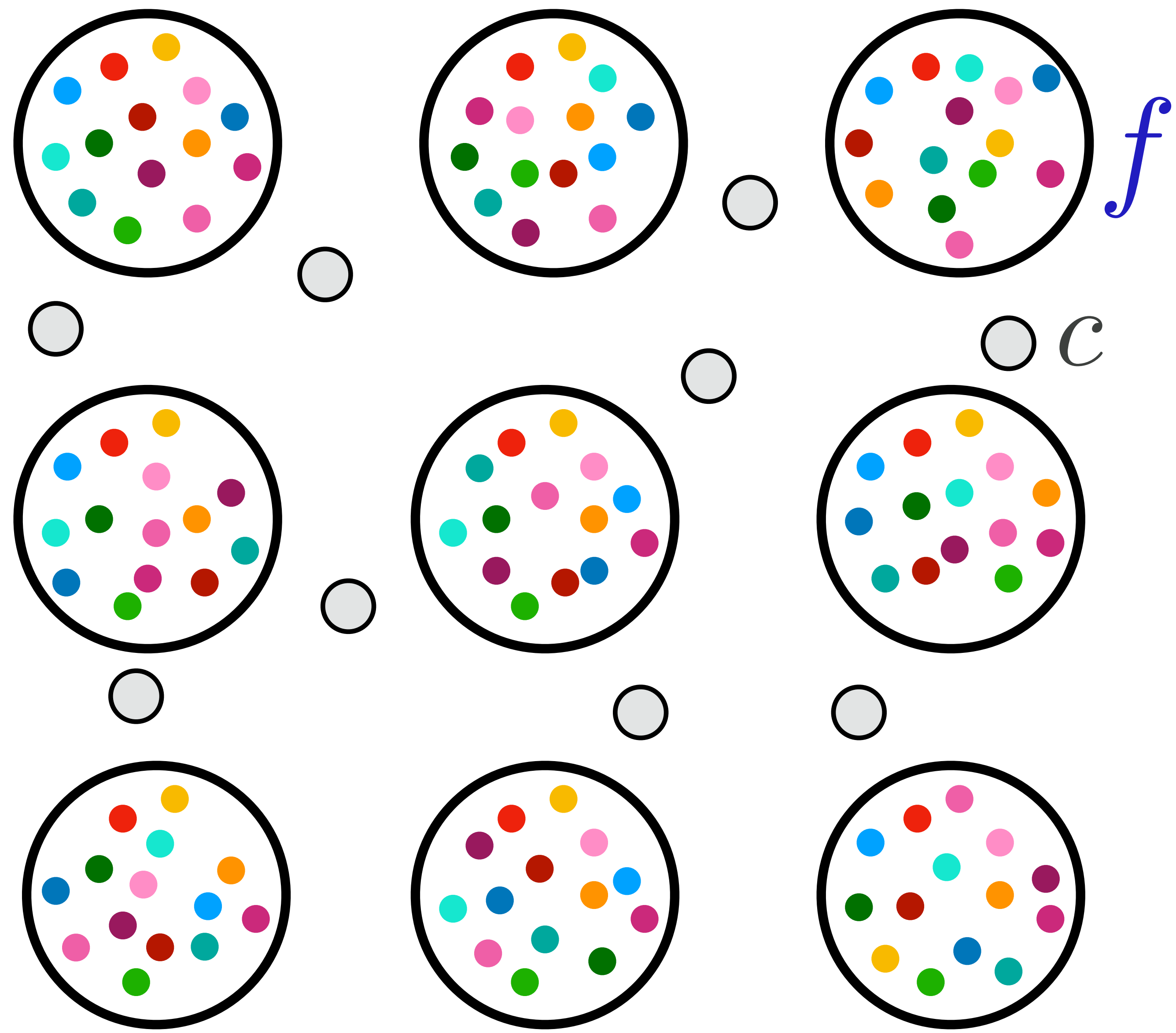
$$\rho \sim \frac{h}{e^2} \left(\frac{T}{E_c} \right) \quad , \quad s = s_0$$

Xue-Yang Song, Chao-Ming Jian, and L. Balents, PRL **119**, 216601 (2017)
See also A. Georges and O. Parcollet PRB **59**, 5341 (1999)

A Kondo-SYK model

Mobile electrons (c) coupled to SYK quantum islands (f) with exchange interactions.

Has a regime where the c electrons form a marginal Fermi liquid with a linear-in-T resistivity dependent upon interaction strength, and a small Fermi surface which does not count the f electrons.



Similar results for many earlier `marginal Fermi liquid' and holographic models

Debanjan Chowdhury, Yochai Werman, Erez Berg, T. Senthil, PRX **8**, 031024 (2018)
Aavishkar A. Patel, John McGreevy, Daniel P. Arovas, Subir Sachdev, PRX **8**, 021049 (2018)

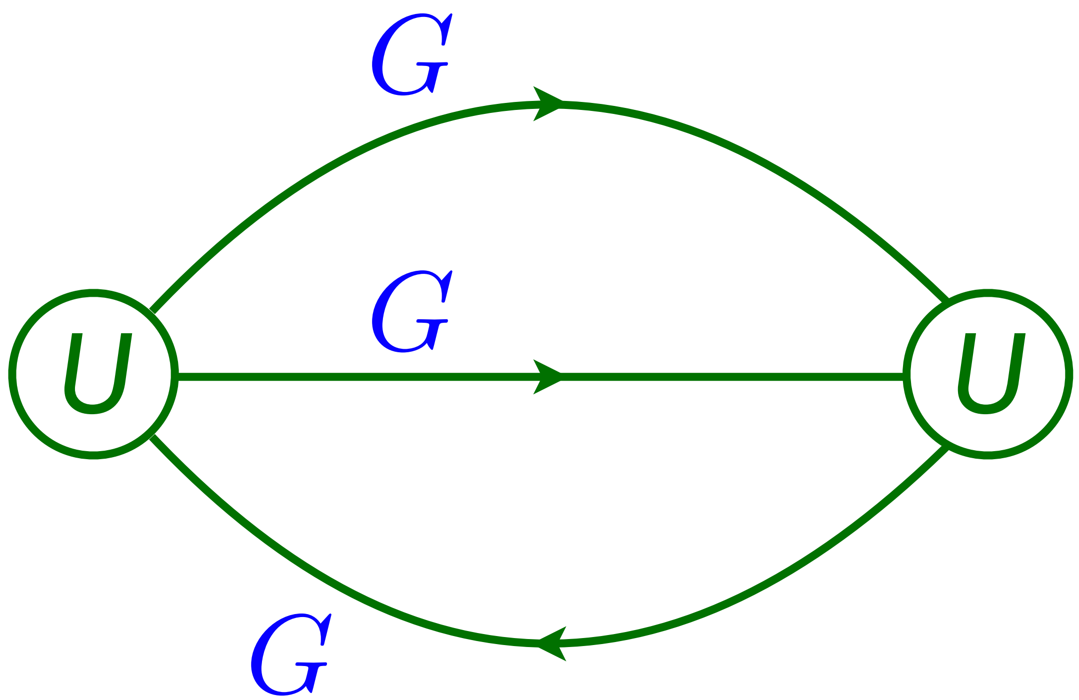
Generalized SYK models

$$H = \frac{1}{(2N)^{3/2}} \sum_{k_a} \sum_{\alpha,\beta,\gamma,\delta=1}^{N} U_{\alpha\beta;\gamma\delta}(k_a)\, c^\dagger_{k_1\alpha} c^\dagger_{k_2\beta} c_{k_3\gamma} c_{k_4\delta}$$
$$+ \sum_{k\alpha} \epsilon_k c^\dagger_{k\alpha} c_{k\alpha}$$

$U_{\alpha\beta;\gamma\delta}(k_a)$ is a random function of $\alpha\beta\gamma\delta$ (as before)
ϵ_k has a range of values of width W.

The large N limit is still given by the sum of "melon" diagrams.

$$G(k, i\omega) = \frac{1}{i\omega - \epsilon_k - \Sigma(k, i\omega)}$$

$\Sigma =$ 

Generalized SYK models

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$$+ \sum_{k\alpha} \epsilon_k c_{k\alpha}^\dagger c_{k\alpha}$$

$U_{\alpha\beta;\gamma\delta}(k_a)$ is a random function of $\alpha\beta\gamma\delta$ (as before)
ϵ_k has a range of values of width W.

The large N limit is still given by the sum of "melon" diagrams.

For many generic models in this class, $\hbar\omega/(k_B T)$ scaling of SYK holds for $W^2/U \ll k_B T \ll U$, and Fermi liquid theory is recovered for $k_B T \ll W^2/U$.

Xue-Yang Song, Chao-Ming Jian, and L. Balents, PRL **119**, 216601 (2017)
Pengfei Zhang, PRB **96**, 205138 (2017)
Debanjan Chowdhury, Yochai Werman, Erez Berg, T. Senthil, PRX **8**, 031024 (2018)
Aavishkar A. Patel, John McGreevy, Daniel P. Arovas, Subir Sachdev, PRX **8**, 021049 (2018)
See also Antoine Georges and Olivier Parcollet PRB **59**, 5341 (1999)

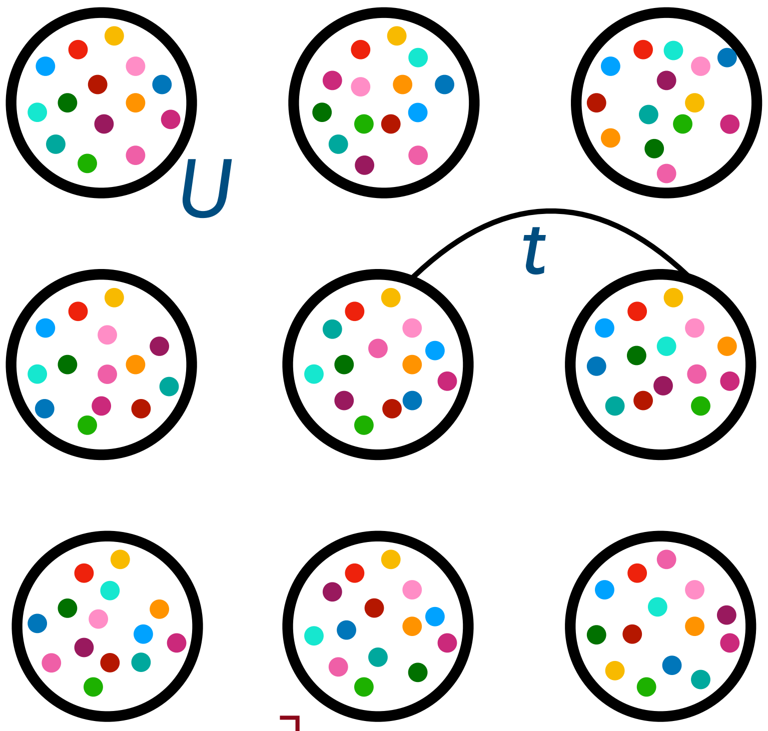
A lattice SYK model

$$H = \frac{1}{(2N)^{3/2}} \sum_{k_a} \sum_{\alpha,\beta,\gamma,\delta=1}^{N} U_{\alpha\beta;\gamma\delta}(k_a)\, c^\dagger_{k_1\alpha} c^\dagger_{k_2\beta} c_{k_3\gamma} c_{k_4\delta}$$

$$+ \sum_{k\alpha} \epsilon_k c^\dagger_{k\alpha} c_{k\alpha}$$

$U_{\alpha\beta;\gamma\delta}(k_a)$ is a random function of $\alpha\beta\gamma\delta$

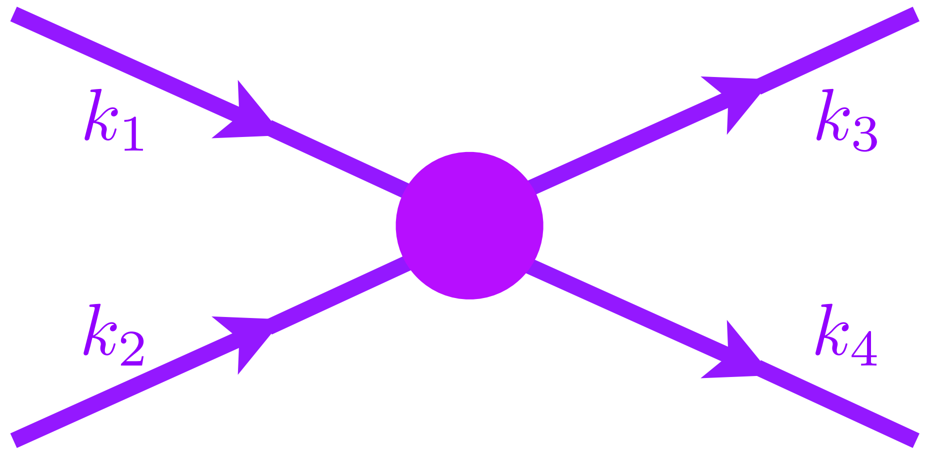
ϵ_k has a bandwidth W.



Rewriting of lattice model of incoherent and bad metal in momentum space

$$\overline{U(k_1,k_2,k_3,k_4)U^*(k_5,k_6,k_7,k_8)} =$$
$$U^2\left[\delta(k_1+k_2-k_3-k_4-k_5-k_6+k_7+k_8)\right]$$

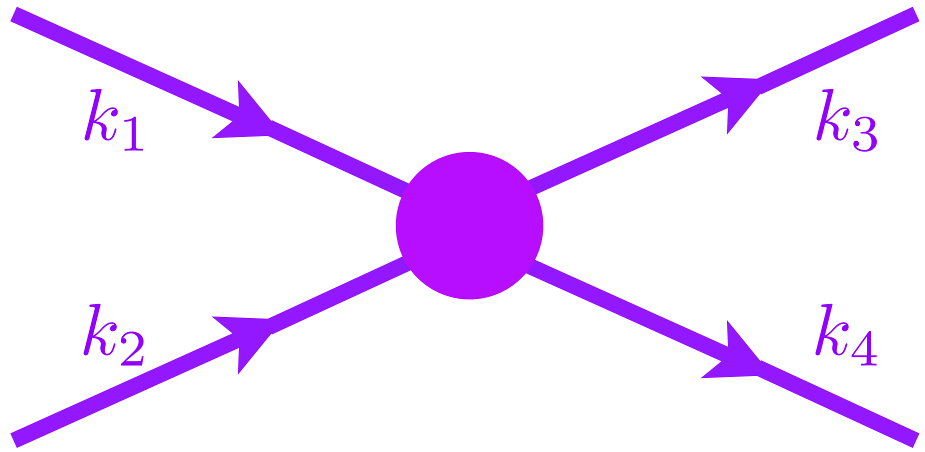
Resonant SYK model



Interactions with $\epsilon_{k_1} + \epsilon_{k_2} \neq \epsilon_{k_3} + \epsilon_{k_4}$ are non-resonant: we "integrate these out" in a RG procedure, and assume that their main effect is a renormalization of the quasiparticle dispersion ϵ_k, which we have already accounted for.

A. A. Patel and S. Sachdev, PRL **123**, 066601 (2019)

Resonant SYK model

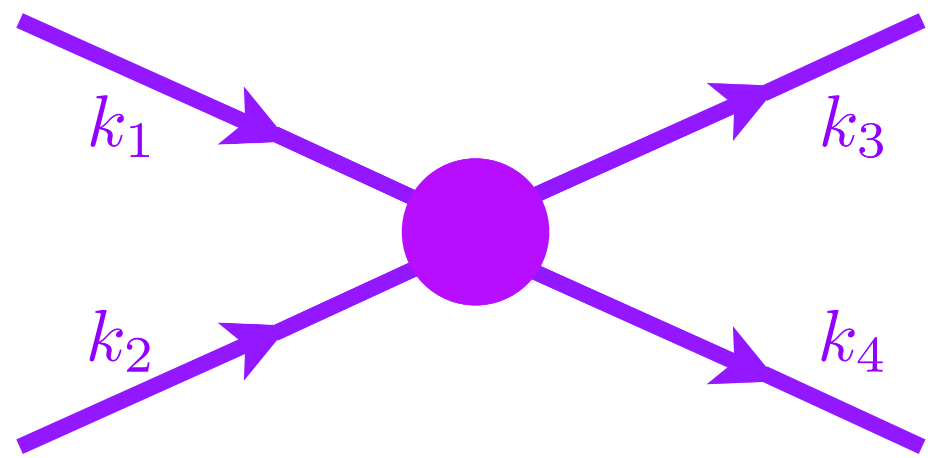


Interactions with $\epsilon_{k_1} + \epsilon_{k_2} \neq \epsilon_{k_3} + \epsilon_{k_4}$ are non-resonant: we "integrate these out" in a RG procedure, and assume that their main effect is a renormalization of the quasiparticle dispersion ϵ_k, which we have already accounted for.

Keep only the **interactions resonant in the bare quasiparticle energy** with $\epsilon_{k_1} + \epsilon_{k_2} = \epsilon_{k_3} + \epsilon_{k_4}$ and account for them with a self-consistent SYK-like analysis.

A. A. Patel and S. Sachdev, PRL **123**, 066601 (2019)

Resonant SYK model



Interactions with $\epsilon_{k_1} + \epsilon_{k_2} \neq \epsilon_{k_3} + \epsilon_{k_4}$ are non-resonant: we "integrate these out" in a RG procedure, and assume that their main effect is a renormalization of the quasiparticle dispersion ϵ_k, which we have already accounted for.

Keep only the **interactions resonant in the bare quasiparticle energy** with $\epsilon_{k_1} + \epsilon_{k_2} = \epsilon_{k_3} + \epsilon_{k_4}$ and account for them with a self-consistent SYK-like analysis.

This is precisely the effective Hamiltonian method, when low energy states are separated from high energy states by a gap; we are assuming it can also apply in a gapless system.

A. A. Patel and S. Sachdev, PRL **123**, 066601 (2019)

Resonant SYK model

$$H = \frac{1}{(2N)^{3/2}} \sum_{k_a} \sum_{\alpha,\beta,\gamma,\delta=1}^{N} U_{\alpha\beta;\gamma\delta}(k_a)\, c^\dagger_{k_1\alpha} c^\dagger_{k_2\beta} c_{k_3\gamma} c_{k_4\delta}$$

$$+ \sum_{k\alpha} \epsilon_k c^\dagger_{k\alpha} c_{k\alpha}$$

$U_{\alpha\beta;\gamma\delta}(k_a)$ is a random function of $\alpha\beta\gamma\delta$ (as before)
The random k_i dependence of U allows only
interactions resonant in the bare quasiparticle energies
with $\underline{\epsilon_{k_1} + \epsilon_{k_2} = \epsilon_{k_3} + \epsilon_{k_4}}$.

$$\overline{U(k_1, k_2, k_3, k_4) U^*(k_5, k_6, k_7, k_8)} =$$
$$U^2 \Big[\delta(k_1 + k_2 - k_3 - k_4 - k_5 - k_6 + k_7 + k_8)\Big]$$
$$\times \Big[\delta(\epsilon_{k_1} + \epsilon_{k_2} - \epsilon_{k_3} - \epsilon_{k_4}) + \delta(\epsilon_{k_5} + \epsilon_{k_6} - \epsilon_{k_7} - \epsilon_{k_8})\Big]$$

Aavishkar Patel

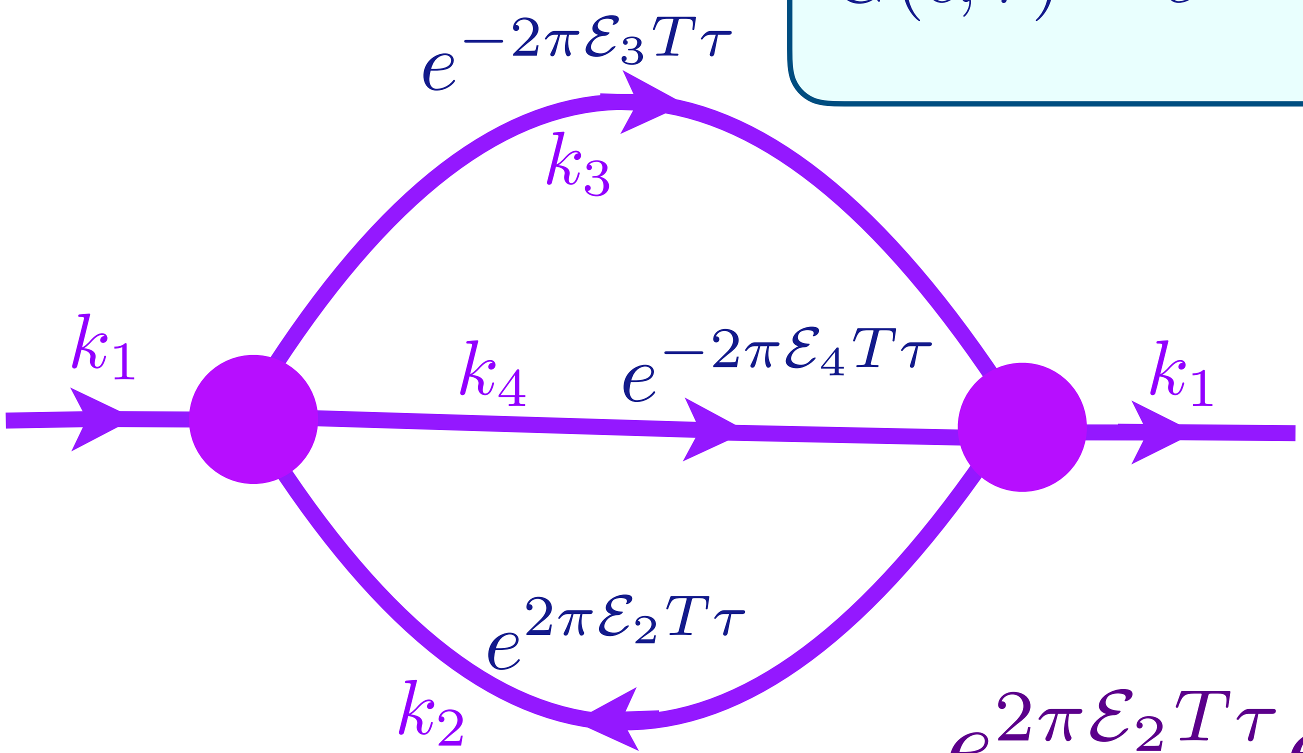
This implies off-site interactions with correlations
which decay with a power-law in space.

A. A. Patel and S. Sachdev, PRL **123**, 066601 (2019)

Resonant SYK model

Conformal Green's function at $T > 0$ must have the form

$$\boxed{G(\epsilon, \tau) \sim e^{-2\pi \mathcal{E} T \tau} \left(\frac{T}{\sin(\pi T \tau)} \right)^{1/2}} \quad , \quad 0 < \tau < 1/T \, .$$



$$e^{2\pi\mathcal{E}_2 T\tau} e^{-2\pi\mathcal{E}_3 T\tau} e^{-2\pi\mathcal{E}_4 T\tau} = e^{-2\pi\mathcal{E}_1 T\tau}$$

if

$$\mathcal{E}_a = \mathbb{C}\epsilon_a / U$$

and

$$\epsilon_1 + \epsilon_2 = \epsilon_3 + \epsilon_4$$

SYK behavior in a
Planckian metal as $T \to 0$
with a remnant Fermi surface:
$$G(k, \omega) = G_{\rm SYK}(\epsilon_k, \hbar\omega/(k_B T)),$$
with $\mathcal{E}_k = \mathbb{C}\,\epsilon_k / U$

A.A. Patel and S. Sachdev, PRL **123**, 066601 (2019)

Incoherent metal

For long times $\tau > 0$

$$\left\langle c_k(\tau) c_k^\dagger(0) \right\rangle = e^{\pi \mathcal{E}} \frac{A}{\sqrt{\tau}}$$

$$\left\langle c_k^\dagger(\tau) c_k(0) \right\rangle = e^{-\pi \mathcal{E}} \frac{A}{\sqrt{\tau}}$$

The parameter $\mathcal{E}$ is independent of k, and determined by the total density

Planckian metal with remnant Fermi surface

For long times $\tau > 0$

$$\left\langle c_k(\tau) c_k^\dagger(0) \right\rangle = e^{\pi \mathbb{C} \epsilon_k / U} \frac{A}{\sqrt{\tau}}$$

$$\left\langle c_k^\dagger(\tau) c_k(0) \right\rangle = e^{-\pi \mathbb{C} \epsilon_k / U} \frac{A}{\sqrt{\tau}}$$

The particle-hole asymmetry changes as we cross the Fermi surface

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A. A. Patel and S. Sachdev, PRL **123**, 066601 (2019)

The complex SYK model

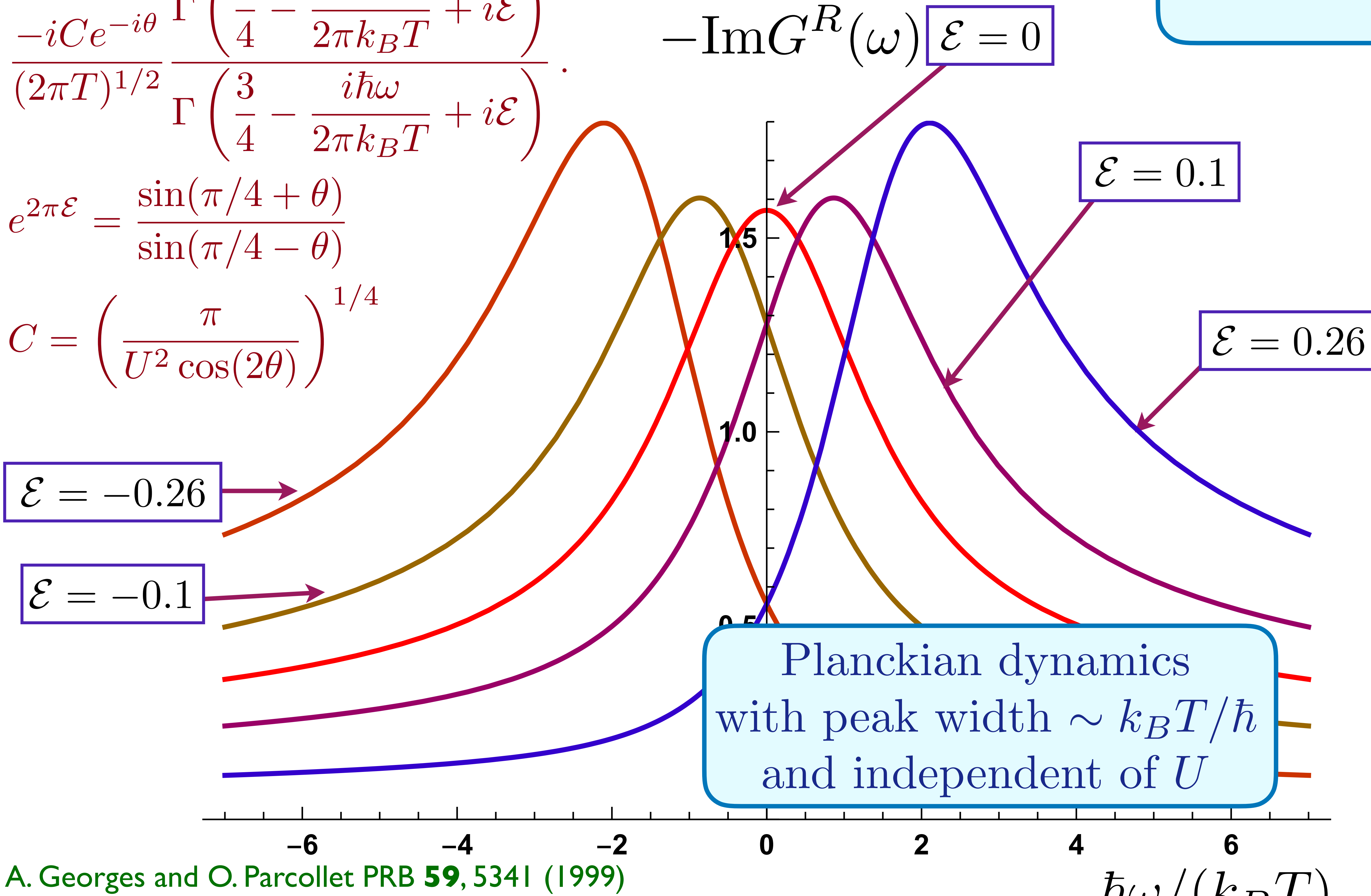
$$\mathcal{E} = \mathbb{C}\frac{\epsilon}{U}$$

$$G^R_{\rm SYK}(\epsilon, \hbar\omega/(k_B T)) =$$

$$\frac{-iCe^{-i\theta}}{(2\pi T)^{1/2}} \frac{\Gamma\left(\frac{1}{4} - \frac{i\hbar\omega}{2\pi k_B T} + i\mathcal{E}\right)}{\Gamma\left(\frac{3}{4} - \frac{i\hbar\omega}{2\pi k_B T} + i\mathcal{E}\right)} .$$

$$e^{2\pi\mathcal{E}} = \frac{\sin(\pi/4 + \theta)}{\sin(\pi/4 - \theta)}$$

$$C = \left(\frac{\pi}{U^2 \cos(2\theta)}\right)^{1/4}$$



Planckian dynamics with peak width $\sim k_B T/\hbar$ and independent of U

A. Georges and O. Parcollet PRB **59**, 5341 (1999)
S. Sachdev, PRX **5**, 041025 (2015)

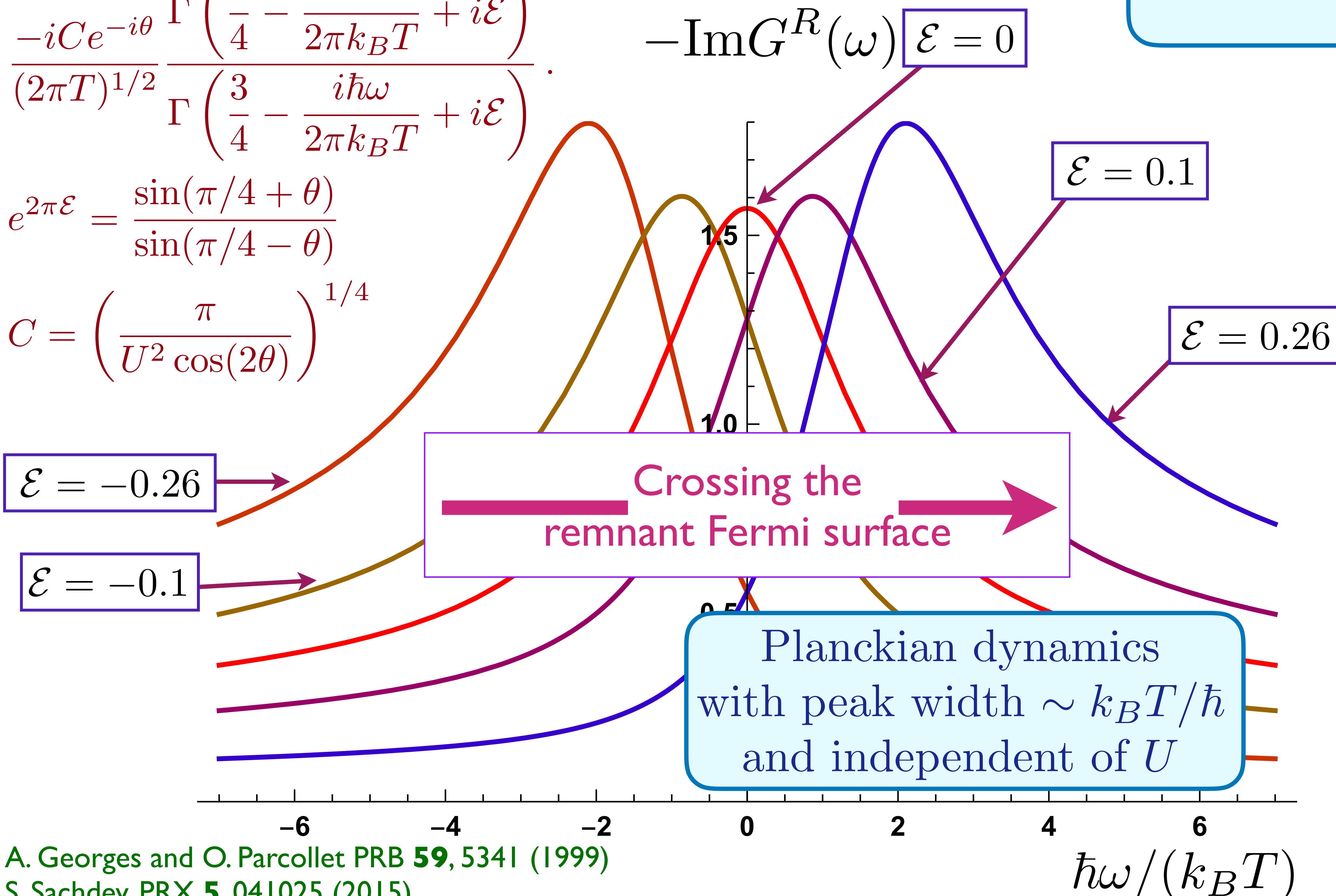
The complex SYK model

$$\mathcal{E} = \mathbb{C}\frac{\epsilon}{U}$$

$$G^R_{\rm SYK}(\epsilon, \hbar\omega/(k_BT)) =$$
$$\frac{-iCe^{-i\theta}}{(2\pi T)^{1/2}} \frac{\Gamma\left(\dfrac{1}{4} - \dfrac{i\hbar\omega}{2\pi k_B T} + i\mathcal{E}\right)}{\Gamma\left(\dfrac{3}{4} - \dfrac{i\hbar\omega}{2\pi k_B T} + i\mathcal{E}\right)}.$$

$$e^{2\pi\mathcal{E}} = \frac{\sin(\pi/4+\theta)}{\sin(\pi/4-\theta)}$$

$$C = \left(\frac{\pi}{U^2\cos(2\theta)}\right)^{1/4}$$



A. Georges and O. Parcollet PRB **59**, 5341 (1999)
S. Sachdev, PRX **5**, 041025 (2015)

Resonant SYK model

$U_{\alpha\beta;\gamma\delta}(k_a)$ is a random function of $\alpha\beta\gamma\delta$ (as before)

The random k_i dependence of U allows only

interactions resonant in the bare quasiparticle energies

with $\epsilon_{k_1} + \epsilon_{k_2} = \epsilon_{k_3} + \epsilon_{k_4}$.

Resistivity of a Planckian metal as $T \to 0$

From the Kubo formula, in the large N limit

$$\sigma = \frac{N e^2 m^* v_F^2}{2T} \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{4\pi} \left[\mathrm{Im}\, G^R_{\mathrm{SYK}} \left(\epsilon, \frac{\omega}{T} \right) \right]^2 \mathrm{sech}^2 \left(\frac{\omega}{2T} \right)$$

A. A. Patel and S. Sachdev, PRL **123**, 066601 (2019)

Resonant SYK model

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$$\boxed{\rho = \frac{m^*}{ne^2} \, 2.71 \mathbb{C} \, \frac{k_B T}{\hbar}} \quad , \quad \text{using } \mathcal{E} = \mathbb{C}\epsilon/U \,,$$

where

$$m^* = \frac{d\, V_{FS}}{\oint_{FS} |\boldsymbol{v}_F|} \,,$$

where d is spatial dimensionality and V_{FS} is the volume enclosed by the Fermi surface. For a circular Fermi surface, this is the usual m^*.

Resonant SYK model

Resistivity of a Planckian metal as $T \to 0$

$$\boxed{\rho = \frac{m^*}{ne^2}\, 2.71\mathbb{C}\, \frac{k_B T}{\hbar}}$$

Note that all explicit dependence on U has cancelled out!

The number $\mathbb{C}$ is defined by $\mathcal{E}_k = \mathbb{C}\,\epsilon_k/U$ as $|\epsilon_k| \to 0$. This is determined by UV physics, and is very weakly dependent upon the ratio of the energy width of the interactions, W_U, to U.

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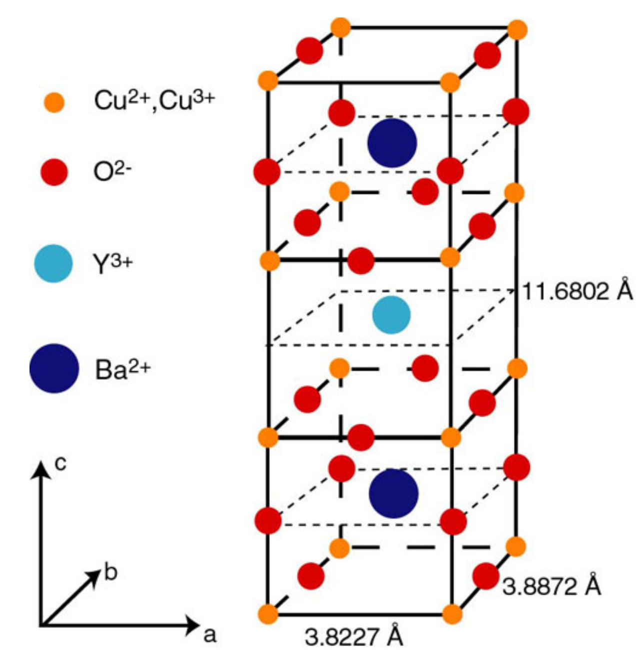


A. A. Patel and S. Sachdev, PRL **123**, 066601 (2019)

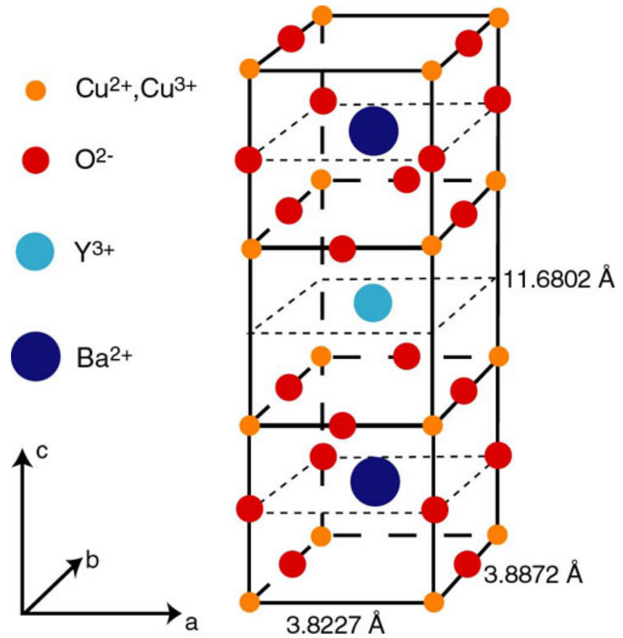
Resonant SYK model

Take the independent momentum shell limit, $W_U/U \to 0$,

$$\overline{U(k_1,k_2,k_3,k_4)U^*(k_5,k_6,k_7,k_8)} = U^2\Big[\delta(k_1+k_2-k_3-k_4-k_5-k_6+k_7+k_8)\Big]$$
$$\times\Big[\delta(\epsilon_{k_1}-\epsilon_{k_2})\delta(\epsilon_{k_2}-\epsilon_{k_3})\delta(\epsilon_{k_3}-\epsilon_{k_4})+\delta(\epsilon_{k_5}-\epsilon_{k_6})\delta(\epsilon_{k_6}-\epsilon_{k_7})\delta(\epsilon_{k_7}-\epsilon_{k_8})\Big]$$

$\mathbb{C} = 0.41$ as in a single SYK model,
and we obtain a Planckian metal with

$$\boxed{\rho = \frac{m^*}{ne^2}\,1.11\,\frac{k_B T}{\hbar}}$$



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A. A. Patel and S. Sachdev, PRL **123**, 066601 (2019)

Planckian metals with a remnant Fermi surface

- Resonant SYK models are compressible and dispersive quantum systems with $\hbar\omega/(k_B T)$ scaling as $T \to 0$.

- The resonance condition is supported by a RG argument: non-resonant interactions mainly renormalize the underlying quasi-particle dispersion ϵ_k, while resonant interactions have to be treated self-consistently.

- The resonance is a single 'fine-tuning' condition designed to obtain $\hbar\omega/(k_B T)$ scaling as $T \to 0$. However, then many other nice features follow: we obtain a Planckian metal with remnant large Fermi surface at $\epsilon_k = 0$, and an effective mass m^* defined by the dispersion of ϵ_k, with a resistivity $\rho \sim (m^*/(ne^2))k_B T/\hbar$ independent of the strength of interactions.

Aavishkar Patel (Harvard $\longrightarrow$ Miller Fellow at Berkeley)

A. A. Patel and S. Sachdev, PRL **123**, 066601 (2019)

1. Quantum matter with quasiparticles:
random matrix model

2. Quantum matter without quasiparticles:
the complex SYK model

3. Fluctuations, and the Schwarzian

S. Sachdev, Phys. Rev. Lett. **105**, 151602 (2010)
S. Sachdev, PRX **5**, 041025 (2015)

4. Models of strange metals

5. Einstein-Maxwell theory of charged
black holes in AdS space

Quantum Black holes

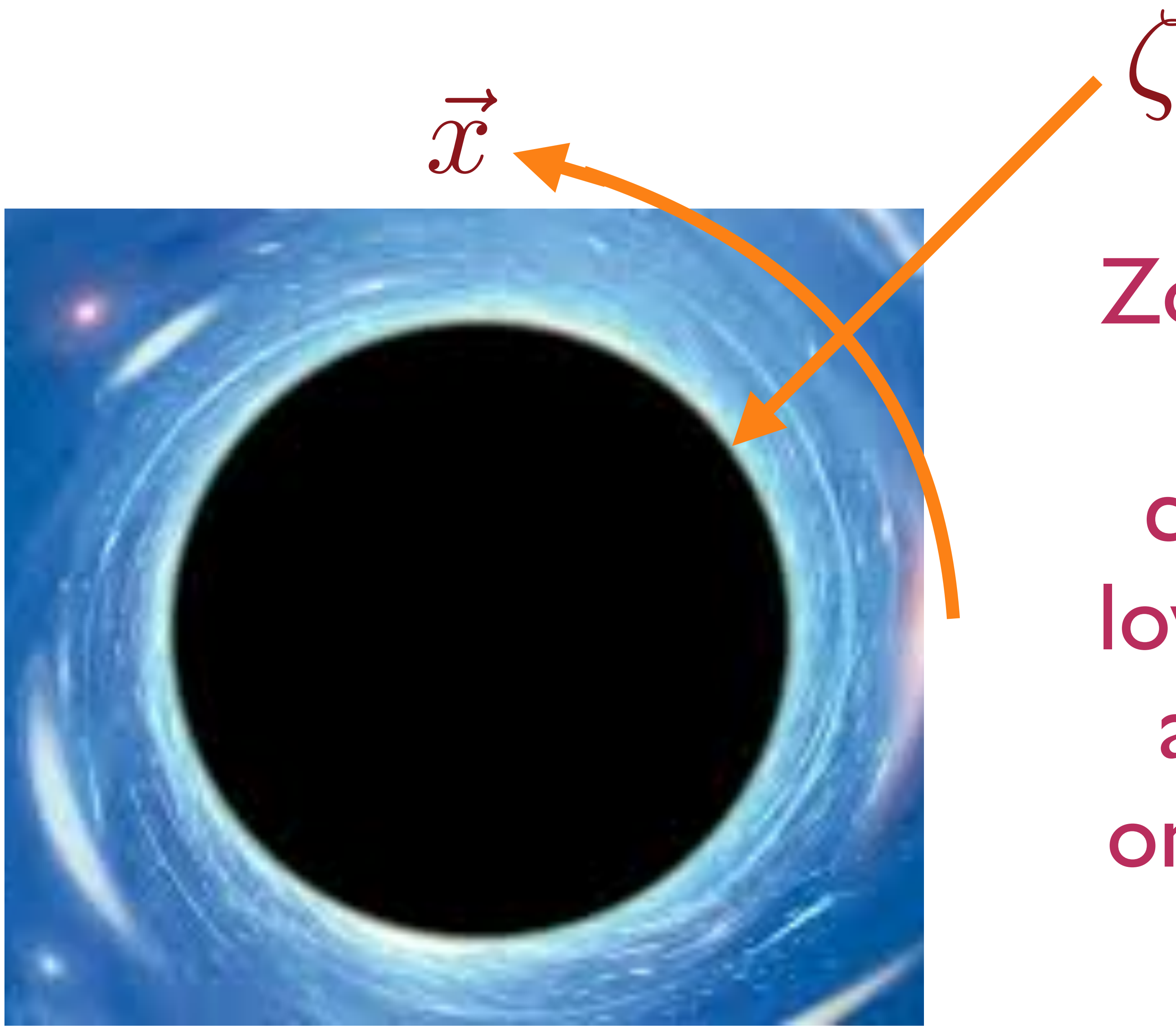
- Black holes have an entropy and a temperature, T_H
- The entropy is proportional to their surface area.
- They relax to thermal equilibrium in a Planckian time $\sim \hbar/(k_B T_H)$.

Holography:
Quantum black holes "look like" quantum many-particle systems without quasiparticle excitations, residing "on" the surface of the black hole

J. Maldacena, IJTP **38**, 1113 (1999); S.S. Gubser, I.R. Klebanov, and A.M. Polyakov Phys. Lett. B **428**, 105 (1998); E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998)

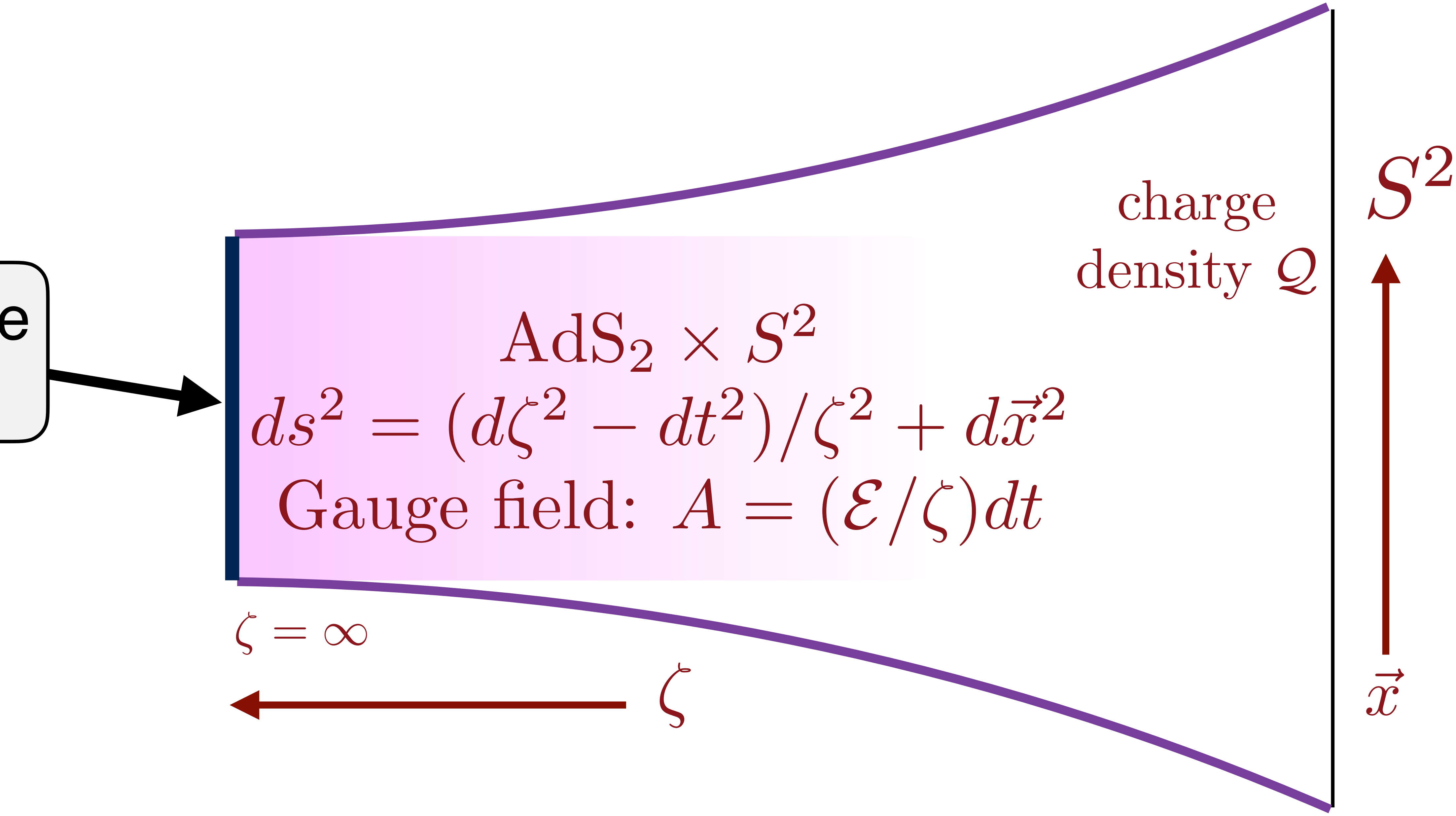
Work with a theory of Maxwell's electromagnetism and Einstein's general relativity. Include a negative cosmological constant, and examine black hole solutions with a net charge

Work with a theory of Maxwell's electromagnetism and Einstein's general relativity. Include a negative cosmological constant, and examine black hole solutions with a net charge



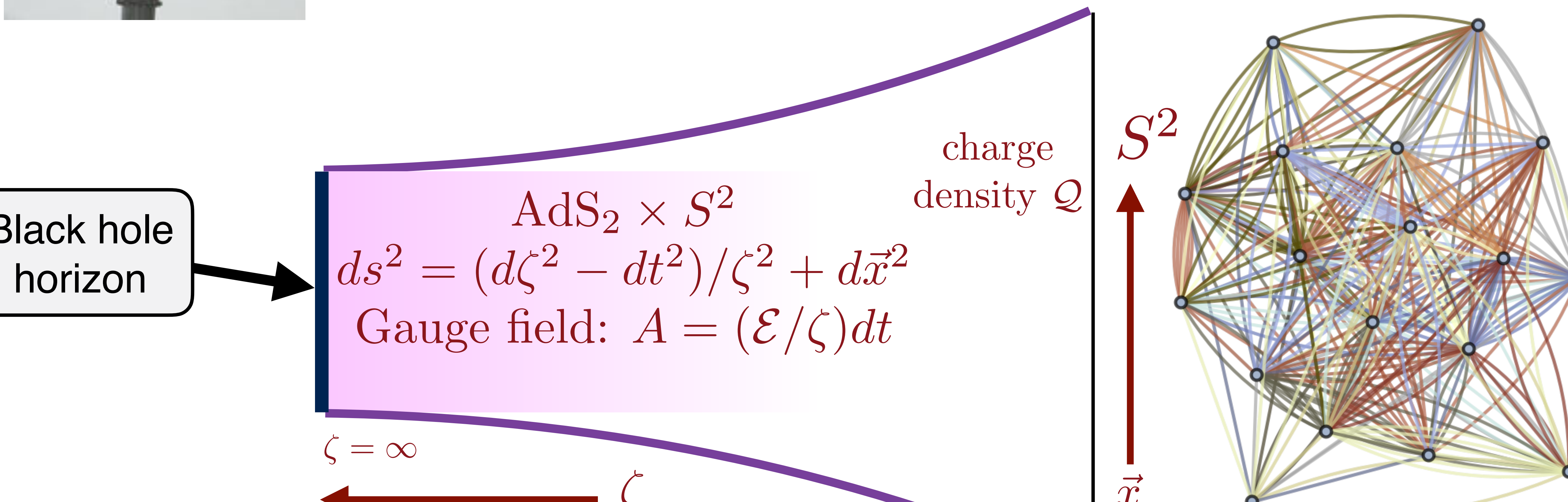
Zooming into the near-horizon region of a charged black hole at low temperature, yields a quantum theory in one space (ζ) and one time dimension

SYK model and charged black holes



The near-horizon region of a charged black hole has the geometry of (1+1)-dimensional anti-de Sitter spacetime. By holography, this should map to a zero-dimensional quantum system: this turns out to be the SYK model

SYK model and charged black holes



Bekenstein-Hawking entropy of AdS_2 horizon at $T = 0 \Leftrightarrow N s_0$ entropy of SYK model.

$\dfrac{\partial s_0}{\partial \mathcal{Q}} = 2\pi\mathcal{E}$ holds for both the black hole and the SYK model, where $\mathcal{E}$ determines identical fermion spectral functions.

S. Sachdev, Phys. Rev. Lett. **105**, 151602 (2010); PRX **5,** 041025 (2015)

Charged black holes

Probe fermion in the AdS_2 near horizon

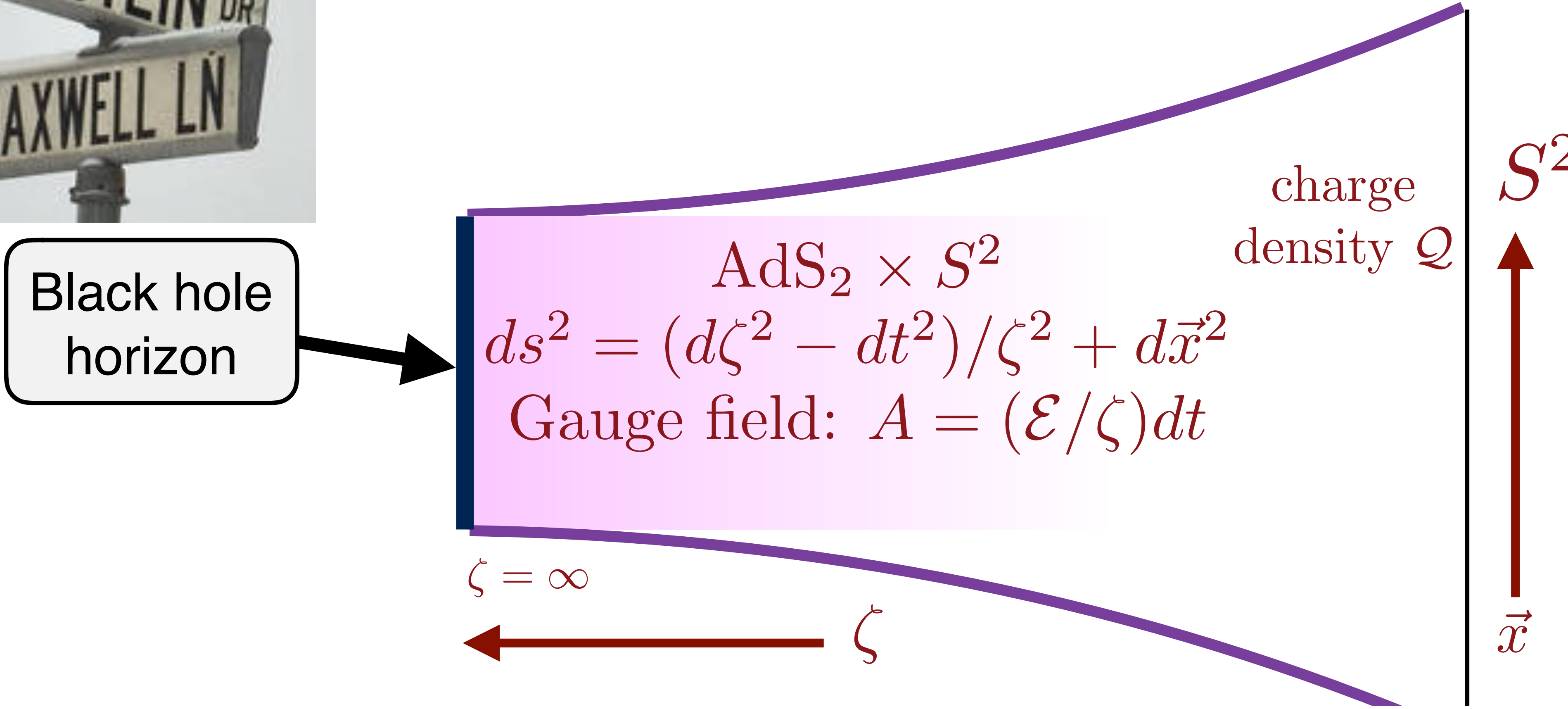
- A probe fermion has a near-horizon Green's function with a conformal structure

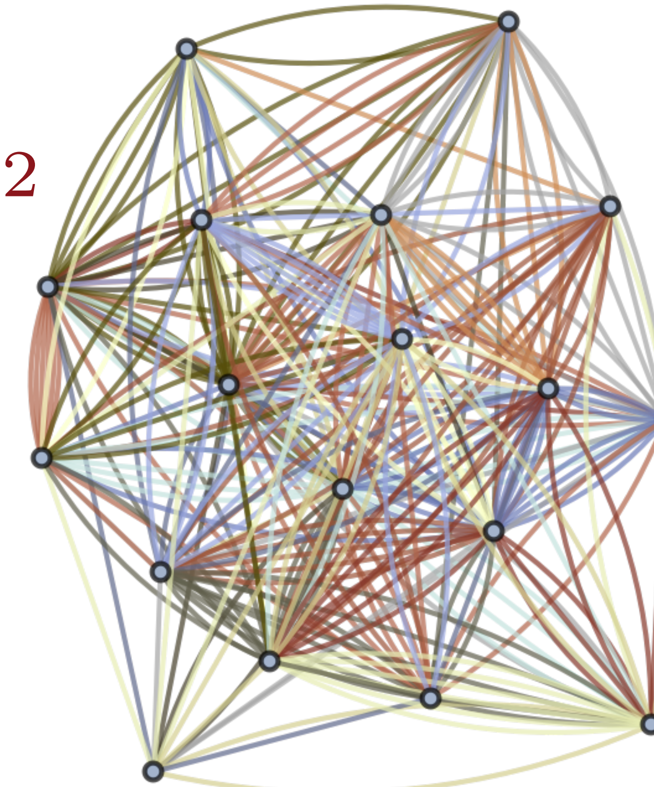
$$G(\tau) = -A \frac{e^{-2\pi \mathcal{E} T \tau}}{\sqrt{1+e^{-4\pi\mathcal{E}}}} \left(\frac{T}{\sin(\pi T \tau)} \right)^{1/2} \quad , \quad 0 < \tau < 1/T \,,$$

where the 'particle-hole asymmetry' is determined by $\mathcal{E}$. This is identical to the complex SYK model.

T. Faulkner, Hong Liu, J. McGreevy, D. Vegh, PRD **83**, 125002 (2011)

SYK model and charged black holes





PHYSICAL REVIEW LETTERS

105, 151602 (2010)

Holographic Metals and the Fractionalized Fermi Liquid

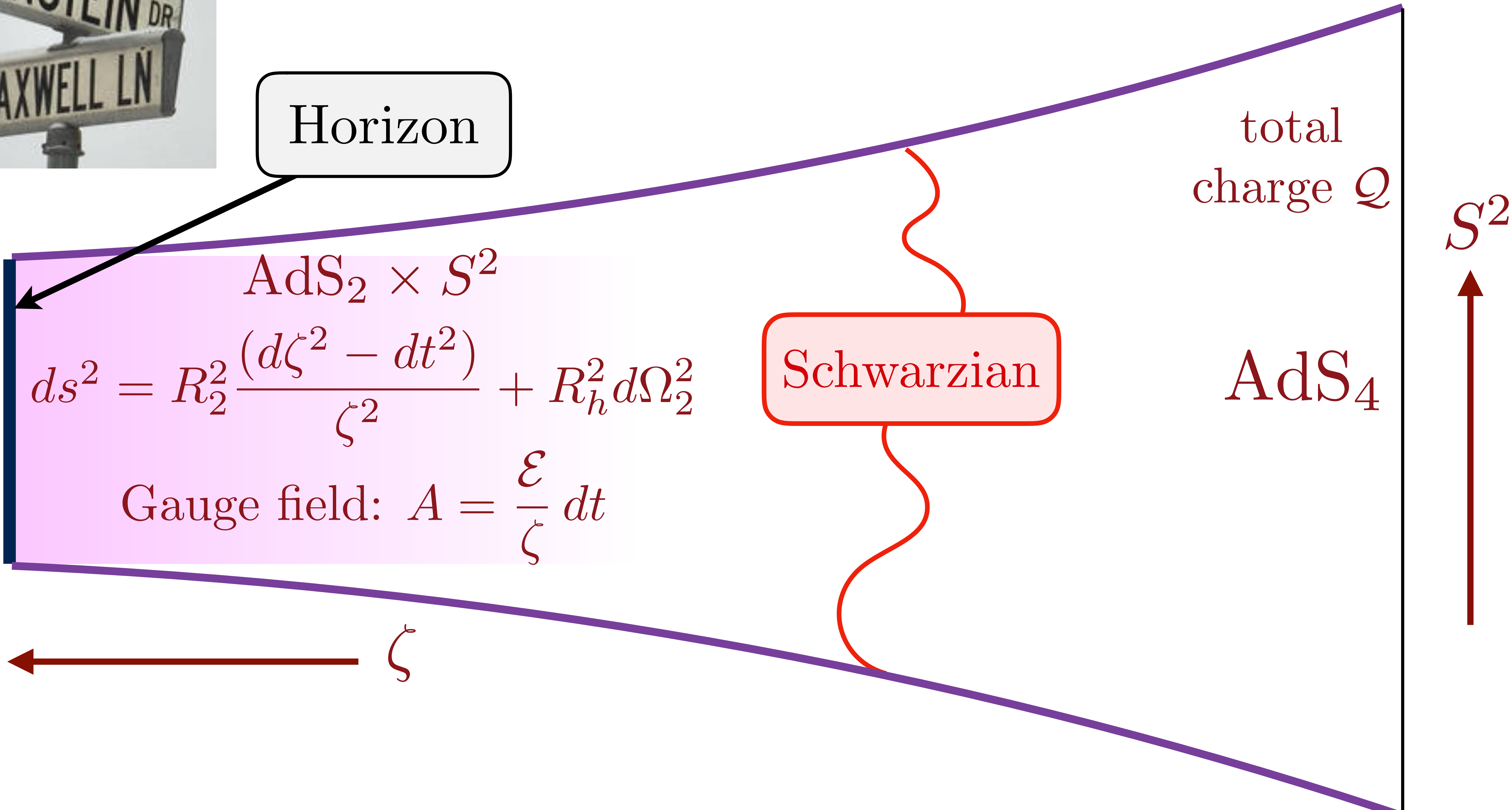
Subir Sachdev

Department of Physics, Harvard University, Cambridge, Massachusetts 02138, USA

(Received 23 June 2010; published 4 October 2010)

We show that there is a close correspondence between the physical properties of holographic metals near charged black holes in anti–de Sitter (AdS) space, and the fractionalized Fermi liquid phase of the lattice Anderson model. The latter phase has a "small" Fermi surface of conduction electrons, along with a spin liquid of local moments. This correspondence implies that certain mean-field gapless spin liquids are states of matter at nonzero density realizing the near-horizon, $\mathrm{AdS}_2 \times \mathrm{R}^2$ physics of Reissner-Nordström black holes.

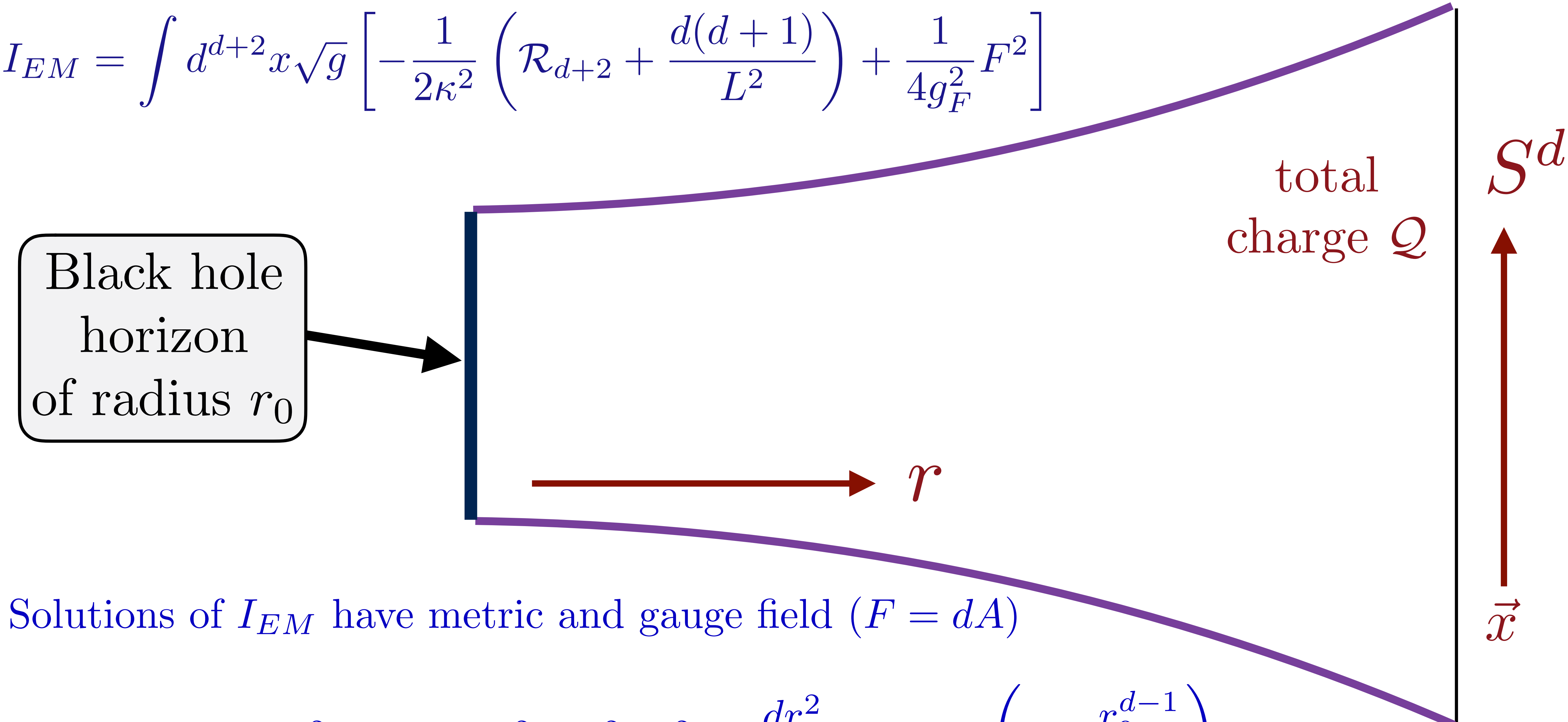
SYK model and charged black holes



Remarkably, the correspondence between charged black holes and the SYK model also holds for the leading fluctuations at higher temperatures: both are described by a 'Schwarzian' theory with emergent SL(2,R) and U(1) gauge symmetries. For the black hole, the Schwarzian describes the fluctuations of the boundary between AdS_2 and AdS_4.

Charged black holes

$$I_{EM} = \int d^{d+2}x\sqrt{g}\left[-\frac{1}{2\kappa^2}\left(\mathcal{R}_{d+2} + \frac{d(d+1)}{L^2}\right) + \frac{1}{4g_F^2}F^2\right]$$



Solutions of I_{EM} have metric and gauge field ($F = dA$)

$$ds^2 = V(r)d\tau^2 + r^2 d\Omega_d^2 + \frac{dr^2}{V(r)} \quad , \quad i\mu\left(1 - \frac{r_0^{d-1}}{r^{d-1}}\right)d\tau$$

$$V(r) = 1 + \frac{r^2}{L^2} + \frac{\Theta^2}{r^{2d-2}} - \frac{M}{r^{d-1}}.$$

where $d\Omega_d^2$ is the metric of the d-sphere. All parameters of the solution are determined in terms of the chemical potential μ, and the Hawking temperature of horizon, T.

A. Chamblin, R. Emparan, C.V. Johnson, and R.C. Myers, PRD **60**, 064018 (1999)

Charged black holes

In the $T \to 0$ limit, at fixed μ, we obtain a charged black hole solution with radius $r_0(T \to 0, \mu) = R_h$. All properties of this black hole can be expressed in terms of R_h

- The total charge in the black hole is

$$\mathcal{Q} = \frac{R_h^{d-1}\sqrt{2d\left[(d+1)R_h^2 + (d-1)L^2\right]}}{\kappa^2 g_F}$$

A. Chamblin, R. Emparan, C.V. Johnson, and R.C. Myers, PRD **60**, 064018 (1999)

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$$\mathcal{Q} = \frac{R_h^{d-1}\sqrt{2d\left[(d+1)R_h^2 + (d-1)L^2\right]}}{\kappa^2 g_F}$$

- The Bekenstein-Hawking entropy remains finite as $T \to 0$ (s_d is the area of the d-dimensional surface of a unit sphere)

$$S(T \to 0) = s_0 + \ldots \quad , \quad s_0 = \frac{2\pi s_d}{\kappa^2} R_h^d$$

A. Chamblin, R. Emparan, C.V. Johnson, and R.C. Myers, PRD **60**, 064018 (1999)

Charged black holes

In the $T \to 0$ limit, at fixed μ, we obtain a charged black hole solution with radius $r_0(T \to 0, \mu) = R_h$. All properties of this black hole can be expressed in terms of R_h

- In the near-horizon region, we change co-ordinates from r to ζ so that

$$r - R_h = \frac{R_2^2}{\zeta} \quad , \quad R_2 = \frac{L R_h}{\sqrt{d(d+1)R_h^2 + (d-1)^2 L^2}}.$$

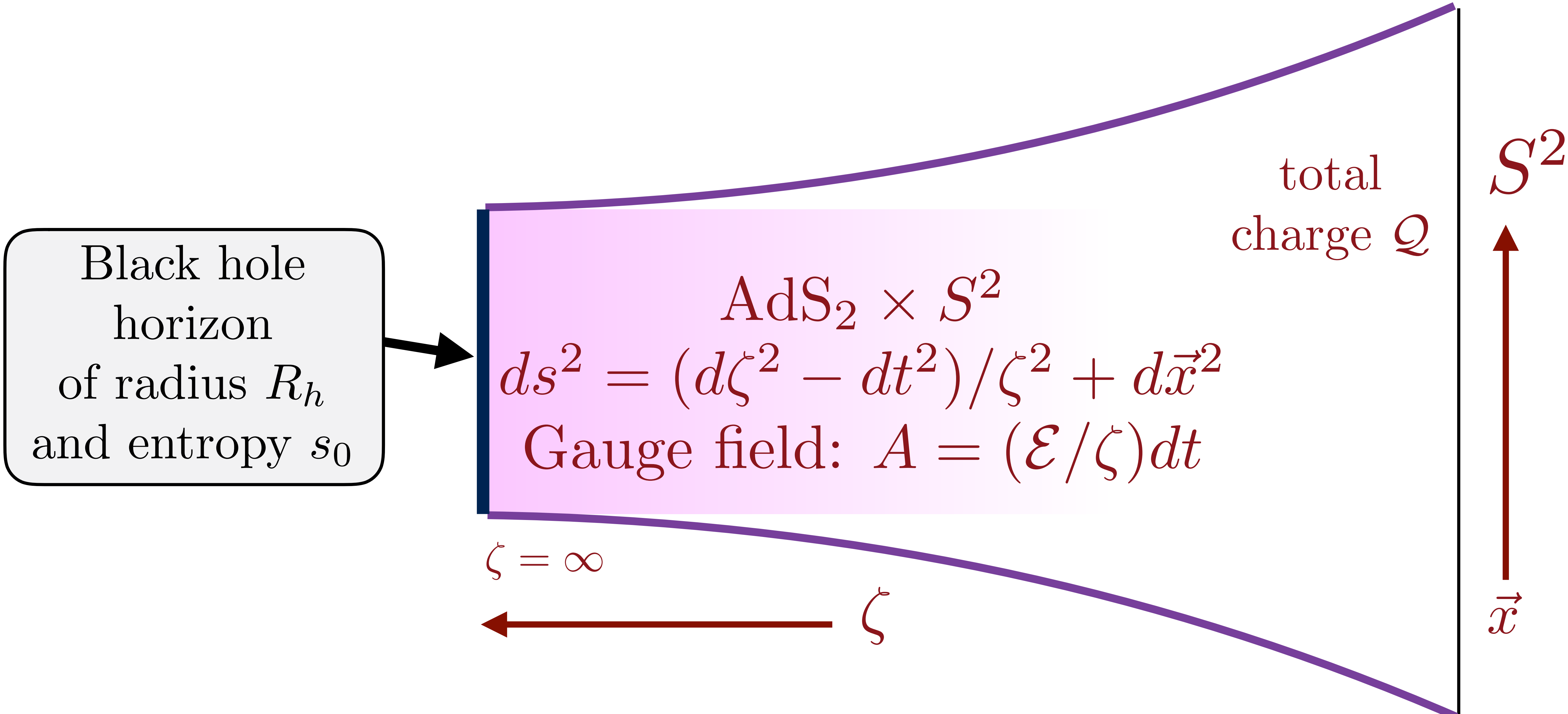
Then the near-horizon metric becomes $\mathrm{AdS}_2 \times \mathrm{S}_d$, with

$$ds^2 = R_2^2 \left[\frac{-dt^2 + d\zeta^2}{\zeta^2} \right] + R_h^2 d\Omega_d^2 \quad , \quad A = \frac{\mathcal{E}}{\zeta} dt \, .$$

where the dimensionless electric field $\mathcal{E}$ is

$$\mathcal{E} = \frac{g_F R_h \sqrt{2d\left[(d+1)R_h^2 + (d-1)L^2\right]}}{2\left[d(d+1)R_h^2 + (d-1)^2 L^2\right]}.$$

Charged black holes



- The entropy s_0, the charge $\mathcal{Q}$, and the dimensionless electric field $\mathcal{E}$ obey

$$\boxed{\frac{ds_0}{d\mathcal{Q}} = 2\pi\mathcal{E}}$$

A. Sen, JHEP **0509**, 038 (2005)

The Schwarzian theory and black holes

- Reparameterization invariance is a defining property of Einstein's theory of gravity

- In imaginary time, AdS_2 is the homogeneous hyperbolic space: two-dimensional surface of constant negative curvature. Its metric is invariant under SL(2,R)

$ds^2 = (d\tau^2 + d\zeta^2)/\zeta^2$ is invariant under

$$\tau' + i\zeta' = \frac{a(\tau + i\zeta) + b}{c(\tau + i\zeta) + d} \text{ with } ad - bc = 1.$$

A. Kitaev, 2015
J. Maldacena, D. Stanford, and Zhenbin Yang, arXiv:1606.01857

The Schwarzian theory and black holes

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- In imaginary time, AdS_2 is the homogeneous hyperbolic space: two-dimensional surface of constant negative curvature. Its metric is invariant under SL(2,R)

Semiclassical fluctuations about the saddle-point of Einstein-Maxwell theory of a charged black holes in $d \geq 2$ spatial dimensions lead to the same Schwarzian+phase theory of fluctuations.

P. Nayak, A. Shukla, R.M. Soni, S.P. Trivedi, and V. Vishal, arXiv:1802.09547
U. Moitra, S. P. Trivedi, and V. Vishal, arXiv:1808.08239
P. Chaturvedi, Yingfei Gu, Wei Song, Boyang Yu, arXiv:1808.08062
A. Gaikwad, L.K. Joshi, G. Mandal, and S.R. Wadia, arXiv:1802.07746

Charged black holes

We write the $(d+2)$-dimensional metric g of I_{EM} in terms of a two-dimensional metric h and a scalar field Φ:

$$ds^2 = \frac{ds_2^2}{\Phi^{d-1}} + \Phi^2 \, d\Omega_d^2 \,.$$

The Einstein-Maxwell and Gibbons-Hawking actions reduce to and extension of Jackiw-Tietelbaum gravity $(x \equiv (\tau, \zeta))$

$$\begin{aligned} I_{EM} &= \int d^2x \sqrt{h} \left[-\frac{s_d}{2\kappa^2} \, \Phi^d \, \mathcal{R}_2 + U(\Phi) + \frac{Z(\Phi)}{4g_F^2} F^2 \right] \\ I_{GH} &= -\frac{s_d}{\kappa^2} \int_\partial dx \sqrt{h_b} \Phi^d \, \mathcal{K}_1 \end{aligned}$$

The explicit forms of the potentials $U(\Phi)$ and $Z(\Phi)$ are,

$$U(\Phi) = -\frac{s_d}{2\kappa^2} \left(\frac{d(d-1)}{\Phi} + \frac{d(d+1)\Phi}{L^2} \right) \quad , \quad Z(\Phi) = s_d \Phi^{2d-1} \,.$$

Charged black holes

The exact saddle point of Φ relates to R_h the horizon radius at $T = 0$

$$\Phi(\zeta) = R_h + \frac{R_2^2}{\zeta} \quad , \quad R_h \equiv \frac{L}{g_F}\left[\frac{(d-1)(\mu_0^2\kappa^2(d-1) - d g_F^2)}{d(d+1)}\right]^{1/2} ,$$

while the near-horizon, low $T \ll 1/R_h$ metric is AdS_2

$$ds_2^2 = \frac{R_2^2 R_h^{d-1}}{\zeta^2}\left[(1 - 4\pi^2 T^2 \zeta^2) d\tau^2 + \frac{d\zeta^2}{1 - 4\pi^2 T^2 \zeta^2}\right] ,$$

where

$$R_2 = \frac{L R_h}{\sqrt{d(d+1) R_h^2 + (d-1)^2 L^2}}$$

The field coupling to $\mathcal{R}_2$ is Φ^d

$$[\Phi(\zeta)]^d = R_h^d + \frac{\Phi_1}{\zeta} + \ldots \quad , \quad \Phi_1 = d R_h^{d-1} R_2^2,$$

S. Sachdev, arXiv:1902.04078

Charged black holes

The field coupling to $\mathcal{R}_2$ is Φ^d

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We choose the boundary of the AdS_2 region at bulk co-ordinates $(f(\tau), \zeta(\tau))$ with the induced boundary metric fixed at $(R_2^2 R_h^{d-1}/\zeta_b^2) d\tau^2$ by choosing

$$\zeta(\tau) = \zeta_b f'(\tau) + \zeta_b^3 \left(\frac{[f''(\tau)]^2}{2 f'(\tau)} - 2\pi^2 T^2 \left[f'(\tau) \right]^3 \right) + \ldots .$$

Finally, we evaluate I_{GH} along this boundary curve

$$I_1[f] = -\frac{\gamma}{4\pi^2} \int_0^{1/T} d\tau \left\{ \tan(\pi T f(\tau)), \tau \right\} ,$$

where

$$\gamma = \frac{4\pi^2 s_d \Phi_1}{\kappa^2} ,$$

matches the linear-in-T co-efficient of the specific heat of the full Reissner-Nördstorm solution in $d+2$ dimensions.

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The Schwarzian theory and black holes

- The Einstein-Maxwell theory leads to the following parameters for the Schwarzian+phase theory

$$K = \frac{d\mathcal{Q}}{d\mu}\bigg|_{T=0} = \frac{2(d-1)L^2 s_d R_h^{d-3}\left[d(d+1)R_h^2 + (d-1)^2L^2\right]}{(d+1)g_F^2\kappa^2}$$

$$S(T \to 0, \mathcal{Q}) = s_0 + \gamma T + \ldots$$

$$\gamma = \frac{4\pi^2 d s_d L^2 R_h^{d+1}}{\kappa^2(d(d+1)R_h^2 + (d-1)^2L^2)}\,.$$

Quantum matter without quasiparticles

- Planckian dynamics is realized in the 'solvable' SYK models
- Black holes thermalize in a time $\sim \hbar/(k_B T_H)$, where T_H is the Hawking temperature.
- A Schwarzian theory of a time reparameterization mode, with SL(2,R) symmetry, describes the quantum dynamics of
  - the SYK models
  - black holes with near-extremal AdS_2 horizons